A first integral form of the energy-momentum equations for viscous flow, with comparisons drawn to classical fluid flow theory

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Abstract

An elegant four-dimensional Lorentz-invariant first-integral of the energy-momentum equations for viscous flow, comprised of a single tensor equation, is derived assuming a flat space-time and that the energy momentum tensor is symmetric. It represents a generalisation of corresponding Galilei-invariant theory associated with the classical incompressible Navier-Stokes equations, with the key features that the first-integral: (i) while taking the same form, overcomes the incompressibility constraint associated with its two- and three-dimensional incompressible Navier-Stokes counterparts; (ii) does not depend at outset on the constitutive fluid relationship forming the energy-momentum tensor. Starting from the resulting first integral: (iii) a rigorous asymptotic analysis shows that it reduces to one representing unsteady compressible viscous flow, from which the corresponding classical Galilei-invariant field equations are recovered; (iv) its use as a rigorous platform from which to solve viscous flow problems is demonstrated by applying the new general theory to the case of propagating acoustic waves, with and without viscous damping, and is shown to recover the well-known classical expressions for sound speed and damping rate consistent with those available in the open literature, derived previously as solutions of the linearised Navier-Stokes equations.

Keywords: Viscous flow theory, Potential fields, Integrability, Gauge criteria, Acoustic waves, Lorentz invariance

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1. Introduction and background

In various branches of physics the judicious use of potential fields has become indispensable for the reformulation and elegant solution of a wide variety of problems, with Maxwell's equations for the electric field, E_i, and magnetic flux density, B_i in a vacuum being a prime example. Following the introduction of scalar and vector potentials φ and A_i, respectively, such that $B_i = \varepsilon_{ijk}\partial_j A_k$ and $E_i = -\partial_i\varphi - \partial_t A_i$, two of the four equations are fulfilled identically, the remaining two being transformed into second-order partial differential equations (PDEs). Furthermore, recognition that both potentials can be redefined, the observables B_i and E_i remaining invariant, via the introduction of an arbitrary scalar field χ and employing the following gauge transformations:

$$A_i \to A_i' = A_i + \partial_i\chi \,, \tag{1}$$

$$\varphi \to \varphi' = \varphi - \partial_t\chi \,, \tag{2}$$

enables the mathematical form of the remaining equations to be manipulated in a beneficial way. For example the Lorenz gauge [1] enables these PDEs to be expressed in decoupled, self-adjoint form as a pair of d'Alembert equations, a form that has resonance with the fluid mechanics community.

Other useful analogies that can be drawn between Maxwell theory and that underpinning fluid flows, for the reduction of the field equations representing the latter, can be found in Marner [2]; the focus here is that of viscous flow. For an incompressible fluid the classical governing Navier-Stokes (NS) and continuity equations are:

$$\varrho\partial_t u_i + \varrho u_j\partial_j u_i = -\partial_i\left[p + U\right] + \eta\partial_j\partial_j u_i \,, \tag{3}$$

$$\partial_i u_i = 0 \,, \tag{4}$$

where u_i is the velocity field, p the pressure, η the dynamic viscosity and U the potential energy density of an external conservative force. Restricting the analysis to that of a conservative external force is sufficient for many fluid flow problems of general interest. To date, two strands of attack have emerged.

The first, of secondary interest in the present context but mentioned for historical completeness, is the so-called Clebsch transformation [3, 4] and related methodologies; its initial purpose being the reduction of Euler's equation (Re $\to \infty$; zero viscosity limit) to that of a generalised Bernoulli equation plus two transport equations for the Clebsch potentials. The approach has subsequently been generalised to encompass baroclinic flow [5], compressible flow involving volume viscosity [6] and, of late, shear viscosity [7, 8].

The second underpins the present work. From the standpoint of two-dimensional (2D) flow, i, j having values 1 and 2, then starting with Stokes flow (Re $\to 0$; zero inertia limit) a method of solution based on the complex-valued Goursat representation of the streamfunction, in terms of two analytic functions, has been progressively generalised – beginning with Legendre [9] and followed by

Coleman [10], Ranger [11], Marner et al. [12]. For 2D viscous flow, Ranger [11] formulated a first integral of equations (3, 4) based on the complex variable transformation $\xi := x_1 + \mathrm{i}x_2$, with $\mathrm{i} = \sqrt{-1}$, together with the introduction of a complex velocity field, $u := u_1 + \mathrm{i}u_2$, and scalar potential, Φ; an approach used to study the vortical structures that arise in shear-driven flows involving wavy substrate and free surface film flows over topographically patterned surfaces. Recently, Marner et al. [13] generalised the procedure to unsteady 2D flow by rewriting Φ in complex form, χ, resulting in two coupled complex field equations:

$$\varrho \frac{u^2}{2} = -4 \frac{\partial^2 \chi}{\partial \bar{\xi}^2} , \tag{5}$$

$$-\mathrm{i} \left[\varrho \frac{\partial \Psi}{\partial t} - 4\eta \frac{\partial^2 \Psi}{\partial \bar{\xi} \partial \xi} \right] + p + \varrho \frac{\bar{u} u}{2} + U = 4 \frac{\partial^2 \chi}{\partial \bar{\xi} \partial \xi} , \tag{6}$$

with the velocity field expressed as $u = -2\mathrm{i}\partial\Psi/\partial\bar{\xi}$ in terms of the streamfunction, Ψ, fulfilling the continuity equation (4) identically. The usefulness of these formulations has been demonstrated by investigating Couette flow confined between translating wavy rigid surfaces in order to study inertial [14] and unsteady [13] effects by analytical, semi-analytical and numerical means. In parallel, a variational formulation of the field equations is available that can be utilised for developing novel solution methods; for example it has been used to great effect for computing steady, thin film coating flows over hemispherical and conically shaped substate [15].

The obvious limitation of the second approach is its extension beyond two dimensions, since for three-dimensional (3D) viscous flow a corresponding complex first integral formulation is, by definition, not possible; however, a real-valued tensor form is. In two dimensions such a form can be established by decomposing both (5, 6) into real and imaginary parts and taking the linear combinations Re(6)±Re(5), leading to four real-valued equations:

$$\varrho u_1^2 + p + U = 2\partial_2^2 \mathrm{Re}\chi + 2\partial_1\partial_2 \mathrm{Im}\chi , \tag{7}$$

$$\varrho u_2^2 + p + U = 2\partial_1^2 \mathrm{Re}\chi - 2\partial_1\partial_2 \mathrm{Im}\chi , \tag{8}$$

$$\varrho u_1 u_2 = - \left\{ \partial_1^2 - \partial_2^2 \right\} \mathrm{Im}\chi - 2\partial_1\partial_2 \mathrm{Re}\chi , \tag{9}$$

$$\varrho \partial_t \Psi = - \nabla^2 \left[\mathrm{Im}\chi - \eta\Psi \right] , \tag{10}$$

which, in the context of extension to higher dimensions, can be written in a convenient and compact tensor form:

$$\varrho u_i u_j + (p + U)\delta_{ij} = -\partial_k \partial_k \tilde{a}_{ij} + \partial_i A_j + \partial_j A_i - \partial_k A_k \delta_{ij}, \tag{11}$$

with the tensor and vector fields given by $\tilde{a}_{ij} = -\mathrm{Re}\chi\delta_{ij}$ and $A_k = \partial_i \tilde{a}_{ij} + \varepsilon_{jk}\partial_k \mathrm{Im}\chi$, respectively. In equation (11) and subsequently, the following *Einstein notation* is utilised: an index variable appearing twice in a single term implies summation of that term over all values of the index. In addition, the abbreviations $\partial_k := \partial/\partial x_k$ and $\partial_t := \partial/\partial t$ imply partial differentiation, while

δ_{ij} and ε_{ij} denote the Kronecker delta function and the two-dimensional Levi-Civita symbol, respectively.

Only recently [16], was the corresponding leap made to three dimensions, an essential underpinning being analogies drawn with the methodical reduction of Maxwell's equations [2]; The continuity equation, (4), is shown to be fulfilled identically following the introduction a vector potential Ψ_k for the velocity, in accordance with $u_i = \varepsilon_{ijk}\partial_j\Psi_k$ and ε_{ijk} denoting the corresponding 3D Levi-Civita symbol, representing a 3D generalisation of the 2D streamfunction [17]. Subsequent parallels drawn with Maxwell's theory, involving the introduction of a tensor potential a_{ij} and a vector potential φ_i, facilitates the formulation of an exact first integral of the full, unsteady, incompressible NS equations in their most general, *preliminary*, form:

$$\varrho u_i u_j + (p+U)\delta_{ij} = \frac{1}{2}\varepsilon_{ilk}\varepsilon_{jpq}\partial_l\partial_p\left[a_{kq}+a_{qk}\right] + \partial_i\left(\varepsilon_{jlk}\partial_l\varphi_k\right) + \partial_j\left(\varepsilon_{ilk}\partial_l\varphi_k\right), \quad (12)$$

$$\varrho\partial_t\Psi_n = \partial_n\partial_k\left[\frac{1}{2}\varepsilon_{kqp}a_{pq} + \varphi_k - \eta\Psi_k\right] - \partial_k\partial_k\left[\varphi_n - \eta\Psi_n\right], \quad (13)$$

the structure of which correspond to the complex field equations (5, 6) for 2D flow. Taking the divergence of (12), the curl of (13) and eliminating all potential fields via linear combination, recovers the NS equations (3), see [16]. Furthermore, since the product $\varepsilon_{ikl}\varepsilon_{jpq}$ can be expressed as $\delta_{ij}\delta_{kp}\delta_{lq}+\delta_{ip}\delta_{kq}\delta_{lj}+\delta_{iq}\delta_{kj}\delta_{lp}-\delta_{ip}\delta_{kj}\delta_{lq}-\delta_{ij}\delta_{kq}\delta_{lp}-\delta_{iq}\delta_{kp}\delta_{lj}$ [18], equation (12) can be reformulated to take the more *utilitarian* form (11), involving now the modified symmetric tensor potential $\tilde{a}_{ij} := \frac{1}{2}\left[a_{ij}+a_{ji}-a_{kk}\delta_{ij}\right]$ and the auxiliary vector field $A_j := \partial_k\tilde{a}_{kj} + \varepsilon_{jlk}\partial_l\varphi_k$; with the indices i, j taking values from 1 to 3.

Starting from the above exact integration of the NS equations, rather than the NS equations themselves, solutions (both analytical and numerical) have been obtained to a hierarchy of classical flow problems available in the open literature showing they are recovered identically [16, 2].

A notable feature of the proceeding analysis is that the tensor equations (11) has the same structure in both two and three dimensions; in the same manner the 3D vector equations (13) corresponds to the scalar equation (10) of the 2D theory. A key question that arises and is addressed below is whether a first integral leading to a tensor form in accordance with (11) can similarly be established for the four-dimensional (4D) energy-momentum equations for viscous flow; completing the picture and pointing the way forward for the future adoption of mathematical/solution techniques unavailable in three dimensions.

The remainder of the paper proceeds as follows. In Section 2 a 4D first integral of the energy-momentum equations for viscous flow is derived in full; an asymptotic analysis of which in the Newtonian limit leads to the well-known field equations describing the classical problem of compressible viscous flow. Use of the new formulation is demonstrated via solution of the problem of propagating undamped and damped acoustic waves in Section 3, starting from the 4D first integral and recovering exactly the results of classical theory. Finally, conclusions, together with an accompanying outlook in relation to the techniques mentioned above, are provided in Section 4. A complementary 4D-reformulation of the incompressible NS equations is provided in Appendix A.

2. Relativistic generalisation

The methodology applied successfully to the 2D and 3D incompressible NS equations is now employed to address the case of 4D Lorentz-invariant viscous flow in the form of the energy-momentum equations. For this purpose a flat space-time[1] is assumed and use made of the usual relativistic notation with metric signature $(+,-,-,-)$, that is, $(\eta_{\mu\nu}) = (\eta^{\mu\nu}) = \mathrm{diag}(1,-1,-1,-1)$, the coordinate vector $(x^\mu) = (ct, x_1, x_2, x_3)$, together with the respective 4-gradient in covariant form given by $(\partial_\mu) = (c^{-1}\partial_t, \partial_1, \partial_2, \partial_3)$ and in contra-variant form by $(\partial^\mu) = (\eta^{\mu\nu}\partial_\nu) = (c^{-1}\partial_t, -\partial_1, -\partial_2, -\partial_3)$.

The basic equations of motion take the form of a combined energy-momentum balance:

$$\partial_\alpha T^{\alpha\beta} = 0\,, \tag{14}$$

supplemented by the continuity equation $\partial_\alpha(nu^\alpha) = 0$, with particle density n, and a balance equation for the specific entropy s. $(u^\alpha) = \gamma\,(1, u_1/c, u_2/c, u_3/c)$, with $\gamma^{-1} = \sqrt{1-(u_1^2+u_2^2+u_3^2)/c^2}$, is used to denote the 4-velocity, while the energy-momentum tensor $T^{\alpha\beta}$ takes [19] the general form:

$$T^{\alpha\beta} = \left(nmc^2 + ne + p\right) u^\alpha u^\beta - p\eta^{\alpha\beta} + R^{\alpha\beta}, \tag{15}$$

where $e = e(n, s)$ is the internal energy density, $p = n^2\partial e/\partial n$ is the pressure and $R^{\alpha\beta}$ a tensor taking irreversible effects such as viscosity and heat conduction into account. The latter two depend on the constitutive relationships chosen to underpin the fluid model.

Although peripheral to the present work, various models have been proposed for embedding viscous effects in relativistic flows, most of them being further developments of classical theories – see for example Landau and Lifshits [20] or Mueller and Ruggeri [21]. The requirement being that they overcome the fundamental causality problem associated with the classical theory of viscous flow due to the related field equations in this case being parabolic, as opposed to hyperbolic, and therefore raising the paradox of infinite signal speed as, for example, discussed by Andersson and Comer [22], Freistühler and Temple [19], Fouxon and Oz [23].

A key feature associated with the first integral of equations (14), as derived below, is that it applies to any physical model; the only necessary assumption

[1] A flat space-time implies that the curvature of space is negligibly small, allowing the metric tensor to be written as $(\eta_{\mu\nu}) = (\eta^{\mu\nu}) = \mathrm{diag}(1,-1,-1,-1)$. Otherwise this tensor would be a field.

is one of symmetry $T^{\alpha\beta} = T^{\beta\alpha}$. In the accompanying analysis parentheses around an index pair denotes symmetrisation, e.g. $\partial_{(\mu}u_{\nu)} = (\partial_\mu u_\nu + \partial_\nu u_\mu)/2$, with square brackets indicating skew-symmetrisation.

2.1. First integral: preliminary form

Proceeding as in [16], a corresponding third rank tensor potential, ${M^\beta}_{\nu\lambda}$, satisfying:

$$T^{\alpha\beta} = \varepsilon^{\alpha\mu\nu\lambda}\partial_\mu {M^\beta}_{\nu\lambda} \tag{16}$$

is employed which fulfils equation (14) identically. Since the Levi-Civita symbol $\varepsilon^{\alpha\mu\nu\lambda}$ is fully skew-symmetric [18] the symmetric part, ${M^\beta}_{(\nu\lambda)}$, of ${M^\beta}_{\nu\lambda}$ is eliminated from (16); ${M^\beta}_{\nu\lambda}$ can thus be assumed skew-symmetric with respect to $\nu\lambda \leftrightarrow \lambda\nu$ from outset.

Next, making use of the symmetry of the energy-momentum tensor its transpose must also fulfil equations (14), that is:

$$0 = \partial_\beta T^{\alpha\beta} = \partial_\beta \left[\varepsilon^{\alpha\mu\nu\lambda}\partial_\mu {M^\beta}_{\nu\lambda}\right] = \varepsilon^{\alpha\mu\nu\lambda}\partial_\mu \left[\partial_\beta {M^\beta}_{\nu\lambda}\right] ,$$

implying that $\partial_\beta {M^\beta}_{\nu\lambda}$ must be the skew-symmetric part of a gradient of a 4-vector, φ_λ:

$$\partial_\beta {M^\beta}_{\nu\lambda} = \partial_\nu\varphi_\lambda - \partial_\lambda\varphi_\nu .$$

By writing the above equation in the form:

$$\partial_\beta \left[{M^\beta}_{\nu\lambda} - {\delta^\beta}_\nu\varphi_\lambda + {\delta^\beta}_\lambda\varphi_\nu\right] = 0 , \tag{17}$$

the expression inside the square brackets can be expressed in terms of a fourth rank tensor potential, $a_{\nu\lambda\kappa\rho}$, and replaced by $\varepsilon^{\beta\gamma\kappa\rho}\partial_\gamma a_{\nu\lambda\kappa\rho}$, which when inserted into equation (16) renders:

$$T^{\alpha\beta} = \varepsilon^{\alpha\mu\nu\lambda}\varepsilon^{\beta\gamma\kappa\rho}\partial_\mu\partial_\gamma a_{\nu\lambda\kappa\rho} + \varepsilon^{\alpha\mu\beta\lambda}\partial_\mu\varphi_\lambda - \varepsilon^{\alpha\mu\nu\beta}\partial_\mu\varphi_\nu . \tag{18}$$

Symmetry considerations dictate that the last two terms in expression (18) have to vanish and it follows that $\varphi_\lambda = 0$, leading to the following *preliminary* form of the first integral of the relativistic equations of motion (14):

$$T^{\alpha\beta} = \varepsilon^{\alpha\mu\nu\lambda}\varepsilon^{\beta\gamma\kappa\rho}\partial_\mu\partial_\gamma a_{\nu\lambda\kappa\rho}, \tag{19}$$

expressed in terms of second order derivatives of the fourth rank tensor potential $a_{\nu\lambda\kappa\rho}$ which has to be symmetric with respect to the exchange $\nu\lambda \leftrightarrow \kappa\rho$, that is:

$$a_{\nu\lambda\kappa\rho} = a_{\kappa\rho\nu\lambda}. \tag{20}$$

Any contributions being symmetric with respect to the exchange $\nu \leftrightarrow \lambda$ or $\kappa \leftrightarrow \rho$ are filtered out by the Levi-Civita symbols, thus one can write:

$$a_{(\nu\lambda)\kappa\rho} = a_{\nu\lambda(\kappa\rho)} = 0 . \tag{21}$$

Considering both symmetry expressions, (20, 21), it follows that the tensor $a_{\nu\lambda\kappa\rho}$ contains $6 \cdot (6+1)/2 = 21$ independent entries.

The preliminary form (19) corresponds to equation (12) apropos the first integral of the 3D-NS equations. Similar to the gauge transformation of the second rank tensor potential associated with the 3D theory [16] (see also Eq. (A.3) in Appendix A.1) an analogous gauge symmetry exists for the fourth rank tensor potential $a_{\nu\lambda\kappa\rho}$. While it is obvious that equation (19) remains unchanged by adding a gradient of the form $\partial_\nu \zeta_{\lambda\kappa\rho}$ to $a_{\nu\lambda\kappa\rho}$, where $\zeta_{\lambda\kappa\rho}$ is a third-rank tensor field, the gradient has to be symmetrised according to:

$$a_{\nu\lambda\kappa\rho} \to a_{\nu\lambda\kappa\rho} + \partial_\nu \zeta_{\lambda\kappa\rho} - \partial_\lambda \zeta_{\nu\kappa\rho} + \partial_\kappa \zeta_{\rho\nu\lambda} - \partial_\rho \zeta_{\kappa\nu\lambda} \,, \tag{22}$$

in order to adhere to constraints (20, 21). Additionally, the gauge field tensor $\zeta_{\lambda\kappa\rho}$ has to fulfil the constraint:

$$\zeta_{\lambda(\kappa\rho)} = 0 \,, \tag{23}$$

implying that it contains $4 \cdot 6 = 24$ independent entries.

Without a deeper analysis it is not immediately obvious how many independent entries in the fourth rank tensor potential $a_{\nu\lambda\kappa\rho}$ really contribute to the ten equations in (19) and how many independent entries in the gauge field tensor $\zeta_{\lambda\kappa\rho}$ are really relevant. The corresponding analysis is provided below and involves re-ordering equation (19) in order to eliminate redundant fields leading, as in the case of the 3D-NS equations and without loss of generality, to a more *utilitarian* form of the first integral.

2.2. Re-ordered, final utilitarian form

A re-ordering of the terms on the right hand of equations (19) renders a much more convenient and conclusive form of the first integral. Firstly, the identity [22]:

$$\begin{aligned}
\varepsilon^{\alpha\mu\nu\lambda}\varepsilon^{\beta\gamma\kappa\rho} =& \eta^{\beta\beta'}\eta^{\gamma\gamma'}\eta^{\kappa\kappa'}\eta^{\rho\rho'}\varepsilon^{\alpha\mu\nu\lambda}\varepsilon_{\beta'\gamma'\kappa'\rho'} \\
=& -4!\eta^{\beta\beta'}\eta^{\gamma\gamma'}\eta^{\kappa\kappa'}\eta^{\rho\rho'}\delta^{[\alpha}{}_{\beta'}\delta^{\mu}{}_{\gamma'}\delta^{\nu}{}_{\kappa'}\delta^{\lambda]}{}_{\rho'}
\end{aligned}$$

is used to rewrite equation (19) as:

$$\begin{aligned}
T^{\alpha\beta} =& -4!\eta^{\beta\beta'}\eta^{\gamma\gamma'}\eta^{\kappa\kappa'}\eta^{\rho\rho'}\delta^{[\alpha}{}_{\beta'}\delta^{\mu}{}_{\gamma'}\delta^{\nu}{}_{\kappa'}\delta^{\lambda]}{}_{\rho'}\partial_\mu\partial_\gamma a_{\nu\lambda\kappa\rho} \\
=& -4!\eta^{\beta\beta'}\delta^{[\alpha}{}_{\beta'}\delta^{\mu}{}_{\gamma'}\delta^{\nu}{}_{\kappa'}\delta^{\lambda]}{}_{\rho'}\partial_\mu\partial^{\gamma'} a_{\nu\lambda}{}^{\kappa'\rho'},
\end{aligned}$$

where $a_{\nu\lambda}{}^{\kappa'\rho'} = \eta^{\kappa\kappa'}\eta^{\rho\rho'}a_{\nu\lambda\kappa\rho}$. Writing out the right hand side explicitly:

$$\begin{aligned}
&4!\delta^{[\alpha}{}_{\beta'}\delta^{\mu}{}_{\gamma}\delta^{\nu}{}_{\kappa}\delta^{\lambda]}{}_{\rho} \\
=&\delta^{\alpha}{}_{\beta'}\left[\delta^{\mu}{}_{\gamma}\delta^{\nu}{}_{\kappa}\delta^{\lambda}{}_{\rho}+\delta^{\lambda}{}_{\gamma}\delta^{\mu}{}_{\kappa}\delta^{\nu}{}_{\rho}+\delta^{\nu}{}_{\gamma}\delta^{\lambda}{}_{\kappa}\delta^{\mu}{}_{\rho}-\delta^{\lambda}{}_{\gamma}\delta^{\nu}{}_{\kappa}\delta^{\mu}{}_{\rho}-\delta^{\mu}{}_{\gamma}\delta^{\lambda}{}_{\kappa}\delta^{\nu}{}_{\rho}-\delta^{\nu}{}_{\gamma}\delta^{\mu}{}_{\kappa}\delta^{\lambda}{}_{\rho}\right] \\
&-\delta^{\mu}{}_{\beta'}\left[\delta^{\alpha}{}_{\gamma}\delta^{\nu}{}_{\kappa}\delta^{\lambda}{}_{\rho}+\delta^{\lambda}{}_{\gamma}\delta^{\alpha}{}_{\kappa}\delta^{\nu}{}_{\rho}+\delta^{\nu}{}_{\gamma}\delta^{\lambda}{}_{\kappa}\delta^{\alpha}{}_{\rho}-\delta^{\lambda}{}_{\gamma}\delta^{\nu}{}_{\kappa}\delta^{\alpha}{}_{\rho}-\delta^{\alpha}{}_{\gamma}\delta^{\lambda}{}_{\kappa}\delta^{\nu}{}_{\rho}-\delta^{\nu}{}_{\gamma}\delta^{\alpha}{}_{\kappa}\delta^{\lambda}{}_{\rho}\right] \\
&+\delta^{\nu}{}_{\beta'}\left[\delta^{\alpha}{}_{\gamma}\delta^{\mu}{}_{\kappa}\delta^{\lambda}{}_{\rho}+\delta^{\lambda}{}_{\gamma}\delta^{\alpha}{}_{\kappa}\delta^{\mu}{}_{\rho}+\delta^{\mu}{}_{\gamma}\delta^{\lambda}{}_{\kappa}\delta^{\alpha}{}_{\rho}-\delta^{\lambda}{}_{\gamma}\delta^{\mu}{}_{\kappa}\delta^{\alpha}{}_{\rho}-\delta^{\alpha}{}_{\gamma}\delta^{\lambda}{}_{\kappa}\delta^{\mu}{}_{\rho}-\delta^{\mu}{}_{\gamma}\delta^{\alpha}{}_{\kappa}\delta^{\lambda}{}_{\rho}\right] \\
&-\delta^{\lambda}{}_{\beta'}\left[\delta^{\alpha}{}_{\gamma}\delta^{\mu}{}_{\kappa}\delta^{\nu}{}_{\rho}+\delta^{\nu}{}_{\gamma}\delta^{\alpha}{}_{\kappa}\delta^{\mu}{}_{\rho}+\delta^{\mu}{}_{\gamma}\delta^{\nu}{}_{\kappa}\delta^{\alpha}{}_{\rho}-\delta^{\nu}{}_{\gamma}\delta^{\mu}{}_{\kappa}\delta^{\alpha}{}_{\rho}-\delta^{\alpha}{}_{\gamma}\delta^{\nu}{}_{\kappa}\delta^{\mu}{}_{\rho}-\delta^{\mu}{}_{\gamma}\delta^{\alpha}{}_{\kappa}\delta^{\nu}{}_{\rho}\right],
\end{aligned}$$

leads to:

$$\begin{aligned}
T^{\alpha\beta} =& -4!\eta^{\beta\beta'}\delta^{[\alpha}{}_{\beta'}\delta^{\mu}{}_{\gamma}\delta^{\nu}{}_{\kappa}\delta^{\lambda]}{}_{\rho}\partial_\mu\partial^\gamma a_{\nu\lambda}{}^{\kappa\rho} \\
=& -\eta^{\beta\alpha}\partial_\mu\left[\partial^\mu a_{\nu\lambda}{}^{\nu\lambda} + \partial^\lambda a_{\nu\lambda}{}^{\mu\nu} + \partial^\nu a_{\nu\lambda}{}^{\lambda\mu} - \partial^\lambda a_{\nu\lambda}{}^{\nu\mu} - \partial^\mu a_{\nu\lambda}{}^{\lambda\nu} - \partial^\nu a_{\nu\lambda}{}^{\mu\lambda}\right] \\
&+ \eta^{\beta\mu}\partial_\mu\left[\partial^\alpha a_{\nu\lambda}{}^{\nu\lambda} + \partial^\lambda a_{\nu\lambda}{}^{\alpha\nu} + \partial^\nu a_{\nu\lambda}{}^{\lambda\alpha} - \partial^\lambda a_{\nu\lambda}{}^{\nu\alpha} - \partial^\alpha a_{\nu\lambda}{}^{\lambda\nu} - \partial^\nu a_{\nu\lambda}{}^{\alpha\lambda}\right] \\
&- \eta^{\beta\nu}\partial_\mu\left[\partial^\alpha a_{\nu\lambda}{}^{\mu\lambda} + \partial^\lambda a_{\nu\lambda}{}^{\alpha\mu} + \partial^\mu a_{\nu\lambda}{}^{\lambda\alpha} - \partial^\lambda a_{\nu\lambda}{}^{\mu\alpha} - \partial^\alpha a_{\nu\lambda}{}^{\lambda\mu} - \partial^\mu a_{\nu\lambda}{}^{\alpha\lambda}\right] \\
&+ \eta^{\beta\lambda}\partial_\mu\left[\partial^\alpha a_{\nu\lambda}{}^{\mu\nu} + \partial^\nu a_{\nu\lambda}{}^{\alpha\mu} + \partial^\mu a_{\nu\lambda}{}^{\nu\alpha} - \partial^\nu a_{\nu\lambda}{}^{\mu\alpha} - \partial^\alpha a_{\nu\lambda}{}^{\nu\mu} - \partial^\mu a_{\nu\lambda}{}^{\alpha\nu}\right] \\
=& -\eta^{\beta\alpha}\partial_\mu\left[2\partial^\mu a_{\nu\lambda}{}^{\nu\lambda} - 4\partial^\lambda a_{\nu\lambda}{}^{\nu\mu}\right] + \partial^\beta\left[2\partial^\alpha a_{\nu\lambda}{}^{\nu\lambda} - 4\partial^\lambda a_{\nu\lambda}{}^{\nu\alpha}\right] \\
&+ \eta^{\beta\nu}\partial_\mu\left[4\partial^\mu a_{\lambda\nu}{}^{\lambda\alpha} + 4\partial^\lambda a_{\lambda\nu}{}^{\alpha\mu} - 4\partial^\alpha a_{\lambda\nu}{}^{\lambda\mu}\right] \\
=& \Box\left[4\eta^{\beta\nu}a_{\lambda\nu}{}^{\lambda\alpha} - \eta^{\alpha\beta}a_{\nu\lambda}{}^{\nu\lambda}\right] + \eta^{\alpha\beta}\partial_\mu\left[4\partial^\lambda a_{\nu\lambda}{}^{\nu\mu} - \partial^\mu a_{\nu\lambda}{}^{\nu\lambda}\right] + 4\eta^{\beta\nu}\partial_\mu\partial^\lambda a_{\lambda\nu}{}^{\alpha\mu} \\
&- \partial^\alpha\left[4\eta^{\beta\nu}\partial_\mu a_{\lambda\nu}{}^{\lambda\mu} - \partial^\beta a_{\nu\lambda}{}^{\nu\lambda}\right] - \partial^\beta\left[4\partial^\lambda a_{\nu\lambda}{}^{\nu\alpha} - \partial^\alpha a_{\nu\lambda}{}^{\nu\lambda}\right],
\end{aligned} \tag{24}$$

with the *d'Alembertian* defined as:

$$\Box := \partial_\mu\partial^\mu = \eta^{\mu\nu}\partial_\mu\partial_\nu = \frac{1}{c^2}\partial_t^2 - \nabla^2\,. \tag{25}$$

Next, defining a symmetric second rank tensor field $\tilde{a}^{\nu\mu}$ and a vector field A^μ as follows:

$$\tilde{a}^{\alpha\beta} := 4\eta^{\beta\nu}a_{\lambda\nu}{}^{\lambda\alpha} - \eta^{\alpha\beta}a_{\nu\lambda}{}^{\nu\lambda}, \tag{26}$$

$$A^\mu := \partial_\kappa\tilde{a}^{\kappa\mu} = 4\partial^\lambda a_{\nu\lambda}{}^{\nu\mu} - \partial^\mu a_{\nu\lambda}{}^{\nu\lambda}, \tag{27}$$

enables equation (24) to be written in the following concise re-ordered form:

$$T^{\alpha\beta} = \Box\tilde{a}^{\alpha\beta} - \partial^\alpha A^\beta - \partial^\beta A^\alpha + \eta^{\beta\alpha}\partial_\mu A^\mu + 4\eta^{\beta\lambda}\partial_\mu\partial^\nu a_{\nu\lambda}{}^{\alpha\mu}\,; \tag{28}$$

which corresponds to equation (11) of the related 3D-NS theory, since it too is based on a symmetric second rank tensor and its divergence. However, in contrast, an additional second rank tensor $b^{\alpha\beta} := \eta^{\beta\lambda}\partial_\mu\partial^\nu a_{\nu\lambda}{}^{\alpha\mu}$ appears in the equations. Furthermore, since $a_{\nu\lambda}{}^{(\alpha\mu)} = 0$, it follows that:

$$\partial_\alpha b^{\alpha\beta} = \eta^{\beta\lambda}\partial_\alpha\partial_\mu\partial^\nu a_{\nu\lambda}{}^{\alpha\mu} = \eta^{\beta\lambda}\partial^\nu\left[\partial_\alpha\partial_\mu a_{\nu\lambda}{}^{\alpha\mu}\right] = 0\,,$$

raising the question as to whether the last term in equation (28) is redundant. Indeed, by applying the gauge transformation (22) for the choice $\zeta_{\lambda\kappa\rho} = \partial_{[\rho}Z_{\kappa]\lambda} = \frac{1}{2}\left[\partial_\rho Z_{\kappa\lambda} - \partial_\kappa Z_{\rho\lambda}\right]$, the tensor $b_{\kappa\lambda} = \partial^\rho\partial^\nu a_{\nu\lambda\kappa\rho}$ is transformed according to:

$$b_{\kappa\lambda} \to b_{\kappa\lambda} + \Box^2 Z_{(\lambda\kappa)} - \Box\left[\partial_\kappa\partial^\nu Z_{(\nu\lambda)} + \partial_\lambda\partial^\nu Z_{(\nu\kappa)}\right] + \partial_\lambda\partial_\kappa\partial^\nu\partial^\rho Z_{(\nu\rho)}\,, \tag{29}$$

and since the field $Z_{\kappa\lambda}$ can be chosen arbitrarily, elimination of the tensor $b_{\kappa\lambda}$ is possible, yielding:

$$T^{\alpha\beta} = \Box\tilde{a}^{\alpha\beta} - \partial^\alpha A^\beta - \partial^\beta A^\alpha + \eta^{\beta\alpha}\partial_\mu A^\mu \,, \tag{30}$$

as the final, *utilitarian*, form of the first integral containing only the second rank tensor $\tilde{a}^{\alpha\beta}$ and its divergence $A^\mu = \partial_\nu \tilde{a}^{\nu\mu}$. The original fourth rank tensor $a_{\nu\lambda\kappa\rho}$ need be considered no longer, reducing the relevant number of potential fields from 21 to 10; with the structure of the reduced equations (30) corresponding to equation (11) for the case of 2D and 3D incompressible viscous flow.

In the course of reducing the number of potentials, the degrees of freedom for gauging are reduced correspondingly. Via the choice $\zeta_{\lambda\kappa\rho} = -\frac{1}{8}\left[\eta_{\lambda\kappa}\xi_\rho - \eta_{\lambda\rho}\xi_\kappa\right]$ for the gauge field in (22) with an arbitrary vector field ξ_κ, the modified potential $\tilde{a}^{\alpha\beta}$ and the vector field A^ν are transformed as follows:

$$\tilde{a}^{\alpha\beta} \to \tilde{a}^{\alpha\beta} + \partial^\alpha\xi^\beta + \partial^\beta\xi^\alpha - \frac{1}{2}\eta^{\alpha\beta}\partial_\nu\xi^\nu \,, \tag{31}$$

$$A^\mu \to A^\mu + \Box\xi^\mu + \frac{1}{2}\partial^\mu\partial_\nu\xi^\nu \,, \tag{32}$$

equivalent to the associated gauge transformations commensurate with the corresponding analysis of the NS equations in the 3D Newtonian case [16] – see also Eqs. (A.5, A.6) and (A.12, A.13) in Appendix A.

Overall the first integral (30) is comprised of ten independent equations, six more than the original energy-momentum equations (14). However, via the modified tensor potential $\tilde{a}^{\alpha\beta}$ ten additional potential fields enter the equations but as a consequence of the gauge symmetry group (31, 32), with ξ^μ an arbitrary vector field, effectively only six additional degrees of freedom count. Thus, six additional degrees of freedom are balanced by six additional equations, ensuring that the formulation is well-posed.

A noteworthy difference between the Newtonian 3D formulation, equations (12) and (13) of Section 1, and the relativistic one is the occurrence of an additional vector potential φ_k in the former. This is explainable within the framework of a non-relativistic 4-formulation of the first integral (12, 13) of the incompressible Navier-Stokes equations, as derived in Appendix A.2, identifying the vector field $\varphi_k - \eta\Psi_k$ and the tensor potential $\tilde{a}_{kq}$ as partitions of a 4-matrix corresponding to the tensor $\tilde{a}^{\alpha\beta}$ in relativistic theory.

2.3. Recovery of the original field equations

For completeness, it is easily demonstrated that the energy-momentum equations (14) can be recovered from the reduced, utilitarian form of the first integral, (30) by simply taking the divergence of the latter leading to:

$$\partial_\alpha T^{\alpha\beta} = \Box\underbrace{\partial_\alpha\tilde{a}^{\alpha\beta}}_{A^\beta} - \underbrace{\partial_\alpha\partial^\alpha}_{\Box} A^\beta - \partial^\beta\partial_\alpha A^\alpha + \underbrace{\eta^{\beta\alpha}\partial_\alpha}_{\partial^\beta}\partial_\mu A^\mu = 0,$$

and reproducing equations (14) exactly.

In addition the recovery of the 3D Newtonian theory from (30) is similarly desirable. This is achieved below via an asymptotic analysis of the relativistic formulation, expanded in powers of $1/c$ assuming the velocities are small, $u^\alpha \ll c$; the alternative of applying the limit $c \to \infty$ directly in going from a relativistic to a Newtonian regime is problematic as reported, for example, by Prix [24], Carter and Khalatnikov [25].

2.4. Asymptotic expansion of the relativistic first integral

For convenience, the inviscid case is analysed first: for $R^{\alpha\beta} = 0$ the following asymptotic expressions for the entries in the energy-momentum tensor (15) arise:

$$T^{00} = \left(nmc^2 + ne + p\right)\gamma^2 - p \approx \varrho c^2 + ne + \mathcal{O}\left(c^{-2}\right), \tag{33}$$

$$T^{0i} = T^{i0} = \left(nmc^2 + ne + p\right)\gamma^2 \frac{u_i}{c} \approx \varrho c u_i + \mathcal{O}\left(c^{-1}\right), \tag{34}$$

$$T^{ij} = \left(nmc^2 + ne + p\right)\gamma^2 \frac{u_i u_j}{c^2} - p\eta^{ij} \approx \varrho u_i u_j + p\delta_{ij} + \mathcal{O}\left(c^{-2}\right), \tag{35}$$

where the classical mass density is recovered via $\varrho = nm$. The above point to analogous expansions for the tensor potential according to:

$$\tilde{a}^{00} \approx c^2 \tilde{a}_2^{00} + \tilde{a}_0^{00} + \mathcal{O}\left(c^{-2}\right), \tag{36}$$

$$\tilde{a}^{0i} = \tilde{a}^{i0} \approx c\tilde{a}_1^{0i} + \tilde{a}_0^{0i} + \mathcal{O}\left(c^{-1}\right), \tag{37}$$

$$\tilde{a}^{ij} \approx \tilde{a}_0^{ij} + \mathcal{O}\left(c^{-2}\right), \tag{38}$$

implying, for the auxiliary vector field (27), that:

$$A^0 \approx c\overbrace{\left[\partial_t \tilde{a}_2^{00} + \partial_i \tilde{a}_1^{0i}\right]}^{=:A_1^0} + \overbrace{\partial_i \tilde{a}_0^{0i}}^{=:A_0^0} + \mathcal{O}\left(c^{-1}\right), \tag{39}$$

$$A^i \approx \underbrace{\partial_t \tilde{a}_1^{0i} + \partial_j \tilde{a}_0^{ji}}_{=:A_0^i} + \mathcal{O}\left(c^{-1}\right). \tag{40}$$

As a consequence equations (30) read asymptotically as:

$$\varrho c^2 + ne = -c^2 \Delta \tilde{a}_2^{00} - \Delta \tilde{a}_0^{00} - \partial_t A_1^0 + \partial_k A_0^k + \mathcal{O}\left(c^{-1}\right), \tag{41}$$

$$\varrho c u_i = -c\Delta \tilde{a}_1^{0i} + c\partial_i A_1^0 - \Delta \tilde{a}_0^{0i} + \partial_i A_0^0 + \mathcal{O}\left(c^{-1}\right), \tag{42}$$

$$\varrho u_i u_j + p\delta_{ij} = -\Delta \tilde{a}_0^{ij} \partial_i A_0^j + \partial_j A_0^i + \left[\partial_t A_1^0 - \partial_k A_0^k\right]\delta_{ij} + \mathcal{O}\left(c^{-1}\right). \tag{43}$$

Sorting terms for the above expressions in powers of c, produces a hierarchical set of equations. With respect to c^2 equation (41) leads to:

$$\varrho = -\Delta \tilde{a}_2^{00}, \tag{44}$$

while for c^1 the identity:

$$\varrho u_i = -\Delta \tilde{a}_1^{0i} + \partial_i A_1^0 = \partial_t \partial_i \tilde{a}_2^{00} + \partial_i \partial_j \tilde{a}_1^{0j} - \Delta \tilde{a}_1^{0i}, \tag{45}$$

results from equation (42). Finally, collecting all terms to the power c^0 in equations (41-43) yields, respectively:

$$ne = -\Delta\tilde{a}_0^{00} - \partial_t A_1^0 + \partial_k A_0^k\,, \tag{46}$$

$$0 = -\Delta\tilde{a}_0^{0i} + \partial_i A_0^0\,, \tag{47}$$

$$\varrho u_i u_j + p\delta_{ij} = -\Delta\tilde{a}_0^{ij} + \partial_i A_0^j + \partial_j A_0^i + \left[\partial_t A_1^0 - \partial_k A_0^k\right]\delta_{ij}\,. \tag{48}$$

Equations (44, 45) provide new potential representations for the mass density and the mass flux density, respectively, while taking the derivative of (44) with respect to time and combining it with the divergence of (45), leads to the classical continuity equation:

$$\partial_t\varrho + \partial_i\left(\varrho u_i\right) = -\partial_t\Delta\tilde{a}_2^{00} + \partial_t\Delta\tilde{a}_2^{00} + \Delta\partial_j\tilde{a}_1^{0j} - \Delta\partial_i\tilde{a}_1^{0i} = 0, \tag{49}$$

for *compressible flow*. Thus, within the terminology of the present work, the two equations (44, 45) represent a first integral of the continuity equation. Furthermore the divergence of (48) yields:

$$\begin{aligned}\partial_i\left[\varrho u_i u_j + p\delta_{ij}\right] =& \Delta\overbrace{\left[A_0^j - \partial_i\tilde{a}_0^{ij}\right]}^{\partial_t\tilde{a}_1^{0i}} + \partial_j\partial_i A_0^i + \partial_j\left[\partial_t A_1^0 - \partial_k A_0^k\right]\\ =& -\partial_t\left[-\Delta\tilde{a}_1^{0i} + \partial_j A_1^0\right],\end{aligned}$$

which together with (45) proves to be the classical momentum balance:

$$\partial_t\left(\varrho u_i\right) + \partial_i\left[\varrho u_i u_j + p\delta_{ij}\right] = 0, \tag{50}$$

for inviscid flow in the absence of external forces.

In summary, the full set of equations (44-48) comprise a first integral of the classical equations of motion for an inviscid compressible flow.

The above asymptotic analysis is readily extendable to the case of viscous flow, provided the friction tensor $R^{\alpha\beta}$ in (15) fulfils certain asymptotic relationships. For example, Fouxon and Oz [23] proposed the following form for the dissipation tensor:

$$R^{\alpha\beta} = 2c\eta\left[\partial^{(\alpha}u^{\beta)} + u^\nu\partial_\nu u^{(\alpha}u^{\beta)} - \frac{1}{3}\partial_\nu u^\nu\left(u^\alpha u^\beta - \eta^{\alpha\beta}\right)\right], \tag{51}$$

where η is the shear viscosity; no bulk viscosity is included due to the underlying conformal field theory. A subsequent model reported by Freistühler and Temple [26] includes both bulk viscosity and heat conduction. A rigorous asymptotic analysis of equation (51) yields the following relationships:

$$\begin{aligned}R^{00} &= \mathcal{O}\left(c^{-2}\right)\,,\\ R^{0i} &= \mathcal{O}\left(c^{-2}\right)\,,\\ R^{ij} &= -\eta\left[\partial_i u_j + \partial_j u_i - \frac{2}{3}\partial_k u_k\delta_{ij}\right] + \mathcal{O}\left(c^{-2}\right)\,,\end{aligned} \tag{52}$$

and therefore a viscous supplement to the field equations (48), taking now the form:

$$\begin{aligned} &\varrho u_i u_j + p\delta_{ij} - \eta\left[\partial_i u_j + \partial_j u_i - \frac{2}{3}\partial_k u_k \delta_{ij}\right] \\ &= -\Delta\tilde{a}_0^{ij} + \partial_i A_0^j + \partial_j A_0^i + \left[\partial_t A_1^0 - \partial_k A_0^k\right]\delta_{ij}\,, \end{aligned} \tag{53}$$

while all other equations (44-47) remain unchanged. Together with (53) they yield a first integral of the equations of motion for compressible viscous flow, opening up the possibility of applications of the first integral approach to classical compressible flow problems with friction.

Note that the Newtonian 3D formulation, equations (12) and (13) of Section 1, cannot be recovered directly by asymptotic analysis. The reason for this is a conceptual conflict between incompressibility and relativity for two reasons: (i) in the limit of infinite compression modulus, $K \to \infty$, the speed of sound tends to infinity in conflict with the speed limit c for any kind of signal; (ii) the incompressibility condition $0 = \partial_i u_i = \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3$ for a classic flow is not invariant with respect to a Lorentz boost, i.e. a flow being incompressible in a given coordinate frame will probably be compressible in a different coordinate frame moving with a constant velocity with respect to the first one. Consequently, the constraint of incompressibility can only be applied *after* the above asymptotic analysis by setting $\varrho =$ const and introducing a streamfunction vector according to $u_i = \varepsilon_{ijk}\partial_j\Psi_k$; thus making the two equations (44, 45) redundant and recovering, after some additional steps, the structure of the first integral for incompressible viscous flow.

3. Acoustic waves

The problem of propagating acoustic waves, the solution of which, both with and without damping, has been obtained to-date starting with the linearised NS equations of fluid motion, is considered below starting instead from the 4D first integral of the energy momentum equations derived above.

3.1. General considerations

Physically, acoustic waves are small fluctuations from an equilibrium state characterised as a fluid medium at rest, $u_1 = u_2 = u_3 = 0$, with constant particle density $n = n_0$ and specific entropy $s = s_0$. Accordingly, the corresponding equilibrium specific inner energy and pressure are defined as $e = e_0 := e(n_0, s_0)$ and $p = p_0 := p(n_0, s_0)$, while the energy-momentum tensor takes the simple form $\left(T^{\alpha\beta}\right) = \mathrm{diag}\left(T_0^{00}, p_0, p_0, p_0\right)$ with $T_0^{00} := n_0 mc^2 + n_0 e_0$. The tensor potential satisfying the field equations (30) for the specified reference state is taken to be $\tilde{a}_0^{\alpha\beta}$.

For the fluctuations, a unidirectional flow geometry $u_2 = u_3 = 0$ and $u_1 = u(x,t)$ with $t = x^0/c$ and $x = x^1$ is assumed corresponding to planar longitudinal waves; while for particle density, specific entropy and tensor potentials, the substitutions:

$$n = n_0(1+\varepsilon) \tag{54}$$
$$s = s_0(1+\sigma) \tag{55}$$
$$\tilde{a}^{\alpha\beta} = \tilde{a}_0^{\alpha\beta} + a^{\alpha\beta} \tag{56}$$

are utilised, where ε, σ and $a^{\alpha\beta}$ are small perturbations from the equilibrium state. A corresponding formula for the pressure is obtained via a Taylor expansion:

$$p = p_0 + \left[n_0 \frac{\partial p}{\partial n}\varepsilon + s_0 \frac{\partial p}{\partial s}\sigma \right]_{n=n_0, s=s_0} = p_0 + \underbrace{K\varepsilon - s_0 n_0 \beta^{-1}\sigma}_{p_*} \tag{57}$$

with compression modulus and inverse thermal expansion coefficient:

$$K := n_0 \left. \frac{\partial p}{\partial n} \right|_{n=n_0, s=s_0} , \tag{58}$$
$$\beta^{-1} := -\frac{1}{n_0} \left. \frac{\partial p}{\partial s} \right|_{n=n_0, s=s_0} = -n_0 \left. \frac{\partial^2 e}{\partial n \partial s} \right|_{n=n_0, s=s_0} = -n_0 \left. \frac{\partial T}{\partial n} \right|_{n=n_0, s=s_0} , \tag{59}$$

respectively, and the temperature given by $T = \partial e / \partial s$. In accordance with the assumed unidirectional flow geometry, the following simplified form of the tensor potential is assumed:

$$\left(a^{\alpha\beta}\right) = \begin{pmatrix} c^2 f_0(x,t) & c f_1(x,t) & 0 & 0 \\ c f_1(x,t) & f_2(x,t) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{60}$$

containing the three unknown functions $f_0(x,t)$, $f_1(x,t)$ and $f_2(x,t)$.

3.2. Acoustic waves without damping

Neglecting friction, $R^{\alpha\beta} = 0$, the components of the energy-momentum tensor (15), following linearisation, read:

$$\begin{aligned} T^{00} &= nmc^2 + ne \\ &\approx n_0 mc^2 + n_0 e_0 + n_0 \left[mc^2 \varepsilon + (1+\varepsilon) e(n_0 + n_0\varepsilon, s_0 + s_0\sigma) - e_0 \right], \\ &= T_0^{00} + \underbrace{\left[n_0 mc^2 + n_0 e_0 + p_0 \right] \varepsilon + n_0 T_0 s_0 \sigma}_{T_*^{00}}, \end{aligned} \tag{61}$$

$$T^{01} \approx \left[n_0 mc^2 + n_0 e_0 + p_0 \right] \frac{u}{c}, \tag{62}$$

and $T^{11} = T^{22} = T^{33} = p_0 + p_*$, while the remaining components vanish. Inserting all of the above relationships, beginning from (54), into the first integral (30), results in the following four independent equations:

$$n_0\frac{\partial p}{\partial n}\varepsilon + s_0\frac{\partial p}{\partial s}\sigma = -\partial_t^2 f_0 - \partial_x\left[2\partial_t f_1 + \partial_x f_2\right], \tag{63}$$

$$\left[n_0 mc^2 + n_0 e_0 + p_0\right]\varepsilon + n_0 T_0 s_0 \sigma = \partial_x^2\left[f_2 - c^2 f_0\right], \tag{64}$$

$$\left[n_0 mc^2 + n_0 e_0 + p_0\right] u = -\partial_t\partial_x\left[f_2 - c^2 f_0\right], \tag{65}$$

$$0 = \frac{1}{c^2}\partial_t^2 f_2 + \partial_x\left[2\partial_t f_1 + \partial_x f_2\right], \tag{66}$$

complemented with the continuity equation, taking the linearised form:

$$0 = \partial_\alpha\left(nu^\alpha\right) = \partial_\alpha\left[n_0(1+\varepsilon)u^\alpha\right] \approx n_0\partial_0\varepsilon + \frac{n_0}{c}\partial_x u,$$

implying:

$$\partial_t\varepsilon + \partial_x u = 0\,. \tag{67}$$

By taking the sum of (63) and (66), the following equation:

$$n_0\frac{\partial p}{\partial n}\varepsilon + s_0\frac{\partial p}{\partial s}\sigma = \frac{1}{c^2}\partial_t^2\left[f_2 - c^2 f_0\right], \tag{68}$$

is obtained. Thus, only the combination:

$$f(x,t) := f_2(x,t) - c^2 f_0(x,t), \tag{69}$$

enters equations (64, 65) and (68), f_1 having been eliminated. Next, taking the sum ∂_t(64)+∂_x(65), leads to the identity:

$$\left[n_0 mc^2 + n_0 e_0 + p_0\right]\underbrace{\left[\partial_t\varepsilon + \partial_x u\right]}_{0} + n_0 T_0 s_0 \partial_t\sigma = 0, \tag{70}$$

implying that the entropy fluctuation σ has to vanish and that the entire process is therefore isentropic, i.e. $\sigma = 0$. Accordingly, equations (64, 68) simplify to:

$$\varepsilon = \frac{\partial_x^2 f}{n_0 mc^2 + n_0 e_0 + p_0}\,, \tag{71}$$

$$\varepsilon = \frac{\partial_t^2 f}{Kc^2}\,. \tag{72}$$

Finally, subtracting the above two equations yields:

$$\left\{\frac{n_0 mc^2 + n_0 e_0 + p_0}{Kc^2}\partial_t^2 - \partial_x^2\right\} f = 0, \tag{73}$$

a d'Alembert type equation, implying the existence of dispersion-free undamped acoustic waves with the speed of sound given by:

$$c_{\mathrm{s}} = \sqrt{\frac{Kc^2}{n_0 mc^2 + n_0 e_0 + p_0}}. \tag{74}$$

Since, under moderate conditions, the inner energy e_0 of particles is very small compared to their rest energy mc^2, the quotient e_0 over mc^2 is negligibly small in the above formula. Furthermore, for dense and condensed matter, the quotient p_0/n_0 over mc^2 is negligible also, yielding the non-relativistic result:

$$c_{\rm s} \approx \sqrt{\frac{K}{n_0 m}}, \tag{75}$$

which is well-known from the classical theory of acoustic waves [27, 28]. At the other extreme, for some cosmological flow problems, the specific enthalpy $e_0 + p_0/n_0$ may reach high values compared to mc^2, in which case (74) reflects the associated relativistic correction to the sound speed. Under mid-relativistic and especially ultra-relativistic conditions, where $p \approx ne/3$ can be assumed [20] while mc^2 is negligible compared to the enthalpy, the compression modulus, according to (58), takes the form:

$$K \approx n_0 \frac{\partial}{\partial n}\left(\frac{ne}{3}\right)\bigg|_{n=n_0} = \frac{1}{3}\left[n_0 e_0 + \frac{\partial e}{\partial n}\bigg|_{n=n_0}\right] = \frac{1}{3}\left[n_0 e_0 + p_0\right] ,$$

leading finally to the well-known result:

$$c_{\rm s} \approx \frac{c}{\sqrt{3}} ,$$

as reported, for example, by Landau and Lifshits [20], Fouxon and Oz [23].

The above results are obtained from equations (14) together with the continuity equation and the constitutive model (15) for $R^{\alpha\beta} = 0$ by undertaking the same perturbation analysis entirely at the level of velocity and stress tensor [20]. However, in terms of the first integral formulation the analysis is simplified since the physical fluctuations ε, σ and u appear directly in equations (63–66) and not as spatial or temporal derivatives as in the original field equations (14); allowing for successive elimination of unknowns, as demonstrated above for undamped waves and similarly below for damped waves.

3.3. Acoustic waves with viscous damping

For acoustic waves transmitted through a medium over a long distance, damping due to dissipation becomes relevant. Two competing mechanisms for damping exist based on thermal conductivity and on viscosity. For certain special fluids, such as pure water, the thermal conductivity can be neglected [28], a more general treatment including heat conduction is possible, e.g. by considering the more general form of the energy-momentum tensor and particle flux density provided by Freistühler and Temple [26]. However, since the primary aim here is to demonstrate application of the first integral approach for a simple test problem, the former modelling assumption is adopted and use made of the already linearised form (52) of the friction tensor $R^{\alpha\beta}$ proposed by Fouxon and

Oz [23], considering shear viscosity only. Based on the same unidirectional flow geometry, the linear part of the energy-momentum tensor has the form:

$$\begin{pmatrix} T^{00}_{*} & T^{00}_{0}\frac{u}{c} & 0 & 0 \\ T^{00}_{0}\frac{u}{c} & p_{*} - \frac{4}{3}\eta\partial_x u & 0 & 0 \\ 0 & 0 & p_{*} + \frac{2}{3}\eta\partial_x u & 0 \\ 0 & 0 & 0 & p_{*} + \frac{2}{3}\eta\partial_x u \end{pmatrix},$$

supplementing equations (63, 66) as follows:

$$K\varepsilon + s_0\frac{\partial p}{\partial s}\sigma + \frac{2}{3}\eta\partial_x u = -\partial_t^2 f_0 - \partial_x\left[2\partial_t f_1 + \partial_x f_2\right], \tag{76}$$

$$-2\eta\partial_x u = \frac{1}{c^2}\partial_t^2 f_2 + \partial_x\left[2\partial_t f_1 + \partial_x f_2\right]. \tag{77}$$

Equations (64, 65) and the linearised continuity equation (67) remain unchanged together with the identity (70) resulting from them, implying $\sigma = 0$. Again it is useful to add (77) to (76) in order to eliminated the f_1-dependence, leading to:

$$K\varepsilon - \frac{4}{3}\eta\partial_x u = \frac{1}{c^2}\partial_t^2 f, \tag{78}$$

with f given according to (69).

Finally, equations (64, 65) can be used to express ε and u as derivatives of f, resulting in the following third order PDE:

$$\left\{\frac{1}{c_{\mathrm{s}}^2}\partial_t^2 - \partial_x^2\right\} f - \frac{4}{3}\frac{\eta}{K}\partial_t\partial_x^2 f = 0, \tag{79}$$

with c_{s} given by (74). Compared to the d'Alembert type equation (73) for undamped acoustic waves an additional term arises. Assuming solutions of the form:

$$f(x,t) = \hat{f}\exp\left(\mathrm{i}\left[kx - \omega t\right]\right), \tag{80}$$

the following dispersion relationship is obtained:

$$\left[1 - \mathrm{i}\frac{4\eta\omega}{3K}\right]k^2 = \frac{\omega^2}{c_{\mathrm{s}}^2}. \tag{81}$$

For a given circular frequency $\omega > 0$, two corresponding wave numbers:

$$k_{1,2} = \pm\frac{\omega}{c_{\mathrm{s}}}\sqrt{\frac{1 + \mathrm{i}\frac{4\eta\omega}{3K}}{1 + \frac{16\eta^2\omega^2}{9K^2}}}, \tag{82}$$

result for waves propagating in the forward/backward direction; the nonlinear dependence on ω, indicates the level of dispersion and, according to whether $\mathrm{Im}\,k_1 > 0$ and $\mathrm{Im}\,k_2 < 0$, the damping of the waves in the direction of propagation. For weakly damped waves, $\eta\omega/K \ll 1$, a Taylor expansion of the above dispersion relationship gives:

$$k_{1,2} = \pm\frac{\omega}{c_{\mathrm{s}}}\left(1 + \mathrm{i}\frac{2\eta\omega}{3K}\right), \tag{83}$$

leading to the fundamental solutions:

$$f = \hat{f} \exp\left(\mp\frac{2\eta\omega^2}{3Kc_\mathrm{s}}x\right)\exp\left(\mathrm{i}\left[\pm\frac{\omega}{c_\mathrm{s}}x - \omega t\right]\right), \tag{84}$$

for forward moving (upper sign) and backward moving (lower sign) damped harmonic waves with frequency-dependent damping coefficient $2\eta\omega^2/3Kc_\mathrm{s}$. This corresponds to existing results for the non-relativistic case – see for example [27, 28, 29].

4. Conclusions and outlook

The novel first integral of the 4D Lorentz-invariant energy-momentum equations for viscous flow, derived in Section 2 utilising a tensor potential, is significant for a number of reasons; not least from a mathematical view point. To date, the well-developed theory of Helmholtz, Maxwell, and Hodge for the decomposition of fields into various potential representations has, on the whole, been targeted at vector fields and differential forms: the analysis for more general tensors, as in the present work, has rarely been considered – the classical work of Berger and Ebin [30] being an exception. Accordingly, the utilitarian form (30) represents an appreciable advance in this respect by providing a rigorous platform to future studies.

In the work reported here, this is exemplified by solving the classical problem, involving compressibility, of propagating acoustic waves starting from the above 4D generalisation. The associated analysis reveals that: (i) in the case of inviscid flow a d'Alembert type equation results directly, via a simplified procedure compared to the classical approach, as a linear combination of the linearised field equations with the expression obtained for the speed of sound being in accordance with the same reported in the open literature; (ii) when viscous effects are present, spatial exponential damping of the sound waves arises, once more in full agreement with earlier results.

Furthermore, the relativistic 4D formulation overcomes a basic restriction associated with the 2D/3D Galilei-invariant theory: namely the incompressibility constraint which is indispensable when formulating a first integral (13, 11) and the consequent necessary introduction of a streamfunction vector. A first integral formulation for classical compressible viscous flow emerges following a rigorous asymptotic expansion of equation (30), elegantly resolving the challenging issue of finding the Newtonian limit $c \to \infty$ compared to classical approaches [24, 25]. In this sense, equations (44-47) and (53), constitute a valuable contribution to non-relativistic compressible flow problems in general. Moreover, the first integral approach applies to any established physical frame since the constitutive relation expressing the energy-momentum tensor $T^{\alpha\beta}$ in (14) in terms of the inner energy, the pressure, the four-velocity field and its derivatives is not subject to any restrictions apart from symmetry $T^{[\alpha\beta]} = 0$; enabling one to incorporate, for example, non-Newtonian and Knudsen number effects [23].

As outlined below, several tantalising possibilities now exist for extending and utilising the first integral approach much further still.

It is incontrovertible that the new approach has considerable potential from the standpoint of research routes encompassing a wide range of flow problems; for example; more challenging relativistic astrophysical flows. Historically, starting from the energy-momentum equations (14), an inviscid flow assumption has predominated [31, 32] due to the very low density of matter in the interstellar medium as a whole. However, viscous forces become important where the gathering of mass due to gravity occurs: for example in accretion disks [33, 34]. Similarly, a combination of both viscous and relativistic effects arises in such disks around neutron stars and black holes [35, 36, 37, 38]. Overcoming of the incompressibility restriction as discussed above also provides a basis for investigation, via the same methodology, of non-relativistic compressible flow problems of general engineering interest.

What is more, the 4D first integral of the energy-momentum equations points to the future use of mathematical techniques and methods of solutions not currently applicable to the field equations in their original form; in this context the reader is referred to the historical example of Maxwell's introduction of a scalar and a vector potential in Electrodynamics, the major benefit of which was the self-adjointness and decoupling of the resulting field equations, making use of a particular gauge of the potentials. Much later these potentials proved to be an essential element in the formulation of relativistic quantum theory. Two promising ways forward in relation to the potential-based 4D first integral approach are outlined below.

The first concerns the use of matrix structures within the framework of Clifford algebra, based on quaternions or Dirac matrices with the goal of developing highly efficient methods of solution. Having mapped the entire problem to a matrix-algebra framework, the limit $c \to \infty$ could be applied in order to provide efficient solutions of the classical NS equations. Alternatively, such techniques can similarly be applied to the 4D-reformulation of the Galilei-invariant first integral of the incompressible NS equations as provided, for completeness, in Appendix A. Although implementation of such a matrix-algebra approach remains speculative, it deserves investigation since its utilisation can lead to significant economic gains in the computation of fluid flows, in a similar fashion to the use of quaternions representing spatial rotation operations [39, 40], potentially leading to the formulation of highly efficient and predictive CFD software.

The second relates to the formulation of a variational principle for viscous flow. For classical continuum physics, being invariant with respect to the Galilei group, Scholle [41] has reported general rules for Lagrangians, providing a basis for a variational principle with a discontinuous Lagrangian [8]; the variation of which leads to a set of equations allowing for the recovery of the NS equations by time averaging [29]. It remains an intriguing and open question as to whether an analogous systematic approach applies to the Lorentz invariant 4-formulation of the first integral of the NS equations. Having once formulated such a variational principle, a variety of new avenues leading to efficient solution methods such as novel FE-approaches and semi-analytical implementation of Ritz's direct method become a reality.

The above and other considerations, although currently at a formative stage, will be developed and explored in forthcoming articles.

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Appendix A. 4D formulation of the incompressible NS equations

Against the background of the covariant utilitarian form of the 4D first integral of the energy-momentum equations, (30), it is shown below that a comparable elegant form can similarly be derived as a first integral of the incompressible NS equations by merging the 3D vector equation (13) and the 3D form of the tensor equation (11) to produce a 4D tensor equation. The basic requirement for achieving this is a unified form for both sets of field equations, which in turn necessitates a particular gauging of the potentials and consequently, in the first instance, a corresponding careful analysis of the gauge group and the prospects for manipulating the field equations in a beneficial way.

Appendix A.1. Gauge freedom analysis

Since the gauge transformation of a given set of potentials replaces them by an equivalent set leading to identical observables, the same approach can be used to simplify the corresponding field equations for the potentials with respect to their mathematical structure as well as to the number of potentials involved. In earlier work [16] the gauge freedoms of Ψ_n, a_{pq} and φ_n were analysed for a symmetric tensor potential, $a_{pq} = a_{qp}$, only; below a detailed analysis is provided for the general case.

First, the streamfunction vector can be gauged by the gradient of an arbitrary scalar field χ as in (1), that is:

$$\Psi_k \longrightarrow \Psi_k + \partial_k \chi \,; \tag{A.1}$$

leading, according to the definition of u_i, to the same velocity field. In the same manner the vector potential can be gauged:

$$\varphi_k \longrightarrow \varphi_k + \partial_k \zeta \,, \tag{A.2}$$

with an arbitrary scalar field ζ, having no effect on the vector equations (13) and the tensor equations (11), since the auxiliary vector field A_j as defined remains invariant. As to the tensor potential, by performing the operation:

$$a_{kq} \longrightarrow a_{kq} + \partial_q \xi^1_k + \partial_k \xi^2_q + \varepsilon_{kqn} \partial_n \xi^0 \,, \tag{A.3}$$

for two arbitrary vector fields $\xi_q^{1,2}$ and a scalar field ξ^0, equation (12) is unaffected: which is obviously true also for its equivalent form (11). In contrast, a combined transformation consisting of (A.1) and (A.3) enables equation (13) to be written as follows:

$$\varrho\partial_t\left[\Psi_n + \partial_n\chi\right] = \partial_n\partial_k\left[\frac{1}{2}\varepsilon_{kqp}a_{pq} - \partial_l\partial_l\xi^0 + \varphi_k - \eta\Psi_k\right] - \partial_k\partial_k\left[\varphi_n - \eta\Psi_n\right] ,$$

the invariance of (13) being guaranteed provided ξ^0 and χ fulfil the following PDE:

$$\varrho\partial_t\chi + \partial_l\partial_l\xi^0 = 0 . \tag{A.4}$$

By applying the general gauge transformation (A.1, A.2, A.3) to the tensor $\tilde{a}_{kq}$ and the vector A_j, the following transformation rules emerge:

$$\tilde{a}_{kq} \longrightarrow \tilde{a}_{kq} + \partial_q\xi_k^+ + \partial_k\xi_q^+ - \partial_l\xi_l^+\delta_{kq} , \tag{A.5}$$

$$A_j \longrightarrow A_j + \partial_k\partial_k\xi_j^+ , \tag{A.6}$$

with

$$\xi_q^\pm := \frac{1}{2}\left[\xi_q^1 \pm \xi_q^2\right] . \tag{A.7}$$

The parameter ξ_q^- does not occur in the above formulae and therefore has no influence on equation (11); it only affects the tensor potential via:

$$a_{kq} \longrightarrow a_{kq} + \partial_q\xi_k^- - \partial_k\xi_q^- , \tag{A.8}$$

or, more precisely, the skew-symmetric part of a_{kq}. It is therefore more convenient to consider $\varepsilon_{kqp}a_{pq}$ rather than a_{pq} itself; including the effect due to the parameter ξ^0, the entire gauge group for $\varepsilon_{kqp}a_{pq}$ reads:

$$\varepsilon_{kqp}a_{pq} \longrightarrow \varepsilon_{kqp}a_{pq} + 2\varepsilon_{kqp}\partial_q\xi_p^- - 2\partial_k\xi^0 , \tag{A.9}$$

showing that the vector field $\varepsilon_{kqp}a_{pq}$ is supplemented by the curl of an arbitrary vector field ξ_p^- and the gradient of an arbitrary scalar field ξ^0. Accordingly, the skew-symmetric part of the tensor potential can be manipulated arbitrarily via gauging and set to any purposeful value. For example it can be set to zero as assumed from the outset by Scholle et al. [16], reducing the number of fields with the remaining rules being utilised subsequently to establish bona fide gauging scenarios leading to partially decoupled or self-adjoint equations. Using the more general rules provided above, a 4D-formulation of the field equations can be established as another gauging scenario as shown below.

Appendix A.2. Gauge reformulation of the equations

It is useful for the purpose of the following analysis to define:

$$\tilde{a}_{n0} := \varphi_n - \eta\Psi_n , \tag{A.10}$$

$$A_0 := \partial_k\left[\frac{1}{2}\varepsilon_{kqp}a_{pq} + \varphi_k - \eta\Psi_k\right] = \partial_k\tilde{a}_{k0} + \frac{1}{2}\varepsilon_{pkq}\partial_k a_{pq} , \tag{A.11}$$

as auxiliary fields which, with respect to the general gauge transformation (A.1, A.2, A.3), fulfil the following transformation rules:

$$\tilde{a}_{k0} \longrightarrow \tilde{a}_{k0} + \partial_k \left[\zeta - \eta\chi\right] \,, \tag{A.12}$$

$$A_0 \longrightarrow A_0 + \partial_k \partial_k \left[\zeta - \eta\chi - \xi^0\right] \,. \tag{A.13}$$

The two equation sets, (11) and (13), can be written in a unified form, leading to an elegant 4-formulation, by making use of definitions (A.10, A.11); enabling equation (13) to be expressed as:

$$\varrho \partial_t \Psi_n = -\partial_k \partial_k \tilde{a}_{n0} + \partial_n A_0 \,, \tag{A.14}$$

having a compatible structure to that of the tensor equation (11) and suggesting the following combinations: the 3-vector A_j with the scalar A_0 to form a 4-vector, A^μ, and the 3-tensor $\tilde{a}_{ij}$ with the 3-vector $\tilde{a}_{n0} = \varphi_n - \eta\Psi_n$ to yield a 4-tensor, $\tilde{a}^{\mu\nu}$, namely:

$$(A^\mu) = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix} , \quad (\tilde{a}^{\mu\nu}) = \begin{pmatrix} \tilde{a}^{00} & \varphi_1 - \eta\Psi_1 & \varphi_2 - \eta\Psi_2 & \varphi_3 - \eta\Psi_3 \\ \varphi_1 - \eta\Psi_1 & \tilde{a}_{11} & \tilde{a}_{12} & \tilde{a}_{13} \\ \varphi_2 - \eta\Psi_2 & \tilde{a}_{12} & \tilde{a}_{22} & \tilde{a}_{23} \\ \varphi_3 - \eta\Psi_3 & \tilde{a}_{13} & \tilde{a}_{23} & \tilde{a}_{33} \end{pmatrix} , \tag{A.15}$$

where the entry $\tilde{a}^{00}$ remains to be identified.

In the following the Δ notation is employed in preference to $\partial_k \partial_k$ for $\partial_1^2 + \partial_2^2 + \partial_3^2$ in order to avoid confusion and delineate between 3D and 4D forms. Greek indices run from 0 to 3, while latin ones run from 1 to 3. Accordingly, equations (11, A.14) take the following form:

$$0 = -\,\Delta \tilde{a}^{n0} - \partial^n A^0 - \partial^0 \left(\varrho c \Psi_n\right) \,, \tag{A.16}$$

$$\varrho u_i u_j + (p+U)\delta_{ij} = -\,\Delta \tilde{a}^{ij} - \partial^i A^j - \partial^j A^i + \partial_k A^k \eta^{ij} \,, \tag{A.17}$$

revealing that a consistent formulation in terms of 4-matrices requires the identity:

$$\varrho c \Psi_n = A^n \,, \tag{A.18}$$

to hold. Since, according to (A.6) and (A.13), a free manipulation of the vector A^n is possible by a proper choice of ξ_j^+ and ξ, the gauge (A.6) can be realised and a matrix equation with the desired structure obtained from (A.16, A.17); it can uniformly be written as:

$$T_\infty^{\mu\nu} = -\Delta \tilde{a}^{\mu\nu} - \partial^\mu A^\nu - \partial^\nu A^\mu + \left(\partial_k A^k\right) \eta^{\mu\nu} \,, \tag{A.19}$$

where the 4-matrix $T_\infty^{\mu\nu}$ is given by:

$$(T_\infty^{\mu\nu}) = \begin{pmatrix} T_\infty^{00} & 0 & 0 & 0 \\ 0 & \varrho u_1^2 + p + U & \varrho u_1 u_2 & \varrho u_1 u_3 \\ 0 & \varrho u_1 u_2 & \varrho u_2^2 + p + U & \varrho u_2 u_3 \\ 0 & \varrho u_1 u_3 & \varrho u_2 u_3 & \varrho u_3^2 + p + U \end{pmatrix} . \tag{A.20}$$

Note that the 00-entry of equation (A.19), namely:

$$T_{\infty}^{00} = -\Delta\tilde{a}^{00} - 2\partial^0 A^0 + \partial_k A^k, \tag{A.21}$$

is an independent equation containing $\tilde{a}^{00}$, viz. equation (A.15), as a new degree of freedom, which does not appear as part of the preliminary first integral form developed above and has no influence on the remaining field equations (A.16, A.17); at this stage it is retained for formalistic reasons. In order to develop solution methods based on the above 4-scheme structure, the yet undetermined component T_{∞}^{00} in (A.20) can be chosen conveniently and as appropriate. Note that the above first integral written as a 4D-formulation, (A.19), by means of gauging condition (A.18), does not change the fundamental character of the underlying Galilei-invariant physics.

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