

DEPOSITION, DIFFUSION, AND NUCLEATION ON AN INTERVAL

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Motivated by nanoscale growth of ultra-thin films, we study a model of deposition, on an interval substrate, of particles that perform Brownian motions until any two meet, when they nucleate to form a static island, which acts as an absorbing barrier to subsequent particles. This is a continuum version of a lattice model studied in the applied literature. We show that the associated interval-splitting process converges in the sparse deposition limit to a Markovian process (in the vein of Brennan and Durrett) governed by a splitting density with a compact Fourier series expansion but, apparently, no simple closed form. We show that the same splitting density governs the fixed deposition rate, large time asymptotics of the normalized gap distribution, so these asymptotics are independent of deposition rate. The splitting density is derived by solving an exit problem for planar Brownian motion from a right-angled triangle, extending work of Smith and Watson.

1. Introduction. Surface phenomena are important in chemistry, physics, and materials science. Our probabilistic model originates with the growth of ultra-thin films. The nonequilibrium dynamics of these self-organized growth processes are of central importance in understanding the construction of nanomaterials by deposition of monomers onto a solid substrate. The materials involved may be crystals, metals, or semiconductors, for example, deposition may be via vapour, chemical methods, or cathodic sputtering, and surface binding may be chemical (chemisorption) or physical (physisorption). In certain contexts, thin film growth is known as “epitaxy”. Nanoscale growth is important in the development of many technological devices reliant on the remarkable electrical, optical, and thermal properties of thin films, and developments in construction of nanomaterials and in atomic-scale experimental observation have fuelled interest over the last couple of decades. We refer to [1–3, 16, 36, 39, 40] for scientific background and technological applications.

Under certain energetic conditions, the early stages of submonolayer growth are described by so-called Volmer–Weber dynamics. Particles are deposited onto a substrate and undergo diffusion until sufficiently many particles come into close proximity, when they “nucleate” to form static islands, which form absorbing barriers with respect to the diffusion of other particles. The nucleation threshold (i.e., the number of particles that must come together to nucleate) increases with temperature. As time goes on, more islands form by nucleation, and these islands grow by the accumulation of additional diffusing particles. Eventually, as coverage increases, monomers will tend to aggregate on growing islands rather than initiate new islands, and these growing islands will coalesce into larger structures. Many interesting aspects of these dynamics are discussed for example, in [3, 4, 7–10, 14, 17, 19, 21, 32–35] and references therein.

In the present paper we study a one-dimensional model on an interval substrate which is both space- and time-continuous, in which two particles suffice for nucleation (“binary nucleation”). Ours is a continuum relative of a type of lattice model that has been widely used

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in the applied literature, for example, by Bartelt and Evans [4] and by Blackman and Mulheran [9], for both simulation and theoretical investigations; see also [29] for a related early Monte Carlo study. The model neglects both the spatial extent of islands (this “point island” assumption is reasonable at low coverage) and also any potential evaporation of particles.

Informally, the model is as follows. At time $t = 0$, there are no active particles and the initial island locations are $\{0, 1\}$, the endpoints of the interval.

- *Deposition.* Particles are deposited on $[0, 1]$ according to a space–time Poisson process on $[0, 1] \times \mathbb{R}_+$ with intensity $\lambda > 0$.
- *Diffusion and nucleation.* Each deposited particle performs an independent Brownian motion until it either (i) hits an existing island, or (ii) meets another diffusing particle. In case (i), the particle is absorbed by the island. In case (ii), we say that nucleation has occurred, and a new island is formed at the collision site. In either case, the particle’s position becomes fixed for all subsequent time.

In Section 2 we present our main results for the nucleation process, which can be understood with the informal definition of the process given above. The first (Theorem 2.2) is a description of the $\lambda \rightarrow 0$ limit as a particular Markovian interval-splitting process, characterized in part by a splitting density ϕ_0 on $[0, 1]$. In contrast to previous applied work, which proposed various Beta distributions in this role, our ϕ_0 does not seem to have a simple expression in terms of elementary functions. Our second main result (Theorem 2.3) treats long-time statistics of the fixed- λ process, in particular, the normalized gap distribution. It turns out that the large-time statistics of the fixed- λ process can be described via the $\lambda \rightarrow 0$ density ϕ_0 , and so, in particular, the limits are independent of λ . In Section 3 we make some comparisons with previous work (which mostly lies outside the probability literature) and comment on possible extensions. A formal construction of our process is presented in Section 4, along with some fundamental initial observations. The key ingredient in our limit theorems is a quantitative approximation of the evolution of our process via an interval-splitting kernel; this is derived in Section 5. This approximation is then used to derive our $\lambda \rightarrow 0$ results (in Section 6) and our fixed- λ , long-time results (in Section 7). The splitting kernel requires evaluation of the density ϕ_0 , which we reduce to a problem of the exit position of planar Brownian motion from a right-angled triangle, started from an arbitrary interior point: the solution to this problem, which extends old work of Smith and Watson [37], is presented in Section 8. In Section 9 we collect necessary analytic properties of the splitting density ϕ_0 , as well as some numerical approximations. Finally, in Section 10 we apply results of Brennan and Durrett [12, 13] to derive normalized gap-distribution statistics for interval-splitting processes; this forms an ingredient to our results but is presented in some generality so as to facilitate comparison with the various other interval-splitting parameters that have been proposed in the literature for related nucleation problems.

We mention briefly that there has been much recent interest in the probability literature in systems of interacting diffusing particles: see, for example, [5, 15, 38]. Several of these models include deposition or particle birth, and coalescence of diffusing particles, but coalescing particles continue to diffuse, rather than nucleate.

2. Main results. We are interested in the interval fragmentation process induced by our model. We defer a formal construction of the model (based on a marked Poisson point process) to Section 4 below. Let I_t denote the number of interior islands at time $t \in \mathbb{R}_+ := [0, \infty)$, so $I_0 = 0$. Set $\nu_0 := 0$, and for $n \in \mathbb{N} := \{1, 2, \dots\}$ denote the time of the n th nucleation by

$$(2.1) \quad \nu_n := \inf\{t \in \mathbb{R}_+ : I_t = n\};$$

throughout the paper, we adopt the usual convention that $\inf \emptyset := +\infty$. The proof of the following fact will be given in Section 4.

LEMMA 2.1. For all $\lambda > 0$, $v_n < \infty$ a.s. for all $n \in \mathbb{N}$, and $\lim_{n \rightarrow \infty} v_n = \infty$, a.s.

Let \mathcal{Z}_n denote the vector of island locations in $[0, 1]$, listed left to right, at time v_n , so $\mathcal{Z}_n \in \Delta_n$ where

$$\Delta_n := \{(z_0, z_1, \dots, z_{n+1}) \in [0, 1]^{n+2} : 0 = z_0 < z_1 < \dots < z_n < z_{n+1} = 1\}.$$

Consider the process $\mathcal{Z} := (\mathcal{Z}_0, \mathcal{Z}_1, \mathcal{Z}_2, \dots)$. At time v_n , the law of \mathcal{Z}_{n+1} is not determined by \mathcal{Z}_n alone, since there may still be active particles in the system. However, our first main result (Theorem 2.2) shows that as $\lambda \rightarrow 0$, the process \mathcal{Z} converges to a Markovian interval-splitting process. We next describe the limiting process.

Let \mathcal{B} denote the Borel subsets of $[0, 1]$, and for $n \in \mathbb{N}$ set $[n] := \{1, 2, \dots, n\}$. Take a function $r : [0, 1] \rightarrow \mathbb{R}_+$ and a probability measure Φ on $([0, 1], \mathcal{B})$. Assume that $r(\ell) > 0$ for all $\ell > 0$, and $\Phi(\{0\}) = \Phi(\{1\}) = 0$. Then define for each $n \in \mathbb{Z}_+ := \{0\} \cup \mathbb{N}$ a splitting map $\Gamma_n : \Delta_n \times [n + 1] \times (0, 1) \rightarrow \Delta_{n+1}$ by

$$(2.2) \quad (\Gamma_n(z; j, v))_i := \begin{cases} z_i & \text{if } i < j, \\ z_{j-1} + v(z_j - z_{j-1}) & \text{if } i = j, \\ z_{i-1} & \text{if } i > j, \end{cases}$$

for $z = (z_0, z_1, \dots, z_{n+1}) \in \Delta_n$, $j \in [n + 1]$, and $v \in (0, 1)$. We say that the process $\mathcal{S} := (\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \dots)$, with $\mathcal{S}_n = (S_{n,0}, S_{n,1}, \dots, S_{n,n+1}) \in \Delta_n$ for all n , is an *interval-splitting process with parameters r and Φ* , if, for all $n \in \mathbb{Z}_+$, $j \in [n + 1]$, and $B \in \mathcal{B}$,

$$(2.3) \quad \mathbb{P}(S_{n+1} \in \Gamma_n(S_n; j, B) \mid \mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_n) = \frac{r(S_{n,j} - S_{n,j-1})}{\sum_{i \in [n+1]} r(S_{n,i} - S_{n,i-1})} \Phi(B), \quad \text{a.s.}$$

The sequence of kernels (2.3) and the initial value $\mathcal{S}_0 = (0, 1) \in \Delta_0$ determine the finite-dimensional distributions of \mathcal{S} , and hence the law of \mathcal{S} as a random element of the product space $\Delta_0 \times \Delta_1 \times \dots$ with the usual (Borel) product topology. In words, the transition from \mathcal{S}_n to \mathcal{S}_{n+1} is achieved by choosing the interval to be split randomly with probabilities proportional to the function r of each interval length, and the chosen interval is split into two by choosing a point in the interval according to the distribution Φ . Interval-splitting processes in this generality were studied by Brennan and Durrett [12, 13].

Our $\lambda \rightarrow 0$ limit of \mathcal{Z} turns out to be an interval-splitting process with a particular r and Φ . To describe the Φ that arises in our limit, we need some more notation. Define

$$(2.4) \quad \psi(z) := \frac{24}{\pi^4} \sum_{n \text{ odd}} a_n \sin n\pi z, \quad \text{where } a_n := \frac{4}{n^4} \tanh\left(\frac{n\pi}{2}\right) - \frac{\pi}{n^3};$$

where “ n odd” means $n \in \{1, 3, 5, \dots\}$. Note that $a_1 > 0$, but $a_n < 0$ for $n \geq 3$. In Section 9 we will use a representation of ψ involving a special function related to the *Clausen function* to show that ψ is twice continuously differentiable on $[0, 1]$, to give a more rapidly converging series approximation, and to show that $\psi(z) \sim 3z^2$ as $z \rightarrow 0$, a property that has important consequences for some of our results, but which is well-hidden in the series representation of (2.4). The probabilistic meaning of ψ is as a (defective) density arising from an exit problem for Brownian motion in a right-angled triangle: see Section 8. In particular, although not obvious from (2.4), $\psi(z) > 0$ for all $z \in (0, 1)$. Also set

$$(2.5) \quad \mu := \int_0^1 \psi(z) dz = \frac{48}{\pi^5} \sum_{n \text{ odd}} \frac{a_n}{n} = \frac{48}{\pi^4} \sum_{n \text{ odd}} \frac{\operatorname{sech}^2(\frac{n\pi}{2})}{n^4};$$

the first series follows directly from (2.4), while the second is established in Section 9. The second series representation in (2.5) is useful for numerical evaluation of μ , because

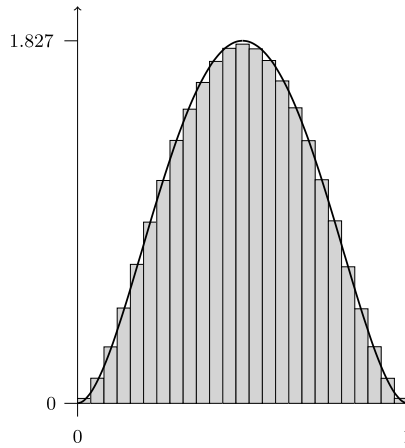


FIG. 1. The smooth curve is a numerical estimate of the density ϕ_0 using the approximant $\phi_0^{k,m}$ for $k = 9, m = 5$ (see Section 9 for a definition) which is accurate to within 10^{-10} for all $x \in [0, 1]$. The histogram is a simulation estimate for the distribution of the location of the first nucleation at $\lambda = 0.1$ based on 10^6 samples of a discrete version of the model on the lattice $\{0, \frac{1}{100}, \frac{2}{100}, \dots, 1\}$, in which any active particle performs continuous-time simple random walk at rate 100^2 , and the Poisson deposition rate at each site is $\lambda/100$.

$\text{sech}^2(\frac{n\pi}{2})$ decays exponentially in n . Indeed, taking only the terms $n = 1, 3$ in the final sum in (2.5) suffices to evaluate the first 8 decimal digits of $\mu \approx 0.07826895$ (see Section 9 for a justification).

Let ϕ_0 be ψ normalized to be a probability density, and let Φ_0 be the corresponding probability measure, that is,

$$(2.6) \quad \Phi_0(B) := \int_B \phi_0(z) dz := \frac{1}{\mu} \int_B \psi(z) dz \quad \text{for } B \in \mathcal{B}.$$

See Figure 1 for an illustration of a numerical approximation to ϕ_0 , and see Section 9 for a discussion of the numerics. We can now state our first main result.

THEOREM 2.2. *As $\lambda \rightarrow 0$, the process \mathcal{Z} converges, in the sense of total-variation convergence of finite-dimensional distributions, to an interval-splitting process with parameters r_0 and Φ_0 , where $r_0(\ell) = \ell^4$ and Φ_0 is given by (2.6).*

When $\lambda \in (0, \infty)$ is fixed, there is not such a neat description of the interval-splitting process. However, after a long time, when all intervals become very small, scaling arguments show that diffusion again dominates deposition (we give details below). Roughly speaking, this means that certain large-time statistics of the fixed- λ process can be asymptotically described in terms of the $\lambda \rightarrow 0$ limit given in Theorem 2.2. To state the result, we need to introduce some notation for the statistics that we wish to consider.

Let $(L_{n,1}, L_{n,2}, \dots, L_{n,n+1})$ denote the gap lengths associated with \mathcal{Z}_n , so if $\mathcal{Z}_n = (Z_{n,0}, Z_{n,1}, \dots, Z_{n,n+1}) \in \Delta_n$, then

$$L_{n,i} := Z_{n,i} - Z_{n,i-1} \quad \text{for } 1 \leq i \leq n + 1.$$

For $x \in [0, 1]$, denote the number of interior islands in $[0, x]$ after n nucleations by

$$N_n(x) := \max\{i \in \{0, 1, \dots, n\} : Z_{n,i} \leq x\};$$

the total number of interior islands is $N_n(1) = I_{v_n} = n$. For U_n uniform on $[n + 1]$, set

$$\tilde{L}_n := \frac{L_{n,U_n}}{\mathbb{E}(L_{n,U_n})} = (n + 1)L_{n,U_n},$$

the length of a randomly chosen gap, normalized to have unit mean. Denote the empirical gap size distribution, also normalized, by

$$\mathcal{E}_n(x) := \frac{1}{n+1} \sum_{i=1}^{n+1} \mathbb{1} \left\{ L_{n,i} \leq \frac{x}{n+1} \right\} \quad \text{for } x \in \mathbb{R}_+.$$

Here is our main result in the case of fixed λ . Note that, for the reasons previously indicated, the limit distributions do not depend on λ . For a positive function g , we write $f(x) \sim g(x)$ to mean the ratio $f(x)/g(x)$ tends to 1.

THEOREM 2.3. *Let $\lambda \in (0, \infty)$.*

- (i) *We have that $\lim_{n \rightarrow \infty} \sup_{x \in [0,1]} |n^{-1} N_n(x) - x| = 0$, a.s.*
- (ii) *There exists a continuous probability density function g_0 on \mathbb{R}_+ , which can be described in terms of r_0 and ϕ_0 appearing in Theorem 2.2, such that, for all $x \in \mathbb{R}_+$,*

$$\lim_{n \rightarrow \infty} \mathbb{P}(\tilde{L}_n \leq x) = \int_0^x g_0(y) dy, \quad \text{and, a.s.,} \quad \lim_{n \rightarrow \infty} \mathcal{E}_n(x) = \int_0^x g_0(y) dy.$$

Moreover, there exist constants $c_{g,0}, c_{g,\infty}, \theta \in (0, \infty)$ such that

$$g_0(x) \sim c_{g,0} x^2 \quad \text{as } x \rightarrow 0 \quad \text{and} \quad g_0(x) \sim \frac{c_{g,\infty}}{x^2} \exp(-\theta x^4) \quad \text{as } x \rightarrow \infty.$$

We do not have an explicit expression for g_0 , but g_0 can be characterized in terms of r_0 and ϕ_0 via a distributional fixed-point equation derived in [13]: see Section 10. In Section 10 we demonstrate, in a context of more general interval-splitting processes, the key properties of r_0 and ϕ_0 that lead to the tail asymptotics for g_0 stated here.

3. Discussion. A Web of Science topic search in May 2020 for “epitaxy” produces over 90,000 titles, covering articles in chemistry, physics, materials science, and so on (for comparison, “percolation” produces about half that number). While, as far as we are aware, our continuum model does not seem to have been considered before, closely related discrete models have generated significant interest, and have been studied both via simulations and various interesting, but not fully rigorous, analytical approaches (see, e.g., [4, 9, 21, 33, 35]). Our model corresponds to a specific case of the models of submonolayer deposition considered in [9] and elsewhere: here we focus on one dimension, on binary nucleation, and on regimes where active particles are sparse.

It is natural to seek to extend our model in the following four important ways.

- (a) Take the nucleation threshold to be an integer $\alpha \geq 2$ (our case is $\alpha = 2$).
- (b) Allow the deposition rate λ to depend on time or on the current number of islands, with $\lambda \rightarrow \infty$.
- (c) Consider substrates in higher dimensions, so that, for example, monomers live in $[0, 1]^d$, $d \in \mathbb{N}$ (the case $d = 2$ being the most physically relevant).
- (d) Permit islands to have spatial extent as an increasing function of the number of monomers that they have captured.

In discrete models, as α increases nucleations become much rarer, and quantitative differences are predicted by existing theory (e.g., [21, 34]). In the continuum context, due to the impossibility of multiple simultaneous Brownian collisions, a meaningful model with $\alpha \geq 3$ in one dimension (or $\alpha \geq 2$ in dimension $d \geq 2$) seems to require an addition of an interaction radius $\delta > 0$ for particles. Thus addressing (a) and/or (c) may simultaneously require dealing with (d).

We raise point (b) because a key feature of the analysis in the present paper is that the density of active particles is low, and tends to zero as time goes on. On the other hand, much existing work is concerned with regimes in which, at a typical time, there are many active particles in the system, and the statistics of the system are driven by a “quasiequilibrium” between particle deposition and capture by islands [8, 9, 19]. Both regimes are potentially relevant for physical applications [36], Section 11.2. While it seems likely that the results of the present paper could be extended to allow λ to grow slowly with time, the methods used here will not fully extend to the case where the average density of active particles remains bounded above zero. Suitable models with any/all of the features (a)–(d) provide much scope for probabilistic investigation.

We discuss some specific points of comparison between our results and earlier work. For their model, Blackman and Mulheran [9], Section V, consider analogues of the parameters r_0 and Φ_0 in our Theorem 2.2, and argue that:

- their analogue of $r_0(\ell)$ scales as ℓ^5 , rather than our ℓ^4 ;
- their analogue of Φ_0 is the Beta(3, 3) distribution, which has density proportional to $z^2(1-z)^2$ over $z \in [0, 1]$; this approaches zero as z^2 , like our ϕ_0 .

O’Neill et al. describe arguments for both ℓ^3 and ℓ^5 scaling for the splitting exponent, and report simulation estimates that fall between the two [34]. As mentioned above, the arguments in [9, 34] have many active particles in the system when a nucleation happens, so their results are not necessarily comparable with ours.

Statistics of the (normalized) gap distribution, such as studied in our Theorem 2.3, have received a lot of attention, along with the closely-related *capture-zone distributions*, that is, the sizes of the Voronoi intervals associated with the islands [9, 21, 33, 35]. Stretched Gamma distributions of the form $g(x) \approx x^{\theta_1} \exp(-cx^{\theta_2})$ have been considered (sometimes called the *generalized Wigner surmise* [35]), but it has since been accepted that such distributions do not capture simultaneously the $x \rightarrow 0$ and $x \rightarrow \infty$ asymptotics. For example, Blackman and Mulheran [9] argue that, in the regime they are considering, the asymptotic density should look like

$$(3.1) \quad g(x) \approx x^2 \quad \text{as } x \rightarrow 0 \quad \text{and} \quad g(x) \approx \frac{1}{x^2} \exp(-\theta x^5) \quad \text{as } x \rightarrow \infty.$$

The predictions of (3.1) are reproduced by a fragmentation approximation [21], while an alternative approach based on distributional fixed-point equations apparently reproduces the asymptotics in (3.1) at 0 but not at ∞ [33], Section III. The exponent 5 in (3.1) comes from Blackman and Mulheran’s predicted splitting exponent. In Theorem 10.1 we give a general result deriving tail asymptotics for the normalized gap distribution in general interval-splitting processes, providing a range of asymptotics like (3.1).

4. Construction, regeneration, and scaling. It is convenient to generalize our model so that the substrate is $[0, \ell]$ for $\ell \in (0, \infty)$. Let $\mathcal{C} := \mathcal{C}(\mathbb{R}_+, \mathbb{R})$, the collection of all continuous functions from \mathbb{R}_+ to \mathbb{R} , and let $\mathcal{C}_0 := \{f \in \mathcal{C} : f(0) = 0\}$. Let W denote the standard Wiener (probability) measure on \mathcal{C}_0 , so that W is the law of standard Brownian motion on \mathbb{R} started at the origin.

We build our process from $\mathcal{P}_{\ell, \lambda}$, a homogeneous Poisson point process of intensity $\lambda > 0$ on $[0, \ell] \times \mathbb{R}_+$, where each Poisson point carries an independent \mathcal{C}_0 -valued random mark distributed according to W . With probability one, all the \mathbb{R}_+ -coordinates of the process are distinct, and then we may (and do) list the points of $\mathcal{P}_{\ell, \lambda}$ in order of increasing \mathbb{R}_+ -coordinate as Ξ_1, Ξ_2, \dots with $\Xi_i = (\xi_i, s_i, b_i)$, where $\xi_i \in [0, \ell]$, $b_i = (b_i(r), r \in \mathbb{R}_+) \in \mathcal{C}_0$, and $0 <$

$s_1 < s_2 < \dots$. We interpret s_i as the time of deposition of the i th particle, which arrives at location $\xi_i \in [0, \ell]$. Set

$$(4.1) \quad x_i(r) := \begin{cases} \partial & \text{if } 0 \leq r < s_i, \\ \xi_i + b_i(r - s_i) & \text{if } r \geq s_i, \end{cases}$$

where $x_i(r) = \partial$ is to be interpreted as particle i having not yet arrived by time r , and $x_i(r) \in \mathbb{R}$ is the position of the i th particle at time $r \geq s_i$, ignoring interactions. Let $\bar{\mathbb{R}} := \mathbb{R} \cup \{\partial\}$.

Let \mathbb{I} be the set of all finite subsets of $[0, \ell]$ (the set of possible island locations), let \mathbb{A} denote the set of all finite (or empty) subsets of \mathbb{N} (possible labels of active particles), and let $\mathbb{X} := \bar{\mathbb{R}}^{\mathbb{N}}$ (locations of the particles, neglecting interactions). From the marked Poisson process $\mathcal{P}_{\ell, \lambda}$, we will construct the process $\mathcal{Y} := (\mathcal{Y}_t, t \in \mathbb{R}_+)$ where $\mathcal{Y}_t = (\mathcal{I}_t, \mathcal{A}_t, \mathcal{X}_t)$ with $\mathcal{I}_t \in \mathbb{I}$, $\mathcal{A}_t \in \mathbb{A}$, and $\mathcal{X}_t := (x_1(t), x_2(t), \dots) \in \mathbb{X}$. The system described informally in Section 1 is captured by \mathcal{I}_t , the locations of the interior islands, and $(x_j(t), j \in \mathcal{A}_t)$, the locations of the active particles.

Here is the algorithm to construct \mathcal{Y} , starting from $\mathcal{I}_0 = \mathcal{A}_0 = \emptyset$, and using $(\mathcal{X}_t, t \in \mathbb{R}_+)$ as defined by (4.1).

1. Suppose we have constructed $\mathcal{Y}_s, s \in [0, t]$. Let $i \geq 0$ be such that $s_i \leq t < s_{i+1}$ (where $s_0 := 0$). At time t , let \mathcal{I}_t be the set of interior islands, and let \mathcal{A}_t be the set of indices of the active particles. For $j, k \in \mathcal{A}_t, j < k$, let

$$T_{j,k} := \inf\{r \geq t : x_j(r) = x_k(r)\},$$

and, for $j \in \mathcal{A}_t$, set

$$T_j := \inf\{r \geq t : x_j(r) \in \mathcal{I}_t \cup \{0, \ell\}\}.$$

Let $a_1 < a_2$ be the (a.s. unique) indices such that $T_{a_1, a_2} = \min_{j,k: j < k} T_{j,k}$, and let a_0 be the (a.s. unique) index such that $T_{a_0} = \min_j T_j$. Let $T = \min\{T_{a_0}, T_{a_1, a_2}\}$.

2. If $T > s_{i+1}$ then the next arrival occurs before any nucleation or absorption, and we set $\mathcal{I}_s = \mathcal{I}_t$ for all $s \in (t, s_{i+1}]$, $\mathcal{A}_s = \mathcal{A}_t$ for all $s \in (t, s_{i+1})$, and $\mathcal{A}_{s_{i+1}} = \mathcal{A}_t \cup \{i + 1\}$. Update $t \mapsto s_{i+1}$ and return to Step 1.

3. On the other hand, if $T < s_{i+1}$, we set $\mathcal{I}_s = \mathcal{I}_t$ and $\mathcal{A}_s = \mathcal{A}_t$ for all $s \in (t, T)$, and proceed as follows at time T .

- If $T_{a_1, a_2} < T_{a_0}$, nucleation of particles a_1, a_2 occurs at time T , and we set $\mathcal{I}_T = \mathcal{I}_t \cup \{x_{a_1}(T)\}$, and $\mathcal{A}_T = \mathcal{A}_t \setminus \{a_1, a_2\}$.
- If $T_{a_0} < T_{a_1, a_2}$, particle a_0 is captured by an existing island at time T , and we set $\mathcal{I}_T = \mathcal{I}_t$ and $\mathcal{A}_T = \mathcal{A}_t \setminus \{a_0\}$.

Then update $t \mapsto T$ and return to Step 1.

LEMMA 4.1. *The above construction defines a Markov process \mathcal{Y} for all time.*

PROOF. The number of Poisson arrivals in time interval $[0, t]$ is a.s. finite, so the number of active particles at any time is a.s. finite, as is the number of islands. Given a finite number of active particles and islands at distinct locations, the independence property of the Poisson process and the Markov property of the Brownian motions imply that the evolution until the next event (either nucleation, deposition, or adsorption) is Markovian. The point-transience of planar Brownian motion implies the following facts about multiple independent one-dimensional Brownian motions with generic starting points: two Brownian motions never visit a given point at the same time, three Brownian motions never meet simultaneously, and two pairs of Brownian motions have two different first meeting times. Together

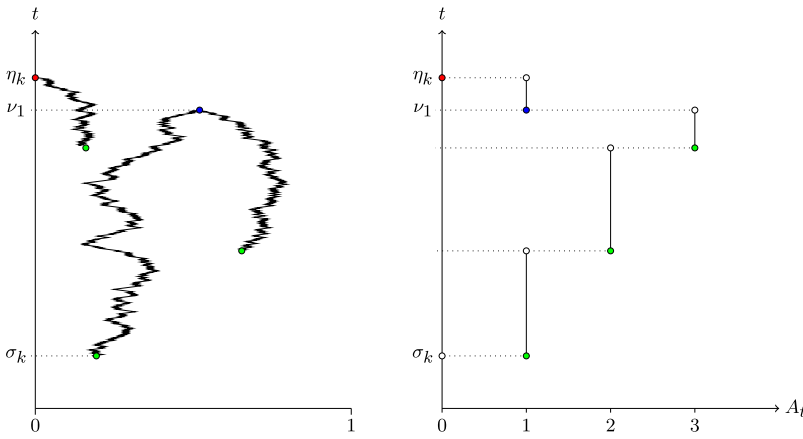


FIG. 2. A cycle $[\sigma_k, \eta_k]$ which starts at time σ_k with deposition of a particle into $[0, 1]$, having no interior islands. The cycle contains two subsequent depositions, the first nucleation (at time ν_1), and a capture of an active particle by an existing island.

with the fact that deposition locations a.s. never coincide with the locations of any currently active particles or islands, this means that there is a.s. a well-defined next event, and, up to and including the time of that next event, active particles and islands are always at distinct locations. The above algorithm thus gives a well-defined construction from each event to the next. Thus the process is well-defined for all time, and inherits the Markov property from the properties of the Poisson process and the Brownian motions. \square

We denote by $\mathbb{P}_{\ell, \lambda}$ the probability measure associated with the process \mathcal{Y} constructed above. In the special case $\ell = 1$, we write simply \mathbb{P}_λ . For the corresponding expectations we use $\mathbb{E}_{\ell, \lambda}$ and \mathbb{E}_λ . Let $A_t := |\mathcal{A}_t|$ denote the number of active particles at time t , and let $I_t := |\mathcal{I}_t|$ denote the number of interior islands at time t . Initially, $A_0 = I_0 = 0$. Let $\mathcal{F}_t := \sigma(\mathcal{Y}_s, 0 \leq s \leq t)$ denote the σ -algebra generated by the process up to time $t \in \mathbb{R}_+$.

Define $\eta_0 := 0$. Also for $k \in \mathbb{N}$ define stopping times

$$(4.2) \quad \sigma_k := \inf\{t > \eta_{k-1} : A_t = 1\} \quad \text{and} \quad \eta_k := \inf\{t > \sigma_k : A_t = 0\}.$$

Lemma 4.2 below shows that all these stopping times are finite, a.s.; for $k \in \mathbb{N}$, we call the time interval $[\sigma_k, \eta_k]$ the k th cycle. See Figure 2 for an illustration.

By definition, $A_t = 0$ for $t \in [\eta_{k-1}, \sigma_k)$, so nucleation can only occur during the cycles $[\sigma_k, \eta_k]$. Up until the first nucleation, the cycles $[\sigma_k, \eta_k]$ encode a *regeneration* structure that we will exploit. Let $M_t := \max_{1 \leq i \leq I_t+1} L_{I_t, i}$, the length of the longest gap at time t . The next lemma is somewhat technical, but important: it gives a tail bound for the duration of a cycle. The intuition is that active particles are captured rather rapidly by existing islands, and faster still if the gaps between islands are small.

LEMMA 4.2.

- (i) For any $\lambda > 0$, we have that $\eta_k, \sigma_k < \infty$ for all $k \in \mathbb{N}$, \mathbb{P}_λ -a.s.
- (ii) For all $\lambda_0 \in (0, \infty)$ there exist constants $\delta = \delta(\lambda_0) > 0$ and $C_1 = C_1(\lambda_0) < \infty$ such that, for all $\lambda \in (0, \lambda_0]$, for all $k \in \mathbb{N}$ and all $t \in \mathbb{R}_+$,

$$(4.3) \quad \mathbb{P}_\lambda(\eta_k - \sigma_k \geq t \mid \mathcal{F}_{\sigma_k}) \leq C_1 \exp(-\delta M_{\sigma_k}^{-1} t^{1/2}) \quad \mathbb{P}_\lambda\text{-a.s.}$$

PROOF. For $\lambda > 0$, it is easy to see that $\eta_k < \infty$ implies $\sigma_{k+1} < \infty$, \mathbb{P}_λ -a.s.; since $\eta_0 = 0$ this means $\sigma_1 < \infty$. Thus to show that $\eta_k, \sigma_k < \infty$ for all k , it suffices to fix $k \in \mathbb{N}$ and to establish (4.3) supposing that $\sigma_k < \infty$, a.s. This is how we proceed.

Let $\lambda_0 \in (0, \infty)$ and $\lambda \in (0, \lambda_0]$. Write $M := M_{\sigma_k} \in (0, 1]$. For $t \in \mathbb{R}_+$, set

$$\tilde{\mathcal{I}}_t := \mathcal{I}_{\sigma_k + M^2 t}, \quad \tilde{\mathcal{A}}_t := \mathcal{A}_{\sigma_k + M^2 t}, \quad \tilde{A}_t := A_{\sigma_k + M^2 t}, \quad \tilde{\mathcal{F}}_t := \mathcal{F}_{\sigma_k + M^2 t},$$

and $\tilde{x}_j(t) := x_j(\sigma_k + M^2 t)$. In the rest of this proof, when we refer to “time” we mean the value of t in the index $\sigma_k + M^2 t$. Let $B_t(j)$ be the event that $\tilde{x}_j(s)$, $j \in \tilde{\mathcal{A}}_t$, hits $\tilde{\mathcal{I}}_t$ at some time $s \in [t, t + 1]$. If $B_t(j)$ occurs, then particle j is no longer active at time $t + 1$, because either it has been captured by an existing island, or it has collided with another active particle in the meantime. Also, $|\tilde{\mathcal{A}}_{t+1} \setminus \tilde{\mathcal{A}}_t|$, the number of new active particles at time $t + 1$ compared to time t , is bounded by the number \tilde{Z}_t of Poisson arrivals in time interval $[t, t + 1]$. Thus

$$\tilde{A}_{t+1} - \tilde{A}_t \leq \tilde{Z}_t - \sum_{j \in \tilde{\mathcal{A}}_t} \mathbb{1}_{B_t(j)},$$

and, by construction, \tilde{Z}_t and the $B_t(j)$ are conditionally independent, given $\tilde{\mathcal{F}}_t$. (This bound ignores nucleations, which can also eliminate active particles.) For $\delta > 0$,

$$\mathbb{E}_\lambda(e^{\delta(\tilde{A}_{t+1} - \tilde{A}_t)} \mid \tilde{\mathcal{F}}_t) \leq \mathbb{E}_\lambda(e^{\delta \tilde{Z}_t} \mid \tilde{\mathcal{F}}_t) \prod_{j \in \tilde{\mathcal{A}}_t} \mathbb{E}_\lambda(e^{-\delta \mathbb{1}_{B_t(j)}} \mid \tilde{\mathcal{F}}_t).$$

Given $\tilde{\mathcal{F}}_t$, \tilde{Z}_t is Poisson with mean λM^2 , so $\mathbb{E}_\lambda(e^{\delta \tilde{Z}_t} \mid \tilde{\mathcal{F}}_t) = \exp(\lambda M^2 (e^\delta - 1))$, while

$$\mathbb{E}_\lambda(e^{-\delta \mathbb{1}_{B_t(j)}} \mid \tilde{\mathcal{F}}_t) = 1 - (1 - e^{-\delta}) \mathbb{P}_\lambda(B_t(j) \mid \tilde{\mathcal{F}}_t).$$

We claim that there is a constant $q > 0$ such that

$$(4.4) \quad \mathbb{P}_\lambda(B_t(j) \mid \tilde{\mathcal{F}}_t) \geq q, \quad \text{a.s., for all } t \in \mathbb{R}_+ \text{ and all } j \in \tilde{\mathcal{A}}_t.$$

Indeed, if $(w_t, t \geq 0)$ is standard Brownian motion on \mathbb{R} with $w_0 = 0$, then the claim (4.4) holds with $q = \mathbb{P}(\sup_{0 \leq t \leq 1} w_t \geq 1) = \mathbb{P}(\sup_{0 \leq t \leq M^2} w_t \geq M)$, since $\tilde{x}_j(t)$ has at least one island within distance M at time t . By the reflection principle for Brownian motion [31], page 45, $q = 2\mathbb{P}(w_1 \geq 1) \approx 0.317$.

Using (4.4), since $1 - z \leq e^{-z}$ and $\lambda M^2 \leq \lambda_0$, we get

$$(4.5) \quad \mathbb{E}_\lambda(e^{\delta(\tilde{A}_{t+1} - \tilde{A}_t)} \mid \tilde{\mathcal{F}}_t) \leq \exp(\lambda_0(e^\delta - 1) - q \tilde{A}_t(1 - e^{-\delta})).$$

There is an absolute constant δ_0 such that $e^\delta - 1 \leq 2\delta$ and $1 - e^{-\delta} \geq \delta/2$ for all $\delta \in [0, \delta_0]$. Fix $\delta \in [0, \delta_0]$, and let $a_0 := \lceil 6\lambda_0/q \rceil$, so $a_0 \in \mathbb{N}$. If $\tilde{A}_t \geq a_0$, then

$$\lambda_0(e^\delta - 1) - q \tilde{A}_t(1 - e^{-\delta}) \leq 2\delta\lambda_0 - \frac{qa_0\delta}{2} \leq -\delta\lambda_0.$$

Thus we obtain from (4.5) that

$$(4.6) \quad \mathbb{E}_\lambda(e^{\delta(\tilde{A}_{t+1} - \tilde{A}_t)} \mid \tilde{\mathcal{F}}_t) \leq \exp(-\delta\lambda_0) \quad \text{on } \{\tilde{A}_t \geq a_0\};$$

$$(4.7) \quad \mathbb{E}_\lambda(e^{\delta(\tilde{A}_{t+1} - \tilde{A}_t)} \mid \tilde{\mathcal{F}}_t) \leq \exp(2\delta\lambda_0) \quad \text{on } \{\tilde{A}_t < a_0\}.$$

Set $\tau_0 := 0$ and define, for $r \in \mathbb{N}$, the stopping times

$$\gamma_r := \inf\{t \in \mathbb{R}_+ : t > \tau_{r-1} + 1, \tilde{A}_t \geq a_0\} \quad \text{and} \quad \tau_r := \inf\{t \in \mathbb{R}_+ : t > \gamma_r, \tilde{A}_t < a_0\}.$$

Also define $\tau'_r := \min\{\gamma_r + n : n \in \mathbb{N}, \tilde{A}_{\gamma_r + n} < a_0\}$. Then, a.s., $\tilde{A}_{\tau_0} = A_{\sigma_k} = 1$ and $\tau_r \leq \tau'_r$; also, \tilde{A}_{γ_r} is bounded above by a_0 plus a Poisson random variable with mean λ_0 . The Foster–Lyapunov drift bounds (4.6) and (4.7) show that we may apply Theorem 2.3 of [22] to the discrete-time process $\tilde{A}_{\gamma_r}, \tilde{A}_{\gamma_{r+1}}, \dots$ and its stopping time $\tau'_r - \gamma_r$ to show that, for some constants $\theta > 0$ and $C < \infty$, depending only on λ_0 ,

$$(4.8) \quad \mathbb{E}_\lambda(e^{\theta(\tau_r - \gamma_r)} \mid \tilde{\mathcal{F}}_{\gamma_r}) \leq \mathbb{E}_\lambda(e^{\theta(\tau'_r - \gamma_r)} \mid \tilde{\mathcal{F}}_{\gamma_r}) \leq C \quad \text{a.s., for all } r \in \mathbb{N}.$$

Next, with \tilde{Z}_t again the number of depositions during time interval $[t, t + 1]$, we have that a sufficient condition for $\tilde{A}_{t+1} = 0$ is that $\tilde{Z}_t = 0$ (no new arrivals) and all the active particles at time t become inactive before time $t + 1$; hence, by (4.4),

(4.9)

$$\mathbb{P}_\lambda(\tilde{A}_{t+1} = 0 \mid \tilde{\mathcal{F}}_t) \geq \mathbb{P}_\lambda\left(\{\tilde{Z}_t = 0\} \cap \bigcap_{j \in \tilde{A}_t} B_t(j) \mid \tilde{\mathcal{F}}_t\right) \geq q^{a_0} e^{-\lambda_0} \quad \text{on } \{\tilde{A}_t \leq a_0\}.$$

Define for $r \in \mathbb{N}$ and $m \in \mathbb{N}$ the event

$$E(r, m) = \{M^{-2}(\eta_k - \sigma_k) > \tau_{r-1} + m, \gamma_r - \tau_{r-1} > m\} \in \tilde{\mathcal{F}}_{\tau_{r-1}+m}.$$

Then, since $E(r, m + 1) \subseteq E(r, m)$,

$$\mathbb{P}_\lambda(E(r, m + 1) \mid \tilde{\mathcal{F}}_{\tau_{r-1}}) = \mathbb{E}_\lambda[\mathbb{P}_\lambda(E(r, m + 1) \mid \tilde{\mathcal{F}}_{\tau_{r-1}+m}) \mathbb{1}_{E(r, m)} \mid \tilde{\mathcal{F}}_{\tau_{r-1}}],$$

where, since $\tilde{A}_{\tau_{r-1}+m+1} = 0$ implies that $M^{-2}(\eta_k - \sigma_k) \leq \tau_{r-1} + m + 1$,

$$\mathbb{P}_\lambda(E(r, m + 1) \mid \tilde{\mathcal{F}}_{\tau_{r-1}+m}) \leq 1 - \mathbb{P}_\lambda(\tilde{A}_{\tau_{r-1}+m+1} = 0 \mid \tilde{\mathcal{F}}_{\tau_{r-1}+m}).$$

Thus, by (4.9) and the fact that $E(r, m)$ implies $\tilde{A}_{\tau_{r-1}+m} < a_0$, with $\varepsilon_0 = q^{a_0} e^{-\lambda_0}$,

$$\mathbb{P}_\lambda(E(r, m + 1) \mid \tilde{\mathcal{F}}_{\tau_{r-1}}) \leq \mathbb{E}_\lambda[(1 - \varepsilon_0) \mathbb{1}_{E(r, m)} \mid \tilde{\mathcal{F}}_{\tau_{r-1}}] \leq e^{-\varepsilon_0} \mathbb{P}_\lambda(E(r, m) \mid \tilde{\mathcal{F}}_{\tau_{r-1}}).$$

Iterating this bound gives $\mathbb{P}_\lambda(E(r, m) \mid \tilde{\mathcal{F}}_0) \leq e^{-\varepsilon_0(m-1)}$, a.s., for all $m \in \mathbb{N}$.

Fix $t \in \mathbb{N}$. Let $K = \max\{r : \tau_r \leq M^{-2}(\eta_k - \sigma_k)\}$ and $L = \min\{r : \gamma_r - \tau_{r-1} > t\}$. Then $\tau_K \leq M^{-2}(\eta_k - \sigma_k)$ and we cannot have $\gamma_{K+1} \leq M^{-2}(\eta_k - \sigma_k)$, or else we would also have $\tau_{K+1} \leq M^{-2}(\eta_k - \sigma_k)$ too. Thus $M^{-2}(\eta_k - \sigma_k) \leq \gamma_{K+1}$, so $M^{-2}(\eta_k - \sigma_k) \leq \tau_K + (\gamma_{K+1} - \tau_K)$. For $r < L$ we have

$$\tau_r = \sum_{j=1}^r (\tau_j - \gamma_j) + \sum_{j=1}^r (\gamma_j - \tau_{j-1}) \leq rt + \sum_{j=1}^r (\tau_j - \gamma_j).$$

On the event

$$\{K < r\} \cap \left\{ \sum_{j=1}^r (\tau_j - \gamma_j) \leq t \right\} \cap \bigcap_{j=1}^r (\{M^{-2}(\eta_k - \sigma_k) \leq \tau_{j-1} + t\} \cup \{\gamma_j - \tau_{j-1} \leq t\}),$$

we have that either $L > r$, in which case

$$M^{-2}(\eta_k - \sigma_k) \leq \gamma_{K+1} \leq \gamma_r \leq \tau_{r-1} + \max_{1 \leq j \leq r} (\gamma_j - \tau_{j-1}) \leq (r + 1)t,$$

or else $L \leq r$ and $M^{-2}(\eta_k - \sigma_k) \leq \tau_{L-1} + t \leq (r + 1)t$ also. Thus

(4.10)

$$\begin{aligned} &\mathbb{P}_\lambda(\eta_k - \sigma_k > (r + 1)M^2 t \mid \mathcal{F}_{\sigma_k}) \\ &\leq \mathbb{P}_\lambda(K \geq r \mid \mathcal{F}_{\sigma_k}) + \mathbb{P}_\lambda\left(\sum_{j=1}^r (\tau_j - \gamma_j) \geq t \mid \mathcal{F}_{\sigma_k}\right) + \mathbb{P}_\lambda\left(\bigcup_{j=1}^r E(j, t) \mid \mathcal{F}_{\sigma_k}\right). \end{aligned}$$

For the first term on the right-hand side of (4.10), we have

$$\begin{aligned} \mathbb{P}_\lambda(K \geq m + 1 \mid \mathcal{F}_{\sigma_k}) &= \mathbb{E}_\lambda[\mathbb{P}_\lambda(K \geq m + 1 \mid \tilde{\mathcal{F}}_{\tau_m}) \mathbb{1}\{K \geq m\} \mid \tilde{\mathcal{F}}_0] \\ &\leq \mathbb{E}_\lambda[(1 - \mathbb{P}_\lambda(\tilde{A}_{\tau_{m+1}} = 0 \mid \tilde{\mathcal{F}}_{\tau_m})) \mathbb{1}\{K \geq m\} \mid \tilde{\mathcal{F}}_0], \end{aligned}$$

since $\tilde{A}_{\tau_{m+1}} = 0$ implies $M^{-2}(\eta_k - \sigma_k) \leq \tau_m + 1 < \tau_{m+1}$. By (4.9), $\mathbb{P}_\lambda(K \geq m + 1 \mid \mathcal{F}_{\sigma_k}) \leq (1 - \varepsilon_0) \mathbb{P}_\lambda(K \geq m \mid \mathcal{F}_{\sigma_k})$, and $\mathbb{P}_\lambda(K \geq m \mid \mathcal{F}_{\sigma_k}) \leq e^{-\varepsilon_0 m}$, a.s., for all $m \in \mathbb{Z}_+$.

For the second term on the right-hand side of (4.10), we have from (4.8) that

$$\begin{aligned} \mathbb{E}_\lambda[e^{\theta \sum_{j=1}^r (\tau_j - \gamma_j)} \mid \tilde{\mathcal{F}}_0] &= \mathbb{E}_\lambda[e^{\theta \sum_{j=1}^{r-1} (\tau_j - \gamma_j)} \mathbb{E}_\lambda(e^{\theta(\tau_r - \gamma_r)} \mid \tilde{\mathcal{F}}_{\tau_{r-1}}) \mid \tilde{\mathcal{F}}_0] \\ &\leq C \mathbb{E}_\lambda[e^{\theta \sum_{j=1}^{r-1} (\tau_j - \gamma_j)} \mid \tilde{\mathcal{F}}_0], \end{aligned}$$

where θ, C are as in (4.8), and depend only on λ_0 . Iterating this argument gives, for all $r \in \mathbb{N}$ and a constant $D < \infty$, $\mathbb{E}_\lambda(e^{\theta \sum_{j=1}^r (\tau_j - \gamma_j)} \mid \mathcal{F}_{\sigma_k}) \leq e^{Dr}$, a.s. By Markov's inequality,

$$\mathbb{P}_\lambda\left(\sum_{j=1}^r (\tau_j - \gamma_j) \geq t \mid \mathcal{F}_{\sigma_k}\right) \leq e^{Dr - \theta t}.$$

Choose $r = \lfloor \frac{\theta}{2D} \rfloor t$. Putting all the bounds together, we obtain from (4.10) that

$$\mathbb{P}_\lambda(\eta_k - \sigma_k > cM^2 t^2 \mid \mathcal{F}_{\sigma_k}) \leq e^{-\varepsilon t},$$

for some constants $c > 0$ and $\varepsilon > 0$. Then (4.3) follows. This completes the proof of (ii), and hence (i), as explained at the start of this proof. \square

The next result shows that the cycles $[\sigma_k, \eta_k]$ do not accumulate in finite time.

LEMMA 4.3. *Let $\lambda \in (0, \infty)$. As $k \rightarrow \infty$, $\nu_k, \eta_k, \sigma_k \rightarrow \infty$, \mathbb{P}_λ -a.s.*

PROOF. By construction, $\sigma_1, \sigma_2, \dots$ is a subsequence of the Poisson arrival times s_1, s_2, \dots , and hence $\eta_k \geq \sigma_k \geq s_k$ for all $k \in \mathbb{N}$. Similarly, the k th nucleation can only occur after $2k$ particles have been deposited, so $\nu_k \geq s_{2k}$ for all $k \in \mathbb{N}$. But $\lim_{k \rightarrow \infty} s_k = \infty$, a.s. \square

Lemma 4.3 shows that ν_0, ν_1, \dots does not have a finite accumulation point, so we can talk about the *first* nucleation in any time interval which contains nucleations; we have not yet proved that ν_k is finite for all k , but we will do so later in this section. Let E_k be the event that at least one nucleation occurs in time interval $[\sigma_k, \eta_k]$. For $k \in \mathbb{N}$, let

$$\alpha_k := \min\{j \in \mathbb{N} : E_{k+j} \text{ occurs}\},$$

the number of cycles after η_k until the first nucleation in time interval (η_k, ∞) , where $\alpha_k = \infty$ if and only if there is no nucleation after time η_k .

On the event E_1 , the first nucleation occurs at some location $\zeta \ell$ for $\zeta \in (0, 1)$. Recall that \mathcal{B} denotes the Borel subsets of $[0, 1]$, and denote the probability that the first nucleation occurs during the first cycle and at spatial location in ℓB by

$$(4.11) \quad \nu(\ell, \lambda; B) := \mathbb{P}_{\ell, \lambda}(E_1 \cap \{\zeta \in B\}) \quad \text{for } B \in \mathcal{B},$$

and set $\nu(\lambda; B) := \nu(1, \lambda; B)$. The probability of nucleation during the first cycle is

$$\mu(\ell, \lambda) := \mathbb{P}_{\ell, \lambda}(E_1) = \nu(\ell, \lambda; [0, 1]) \quad \text{and} \quad \mu(\lambda) := \mathbb{P}_\lambda(E_1) = \nu(\lambda; [0, 1]) = \mu(1, \lambda).$$

The next result gives an important scaling property, which is a consequence of the scaling properties of the Poisson process and of Brownian motion. For a scalar $a > 0$, let $a\mathcal{Y}_t = (a\mathcal{I}_t, \mathcal{A}_t, a\mathcal{X}_t)$, that is, scalar multiplication of all spatial variables.

LEMMA 4.4. *Let $\ell, \lambda \in (0, \infty)$. Then*

$$(4.12) \quad \mathbb{P}_{\ell, \lambda}((\ell^{-1}\mathcal{Y}_{\ell^2 t})_{t \geq 0} \in \cdot) = \mathbb{P}_{\ell^3 \lambda}((\mathcal{Y}_t)_{t \geq 0} \in \cdot).$$

As a consequence, for all $\ell, \lambda \in (0, \infty)$,

$$(4.13) \quad \nu(\ell, \lambda; \cdot) = \nu(\ell^3 \lambda; \cdot).$$

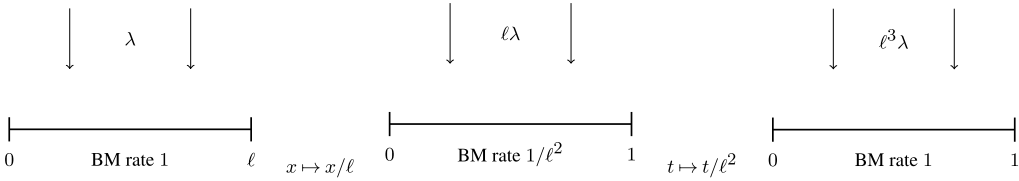


FIG. 3. Illustration of the scaling argument in the proof of Lemma 4.4.

PROOF. For $\ell > 0$, define the space–time scaling operation $S_\ell : \mathcal{C}_0 \rightarrow \mathcal{C}_0$ by $S_\ell(f)(t) = \ell^{-1} f(\ell^2 t)$, $t \in \mathbb{R}_+$. Then define the function $T_\ell : [0, \ell] \times \mathbb{R}_+ \times \mathcal{C}_0 \rightarrow [0, 1] \times \mathbb{R}_+ \times \mathcal{C}_0$ by

$$T_\ell(x, s, f) = \left(\frac{x}{\ell}, \frac{s}{\ell^2}, S_\ell(f) \right).$$

We claim that

$$(4.14) \quad T_\ell(\mathcal{P}_{\ell,\lambda}) \text{ has the same law as } \mathcal{P}_{1,\ell^3\lambda}.$$

To see this, view $\mathcal{P}_{\ell,\lambda}$ as a Poisson point process with intensity measure $\lambda \Lambda_\ell \otimes W$, where Λ_ℓ is Lebesgue measure on $[0, \ell] \times \mathbb{R}_+$ and W is Wiener measure on \mathcal{C}_0 . Brownian scaling ([31], page 12) shows that W is preserved by the transformation S_ℓ , since b is standard Brownian motion on \mathbb{R} if and only if $S_\ell(b)$ is too. Thus, by the mapping theorem [26], page 38, $T_\ell(\mathcal{P}_{\ell,\lambda})$ is a Poisson point process with intensity measure $\ell^3 \lambda \Lambda_1 \otimes W$. This verifies (4.14).

Let $(\mathcal{Y}_s, s \in \mathbb{R}_+)$ and $(\mathcal{Y}'_t, t \in \mathbb{R}_+)$ denote the processes constructed from $\mathcal{P}_{\ell,\lambda}$ and $T_\ell(\mathcal{P}_{\ell,\lambda})$ using the algorithm described above. The action of the map T_ℓ shows that

$$(4.15) \quad \mathcal{Y}'_t = \frac{1}{\ell} \mathcal{Y}_{\ell^2 t} = \left(\frac{1}{\ell} \mathcal{I}_{\ell^2 t}, \mathcal{A}_{\ell^2 t}, \frac{1}{\ell} \mathcal{X}_{\ell^2 t} \right),$$

since T_ℓ scales space by $1/\ell$ and time by $1/\ell^2$. For instance, the arrival time of the i th particle in \mathcal{Y}' is $s'_i = \ell^{-2} s_i$, and the i th particle’s trajectory x'_i is given for $r \geq s'_i$ by

$$x'_i(r) = \frac{1}{\ell} \xi_i + S_\ell(b_i)(r - \ell^{-2} s_i) = \frac{1}{\ell} \xi_i + \frac{1}{\ell} b_i(\ell^2 r - s_i),$$

so that $x'_i(r) = \ell^{-1} x_i(\ell^2 r)$. Combining (4.15) with (4.14), we see that $\ell^{-1} \mathcal{Y}_{\ell^2 t}$ under $\mathbb{P}_{\ell,\lambda}$ has the same law as \mathcal{Y}'_t under $\mathbb{P}_{1,\ell^3\lambda}$. This proves (4.12). See Figure 3 for a schematic.

The event defining $\nu(\ell, \lambda; \cdot)$ in (4.11), namely $E_1 \cap \{\zeta \in B\}$, is invariant under time-scaling, and ζ is already scaled so as to be in $[0, 1]$. Then, by (4.12), $\mathbb{P}_{\ell,\lambda}(E_1 \cap \{\zeta \in B\}) = \mathbb{P}_{\ell^3\lambda}(E_1 \cap \{\zeta \in B\})$, which establishes (4.13). \square

Fix $t \in \mathbb{R}_+$. Let $\zeta'_t \in (0, \ell)$ denote the location of the earliest nucleation in the time interval (t, ∞) , if there is one, otherwise set $\zeta'_t = \infty$; we will shortly be able to prove Lemma 2.1, which says that there will a.s. always be such a nucleation. At time t there are $I_t + 1$ gaps, and gap $j \in [I_t + 1]$ is given by $[Z_{I_t, j-1}, Z_{I_t, j}]$. Define

$$\zeta_t := \sum_{j \in [I_t + 1]} \mathbb{1}\{\zeta'_t \in (Z_{I_t, j-1}, Z_{I_t, j})\} \left(\frac{\zeta'_t - Z_{I_t, j-1}}{Z_{I_t, j} - Z_{I_t, j-1}} \right),$$

so $\zeta_t \in (0, 1)$ as long as ζ'_t is finite. For $t \in \mathbb{R}_+$, $j \in [I_t + 1]$, and $B \in \mathcal{B}$, define the event

$$(4.16) \quad D_t(j, B) := \{\zeta'_t \in [Z_{I_t, j-1}, Z_{I_t, j}], \zeta_t \in B\},$$

which says that the next nucleation after time t occurs in gap j and at relative location in B .

By the strong Markov property, there is a measurable π_λ such that, \mathbb{P}_λ -a.s.,

$$(4.17) \quad \mathbb{P}_\lambda(\{\alpha_k = 1\} \cap D_{\eta_k}(j, B) \mid \mathcal{F}_{\eta_k}) = \pi_\lambda(\mathcal{Z}_{I_{\eta_k}}; j, B).$$

Similarly, there is a measurable Π_λ such that

$$(4.18) \quad \mathbb{P}_\lambda(\alpha_k = 1 \mid \mathcal{F}_{\eta_k}) = \Pi_\lambda(\mathcal{Z}_{I_{\eta_k}}) = \sum_{j \in [I_{\eta_k} + 1]} \pi_\lambda(\mathcal{Z}_{I_{\eta_k}}; j, [0, 1]).$$

The next result gives a lower bound for π_λ on a certain set; in particular, it shows that $\Pi_\lambda(\mathcal{Z}_{I_{\eta_k}}) > 0$. We will see in Section 6 that this bound is of the correct order as $\lambda \rightarrow 0$.

LEMMA 4.5. *Let $B_0 := [1/8, 7/8]$. For any $\lambda_0 \in (0, \infty)$ there exists a constant $\varepsilon_0 = \varepsilon_0(\lambda_0) > 0$ such that, for all $\lambda \in (0, \lambda_0)$ and all $k \in \mathbb{Z}_+$,*

$$\pi_\lambda(\mathcal{Z}_{\eta_k}; j, B_0) \geq \varepsilon_0 \lambda L_{I_{\eta_k}, j}^4 \quad \text{for all } j \in [I_{\eta_k} + 1].$$

PROOF. Fix $k \in \mathbb{Z}_+$. To simplify notation, write $I = I_{\eta_k}$ for the number of interior islands, and, for $1 \leq j \leq I + 1$, $Z_j = Z_{I, j}$ for the island locations and $L_j = L_{I, j}$ for the lengths of the gaps. Take $j \in [I + 1]$. Define nested subintervals of $[Z_{j-1}, Z_j]$ by

$$\Lambda_{j, k} = \left[Z_{j-1} + \frac{k+1}{8} L_j, Z_j - \frac{k+1}{8} L_j \right] \quad \text{for } k \in \{0, 1, 2\}.$$

We will define a series of events whose intersection implies that nucleation occurs in $\Lambda_{j, 0}$. Let $T := \min\{i \in \mathbb{Z}_+ : s_i \geq \eta_k\}$. Take a constant $t_0 \in (1, \infty)$ to be chosen later. Let

$$F_1 := \{\xi_T \in \Lambda_{j, 2}\} \cap \left\{ \sup_{0 \leq t \leq L_j^2} |x_T(s_T + t) - \xi_T| < \frac{1}{8} L_j \right\},$$

the event that the next deposition occurs in $\Lambda_{j, 2}$, and that x_T stays in $\Lambda_{j, 1}$ through time interval $[s_T, s_T + L_j^2]$. Define the event

$$F_2 := \{s_{T+1} \leq s_T + L_j^2\} \cap \{\xi_{T+1} \in \Lambda_{j, 1}\} \cap \{s_{T+2} > s_T + t_0 L_j^2\},$$

that a single arrival occurs during time interval $(s_T, s_T + L_j^2]$ and at location in $\Lambda_{j, 1}$, and no arrival occurs during time interval $(s_T + L_j^2, s_T + t_0 L_j^2]$.

On $F_1 \cap F_2$, at time s_{T+1} both particles $T, T + 1$ are active and are at locations in $\Lambda_{j, 1}$, since neither can have encountered another active particle or an existing island. Suppose (without loss of generality) that the leftmost of the two active particles $T, T + 1$ at time s_{T+1} is the particle labelled T : that is, $x_T(s_{T+1}) < x_{T+1}(s_{T+1})$. Let F_3 denote the event that both x_T visits $[Z_j - \frac{1}{8} L_j, Z_j]$ before visiting $[Z_{j-1}, Z_{j-1} + \frac{1}{8} L_j]$, and x_{T+1} visits $[Z_{j-1}, Z_{j-1} + \frac{1}{8} L_j]$ before visiting $[Z_j - \frac{1}{8} L_j, Z_j]$. Also let F_4 denote the event that both x_T and x_{T+1} exit the interval $\Lambda_{j, 0}$ before time $s_{T+1} + t_0 L_j^2$. If $F_3 \cap F_4$ occurs, then the particles $T, T + 1$ must meet in the interval $\Lambda_{j, 0}$ before time $s_{T+1} + t_0 L_j^2$, and, still being active, nucleate. Therefore,

$$(4.19) \quad \pi_\lambda(\mathcal{Z}_I; j, B_0) \geq \mathbb{P}_\lambda(F_1 \cap F_2 \cap F_3 \cap F_4 \mid \mathcal{F}_{\eta_k}).$$

We bound the probability on the right-hand side of (4.19): for concreteness, we give a quantitative estimate, although we make no attempt to optimize the constants. We have

$$\mathbb{P}_\lambda(F_1 \mid \mathcal{F}_{s_T}) \geq \frac{4}{\pi} \exp\{-8\pi^2\} - \frac{4}{3\pi} \exp\{-72\pi^2\} =: q_1 > 10^{-35} \quad \text{on } \{\xi_T \in \Lambda_{j, 2}\},$$

using bounds on two-sided exit times from for example, [23], page 1047. Also,

$$\mathbb{P}_\lambda(F_2 \mid \mathcal{F}_{s_T}) \geq \frac{1}{2}L_j \cdot \lambda L_j^2 e^{-L_j^2 \lambda} \cdot e^{-(t_0-1)L_j^2 \lambda} \geq p_\lambda L_j^3,$$

where $p_\lambda := \frac{\lambda}{2}e^{-\lambda t_0}$, and, given \mathcal{F}_{s_T} , F_1 and F_2 are independent. So

$$\mathbb{P}_\lambda(F_1 \cap F_2 \mid \mathcal{F}_{\eta_k}) = \mathbb{E}_\lambda[\mathbb{P}_\lambda(F_1 \mid \mathcal{F}_{s_T}) \mathbb{P}_\lambda(F_2 \mid \mathcal{F}_{s_T}) \mathbb{1}\{\xi_T \in \Lambda_{j,2}\} \mid \mathcal{F}_{\eta_k}] \geq \frac{q_1 p_\lambda}{4} L_j^4.$$

Brownian motion started at $x \in (a, b)$ hits b before a with probability $\frac{x-a}{b-a}$, so

$$(4.20) \quad \mathbb{P}_\lambda(F_3 \mid \mathcal{F}_{s_{T+1}}) = \left(\frac{8x_T(s_{T+1}) - 8Z_{j-1} - L_j}{6L_j} \right) \left(\frac{8Z_j - 8x_{T+1}(s_{T+1}) - L_j}{6L_j} \right),$$

which is at least $1/36$ on $F_1 \cap F_2$. Let τ be the first exit time of a Brownian motion started at $x \in [2/8, 6/8]$ from the interval $[1/8, 7/8]$. The minimal distance from x to the set $\{1/8, 7/8\}$ is at most $3/8$, so, if w is Brownian motion on \mathbb{R} ,

$$\mathbb{P}(\tau \geq t) \leq \mathbb{P}\left(\sup_{0 \leq s \leq t} w_s \leq \frac{3}{8}\right) = 1 - 2\mathbb{P}\left(w_1 \geq \frac{3}{8\sqrt{t}}\right),$$

by the reflection principle and scaling. Hence

$$\mathbb{P}(\tau \geq t) \leq 2 \int_0^{3/(8\sqrt{t})} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \leq \frac{3}{4\sqrt{6t}}.$$

Taking $t = t_0 = 1944$ ensures that $\mathbb{P}(\tau \geq t_0) \leq 1/144$, so that, by Brownian scaling,

$$(4.21) \quad \mathbb{P}_\lambda(F_4^c \mid \mathcal{F}_{s_{T+1}}) \leq 2\mathbb{P}(\tau \geq t_0) \leq \frac{1}{72} \quad \text{on } F_1 \cap F_2.$$

Combining (4.20) and (4.21) we get

$$\mathbb{P}_\lambda(F_3 \cap F_4 \mid \mathcal{F}_{s_{T+1}}) \geq \mathbb{P}_\lambda(F_3 \mid \mathcal{F}_{s_{T+1}}) - \mathbb{P}_\lambda(F_4^c \mid \mathcal{F}_{s_{T+1}}) \geq \frac{1}{72} \quad \text{on } F_1 \cap F_2.$$

Hence we conclude that

$$\begin{aligned} \mathbb{P}_\lambda(F_1 \cap F_2 \cap F_3 \cap F_4 \mid \mathcal{F}_{\eta_k}) &\geq \mathbb{E}_\lambda[\mathbb{P}_\lambda(F_3 \cap F_4 \mid \mathcal{F}_{s_{T+1}}) \mathbb{1}_{F_1 \cap F_2} \mid \mathcal{F}_{\eta_k}] \\ &\geq \frac{1}{72} \mathbb{P}_\lambda(F_1 \cap F_2 \mid \mathcal{F}_{\eta_k}) \geq \frac{q_1 p_\lambda}{288} L_j^4, \end{aligned}$$

which, with (4.19), completes the proof on setting $\varepsilon_0 = \frac{q_1}{576} e^{-1944\lambda_0}$. \square

Now we can complete the proof of Lemma 2.1.

PROOF OF LEMMA 2.1. Fix $\lambda_0 = \lambda > 0$, and let ε_0 be as in Lemma 4.5. By (4.18),

$$\mathbb{P}_\lambda(\alpha_k = 1 \mid \mathcal{F}_{\eta_k}) = \Pi_\lambda(\mathcal{Z}_{I_{\eta_k}}) \geq \sum_{j \in [I_{\eta_k}+1]} \pi_\lambda(\mathcal{Z}_{I_{\eta_k}}; j, B_0) \geq \varepsilon_0 \lambda \sum_{j \in [I_{\eta_k}+1]} L_{I_{\eta_k}, j}^4,$$

by Lemma 4.5. Then, by Jensen’s inequality, $\mathbb{P}_\lambda(\alpha_k = 1 \mid \mathcal{F}_{\eta_k}) \geq \varepsilon_0 \lambda (1 + I_{\eta_k})^{-3}$, since $\sum_{j \in [I_{\eta_k}+1]} L_{I_{\eta_k}, j} = 1$. Also, $\{\alpha_k = 1\} \in \mathcal{F}_{\eta_{k+1}}$. By Lévy’s extension of the Borel–Cantelli lemma (e.g., [24], Corollary 7.20), it follows that $\sum_{k \in \mathbb{Z}_+} (1 + I_{\eta_k})^{-3} = \infty$ implies that $\alpha_k = 1$ for infinitely many k . On the other hand, if $\sum_{k \in \mathbb{Z}_+} (1 + I_{\eta_k})^{-3} < \infty$, then $I_{\eta_k} \rightarrow \infty$. In either case, there are infinitely many nucleations. \square

The next result shows how the regenerative structure leads to a description of the joint distribution of the gap which nucleates and the nucleation location in terms of single-cycle distributions. Recall the definition of $D_t(j, B)$ from (4.16).

LEMMA 4.6. *Let $\lambda \in (0, \infty)$. For all $k \in \mathbb{Z}_+$, all $j \in [I_{\eta_k} + 1]$, and all $B \in \mathcal{B}$,*

$$\mathbb{P}_\lambda(D_{\eta_k}(j, B) \mid \mathcal{F}_{\eta_k}) = \frac{\pi_\lambda(\mathcal{Z}_{I_{\eta_k}}; j, B)}{\Pi_\lambda(\mathcal{Z}_{I_{\eta_k}})}, \quad \mathbb{P}_\lambda\text{-a.s.}$$

PROOF. Fix $k \in \mathbb{Z}_+$ and write $I = I_{\eta_k}$. For $m \in \mathbb{Z}_+$,

$$\begin{aligned} & \mathbb{P}_\lambda(\{\alpha_k = m + 1\} \cap D_{\eta_k}(j, B) \mid \mathcal{F}_{\eta_k}) \\ (4.22) \quad &= \mathbb{E}_\lambda[\mathbb{P}_\lambda(\{\alpha_k = m + 1\} \cap D_{\eta_{k+m}}(j, B) \mid \mathcal{F}_{\eta_{k+m}}) \mathbb{1}\{\alpha_k > m\} \mid \mathcal{F}_{\eta_k}] \\ &= \mathbb{E}_\lambda[\pi_\lambda(\mathcal{Z}_I; j, B) \mathbb{1}\{\alpha_k > m\} \mid \mathcal{F}_{\eta_k}] \\ &= \pi_\lambda(\mathcal{Z}_I; j, B) \mathbb{P}_\lambda(\alpha_k > m \mid \mathcal{F}_{\eta_k}), \end{aligned}$$

using the regeneration at time η_{k+m} and (4.17). Taking $B = [0, 1]$ and summing over $j \in [I + 1]$, we get $\mathbb{P}_\lambda(\alpha_k = m + 1 \mid \mathcal{F}_{\eta_k}) = \Pi_\lambda(\mathcal{Z}_I) \mathbb{P}_\lambda(\alpha_k > m \mid \mathcal{F}_{\eta_k})$. In other words,

$$\begin{aligned} \mathbb{P}_\lambda(\alpha_k > m + 1 \mid \mathcal{F}_{\eta_k}) &= \mathbb{P}_\lambda(\alpha_k > m \mid \mathcal{F}_{\eta_k}) - \mathbb{P}_\lambda(\alpha_k = m + 1 \mid \mathcal{F}_{\eta_k}) \\ &= (1 - \Pi_\lambda(\mathcal{Z}_I)) \mathbb{P}_\lambda(\alpha_k > m \mid \mathcal{F}_{\eta_k}). \end{aligned}$$

Iterating this gives $\mathbb{P}_\lambda(\alpha_k > m \mid \mathcal{F}_{\eta_k}) = (1 - \Pi_\lambda(\mathcal{Z}_I))^m$. Thus, by (4.22),

$$\mathbb{P}_\lambda(\{\alpha_k = m + 1\} \cap D_{\eta_k}(j, B) \mid \mathcal{F}_{\eta_k}) = (1 - \Pi_\lambda(\mathcal{Z}_I))^m \pi_\lambda(\mathcal{Z}_I; j, B).$$

Summing over $m \in \mathbb{Z}_+$ gives the result. \square

5. Splitting distribution estimates. Define $\kappa_n : \Delta_n \times [n + 1] \times \mathcal{B} \rightarrow [0, 1]$, $n \in \mathbb{Z}_+$, by

$$(5.1) \quad \kappa_n(z; j, B) := \frac{(z_j - z_{j-1})^4}{\sum_{i \in [n+1]} (z_i - z_{i-1})^4} \Phi_0(B),$$

the interval-splitting kernel in (2.3) specialized to the parameters r_0 and Φ_0 as appearing in Theorem 2.2. The main result of this section, as follows, shows that the evolution of the island locations in our nucleation process is approximated by the kernel (5.1). This result will serve both for fixed time as $\lambda \rightarrow 0$, and for fixed λ in the long-time limit. In the supremum in Proposition 5.1, and subsequent similar instances, $B_j \in \mathcal{B}$ for each j .

PROPOSITION 5.1. *For any $\lambda_0 \in (0, \infty)$ there exists a constant $C_2 = C_2(\lambda_0) < \infty$ such that, for all $\lambda \in (0, \lambda_0]$ and all $k \in \mathbb{Z}_+$, \mathbb{P}_λ -a.s.,*

$$\sup_{B_1, \dots, B_{I_{\eta_k}+1}} \left| \sum_{j \in [I_{\eta_k}+1]} \mathbb{P}_\lambda(D_{\eta_k}(j, B_j) \mid \mathcal{F}_{\eta_k}) - \sum_{j \in [I_{\eta_k}+1]} \kappa_{I_{\eta_k}}(\mathcal{Z}_{I_{\eta_k}}; j, B_j) \right| \leq C_2 \lambda^{1/2} M_{\eta_k}^{3/2}.$$

The rest of this section will develop the proof of Proposition 5.1, which is built on the regeneration structure in Lemma 4.6. First, we explain the origin of Φ_0 .

Recall from (4.11) that $\nu(\lambda; B) = \mathbb{P}_\lambda(E_1 \cap \{\zeta \in B\})$, where E_1 is the event that at least one nucleation occurs in time interval $[\sigma_1, \eta_1]$, and ζ is the location of the first nucleation. For the $\lambda \rightarrow 0$ asymptotics of $\nu(\lambda; B)$ we need some more notation.

Let w denote standard Brownian motion on \mathbb{R} , started at $x \in [0, 1]$, and set $\tau := \inf\{t \in \mathbb{R}_+ : w_t \notin (0, 1)\}$, the first exit time from the interval $(0, 1)$. Then for $B \in \mathcal{B}$,

$$\mathbb{P}(w_t \in B, t \leq \tau \mid w_0 = x) = \int_B q_t(x, y) dy,$$

where

$$(5.2) \quad q_t(x, y) := \frac{1}{\sqrt{2\pi t}} \sum_{k \in \mathbb{Z}} \left\{ \exp\left(-\frac{(y-x+2k)^2}{2t}\right) - \exp\left(-\frac{(y+x+2k)^2}{2t}\right) \right\};$$

see, for example, [18], pages 341–342, or [11], pages 122, 174. The density $q_t(x; \cdot)$ corresponds to a (defective) distribution with total mass $\mathbb{P}(t \leq \tau \mid w_0 = x)$.

Let W denote a standard Brownian motion in \mathbb{R}^2 given in components as $W_t = (W_t^{(1)}, W_t^{(2)})$, and let $S := \partial[0, 1]^2$ and $D := \{(x, y) \in [0, 1]^2 : x = y\}$ denote the boundary and diagonal of the unit square, respectively. For measurable $A \subseteq \mathbb{R}^2$, define $\tau_A := \inf\{t \in \mathbb{R}_+ : W_t \in A\}$. For $u, v \in [0, 1]^2$ and $B \in \mathcal{B}$, set

$$(5.3) \quad H(u, v; B) := \mathbb{P}(\tau_D < \tau_S, W_{\tau_D}^{(1)} \in B \mid W_0 = (u, v)),$$

so that $H(u, v; \cdot)$ is a measure on $([0, 1], \mathcal{B})$ with total mass $H(u, v; [0, 1]) = \mathbb{P}(\tau_D < \tau_S \mid W_0 = (u, v))$. Define

$$(5.4) \quad \Phi_1(B) := \int_0^1 dz \int_0^1 dy \int_0^1 dx \int_0^\infty q_t(x, y) H(y, z; B) dt.$$

The proof of the following result is given in Section 8.

PROPOSITION 5.2. *We have that $\Phi_1 = \mu \Phi_0$, where μ is given by (2.5) and Φ_0 is defined at (2.6). In particular, $\Phi_1([0, 1]) = \mu$.*

We will use the simple fact that if Z is Poisson with mean $\theta \in \mathbb{R}_+$, then for all $k \in \mathbb{N}$,

$$(5.5) \quad \begin{aligned} k\mathbb{P}(Z \geq k) &\leq \mathbb{E}[Z \mathbb{1}\{Z \geq k\}] = e^{-\theta} \sum_{\ell=k-1}^\infty \frac{\theta^{\ell+1}}{\ell!} \\ &\leq \theta^k e^{-\theta} \sum_{\ell=k-1}^\infty \frac{\theta^{\ell-k+1}}{(\ell-k+1)!} = \theta^k. \end{aligned}$$

The next result shows how Φ_1 , and hence, by Proposition 5.2, Φ_0 , arises in our model.

LEMMA 5.3. *For any $\lambda_0 \in (0, \infty)$ there is a constant $C_3 = C_3(\lambda_0) < \infty$ such that*

$$(5.6) \quad \sup_{B \in \mathcal{B}} |\nu(\lambda; B) - \lambda \Phi_1(B)| \leq C_3 \lambda^{5/3} \quad \text{for all } \lambda \in (0, \lambda_0].$$

PROOF. In order for E_1 to occur, the particle that arrives at time $s_1 = \sigma_1$ must remain active until the second particle arrives at time s_2 (or else the number of active particles would fall to zero). Define events $F_1(B) = \{1 \in \mathcal{A}_{s_2}, x_1(s_2) \in B\}$, and $F_1 = F_1((0, 1))$, the event that the first particle is still active when the second one arrives. We have

$$\mathbb{P}_\lambda(F_1(B)) = \mathbb{E}_\lambda[q_{s_2-s_1}(\xi_1, B)] = \int_B dy \int_0^1 dx \int_0^\infty \lambda e^{-\lambda t} q_t(x, y) dt,$$

since ξ_1 is uniform on $[0, 1]$, $s_2 - s_1$ is exponential with parameter λ , and the two are independent. From time s_2 , on F_1 , there are active particles at $x_1(s_2) = y$ (say) and $x_2(s_2) = \xi_2 = z$ (say); if these two particles meet in B before either exits $[0, 1]$ (call this event $F_2(y, z; B)$), and no other particle is deposited in the meantime, then $E_1 \cap \{\zeta \in B\}$ occurs. Any other way for $E_1 \cap \{\zeta \in B\}$ to occur requires that a third particle arrive before time η_1 . Thus if $F_3 = \{s_3 > \eta_1\}$, we have

$$F_1 \cap F_2(x_1(s_2), \xi_2; B) \cap F_3 \subseteq E_1 \cap \{\zeta \in B\} \subseteq (F_1 \cap F_2(x_1(s_2), \xi_2; B)) \cup F_3^c.$$

It follows that

$$(5.7) \quad |\mathbb{P}_\lambda(E_1 \cap \{\zeta \in B\}) - \mathbb{P}_\lambda(F_1 \cap F_2(x_1(s_2), \xi_2; B))| \leq \mathbb{P}_\lambda(F_3^c).$$

Here

$$\mathbb{P}_\lambda(F_1 \cap F_2(x_1(s_2), \xi_2; B)) = \int_0^1 dz \int_0^1 \mathbb{P}_\lambda(F_1(dy)) \mathbb{P}_\lambda(F_2(y, z; B)),$$

using the Markov property at time s_2 , and the fact that ξ_2 is uniform on $[0, 1]$. Thus

$$\mathbb{P}_\lambda(F_1 \cap F_2(x_1(s_2), \xi_2; B)) = \lambda \int_0^1 dz \int_0^1 dy \int_0^1 dx \int_0^\infty e^{-\lambda t} q_t(x, y) H(y, z; B) dt,$$

and hence, by (5.4) and the fact that $H(y, z; B) \leq 1$ and $1 - e^{-z} \leq z$,

$$(5.8) \quad \begin{aligned} & \sup_{B \in \mathcal{B}} |\mathbb{P}_\lambda(F_1 \cap F_2(x_1(s_2), \xi_2; B)) - \lambda \Phi_1(B)| \\ & \leq \lambda \int_0^1 dx \int_0^\infty (1 - e^{-\lambda t}) \mathbb{P}(\tau \geq t \mid w_0 = x) dt \\ & \leq \lambda^2 \int_0^1 \mathbb{E}(\tau^2 \mid w_0 = x) dx, \end{aligned}$$

which is $O(\lambda^2)$. Let $\varepsilon \in (0, 1)$. If Z is the number of arrivals in time interval $(\sigma_1, \sigma_1 + \lambda^{-\varepsilon}]$, then, since Z is Poisson with mean $\lambda^{1-\varepsilon}$, $\mathbb{P}_\lambda(Z \geq 2) \leq \lambda^{2-2\varepsilon}$ by (5.5), and

$$(5.9) \quad \mathbb{P}_\lambda(F_3^c) \leq \mathbb{P}_\lambda(\eta_1 - \sigma_1 \geq \lambda^{-\varepsilon}) + \mathbb{P}_\lambda(Z \geq 2) \leq C_1 \exp(-\delta \lambda^{-\varepsilon/2}) + \lambda^{2-2\varepsilon},$$

by Lemma 4.2. The result follows from (5.7), (5.8), and (5.9). \square

Consider the end of a cycle at time η_k . Denote by $J_k \in [I_{\eta_k} + 1]$ the index such that the arrival at time σ_{k+1} lands in gap $[Z_{I_{\eta_k}, J_k-1}, Z_{I_{\eta_k}, J_k}]$. Let \mathcal{F}'_{η_k} denote the σ -algebra generated by \mathcal{F}_{η_k} and the value J_k , so \mathcal{F}'_{η_k} identifies the gap occupied by the first arrival after η_k , but not that arrival's location in the gap.

Let $G_k(j, s)$ be the event that during time interval $[\sigma_{k+1}, \sigma_{k+1} + s]$ at least one nucleation occurs in gap $j \in [I_{\eta_k} + 1]$. The next result gives an upper bound on nucleation occurring outside gap J_k during a fixed time horizon.

LEMMA 5.4. *Let $\lambda_0 \in (0, \infty)$. There exists a constant $C_4 = C_4(\lambda_0) < \infty$ such that, for all $\lambda \in (0, \lambda_0]$, all $s \in [0, \frac{1}{2\lambda}]$, all $k \in \mathbb{Z}_+$, and all $j \in [I_{\eta_k} + 1] \setminus \{J_k\}$,*

$$\mathbb{P}_\lambda(G_k(j, s) \mid \mathcal{F}'_{\eta_k}) \leq C_4 \lambda^2 s L_{I_{\eta_k}, j}^4, \quad \mathbb{P}_\lambda\text{-a.s.}$$

PROOF. Fix $k \in \mathbb{Z}_+$ and write $I = I_{\eta_k}$, $J = J_k$, and, for $1 \leq j \leq I + 1$, $Z_j = Z_{I, j}$ and $L_j = L_{I, j}$. Given \mathcal{F}'_{η_k} , take $j \in [I + 1] \setminus \{J\}$. For the process restricted to the interval $[Z_{j-1}, Z_j]$, define ‘‘local cycles’’ $[\sigma_{j, \ell}, \eta_{j, \ell}]$ by $\eta_{j, 0} := \sigma_{k+1}$ (at which point there are no active particles in gap j) and, for $\ell \in \mathbb{N}$,

$$\begin{aligned} \sigma_{j, \ell} &= \inf \left\{ t > \eta_{j, \ell-1} : \sum_{i \in \mathcal{A}_t} \mathbb{1}\{x_i(t) \in [Z_{j-1}, Z_j]\} = 1 \right\}, \\ \eta_{j, \ell} &= \inf \left\{ t > \sigma_{j, \ell} : \sum_{i \in \mathcal{A}_t} \mathbb{1}\{x_i(t) \in [Z_{j-1}, Z_j]\} = 0 \right\}. \end{aligned}$$

Nucleation in $[Z_{j-1}, Z_j]$ can only occur during time intervals $[\sigma_{j, \ell}, \eta_{j, \ell}]$. In order for $G_k(j, s)$ to occur via nucleation during $[\sigma_{j, \ell}, \eta_{j, \ell}]$, there must have been at least ℓ arrivals in

$[Z_{j-1}, Z_j]$ during time interval $[\sigma_{k+1}, \sigma_{k+1} + s]$, and then nucleation must occur during that cycle, an event of probability $\mu(L_j, \lambda)$. Together with (5.5) this gives

$$\mathbb{P}_\lambda(G_k(j, s) \mid \mathcal{F}'_{\eta_k}) \leq \sum_{\ell=1}^\infty \lambda^\ell L_j^\ell s^\ell \mu(L_j, \lambda) = \lambda L_j s \mu(L_j^3 \lambda) \sum_{\ell=0}^\infty \lambda^\ell L_j^\ell s^\ell,$$

by (4.13). Here $\lambda L_j s \leq 1/2$, provided $s \leq \frac{1}{2\lambda}$. Then $\mathbb{P}_\lambda(G_k(j, s) \mid \mathcal{F}'_{\eta_k}) \leq 2\lambda L_j s \mu(L_j^3 \lambda)$, and Lemma 5.3 completes the proof. \square

Now we can give the proof of Proposition 5.1.

PROOF OF PROPOSITION 5.1. Fix $k \in \mathbb{Z}_+$ and write $I = I_{\eta_k}$, $J = J_k$, $M = M_{\eta_k}$, and, for $1 \leq j \leq I + 1$, $Z_j = Z_{I,j}$ and $L_j = L_{I,j}$. The new arrival at time σ_{k+1} is deposited in gap J . We show that the main contribution to $\mathbb{P}_\lambda(D_{\eta_k}(j, B) \mid \mathcal{F}_{\eta_k})$ comes from $J = j$.

Fix $\lambda_0 \in (0, \infty)$. Define event $G_k(s) := \bigcup_{j \in [I+1] \setminus \{J\}} G_k(j, s)$, that there is at least one nucleation outside interval $[Z_{J-1}, Z_J]$ during time interval $[\sigma_{k+1}, \sigma_{k+1} + s]$. By Lemma 5.4,

$$\begin{aligned} \mathbb{P}_\lambda(G_k(s) \mid \mathcal{F}_{\eta_k}) &\leq \mathbb{E}_\lambda \left[\sum_{j \in [I+1] \setminus \{J\}} \mathbb{P}(G_k(j, s) \mid \mathcal{F}'_{\eta_k}) \mid \mathcal{F}_{\eta_k} \right] \\ &\leq C_4 \lambda^2 s \mathbb{E}_\lambda \left[\sum_{j \in [I+1] \setminus \{J\}} L_j^4 \mid \mathcal{F}_{\eta_k} \right] \leq C_4 \lambda^2 s \sum_{j \in [I+1]} L_j^4, \end{aligned}$$

for all $\lambda \in (0, \lambda_0]$ and all $s \leq \frac{1}{2\lambda}$. Moreover, from Lemma 4.2 we have that

$$\begin{aligned} \mathbb{P}_\lambda(\eta_{k+1} - \sigma_{k+1} \geq s \mid \mathcal{F}_{\eta_k}) &= \mathbb{E}_\lambda[\mathbb{P}_\lambda(\eta_{k+1} - \sigma_{k+1} \geq s \mid \mathcal{F}_{\sigma_{k+1}}) \mid \mathcal{F}_{\eta_k}] \\ &\leq C_1 \mathbb{E}_\lambda[\exp(-\delta M_{\sigma_{k+1}}^{-1} s^{1/2}) \mid \mathcal{F}_{\eta_k}]. \end{aligned}$$

Then, if $G_\star := G_k(\eta_{k+1} - \sigma_{k+1})$, since $M_{\sigma_{k+1}} \leq M_{\eta_k} = M$, we get

$$\begin{aligned} \mathbb{P}_\lambda(G_\star \mid \mathcal{F}_{\eta_k}) &\leq \mathbb{P}_\lambda(\eta_{k+1} - \sigma_{k+1} \geq s \mid \mathcal{F}_{\eta_k}) + \mathbb{P}_\lambda(G_k(s) \mid \mathcal{F}_{\eta_k}) \\ (5.10) \quad &\leq C_1 \exp(-\delta M^{-1} s^{1/2}) + C_4 \lambda^2 s \sum_{i \in [I+1]} L_i^4, \end{aligned}$$

for all $\lambda \in (0, \lambda_0]$ and all $s \leq \frac{1}{2\lambda}$. For $\varepsilon \in (0, 1)$, take $s = M^{2-\varepsilon} \min(\lambda^{-1/2}, \frac{1}{2}\lambda^{-1})$. By (5.10) and noting that $\sum_{i \in [I+1]} L_i^4 \geq M^4$, we get, for some $C < \infty$ and all $\lambda \in (0, \lambda_0]$,

$$(5.11) \quad \mathbb{P}_\lambda(G_\star \mid \mathcal{F}_{\eta_k}) \leq C \lambda^{3/2} M^{2-\varepsilon} \sum_{i \in [I+1]} L_i^4 \quad \text{a.s.}$$

For $t \in \mathbb{R}_+$, let A'_t denote the number of active particles in gap $[Z_{J-1}, Z_J]$ at time $\sigma_{k+1} + t$, and let $\eta' := \inf\{t > 0 : A'_t = 0\}$; note $A'_0 = 1$. Let $\zeta^\star \in (0, 1)$ denote the relative location of the first nucleation for the process restricted to gap $[Z_{J-1}, Z_J]$. Observe that on the event G_\star^c , we have $\zeta_{\eta_k} = \zeta^\star$ and nucleation occurs in gap J . Let E' be the event that nucleation occurs in gap $[Z_{J-1}, Z_J]$ during time interval $[\sigma_{k+1}, \sigma_{k+1} + \eta']$. Then

$$\begin{aligned} &\left| \sum_{j \in [I+1]} \mathbb{P}_\lambda(\{\alpha_k = 1\} \cap D_{\eta_k}(j, B_j) \mid \mathcal{F}_{\eta_k}) \right. \\ (5.12) \quad &\quad \left. - \sum_{j \in [I+1]} \mathbb{P}_\lambda(E' \cap \{J = j\} \cap \{\zeta^\star \in B_j\} \mid \mathcal{F}_{\eta_k}) \right| \\ &\leq \mathbb{P}_\lambda(G_\star \mid \mathcal{F}_{\eta_k}). \end{aligned}$$

Here

$$\mathbb{P}_\lambda(E' \cap \{J = j\} \cap \{\zeta^* \in B_j\} \mid \mathcal{F}_{\eta_k}) = \mathbb{E}_\lambda[\mathbb{P}_\lambda(E' \cap \{\zeta^* \in B_j\} \mid \mathcal{F}'_{\eta_k}) \mathbb{1}\{J = j\} \mid \mathcal{F}_{\eta_k}].$$

The event $E' \cap \{\zeta^* \in B\}$ depends only on the process restricted to the interval $[Z_{J-1}, Z_J]$ after time σ_{k+1} , which has the same law as the process on interval $[0, L_J]$ after time σ_1 , for which the event $E' \cap \{\zeta^* \in B\}$ translates as $E_1 \cap \{\zeta \in B\}$. Thus,

$$\mathbb{P}_\lambda(E' \cap \{\zeta^* \in B\} \mid \mathcal{F}'_{\eta_k}) = \mathbb{P}_{L_J, \lambda}(E_1 \cap \{\zeta \in B\}) = \nu(L_J, \lambda; B),$$

by (4.11). Then, since $\nu(L_j, \lambda; B)$ is \mathcal{F}_{η_k} -measurable and $\mathbb{P}_\lambda(J = j \mid \mathcal{F}_{\eta_k}) = L_j$, we obtain

$$(5.13) \quad \mathbb{P}_\lambda(E' \cap \{J = j\} \cap \{\zeta^* \in B_j\} \mid \mathcal{F}_{\eta_k}) = L_j \nu(L_j, \lambda; B_j).$$

Then from (4.17) with (5.11), (5.12), (5.13), and the scaling property (4.13), there is a constant $C < \infty$ such that, a.s., for all $\lambda \in (0, \lambda_0]$,

$$(5.14) \quad \begin{aligned} & \sup_{B_1, \dots, B_{I+1}} \left| \sum_{j \in [I+1]} \pi_\lambda(\mathcal{Z}_I; j, B_j) - \sum_{j \in [I+1]} L_j \nu(L_j^3 \lambda; B_j) \right| \\ & \leq C \lambda^{3/2} M^{2-\varepsilon} \sum_{j \in [I+1]} L_j^4. \end{aligned}$$

Now applying (5.6), we have from (5.14) that, for all $\lambda \in (0, \lambda_0]$,

$$(5.15) \quad \begin{aligned} & \sup_{B_1, \dots, B_{I+1}} \left| \sum_{j \in [I+1]} \pi_\lambda(\mathcal{Z}_I; j, B_j) - \lambda \sum_{j \in [I+1]} L_j^4 \Phi_1(B_j) \right| \\ & \leq C_3 \lambda^{5/3} \sum_{j \in [I+1]} L_j^6 + C \lambda^{3/2} M^{2-\varepsilon} \sum_{j \in [I+1]} L_j^4 \\ & \leq C \lambda^{3/2} M^{2-\varepsilon} \sum_{j \in [I+1]} L_j^4, \end{aligned}$$

redefining $C < \infty$ as necessary, since $L_j^6 \leq M^2 L_j^4$. Taking all the $B_j = [0, 1]$ in (5.15), and using the fact that $\Phi_1([0, 1]) = \mu$ (see Proposition 5.2), we get

$$(5.16) \quad \left| \Pi_\lambda(\mathcal{Z}_I) - \mu \lambda \sum_{j \in [I+1]} L_j^4 \right| \leq C \lambda^{3/2} M^{2-\varepsilon} \sum_{j \in [I+1]} L_j^4.$$

For the constants $\varepsilon > 0$ and $C < \infty$ as appearing in (5.15) and (5.16), let $\varepsilon_0 = \frac{\mu}{2C}$ and define the event $F := \{\lambda^{1/2} M^{2-\varepsilon} \leq \varepsilon_0\}$. Then, from (5.16),

$$(5.17) \quad \Pi_\lambda(\mathcal{Z}_I) \geq \frac{\mu \lambda}{2} \sum_{j \in [I+1]} L_j^4 \quad \text{on } F.$$

Thus we have from (5.15) and (5.17) that, for all $\lambda \in (0, \lambda_0]$,

$$\sup_{B_1, \dots, B_{I+1}} \left| \frac{\sum_{j \in [I+1]} \pi_\lambda(\mathcal{Z}_I; j, B_j)}{\Pi_\lambda(\mathcal{Z}_I)} - \frac{\lambda \sum_{j \in [I+1]} L_j^4 \Phi_1(B_j)}{\Pi_\lambda(\mathcal{Z}_I)} \right| \leq C \lambda^{1/2} M^{2-\varepsilon} \quad \text{on } F.$$

Moreover, since $\sum_{j \in [I+1]} L_j^4 \Phi_1(B_j) \leq \mu \sum_{j \in [I+1]} L_j^4$, we have that, on F ,

$$\begin{aligned} \left| \frac{\lambda \sum_{j \in [I+1]} L_j^4 \Phi_1(B_j)}{\Pi_\lambda(\mathcal{Z}_I)} - \frac{\lambda \sum_{j \in [I+1]} L_j^4 \Phi_1(B_j)}{\mu \lambda \sum_{j \in [I+1]} L_j^4} \right| & \leq \left| \frac{\Pi_\lambda(\mathcal{Z}_I) - \mu \lambda \sum_{j \in [I+1]} L_j^4}{\Pi_\lambda(\mathcal{Z}_I)} \right| \\ & \leq C \lambda^{1/2} M^{2-\varepsilon}, \end{aligned}$$

by (5.16) and (5.17). Combining the bounds in the last two displays we get

$$\sup_{B_1, \dots, B_{I+1}} \left| \frac{\sum_{j \in [I+1]} \pi_\lambda(\mathcal{Z}_I; j, B_j)}{\Pi_\lambda(\mathcal{Z}_I)} - \frac{\sum_{j \in [I+1]} L_j^4 \Phi_1(B_j)}{\mu \sum_{j \in [I+1]} L_j^4} \right| \leq C\lambda^{1/2} M^{2-\varepsilon} \quad \text{on } F.$$

Lemma 4.6 and the fact that $\Phi_1(B) = \mu \Phi_0(B)$ (Proposition 5.2) finish the proof. \square

6. Sparse deposition regime. In this section we focus on the $\lambda \rightarrow 0$ regime, and prove Theorem 2.2. Proposition 5.1 refers to the next nucleation after time η_k . For the convergence of finite-dimensional distributions in Theorem 2.2, we need to consider the next nucleation after time ν_n , the previous nucleation time. This is the purpose of the next result.

LEMMA 6.1. *For any $n \in \mathbb{Z}_+$, we have*

$$\lim_{\lambda \rightarrow 0} \mathbb{E}_\lambda \sup_{B_1, \dots, B_{n+1}} \left| \sum_{j \in [n+1]} \mathbb{P}_\lambda(D_{\nu_n}(j, B_j) \mid \mathcal{F}_{\nu_n}) - \sum_{j \in [n+1]} \kappa_n(\mathcal{Z}_n; j, B_j) \right| = 0.$$

For the proof of this result, and later, it is useful to define

$$(6.1) \quad k_n := \min\{k \in \mathbb{Z}_+ : \eta_k \geq \nu_n\} \quad \text{for } n \in \mathbb{Z}_+.$$

Then $k_0 = 0$, and, for all $n \in \mathbb{N}$, $\sigma_{k_n} < \nu_n \leq \eta_{k_n}$ for $k_n \in \mathbb{N}$. Note that η_{k_n} is a stopping time, but σ_{k_n} , $n \in \mathbb{N}$, is not a stopping time.

PROOF OF LEMMA 6.1. With k_n as defined at (6.1), we have

$$\mathbb{P}_\lambda(D_{\nu_n}(j, B_j) \mid \mathcal{F}_{\nu_n}) = \mathbb{E}_\lambda[\mathbb{P}_\lambda(D_{\nu_n}(j, B_j) \mid \mathcal{F}_{\eta_{k_n}}) \mid \mathcal{F}_{\nu_n}].$$

Let F_k be the event that there are two or more nucleations in time interval $[\sigma_k, \eta_k]$. On $F_{k_n}^c$, there is no nucleation in the interval $(\nu_n, \eta_{k_n}]$, and so $D_{\nu_n}(j, B) = D_{\eta_{k_n}}(j, B)$. Thus

$$(6.2) \quad \sup_{B_1, \dots, B_{n+1}} \left| \sum_{j \in [n+1]} \mathbb{P}_\lambda(D_{\nu_n}(j, B_j) \mid \mathcal{F}_{\eta_{k_n}}) - \sum_{j \in [n+1]} \kappa_n(\mathcal{Z}_n; j, B_j) \right| \leq C_2 \lambda^{1/2} + \mathbb{1}_{F_{k_n}},$$

by Proposition 5.1 and the fact that $I_{\eta_{k_n}} = n$ on $F_{k_n}^c$. If $G_k = \bigcup_{i=1}^k F_i$, then

$$(6.3) \quad \sup_{B_1, \dots, B_{n+1}} \left| \sum_{j \in [n+1]} \mathbb{P}_\lambda(D_{\nu_n}(j, B_j) \mid \mathcal{F}_{\nu_n}) - \sum_{j \in [n+1]} \kappa_n(\mathcal{Z}_n; j, B_j) \right| \leq C_2 \lambda^{1/2} + \mathbb{P}_\lambda(G_{k_n} \mid \mathcal{F}_{\nu_n}),$$

by (6.2) and the fact that $\kappa_n(\mathcal{Z}_n; j, B)$ is \mathcal{F}_{ν_n} -measurable. Next, we bound $\mathbb{P}_\lambda(F_k)$ and hence $\mathbb{P}_\lambda(G_{k_n})$. In order for there to be two (or more) nucleations in time interval $[\sigma_k, \eta_k]$, there must be at least three deposition events during time interval $(\sigma_k, \eta_k]$. Let Z denote the number of deposition events during time $(\sigma_k, \sigma_k + \lambda^{-1/6}]$. Then,

$$\mathbb{P}_\lambda(F_k \mid \mathcal{F}_{\sigma_k}) \leq \mathbb{P}_\lambda(\eta_k - \sigma_k > \lambda^{-1/6} \mid \mathcal{F}_{\sigma_k}) + \mathbb{P}_\lambda(Z \geq 3 \mid \mathcal{F}_{\sigma_k}),$$

and, since, given \mathcal{F}_{σ_k} , Z is Poisson with mean $\lambda^{5/6}$, $\mathbb{P}_\lambda(Z \geq 3 \mid \mathcal{F}_{\sigma_k}) \leq \lambda^{5/2}$, by (5.5). Together with the tail bound in Lemma 4.2, this shows that $\mathbb{P}_\lambda(F_k \mid \mathcal{F}_{\sigma_k}) \leq C\lambda^{5/2}$, for some $C < \infty$ and all $\lambda \in (0, 1]$, say. For fixed $n \in \mathbb{Z}_+$ and $\varepsilon > 0$, choose k sufficiently large so that $\mathbb{P}_\lambda(k_n > k) \leq \varepsilon$. Then

$$\mathbb{P}_\lambda(G_{k_n}) \leq \mathbb{P}_\lambda(k_n > k) + \sum_{i=1}^k \mathbb{P}_\lambda(F_i) \leq \varepsilon + Ck\lambda^{5/2}.$$

Thus, for fixed n and $\varepsilon > 0$, we may choose λ small enough so that $\mathbb{P}_\lambda(G_{k_n}) \leq 2\varepsilon$. Hence $\lim_{\lambda \rightarrow 0} \mathbb{P}_\lambda(G_{k_n}) = 0$. Together with (6.3), this completes the proof. \square

Now we are ready to prove Theorem 2.2. Recall the definition of the splitting function Γ_n from (2.2), and that, from (2.3) and (5.1), the interval-splitting process $\mathcal{S} = (\mathcal{S}_0, \mathcal{S}_1, \dots)$ with parameters r_0 and Φ_0 has

$$\mathbb{P}(\mathcal{S}_{n+1} \in \Gamma_n(\mathcal{S}_n; j, B) \mid \mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_n) = \kappa_n(\mathcal{S}_n; j, B).$$

Define the transition kernel $P_n : \Delta_n \times \mathcal{B}_{n+1} \rightarrow [0, 1]$ by $\mathbb{P}(\mathcal{S}_{n+1} \in A \mid \mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_n) = P_n(\mathcal{S}_n, A)$, where $A \in \mathcal{B}_{n+1}$ and \mathcal{B}_n denotes the Borel sets on Δ_n . Then, if $\Gamma_n^{-1}(z; j, A) := \{v \in [0, 1] : \Gamma_n(z; j, v) \in A\}$ for $A \in \mathcal{B}_{n+1}$, we have from (5.1) that

$$P_n(z, A) = \sum_{j \in [n+1]} \kappa_n(z; j, \Gamma_n^{-1}(z; j, A)) = \frac{\sum_{j \in [n+1]} (z_j - z_{j-1})^4 \Phi_0(\Gamma_n^{-1}(z; j, A))}{\sum_{j \in [n+1]} (z_j - z_{j-1})^4},$$

where $z = (z_0, \dots, z_{n+1}) \in \Delta_n$.

PROOF OF THEOREM 2.2. Define for $n \in \mathbb{N}$ and $A_1 \in \Delta_1, \dots, A_n \in \Delta_n$,

$$K_n(A_1, \dots, A_n) := \mathbb{P}(\mathcal{S}_1 \in A_1, \dots, \mathcal{S}_n \in A_n),$$

where \mathcal{S} is the interval-splitting process with parameters r_0 and Φ_0 . Then $K_1(A) = P_0(\mathcal{Z}_0, A)$ and, for $n \in \mathbb{N}$,

$$\begin{aligned} K_{n+1}(A_1, \dots, A_{n+1}) &= \mathbb{E}[\mathbb{1}_{\{\mathcal{S}_1 \in A_1, \dots, \mathcal{S}_n \in A_n\}} \mathbb{P}(\mathcal{S}_{n+1} \in A_{n+1} \mid \mathcal{S}_0, \dots, \mathcal{S}_n)] \\ (6.4) \qquad \qquad \qquad &= \mathbb{E}[\mathbb{1}_{\{\mathcal{S}_1 \in A_1, \dots, \mathcal{S}_n \in A_n\}} P_n(\mathcal{S}_n, A_{n+1})] \\ &= \int_{A_n} K_n(A_1, \dots, A_{n-1}, dz) P_n(z, A_{n+1}). \end{aligned}$$

We wish to prove that for any $n \in \mathbb{N}$,

$$(6.5) \qquad \lim_{\lambda \rightarrow 0} \sup_{A_1, \dots, A_n} |\mathbb{P}_\lambda(\mathcal{Z}_1 \in A_1, \dots, \mathcal{Z}_n \in A_n) - K_n(A_1, \dots, A_n)| = 0,$$

the supremum over $A_1 \in \mathcal{B}_1, \dots, A_n \in \mathcal{B}_n$. We establish (6.5) by induction on n . First,

$$(6.6) \qquad \begin{aligned} &|\mathbb{P}_\lambda(\mathcal{Z}_{n+1} \in A \mid \mathcal{F}_{v_n}) - P_n(\mathcal{Z}_n, A)| \\ &= \left| \sum_{j \in [n+1]} \mathbb{P}_\lambda(D_{v_n}(j, \Gamma_n^{-1}(\mathcal{Z}_n; j, A)) \mid \mathcal{F}_{v_n}) - \sum_{j \in [n+1]} \kappa_n(\mathcal{Z}_n; j, \Gamma_n^{-1}(\mathcal{Z}_n; j, A)) \right|. \end{aligned}$$

By (6.6), a consequence of Lemma 6.1 is that, for all $n \in \mathbb{Z}_+$,

$$(6.7) \qquad \lim_{\lambda \rightarrow 0} \mathbb{E}_\lambda \sup_{A \in \mathcal{B}_{n+1}} |\mathbb{P}_\lambda(\mathcal{Z}_{n+1} \in A \mid \mathcal{F}_{v_n}) - P_n(\mathcal{Z}_n, A)| = 0.$$

In particular, taking $n = 0$ in (6.7), we get the $n = 1$ case of (6.5), the basis for the induction. For the inductive step, suppose that (6.5) holds for some given $n \in \mathbb{N}$. Then

$$\mathbb{P}_\lambda(\mathcal{Z}_1 \in A_1, \dots, \mathcal{Z}_{n+1} \in A_{n+1}) = \mathbb{E}_\lambda[\mathbb{1}_{\{\mathcal{Z}_1 \in A_1, \dots, \mathcal{Z}_n \in A_n\}} \mathbb{P}_\lambda(\mathcal{Z}_{n+1} \in A_{n+1} \mid \mathcal{F}_{v_n})].$$

By (6.7), it follows that

$$\begin{aligned} &\lim_{\lambda \rightarrow 0} \sup_{A_1, \dots, A_{n+1}} |\mathbb{P}_\lambda(\mathcal{Z}_1 \in A_1, \dots, \mathcal{Z}_{n+1} \in A_{n+1}) \\ &\quad - \mathbb{E}_\lambda[\mathbb{1}_{\{\mathcal{Z}_1 \in A_1, \dots, \mathcal{Z}_n \in A_n\}} P_n(\mathcal{Z}_n, A_{n+1})]| = 0. \end{aligned}$$

Now by inductive hypothesis (6.5) and the relationship between total-variation distance and coupling, for any $\varepsilon > 0$ we can choose $\lambda > 0$ sufficiently small, and work on a suitable probability space in which $\mathbb{P}((X_1, \dots, X_n) \neq (Y_1, \dots, Y_n)) \leq \varepsilon$ and (X_1, \dots, X_n) has law $\mathbb{P}_\lambda(\mathcal{Z}_1 \in \cdot, \dots, \mathcal{Z}_n \in \cdot)$ and (Y_1, \dots, Y_n) has law K_n . Hence

$$\sup_{A_1, \dots, A_{n+1}} |\mathbb{E}_\lambda[\mathbb{1}_{\{\mathcal{Z}_1 \in A_1, \dots, \mathcal{Z}_n \in A_n\}} P_n(\mathcal{Z}_n, A_{n+1})] - \mathbb{E}[\mathbb{1}_{\{Y_1 \in A_1, \dots, Y_n \in A_n\}} P_n(Y_n, A_{n+1})]| \leq \varepsilon,$$

and the expectation involving the Y_i s is, by (6.4), equal to $K_{n+1}(A_1, \dots, A_{n+1})$. This completes the inductive step. \square

7. Fixed-rate deposition regime. In this section, we will prove Theorem 2.3, which says that, roughly speaking, the long-term asymptotics of the fixed- λ process are governed by the interval-splitting process that arises as the $\lambda \rightarrow 0$ limit established in Theorem 2.2. The intuition for this is that as time goes on, the gaps get smaller and so capture of active particles by existing islands gets faster, which has a similar effect as driving down the deposition rate.

The proof of Theorem 2.3 uses coupling based on the following result.

PROPOSITION 7.1. *For any $\lambda \in (0, \infty)$,*

$$\mathbb{E}_\lambda \sum_{n \in \mathbb{Z}_+} \sup_{B_1, \dots, B_{n+1}} \left| \sum_{j \in [n+1]} \mathbb{P}_\lambda(D_{v_n}(j, B_j) \mid \mathcal{F}_{v_n}) - \sum_{j \in [n+1]} \kappa_n(\mathcal{Z}_n; j, B_j) \right| < \infty.$$

To obtain Proposition 7.1, we need an improved version of the bound in Lemma 6.1, and this requires control of the chance of additional nucleations occurring in time interval $(v_n, \eta_{k_n}]$, where k_n is as defined at (6.1). This is the purpose of the next lemma.

Let χ_k denote the number of nucleations during time interval $[\sigma_k, \eta_k]$.

LEMMA 7.2. *For any $\lambda_0 \in (0, \infty)$, there exists $C_5 = C_5(\lambda_0) < \infty$ such that, for all $\lambda \in (0, \lambda_0]$ and all $k \in \mathbb{Z}_+$,*

$$\mathbb{E}_\lambda(\chi_{k+1} \mathbb{1}\{\chi_{k+1} \geq 2\} \mid \mathcal{F}_{\eta_k}) \leq C_5 \lambda^{5/2} M_{\eta_k}^{13/2}, \quad \mathbb{P}_\lambda\text{-a.s.}$$

PROOF. Fix $k \in \mathbb{Z}_+$. Write $I = I_{\eta_k}$, $J = J_k$, $M = M_{\eta_k}$, and $L_j = L_{I,j}$ for $1 \leq j \leq I + 1$. Fix $s > 0$. During time interval $(\sigma_{k+1}, \sigma_{k+1} + s]$, let Y denote the number of depositions in gap J , and let Y' denote the number elsewhere in the interval. Given \mathcal{F}'_{η_k} , Y and Y' are independent Poisson random variables with $\mathbb{E}(Y \mid \mathcal{F}'_{\eta_k}) = \lambda s L_J \leq \lambda s M$ and $\mathbb{E}(Y' \mid \mathcal{F}'_{\eta_k}) \leq \lambda s$. Since each nucleation consumes two active particles in the same interval, in order for there to be (at least) two nucleations during time interval $(\sigma_{k+1}, \sigma_{k+1} + s]$, we must either have (i) at least 3 depositions in gap J , (ii) at least one deposition in gap J , and at least two depositions elsewhere, or (iii) no depositions in gap J , and at least 4 depositions elsewhere. In any case, the number of nucleations is not more than the number of depositions. Hence

$$\begin{aligned} & \mathbb{E}_\lambda(\chi_{k+1} \mathbb{1}\{\chi_{k+1} \geq 2, \eta_{k+1} - \sigma_{k+1} \leq s\} \mid \mathcal{F}'_{\eta_k}) \\ & \leq \mathbb{E}_\lambda((Y + Y') \mathbb{1}\{Y \geq 3\} \mid \mathcal{F}'_{\eta_k}) + \mathbb{E}_\lambda((Y + Y') \mathbb{1}\{Y \geq 1, Y' \geq 2\} \mid \mathcal{F}'_{\eta_k}) \\ & \quad + \mathbb{E}_\lambda(Y' \mathbb{1}\{Y' \geq 4\} \mid \mathcal{F}'_{\eta_k}) \\ (7.1) \quad & \leq \mathbb{E}_\lambda(Y \mathbb{1}\{Y \geq 3\} \mid \mathcal{F}'_{\eta_k}) + \mathbb{E}_\lambda(Y' \mid \mathcal{F}'_{\eta_k}) \mathbb{P}_\lambda(Y \geq 3 \mid \mathcal{F}'_{\eta_k}) \\ & \quad + 2 \mathbb{E}_\lambda(Y \mathbb{1}\{Y \geq 1\} \mid \mathcal{F}'_{\eta_k}) \mathbb{E}_\lambda(Y' \mathbb{1}\{Y' \geq 2\} \mid \mathcal{F}'_{\eta_k}) + \mathbb{E}_\lambda(Y' \mathbb{1}\{Y' \geq 4\} \mid \mathcal{F}'_{\eta_k}) \\ & \leq (\lambda s M)^3 + \lambda s \cdot (\lambda s M)^3 + 2 \lambda s M \cdot (\lambda s)^2 + (\lambda s)^4 \\ & \leq 3 \lambda^3 s^3 M + 2 \lambda^4 s^4, \end{aligned}$$

by (5.5) and the fact that $M \leq 1$. On the other hand, by Cauchy–Schwarz,

$$\mathbb{E}_\lambda(\chi_{k+1} \mathbb{1}\{\eta_{k+1} - \sigma_{k+1} > s\} \mid \mathcal{F}_{\eta_k}) \leq (\mathbb{E}_\lambda(\chi_{k+1}^2 \mid \mathcal{F}_{\eta_k}))^{1/2} (\mathbb{P}_\lambda(\eta_{k+1} - \sigma_{k+1} > s \mid \mathcal{F}_{\eta_k}))^{1/2}.$$

If Y'' denotes the number of Poisson arrivals during time interval $(\sigma_{k+1}, \sigma_{k+1} + x]$, then

$$\mathbb{P}_\lambda(\chi_{k+1} \geq 3 \lceil \lambda_0 x \rceil \mid \mathcal{F}_{\eta_k}) \leq \mathbb{P}(\eta_{k+1} - \sigma_{k+1} \geq x \mid \mathcal{F}_{\eta_k}) + \mathbb{P}(Y'' \geq 3 \lceil \lambda_0 x \rceil \mid \mathcal{F}_{\eta_k}).$$

The Poisson variable Y'' has $\mathbb{E}_\lambda(e^{Y''} \mid \mathcal{F}_{\eta_k}) \leq e^{2\lambda_0 x}$, a.s., so, by Lemma 4.2 and Markov’s inequality, $\mathbb{P}_\lambda(\chi_{k+1} \geq 3 \lceil \lambda_0 x \rceil \mid \mathcal{F}_{\eta_k}) \leq C_1 \exp(-cx^{1/2}) + \exp(-\lambda_0 x)$, where $c > 0$. It follows that $\mathbb{E}_\lambda(\chi_{k+1}^2 \mid \mathcal{F}_{\eta_k}) \leq C$ for some $C < \infty$. With Lemma 4.2, this shows that

$$\mathbb{E}_\lambda(\chi_{k+1} \mathbb{1}\{\eta_{k+1} - \sigma_{k+1} > s\} \mid \mathcal{F}_{\eta_k}) \leq C \exp(-cM^{-1}s^{1/2}),$$

where the constants $C < \infty$ and $c > 0$ depend on λ_0 . Thus we obtain

$$\mathbb{E}_\lambda(\chi_{k+1} \mathbb{1}\{\chi_{k+1} \geq 2\} \mid \mathcal{F}_{\eta_k}) \leq 3\lambda^3 s^3 M + 2\lambda^4 s^4 + C \exp(-cM^{-1}s^{1/2}),$$

provided $\lambda \leq \lambda_0$. Taking $s = \lambda^{-1/6} M^{11/6}$ we get the result. \square

The next result shows that $M_{v_n} \rightarrow 0$, a.s., and gives some quantification of the rate. One expects that M_{v_n} , the length of the largest gap when there are n interior islands, is not much greater than $1/n$, and Lemma 7.3 is, in a rough sense, a bound of $O(n^{\varepsilon-(3/4)})$. On the basis of the upper tail of g_0 in Theorem 2.3, we conjecture that the correct order for M_{v_n} is $(\log n)^{1/4}/n$.

LEMMA 7.3. *For any $\lambda \in (0, \infty)$ and any $\gamma > 4/3$, we have $\mathbb{E}_\lambda \sum_{n=0}^\infty M_{v_n}^\gamma < \infty$.*

PROOF. Consider the process $(W_t, t \in \mathbb{R}_+)$ defined by

$$W_t = \sum_{i \in [I_t+1]} L_{I_t, i}^\alpha,$$

where $\alpha > 1$. Note that $0 < W_t \leq W_s \leq W_0 = 1$ for $0 \leq s \leq t < \infty$. Lemma 4.5 says

$$\mathbb{P}_\lambda(\{\alpha_k = 1\} \cap D_{\eta_k}(j, B_0) \mid \mathcal{F}_{\eta_k}) \geq \varepsilon_0 L_{I_{\eta_k}, j}^4,$$

where $B_0 = [\frac{1}{8}, \frac{7}{8}]$ and $\varepsilon_0 = \varepsilon_0(\lambda) > 0$. On $\{\alpha_k = 1\} \cap D_{\eta_k}(j, B_0)$, nucleation occurs during time interval $[\eta_k, \eta_{k+1}]$ at relative location $v \in B_0$ in gap $j \in [I_{\eta_k} + 1]$, and any subsequent nucleation before time η_{k+1} only decreases $W_{\eta_{k+1}}$. Hence, on $\{\alpha_k = 1\} \cap D_{\eta_k}(j, B_0)$,

$$(7.2) \quad W_{\eta_{k+1}} - W_{\eta_k} \leq - \inf_{v \in B_0} \Lambda(v, \alpha) L_{I_{\eta_k}, j}^\alpha, \quad \text{where } \Lambda(v, \alpha) := 1 - v^\alpha - (1 - v)^\alpha.$$

Here $\inf_{v \in B_0} \Lambda(v, \alpha) = \delta > 0$ depending only on $\alpha > 1$. From Lemma 4.5 with (7.2),

$$(7.3) \quad \mathbb{E}_\lambda(W_{\eta_{k+1}} - W_{\eta_k} \mid \mathcal{F}_{\eta_k}) \leq -\delta \varepsilon_0 \sum_{i \in [I_{\eta_k}+1]} L_{I_{\eta_k}, i}^{4+\alpha} \leq -\delta \varepsilon_0 M_{\eta_k}^{4+\alpha}.$$

Taking expectations and summing, since $W_0 = 1$, we obtain, for every $\alpha > 1$,

$$(7.4) \quad \mathbb{E}_\lambda \sum_{k \in \mathbb{Z}_+} M_{\eta_k}^{4+\alpha} \leq \frac{1}{\delta \varepsilon_0} < \infty.$$

In particular, (7.4) shows that $\lim_{t \rightarrow \infty} M_t = 0$, a.s. As in the proof of Lemma 6.1, let $F_k = \mathbb{1}\{\chi_k \geq 2\}$, the event that there are two or more nucleations during $[\sigma_k, \eta_k]$. Recall the definition of k_n at (6.1). From Lemma 7.2 and (7.4) (take $\alpha = 5/2$), we have

$$(7.5) \quad \sum_{n \in \mathbb{N}} \mathbb{P}_\lambda(F_{k_n}) = \sum_{k \in \mathbb{N}} \mathbb{E}_\lambda \sum_{n: k_n=k} \mathbb{1}\{\chi_k \geq 2\} = \mathbb{E}_\lambda \sum_{k \in \mathbb{N}} \chi_k \mathbb{1}\{\chi_k \geq 2\} < \infty,$$

since $k_n = k \in \mathbb{N}$ if and only if $\sigma_k \leq v_n \leq \eta_k$.

Now we extend the argument to get the statement in the lemma. Take $\alpha = 4$ in the definition of W_t . For $k \in \mathbb{Z}_+$ let $\rho_k = \min\{n \in \mathbb{N} : v_n > \eta_k\}$. As above, we have that $\Lambda(v, 4) \geq \delta > 0$ for all $v \in B_0$. On the event $D_{\eta_k}(j, B_0)$, we have $W_{v_{\rho_k}} - W_{\eta_k} \leq -\delta L_{I_{\eta_k}, j}^4$. If v_1 and v_2 are finite measures on a countable set S , then, for $A = \{j \in S : v_1(j) \geq v_2(j)\}$, supposing, without loss of generality, that $v_1(S) \geq v_2(S)$,

$$(7.6) \quad \begin{aligned} \sum_{j \in S} |v_1(j) - v_2(j)| &= 2(v_1(A) - v_2(A)) + v_2(S) - v_1(S) \\ &\leq 2 \sup_{\mathcal{J} \subseteq S} |v_1(\mathcal{J}) - v_2(\mathcal{J})|. \end{aligned}$$

From Proposition 5.1, taking $B_j = B$ for all $j \in \mathcal{J}$ and $B_j = \emptyset$ for $j \notin \mathcal{J}$, we obtain

$$\sup_{B \in \mathcal{B}} \left| \sum_{j \in \mathcal{J}} \mathbb{P}_\lambda(D_{\eta_k}(j, B) \mid \mathcal{F}_{\eta_k}) - \sum_{j \in \mathcal{J}} \kappa_{I_{\eta_k}}(\mathcal{Z}_{I_{\eta_k}}; j, B) \right| \leq C_2 \lambda^{1/2} M_{\eta_k}^{3/2}.$$

Then using (5.1) and (7.6), it follows that

$$\sum_{j \in [I_{\eta_k} + 1]} \left| \mathbb{P}_\lambda(D_{\eta_k}(j, B_0) \mid \mathcal{F}_{\eta_k}) L_{I_{\eta_k}, j}^4 - \frac{L_{I_{\eta_k}, j}^8 \Phi_0(B_0)}{\sum_{i \in [I_{\eta_k} + 1]} L_{I_{\eta_k}, i}^4} \right| \leq 2C_2 \lambda^{1/2} M_{\eta_k}^{11/2}.$$

Consequently, for $\varepsilon = \Phi_0(B_0) > 0$,

$$\mathbb{E}_\lambda(W_{v_{\rho_k}} - W_{\eta_k} \mid \mathcal{F}_{\eta_k}) \leq 2C_2 \lambda^{1/2} W_{\eta_k}^{11/8} - \delta \varepsilon \frac{\sum_{i \in [I_{\eta_k} + 1]} L_{I_{\eta_k}, i}^8}{\sum_{i \in [I_{\eta_k} + 1]} L_{I_{\eta_k}, i}^4}.$$

Let A_n denote the event that $v_{n+1} > \eta_{k_n}$; on A_n , $I_{\eta_{k_n}} = I_{v_n} = n$ and $\rho_{k_n} = n + 1$. Then taking $k = k_n$ (noting that η_{k_n} is a stopping time) and using the monotonicity of W_t ,

$$\begin{aligned} \mathbb{E}_\lambda(W_{v_{n+1}} - W_{v_n} \mid \mathcal{F}_{\eta_{k_n}}) &\leq \mathbb{E}_\lambda(W_{v_{\rho_{k_n}}} - W_{\eta_{k_n}} \mid \mathcal{F}_{\eta_{k_n}}) \mathbb{1}_{A_n} \\ &\leq 2C_2 \lambda^{1/2} W_{v_n}^{11/8} - \delta \varepsilon \frac{\sum_{i \in [n+1]} L_{n,i}^8}{\sum_{i \in [n+1]} L_{n,i}^4} \mathbb{1}_{A_n}. \end{aligned}$$

Since $\sum_{i \in [n+1]} L_{n,i} = 1$, Jensen’s inequality gives $\sum_{i \in [n+1]} L_{n,i}^8 \geq W_{v_n}^{7/3}$. Hence

$$\mathbb{E}_\lambda(W_{v_{n+1}} - W_{v_n} \mid \mathcal{F}_{\eta_{k_n}}) \leq C W_{v_n}^{11/8} - \delta \varepsilon W_{v_n}^{4/3} \mathbb{1}_{A_n},$$

where $C < \infty$. Since $4/3 < 11/8$, and $W_{v_{n+1}} \leq W_{v_n} \leq 1$, there exists $\varepsilon > 0$ such that, a.s.,

$$(7.7) \quad \mathbb{E}_\lambda(W_{v_{n+1}} - W_{v_n} \mid \mathcal{F}_{\eta_{k_n}}) \leq -\varepsilon W_{v_n}^{4/3} \mathbb{1}\{W_{v_n} < \varepsilon\} \mathbb{1}_{A_n}.$$

Let $\beta \in (0, 1)$. Then $(1 + x)^\beta \leq 1 + \beta x$ for all $x \in [-1, 0]$, so

$$\begin{aligned} \mathbb{E}_\lambda(W_{v_{n+1}}^\beta - W_{v_n}^\beta \mid \mathcal{F}_{\eta_{k_n}}) &= W_{v_n}^\beta \mathbb{E}_\lambda \left[\left(1 + \frac{W_{v_{n+1}} - W_{v_n}}{W_{v_n}} \right)^\beta - 1 \mid \mathcal{F}_{\eta_{k_n}} \right] \\ &\leq \mathbb{1}_{A_n^c} - \beta \varepsilon W_{v_n}^{\beta+(1/3)} \mathbb{1}\{W_{v_n} < \varepsilon\}, \end{aligned}$$

by (7.7). Taking expectations, summing, and using the fact that $W_0 = 1$, we get

$$\beta \varepsilon \sum_{m=0}^{n-1} \mathbb{E}_\lambda(W_{v_m}^{\beta+(1/3)} \mathbb{1}\{W_{v_m} < \varepsilon\}) \leq 1 + \sum_{m=0}^{n-1} \mathbb{P}_\lambda(A_m^c) \quad \text{for all } n \in \mathbb{N}.$$

Now $A_n^c \subseteq F_{k_n}$, so, by (7.5), $\sum_{n=0}^\infty \mathbb{P}_\lambda(A_n^c) < \infty$. Thus

$$(7.8) \quad \sum_{n=0}^\infty \mathbb{E}_\lambda(W_{v_n}^{\beta+(1/3)} \mathbb{1}\{W_{v_n} < \varepsilon\}) < \infty.$$

On the other hand, since $W_{v_n} \in [0, 1]$ and $\beta > 0$,

$$\begin{aligned} \sum_{n=0}^\infty \mathbb{E}_\lambda(W_{v_n}^{\beta+(1/3)} \mathbb{1}\{W_{v_n} \geq \varepsilon\}) &\leq \varepsilon^{-2} \sum_{n=0}^\infty \mathbb{E}_\lambda(W_{v_n}^{7/3}) \\ &\leq \varepsilon^{-2} \mathbb{E}_\lambda \sum_{k \in \mathbb{Z}_+} \sum_{n: k_n=k} (W_{\eta_k}^{7/3} \mathbb{1}\{\chi_k = 1\} + \mathbb{1}\{\chi_k \geq 2\}), \end{aligned}$$

since $W_{v_n} = W_{\eta_{k_n}}$ on $\{\chi_{k_n} = 1\}$. Here we have that

$$\begin{aligned} \mathbb{E}_\lambda \sum_{k \geq 1} \sum_{n: k_n=k} (W_{\eta_k}^{7/3} \mathbb{1}\{\chi_k = 1\} + \mathbb{1}\{\chi_k \geq 2\}) &\leq \mathbb{E}_\lambda \sum_{k \geq 1} W_{\eta_k}^{7/3} + \mathbb{E}_\lambda \sum_{k \geq 1} \chi_k \mathbb{1}\{\chi_k \geq 2\} \\ &\leq \mathbb{E}_\lambda \sum_{k \geq 1} M_{\eta_k}^7 + C \mathbb{E}_\lambda \sum_{k \geq 1} M_{\eta_k}^{13/2}, \end{aligned}$$

by Lemma 7.2 and the fact that $W_{\eta_k} \leq M_{\eta_k}^3$. Then, by (7.4), since $7 > \frac{13}{2} > 5$,

$$(7.9) \quad \sum_{n=0}^\infty \mathbb{E}_\lambda(W_{v_n}^{\beta+(1/3)} \mathbb{1}\{W_{v_n} \geq \varepsilon\}) < \infty.$$

Combining (7.8) and (7.9), since $W_{v_n} \geq M_{v_n}^4$ we conclude

$$\mathbb{E}_\lambda \sum_{n \in \mathbb{Z}_+} M_{v_n}^{4\beta+(4/3)} \leq \mathbb{E}_\lambda \sum_{n \in \mathbb{Z}_+} W_{v_n}^{\beta+(1/3)} < \infty,$$

which gives the result, since $\beta \in (0, 1)$ was arbitrary. \square

Now we can give the proof of Proposition 7.1.

PROOF OF PROPOSITION 7.1. On $F_{k_n}^c$, $I_{\eta_{k_n}} = I_{v_n} = n$, and so, by Proposition 5.1,

$$\sup_{B_1, \dots, B_{n+1}} \left| \sum_{j \in [n+1]} \mathbb{P}_\lambda(D_{v_n}(j, B_j) \mid \mathcal{F}_{\eta_{k_n}}) - \sum_{j \in [n+1]} \kappa_n(\mathcal{Z}_n; j, B_j) \right| \leq \mathbb{1}_{F_{k_n}^c} + C_2 \lambda^{1/2} M_{\eta_{k_n}}^{3/2}.$$

Since $M_{\eta_{k_n}} \leq M_{v_n}$, it follows on taking conditional expectations given \mathcal{F}_{v_n} that

$$\begin{aligned} \sup_{B_1, \dots, B_{n+1}} \left| \sum_{j \in [n+1]} \mathbb{P}_\lambda(D_{v_n}(j, B_j) \mid \mathcal{F}_{v_n}) - \sum_{j \in [n+1]} \kappa_n(\mathcal{Z}_n; j, B_j) \right| \\ \leq \mathbb{P}_\lambda(F_{k_n} \mid \mathcal{F}_{v_n}) + C_2 \lambda^{1/2} M_{v_n}^{3/2}. \end{aligned}$$

Here we know from Lemma 7.3 that $\mathbb{E}_\lambda \sum_{n \in \mathbb{Z}_+} M_{v_n}^{3/2} < \infty$, and we know from (7.5) that $\sum_{n \in \mathbb{Z}_+} \mathbb{P}_\lambda(F_{k_n}) < \infty$. The result follows. \square

Finally we present the proof of Theorem 2.3. This is based on a coupling argument, using Proposition 7.1, together with appropriate asymptotic results for interval-splitting processes, which we defer to Section 10.

PROOF OF THEOREM 2.3. Recall from the proof of Theorem 2.2 that $P_n(\mathcal{S}_n, A) = \mathbb{P}(\mathcal{S}_{n+1} \in A \mid \mathcal{S}_0, \dots, \mathcal{S}_n)$, $A \in \mathcal{B}_{n+1}$, is the kernel associated with the interval-splitting process $\mathcal{S} = (\mathcal{S}_0, \mathcal{S}_1, \dots)$ with parameters r_0 and Φ_0 . Proposition 7.1 with (6.6) shows that

$$(7.10) \quad \mathbb{E}_\lambda \sum_{n \in \mathbb{Z}_+} \sup_{A \in \mathcal{B}_{n+1}} |\mathbb{P}_\lambda(\mathcal{Z}_{n+1} \in A \mid \mathcal{F}_{v_n}) - P_n(\mathcal{Z}_n, A)| < \infty.$$

Let $\mathcal{F}_n^{\mathcal{Z}} := \sigma(\mathcal{Z}_0, \mathcal{Z}_1, \dots, \mathcal{Z}_n)$. Since $\mathbb{P}_\lambda(\mathcal{Z}_{n+1} \in A \mid \mathcal{F}_n^{\mathcal{Z}}) = \mathbb{E}_\lambda[\mathbb{P}_\lambda(\mathcal{Z}_{n+1} \in A \mid \mathcal{F}_{v_n}) \mid \mathcal{F}_n^{\mathcal{Z}}]$,

$$(7.11) \quad \begin{aligned} & \mathbb{E}_\lambda \sum_{n \in \mathbb{Z}_+} \sup_{A \in \mathcal{B}_{n+1}} |\mathbb{P}_\lambda(\mathcal{Z}_{n+1} \in A \mid \mathcal{F}_n^{\mathcal{Z}}) - P_n(\mathcal{Z}_n, A)| \\ & \leq \mathbb{E}_\lambda \sum_{n \in \mathbb{Z}_+} \sup_{A \in \mathcal{B}_{n+1}} \mathbb{E}_\lambda[|\mathbb{P}_\lambda(\mathcal{Z}_{n+1} \in A \mid \mathcal{F}_{v_n}) - P_n(\mathcal{Z}_n, A)| \mid \mathcal{F}_n^{\mathcal{Z}}] < \infty, \end{aligned}$$

by (7.10). Fix $n_0 \in \mathbb{Z}_+$. We couple $(\mathcal{Z}_0, \mathcal{Z}_1, \dots)$ and $(\mathcal{S}_{n_0}, \mathcal{S}_{n_0+1}, \dots)$, an interval-splitting process with parameters r_0 and Φ_0 and initial configuration $\mathcal{S}_{n_0} = \mathcal{Z}_{n_0}$. On a common probability space, where we still denote probabilities by \mathbb{P}_λ , construct $(\mathcal{Z}_0, \dots, \mathcal{Z}_{n_0})$ according to the law given by \mathbb{P}_λ , and then set $\mathcal{S}_{n_0} = \mathcal{Z}_{n_0}$. Let $n \geq n_0$. Let $\mathcal{F}_n^{\mathcal{Z}, \mathcal{S}} = \sigma(\mathcal{Z}_0, \dots, \mathcal{Z}_n, \mathcal{S}_{n_0}, \dots, \mathcal{S}_n)$. Given $\mathcal{F}_n^{\mathcal{Z}, \mathcal{S}}$, if $\mathcal{Z}_n \neq \mathcal{S}_n$, then generate \mathcal{S}_{n+1} independently of \mathcal{Z}_{n+1} according to $P_n(\mathcal{S}_n, \cdot)$. If $\mathcal{Z}_n = \mathcal{S}_n$, then generate $(\mathcal{Z}_{n+1}, \mathcal{S}_{n+1})$ by maximal coupling of $\mathbb{P}_\lambda(\mathcal{Z}_{n+1} \in \cdot \mid \mathcal{F}_n^{\mathcal{Z}})$ and $P_n(\mathcal{Z}_n, \cdot)$. Then

$$(7.12) \quad \mathbb{P}_\lambda(\mathcal{Z}_n = \mathcal{S}_n \text{ for all } n \geq n_0) \geq 1 - \varepsilon_{n_0},$$

where

$$\varepsilon_{n_0} = \mathbb{E}_\lambda \sum_{n \geq n_0} \sup_{A \in \mathcal{B}_{n+1}} |\mathbb{P}_\lambda(\mathcal{Z}_{n+1} \in A \mid \mathcal{F}_n^{\mathcal{Z}}) - P_n(\mathcal{Z}_n, A)|.$$

By (7.11), for any $\varepsilon > 0$ we can choose n_0 large enough that $\varepsilon_{n_0} < \varepsilon$; fix such an n_0 .

We apply Theorem 10.1 to the interval-splitting process with parameters r_0 and Φ_0 ; in the hypotheses we take $\alpha = 4$ and $\beta = 2$, using Lemma 9.6 for the behaviour of ϕ_0 near zero. Theorem 10.1(i) shows that $\mathbb{P}_\lambda(n^{-1}C_n(x) \rightarrow x \mid \mathcal{S}_{n_0} = z) = 1$ for all z , and hence $\mathbb{P}_\lambda(n^{-1}C_n(x) \rightarrow x \mid \mathcal{F}_{n_0}^{\mathcal{Z}, \mathcal{S}}) = 1$, a.s. On the event $\mathcal{Z}_n = \mathcal{S}_n$ for all $n \geq n_0$, we have $n^{-1}|N_n(x) - C_n(x)| \rightarrow 0$, a.s. Hence

$$\begin{aligned} \mathbb{P}_\lambda(n^{-1}N_n(x) \rightarrow x \mid \mathcal{F}_{n_0}^{\mathcal{Z}, \mathcal{S}}) & \geq \mathbb{P}_\lambda(n^{-1}C_n(x) \rightarrow x, \mathcal{Z}_n = \mathcal{S}_n \text{ for all } n \geq n_0 \mid \mathcal{F}_{n_0}^{\mathcal{Z}, \mathcal{S}}) \\ & = \mathbb{P}_\lambda(\mathcal{Z}_n = \mathcal{S}_n \text{ for all } n \geq n_0 \mid \mathcal{F}_{n_0}^{\mathcal{Z}, \mathcal{S}}). \end{aligned}$$

Taking expectations and using (7.12), we get $\mathbb{P}_\lambda(n^{-1}N_n(x) \rightarrow x) \geq 1 - \varepsilon_{n_0} \geq 1 - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we establish part (i) of the theorem. The a.s. convergence in part (ii) is deduced from Theorem 10.1(ii) in a similar way, and convergence of $\mathbb{P}_\lambda(\bar{L}_n \leq x) = \mathbb{E}_\lambda \mathcal{E}_n(x)$ follows from the bounded convergence theorem. The asymptotics for g_0 follow from Theorem 10.1(iii), noting that there $\alpha = 4$, $\beta = 2$, and $a = 0$. \square

8. Brownian motion exiting a right-angled triangle. This section provides a proof of Proposition 5.2. Recall the notation $S = \partial[0, 1]^2$ and $D = \{(x, y) \in [0, 1]^2 : x = y\}$ for the boundary and diagonal of the unit square, and the definition of $H(u, v; B)$ from (5.3). Let $U := \{(x, y) \in [0, 1]^2 : x \geq y\}$, the right-angled triangle with side-lengths 1, 1, $\sqrt{2}$. Then for $B \in \mathcal{B}$ we can write

$$H(u, v; B) = \mathbb{P}(\tau_{\partial U} = \tau_D, W_{\tau_D}^{(1)} \in B \mid W_0 = (u, v)) \quad \text{for } (u, v) \in U;$$

the symmetry $H(u, v; B) = H(v, u; B)$ gives $H(u, v; B)$ for all $(u, v) \in [0, 1]^2$. An old result of Smith and Watson [37] states that the probability that planar Brownian motion started from a *uniform random* point in U exits via the diagonal is given by

$$\begin{aligned}
 (8.1) \quad 2 \int_U H(u, v; [0, 1]) \, du \, dv &= \int_{[0,1]^2} H(u, v; [0, 1]) \, du \, dv \\
 &= 1 - \frac{16}{\pi^3} \sum_{n \text{ odd}} \frac{\coth(n\pi/2)}{n^3} \\
 &\approx 0.41063 \text{ to 5 decimal places.}
 \end{aligned}$$

(The term $(\coth(3\pi/2) - 1)/9$ on page 484 of [37] should be $(\coth(3\pi/2) - 1)/27$, which leads to an error in the 5th decimal place of their numerical approximation.) The method of [37] could be adapted to find $\int_0^1 \int_0^1 H(u, v; B) \, du \, dv$, but we want to evaluate $H(u, v; B)$ integrated against a different measure, as in (5.4). We use a variation on the classical *method of images* to evaluate $H(u, v; B)$ explicitly for *fixed* u, v . This is the content of Theorem 8.1, which appears to be new, and from which we deduce Proposition 5.2.

For $n \in \mathbb{Z}$, set $s_n(x, y) := \sin(n\pi x) \sinh(n\pi y)$. For $x, y \in (0, 1)$ and $z \in [0, 1]$, define

$$\begin{aligned}
 h(x, y, z) &:= \sum_{n \in \mathbb{N}} \frac{2 \sin(n\pi(1-z))}{\sinh(n\pi)} (s_n(x, y) + s_n(1-x, 1-y) - s_n(y, x) - s_n(1-y, 1-x));
 \end{aligned}$$

provided $x, y \in (0, 1)$, the sum here converges absolutely, uniformly for $z \in [0, 1]$.

THEOREM 8.1. *For all $(u, v) \in U \setminus D$ and all $B \in \mathcal{B}$,*

$$(8.2) \quad H(u, v; B) = \int_B h\left(\frac{u+v}{2}, \frac{u-v}{2}, w\right) \, dw.$$

REMARK 8.2. As a corollary to the theorem, following a similar (but simpler) series of calculations to those in the proof of Proposition 5.2 below, one can derive

$$\begin{aligned}
 (8.3) \quad 2 \int_U H(u, v; [0, 1]) \, du \, dv &= \int_{[0,1]^2} H(u, v; [0, 1]) \, du \, dv \\
 &= \frac{32}{\pi^3} \sum_{n \text{ odd}} \frac{(-1)^{\frac{n-1}{2}}}{n^3} \operatorname{sech}(n\pi/2),
 \end{aligned}$$

which converges much faster than (8.1). Equality of (8.1) and (8.3) entails the identity

$$1 - \frac{16}{\pi^3} \sum_{n \text{ odd}} \frac{\coth(n\pi/2)}{n^3} = \frac{32}{\pi^3} \sum_{n \text{ odd}} \frac{(-1)^{\frac{n-1}{2}}}{n^3} \operatorname{sech}(n\pi/2),$$

for which we have not been able to find a reference.

We prove Theorem 8.1 by solving an appropriate Dirichlet problem. For a domain $\mathcal{D} \subset \mathbb{R}^2$ with boundary $\partial\mathcal{D}$ and $g : \partial\mathcal{D} \rightarrow \mathbb{R}$, a twice-differentiable $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ solves the Dirichlet problem (\mathcal{D}, g) if $\nabla^2 f = 0$ on \mathcal{D} and $f = g$ on $\partial\mathcal{D}$. We will show that H solves the Dirichlet problem (U, g) where g depends on u, v , and B . Since g is not continuous, we approximate it by continuous functions. Then we appeal to the explicit eigenfunctions of the Laplacian on $[0, 1]^2$, and an application of the method of images [25], to solve the modified Dirichlet problem, and then take a limit. While we believe that Theorem 8.1 is new, the idea has a long history, and we refer to [30] for some similar examples.

PROOF OF THEOREM 8.1. Provided $(u, v) \in U \setminus D$, both $(u \pm v)/2$ are in $(0, 1)$. Note also that $h((1 + v)/2, (1 - v)/2, w) = h(u/2, u/2, w) = 0$ because of the antisymmetries $h(x, y, z) = -h(y, x, z) = -h(1 - y, 1 - x, z)$, so the integral in (8.2) is 0 if $u = 1$ or $v = 0$. Thus it remains to prove (8.2) for $(u, v) \in U \setminus \partial U$. Moreover, since Brownian motion started in the interior of U hits $(0, 0)$ or $(1, 1)$ with probability 0, and the value of the integral is unaffected by the addition of points 0 or 1 to B , it suffices to suppose that $B \subseteq (0, 1)$.

Set $V := \{(x, y) \in [0, 1]^2 : x + y \leq 1, x \geq y\}$. Define the matrix M and associated linear transformation m by

$$M := \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad m(u, v) := M \begin{pmatrix} u \\ v \end{pmatrix}.$$

Then m maps U to V , and $m(x, x) = (x, 0)$. Since $MM^T = \frac{1}{2}I$, where I is the identity, the process MW is a constant time-change of Brownian motion, so that

$$(8.4) \quad H(u, v; B) = \mathbb{P}(W_{\tau_{\partial V}}^{(1)} \in B, W_{\tau_{\partial V}}^{(2)} = 0 \mid W_0 = m(u, v)).$$

Take $\gamma : [0, 1] \rightarrow [0, 1]$ continuous with $\gamma(0) = \gamma(1) = 0$. Then $g_V : \partial V \rightarrow [0, 1]$ with $g_V(x, y) = 0$ for $y > 0$ and $g_V(x, 0) = \gamma(x)$ is continuous on ∂V . Define $g : S \rightarrow [0, 1]$ by

$$(8.5) \quad g(x, y) = \begin{cases} \gamma(x) & \text{if } y = 0, \\ \gamma(1 - x) & \text{if } y = 1, \\ -\gamma(y) & \text{if } x = 0, \\ -\gamma(1 - y) & \text{if } x = 1, \end{cases}$$

then g is continuous on S and satisfies $g(x, y) = -g(y, x) = g(1 - x, 1 - y)$ for all $(x, y) \in S$. There is a unique twice-differentiable function $f : [0, 1]^2 \rightarrow [0, 1]$ that solves the Dirichlet problem $([0, 1]^2, g)$. Moreover, f inherits from g the symmetries $f(x, y) = -f(y, x) = f(1 - x, 1 - y)$ for all $(x, y) \in [0, 1]^2$ (to see this, note, for instance, that $f(x, y) + f(y, x)$ solves the Dirichlet problem with zero boundary condition, and hence is identically zero). In particular, $f(x, x) = f(x, 1 - x) = 0$ for all $x \in [0, 1]$. Hence the function $f_V := f|_V$ solves the Dirichlet problem (V, g_V) .

For the simple region $[0, 1]^2$, solutions to the Dirichlet problem can easily be described in terms of combinations of functions $\sin(n\pi x) \sinh(n\pi y)$ for $n \in \mathbb{N}$ and their images under the transformations $x \leftrightarrow y$ and $x \leftrightarrow 1 - y$. In particular, the solution to the Dirichlet problem $([0, 1]^2, g)$, where g is of the form (8.5), is f given by

$$f(x, y) = \sum_{n \in \mathbb{N}} \frac{A_n s_n(x, y) + B_n s_n(1 - x, 1 - y) + C_n s_n(y, x) + D_n s_n(1 - y, 1 - x)}{\sinh(n\pi)},$$

where the boundary condition gives $A_n = B_n = -C_n = -D_n = 2 \int_0^1 \gamma(z) \sin(n\pi(1 - z)) dz$.

Given $B \in \mathcal{B}$, with $B \cap \{0, 1\} = \emptyset$, consider a sequence γ_k of bounded continuous functions on $[0, 1]$ with $\gamma_k(0) = \gamma_k(1) = 0$ and $\lim_{k \rightarrow \infty} \gamma_k(x) = \mathbb{1}_B(x)$ for every $x \in [0, 1]$. Then let f_k denote the solution to the Dirichlet problem $([0, 1]^2, g_k)$, where g_k is constructed from γ_k according to (8.5). Then

$$f_k(x, y) = \sum_{n \in \mathbb{N}} \frac{A_{k,n}}{\sinh(n\pi)} (s_n(x, y) + s_n(1 - x, 1 - y) - s_n(y, x) - s_n(1 - y, 1 - x)),$$

where $A_{k,n} = 2 \int_0^1 \gamma_k(z) \sin(n\pi(1 - z)) dz$. As described above, restricting f_k to V gives the (unique) solution to the Dirichlet problem $(V, g_{V,k})$, where $g_{V,k}(x, y) = \gamma_k(x) \mathbb{1}\{y = 0\}$.

By the connection between the Dirichlet problem with continuous boundary conditions and stopped Brownian motion (see, e.g., Theorem 3.12 of [31]) we have

$$\mathbb{E}(g_{V,k}(W_{\tau_{\partial V}}) \mid W_0 = (x, y)) = f_k(x, y) \quad \text{for all } (x, y) \in V.$$

Now, by choice of γ_k ,

$$\lim_{k \rightarrow \infty} g_{V,k}(W_{\tau_{\partial V}}) = \mathbb{1}\{W_{\tau_{\partial V}}^{(1)} \in B, W_{\tau_{\partial V}}^{(2)} = 0\}, \quad \text{a.s.,}$$

so, by bounded convergence,

$$(8.6) \quad \mathbb{P}(W_{\tau_{\partial V}}^{(1)} \in B, W_{\tau_{\partial V}}^{(2)} = 0 \mid W_0 = (x, y)) = \lim_{k \rightarrow \infty} f_k(x, y) \quad \text{for all } (x, y) \in V.$$

By bounded convergence, $\lim_{k \rightarrow \infty} A_{k,n} = 2 \int_B \sin(n\pi(1 - z)) dz$. For fixed $(x, y) \in (0, 1)^2$, the series expression for f_k is absolutely convergent, and $f(x, y) = \lim_{k \rightarrow \infty} f_k(x, y)$ satisfies $f(x, y) = \int_B h(x, y, z) dz$, with h as defined in the display above Theorem 8.1. Then combining (8.4) and (8.6), we obtain the result. \square

Now we can complete the proof of Proposition 5.2.

PROOF OF PROPOSITION 5.2. Recall the definition of Φ_1 from (5.4). For $y, z \in (0, 1)$, $y \neq z$, we have $H(y, z; B) = H(y \vee z, y \wedge z; B)$ is given by the formula (8.2), so that

$$\Phi_1(B) = \int_B dw \int_0^1 dz \int_0^1 dy \int_0^1 dx \int_0^\infty q_t(x, y) h\left(\frac{y+z}{2}, \frac{(y \vee z) - (y \wedge z)}{2}, w\right) dt,$$

where we may define $h(x, 0, w)$ and $h(1, y, w)$ arbitrarily. We now proceed to show that $\Phi_1(B) = \int_B \psi(w) dw$, where ψ is given by (2.4). Let

$$Q(y) := \int_0^1 dx \int_0^\infty q_t(x, y) dt.$$

Then

$$\begin{aligned} \Phi_1(B) &= \int_B dw \int_0^1 dy \int_0^y Q(y) h\left(\frac{y+z}{2}, \frac{y-z}{2}, w\right) dz \\ &\quad + \int_B dw \int_0^1 dz \int_0^z Q(y) h\left(\frac{y+z}{2}, \frac{z-y}{2}, w\right) dy \\ &= \int_B dw \int_0^1 dy \int_0^y [Q(y) + Q(z)] h\left(\frac{y+z}{2}, \frac{y-z}{2}, w\right) dz. \end{aligned}$$

Changing variables from $(y, z) \in U$ to $(u, v) = (\frac{y+z}{2}, \frac{y-z}{2}) \in V$ we get

$$\Phi_1(B) = 2 \int_B dw \int_0^1 du \int_0^{u \wedge (1-u)} [Q(u+v) + Q(u-v)] h(u, v, w) dv.$$

A useful alternative expression for q_t (see [11], page 122) is the spectral representation

$$q_t(x, y) = 2 \sum_{m \in \mathbb{N}} \exp\left(-\frac{m^2 \pi^2 t}{2}\right) \sin(m\pi x) \sin(m\pi y).$$

Hence

$$Q(y) = \frac{8}{\pi^3} \sum_{m \text{ odd}} \frac{\sin(m\pi y)}{m^3} = y(1 - y);$$

see (9.9) below. It follows that

$$\Phi_1(B) = 4 \int_B dw \int_0^1 du \int_0^{u \wedge (1-u)} [u(1-u) - v^2] h(u, v, w) dv.$$

Decomposing $h(u, v, w)$ into sums over even and odd n , and using that $h_{\text{even}}(u, v, w) = -h_{\text{even}}(1-u, v, w)$ and $h_{\text{odd}}(u, v, w) = h_{\text{odd}}(1-u, v, w) = h_{\text{odd}}(u, v, 1-w)$, gives

$$\begin{aligned} & \int_0^1 du \int_0^{u \wedge (1-u)} [u(1-u) - v^2] h(u, v, w) dv \\ &= 2 \int_0^{1/2} du \int_0^u [u(1-u) - v^2] h_{\text{odd}}(u, v, 1-w) dv, \end{aligned}$$

so that $\Phi_1(B) = 16 \int_B \sum_{n \text{ odd}} \frac{I_n}{\sinh(n\pi)} \sin(n\pi w) dw$, where

$$\begin{aligned} I_n &= \int_0^{1/2} du \int_0^u [u(1-u) - v^2] \\ &\quad \times (s_n(u, v) + s_n(1-u, 1-v) - s_n(v, u) - s_n(1-v, 1-u)) dv. \end{aligned}$$

It remains to evaluate the integral I_n for n odd. To simplify the calculation, observe that for n odd, the angle-sum formulae for the trigonometric and hyperbolic sines imply

$$s_n(1/2 - x, 1/2 - y) + s_n(1/2 + x, 1/2 + y) = 2s_n(1/2, 1/2)c_n(x, y),$$

where $c_n(x, y) := \cos(n\pi x) \cosh(n\pi y)$. Hence, changing variables from (u, v) to $(x, y) = (1/2 - v, 1/2 - u)$, the integral I_n becomes

$$I_n = 2s_n(1/2, 1/2) \int_0^{1/2} dx \int_0^x [x(1-x) - y^2] (c_n(y, x) - c_n(x, y)) dy.$$

We can write $I_n = 2s_n(1/2, 1/2)(I_{n,1} + I_{n,2})$ where, for $\tilde{c}_n(x, y) := c_n(y, x) - c_n(x, y)$,

$$I_{n,1} := \int_0^{1/2} dx \int_0^x x(1-2x)\tilde{c}_n(x, y) dy \quad \text{and} \quad I_{n,2} := \int_0^{1/2} dx \int_0^x (x^2 - y^2)\tilde{c}_n(x, y) dy.$$

Then, since $n\pi \int_0^x \tilde{c}_n(x, y) dy = \Re((1+i) \sin(n\pi(1+i)x))$, integration by parts of the (complex) integral $\int_0^{1/2} x(1-2x) \sin(n\pi(1+i)x) dx$ yields

$$I_{n,1} = \frac{\sin(n\pi/2)}{\pi^4} \left(\frac{2 \sinh(n\pi/2)}{n^4} - \frac{\pi \cosh(n\pi/2)}{2n^3} \right) \quad \text{for odd } n.$$

For $I_{n,2}$, notice that $-\int_0^{1/2} dx \int_0^x y^2 \tilde{c}_n(x, y) dy = \int_0^{1/2} dy \int_0^y x^2 \tilde{c}_n(x, y) dx$, so that $I_{n,2} = \int_0^{1/2} \int_0^{1/2} x^2 \tilde{c}_n(x, y) dx dy$, and integration by parts yields

$$I_{n,2} = \frac{\sin(n\pi/2)}{\pi^4} \left(\frac{4 \sinh(n\pi/2)}{n^4} - \frac{\pi \cosh(n\pi/2)}{n^3} \right) \quad \text{for odd } n.$$

Hence, using that $\sin^2(n\pi/2) = 1$ for n odd, we have

$$\frac{16I_n}{\sinh(n\pi)} = \frac{32 \sinh(n\pi/2)}{\pi^4 \sinh(n\pi)} \left(\frac{6 \sinh(n\pi/2)}{n^4} - \frac{3\pi \cosh(n\pi/2)}{2n^3} \right) = \frac{24}{\pi^4} a_n,$$

for odd n , where a_n is given by (2.4), and therefore $\Phi_1(B) = \int_B \psi(w) dw$. \square

9. Analysis of the splitting density. In this section we present analytical and numerical results on the probability density ϕ_0 appearing in (2.6). We start by discussing efficient numerical approximation of the function ψ defined at (2.4) and the constant μ defined at (2.5).

First we establish the final equality in (2.5). This will follow from the identity

$$(9.1) \quad 4 \sum_{n \text{ odd}} \frac{\tanh(n\pi/2)}{n^5} = \frac{\pi^5}{96} + \pi \sum_{n \text{ odd}} \frac{\operatorname{sech}^2(n\pi/2)}{n^4}.$$

The equality (9.1) may be deduced from the fact that, for $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$,

$$(9.2) \quad \alpha^{-2} \sum_{n \text{ odd}} \frac{\tanh(n\alpha/2)}{n^5} - (-\beta)^{-2} \sum_{n \text{ odd}} \frac{\tanh(n\beta/2)}{n^5} = \frac{\alpha\beta(\beta - \alpha)}{192},$$

a formula attributed to de Saint-Venant in 1856 ([6], page 294). It follows from (9.2) that

$$\frac{\alpha + \beta}{\pi^4} \sum_{n \text{ odd}} \frac{\tanh(n\alpha/2)}{n^5} - \frac{1}{\beta^2} \sum_{n \text{ odd}} \frac{\tanh(n\beta/2) - \tanh(n\alpha/2)}{n^5(\beta - \alpha)} = \frac{\pi^2}{192}.$$

Taking $\alpha - \beta \rightarrow 0$ gives (9.1). Then from the first series in (2.5) with (2.4), we have that

$$\mu = \frac{192}{\pi^5} \sum_{n \text{ odd}} \frac{\tanh(n\pi/2)}{n^5} - \frac{48}{\pi^4} \sum_{n \text{ odd}} \frac{1}{n^4}.$$

Writing $\zeta(s) := \sum_{n \in \mathbb{N}} n^{-s}$, note that $\sum_{n \text{ odd}} n^{-s} = (1 - 2^{-s})\zeta(s)$ for $s > 1$. Thus we obtain the final series in (2.5), using (9.1) and the fact that $\zeta(4) = \pi^4/90$.

Truncating the second series in (2.5), we can write, for any odd integer n ,

$$\mu = \mu_n + \frac{48}{\pi^4} r_n, \quad \text{where } \mu_n := \frac{48}{\pi^4} \sum_{\substack{k \leq n \\ k \text{ odd}}} \frac{\operatorname{sech}^2(k\pi/2)}{k^4} \text{ and } r_n := \sum_{\substack{k > n \\ k \text{ odd}}} \frac{\operatorname{sech}^2(k\pi/2)}{k^4}.$$

Since $\operatorname{sech} x \leq 2e^{-x}$ for all $x \in \mathbb{R}$ we have, for n odd,

$$(9.3) \quad r_n \leq \frac{4}{(n+2)^4} \sum_{\substack{k \geq n+2 \\ k \text{ odd}}} e^{-k\pi} \leq \frac{4e^{-(n+2)\pi}}{(n+2)^4} \cdot \frac{1}{1 - e^{-2\pi}}.$$

In particular, the bound $r_3 < 10^{-9}$ guarantees that μ is approximated by μ_3 to within 5×10^{-10} . Since $\mu_3 \approx 0.078268954659$, this suffices to evaluate the first 8 decimal digits of μ as $\mu \approx 0.07826895$.

This idea can be extended to compute moments of Φ_0 . Set $m_k := \int_0^1 z^k \phi_0(z) dz$.

PROPOSITION 9.1. *We have that $m_1 = 1/2$,*

$$m_2 = \frac{1}{2} - \frac{1}{60\mu}, \quad m_3 = \frac{1}{2} - \frac{1}{40\mu} \quad \text{and} \quad m_4 = \frac{1}{2} - \frac{11}{280\mu} + \frac{576}{\mu\pi^8} \sum_{n \text{ odd}} \frac{\operatorname{sech}^2(n\pi/2)}{n^8}.$$

PROOF. If

$$\omega_{k,n} := \int_0^1 z^k \sin(n\pi z) dz,$$

then we have from (2.4) and the fact that $\phi_0(z) = \mu^{-1}\psi(z)$ that $m_k = \frac{24}{\mu\pi^4} \sum_{n \text{ odd}} a_n \omega_{k,n}$. For example, $\omega_{1,n} = \frac{1}{n\pi}(-1)^{n+1}$ so $m_1 = \frac{24}{\mu\pi^5} \sum_{n \text{ odd}} \frac{a_n}{n} = \frac{1}{2}$, by (2.5), as is to be expected, due to the symmetry of ϕ_0 around $1/2$. Also, $\omega_{2,n} = \frac{1}{n\pi} - \frac{4}{n^3\pi^3}$ for odd n , so

$$m_2 = \frac{1}{2} - \frac{96}{\mu\pi^7} \sum_{n \text{ odd}} \frac{a_n}{n^3} = \frac{1}{2} - \frac{384}{\mu\pi^7} \sum_{n \text{ odd}} \frac{\tanh(n\pi/2)}{n^7} + \frac{96}{\mu\pi^6} \cdot \zeta(6) \cdot \frac{63}{64}.$$

Since $\sum_{n \text{ odd}} n^{-7} \tanh(\frac{n\pi}{2}) = \frac{7\pi^7}{23040}$ [6], page 293, and $\zeta(6) = \frac{\pi^6}{945}$, we get the claimed formula for m_2 . The formula for m_3 follows from those for m_1 and m_2 by symmetry of ϕ_0 .

Finally, for n odd, $\omega_{4,n} = \frac{1}{n\pi} - \frac{12}{n^3\pi^3} + \frac{48}{n^5\pi^5}$, and, similar to before,

$$m_4 = \frac{1}{2} - \frac{6}{35\mu} + \frac{4608}{\mu\pi^9} \sum_{n \text{ odd}} \frac{\tanh(n\pi/2)}{n^9}.$$

The claimed formula for m_4 now follows from the identity

$$4096 \sum_{n \text{ odd}} \frac{\tanh(n\pi/2)}{n^9} = \frac{37\pi^9}{315} + 512\pi \sum_{n \text{ odd}} \frac{\operatorname{sech}^2(n\pi/2)}{n^8},$$

which can be obtained in a similar fashion to (9.1), but replacing (9.2) by the appropriate higher-order analogue from [6], page 294. \square

The formulae in Proposition 9.1 give m_2 and m_4 to 10 decimal places as

$$(9.4) \quad m_2 \approx 0.2870590372 \quad \text{and} \quad m_4 \approx 0.1212564646.$$

COROLLARY 9.2. *The distribution Φ_0 is not a Beta distribution.*

PROOF. For $\beta > 0$, the $\text{Beta}(\beta, \beta)$ distribution has density proportional to $x^{\beta-1}(1-x)^{\beta-1}$ for $x \in [0, 1]$, and its k th moment is $m_{\beta,k} = \prod_{j=0}^{k-1} \frac{\beta+j}{2\beta+j}$. Thus $m_{\beta,1} = 1/2$. To fit $m_{\beta,2} = m_2$ as given by (9.4) requires that $\beta = \beta_\star \approx 2.8729963811$. But the $\text{Beta}(\beta_\star, \beta_\star)$ distribution has $m_{\beta_\star,4} \approx 0.1212665009$, which fails to match m_4 from (9.4). \square

Now we turn to analysis of the density ϕ_0 . It is useful to write

$$a_n = b_n - \frac{4}{n^4}d_n, \quad \text{where } b_n := \frac{4-n\pi}{n^4} \quad \text{and} \quad d_n := 1 - \tanh(n\pi/2) \quad \text{for } n \in \mathbb{N}.$$

Note that $0 < d_n < 2e^{-n\pi}$. For $k \in \mathbb{N}$, define the functions

$$(9.5) \quad S_k(x) := \sum_{n=1}^{\infty} \frac{\sin nx}{n^k} \quad \text{and} \quad C_k(x) := \sum_{n=1}^{\infty} \frac{\cos nx}{n^k}.$$

It is known (see, e.g., equation 1.443.1 of [20], page 47) that

$$(9.6) \quad S_3(x) = \frac{\pi^2}{6}x - \frac{\pi}{4}x^2 + \frac{1}{12}x^3 \quad \text{for } 0 \leq x \leq 2\pi.$$

There is no closed form for S_2 or S_4 , which are relatives of the *Clausen function* [27]. We will express ψ in terms of the function

$$(9.7) \quad S(x) := \sum_{n \text{ odd}} \frac{\sin n\pi x}{n^4} = S_4(\pi x) - \frac{1}{16}S_4(2\pi x).$$

LEMMA 9.3. *We have that*

$$(9.8) \quad \psi(x) = \frac{96}{\pi^4} S(x) - 3x(1-x) - \frac{96}{\pi^4} \sum_{n \text{ odd}} \frac{d_n}{n^4} \sin n\pi x.$$

Moreover, ψ is twice continuously differentiable on $[0, 1]$.

REMARK 9.4. The third derivative of ψ diverges as $x \rightarrow 0$ (to $-\infty$) and $x \rightarrow 1$ (to $+\infty$).

PROOF OF LEMMA 9.3. By rearranging (2.4), we can write

$$\psi(x) = \frac{96}{\pi^4} \sum_{n \text{ odd}} \frac{\sin n\pi x}{n^4} - \frac{24}{\pi^3} \sum_{n \text{ odd}} \frac{\sin n\pi x}{n^3} - \frac{96}{\pi^4} \sum_{n \text{ odd}} \frac{d_n}{n^4} \sin n\pi x.$$

From (9.6) we have that, for $0 \leq x \leq 1$,

$$(9.9) \quad \sum_{n \text{ odd}} \frac{\sin n\pi x}{n^3} = S_3(\pi x) - \frac{1}{8} S_3(2\pi x) = \frac{\pi^3}{8} x(1-x).$$

This yields (9.8). The series expression for $S(x)$ is evidently twice continuously differentiable, and hence the same is true for ψ . \square

Although S_4 has no closed form, it has some numerically efficient series representations. We use one of these to obtain an efficient approximation for ψ , and hence ϕ_0 . The (absolute) Bernoulli numbers are $|B(2\ell)| := 2\zeta(2\ell)(2\ell)!/(2\pi)^{2\ell}$. For $k, m \in \mathbb{Z}_+$ and $x \in [0, 1]$, define

$$\begin{aligned} \psi^{k,m}(x) := & \frac{84}{\pi^3} x\zeta(3) + \frac{8}{\pi} x^3 \log(\pi x) - \frac{8}{\pi} \left(\frac{11}{6} + \log 2 \right) x^3 - 3x(1-x) \\ & + 48\pi x^5 \sum_{n=0}^k \frac{|B(2n+2)|(2^{2n+1}-1)}{(n+1)(2n+5)!} \pi^{2n} x^{2n} - \frac{96}{\pi^4} \sum_{\substack{n \text{ odd} \\ n \leq m}} \frac{d_n}{n^4} \sin n\pi x. \end{aligned}$$

It turns out that $\psi^{k,m} \rightarrow \psi$ as $k, m \rightarrow \infty$, but the convergence is poor as x approaches 1. Thus we make use of the symmetry of ψ and consider the symmetrization

$$\phi_0^{k,m}(x) := \begin{cases} \frac{1}{\mu} \psi^{k,m}(x) & \text{if } 0 \leq x \leq 1/2, \\ \frac{1}{\mu} \psi^{k,m}(1-x) & \text{if } 1/2 < x \leq 1. \end{cases}$$

Then $\phi_0^{k,m}$ converges rather rapidly to ϕ_0 , as shown by the following estimate.

LEMMA 9.5. *For all $k, m \in \mathbb{Z}_+$, with m odd,*

$$\sup_{0 \leq x \leq 1} |\phi_0^{k,m}(x) - \phi_0(x)| \leq \frac{4^{-k} \zeta(2k+4)}{\pi \mu (2k+4)^4} + \frac{2e^{-(m+2)\pi}}{\mu(m+2)^4}.$$

For example, $\sup_{0 \leq x \leq 1} |\phi_0^{9,5}(x) - \phi_0(x)| < 10^{-10}$.

PROOF. A standard series expansion, valid for $0 \leq x < 2\pi$, is

$$(9.10) \quad S_2(x) = x - x \log x + \frac{x^3}{2} \sum_{n=0}^{\infty} \frac{|B(2n+2)|}{(n+1)(2n+3)!} x^{2n};$$

see, for example, equation (4.28) of [27] or Proposition 3.1 of the more readily accessible [28]. Differentiation in (9.5) gives $S'_4(x) = C_3(x)$ and $C'_3(x) = -S_2(x)$, so we may integrate (9.10) twice, term by term, using the initial values $C_3(0) = \zeta(3)$ and $S_4(0) = 0$, to get

$$(9.11) \quad S_4(x) = x\zeta(3) + \frac{x^3}{6} \log x - \frac{11}{36}x^3 - \frac{x^5}{2} \sum_{n=0}^{\infty} \frac{|B(2n+2)|}{(n+1)(2n+5)!} x^{2n},$$

for $0 \leq x < 2\pi$. It follows from (9.11) that, for $0 \leq x < 1$,

$$(9.12) \quad \begin{aligned} S_4(\pi x) - \frac{1}{16}S_4(2\pi x) &= \frac{7}{8}\pi x\zeta(3) + \frac{\pi^3}{12}x^3 \log(\pi x) - \frac{\pi^3}{12} \left(\frac{11}{6} + \log 2 \right) x^3 \\ &\quad + \frac{\pi^5 x^5}{2} \sum_{n=0}^{\infty} \frac{|B(2n+2)|(2^{2n+1}-1)}{(n+1)(2n+5)!} \pi^{2n} x^{2n}. \end{aligned}$$

Then substituting (9.12) for $S(x)$ in (9.7) and (9.8), we get for $0 \leq x < 1$,

$$(9.13) \quad \begin{aligned} \psi(x) &= \frac{84}{\pi^3}x\zeta(3) + \frac{8}{\pi}x^3 \log(\pi x) - \frac{8}{\pi} \left(\frac{11}{6} + \log 2 \right) x^3 - 3x(1-x) \\ &\quad + 48\pi x^5 \sum_{n=0}^{\infty} \frac{|B(2n+2)|(2^{2n+1}-1)}{(n+1)(2n+5)!} \pi^{2n} x^{2n} - \frac{96}{\pi^4} \sum_{n \text{ odd}} \frac{d_n}{n^4} \sin n\pi x. \end{aligned}$$

Since $\phi_0(x) = \mu^{-1}\psi(x)$, it follows that

$$\sup_{0 \leq x \leq 1/2} |\phi_0(x) - \phi_0^{k,m}(x)| \leq \frac{3\pi}{\mu} \sum_{n>k} \frac{|B(2n+2)|\pi^{2n}}{(n+1)(2n+5)!} + \frac{96}{\pi^4\mu} \sum_{\substack{n \text{ odd} \\ n>m}} \frac{d_n}{n^4}.$$

Here, since $|B(2\ell)| = 2\zeta(2\ell)(2\ell)!/(2\pi)^{2\ell}$,

$$\frac{3\pi}{\mu} \sum_{n>k} \frac{|B(2n+2)|\pi^{2n}}{(n+1)(2n+5)!} \leq \frac{3}{\pi\mu} \sum_{n>k} \frac{\zeta(2n+2)}{(2n+2)^4} 2^{-2n} \leq \frac{3}{\pi\mu} \frac{\zeta(2k+4)}{(2k+4)^4} \sum_{n=k+1}^{\infty} 4^{-n},$$

since $\zeta(\cdot)$ is decreasing. Moreover, a similar bound to (9.3) gives

$$\sum_{\substack{n \text{ odd} \\ n>m}} \frac{d_n}{n^4} \leq 2 \sum_{\substack{n \text{ odd} \\ n>m}} \frac{e^{-n\pi}}{n^4} \leq \frac{2}{1 - e^{-2\pi}} \frac{e^{-(m+2)\pi}}{(m+2)^4}.$$

With the numerical bound $96 < \pi^4(1 - e^{-2\pi})$, this completes the proof. \square

Important for the asymptotics of the normalized gap distribution given in Theorem 2.3 is the behaviour of $\phi_0(x)$ as $x \rightarrow 0$ (see Theorem 10.1 below). Here the expression (9.13) is misleading at first glance, as the next result shows.

LEMMA 9.6. *We have that $\phi_0(x) \sim (3/\mu)x^2$ as $x \rightarrow 0$.*

PROOF. First note that, using [6], page 287, to evaluate the sum involving tanh,

$$(9.14) \quad \sum_{n \text{ odd}} na_n = 4 \sum_{n \text{ odd}} \frac{\tanh(n\pi/2)}{n^3} - \pi \sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^3}{8} - \frac{3\pi}{4}\zeta(2) = 0.$$

Then from (2.4) with (9.7), (9.9), and (9.14) we obtain, as an alternative to (9.8),

$$\begin{aligned}
 \psi(x) &= \frac{24}{\pi^4} \sum_{n \text{ odd}} a_n (\sin n\pi x - n\pi x) \\
 (9.15) \qquad &= \frac{96}{\pi^4} S(x) + 3x^2 - \frac{84\zeta(3)}{\pi^3} x - \frac{96}{\pi^4} \sum_{n \text{ odd}} \frac{d_n}{n^4} (\sin n\pi x - n\pi x).
 \end{aligned}$$

Here $S(x) = \frac{7}{8}\pi x\zeta(3) + o(x^2)$ as $x \rightarrow 0$, by (9.12). Since $|y - \sin y| = O(y^3)$ as $y \rightarrow 0$, and $d_n = O(e^{-n\pi})$, the final sum in (9.15) is absolutely convergent, uniformly for $x \in [0, 1]$, and hence is $O(x^3)$. Thus (9.15) gives $\psi(x) \sim 3x^2$ as $x \rightarrow 0$. \square

10. Limiting gap statistics. This section contributes to the proof of Theorem 2.3, by establishing the corresponding limit statements for the approximating interval-splitting process appearing in Theorem 2.2, building on work of Brennan and Durrett [12, 13]. We work in a more general setting to emphasize which elements of r_0 and ϕ_0 contribute to the tail asymptotics of the normalized gap density g_0 . Also, because the approximation between the nucleation process and the interval-splitting limit works well only for large times (see Section 7), we derive our results on the interval-splitting process started from arbitrary initial conditions. To this end, for $n_0 \in \mathbb{Z}_+$ and $z \in \Delta_{n_0}$, we write $\mathbb{P}_{n_0,z}^{r,\Phi}$ for the law of the interval-splitting process $\mathcal{S} = (\mathcal{S}_{n_0}, \mathcal{S}_{n_1}, \dots)$ with $\mathcal{S}_{n_0} = z$ and evolving for $n \geq n_0$ according to (2.3) with parameters r and Φ . Here is the main result of this section.

THEOREM 10.1. *Let $\alpha, b \in (0, \infty)$, $\beta \in \mathbb{R}_+$, $r(\ell) = \ell^\alpha$, and ϕ be a bounded probability density on $[0, 1]$ with $\phi(x) = \phi(1 - x)$ for all $x \in [0, 1]$ and $\phi(x) \sim bx^\beta$ as $x \rightarrow 0$. Define $\Phi(B) = \int_B \phi(x) dx$ for all $B \in \mathcal{B}$. Let \mathcal{S} be an interval-splitting process with parameters r and Φ , and let $\ell_{n,i}$, $i \in [n + 1]$, denote the lengths of the gaps in \mathcal{S}_n . For $x \in [0, 1]$, let $C_n(x) = \max\{m \in \{0, 1, \dots, n + 1\} : \sum_{i=1}^m \ell_{n,i} \leq x\}$.*

- (i) *For all $n_0 \in \mathbb{Z}_+$ and all $z \in \Delta_{n_0}$, $\lim_{n \rightarrow \infty} \sup_{x \in [0,1]} |n^{-1} C_n(x) - x| = 0$, $\mathbb{P}_{n_0,z}^{r,\Phi}$ -a.s.*
- (ii) *There exists a continuous probability density function g on \mathbb{R}_+ such that for all $n_0 \in \mathbb{Z}_+$, all $z \in \Delta_{n_0}$, and all $x \in \mathbb{R}_+$,*

$$(10.1) \quad \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{i \in [n+1]} \mathbb{1}\{(n+1)\ell_{n,i} \leq x\} = \int_0^x g(y) dy, \quad \mathbb{P}_{n_0,z}^{r,\Phi}\text{-a.s. and in } L^1.$$

- (iii) *There exist constants $c_{g,0}, c_{g,\infty}, \theta \in (0, \infty)$ such that*

$$g(x) \sim c_{g,0}x^\beta \quad \text{as } x \rightarrow 0, \quad \text{and} \quad g(x) \sim c_{g,\infty}x^{2a-2} \exp(-\theta x^\alpha) \quad \text{as } x \rightarrow \infty,$$

where in the latter case, $a = \lim_{x \rightarrow 0} \phi(x)$.

REMARK 10.2. In the case of a uniform splitting distribution, where $\phi(x) \equiv 1$ for $x \in [0, 1]$, one has the explicit expression (see Remark 10.4 below) that

$$g(x) = \frac{\alpha\Gamma(2/\alpha)}{\Gamma(1/\alpha)^2} \exp\left\{-\left(\frac{\Gamma(2/\alpha)}{\Gamma(1/\alpha)}\right)^\alpha x^\alpha\right\}.$$

There are two parts to the proof of Theorem 10.1. One is to translate the results of [12, 13], which pertain to a continuous-time interval-splitting model started from a unit interval, to our setting, to obtain a characterization of the density g in terms of distributional fixed-point equations. The second part of the proof is an analysis of these fixed-point equations to obtain the tail asymptotics. We start with the second part.

The fixed-point description goes as follows. Let X and T be random variables on \mathbb{R}_+ with probability density functions f_X and f_T respectively, given by

$$f_X(x) = 2e^{-2x}\phi(e^{-x}) \quad \text{and}$$

$$f_T(x) = c_T \int_0^{e^{-x}} s\phi(s) ds, \quad \text{where } \frac{1}{c_T} = \int_0^\infty ue^{-2u}\phi(e^{-u}) du.$$

Define the distribution of random variables Q and Z via the fixed-point equation

$$(10.2) \quad (Q, Z) \stackrel{d}{=} (Ze^{-\alpha T}, Ze^{-\alpha X} + \xi), \quad Q \geq 0, Z \geq 0,$$

where the Z, T, X , and ξ on the right-hand side are independent, and ξ is exponentially distributed with unit mean. The second coordinate equality in (10.2) determines uniquely the distribution of Z by for example, Theorem 1.5(i) and Lemma 1.4(a) of [41]; the first coordinate equality then specifies the distribution of Q . We will show that the g in (10.1) is given in terms of the density q of the random variable $Q^{1/\alpha}$; the next result gives asymptotics for q .

LEMMA 10.3. *Let $\alpha, b \in (0, \infty)$, $\beta \in \mathbb{R}_+$, $r(\ell) = \ell^\alpha$, and ϕ be a bounded probability density on $[0, 1]$ with $\phi(x) = \phi(1 - x)$ for all $x \in [0, 1]$ and $\phi(x) \sim bx^\beta$ as $x \rightarrow 0$. Then the random variable $Q^{1/\alpha}$ whose distribution is characterized by (10.2) has a density q which is continuous on \mathbb{R}_+ , and there exist constants $c_{q,0}, c_{q,\infty} \in (0, \infty)$ such that*

$$q(x) \sim c_{q,0}x^{1+\beta} \quad \text{as } x \rightarrow 0 \quad \text{and} \quad q(x) \sim c_{q,\infty}x^{2\alpha-1} \exp(-x^\alpha) \quad \text{as } x \rightarrow \infty,$$

where in the latter case, $a = \lim_{x \rightarrow 0} \phi(x)$.

PROOF. Let $F_Z(r) := \mathbb{P}(Z \leq r)$. By (10.2), conditioning on ξ and then X , for $r \geq 0$,

$$F_Z(r) = \int_0^r e^{-u} \mathbb{P}(Ze^{-\alpha X} \leq r - u) du$$

$$= 2 \int_0^r du e^{-u} \int_0^\infty e^{-2x} \phi(e^{-x}) F_Z((r - u)e^{\alpha x}) dx.$$

With the change of variable $v = r - u$, this says

$$F_Z(r) = 2e^{-r} \int_0^r dve^v \int_0^\infty e^{-2x} \phi(e^{-x}) F_Z(v e^{\alpha x}) dx,$$

which is continuously differentiable, so $f_Z(r) := F_Z'(r)$ exists and is continuous. Also

$$\mathbb{P}(Ze^{-\alpha X} \leq r) = 2 \int_0^\infty e^{-2x} \phi(e^{-x}) F_Z(r e^{\alpha x}) dx.$$

Since F_Z is continuously differentiable, we can differentiate under the integral to get that $Y := Ze^{-\alpha X}$ has a density f_Y satisfying

$$f_Y(r) = 2 \int_0^\infty e^{(\alpha-2)x} \phi(e^{-x}) f_Z(r e^{\alpha x}) dx.$$

Then since Z is distributed as $Y + \xi$, we can write

$$F_Z(r) = \int_0^r f_Y(y) \mathbb{P}(\xi \leq r - y) dy = 2 \int_0^r dy (1 - e^{-(r-y)}) \int_0^\infty e^{(\alpha-2)x} \phi(e^{-x}) f_Z(y e^{\alpha x}) dx.$$

Differentiating we obtain

$$f_Z(r) = 2e^{-r} \int_0^r dy e^y \int_0^\infty e^{(\alpha-2)x} \phi(e^{-x}) f_Z(y e^{\alpha x}) dx.$$

With the substitution $u = ye^{\alpha x}$, we get, for $r \geq 0$,

$$(10.3) \quad f_Z(r) = \frac{2}{\alpha} e^{-r} \int_0^r dy e^y y^{\frac{2-\alpha}{\alpha}} \int_y^\infty u^{-\frac{2}{\alpha}} \phi((y/u)^{1/\alpha}) f_Z(u) du.$$

We use the relation (10.3) to derive asymptotics of $f_Z(r)$ as $r \rightarrow 0$. Fix $\varepsilon > 0$. For $u \geq Ky$, $K > 1$, we have $y/u \leq 1/K$. Choosing $K > 1$ large enough (depending on ε), this means $\phi((y/u)^{1/\alpha}) \leq (b + \varepsilon)(y/u)^{\beta/\alpha}$ for all $y > 0$ and all $u \geq Ky$. Hence

$$\int_{Ky}^\infty u^{-\frac{2}{\alpha}} \phi((y/u)^{1/\alpha}) f_Z(u) du \leq (b + \varepsilon) y^{\beta/\alpha} \int_{Ky}^\infty u^{-\frac{2+\beta}{\alpha}} f_Z(u) du.$$

On the other hand, let $A := \sup_{x \in [0,1]} \phi(x)$, which is finite. Then, for all $y \in \mathbb{R}_+$,

$$\int_y^{Ky} u^{-\frac{2}{\alpha}} \phi((y/u)^{1/\alpha}) f_Z(u) du \leq AB(Ky) \int_y^{Ky} u^{-\frac{2}{\alpha}} du, \quad \text{where } B(y) := \sup_{0 \leq u \leq y} f_Z(u).$$

It follows from (10.3) that for C a finite constant depending on K , for all $r \in \mathbb{R}_+$,

$$(10.4) \quad f_Z(r) \leq C \int_0^r B(Ky) dy + \frac{2}{\alpha} (b + \varepsilon) \int_0^r dy y^{\frac{2+\beta-\alpha}{\alpha}} \int_{Ky}^\infty u^{-\frac{2+\beta}{\alpha}} f_Z(u) du.$$

We apply (10.4) successively to get a bound. Let $r_k = K^{-k}$. Suppose that for constants $C_k, \gamma_k \in \mathbb{R}_+$ we have $f_Z(r) \leq C_k r^{\gamma_k}$ for $r \in [0, r_k]$. We bound the u -integral in (10.4) via

$$\begin{aligned} \int_{Ky}^\infty u^{-\frac{2+\beta}{\alpha}} f_Z(u) du &\leq C_k \int_{Ky}^{r_k} u^{\gamma_k - \frac{2+\beta}{\alpha}} du + r_k^{-\frac{2+\beta}{\alpha}} \\ &\leq C_{k+1} + C_{k+1} y^{1+\gamma_k - \frac{2+\beta}{\alpha}} \log(1/y), \end{aligned}$$

for some $C_{k+1} < \infty$ and all $y \in [0, r_{k+1}]$. Thus from (10.4) we get, for all $r \in [0, r_{k+1}]$,

$$f_Z(r) \leq C_{k+1} r^{1+\gamma_k} \log(1/r) + C_{k+1} r^{\frac{2+\beta}{\alpha}} \leq C_{k+1} r^{\gamma_{k+1}},$$

$$\text{where } \gamma_{k+1} = \left(\frac{1}{2} + \gamma_k\right) \wedge \left(\frac{2+\beta}{\alpha}\right),$$

redefining C_{k+1} as necessary. Starting with the bound $f_Z(r) \leq C_0 = \sup_{x \in [0,1]} f_Z(x) < \infty$ for $r \in [0, r_0]$, we iterate this argument from $\gamma_0 = 0$ to get, for some finite k ,

$$(10.5) \quad f_Z(r) \leq C_k r^{\frac{2+\beta}{\alpha}} \quad \text{for all } r \in [0, r_k].$$

Using the bound (10.5) now in (10.4) shows that, for all r sufficiently small,

$$(10.6) \quad f_Z(r) \leq \frac{2}{2+\beta} c_1 (b + \varepsilon) r^{\frac{2+\beta}{\alpha}}, \quad \text{where } c_1 = \int_0^\infty u^{-\frac{2+\beta}{\alpha}} f_Z(u) du,$$

the c_1 being a finite positive constant, since (10.5) shows that the integral does not blow up near zero. The other direction is similar: from (10.3) we have

$$\begin{aligned} f_Z(r) &\geq \frac{2}{\alpha} (b - \varepsilon) e^{-r} \int_0^r dy y^{\frac{2+\beta-\alpha}{\alpha}} \int_{Ky}^\infty u^{-\frac{2+\beta}{\alpha}} f_Z(u) du \\ &\geq \frac{2}{2+\beta} c_1 (b - \varepsilon) e^{-r} r^{\frac{2+\beta}{\alpha}} - C \int_0^r dy y^{\frac{2+\beta-\alpha}{\alpha}} \int_0^{Ky} u^{-\frac{2+\beta}{\alpha}} f_Z(u) du. \end{aligned}$$

With the upper bound from (10.6) we get that the negative term here is $O(r^{1+\frac{2+\beta}{\alpha}})$ as $r \rightarrow 0$. Since $\varepsilon > 0$ was arbitrary, we conclude that

$$(10.7) \quad f_Z(r) = (c_{Z,0} + o(1)) r^{\frac{2+\beta}{\alpha}} \quad \text{as } r \rightarrow 0,$$

where $c_{Z,0} := \frac{2bc_1}{2+\beta}$, with c_1 defined in (10.6).

Now we turn to the random variable $Q^{1/\alpha}$. Note that the density f_T of T satisfies $f_T(t) \sim c_{T,\infty} e^{-(2+\beta)t}$ as $t \rightarrow \infty$ where $c_{T,\infty} := \frac{bc_T}{2+\beta} \in (0, \infty)$. It follows that $f_T(t) \leq C e^{-(2+\beta)t}$ for some $C < \infty$ and all $t \in \mathbb{R}_+$. For $r \in \mathbb{R}$ we have from (10.2) that

$$\mathbb{P}(-\log Q > r\alpha) = \mathbb{P}(\alpha T - \log Z > r\alpha) = \int_0^\infty f_T(t) F_Z(e^{(t-r)\alpha}) dt.$$

Since F_Z is continuously differentiable, and since $\mathbb{P}(-\log Q > r\alpha) = \mathbb{P}(Q^{1/\alpha} \leq e^{-r})$, we can differentiate under the integral sign to see that $Q^{1/\alpha}$ has a density q which satisfies

$$\begin{aligned} e^{(\alpha-1)r} q(e^{-r}) &= \alpha \int_0^\infty e^{\alpha t} f_T(t) f_Z(e^{\alpha(t-r)}) dt \\ (10.8) \qquad \qquad \qquad &= e^{\alpha r} \int_{e^{-\alpha r}}^\infty f_T\left(\frac{\log u}{\alpha} + r\right) f_Z(u) du. \end{aligned}$$

Here f_T is bounded and continuous, so the dominated convergence theorem shows that the second integral in (10.8) is continuous over $r \in \mathbb{R}$, and hence $q(r)$ is continuous over $r \in (0, \infty)$. We now use the first integral in (10.8) to derive the asymptotics of q near zero. By (10.7) there exists $C < \infty$ such that $f_Z(r) \leq Cr^{\frac{2+\beta}{\alpha}}$ for all $r \in \mathbb{R}_+$. Thus

$$\int_0^{r/2} e^{\alpha t} f_T(t) f_Z(e^{\alpha(t-r)}) dt \leq C e^{-(2+\beta)r} \int_0^{r/2} e^{\alpha t} dt \leq C e^{\alpha r/2} e^{-(2+\beta)r}.$$

Similarly, for any $\varepsilon > 0$ and all $t > r/2$ with r sufficiently large,

$$\begin{aligned} \int_{r/2}^\infty e^{\alpha t} f_T(t) f_Z(e^{\alpha(t-r)}) dt &\leq (c_{T,\infty} + \varepsilon) \int_{r/2}^\infty e^{\alpha t} e^{-(2+\beta)t} f_Z(e^{\alpha(t-r)}) dt \\ &= (c_{T,\infty} + \varepsilon) e^{\alpha r - (2+\beta)r} \int_{-r/2}^\infty e^{\alpha s - (2+\beta)s} f_Z(e^{\alpha s}) ds, \end{aligned}$$

using the change of variable $s = t - r$. The $r \rightarrow \infty$ limit of the s -integral here converges to c_1/α , with c_1 the integral defined at (10.6). Thus we get, for all r sufficiently large,

$$\int_{r/2}^\infty e^{\alpha t} f_T(t) f_Z(e^{\alpha(t-r)}) dt \leq \left(\frac{c_1 c_{T,\infty}}{\alpha} + \varepsilon\right) e^{\alpha r - (2+\beta)r}.$$

A similar argument in the other direction shows that, for all r sufficiently large,

$$\int_{r/2}^\infty e^{\alpha t} f_T(t) f_Z(e^{\alpha(t-r)}) dt \geq \left(\frac{c_1 c_{T,\infty}}{\alpha} - \varepsilon\right) e^{\alpha r - (2+\beta)r}.$$

It follows from (10.8) and the above estimates that

$$(10.9) \qquad \qquad \qquad q(r) = (c_{q,0} + o(1))r^{1+\beta} \quad \text{as } r \rightarrow 0,$$

where $c_{q,0} := \frac{bc_1 c_T}{2+\beta}$, with c_1 as defined at (10.6).

Next we turn to the upper tail estimates. In this case we will use Brennan and Durrett’s expression for the moment generating function of Z and a Tauberian theorem. Brennan and Durrett also give an expression for the moment generating function of Q , but monotonicity properties, helpful for deducing density asymptotics via the Tauberian argument, are easier to demonstrate for Z . Recalling that Z has the same distribution as $Y + \xi$, for Y, ξ independent and ξ exponential with unit mean, we have

$$\mathbb{P}(Z \leq r) = \int_0^r e^{-s} \mathbb{P}(Y \leq r - s) ds = e^{-r} \int_0^r e^u \mathbb{P}(Y \leq u) du.$$

Differentiation gives $f_Z(r) = \mathbb{P}(Y \leq r) - \mathbb{P}(Z \leq r)$, and so $f'_Z(r) = f_Y(r) - f_Z(r)$. Thus

$$\frac{d}{dr}(e^r f_Z(r)) = e^r (f'_Z(r) + f_Z(r)) = e^r f_Y(r) \geq 0.$$

Hence $e^r f_Z(r)$ is nondecreasing; this is the helpful monotonicity property mentioned above. Using Brennan and Durrett’s formula for the moments of Z ([13], page 114), we see

$$(10.10) \quad m_Z(t) := \mathbb{E}(e^{tZ}) = \sum_{k=0}^{\infty} t^k \prod_{j=1}^k \frac{1}{1 - h(j\alpha)},$$

where

$$h(t) := \int_0^{\infty} e^{-tx} f_X(x) dx = 2 \int_0^{\infty} e^{-(2+t)x} \phi(e^{-x}) dx.$$

Since X is nondegenerate, $h(t) < 1$ for all $t > 0$. Moreover, since $\phi(e^{-x}) \sim \phi(x) \sim bx^\beta$ as $x \rightarrow 0$, we may apply Laplace’s method (see, e.g., [42], pages 55–58) to obtain

$$(10.11) \quad h(t) \sim 2b\Gamma(1 + \beta)t^{-1-\beta} \quad \text{as } t \rightarrow \infty.$$

It follows from (10.11) that $m_Z(t) < \infty$ provided $|t| < 1$; indeed, as we will see, the information we need is contained in the asymptotics of $m_Z(t)$ as $t \uparrow 1$. Consider the Laplace transform \tilde{m}_Z associated with $e^r f_Z(r)$, namely

$$\tilde{m}_Z(t) := \int_0^{\infty} e^{-tx} e^x f_Z(x) dx = m_Z(1 - t),$$

which is finite for $t \in (0, 1)$. We will use a Tauberian theorem to relate the $r \rightarrow \infty$ asymptotics of $e^r f_Z(r)$ to the $t \rightarrow 0$ asymptotics of $\tilde{m}_Z(t)$. From (10.10), we have

$$(10.12) \quad \tilde{m}_Z(t) = \sum_{k=0}^{\infty} (1 - t)^k \exp \sum_{j=1}^k \log \left(\frac{1}{1 - h(j\alpha)} \right).$$

Here we have from (10.11) that, as $j \rightarrow \infty$,

$$\log \left(\frac{1}{1 - h(j\alpha)} \right) = \log \left(1 + \frac{h(j\alpha)}{1 - h(j\alpha)} \right) = 2b\Gamma(1 + \beta)(j\alpha)^{-1-\beta} + O(j^{-2-\beta}).$$

It follows that, as $k \rightarrow \infty$,

$$(10.13) \quad \sum_{j=1}^k \log \left(\frac{1}{1 - h(j\alpha)} \right) = \begin{cases} \frac{2b}{\alpha} \log k + \log c_2 + o(1) & \text{if } \beta = 0, \\ \log c_2 + o(1) & \text{if } \beta > 0, \end{cases}$$

where $c_2 \in (0, \infty)$ is a constant depending on α, β , and ϕ .

If $\beta > 0$, then (10.12) and (10.13) show that $\tilde{m}_Z(t) = \sum_{k=0}^{\infty} (1 - t)^k (c_2 + o(1))$, where the $o(1)$ is as $k \rightarrow \infty$, and is uniform in $t > 0$. It is elementary to deduce that

$$(10.14) \quad \tilde{m}_Z(t) \sim c_2/t \quad \text{as } t \rightarrow 0, \text{ if } \beta > 0.$$

On the other hand, suppose that $\beta = 0$. Then we have from (10.12) and (10.13) that $\tilde{m}_Z(t) = \sum_{k=0}^{\infty} (1 - t)^k (c_2 + o(1))k^{2b/\alpha}$, where the $o(1)$ is as $k \rightarrow \infty$, and is uniform in t . It is a consequence of a standard Abelian theorem for power series that $\sum_{k=0}^{\infty} (1 - t)^k k^\rho \sim \Gamma(1 + \rho)t^{-\rho-1}$ as $t \downarrow 0$. Thus we deduce that

$$(10.15) \quad \tilde{m}_Z(t) \sim \Gamma \left(1 + \frac{2b}{\alpha} \right) c_2 t^{-\frac{2b}{\alpha}-1} \quad \text{as } t \rightarrow 0, \text{ if } \beta = 0.$$

Defining $a := \lim_{x \rightarrow 0} \phi(x)$, so $a = 0$ if $\beta > 0$ and $a = b$ if $\beta = 0$, we can combine the asymptotics (10.14) and (10.15) into the single statement that, for some $\tilde{c}_{Z,0} \in (0, \infty)$,

$$(10.16) \quad \tilde{m}_Z(t) \sim \tilde{c}_{Z,0} t^{-1-(2a/\alpha)} \quad \text{as } t \rightarrow 0.$$

Together with the fact that $e^r f_Z(r)$ is nondecreasing, the asymptotics (10.16) allow us to apply a monotone-density Tauberian theorem (e.g., [18], page 446) to deduce

$$(10.17) \quad f_Z(r) = (c_{Z,\infty} + o(1)) r^{2a/\alpha} e^{-r} \quad \text{as } r \rightarrow \infty,$$

where $c_{Z,\infty} \in (0, \infty)$. Rewriting the first equality in (10.8), we have

$$(10.18) \quad q(r) = \alpha r^{\alpha-1} \int_0^\infty e^{\alpha t} f_T(t) f_Z(e^{\alpha t} r^\alpha) dt.$$

From (10.18), with the change of variable $u = e^{\alpha t}$ and the f_Z asymptotics from (10.17),

$$q(r) = (c_{Z,\infty} + o(1)) r^{2a+\alpha-1} \int_1^\infty f_T(\alpha^{-1} \log u) u^{2a/\alpha} e^{-ur^\alpha} du,$$

as $r \rightarrow \infty$. Since $\lim_{x \rightarrow 0} f_T(x) = c_T/2$ (by symmetry of ϕ), the asymptotics of the latter integral can be obtained by Laplace’s method (e.g., [42], pages 55–58), which gives

$$q(r) = (c_{q,\infty} + o(1)) r^{2a-1} e^{-r^\alpha} \quad \text{as } r \rightarrow \infty,$$

where $c_{q,\infty} := \frac{c_T c_{Z,\infty}}{2}$. This completes the proof. \square

PROOF OF THEOREM 10.1. Define the distribution function

$$(10.19) \quad G(x) := \frac{1}{\rho} \int_0^x \frac{q(y)}{y} dy \quad \text{for } x \in \mathbb{R}_+,$$

where q is the density of the random variable $Q^{1/\alpha}$ and $\rho := \mathbb{E}(Q^{-1/\alpha}) \in (0, \infty)$. Brennan and Durrett [13] consider a continuous-time embedding of the interval-splitting process in which an interval of length ℓ splits at rate $r(\ell) = \ell^\alpha$, and, when it splits, does so according to Φ . Starting at time $t = 0$ with a single gap of length 1, let i_t denote the number of intervals at time $t \in \mathbb{R}_+$ and let $e_{t,i}$, $i \in [i_t]$, denote the lengths of those intervals, listed left to right. For $x \in [0, 1]$, let $c_t(x) := \max\{m \in \{0, 1, \dots, i_t\} : \sum_{i=1}^m e_{t,i} \leq x\}$, the number of intervals wholly contained in $[0, x]$. The result of [13], page 113, says that

$$(10.20) \quad \lim_{t \rightarrow \infty} t^{-1/\alpha} i_t = \rho, \quad \text{a.s.,} \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{1}{i_t} \sum_{i \in [i_t]} \mathbb{1}\{t^{1/\alpha} e_{t,i} \leq x\} = G(x) \quad \text{a.s.,}$$

where G is given by (10.19), while Theorem 1.1 of [12], pages 1027–1028, shows that

$$(10.21) \quad \lim_{t \rightarrow \infty} \frac{c_t(x)}{i_t} = x, \quad \text{a.s., for all } x \in [0, 1].$$

Now we extend the model to permit an arbitrary initial configuration $z \in \Delta_{n_0}$ at time $t = 0$. Then the initial gaps $j \in [n_0 + 1]$ have lengths u_1, \dots, u_{n_0+1} , say, with $\sum_{j=1}^{n_0+1} u_j = 1$. The process evolves independently on each gap. Let $i_{j,t}^z$ denote the number of intervals at time $t \in \mathbb{R}_+$ for the process restricted to initial gap j , and let $i_t^z = \sum_{j=1}^{n_0+1} i_{j,t}^z$ denote the total number of intervals. Also let $e_{j,t,i}^z$, $i \in [i_{j,t}^z]$, denote the interval lengths for the process in interval j . The process in interval j is a copy of the process on the single initial interval $[0, 1]$, but with all lengths scaled by a factor of u_j , which entails a time-scaling of u_j^α ; in

particular, $i_{j,t}^z$ has the same distribution as $i_{u_j^\alpha t}$, and the collection $e_{j,t,i}^z, i \in [i_{j,t}^z]$, has the same distribution as $u_j e_{u_j^\alpha t, i}, i \in [i_{u_j^\alpha t}]$. Thus (10.20) implies that

$$(10.22) \quad \begin{aligned} \lim_{t \rightarrow \infty} t^{-1/\alpha} i_{j,t}^z &= \rho u_j, \quad \text{a.s., and} \\ \lim_{t \rightarrow \infty} \frac{1}{i_{j,t}^z} \sum_{i \in [i_{j,t}^z]} \mathbb{1}\{t^{1/\alpha} e_{j,t,i}^z \leq x\} &= G(x), \quad \text{a.s.} \end{aligned}$$

Also, if $e_{t,i}^z, i \in [i_t^z]$ are the (aggregated) interval lengths, listed left to right, then

$$\frac{1}{i_t^z} \sum_{i \in [i_t^z]} \mathbb{1}\{t^{1/\alpha} e_{t,i}^z \leq x\} = \sum_{j \in [n_0+1]} \frac{i_{t,j}^z}{i_t^z} \frac{1}{i_{t,j}^z} \sum_{i \in [i_{t,j}^z]} \mathbb{1}\{t^{1/\alpha} e_{j,t,i}^z \leq x\}.$$

Since $\sum_{j=1}^{n_0+1} i_{j,t}^z = i_t^z$ and $\sum_{j=1}^{n_0+1} u_j = 1$, we conclude from (10.22) that, for any $z \in \Delta_{n_0}$,

$$(10.23) \quad \lim_{t \rightarrow \infty} t^{-1/\alpha} i_t^z = \rho, \quad \text{a.s. and} \quad \lim_{t \rightarrow \infty} \frac{1}{i_t^z} \sum_{i \in [i_t^z]} \mathbb{1}\{t^{1/\alpha} e_{t,i}^z \leq x\} = G(x), \quad \text{a.s.}$$

If $\tau_0 = 0$ and $\tau_n \in \mathbb{R}_+$ denotes the time of the n th splitting event, then $\mathcal{S}_{n_0}, \mathcal{S}_{n_0+1}, \dots$ is embedded at times τ_0, τ_1, \dots of the continuous-time process stated at $\mathcal{S}_{n_0} = z$. Given $\mathcal{S}_n, n \geq n_0$, let $\ell_{n,1}, \dots, \ell_{n,n+1}$ denote the lengths of the gaps, so $i_{\tau_n}^z = n + 1$ and $\ell_{n,i} = e_{\tau_n,i}^z$. Translating (10.23) into discrete time thus gives

$$(10.24) \quad \begin{aligned} \lim_{n \rightarrow \infty} \tau_n^{-1/\alpha} n &= \rho, \quad \text{a.s., and} \\ \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{i \in [n+1]} \mathbb{1}\{\tau_n^{1/\alpha} \ell_{n,i} \leq x\} &= G(x) \quad \text{a.s.} \end{aligned}$$

Let \mathcal{G}_n denote the σ -algebra generated by $\mathcal{S}_0, \dots, \mathcal{S}_n$ and τ_0, \dots, τ_n . Then, for U_n a uniform random variable on $[n + 1]$, independent of \mathcal{G}_n , set $\tilde{\ell}_n = (n + 1)\ell_{n,U_n}$, so that

$$\mathbb{P}(\tilde{\ell}_n \leq x \mid \mathcal{G}_n) = \frac{1}{n+1} \sum_{i \in [n+1]} \mathbb{1}\{\tau_n^{1/\alpha} \ell_{n,i} \leq x \tau_n^{1/\alpha} (n+1)^{-1}\}.$$

Since, by (10.24), $(n + 1)\tau_n^{-1/\alpha} \rightarrow \rho$, a.s., for any $\varepsilon > 0$ and all n sufficiently large,

$$\mathbb{P}(\tilde{\ell}_n \leq x \mid \mathcal{G}_n) \leq \frac{1}{n+1} \sum_{i \in [n+1]} \mathbb{1}\{\tau_n^{1/\alpha} \ell_{n,i} \leq x(\rho^{-1} + \varepsilon)\},$$

by monotonicity, so that, by (10.24), $\limsup_{n \rightarrow \infty} \mathbb{P}(\tilde{\ell}_n \leq x \mid \mathcal{G}_n) \leq G(x(\rho^{-1} + \varepsilon))$, a.s. By a similar argument in the other direction, and continuity of G given at (10.19), we get, a.s.,

$$(10.25) \quad \begin{aligned} \lim_{n \rightarrow \infty} \mathbb{P}(\tilde{\ell}_n \leq x \mid \mathcal{G}_n) &= \frac{1}{\rho} \int_0^{x/\rho} \frac{q(y)}{y} dy = \int_0^x g(z) dz, \\ \text{where } g(x) &:= \frac{q(x/\rho)}{\rho x}. \end{aligned}$$

This establishes the a.s. convergence result in (10.1) with $g(x)$ as displayed, and the L^1 convergence follows by the bounded convergence theorem. This proves (ii). Moreover, Lemma 10.3 shows that g as defined in (10.25) is continuous on $(0, \infty)$, and satisfies the asymptotics for g given in part (iii) of the theorem, with $c_{g,0} = c_{q,0} \rho^{-2-\beta}$, $c_{g,\infty} = c_{q,\infty} \rho^{-2a}$, and $\theta = \rho^{-\alpha}$. Thus (iii) is also proved.

For part (i), fix $x \in [0, 1]$ and let $j_x = \min\{j \in [n_0 + 1] : \sum_{i=1}^j u_i \geq x\}$, so that j_x is the index of the initial gap that contains x . Let $x' = \sum_{i=1}^{j_x-1} u_i$, so $0 \leq x' \leq x$. Then

$$(10.26) \quad c_t^z(x) := \max \left\{ m \in \{0, 1, \dots, i_t^z\} : \sum_{i=1}^m e_{t,i}^z \leq x \right\} = c_{j_x,t}^z(x) + \sum_{j < j_x} i_{j,t}^z,$$

where $c_{j_x,t}^z(x)$ means the number of intervals at time t contained in initial gap j_x (whose left endpoint is at x') that fall wholly in $[0, x]$. By scaling, $c_{j_x,t}^z(x)$, $i_{j_x,t}^z$ have the same distribution as $c_{u_j^{\alpha_t}}(\frac{x-x'}{u_j})$, $i_{u_j^{\alpha_t}}$, and so we have from (10.21) and (10.22) that

$$i_{j_x,t}^z \sim \rho t^{1/\alpha} u_j \quad \text{for all } j \quad \text{and} \quad c_{j_x,t}^z(x) \sim i_{j_x,t}^z \left(\frac{x-x'}{u_{j_x}} \right) \sim \rho t^{1/\alpha} (x-x').$$

Together with (10.26), this implies that $c_t^z(x) \sim \rho t^{1/\alpha} x$. It follows from (10.23) that $c_t^z(x)/i_t^z \rightarrow x$, a.s., and thus we get (i) after translating the result into discrete time. \square

REMARK 10.4. In the special case where $\phi(x) \equiv 1$ (uniform splitting), the explicit solutions to (10.10), (10.3), and (10.18) are $m_Z(t) = (1-t)^{-\frac{\alpha+2}{\alpha}}$ for $|t| < 1$, and

$$f_Z(r) = \frac{r^{2/\alpha}}{\Gamma(1 + \frac{2}{\alpha})} e^{-r} \quad \text{and} \quad q(r) = \frac{2r}{\Gamma(1 + \frac{2}{\alpha})} e^{-r^\alpha}, \quad r \in \mathbb{R}_+,$$

so that $\rho = \Gamma(1/\alpha) / \Gamma(2/\alpha)$ (cf. [13], page 113), which with (10.25) justifies Remark 10.2.

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