

NEW NON-ARITHMETIC COMPLEX HYPERBOLIC LATTICES II

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ABSTRACT. We describe a general procedure to produce fundamental domains for complex hyperbolic triangle groups. This allows us to produce new non-arithmetic lattices, bringing the number of known non-arithmetic commensurability classes to 22.

1. INTRODUCTION

The goal of this paper is to give a unified construction of various families of lattices in the isometry group $\mathrm{PU}(2, 1)$ of the complex hyperbolic plane $\mathbf{H}_{\mathbb{C}}^2$. We describe a systematic manner to produce fundamental domains that works for all known triangle group lattices, with minor modifications for some pathological cases.

The groups we consider turn out to produce all previously known examples of non-arithmetic lattices in $\mathrm{PU}(2, 1)$. Until recently, all such groups were contained, up to commensurability, in the list of lattices that appears in work of Deligne-Mostow/Thurston, see [6], [23], [44]. In fact, a lot of these groups were discovered over a century ago by Picard [34], and studied by several people including Terada [42]. These groups give nine commensurability classes of non-arithmetic lattices, but the determination of the precise number of commensurability classes required a significant amount of work (see [37], [7], [18], [21]).

In [14], we announced the construction of 12 lattices, giving at least 9 new non-arithmetic commensurability classes. The most difficult part of the result is the proof that the groups are lattices. Indeed, the fact that they are not commensurable to any Deligne-Mostow lattice can be proved by the somewhat rough commensurability invariant given by the field generated by traces in the adjoint representation. The fact that they are not arithmetic follows from a standard application of the complex reflection version of the Vinberg arithmeticity criterion, see for instance [33].

The proof of discreteness relies on the construction of an explicit fundamental domain for each group. There are general ways to produce such fundamental domains, for instance Dirichlet domains, but these often turn out to give overly complicated combinatorics, see for instance [8].

The domains used in [14] are quite simple and natural. Their vertices are all given by (well-chosen) intersections of mirrors of reflections in the group, their 1-faces are all geodesic arcs, and 2-faces are as natural as possible in the context of the non-constant curvature geometry of the complex hyperbolic plane, as they lie on complex lines or Giraud disks.

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The combinatorial construction of the domains was inspired by the fundamental domains constructed by Rich Schwartz in [39], and work related to James Thompson's thesis [43]. The general procedure turns out to be quite elementary, and a lot of it can be described by hand, even though a lot of the computations are much easier to perform with a computer.

The computational heart of our argument is the proof that the geometric realization gives an embedding of our combinatorial fundamental domain into $\overline{\mathbf{H}}_{\mathbb{C}}^2$. This relies on interval arithmetic in conjunction with the rational univariate representation (RUR) for 0-dimensional polynomial systems (see [36]). Software that performs these verifications is publicly available [11]; it is based on the implementation of the RUR that was developed by B. Parisse in `giac`, see [25].

We will review and clarify the construction, and show that it applies to a wide class of complex hyperbolic triangle groups. As a result, we get new fundamental domains for many groups that appeared previously in the literature, including many of the Deligne-Mostow lattices.

Our methods also allow us to treat the 6 sporadic triangle groups left over from [14]. We denote by $\mathcal{S}(p, \tau)$ the sporadic triangle group generated by a complex reflection R_1 with rotation angle $2\pi/p$ and an order 3 isometry J with $\text{Tr}(R_1 J) = \tau$; see Table 3.1 for the meaning of the notation σ_j . The family of groups $\mathcal{S}(p, \bar{\sigma}_4)$ was studied in [14].

Theorem 1.1. *The groups $\mathcal{S}(p, \sigma_1)$ are non-arithmetic lattices for $p = 3, 4, 6$. The groups $\mathcal{S}(p, \sigma_5)$ are non-arithmetic lattices for $p = 3, 4$. They are not commensurable to any Deligne-Mostow lattice, nor to any lattice of the form $\mathcal{S}(p, \bar{\sigma}_4)$.*

Theorem 1.2. *The group $\mathcal{S}(2, \sigma_5)$ is an arithmetic lattice, and so are the groups $\mathcal{S}(p, \sigma_{10})$ for $p = 3, 4, 5, 10$.*

We also consider a slightly different family of lattices $\mathcal{T}(p, \mathbf{T})$, that comes out of James Thompson's thesis (see Table 3.3). We prove that some of them are non-arithmetic and also that they are new, in the sense that they are not commensurable to any Deligne-Mostow lattice, nor to any sporadic triangle group.

Theorem 1.3. *The groups $\mathcal{T}(p, \mathbf{S}_2)$ for $p = 4, 5$ and $\mathcal{T}(3, \mathbf{H}_2)$ are non-arithmetic lattices. They are not commensurable to each other, to any Deligne-Mostow lattice, nor to any sporadic triangle group.*

This statement follows from the analysis of their adjoint trace fields (see section 6.1) and their non-arithmeticity index, see section 6.2. A more detailed analysis, requiring more subtle arguments, shows the following (see section 7, Table A.2 in particular).

Theorem 1.4. *The currently known non-arithmetic lattices in $\text{PU}(2, 1)$ come in 22 commensurability classes.*

It was recently observed [10] that some of these non-arithmetic lattices actually appear in a list of lattices constructed by Couwenberg, Heckman and Looijenga [4], that gave a common generalization of work of Barthel-Hirzebruch-Höfer [1] and Deligne-Mostow [6]. We refer to these lattices as CHL lattices. Note that, apart from Deligne-Mostow lattices,

the CHL lattices contain three families of 2-dimensional lattices, corresponding to line arrangements in $\mathbf{P}_{\mathbb{C}}^2$ of type H_3 , G_{24} and G_{26} . Using the same techniques as the ones in [10], one verifies that these three families correspond to our families $\mathcal{S}(p, \sigma_{10})$, $\mathcal{S}(p, \bar{\sigma}_4)$ and $\mathcal{T}(p, \mathbf{S}_2)$, respectively (see also [9]).

Using the analysis in section 7 (Table A.2), we see that our lattices $\mathcal{S}(p, \sigma_1)$ ($p = 3, 4, 6$), $\mathcal{S}(p, \sigma_5)$ ($p = 3, 4$) and $\mathcal{T}(3, \mathbf{H}_2)$ are non-arithmetic lattices that are not commensurable to any CHL lattice.

We assume the reader is familiar with basic notions of hyperbolic geometry over some base field, and with Coxeter groups. To get a quick idea of the main differences between real and complex hyperbolic geometry, the reader can consult [2]. We will freely use the classification of isometries into elliptic, parabolic and loxodromic elements, sometimes with slight refinements, e.g. a regular elliptic isometry is an elliptic isometry whose matrix representatives have distinct eigenvalues. We refer to [15] for background on complex hyperbolic geometry and bisectors, see also section 2 of [14] for a quick review.

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2. GROUPS GENERATED BY TWO COMPLEX REFLECTIONS

2.1. Subgroups of $\mathrm{PU}(1, 1)$ generated by two elliptic elements. Let b and c be two elliptic elements in $\mathrm{PU}(1, 1)$, which we assume to be primitive of the same order, i.e. they rotate in $\mathbf{H}_{\mathbb{C}}^1$ by an angle $2\pi/p$, $p \in \mathbb{N}$, $p \geq 2$. It is a well known fact that the discreteness of the group generated by b and c is controlled by the product bc , in the following sense.

Proposition 2.1. *If $\langle b, c \rangle$ is a lattice, then bc is non-loxodromic. If bc is elliptic, then the group is a lattice if and only if bc rotates by an angle $4\pi/n$ for some $n \in \mathbb{N}^*$, or by $8\pi/p$.*

The first part follows from a straightforward application of the Poincaré polyhedron theorem. The second one is more subtle, it is a special case of Knapp's theorem, see [19].

Proposition 2.1 will serve as a model for higher dimensional analogues (we will look for simple words in the generators whose behavior determines whether or not the group is a lattice), and it is also important because it explains the behavior of subgroups generated by two complex reflections in $\mathbf{H}_{\mathbb{C}}^2$ (by looking at the projective line of lines through the intersection of the mirrors, possibly in projective space).

A natural analogue of the elliptic elements of $\mathrm{PU}(1, 1)$ for higher dimensions is given by complex reflections in $\mathrm{PU}(n, 1)$, whose representative matrices have an eigenvalue of multiplicity n . Geometrically, such an isometry fixes pointwise a complex projective hyperplane called its mirror, and rotates about it by a certain angle. In the next section, we discuss groups generated by two such complex reflections.

2.2. Subgroups of $SU(2, 1)$ generated by two complex reflections. Let $A, B \in SU(2, 1)$ be complex reflections with angle $2\pi/p$, with distinct mirrors. We assume they each have eigenvalues $u^2, \bar{u}, \bar{u}$, where $u = e^{2\pi i/3p}$. Let $\mathbf{a}$ and $\mathbf{b}$ be polar vectors to the mirrors of A and B respectively; that is $\mathbf{a}$ and $\mathbf{b}$ are u^2 -eigenvectors. Note that $\bar{u}^2$ is an eigenvalue of AB , corresponding to the intersection of the multiple eigenspaces of A and B . Indeed using formulae in [35], see also [29], we can write down the trace of AB .

Lemma 2.2. *Let A and B be as above. Then*

$$\mathrm{tr}(AB) = \left(2 - \frac{|u^3 - 1|^2 |\langle \mathbf{a}, \mathbf{b} \rangle|^2}{\langle \mathbf{a}, \mathbf{a} \rangle \langle \mathbf{b}, \mathbf{b} \rangle} \right) u + \bar{u}^2.$$

We are interested in the case where AB is elliptic of finite order. The following proposition follows easily from Lemma 2.2.

Proposition 2.3. *Let A and B be as above. Then the following are equivalent.*

(1)

$$\mathrm{tr}(AB) = (2 - 4 \cos^2(\theta))u + \bar{u}^2 = -2 \cos(2\theta)u + \bar{u}^2,$$

(2) *AB has eigenvalues $-ue^{2i\theta}$, $-ue^{-2i\theta}$, $\bar{u}^2$,*

(3)

$$\frac{|u^3 - 1|^2 |\langle \mathbf{a}, \mathbf{b} \rangle|^2}{\langle \mathbf{a}, \mathbf{a} \rangle \langle \mathbf{b}, \mathbf{b} \rangle} = 4 \cos^2(\theta).$$

In particular, if AB has finite order then θ is a rational multiple of π .

If the $\bar{u}^2$ -eigenspace of AB is spanned by a negative vector, it corresponds to the intersection of the mirrors of A and B and is a fixed point of AB in $\mathbf{H}_{\mathbb{C}}^2$. If the $\bar{u}^2$ -eigenspace of AB is spanned by a positive vector then it is polar to a complex line preserved by AB which is orthogonal to the mirrors of A and B .

2.3. Braid length. Throughout the paper, we will use the following terminology for **braid relations** between group elements (see Section 2.2 of Mostow [22]). If G is a group and $a, b \in G$, we say that a and b satisfy a braid relation of length $n \in \mathbb{N}^*$ if

$$(1) \quad (ab)^{n/2} = (ba)^{n/2},$$

where powers that are half integers should be interpreted as saying that the corresponding alternating product of a and b should have n factors. For instance, $(ab)^{3/2} = aba$, $(ba)^2 = baba$, $(ab)^{5/2} = ababa$, etc. For short, we will sometimes write the sentence “ a and b satisfy a braid relation of length n ” simply as “ $\mathrm{br}_n(a, b)$ ”.

If a and b satisfy some braid relation, the smallest n such that (1) holds will be called the **braid length** of the pair a, b , which we will denote by $\mathrm{br}(a, b)$.

Remark 2.4. • A braid relation of length 2 simply means a and b commute.

- The classical braid relation $aba = bab$ is a braid relation of length 3.
- If a and b both have order 2, $\mathrm{br}(a, b) = n$ if and only if their product has order n .

- If $\text{br}_n(a, b)$ holds for some integer n , then clearly the relation $\text{br}_{kn}(a, b)$ also holds for every integer $k > 1$. In particular, the relation $\text{br}_n(a, b)$ does not imply $\text{br}(a, b) = n$, but it does imply $\text{br}(a, b)$ divides n .

It will be useful later in the paper to consider in some detail the case where $\text{tr}(AB) = -2u \cos(2\pi/q) + \bar{u}^2$ for some $q \in \mathbb{N}^*$. That is, we take $\theta = \pi/q$ in Proposition 2.3.

Proposition 2.5. *Let $u = e^{2\pi i/3p}$ for some integer $p > 1$. If $\text{tr}(AB) = -2u \cos(2\pi/q) + \bar{u}^2$ for some integer $q > 1$, then $\text{br}(A, B) = q$. Moreover, the group generated by A and B in $\text{PU}(2, 1)$ is a central extension of the rotation subgroup of a triangle group.*

- (1) *If q is odd, then its center is generated by $(AB)^q$, which is a complex reflection with angle $\frac{(q-2)p-2q}{p}\pi$. The corresponding quotient is a $(2, p, q)$ -triangle group.*
- (2) *If q is even, the center is generated by $(AB)^{q/2}$, which is a complex reflection with angle $\frac{(q-2)p-2q}{2p}\pi$. The quotient is a $(\frac{q}{2}, p, p)$ triangle group.*

In particular, if $r = \frac{2pq}{(q-2)p-2q}$ is an integer then the order of AB is the least common multiple of q and r when q is odd and it is the least common multiple of $q/2$ and r when q is even.

We remark that in most cases which we consider, the least common multiple in the last part of this result is r . However, when $p = 12$ and $q = 6$ we have $r = 4$ and the order of AB is 12. This will arise for the group $\mathcal{T}(12, \mathbf{E}_2)$ below.

When mentioning (k, l, m) -triangle groups, we always assume $k, l, m \geq 2$ are integers. By the rotation subgroup of a (k, l, m) -triangle group, we mean the index two subgroup of orientation preserving isometries in the group generated by real reflections in the sides of a triangle with angles $\pi/k, \pi/l, \pi/m$ (note that such a triangle lives in $\mathbf{H}_{\mathbb{R}}^2, \mathbb{R}^2$ or S^2 depending on k, l, m). In other words, it is generated by rotations around the vertices of the triangle, with respective angles $2\pi/k, 2\pi/l, 2\pi/m$. We note that the triangle group in Proposition 2.5 is spherical, Euclidean or hyperbolic whenever $1/p + 1/q - 1/2$ is positive, zero or negative respectively. This happens if and only if the angle $\frac{(q-2)p-2q}{p}\pi$ or $\frac{(q-2)p-2q}{2p}\pi$ is negative, zero or positive respectively.

The equation $r = \frac{2pq}{(q-2)p-2q}$ is equivalent to $2/p + 2/q + 2/r = 1$. We are interested in solutions with $p, q, r \in \mathbb{Z} \cup \{\infty\}$ (and the usual convention that $1/\infty = 0$). Since this is symmetric when we permute p, q and r , it suffices to give the set $\{p, q, r\}$ and allow permutations. Hence we give solutions with $1/r \leq 1/q \leq 1/p < 1$. There is one infinite family of solutions, namely $(2, q, -q)$, as well as the following finite list:

$$\begin{array}{lllll} (3, 3, -6), & (3, 4, -12), & (3, 5, -30), & (3, 6, \infty), & (3, 7, 42), \\ (3, 8, 24), & (3, 9, 18), & (3, 10, 15), & (3, 12, 12), & (4, 4, \infty), \\ (4, 5, 20), & (4, 6, 12), & (4, 8, 8), & (5, 5, 10), & (6, 6, 6). \end{array}$$

3. SUBGROUPS OF $\text{PU}(2, 1)$ GENERATED BY THREE COMPLEX REFLECTIONS

We now wish to analyze groups generated by three complex reflections R_1, R_2 and R_3 in $\text{PU}(2, 1)$. Throughout, we will consider triangle groups whose generators have the same

rotation angle, given by $2\pi/p$. If the triangle group is equilateral, i.e. there is an elliptic isometry cyclically permuting the mirrors of the generators, we write J for that isometry, and order the reflections so that $R_2 = JR_1J^{-1}$, $R_3 = JR_2J^{-1}$. We then write

$$P = R_1J, \quad Q = R_1R_2R_3.$$

It is straightforward to check that, in the equilateral case, $Q = P^3$.

For reasons that will become clear later, we assume that Q has an isolated fixed point. This assumption may seem somewhat unnatural, but the discussion in the previous section should make it more natural in the search for lattices (rather than simply discrete groups).

The central motivating question of this paper is the following:

When is the group generated by R_1 , R_2 and R_3 a lattice?

It is a folklore belief that the discreteness of the group should be controlled by explicit short words in the generators. In the special case where the R_j are involutions, a precise conjectural statement was given by Rich Schwartz in [38], where the conjectural control words actually depend on the triangle. In his Ph.D. thesis, James Thompson gave a conjectural list of the triangle groups (with involutive generators) that were not only discrete, but actually lattices (his work was partly motivated by the example in [8]).

A guiding principle (which is at this stage far from justified rigorously) is that, if the group is to be a lattice, then

- for all $j = 1, 2, 3$, R_j and R_{j+1} should generate a lattice in $\mathrm{PU}(1, 1)$ (or in $\mathrm{PU}(2)$), in particular R_1R_2 , R_2R_3 and R_3R_1 should all be non-loxodromic;
- $R_1R_2R_3$ should be non-loxodromic;
- $R_1R_2R_3R_2^{-1}$, $R_1R_3^{-1}R_2R_3$ and $R_3R_1R_2R_1^{-1}$ should be non-loxodromic.

Throughout the paper, we will use word notation in the generators R_1 , R_2 , R_3 , and denote these group elements simply by $1, 2, 3$. Hoping that no confusion with complex conjugation occurs, we will also denote their inverses by $\bar{1}, \bar{2}, \bar{3}$. In particular, the above control words read $12, 23, 31, 123, 123\bar{2}, 1\bar{3}23, 312\bar{1}$, etc.

3.1. Equilateral triangle groups. The idea in the above guiding principle was used to give a rough sieve of the lattice candidates in [27], [30], whose results we now briefly recall. The basic point is that equilateral triangle groups can be parametrized by the order p of the generators and the complex parameter

$$\tau = \mathrm{Tr}(R_1J).$$

Writing $\mathbf{n}_j$ for a polar vector to the mirror of R_j and $u = e^{2\pi i/3p}$, an equivalent definition of τ is

$$\tau = (u^2 - \bar{u}) \frac{\langle \mathbf{n}_{j+1}, \mathbf{n}_j \rangle}{\|\mathbf{n}_{j+1}\| \|\mathbf{n}_j\|}.$$

The precise statement about parametrizing groups by the pair p, τ is the following.

Proposition 3.1. *Let $p \in \mathbb{N}$, $p \geq 2$ and $\tau \in \mathbb{C}$. We write $u = e^{2\pi i/3p}$, $\alpha = 2 - u^3 - \bar{u}^3$ and $\beta = (\bar{u}^2 - u)\tau$. Then there exists a complex reflection R_1 with rotation angle $2\pi/p$ and a*

regular elliptic element J in $\mathrm{SU}(2, 1)$ such that $\mathrm{Tr}(R_1 J) = \tau$ if and only if

$$(2) \quad \alpha^3 + 2\Re(\beta^3) - 3\alpha|\beta|^2 < 0.$$

In fact, using a basis for $\mathbb{C}^3$ consisting of vectors polar to the mirrors of the reflections R_j , we can write

$$H = \begin{pmatrix} \alpha & \beta & \bar{\beta} \\ \bar{\beta} & \alpha & \beta \\ \beta & \bar{\beta} & \alpha \end{pmatrix}, \quad R_1 = \begin{pmatrix} u^2 & \tau & -u\bar{\tau} \\ 0 & \bar{u} & 0 \\ 0 & 0 & \bar{u} \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and the expression that appears in equation (2) is simply the determinant of H . We denote by $\mathcal{S}(p, \tau)$ the corresponding group (we will always assume that (2) is satisfied). Note that the generating pair is almost uniquely determined by p and τ , in the following sense.

Proposition 3.2. *Let R_1, R'_1 be complex reflections of angle $2\pi/p$, let J, J' be regular elliptic elements of $\mathrm{SU}(2, 1)$. Denote by $\tau = \mathrm{Tr}(R_1 J)$, $\tau' = \mathrm{Tr}(R'_1 J')$. If the pairs (R_1, J) and (R'_1, J') are conjugate in $\mathrm{PU}(2, 1)$, then there exists a cube root of unity ω such that $\tau' = \omega\tau$, or $p = 2$ and there is a cube root of unity ω such that $\tau' = \omega\bar{\tau}$.*

Beware that the groups $\mathcal{S}(p, \tau)$ and $\mathcal{S}(p', \tau')$ may well be conjugate in $\mathrm{PU}(2, 1)$ even when the corresponding generating pairs (R_1, J) , (R'_1, J') are not.

It is difficult to determine the values of the parameters for which the group $\mathcal{S}(p, \tau)$ is lattice, even though, as mentioned above, it is likely that this implies that the pairwise product of generators should be non-loxodromic (see [38], [43]).

In particular, we search for groups such that the eigenvalues of $R_1 J$ and $R_1 R_2$ are all roots of unity (recall that $R_2 = J R_1 J^{-1}$). Note that

$$(3) \quad \mathrm{Tr}(R_1 J) = \tau$$

$$(4) \quad \mathrm{Tr}(R_1 R_2) = u(2 - |\tau|^2) + \bar{u}^2$$

Using Proposition 2.3 we see that when $R_1 R_2$ is elliptic then $|\tau| = 2\cos(\theta)$, or equivalently $|\tau|^2 - 2 = 2\cos(2\theta)$, for some θ .

Now we search for p, τ such that

$$(5) \quad \tau = e^{i\alpha} + e^{i\beta} + e^{-i(\alpha+\beta)}$$

$$(6) \quad |\tau|^2 - 2 = 2\cos 2\theta,$$

where α, β and θ are all rational multiples of π . This allows us to make the crucial observation that our set of equations is in fact equivalent to one that does not involve p . In other words, we need only find the values of τ such that there exist α, β and θ rational multiples of π satisfying (5) and (6). For each such value of τ , any value of $p \geq 2$ gives a group preserving a Hermitian form, but the signature of this form depends on p and τ . We are interested in the case where this signature is $(2, 1)$.

Eliminating τ from (5) and (6) yields

$$(7) \quad \cos(2\theta) - \cos(\alpha - \beta) - \cos(2\alpha + \beta) - \cos(\alpha + 2\beta) = \frac{1}{2},$$

so the question is now reduced to a problem about finding all possible sets of rational multiples of π that satisfy the rational relation (7); as explained in [27], this problem was stated and solved by Conway and Jones (see Theorem 7 of [3]).

Note that τ determines the angles, so we can list the solutions only by giving the values of τ . Moreover, if τ corresponds to a solution, then clearly so do $\omega\tau$ and $\bar{\omega}\tau$, where $\omega = (-1 + i\sqrt{3})/2$ is a primitive cube root of unity; in terms of our geometric motivation, this corresponds to multiplying the group by a scalar matrix of order 3. Also, if τ is a solution, then so is $\bar{\tau}$, so in the list below we will only list one representative for complex conjugate pairs, and avoid repetitions coming from multiplying a given trace τ by a cube root of unity.

Because of the fact that there are many solutions, Conway and Jones only list them up to obvious symmetry in the angles. As a consequence, the application of [3] in this context requires a lot of bookkeeping, and it is quite difficult to achieve it by hand.

It turns out there are two continuous families of solutions, given by

$$(8) \quad \tau = -e^{i\phi/3}, \text{ and}$$

$$(9) \quad \tau = e^{i\phi/6} \cdot 2 \cos(\phi/2).$$

These are of course only seemingly continuous, since we are only interested in solutions where ϕ is a rational multiple of π .

As mentioned in [27], the first family corresponds to Mostow groups, whereas the second family corresponds to certain subgroups of Mostow groups (note that some values of τ lie in both families). We refer to the corresponding (parametrized) curves in the complex plane as the Mostow curve and the Sauter curve, respectively.

For groups with τ on the Mostow or Sauter curves, the list of lattices can be deduced from work of Deligne-Mostow (see [24], [27] and [30]). In order to refer to these groups, we will use the same notation as Mostow, namely

$$\Gamma(p, t)$$

denotes the group generated by reflections of order p and phase-shift $t \in \mathbb{Q}$. This group can also be described as $\mathcal{S}(p, \tau)$ where $\tau = e^{\pi i(\frac{3}{2} + \frac{1}{3p} - \frac{t}{3})}$.

There are also a finite number of solutions that lie neither on the Mostow curve nor on the Sauter curve, which are given in Table 3.1. Note that the last two values were missing in [27], but this has essentially no bearing on the results in [14], since the corresponding lattices turn out to be arithmetic (see the commensurability invariants given in the appendix).

Groups with $\tau = \text{Tr}(R_1 J)$ in Table 3.1 are called **sporadic triangle groups**. A conjectural list of sporadic triangle groups that are lattices was given in [13], and a significant part of that conjecture was proved in [14]. The goal of the present paper is to extend the methods of [14] to a wider class of groups. For one thing, the general method should make some of the ad hoc constructions in [14] more transparent. In particular, we complete the proof of the conjectures from [13].

$$\begin{aligned}
\sigma_1 &= -1 + i\sqrt{2}; \\
\sigma_2 &= -1 + i(\sqrt{5} + 1)/2; \quad \sigma_3 = -1 + i(\sqrt{5} - 1)/2 \\
\sigma_4 &= (-1 + i\sqrt{7})/2 \\
\sigma_5 &= e^{-\pi i/9}(-\bar{\omega} - (1 - \sqrt{5})/2); \quad \sigma_6 = e^{-\pi i/9}(-\bar{\omega} - (1 + \sqrt{5})/2) \\
\sigma_7 &= -e^{-\pi i/9}(\bar{\omega} + 2 \cos \frac{2\pi}{7}); \quad \sigma_8 = -e^{-\pi i/9}(\bar{\omega} + 2 \cos \frac{4\pi}{7}); \quad \sigma_9 = -e^{-\pi i/9}(\bar{\omega} + 2 \cos \frac{6\pi}{7}) \\
\sigma_{10} &= (1 + \sqrt{5})/2 \quad \sigma_{11} = (1 - \sqrt{5})/2
\end{aligned}$$

TABLE 3.1. The list of isolated values of τ that give $R_1 J$ and $R_1 R_2$ of finite order (or possibly parabolic). The list is given only up to complex conjugation, and up to multiplication by a cube root of unity.

τ	Lattice for $p =$
σ_1	3,4,6
$\bar{\sigma}_4$	3,4,5,6,8,12
σ_5	2,3,4
σ_{10}	3,4,5,10

TABLE 3.2. Values of p, τ such that $\mathcal{S}(p, \tau)$ are lattices.

For the other, we exhibit a larger number of lattices, some of them giving new non-arithmetic commensurability classes of lattices (some are not commensurable to any Deligne-Mostow/Thurston groups, nor to any sporadic triangle group).

- We prove that all 12 groups mentioned in [14] are indeed lattices (the proof given there covered six out of the twelve), as well as the four extra sporadic groups.
- We propose an extension of the construction to some non-equilateral lattices, and handle the groups that come out of the analysis in James Thompson's thesis [43].

For concreteness, in Table 3.2 we list the relevant values of the order p of complex reflections, for sporadic families of groups that do indeed contain lattices.

3.2. Non equilateral triangle groups. In this section, we describe the groups that come from Thompson's thesis, since they do not appear anywhere in the literature (in [43] and [17] mainly involutive generators were considered).

The non equilateral triangle groups that appear in this paper will be parametrized by a triple of complex numbers, denoted by $\mathbf{T} = (\rho, \sigma, \tau)$. These three complex numbers generalize τ in the sense that when the triangle is equilateral they are all equal to the parameter τ given above. As before, we assume the three generators rotate by the same angle $2\pi/p$, and denote $u = e^{2i\pi/3p}$. Then

$$\rho = (u^2 - \bar{u}) \frac{\langle \mathbf{n}_2, \mathbf{n}_1 \rangle}{\|\mathbf{n}_2\| \|\mathbf{n}_1\|}, \quad \sigma = (u^2 - \bar{u}) \frac{\langle \mathbf{n}_3, \mathbf{n}_2 \rangle}{\|\mathbf{n}_3\| \|\mathbf{n}_2\|}, \quad \tau = (u^2 - \bar{u}) \frac{\langle \mathbf{n}_1, \mathbf{n}_3 \rangle}{\|\mathbf{n}_1\| \|\mathbf{n}_3\|}.$$

We denote the corresponding group by $\mathcal{T}(p, \mathbf{T})$. Its generators are given by

$$R_1 = \begin{pmatrix} u^2 & \rho & -u\bar{\tau} \\ 0 & \bar{u} & 0 \\ 0 & 0 & \bar{u} \end{pmatrix}; R_2 = \begin{pmatrix} \bar{u} & 0 & 0 \\ -u\bar{\rho} & u^2 & \sigma \\ 0 & 0 & \bar{u} \end{pmatrix}; R_3 = \begin{pmatrix} \bar{u} & 0 & 0 \\ 0 & \bar{u} & 0 \\ \tau & -u\bar{\sigma} & u^2 \end{pmatrix}$$

which preserve the Hermitian form

$$H = \begin{pmatrix} \alpha & \beta_1 & \bar{\beta}_3 \\ \bar{\beta}_1 & \alpha & \beta_2 \\ \beta_3 & \bar{\beta}_2 & \alpha \end{pmatrix}$$

where $\alpha = 2 - u^3 - \bar{u}^3$, $\beta_1 = (\bar{u}^2 - u)\rho$, $\beta_2 = (\bar{u}^2 - u)\sigma$, $\beta_3 = (\bar{u}^2 - u)\tau$. Note that putting $u = -1$ gives the formulae in Section 2.3 of [43] except that H is multiplied by 2.

The triple (R_1, R_2, R_3) is determined up to conjugacy by $|\rho|$, $|\sigma|$, $|\tau|$ and $\arg(\rho\sigma\tau)$; see [35] and [29]:

Proposition 3.3. *For $j = 1, 2, 3$, let R_j, R'_j be complex reflections of angle $2\pi/p$ in $\text{SU}(2, 1)$. Let (ρ, σ, τ) and (ρ', σ, τ') be defined as above. If the triples (R_1, R_2, R_3) and (R'_1, R'_2, R'_3) are conjugate in $\text{PU}(2, 1)$, then*

$$|\rho'| = |\rho|, \quad |\sigma'| = |\sigma|, \quad |\tau'| = |\tau|, \quad \arg(\rho'\sigma'\tau') = \arg(\rho\sigma\tau)$$

or $p = 2$ and

$$|\rho'| = |\rho|, \quad |\sigma'| = |\sigma|, \quad |\tau'| = |\tau|, \quad \arg(\rho'\sigma'\tau') = -\arg(\rho\sigma\tau).$$

Even though the triangle is not equilateral, we take complex reflections that rotate by the same angle, and an important consequence of this is that the condition corresponding to the requirement that short words (123, 123 $\bar{2}$, etc) be non-loxodromic turns out to be independent of that angle.

In particular, in order to determine the relevant values of (ρ, σ, τ) , one can restrict to considering groups generated by reflections of order 2. In that case, the triangle is determined by its angles together with a Cartan angular invariant (see [38] or [43]), and it has become customary to label this triangle according to the orders of 23, 31, 12 and 1 $\bar{3}$ 23 and Schwartz uses p, q, r, n for these orders respectively. Because of the conflict of this notation with the order p of the complex reflections, we choose to write $(a, b, c; d)$ instead. Specifically, we write $(a, b, c; d)$ for the group generated by complex reflections in a triangle with angles $\pi/a, \pi/b, \pi/c$ such that the element corresponding to 1 $\bar{3}$ 23 has order d (more specifically the triangle with sides the mirrors of $R_1, R_3, R_3^{-1}R_2R_3$ has angles $\pi/a, \pi/b, \pi/d$).

Note that the above discussion makes sense only when $a, b, c \geq 3$, since the $(2, b, c)$ triangle groups are rigid in $\text{PU}(2, 1)$. In fact, some of these rigid groups turn out to produce lattices as well, when replacing involutions by reflections of order larger than 2; we will come back to this below (see Table 3.4 for instance).

The traces of the relevant products of reflections are

$$\begin{aligned}
\mathrm{Tr}(R_1 R_2) &= u(2 - |\rho|^2) + \bar{u}^2, \\
\mathrm{Tr}(R_2 R_3) &= u(2 - |\sigma|^2) + \bar{u}^2, \\
\mathrm{Tr}(R_3 R_1) &= u(2 - |\tau|^2) + \bar{u}^2, \\
\mathrm{Tr}(R_1 R_3^{-1} R_2 R_3) &= u(2 - |\sigma\tau - \bar{\rho}|^2) + \bar{u}^2, \\
\mathrm{Tr}(R_1 R_2 R_3) &= 3 - |\rho|^2 - |\sigma|^2 - |\tau|^2 + \rho\sigma\tau.
\end{aligned}$$

Therefore, the analogues of equation (6) are:

$$\begin{aligned}
|\rho|^2 - 2 &= 2 \cos(2\pi/c), \\
|\sigma|^2 - 2 &= 2 \cos(2\pi/a), \\
|\tau|^2 - 2 &= 2 \cos(2\pi/b), \\
|\sigma\tau - \bar{\rho}|^2 - 2 &= 2 \cos(2\pi/d).
\end{aligned}$$

The analogue of the equation (7) in this context turns out to be much harder to solve (it involves a sum of eight cosines rather than four). Rather than solving that equation, Thompson used a computer search to list $(a, b, c; d)$ triangles (still with $a, b, c \geq 3$) such that the short words mentioned above are all elliptic, assuming that a, b, c and d are no larger than 2000.

The corresponding groups are listed in Table 3.3 in terms of ρ, σ, τ . Note that the presence of the symmetries described in [17] allows us to assume that $a \leq b \leq c \leq d$.

	a	b	c	d	$o(123)$	ρ	σ	τ	Lattice for $p =$
S ₁	3	3	4	4	7	$\frac{1+i\sqrt{7}}{2}$	1	1	3, 4, 5, 6, 8, 12
S ₂	3	3	4	5	5	$1 + \omega \frac{1+\sqrt{5}}{2}$	1	1	3, 4, 5
E ₁	3	3	4	6	8	$i\sqrt{2}$	1	1	3, 4, 6
E ₂	3	4	4	4	6	$\sqrt{2}$	$-\bar{\omega}$	$\sqrt{2}$	3, 4, 6, 12
H ₁	3	3	4	7	42	$\frac{-1+i\sqrt{7}}{2}$	$e^{-4i\pi/7}$	$e^{-4i\pi/7}$	2, -7
H ₂	3	3	5	5	15	$-1 - e^{-2i\pi/5}$	$e^{4i\pi/5}$	$e^{4i\pi/5}$	2, 3, 5, 10, -5

TABLE 3.3. Thompson's list of parameters (up to complex conjugation). In the table ω denotes $(-1 + i\sqrt{3})/2$. Negative values of p can also be replaced by their absolute value $|p|$, provided we take the complex conjugate value of the corresponding parameter $\mathbf{T}$, since $\mathcal{T}(p, \mathbf{T}) = \mathcal{T}(-p, \bar{\mathbf{T}})$.

In Table 3.4, we list the corresponding groups coming from rigid triangle groups (these were not considered in [43], but they produce lattices as well).

Remark 3.4. (1) If $\mathbf{T} = (\rho, \sigma, \tau)$ is a valid parameter, then so is $\bar{\mathbf{T}} = (\bar{\rho}, \bar{\sigma}, \bar{\tau})$, and the braid lengths corresponding to control words are the same for both groups (but for the same value of $p > 2$, the groups are usually not conjugate in $\mathrm{PU}(2, 1)$). In the table, we list only one representative for each complex conjugate pair.

	a	b	c	d	$o(123)$	ρ	σ	τ	Lattice for $p =$
$\mathbf{S}_3$	2	3	3	3	4	1	0	1	5, 6, 7, 8, 9, 10, 12, 18
$\mathbf{S}_4$	2	3	4	4	3	$\sqrt{2}$	0	1	4, 5, 6, 8, 12
$\mathbf{S}_5$	2	3	5	5	5	$\frac{1+\sqrt{5}}{2}$	0	1	3, 4, 5, 10
$\mathbf{E}_3$	2	3	6	6	∞	$\sqrt{3}$	0	1	3, 4, 6

TABLE 3.4. Non-equilateral triangle groups, coming from rigid triangle groups.

- (2) It follows from the above discussion that among triangle groups with non-loxodromic control words (of orders less than 2000), the ones in Table 3.3 are characterized up to complex conjugation by
- the order of $R_1 R_2 R_3$ and
 - the braid lengths of pairs of reflections corresponding to control words (provided the braid relation is not too large, i.e. at most 2000), namely the braid length of the pairs (R_j, R_k) , $(R_1, R_2 R_3 R_2^{-1})$, $(R_1, R_3^{-1} R_2 R_3)$, $(R_3, R_1 R_2 R_1^{-1})$.
- (3) Even though the triangles associated to these groups are not equilateral, for some values of $\mathbf{T} = (\rho, \sigma, \tau)$ the groups do possess extra symmetries. For example, when $a = b$ and $c = d$, that is the pairs (R_1, R_2) , $(R_2 R_3)$ and the pairs (R_3, R_1) , $(R_1, R_3^{-1} R_3)$ braid to the same length, as in the case of $\mathbf{H}_2$, we may adjoin a square root of $Q = R_1 R_2 R_3$ conjugating R_1 to $R_1 R_2 R_1^{-1}$, R_2 to R_3 and R_3 to $R_3^{-1} R_1 R_3$. A more interesting symmetry arises for $\mathbf{E}_2$. Consider the map

$$S = \bar{\omega}^{1/3} \begin{pmatrix} \omega & 0 & 0 \\ 0 & 0 & u\bar{\omega} \\ 0 & -\bar{u}\omega & -1 \end{pmatrix}.$$

The map S has the following effect on the generators:

$$S R_1 S^{-1} = R_1, \quad S R_2 S^{-1} = R_3, \quad S R_3 S^{-1} = R_3^{-1} R_2 R_3.$$

Therefore $\langle R_1, R_2, R_3 \rangle$ is a normal subgroup of $\langle R_1, R_2, S \rangle$. Moreover, S is a complex reflection of order 3 whose mirror is orthogonal to the mirror of Q^3 . In particular, S fixes p_0 , the fixed point of Q .

Later in the paper, we will not consider the (non-equilateral) groups for $\mathbf{S}_1$ and $\mathbf{E}_1$, since these are actually conjugate to sporadic (hence equilateral) triangle groups, see section 7.1.1.

We will not give much detail about the rigid Thompson groups. Indeed, we will check that $\mathcal{T}(p, \mathbf{S}_3)$ are Livné lattices, $\mathcal{T}(p, \mathbf{S}_4)$ are all isomorphic to some specific Mostow lattices, $\mathcal{T}(p, \mathbf{S}_5)$ are isomorphic to the corresponding sporadic groups $\mathcal{S}(p, \sigma_{10})$ (see section 7.1.2).

We will not consider the groups of the form $\mathcal{T}(p, \mathbf{E}_3)$ either, because of the following result.

Proposition 3.5. *The lattices $\mathcal{T}(p, \mathbf{E}_3)$, $p = 3, 4, 6$ are arithmetic.*

Proof: One verifies that their adjoint trace fields are $\mathbb{Q}$ (see section 6.1), from which arithmeticity follows (see sections 6.1 and 6.2). $\square$

4. DESCRIPTION OF THE ALGORITHM

4.1. Combinatorial construction. The general goal of this section is to describe the basic building blocks of our fundamental domains, which should be bounded by spherical shells that surround the fixed point of $P = R_1J$ (or $Q = R_1R_2R_3$ in the non-equilateral case).

By a spherical shell, we mean that the corresponding cell complex should be an embedded (piecewise smooth) copy of S^3 , so that it bounds a well-defined 4-ball. Surrounding a point means that we want that point to be in the ball component of the complement of that copy of S^3 .

We will first discuss the construction on the combinatorial level, and defer geometric realization to later in the paper (section 4.2). Both at the combinatorial and the geometrical level, we will refer to 0-faces as **vertices**, 1-faces as **edges**, 2-faces as **ridges**, and 3-faces as **sides**. In section 4.1.1, we explain how sides of our combinatorial domain are obtained from ordered triangles of complex lines. We will then explain how to find a suitable list of triangles, so that the corresponding pyramids form a spherical shell.

Remark 4.1. Calling sides the 3-dimensional faces of our polytopes induces a slight conflict of terminology, since it is customary to talk about the sides of a triangle, which are one (real or complex)-dimensional in nature. From this point on, we will use the word “side” exclusively for 3-dimensional facets, hence we will replace the word “side” by the word “edge” when referring to the 1-dimensional subsets attached to a triangle.

4.1.1. Pyramid associated to an ordered triangle. The basic building blocks for our fundamental domain will be pyramids in bisectors. We start with a simple procedure to build a pyramid with a given triangle as one of its faces, relying on as little geometric information as possible.

Let us start with an ordered triangle in complex hyperbolic space, which we will think of as encoded by its complex edges. We write the complex edges as **a**, **b**, and **c**, and we denote by a , b and c the corresponding complex reflections (all of the same rotation angle $2\pi/p$). In a slight abuse of notation, we will often use the same notation **d** for a complex line, its extension to projective space or its polar vector.

We will call **a** the **base** of the triangle, and we call the intersection point between **b** and **c** the **apex** of the triangle. Note that the apex may or may not lie in complex hyperbolic space, but this will be unimportant until we try to realize the pyramids geometrically.

The action of b and c on the projective line of complex lines through the apex is depicted in Figure 4.1. Both b and c act as rotations by angle $2\pi/p$, and their product bc acts as a rotation as well; see Proposition 2.5. We assume that the latter rotation has finite order, or in other words that b and c satisfy a braid relation of some finite length $n \in \mathbb{N}$ (we will of course assume $n > 1$).

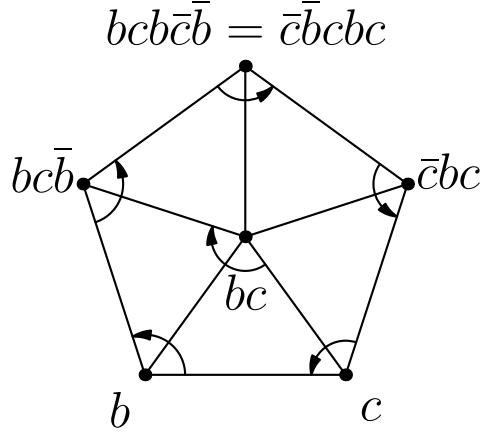


FIGURE 4.1. Triangle group picture, seen in the projective line through the intersection of $\mathbf{b}$ and $\mathbf{c}$.

Note that when going around the picture in Figure 4.1 counter-clockwise, the product of any two successive rotations is equal to the product bc , which gives a mnemonic device for some of the formulas below.

Inspired by the picture in Figure 4.1, if b and c braid to order n , the pyramid associated to the above triangle will have an n -gon as its base, given by the intersection of the base of the triangle with the mirrors of

$$\dots, bcb c^{-1} b^{-1}, bcb^{-1}, b, c, c^{-1}bc, c^{-1}b^{-1}cbc, \dots$$

Hoping that no confusion will arise, we will often use bars to denote inverses, so the above sequence also reads

$$\dots, bcb\bar{c}\bar{b}, b\bar{c}\bar{b}, b, c, \bar{c}bc, \bar{c}\bar{b}cbc, \dots$$

The fact that b and c braid to order n says that the above sequence has period n . The case $n = 5$ is illustrated in Figure 4.2. In particular, the relation $bcb\bar{c}\bar{b} = \bar{c}\bar{b}cbc$ is a consequence of the generalized braid relation $bcbcb = cbcb$.

Note that the sequence has the property that any successive terms in the sequence multiply to the same product bc . This implies that the pyramid would be the same (up to rotational symmetry) if we had started with, say, $a, b\bar{c}\bar{b}, b$ instead of $a; b, c$ (more generally with $a; b_k, b_{k+1}$ with b_k, b_{k+1} consecutive lateral edges of the pyramid). Note in particular that we consider two pyramids the same precisely when their base labels are the same isometry, and the ordered labels of lateral edges are the same up to cyclic permutation.

Remark 4.2. In principle, we allow a slightly degenerate kind of pyramid in the construction, namely when b and c commute, the pyramid has only two lateral edges, or equivalently two base vertices. These “flat” pyramids will actually get discarded from the shell when checking that ridges are on precisely two pyramids, see Section 4.1.5.

One can of course shift a given triangle $\mathbf{a}, \mathbf{b}, \mathbf{c}$ to two other ordered triangles with the same orientation, namely $\mathbf{b}, \mathbf{c}, \mathbf{a}$ and $\mathbf{c}, \mathbf{a}, \mathbf{b}$, but these will, in general, produce pyramids

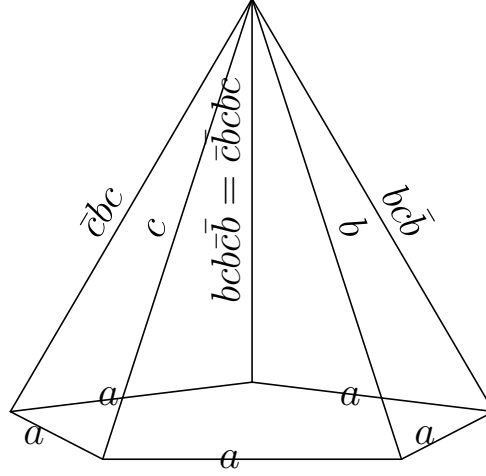


FIGURE 4.2. Pyramid with pentagonal base, corresponding to the braid relation $(bc)^{5/2} = (cb)^{5/2}$. The edges labelled a are referred to as *base edges*, the other ones as *lateral edges*.

that differ combinatorially, since the pairs (a, b) , (b, c) , (c, a) need not braid with the same order. This will be exploited in Section 4.1.4.

4.1.2. Side pairing maps. We now think of the pyramid $a; b, c$ associated to the triangle $\mathbf{a}, \mathbf{b}, \mathbf{c}$, see Section 4.1.1, as encoding a side (i.e. a 3-face) of a fundamental domain for our group. In particular, the sides should come in pairs, so there should be another side isometric to it.

We would like to use the reflection a , or its inverse, as a side-pairing map, and construct a side that has the same base as $a; b, c$. Recall that, by construction, the base of that pyramid is the mirror of the reflection a , so it is fixed by a . There are two natural candidates to create an opposite face, namely those associated to $\mathbf{a}, \mathbf{ab}\bar{\mathbf{a}}, \mathbf{ac}\bar{\mathbf{a}}$ and $\mathbf{a}, \bar{\mathbf{a}}\mathbf{b}\mathbf{a}, \bar{\mathbf{a}}\mathbf{c}\mathbf{a}$.

In order to decide which of the two triangles we choose, we will use the fact that we want to build a spherical shell around the fixed point of $R_1 R_2 R_3$.

Side-pairing selection process (123-rule). We only include the pyramid corresponding to a triangle a, b, c provided either abc or cba is equal to 123. If $abc = 123$, then the corresponding side-pairing map will be a , and if $bca = 123$, the side-pairing map will be $\bar{a}$.

The equality $abc = 123$ is to be understood in the triangle group (not in the free group in three letters). In order to check such a relation, it is enough to reduce the corresponding words according to the braid relations between a, b and c .

The most basic example is the initial pyramid $1; 2, 3$, which is paired by 1 to $1; 12\bar{1}, 13\bar{1}$ (but we do not use $1; \bar{1}21, \bar{1}31$).

4.1.3. Forcing invariance. We want our spherical shell to be P -invariant, so whenever a pyramid from a triangle a, b, c is included, we want to include all its conjugates by either

powers of $P = R_1 J$ when the triangle group is symmetric or powers of $Q = R_1 R_2 R_3$ in the non-symmetric case.

This is easily done using word notation, note that

$$P1\bar{P} = 1J1\bar{J}\bar{1} = 12\bar{1}; \quad P2\bar{P} = 1J2\bar{J}\bar{1} = 13\bar{1}; \quad P3\bar{P} = 1J3\bar{J}\bar{1} = 1.$$

and similarly

$$\bar{P}1P = 3; \quad \bar{P}2P = \bar{3}13; \quad \bar{P}3P = \bar{3}23.$$

4.1.4. Forcing ridge cycles. The discussion in this section is related to the fact that we want our set of pyramids to form a spherical shell. In particular, for each pyramid, its ridges (i.e. 2-faces) should lie on precisely two different pyramids (i.e. 3-faces) in the shell.

At least on the combinatorial level, ridges from two different pyramids are considered the same provided they have the same (cyclically ordered) sets of labels. If we ensured that the existence of side-pairing maps by the selection process explained in section 4.1.2, then the base ridges are on at least two pyramids; in the sequel, we will assume they are on precisely those two.

In fact we want *all* ridges to be on precisely two pyramids of our invariant shell. Applying this to lateral ridges gives a strong restriction to produce the shell. If $a; b, c$ appears in the shell, then it is natural to consider the shifted pyramid $b; c, a$ and $c; a, b$, but only one of them will satisfy the 123-rule (see page 15). Indeed, if $a; b, c$ has been included, then either abc or bca is equal to 123. In the first case, we need to select $c; a, b$, in the second we select $b; c, a$. Of course, for these to yield well-defined pyramids, we need a and b (or c and a , respectively) to braid to some finite order.

For example, we could shift the initial pyramid $1; 2, 3$ to either $2; 3, 1$ or $3; 1, 2$. The first shift gets discarded, since the corresponding products are 231 and 312, neither of which is 123. The second one is kept, since $(1 \cdot 2) \cdot 3 = 123$ and its side-pairing map is $\bar{3}$, which maps $3; 1, 2$ to $3; \bar{3}13, \bar{3}23$.

Another example is the pyramid $2; 1, 23\bar{2}$. We discard $23\bar{2}; 2, 1$, but we keep $1; 23\bar{2}, 2$, whose side-pairing map is 1. More examples appear in section 5.

Provided we use the 123-rule and the corresponding pyramids all have finite braiding order, all lateral ridges in the shell will lie on precisely two pyramids in the shell.

4.1.5. Building an invariant spherical shell. The previous sections suggest a procedure for building an invariant spherical shell. We denote by p_0 the isolated fixed point of $P = R_1 J$ (or of $Q = R_1 R_2 R_3$ in the non-symmetric case).

We say a pyramid $a; b, c$ surrounds p_0 provided $abc = 123$ or $bca = 123$. Note that if $a; b, c$ surrounds p_0 , then so do all of its P -images.

Now start with a set $\mathcal{P}$ of pyramids that all surround p_0 , and force its faces to be paired (see Section 4.1.2), and invariant (see Section 4.1.3).

Consider the ridges of pyramids of $\mathcal{P}$ that lie only on one pyramid; then shift the corresponding triangle according to the rule in Section 4.1.4; if the corresponding apex isometries braid to finite order, enlarge $\mathcal{P}$ to contain the corresponding shifted pyramid.

Assumption 1. The above process never fails (i.e. apex isometries always braid to some finite order), and at some finite stage we get a paired P -invariant shell $\mathcal{P}$, such that every ridge is on precisely two pyramids of the shell.

This may seem like a lot to ask, but this hypothesis holds in many cases, as discussed in Section 5 below. In particular, see Theorem 5.4.

In order to obtain the condition that every ridge is on precisely two pyramids, we need to discard all flat pyramids, i.e. those of the form $a; b, c$ where b and c commute; see Remark 4.2. Such a pyramid collapses to a single triangle, which is also a lateral ridge of pyramids of the form $b; c, a$ and $c; b, a$. Since b and c commute these pyramids both satisfy the 123-rule.

4.2. Geometric realization.

4.2.1. Realizing vertices. The first point is that we want to realize vertices of our pyramids in complex hyperbolic space. Throughout this section, $a; b_1, b_2$ denotes a given pyramid in the invariant shell, one of whose ridges is the triangle with sides $\mathbf{a}$, $\mathbf{b}_1$ and $\mathbf{b}_2$. We will denote by $\mathbf{b}_1, \dots, \mathbf{b}_n$ the ordered set of lateral edges of the pyramid.

Note that the lateral ridges of the pyramids are complex triangles, and two complex lines in $\mathbf{H}_{\mathbb{C}}^2$ may or may not intersect in $\mathbf{H}_{\mathbb{C}}^2$. The basic idea is that the corresponding projective lines always intersect in $\mathbf{P}_{\mathbb{C}}^2$, and that point is unique provided the corresponding complex lines are distinct. This brings forward a genericity assumption:

Assumption 2. For every side $a; b, c$ of a pyramid in $\mathcal{P}$, the mirrors of a , b and c are in general position, by which we mean they are pairwise distinct, and their intersection points are distinct.

If that is the case, the vertices of the pyramids have a natural realization in $\mathbf{P}_{\mathbb{C}}^2$. Of course this is not completely satisfactory in terms of complex hyperbolic geometry, we now explain how to realize our shell in $\mathbf{H}_{\mathbb{C}}^2$.

Each lateral edge will contribute two or three vertices, depending on where various projective lines intersect (inside or outside $\overline{\mathbf{H}_{\mathbb{C}}^2}$).

Recall that complex lines in $\mathbf{H}_{\mathbb{C}}^2$ can be described by a polar vector v in $\mathbb{C}^3$, in which case the complex line corresponds to the set of negative lines in $v^{\perp}$. Moreover, two lines with distinct polar vectors v and w respectively meet in a unique point in $\mathbf{P}_{\mathbb{C}}^2$ denoted by $u = v \boxtimes w$, which is inside $\mathbf{H}_{\mathbb{C}}^2$ if and only if $\langle u, u \rangle < 0$. If they intersect outside $\overline{\mathbf{H}_{\mathbb{C}}^2}$ (i.e. if $\langle u, u \rangle > 0$), then they have a unique common perpendicular complex line, which is simply the complex line polar to u (see [15] for details).

Top vertices

- If $\mathbf{b}_1$ and $\mathbf{b}_2$ intersect inside $\overline{\mathbf{H}_{\mathbb{C}}^2}$ the pyramid will have a single top vertex, given by their intersection point.
- If not, then the intersection point is polar to a complex line $\mathbf{d}$, which with abuse of notation we write as $\mathbf{d} = \mathbf{b}_1 \boxtimes \mathbf{b}_2$ (in fact $\mathbf{b}_k \boxtimes \mathbf{b}_l$ is actually independent of k and l , of course with $k \neq l$). In the latter case, there will be n top vertices, given by

the intersection of $\mathbf{d}$ with the mirrors corresponding to the n lateral edges of the pyramid.

Base and mid vertices

For each k , the k -th lateral edge of the pyramid will contribute two or three vertices, depending on the position in $\mathbf{P}_{\mathbb{C}}^2$ (inside or outside $\overline{\mathbf{H}}_{\mathbb{C}}^2$) of $\mathbf{d}_k = \mathbf{a} \boxtimes \mathbf{b}_k$.

- If $\mathbf{d}_k$ is in $\overline{\mathbf{H}}_{\mathbb{C}}^2$, then there are only two vertices on $\mathbf{b}_k$, namely the top vertex described previously, and $\mathbf{d}_k$ which we call a **bottom vertex**.
- If $\mathbf{d}_k$ is outside $\overline{\mathbf{H}}_{\mathbb{C}}^2$, then it is polar to the common perpendicular complex line to $\mathbf{a}$ and $\mathbf{b}_k$. In that case, this lateral edge will actually contribute three vertices, namely the top vertex, and the two feet of the common perpendicular complex line, which are $\mathbf{d}_k \boxtimes \mathbf{b}_k$ and $\mathbf{d}_k \boxtimes \mathbf{a}$. The point $\mathbf{d}_k \boxtimes \mathbf{b}_k$ will be called a **mid vertex** of the pyramid, and $\mathbf{d}_k \boxtimes \mathbf{a}$ a **bottom vertex**.

4.2.2. *Realizing edges.* The 1-skeleton of the realization of the pyramid is obtained by joining suitable pairs of vertices by geodesic arcs.

- **Top edges** The realization of a pyramid has top edges if and only if the geometric realization of the apex of the pyramid lies outside $\overline{\mathbf{H}}_{\mathbb{C}}^2$. One simply includes a geodesic arc between $\mathbf{d} \cap \mathbf{b}_k$ and $\mathbf{d} \cap \mathbf{b}_{k+1}$, for k modulo n .
- **Top to bottom vertices** If $\mathbf{a} \boxtimes \mathbf{b}_k$ is in $\overline{\mathbf{H}}_{\mathbb{C}}^2$, then we join it either to $\mathbf{b}_k \boxtimes \mathbf{b}_{k+1}$ (if this point is in $\overline{\mathbf{H}}_{\mathbb{C}}^2$), or to $\mathbf{d} \boxtimes \mathbf{b}_k$ where $\mathbf{d}$ is polar to $\mathbf{b}_k \boxtimes \mathbf{b}_{k+1}$.
- **Top to mid vertices** If $\mathbf{a} \boxtimes \mathbf{b}_k$ is *not* in $\overline{\mathbf{H}}_{\mathbb{C}}^2$, let $\mathbf{d}_k$ denote its polar complex line. Then we join the mid vertex $\mathbf{d}_k \boxtimes \mathbf{b}_k$ to the top vertex $\mathbf{b}_k \boxtimes \mathbf{b}_{k+1}$ (if this point is in $\overline{\mathbf{H}}_{\mathbb{C}}^2$), or to the top vertex $\mathbf{d} \boxtimes \mathbf{b}_k$ where $\mathbf{d}$ is polar to $\mathbf{b}_k \boxtimes \mathbf{b}_{k+1}$.
- **Mid to bottom vertices** If $\mathbf{a} \boxtimes \mathbf{b}_k$ is not in $\overline{\mathbf{H}}_{\mathbb{C}}^2$, let $\mathbf{d}_k$ denote its polar complex line. Then we join the mid vertex $\mathbf{d}_k \boxtimes \mathbf{b}_k$ to the bottom vertex $\mathbf{d}_k \boxtimes \mathbf{a}$.
- **Bottom edges** One includes a geodesic arc between $\mathbf{a} \boxtimes \mathbf{b}_k$ (or $\mathbf{a} \boxtimes \mathbf{d}_k$ if the previous point is outside $\overline{\mathbf{H}}_{\mathbb{C}}^2$) and $\mathbf{a} \boxtimes \mathbf{b}_{k+1}$ (or $\mathbf{a} \boxtimes \mathbf{d}_{k+1}$), for k modulo n .

4.2.3. *Realizing ridges.* We make the following

Assumption 3. The (ordered) polygon obtained by taking the bottom edges joining the bottom vertices $\mathbf{a} \boxtimes \mathbf{b}_k$ or $\mathbf{a} \boxtimes \mathbf{d}_k$ is an embedded (piecewise smooth) topological circle in the (closure in $\overline{\mathbf{H}}_{\mathbb{C}}^2$ of the) complex line $\mathbf{a}$, equivalently this polygon bounds a disk in that (closed) complex line.

This allows us to define the bottom ridge.

If $\mathbf{d} = \mathbf{b}_1 \boxtimes \mathbf{b}_2$ is outside complex hyperbolic space, then there is a similar n -gon in $\mathbf{d}$, which will be a ridge as well, which we refer to as the *top ridge*. Just as for the bottom ridge, we assume embeddedness of the top polygon in order to be able to define a top ridge.

The lateral ridges are slightly more difficult to describe, since their combinatorial type depends on the position of intersections of edges in $\mathbf{P}_{\mathbb{C}}^2$. We list the eight possibilities for

the combinatorics of a ridge that contain lateral edges $\mathbf{b}_k$ and $\mathbf{b}_{k+1}$ in Figure 4.3 (we take indices mod n , so when $k = n$, we have $k + 1 = 1$).

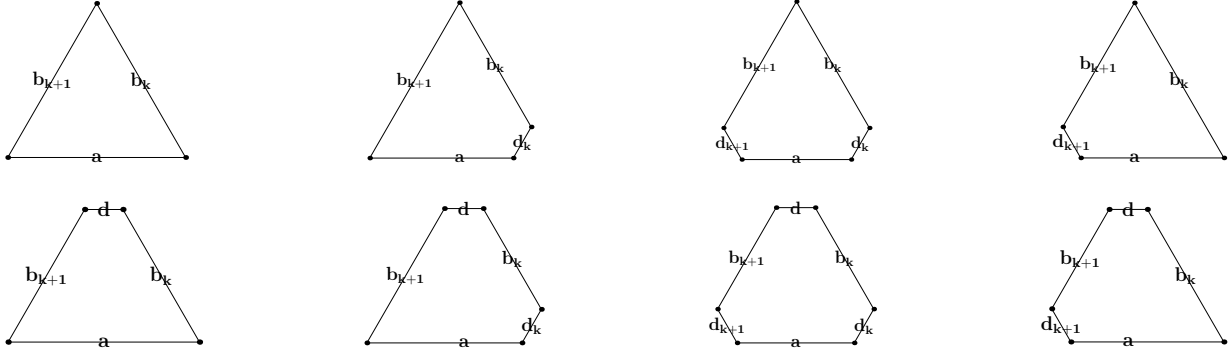


FIGURE 4.3. Combinatorial types of lateral ridges.

We will need to consider triangles with vertices outside complex hyperbolic space, so we start by establishing some terminology. Consider a triple $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of pairwise distinct complex lines (as before, $\mathbf{e}_j$ denotes either a vector in $\mathbb{C}^3$ which is positive with respect to the Hermitian form, or its polar complex line). The *vertices* of the triangle are the intersection points in projective space of its edges, which are given by $\mathbf{v}_i = \mathbf{e}_j \boxtimes \mathbf{e}_k$, where the indices i, j, k are pairwise distinct. We call $\mathbf{v}_i$ the vertex opposite to the edge $\mathbf{e}_i$.

A *complex height* of the triangle through $\mathbf{v}_i$ is a complex geodesic in $\mathbf{H}_{\mathbb{C}}^2$ that is orthogonal to one of the complex edges, and whose extension to projective space contains the opposite vertex. If a complex height through a given vertex $\mathbf{v}_i$ exists, then it is unique (in fact it is given by the complex line polar to $\mathbf{v}_i \boxtimes \mathbf{e}_i$).

If the complex height through $\mathbf{v}_i$ exists, we call its intersection with the edge $\mathbf{e}_i$ the *foot* of the complex height. The foot of the complex height through $\mathbf{v}_i$ is given by $\mathbf{f}_i = \mathbf{v}_i - \frac{\langle \mathbf{v}_i, \mathbf{e}_i \rangle}{\langle \mathbf{e}_i, \mathbf{e}_i \rangle} \mathbf{e}_i$, and one easily checks that the triangle has a complex height through $\mathbf{v}_i$ if and only if $\mathbf{f}_i$ is a negative vector.

Definition 4.3. A complex hyperbolic triangle is a triple of pairwise distinct complex lines that admits three complex heights.

From now on, all triangles are assumed to be complex hyperbolic triangles, by which we mean the edges are pairwise distinct, and there are three well-defined complex heights.

The basic fact that allows us to construct lateral ridges is the following.

Proposition 4.4. Given a complex hyperbolic triangle $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in complex hyperbolic space, there is a unique bisector $\mathcal{B}_{\mathbf{a}}$ such that

- (1) $\mathbf{a}$ is a complex slice of $\mathcal{B}_{\mathbf{a}}$ and
- (2) the extended real spine of $\mathcal{B}_{\mathbf{a}}$ contains $\mathbf{b} \boxtimes \mathbf{c}$.

This result follows from the fact that a bisector is uniquely determined by its real spine. The complex spine of the bisector $\mathcal{B}_{\mathbf{a}}$ in the proposition must be orthogonal to the base $\mathbf{a}$

and it must contain the vertex $\mathbf{b} \boxtimes \mathbf{c}$, so it must be the complex height through that vertex. Its real spine simply joins the foot of the complex height and the corresponding vertex.

Proposition 4.5. *The bisector $\mathcal{B}_{\mathbf{a}}$ from Proposition 4.4 contains the 1-skeleton of the geometric realization for $\mathbf{a}; \mathbf{b}, \mathbf{c}$.*

In order to prove this, we review the following fact, which appears as Lemma 2.3 in [13].

Lemma 4.6. *Let L be a complex line orthogonal to a complex slice of a bisector $\mathcal{B}$. Then $L \cap \mathcal{B}$ is a geodesic, contained in a meridian of $\mathcal{B}$.*

Proof: (of Prop. 4.5) This is only slightly tedious because of the diversity of cases for the combinatorial types of lateral ridges, see Figure 4.3. The bottom edges are in the bisectors because by construction $\mathcal{B}_{\mathbf{a}}$ has $\mathbf{a}$ as one of its slices. The top edges, if any, are also in a slice of $\mathcal{B}_{\mathbf{a}}$, polar to the apex. The fact that the other edges are in the bisector follows from Lemma 4.6. $\square$

Note that the choice of the base $\mathbf{a}$ of the triangle is of course artificial. In fact, as discussed in Section 4.1.4, there should be two sides containing a given ridge, and the other one should be constructed by the same process, but using $\mathbf{b}$ or $\mathbf{c}$ as the base.

We call the bisectors $\mathcal{B}_{\mathbf{a}}$, $\mathcal{B}_{\mathbf{b}}$ and $\mathcal{B}_{\mathbf{c}}$ the *natural bisectors* associated to the triangle. We will say that the triangle $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is *real* if it is the complexification of a triangle in a copy of $\mathbf{H}_{\mathbb{R}}^2$, or equivalently if the three corresponding polar vectors can be scaled so that their pairwise inner products are real.

Proposition 4.7. *Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be a non-real complex hyperbolic triangle. The natural bisectors satisfy the following properties.*

- (1) $\mathcal{B}_{\mathbf{a}} \cap \mathcal{B}_{\mathbf{b}} = \mathcal{B}_{\mathbf{b}} \cap \mathcal{B}_{\mathbf{c}} = \mathcal{B}_{\mathbf{c}} \cap \mathcal{B}_{\mathbf{a}}$.
- (2) *The above intersections have at most two connected components, and each component is a proper smooth disk in $\mathbf{H}_{\mathbb{C}}^2$.*
- (3) *The 1-skeleton of the corresponding ridge is contained in (the closure of) only one of the connected components.*

Remark 4.8. Among the groups studied in this paper, only the sporadic groups with $\tau = \sigma_{10}$ require real triangles. In fact, for $\mathcal{S}(p, \sigma_{10})$ with $p = 3, 4, 5, 10$, one checks by direct computation that the relevant real spines actually intersect inside $\mathbf{H}_{\mathbb{C}}^2$, so the natural bisectors are not cospinal either, and Lemma 9.1.5 of [15] applies. The situation is in fact simpler then, as the intersection is connected; indeed both bisectors are then linear in coordinates centered on the common point of their real spines.

Note also that by construction the 1-skeleton of the corresponding ridge is embedded in a (non-totally geodesic) disk, so it bounds a piecewise smooth disk. In other words, Proposition 4.7 gives a well-defined realization of the lateral ridges.

Proof: (of Prop. 4.7) Since the complex lines are pairwise distinct, the vectors polar to the edges are linearly independent, and we choose them as a basis of $\mathbb{C}^3$. Consequently, we denote by $\mathbf{e}_i$, $i = 1, 2, 3$ the standard basis vectors of $\mathbb{C}^3$, and take these as polar to the

edges of the triangle. We may assume $\langle \mathbf{e}_j, \mathbf{e}_j \rangle = 1$ for all $j = 1, 2, 3$, and $\langle \mathbf{e}_1, \mathbf{e}_2 \rangle = a_{12}\varphi$, $\langle \mathbf{e}_2, \mathbf{e}_3 \rangle = a_{23}\varphi$, $\langle \mathbf{e}_3, \mathbf{e}_1 \rangle = a_{31}\varphi$, with a_{jk} real and $|\varphi| = 1$.

We write

$$\varphi^3 + \bar{\varphi}^3 = 2r$$

for some $r \in [-1, 1]$.

Let H denote the matrix of the relevant Hermitian form in the standard basis, which is given by

$$H = \begin{pmatrix} 1 & a_{12}\varphi & a_{31}\bar{\varphi} \\ a_{12}\bar{\varphi} & 1 & a_{23}\varphi \\ a_{31}\varphi & a_{23}\bar{\varphi} & 1 \end{pmatrix}.$$

Since the Hermitian form must have signature $(2, 1)$, writing $d = \det H$, we must have

$$(10) \quad d = 2ra_{12}a_{23}a_{31} - a_{12}^2 - a_{23}^2 - a_{31}^2 + 1 < 0,$$

which we assume in what follows.

As above, we denote by $\mathbf{e}_i$ the vectors polar to the edges, by $\mathbf{v}_i$ the vertex opposite to $\mathbf{e}_i$, by $\mathbf{f}_i$ the corresponding foot of the complex height, and by $\mathcal{B}_i$ the corresponding natural bisector.

One computes that the vertices are given by

$$\mathbf{v}_1 = \begin{pmatrix} 1 - a_{23}^2 \\ a_{23}a_{31}\varphi^2 - a_{12}\bar{\varphi} \\ a_{12}a_{23}\bar{\varphi}^2 - a_{31}\varphi \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} a_{23}a_{31}\bar{\varphi}^2 - a_{12}\varphi \\ 1 - a_{31}^2 \\ a_{12}a_{31}\varphi^2 - a_{23}\bar{\varphi} \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} a_{12}a_{23}\varphi^2 - a_{31}\bar{\varphi} \\ a_{12}a_{31}\bar{\varphi}^2 - a_{23}\varphi \\ 1 - a_{12}^2 \end{pmatrix}.$$

and the feet of the complex heights are given by

$$\mathbf{f}_1 = \begin{pmatrix} -2ra_{12}a_{23}a_{31} + a_{12}^2 + a_{31}^2 \\ a_{23}a_{31}\varphi^2 - a_{12}\bar{\varphi} \\ a_{12}a_{23}\bar{\varphi}^2 - a_{31}\varphi \end{pmatrix}, \quad \mathbf{f}_2 = \begin{pmatrix} a_{23}a_{31}\bar{\varphi}^2 - a_{12}\varphi \\ -2ra_{12}a_{23}a_{31} + a_{12}^2 + a_{23}^2 \\ a_{12}a_{31}\varphi^2 - a_{23}\bar{\varphi} \end{pmatrix}, \quad \mathbf{f}_3 = \begin{pmatrix} a_{12}a_{23}\varphi^2 - a_{31}\bar{\varphi} \\ a_{12}a_{31}\bar{\varphi}^2 - a_{23}\varphi \\ -2ra_{12}a_{23}a_{31} + a_{23}^2 + a_{31}^2 \end{pmatrix}.$$

The condition that the complex heights be well-defined translates into the following inequalities:

$$(11) \quad \begin{aligned} 2ra_{12}a_{23}a_{31} - a_{12}^2 - a_{23}^2 &< 0 \\ 2ra_{12}a_{23}a_{31} - a_{23}^2 - a_{31}^2 &< 0 \\ 2ra_{12}a_{23}a_{31} - a_{31}^2 - a_{12}^2 &< 0 \end{aligned}$$

We denote by $\mathbf{s}_j$ a vector polar to the complex spine of $\mathcal{B}_j$ (which is also the complex height through $\mathbf{v}_j$). We have

$$\mathbf{s}_1 = \begin{pmatrix} -a_{23}\bar{\varphi} \\ a_{31}\varphi \\ 0 \end{pmatrix}, \quad \mathbf{s}_2 = \begin{pmatrix} 0 \\ -a_{31}\bar{\varphi} \\ a_{12}\varphi \end{pmatrix}, \quad \mathbf{s}_3 = \begin{pmatrix} a_{23}\varphi \\ 0 \\ -a_{12}\bar{\varphi} \end{pmatrix}.$$

Since the real spine of $\mathcal{B}_j$ contains $\mathbf{v}_j$ and $\mathbf{f}_j$, and these two vectors are not orthogonal, the vector $\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$ is on $\mathcal{B}_j$ if and only if the triple Hermitian inner product $\langle \mathbf{z}, \mathbf{v}_j, \mathbf{f}_j \rangle = \langle \mathbf{z}, \mathbf{v}_j \rangle \langle \mathbf{v}_j, \mathbf{f}_j \rangle \langle \mathbf{f}_j, \mathbf{z} \rangle$ is real.

The natural bisectors can then be described by the following equations

$$\begin{aligned}\mathcal{B}_1 &: \Im((a_{31}v_1\bar{v}_3 - a_{12}v_2\bar{v}_1)\varphi) = 0, \\ \mathcal{B}_2 &: \Im((a_{12}v_2\bar{v}_1 - a_{23}v_3\bar{v}_2)\varphi) = 0, \\ \mathcal{B}_3 &: \Im((a_{23}v_3\bar{v}_2 - a_{31}v_1\bar{v}_3)\varphi) = 0.\end{aligned}$$

It follows that $\mathcal{B}_1 \cap \mathcal{B}_2$ is contained in $\mathcal{B}_3$, which proves (1).

We now prove (2), which follows from the fact that the natural bisectors are not cotranchal (see Lemma 9.1.5 in [15]). In order to prove non-cotranchality, we want to prove that $\mathbf{u}$ is not on the real spine of $\mathcal{B}_1$ nor of $\mathcal{B}_2$. Since the intersection of any bisector with its complex spine is precisely its real spine, it is equivalent to show that $\mathbf{u}$ is not on $\mathcal{B}_1$ nor on $\mathcal{B}_2$.

Now consider the intersection in projective space of the extended complex spines of $\mathcal{B}_1$ and $\mathcal{B}_2$, which is represented by $\mathbf{u} = \mathbf{s}_1 \boxtimes \mathbf{s}_2$. One computes

$$\mathbf{u} = \begin{pmatrix} (a_{12}a_{23}\bar{\varphi}^2 - a_{31}\varphi)(a_{23}a_{31}\bar{\varphi}^2 - a_{12}\varphi) \\ (a_{31}a_{12}\varphi^2 - a_{23}\bar{\varphi})(a_{23}a_{31}\varphi^2 - a_{12}\bar{\varphi}) \\ (a_{12}a_{23}\bar{\varphi}^2 - a_{31}\varphi)(a_{31}a_{12}\varphi^2 - a_{23}\bar{\varphi}) \end{pmatrix},$$

and the triple Hermitian inner products

$$\begin{aligned}(12) \quad \langle \mathbf{u}, \mathbf{v}_1, \mathbf{f}_1 \rangle &= d(a_{12}a_{23}\bar{\varphi}^2 - a_{31}\varphi)(a_{12}a_{31}\bar{\varphi}^2 - a_{23}\varphi)(a_{23}a_{31}\bar{\varphi}^2 - a_{12}\varphi)(2ra_{12}a_{23}a_{31} - a_{12}^2 - a_{31}^2)^2, \\ \langle \mathbf{u}, \mathbf{v}_2, \mathbf{f}_2 \rangle &= d(a_{12}a_{23}\varphi^2 - a_{31}\bar{\varphi})(a_{12}a_{31}\varphi^2 - a_{23}\bar{\varphi})(a_{23}a_{31}\varphi^2 - a_{12}\bar{\varphi})(2ra_{12}b_{23}a_{31} - a_{12}^2 - a_{23}^2)^2.\end{aligned}$$

For the meaning of d in (12), see equation (10). This shows that $\mathbf{u}$ is on the real spine of $\mathcal{B}_1$ if and only if it is on the real spine of $\mathcal{B}_2$, and this happens if and only if

$$(a_{12}a_{23}\bar{\varphi}^2 - a_{31}\varphi)(a_{12}a_{31}\bar{\varphi}^2 - a_{23}\varphi)(a_{23}a_{31}\bar{\varphi}^2 - a_{12}\varphi) \in \mathbb{R}.$$

Using the inequalities (11), we see that the last condition is equivalent to the requirement that

$$a_{12}a_{23}a_{31}(\varphi^6 - 1) = 0.$$

This occurs if and only if the triangle is real.

Part (3) follows from the fact that the closures of the components contain at most one point (see Lemma 9.1.5 in [15] again), and the fact that the vertices of the ridge are pairwise distinct. $\square$

We can now state the next assumption.

Assumption 4. The 2-skeleton of the geometric realization of every side $a; b, c$ is embedded in the closure $\overline{\mathcal{B}}_a$ in $\overline{\mathbf{H}}_{\mathbb{C}}^2$ of the bisector of Proposition 4.4.

This allows us to define the geometric realization of sides of the shell, since the realization of the 2-skeleton of a side is a 2-ball, hence it bounds a (piecewise smooth) 3-ball in the closure of the bisector.

As a final embeddedness hypothesis, we require:

Assumption 5. The 3-skeleton of the geometric realization is a manifold homeomorphic to S^3 , embedded in $\overline{\mathbf{H}}_{\mathbb{C}}^2$.

In view of the solution of the Poincaré conjecture, checking the “manifold homeomorphic to S^3 ” part is a completely combinatorial check: we need to check that all links of the corresponding cell complex are spheres, and that its fundamental group is trivial. The embedding part of the assumption can be verified by a large (but finite!) amount of computation, as explained in [14].

Assumption 6. The geometric realization of the invariant shell satisfies the hypotheses of the Poincaré polyhedron theorem for the cosets of the cyclic subgroup generated by $R_1R_2R_3$ (or R_1J in the symmetric case).

Recall that the Poincaré polyhedron theorem for cosets of H in Γ produces a polytope Π that is a fundamental domain only modulo the action of H , i.e. it produces a polytope that tiles $\mathbf{H}_{\mathbb{C}}^2$, but Π is H -invariant and such that if the interior of two images $\gamma_1\Pi$ and $\gamma_2\Pi$ intersect (for $\gamma_j \in \Gamma$), then the cosets γ_1H and γ_2H coincide.

4.3. The Poincaré polyhedron theorem. For the general formulation of the Poincaré polyhedron theorem for coset decompositions, see section 3.2 in [14]. The only additional difficulty, compared with the domains that appear in [14], is that it can happen that some power of $P = R_1J$ or $Q = R_1R_2R_3$ is a complex reflection, in which case it can stabilize some ridges of the polyhedron. For simplicity, in the following discussion, we use only P , but the same applies to Q in the case of non-symmetric triangle groups.

When implementing the Poincaré polyhedron we need to keep track of cycles of ridges, and we now recall this process. A given ridge e_0 is in exactly two sides s_0 and s_1 of the polyhedron. Suppose that the side pairing map corresponding to the side s_1 is γ_1 and that $\gamma_1(e_0)$ is the ridge e_1 . Now, e_1 is in precisely two sides, namely $\gamma_1(s_1)$ and a second side s_2 . Suppose the side pairing map associated to s_2 is γ_2 and the image of e_1 under γ_2 is $e_2 = \gamma_2(e_1) = \gamma_2\gamma_1(e_0)$. Repeating this process gives a sequence of sides $e_j = \gamma_j \cdots \gamma_1(e_0)$ and we call $\gamma_j \cdots \gamma_1$ the partial cycle associated to e_0 . We stop this process whenever e_j is in the P -orbit of the original ridge, i.e. there exists a $k \in \mathbb{N}$ such that $P^k(e_j) = e_0$, in which case the cycle transformation is given by $P^k\gamma_j \cdots \gamma_1$.

In case some ridges have non-trivial stabilizers under the action of $\langle P \rangle$, there is some ambiguity in choosing k as above, and the rotation angles of $P^k\gamma_j \cdots \gamma_1$ and $P^l\gamma_j \cdots \gamma_1$ will of course in general be different.

When this happens, we consider all possible choices of k , and verify that the corresponding images of the polytope D under powers of $A = P^k\gamma_j \cdots \gamma_1$ do not overlap. More precisely, if the interiors of D and $A^j(D)$ overlap, then these should be equal (but A^j need not be the identity, it may correspond to a symmetry of D).

The presence of non-trivial stabilizers of ridges in the action of $\langle P \rangle$ also has some consequences when writing explicit presentations for our lattices in terms of generators and relations, based on the tiling of $\mathbf{H}_{\mathbb{C}}^2$ by images of D .

Specifically, if a ridge e has stabilizer in $\langle P \rangle$ generated by P^k , we need to include a presentation for the group generated by P^k and the corresponding cycle transformation A . For the groups that occur in this paper, this occurs only for complex ridges that are

stabilized by a complex reflection P^k , and we need to include a commutation relation

$$[A, P^k] = id.$$

In the next two sections, we list some important information that can be gathered by applying the Poincaré polyhedron theorem, namely Vertex stabilizers (and more generally facet stabilizers) and singularities of the quotient.

Two different sets of presentations for our lattices in terms of generators and relations will be given in sections 5.5 and 5.6.

4.3.1. Vertex stabilizers, cusps. In the “Vertex stabilizers” tables in the appendix, for each group where the algorithm produces a fundamental domain, we give a list of representatives for vertices of the fundamental domain, under the equivalence relation generated by side pairings. These are obtained by tracking cycles of vertices, in the sense of the Poincaré polyhedron theorem.

The corresponding groups are finite for vertices in $\mathbf{H}_{\mathbb{C}}^2$, and cusps for ideal vertices. For many finite groups, the order of the group can be obtained by Proposition 2.5. In all generality, the full vertex stabilizers are computed from our fundamental polytopes, by tracking orbits of vertices under side-pairing maps and the cyclic group generated by P (or Q). This amounts to constructing a directed graph whose vertices correspond to the vertices in the orbit, with edges labelled by isometries that map the origin to the endpoint; the stabilizer is then generated by the isometries corresponding to generators of the fundamental group of that graph.

In particular, sometimes the stabilizer of a vertex is larger than just the group generated by the complex reflections attached to complex faces of the domain through that point. This happens for instance when some power of P (or Q) is a complex reflection with mirror through that vertex. In the tables, we indicate this phenomenon by an asterisk.

In order to describe the finite groups, we use the Shephard-Todd notation [40] (or product of cyclic groups, when the stabilizer is generated by two complex reflections with orthogonal mirrors), i.e. G_k denotes the k -th group in the Shephard-Todd list, and the $G(m, p, n)$ are so-called *imprimitive* groups, see section 2 of [40].

4.3.2. Singularities of the quotient. In the appendix, we list the singular points of the quotient, for each of the lattice Γ where our algorithm produces a fundamental domain D . The basic observation is that, by the definition of a fundamental domain, for any element $\gamma \in \Gamma$ with a fixed point in $\overline{\mathbf{H}}_{\mathbb{C}}^2$, there is a conjugate $\gamma' \in \Gamma$ that fixes a point on the boundary ∂D of D . Hence, in order to determine conjugacy classes of fixed points of Γ , it is enough to study stabilizers of facets of D .

The second basic tool used to list singular points of the quotient is a theorem of Chevalley, according to which the quotient of $\mathbb{C}^2$ by a finite subgroup of $\mathrm{GL}(2, \mathbb{C})$ is smooth if and only if the group is generated by complex reflections (see chapter 4 in [41]).

Now for each facet f of D , we determine the stabilizer G_f of f (this is done computing cycles in the Poincaré polyhedron theorem), and determine the reflection subgroup R_f , generated by the set of complex reflections in G_f . The quotient has a singular point on f if and only if $R_f \subsetneq G_f$.

Note that all singularities turn out to be cyclic quotient singularities, even though some facet stabilizers are not – in those cases, only the reflection subgroup of the facet stabilizer is slightly complicated.

5. RESULTS

5.1. Good cases. It may seem unlikely that the above assumptions would ever be satisfied, but in fact they turn out to be satisfied for most known geometric constructions of lattices in $\mathrm{PU}(2, 1)$. The three classes of lattices we have in mind are:

- Mostow/Deligne-Mostow groups;
- Sporadic triangle groups;
- Non-symmetric triangle groups from James Thompson’s thesis.

We denote these three families of groups by $\Gamma(p, t)$ (where t is a rational number), $\mathcal{S}(p, \tau)$ (where τ is a complex number), and $\mathcal{T}(p, \mathbf{T})$ (where $\mathbf{T}$ is a triple of complex numbers) respectively.

The values of p such that these groups are lattices are listed, for each value of the parameter, for sporadic groups in Tables 3.2, and for Thompson groups in Tables 3.3, 3.4, respectively.

In the next few sections, we describe the results of the algorithm for the lattices in the above three families, in terms of general structure and combinatorics of the shell (sections 5.2 and 5.3) and the verification of the hypotheses of the Poincaré polyhedron theorem (section 5.4). Technically, we defer part of the proof of discreteness to calculations that are practically impossible to perform by hand, but we have described in quite a bit of detail the necessary computer verifications (see section 4, and also [14]). A computer program that implements the corresponding methods is available to verify our claims, see [11].

5.2. Combinatorial invariant shell. It turns out that all these groups satisfy Assumption 1, which says there exists a finite invariant shell, at least on the combinatorial level, and Assumption 2, which says all triangles are non degenerate. In fact, these assumptions are satisfied for a much wider class of groups, discreteness is by no means necessary at that stage.

We will describe the rough structure of the invariant shell by giving a list of (ordered) triangles, each of which generates a side, by the process described in Section 4.1.1; we write

$$[k] \ a; \ b, \ c$$

to denote a k -gon pyramid with base a and two consecutive alteral edges given by b and c . For each group, we list only side representatives for each P -orbit of sides, and we only list one side for each pair of opposite sides – that is sides that are paired in the sense of the Poincaré polyhedron theorem. The results are listed in the “Combinatorics” tables in the appendix.

5.3. Detailed combinatorics and embeddedness. Even though this rough description of the shell (in particular, the number of sides) depends only on the shape parameter of the group, the detailed combinatorics depend on the order p of the generators. This is

illustrated in the pictures of the appendix, sections A.1 through A.5. The pictures show the 1-skeleton of sides in geographical coordinates, we label ideal vertices by red dots, finite vertices by blue dots, and we label edges in word notation (at least when the label is not too long to keep the pictures readable).

We only list side representatives, i.e. we pick a representative for each P -orbit, and one for each pair of faces with inverse side-pairing transformation.

As explained in [14], the embeddedness of the skeleton of the domain can be reduced to a finite number of computations in the relevant number field. Here the relevant field is the smallest field ℓ such that the group can be represented as a subgroup of $\mathrm{PU}(2, 1, \mathcal{O}_\ell)$, see Section 6.2.

Perhaps surprisingly, even when the triangle group is a lattice, the invariant shell we build is not always embedded in $\overline{\mathbf{H}}_{\mathbb{C}}^2$. By performing the computations as in [14] (one way to do this is to run our computer program [11]), we obtain the following.

Proposition 5.1. *The invariant shells of sporadic lattice triangle groups are all embedded in $\overline{\mathbf{H}}_{\mathbb{C}}^2$. The invariant shells of Thompson lattice triangle groups are all embedded in $\overline{\mathbf{H}}_{\mathbb{C}}^2$ except $\mathcal{T}(5, \overline{\mathbf{H}}_2)$. The invariant shells of Mostow lattices $\Gamma(p, t)$ are all embedded except for those for the groups with parameters $(p, t) = (5, 1/2)$, $(7, 3/14)$ and $(9, 1/18)$.*

For the three Mostow groups that are excluded in Proposition 5.1, Assumption 3 fails. A typical non-embedded 1-skeleton face is shown in Figure 5.1. Note that the fact that

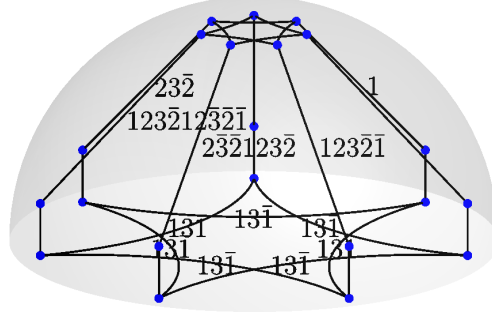


FIGURE 5.1. The 1-skeleton of pyramids is not always embedded, as here in the picture $\Gamma(7, 3/14)$.

these three groups do not fit well in the framework of our paper is not a surprise, it can easily be explained by Deligne-Mostow theory (see [6], [23] or [44]).

In fact, the construction in our paper produces fundamental domains for Deligne-Mostow groups corresponding to 5-tuples of weights that satisfy condition Σ -INT with $\Sigma = S_3$ (for the relation between our groups and Deligne-Mostow theory, see [28] for instance). The Mostow groups where our polytopes are *not* fundamental polytopes are those corresponding to Deligne-Mostow groups corresponding to 5-tuples that satisfy Σ -INT with $\Sigma = S_4$ rather than S_3 . For such groups, one does not expect the quotient to have the same structure, and a fundamental domain should be very different from ours.

5.4. Hypotheses of Poincaré. The hypotheses of the Poincaré polyhedron theorem can be checked as explained in [14]. This is done systematically in our computer code [11], we summarize the result of the computations in the statement of Proposition 5.2. It turns out the theorem applies in most, but not all, cases where the invariant shell is embedded.

Proposition 5.2. *For all sporadic triangle groups and all Mostow groups apart from the Σ -INT examples with 4-fold symmetry, the hypotheses of the Poincaré polyhedron theorem are satisfied.*

The hypotheses also hold for all Thompson triangle groups except for $\mathcal{T}(12, \mathbf{E}_2)$, $\mathcal{T}(7, \bar{\mathbf{H}}_1)$, $\mathcal{T}(10, \mathbf{H}_2)$ and $\mathcal{T}(5, \bar{\mathbf{H}}_2)$.

We now give some detail about how the hypotheses fail for the four problematic Thompson groups listed in Proposition 5.2.

- (1) For $\mathcal{T}(12, \mathbf{E}_2)$, the hypotheses of the Poincaré polyhedron fail, more specifically the local tiling near some ridges give extra overlap. In fact, the ridge which is the intersection of the pyramids $\bar{2}12; 12\bar{1}, 3$ and $\bar{3}13; 12\bar{1}, 3$ gives a cycle transformation $Q^{-1}\bar{2}12 = (12\bar{1}3)^{-1}$, whose fourth power fixes p_0 , without being a power of Q (recall that p_0 denotes the isolated fixed point of Q). What happens in this case is that $(12\bar{1}3)^4 = (SQ)^2$ where S is the extra symmetry given in Remark 3.4 (3). In other words in this case $S = Q^2(12\bar{1}3)^{-4}$ and so $\langle R_1, R_2, R_3 \rangle = \langle R_1, R_2, S \rangle$. In fact, the algorithm does work with the spherical shell given by the algorithm, but with the larger group $\langle S, Q \rangle$ as its stabilizer. One must make several simple modifications, including adjoining more relations when applying the Poincaré polyhedron theorem. Since this group is arithmetic, we will not go into the details of the necessary changes here.
- (2) For $\mathcal{T}(7, \bar{\mathbf{H}}_1)$, the integrality condition fails. Indeed, the complex ridge given by the intersection of the two pyramids $3; 1, 2$ and $123\bar{2}\bar{1}; 1, 2$ has cycle transformation given by 12 . The isometry R_1R_2 is a regular elliptic element with angles $(3\pi/14, \pi)$, and its square is a complex reflection with angle $3\pi/7$, which is not of the form $2\pi/k$ for any $k \in \mathbb{N}$.
- (3) For $\mathcal{T}(10, \mathbf{H}_2)$, the integrality condition fails. More specifically, the ridge on the mirror of $R_2^{-1}R_1R_2$ has cycle transformation given by $Q^2R_2^{-1}R_1R_2$, and this is a complex reflection with rotation angle $2\pi \cdot 3/10$.
- (4) For $\mathcal{T}(5, \bar{\mathbf{H}}_2)$, the spherical shell is not embedded. In fact, in that case, the point p_0 (which is the isolated fixed point of Q) lies on the bottom ridge of $123\bar{2}12\bar{3}\bar{2}\bar{1}; 123\bar{2}\bar{1}, 2$.

For $\mathcal{T}(7, \bar{\mathbf{H}}_1)$ and $\mathcal{T}(5, \bar{\mathbf{H}}_2)$, note that the groups are each conjugate to a Mostow group where the algorithm runs fine, see Proposition 7.2 (this commensurability corresponds to a change of generators). Similarly, the group $\mathcal{T}(10, \mathbf{H}_2)$ has two alternative descriptions that allow to use the algorithm, see Proposition 7.6.

Among Mostow groups, the problematic ones are $\Gamma(5, 1/2)$, $\Gamma(7, 3/14)$, $\Gamma(9, 1/18)$. Every one of these three groups is known to be conjugate to a Mostow group without the extra 4-fold symmetry, by work of Sauter (see [37] and also Corollary 10.18 in [7]).

Specifically, we have that $\Gamma(5, 1/2)$ is conjugate to the group Γ_μ with $\mu = (3, 3, 3, 3, 8)/10$, which gives the same group as $\mu = (2, 3, 3, 3, 9)/10$, which in turn gives the group $\Gamma(5, 7/10)$. Similarly, $\Gamma(7, 3/14)$ corresponds to exponents $(2, 5, 5, 5, 11)/14$, which gives the same group as $(5, 5, 5, 5, 8)/14$, which gives the Mostow group $\Gamma(7, 9/14)$. Finally, $\Gamma(9, 1/18)$ corresponds to $(7, 7, 7, 7, 8)/18$, which gives the same group as $(2, 7, 7, 7, 13)/18$, which is $\Gamma(9, 11/18)$.

5.5. Geometric presentations. In this section we list the presentations obtained from the Poincaré polyhedron theorem (see section 5.6) for each of the groups where the hypotheses of the Poincaré polyhedron theorem are satisfied, see section 5.4.

We will call these presentations *geometric presentations*, as opposed to the *natural presentations*, as described in section 5.6. Rather than listing the actual computer output for presentations (which is available via [11]), we will list slightly modified versions where we have used simple Tietze transformations to make the presentation more readable.

As an example, the presentations for σ_1 groups obtained by the computer are equivalent to the following. For $p = 3$, we get

$$(13) \quad \langle R_1, R_2, R_3, J \mid (R_1 J)^8, J^3, J R_1 J^{-1} R_2^{-1}, J R_2 J^{-1} R_3^{-1}, \\ \text{br}_6(R_1, R_2), \text{br}_3(R_1, R_2 R_3 R_2 R_3^{-1} R_2^{-1}), \\ R_1^3 \rangle.$$

For $p = 4$, we get

$$(14) \quad \langle R_1, R_2, R_3, J \mid (R_1 J)^8, J^3, J R_1 J^{-1} R_2^{-1}, J R_2 J^{-1} R_3^{-1}, \\ \text{br}_6(R_1, R_2), \text{br}_3(R_1, R_2 R_3 R_2 R_3^{-1} R_2^{-1}), \\ R_1^4, (R_1 R_2)^{12} \rangle.$$

For $p = 6$

$$(15) \quad \langle R_1, R_2, R_3, J \mid (R_1 J)^8, J^3, J R_1 J^{-1} R_2^{-1}, J R_2 J^{-1} R_3^{-1}, \\ \text{br}_6(R_1, R_2), \text{br}_3(R_1, R_2 R_3 R_2 R_3^{-1} R_2^{-1}), \\ R_1^6, (R_1 R_2)^6, (R_1 R_2 R_3 R_2^{-1})^{12} \rangle.$$

The three presentations in equations (13) through (15) can be written in a uniform way, see the σ_1 entry in Table 5.1, where by convention the relations giving the order of $R_1 R_2$ and $R_1 R_2 R_3 R_2^{-1}$ can be omitted when the exponents $3p/(p-3)$ or $4p/(p-4)$ are negative or infinite.

All geometric presentations are listed in the left part of Tables 5.1 and 5.2.

5.6. Natural presentations. The list of braid relations that occur in geometric presentations (see section 5.5) may seem strange to the reader. Recall that one of our assumptions is that for every side bounding our polytope, the complex reflections attached to every pair of consecutive lateral edges satisfy a braid relation (see Assumption 1). Not every such braid relation occurs in the geometric presentation, however.

In this section, we give a systematic way to produce a presentation for every lattice in our list (more precisely, lattices where the hypotheses of the Poincaré polyhedron theorem are satisfied). Motivated by the fact that our lattices are essentially uniquely determined by

$\mathcal{S}(\sigma_1, p), p = 3, 4, 6$	
$\left\langle R_1, R_2, R_3, J \mid (R_1 J)^8, J^3, J R_1 J^{-1} R_2^{-1}, J R_2 J^{-1} R_3^{-1}, \right.$ $\quad \text{br}_6(R_1, R_2),$ $\quad \text{br}_3(R_1, R_2 R_3 R_2 R_3^{-1} R_2^{-1}),$ $\quad \left. R_1^p, (R_1 R_2)^{\frac{3p}{p-3}}, (R_1 R_3^{-1} R_2 R_3)^{\frac{4p}{p-4}} \right\rangle$	$\left\langle R_1, R_2, R_3, J \mid (R_1 J)^8, J^3, J R_1 J^{-1} R_2^{-1}, J R_2 J^{-1} R_3^{-1}, \right.$ $\quad \text{br}_6(R_1, R_2),$ $\quad \text{br}_4(R_1, R_3^{-1} R_2 R_3)$ $\quad \left. R_1^p, (R_1 R_2)^{\frac{3p}{p-3}}, (R_1 R_3^{-1} R_2 R_3)^{\frac{4p}{p-4}} \right\rangle$
$\mathcal{S}(\bar{\sigma}_4, p), p = 3, 4, 5, 6, 8, 12$	
$\left\langle R_1, R_2, R_3, J \mid (R_1 J)^7, J^3, J R_1 J^{-1} R_2^{-1}, J R_2 J^{-1} R_3^{-1}, \right.$ $\quad \text{br}_4(R_1, R_2),$ $\quad \left. R_1^p, (R_1 R_2)^{\frac{4p}{p-4}}, (R_1 R_3^{-1} R_2 R_3)^{\frac{6p}{p-6}} \right\rangle$	$\left\langle R_1, R_2, R_3, J \mid (R_1 J)^7, J^3, J R_1 J^{-1} R_2^{-1}, J R_2 J^{-1} R_3^{-1}, \right.$ $\quad \text{br}_4(R_1, R_2),$ $\quad \text{br}_3(R_1, R_3^{-1} R_2 R_3),$ $\quad \left. R_1^p, (R_1 R_2)^{\frac{4p}{p-4}}, (R_1 R_3^{-1} R_2 R_3)^{\frac{6p}{p-6}} \right\rangle$
$\mathcal{S}(\sigma_5, p), p = 2, 3, 4$	
$\left\langle R_1, R_2, R_3, J \mid (R_1 J)^{30}, J^3, J R_1 J^{-1} R_2^{-1}, J R_2 J^{-1} R_3^{-1}, \right.$ $\quad \text{br}_4(R_1, R_2),$ $\quad \text{br}_2((R_1 J)^5, R_2 R_3^{-1} R_2^{-1} R_1 R_2 R_3 R_2^{-1}),$ $\quad R_1^p, (R_1 R_3^{-1} R_2 R_3)^{\frac{10p}{3p-10}},$ $\quad \left. (R_2 R_3^{-1} R_2^{-1} R_1^{-1} R_2 R_3 R_2^{-1} R_1 R_2 R_3)^{\frac{3p}{p-3}} \right\rangle,$	$\left\langle R_1, R_2, R_3, J \mid (R_1 J)^{30}, J^3, J R_1 J^{-1} R_2^{-1}, J R_2 J^{-1} R_3^{-1}, \right.$ $\quad \text{br}_4(R_1, R_2),$ $\quad \text{br}_5(R_1, R_3^{-1} R_2 R_3),$ $\quad R_1^p, (R_1 R_3^{-1} R_2 R_3)^{\frac{10p}{3p-10}},$ $\quad \left. (R_2 R_3^{-1} R_2^{-1} R_1^{-1} R_2 R_3 R_2^{-1} R_1 R_2 R_3)^{\frac{3p}{p-3}} \right\rangle,$
$\mathcal{S}(\sigma_{10}, p), p = 3, 4, 5, 10$	
$\left\langle R_1, R_2, R_3, J \mid (R_1 J)^5, J^3, J R_1 J^{-1} R_2^{-1}, J R_2 J^{-1} R_3^{-1}, \right.$ $\quad \text{br}_5(R_1, R_2),$ $\quad \text{br}_2(R_1, R_3^{-1} R_2^{-1} R_3 R_2 R_3),$ $\quad \left. R_1^p, (R_1 R_2)^{\frac{10p}{3p-10}}, (R_1 R_3^{-1} R_2 R_3)^{\frac{6p}{p-6}} \right\rangle$	$\left\langle R_1, R_2, R_3, J \mid (R_1 J)^5, J^3, J R_1 J^{-1} R_2^{-1}, J R_2 J^{-1} R_3^{-1}, \right.$ $\quad \text{br}_5(R_1, R_2),$ $\quad \text{br}_3(R_1, R_3^{-1} R_2 R_3),$ $\quad \left. R_1^p, (R_1 R_2)^{\frac{10p}{3p-10}}, (R_1 R_3^{-1} R_2 R_3)^{\frac{6p}{p-6}} \right\rangle$

TABLE 5.1. Geometric (left) and natural (right) presentations for all sporadic groups where the hypotheses of the Poincaré polyhedron theorem hold. Apart from inverses, relations involving negative or infinite exponents can be removed from the presentation.

the four braid lengths $\text{br}(R_1, R_2), \text{br}(R_2, R_3), \text{br}(R_3, R_1), \text{br}_d(R_1, R_3^{-1} R_2 R_3)$ (see [43], [17]), one naturally expects that these braid relations should be enough to reconstruct all others in the geometric presentations.

Accordingly, we construct *natural presentations*, having the same generators as the geometric presentations, and the following relations:

- (1) Basic relations between generators (in the equilateral case, J has order 3 and conjugates R_j into R_{j+1} , and in the non-equilateral case, Q is equal to $R_1 R_2 R_3$), and relations giving the order of $P = R_1 J$ or Q ;
- (2) Orders of complex reflections stabilizing each complex face. These correspond to the bases of pyramids, or top faces when they are truncated; see the “Combinatorics” tables in the appendix;

$\mathcal{T}(\mathbf{S}_2, p), p = 3, 4, 5$	
$\left\langle R_1, R_2, R_3 \mid (R_1 R_2 R_3)^5, \right.$ $\text{br}_3(R_2, R_3), \text{br}_3(R_3, R_1), \text{br}_4(R_1, R_2),$ $\left. R_1^p, R_2^p, R_3^p, (R_1 R_2)^{\frac{4p}{p-4}}, (R_1 R_3^{-1} R_2 R_3)^{\frac{10p}{3p-10}} \right\rangle$	$\left\langle R_1, R_2, R_3 \mid (R_1 R_2 R_3)^5, \right.$ $\text{br}_3(R_2, R_3), \text{br}_3(R_3, R_1), \text{br}_4(R_1, R_2),$ $\text{br}_5(R_1, R_3^{-1} R_2 R_3)$ $\left. R_1^p, R_2^p, R_3^p, (R_1 R_2)^{\frac{4p}{p-4}}, (R_1 R_3^{-1} R_2 R_3)^{\frac{10p}{3p-10}} \right\rangle$
$\mathcal{T}(\mathbf{E}_2, p), p = 3, 4, 6$	
$\left\langle R_1, R_2, R_3 \mid (R_1 R_2 R_3)^6, \right.$ $\text{br}_3(R_2, R_3), \text{br}_4(R_3, R_1), \text{br}_4(R_1, R_2),$ $\text{br}_2((R_1 R_2 R_3)^3, R_2^{-1} R_1 R_2),$ $R_1^p, R_2^p, R_3^p, (R_1 R_2)^{\frac{4p}{p-4}}, (R_1 R_3)^{\frac{4p}{p-4}},$ $\left. (R_1 R_3^{-1} R_2 R_3)^{\frac{4p}{p-4}}, (R_3 R_1 R_2 R_1^{-1})^{\frac{3p}{p-3}} \right\rangle$	$\left\langle R_1, R_2, R_3 \mid (R_1 R_2 R_3)^6, \right.$ $\text{br}_3(R_2, R_3), \text{br}_4(R_3, R_1), \text{br}_4(R_1, R_2),$ $\text{br}_4(R_1, R_3^{-1} R_2 R_3),$ $R_1^p, R_2^p, R_3^p, (R_1 R_2)^{\frac{4p}{p-4}}, (R_1 R_3)^{\frac{4p}{p-4}},$ $\left. (R_1 R_3^{-1} R_2 R_3)^{\frac{4p}{p-4}}, (R_3 R_1 R_2 R_1^{-1})^{\frac{3p}{p-3}} \right\rangle$
$\mathcal{T}(\mathbf{H}_1, p), p = 2$	
$\left\langle R_1, R_2, R_3 \mid (R_1 R_2 R_3)^{42}, \right.$ $\text{br}_3(R_2, R_3), \text{br}_3(R_3, R_1), \text{br}_4(R_1, R_2),$ $\text{br}_2((R_1 R_2 R_3)^3, R_2^{-1} R_1 R_2 R_3 R_2^{-1} R_1^{-1} R_2),$ $\left. R_1^p, R_2^p, R_3^p \right\rangle$	$\left\langle R_1, R_2, R_3 \mid (R_1 R_2 R_3)^{42}, \right.$ $\text{br}_3(R_2, R_3), \text{br}_3(R_3, R_1), \text{br}_4(R_1, R_2),$ $\text{br}_7(R_1, R_3^{-1} R_2 R_3),$ $\left. R_1^p, R_2^p, R_3^p \right\rangle$
$\mathcal{T}(\mathbf{H}_2, p), p = 2, 3, 5$	
$\left\langle R_1, R_2, R_3 \mid (R_1 R_2 R_3)^{15}, \right.$ $\text{br}_3(R_2, R_3), \text{br}_3(R_3, R_1),$ $\text{br}_5(R_1, R_3^{-1} R_2 R_3), \text{br}_2((R_1 R_2 R_3)^3, R_2^{-1} R_1 R_2),$ $R_1^p, R_2^p, R_3^p, (R_1 R_2)^{\frac{10p}{3p-10}},$ $\left. (R_1 R_3^{-1} R_2 R_3)^{\frac{10p}{3p-10}}, (R_2 R_1 R_2^{-1} R_3 R_1 R_3^{-1})^{\frac{5p}{2p-5}} \right\rangle$	$\left\langle R_1, R_2, R_3 \mid (R_1 R_2 R_3)^{15}, \right.$ $\text{br}_3(R_2, R_3), \text{br}_3(R_3, R_1), \text{br}_5(R_1, R_2),$ $\text{br}_5(R_1, R_3^{-1} R_2 R_3)$ $R_1^p, R_2^p, R_3^p, (R_1 R_2)^{\frac{10p}{3p-10}},$ $\left. (R_1 R_3^{-1} R_2 R_3)^{\frac{10p}{3p-10}}, (R_2 R_1 R_2^{-1} R_3 R_1 R_3^{-1})^{\frac{5p}{2p-5}} \right\rangle$

TABLE 5.2. Geometric (left) and natural (right) presentations for all Thompson groups where the hypotheses of the Poincaré polyhedron theorem hold.

(3) The following braid relations corresponding to the parameters $(a, b, c; d)$:

$$\text{br}_a(R_1, R_2), \quad \text{br}_b(R_2, R_3), \quad \text{br}_c(R_3, R_1), \quad \text{br}_d(R_1, R_3^{-1} R_2 R_3).$$

The corresponding presentations are listed in the right part of Tables 5.1 and 5.2 and 5.2.

Note that we could use Proposition 2.5 and 5.2 to deduce the order of complex reflections stabilizing the top faces in (2), but we do not know how to deduce these orders directly from the other relations.

At this stage, it is not at all clear that the geometric and the natural presentations for one given group are equivalent, i.e. the corresponding finitely presented groups are isomorphic. We will prove this in section 5.7.

5.7. Equivalence of the geometric and the natural presentations. The goal of this section is to prove that for every lattice where the hypotheses of the Poincaré polyhedron

theorem are satisfied, the natural presentation is indeed a presentation for the lattice, i.e. it is equivalent to the geometric presentation.

Theorem 5.3. *For each lattice where the hypotheses of the Poincaré polyhedron theorem hold, the natural presentation is indeed a presentation of the corresponding lattice.*

We will prove this by a case-by-case analysis, compare the left and right halves of Tables 5.1 and 5.2 to see what we need to prove.

We start by proving that the natural braid relations (together with basic relations between generators) imply all vertex braid relations (we call *vertex braid relations* the braid relations that appear in Assumption 1).

Theorem 5.4. *For each lattice in our list the natural relations imply the appropriate braid relation at each vertex.*

Recall that for every triangle group lattice Γ , the geometric and natural presentations have the same generating set, which we denote by X ($X = \{R_1, R_2, R_3, J\}$ if Γ is a sporadic or Mostow group, and $X = \{R_1, R_2, R_3\}$ if it is a Thompson group). We write G (resp. N) for the set of geometric (resp. natural) relations. If $G \subset N$, then there is nothing to prove, since we know $\langle X|G \rangle$ is a presentation by the Poincaré polyhedron theorem, and we also know that the relations in N hold. If $G \not\subset N$, we will enlarge the set of relations N to a new set N' where $G \subset N'$ and $\langle X|N' \rangle$ is isomorphic to $\langle X|N \rangle$.

It turns out that the calculations needed to prove Theorem 5.3 are essentially the same as those needed to prove Theorem 5.4. In some cases, all the vertices are P -images (or Q -images) of a vertex where we have one of the four braid relations in the natural presentation. In this case there is nothing to prove. To find which extra braid relations are required we refer to the description of the pyramids given in Appendix A.3.

We now summarize what we need to prove in each case and where we do so:

- (1) σ_1 : for both Theorem 5.3 and Theorem 5.4 we need to show that the natural presentation implies $\text{br}_3(R_1, R_2 R_3 R_2 R_3^{-1} R_2^{-1})$ and for Theorem 5.4 we need to show $\text{br}_3(R_1, R_3^{-1} R_2^{-1} R_2 R_3 R_2)$ as well; see Proposition 5.8.
- (2) $\bar{\sigma}_4$: we have $G \subset N$ and there is nothing to prove for either theorem.
- (3) σ_5 : we show the natural presentation implies $\text{br}_2((R_1 J)^5, R_2 R_3^{-1} R_2^{-1} R_1 R_2 R_3 R_2^{-1})$ for Theorem 5.3 and $\text{br}_6(R_2, R_3^{-1} R_2^{-1} R_1^{-1} R_2 R_3 R_2^{-1} R_1 R_2 R_3)$ for Theorem 5.4; see Corollary 5.7 (1).
- (4) σ_{10} : we need to show that the natural presentation implies $\text{br}_2(R_1, R_2 R_3 R_2 R_3^{-1} R_2^{-1})$ for Theorem 5.3; see Proposition 5.9. There is nothing to prove for Theorem 5.4.
- (5) $\mathbf{S}_2$: we have $G \subset N$ and there is nothing to prove for either theorem.
- (6) $\mathbf{E}_2$: we need to show that the natural presentation implies $\text{br}_2((R_1 R_2 R_3)^2, R_2^{-1} R_1 R_2)$ for Theorem 5.3 and $\text{br}_6(R_1 R_2 R_1^{-1}, R_3)$; for Theorem 5.4; see Corollary 5.7 (2).
- (7) $\mathbf{H}_1$: we show the natural presentation implies $\text{br}_2((R_1 R_2 R_3)^3, R_2^{-1} R_1 R_2 R_3 R_2^{-1} R_1^{-1} R_2)$ for Theorem 5.3 and $\text{br}_{14}(R_2^{-1} R_1 R_2, R_3^{-1} R_1 R_2 R_1^{-1} R_3)$ for Theorem 5.4; see Corollary 5.7 (3).

- (8) **H₂**: we need to show that the natural presentation implies $\text{br}_2((R_1 R_2 R_3)^3, R_2^{-1} R_1 R_2)$ for Theorem 5.3 and $\text{br}_{10}(R_1 R_2 R_3 R_2^{-1} R_1^{-1}, R_2)$ for Theorem 5.4; see Corollary 5.7 (4).

Observe that in many cases we need to prove a braid relation of length 2 and a braid relation of length n . It turns out that these are both consequences of a braid relation of length 4, as shown in the following general lemma.

Lemma 5.5. *Let G be a group and suppose $A, B \in G$ satisfy $\text{br}_4(A, B)$.*

- (1) *If $C \in G$ is any element that commutes with B then $\text{br}_2(B, CBA^{-1}B^{-1}A^{-1})$.*
- (2) *If $B^n = \text{Id}$ then $\text{br}_n(A, BAB^{-1})$.*

Proof. We first prove (1). The condition $\text{br}_4(A, B)$ implies B commutes with $(BA)^2$. We can write

$$CBA^{-1}B^{-1}A^{-1} = CB(BA)^{-2}B.$$

Since B commutes with C , B and $(BA)^2$ we see that it commutes with $CBA^{-1}B^{-1}A^{-1}$ as required.

For part (2), observe that $(BA)^2$ lies in the center of $\langle A, B \rangle$. Note that for every integer m , we have

$$\begin{aligned} (ABAB^{-1})^m (BAB^{-1}A)^{-m} &= (ABAB^{-1})^m A^{-1} (ABAB^{-1})^{-m} A \\ &= (B^{-1}(BA)^2 B^{-1})^m A^{-1} (B^{-1}(BA)^2 B^{-1})^{-m} A \\ &= B^{-2m} A^{-1} B^{2m} A. \end{aligned}$$

We used the fact that $(BA)^2$ commutes with A and B on the last line. If n is even then setting $m = n/2$ immediately gives the result. If n is odd then setting $m = (n-1)/2$ we have

$$\begin{aligned} BAB^{-1}(ABAB^{-1})^{(n-1)/2}(BAB^{-1}A)^{-(n-1)/2}A^{-1} &= BAB^{-1}(B^{-(n-1)}A^{-1}B^{n-1}A)A^{-1} \\ &= BAB^{-n}A^{-1}B^{n-1}. \end{aligned}$$

Since B has order n the last line is the identity, which completes the proof. $\square$

Lemma 5.5 has the following geometric interpretation. In all the cases where we use it, the map A will be a complex reflection of angle $2\pi/p$, in which case $\langle A, B \rangle$ is a central extension of a $(2, p, n)$ triangle group, see Lemma 2.5. The apex of a pyramid with A and BAB^{-1} as lateral edges is also fixed by B and there is an n -gon in the projective line of complex lines through the apex (see section 4.1.1). The top ridge will be an n -gon whose center is fixed by B and whose vertices are the intersection points of this complex line with the mirrors of $B^j A B^{-j}$. An example is the top hexagon in the lower right hand pyramid of the pictures in Appendix A.3.

We now describe how to apply this lemma to obtain extra braid relations for some of the families of groups. We show that some power of P , Q or another symmetry braids with length 4 with one of the complex reflections.

Lemma 5.6. (1) *The natural presentation for σ_5 implies $\text{br}_4(R_2, P^{-5})$.*

- (2) Let S satisfy $SR_1S^{-1} = R_1$, $SR_2S^{-1} = R_3$ and $SR_3S^{-1} = R_3^{-1}R_2R_3$ (as the symmetry S of $\mathbf{E}_2$ groups, see Remark 3.4). Then $\text{br}_4(R_1, R_2)$ implies $\text{br}_4(R_3, R_3^{-1}R_2^{-1}R_1^{-1}S^{-1})$.
- (3) The natural presentation for $\overline{\mathbf{H}}_1$ implies $\text{br}_4(R_2^{-1}R_1R_2, Q^{-3})$.
- (4) Let $Q_{1/2}$ satisfy $Q_{1/2}R_1Q_{1/2}^{-1} = R_1R_2R_1^{-1}$, $Q_{1/2}R_2Q_{1/2}^{-1} = R_3$ and $Q_{1/2}R_3Q_{1/2}^{-1} = R_3^{-1}R_1R_3$ (as does the symmetry of $\mathbf{H}_2$ groups described in Remark 3.4). Then the braid relations in the natural presentations for $\mathbf{H}_2$ imply $\text{br}_4(R_1R_2R_3R_2^{-1}R_1^{-1}, Q_{1/2}^{-3})$.
- (5) The natural presentation for Mostow groups implies $\text{br}_4(R_1, P^{-2})$.

Proof. We give the proof of (1). First note that $P^5 = J^{-1}23123 = 12312J^{-1}$. Now using $JR_2 = R_3J$ and $J^{-1}R_2^{-1} = R_1^{-1}J^{-1}$, then $\text{br}_4(R_2, R_3)$, and $\text{br}_5(R_1, R_2R_3R_2^{-1})$ we have:

$$\begin{aligned}
(P^{-5}2P^{-5}2)(P^5\bar{2}P^5\bar{2}) &= (J\bar{2}\bar{1}\bar{3}\bar{2}\bar{1})2(\bar{3}\bar{2}\bar{1}\bar{3}\bar{2}J)2(J^{-1}23123)\bar{2}(12312J^{-1})\bar{2} \\
&= (J\bar{2}\bar{1}\bar{3}\bar{2}\bar{1})2\bar{3}\bar{2}\bar{1}(\bar{3}\bar{2}\bar{3}\bar{2}\bar{3})123\bar{2}(12312\bar{1}J^{-1}) \\
&= (J\bar{2}\bar{1}\bar{3}\bar{2}\bar{1})(2\bar{3}\bar{2}\bar{1}2\bar{3}\bar{2}123\bar{2})(12312\bar{1}J^{-1}) \\
&= (J\bar{2}\bar{1}\bar{3}\bar{2}\bar{1})123\bar{2}\bar{1}2\bar{3}\bar{2}\bar{1}(12312\bar{1}J^{-1}) \\
&= J\bar{2}\bar{1}\bar{2}1212\bar{1}J^{-1}.
\end{aligned}$$

The last line is the identity using $\text{br}_4(R_1, R_2)$. Therefore $\text{br}_4(R_2, P^{-5})$.

Now consider (2). Recall (see Remark 3.4) that $S \in \text{PU}(2, 1)$ satisfies

$$(16) \quad SR_1S^{-1} = R_1, \quad SR_2S^{-1} = R_3, \quad SR_3S^{-1} = R_3^{-1}R_2R_3.$$

First, we simplify and then we use $S^{-1}R_3S = R_2$ and $S^{-1}R_1R_2R_1^{-1}S = R_1R_2R_3R_2^{-1}R_1^{-1}$.

$$\begin{aligned}
(\bar{3}\bar{2}\bar{1}\bar{S}3\bar{3}\bar{2}\bar{1}\bar{S}3)(S123\bar{3}S123\bar{3}) &= \bar{3}\bar{2}\bar{1}\bar{S}\bar{2}\bar{1}(\bar{S}3S)12S12 \\
&= \bar{3}\bar{2}\bar{1}\bar{S}(\bar{2}\bar{1}212)S12 \\
&= \bar{3}\bar{2}\bar{1}(\bar{S}12\bar{1}S)12 \\
&= \bar{3}\bar{2}\bar{1}123\bar{2}\bar{1}12.
\end{aligned}$$

All the terms in the last line cancel.

For (3) we argue similarly using the braid relations.

$$\begin{aligned}
&(Q^{-3}\bar{2}12Q^{-3}\bar{2}12)(Q^3\bar{2}\bar{1}2Q^3\bar{2}\bar{1}2) \\
&\stackrel{\text{br}_4(1,2)}{=} (\bar{3}\bar{2}\bar{1})^2\bar{3}(\bar{2}\bar{1}\bar{2}12)(\bar{3}\bar{2}\bar{1})^2\bar{3}(\bar{2}\bar{1}\bar{2}1212)3(123)^2(\bar{2}\bar{1}212)3(123)^2\bar{2}\bar{1}2 \\
&\stackrel{\text{br}_3(1,3)}{=} (\bar{3}\bar{2}\bar{1})^2\bar{3}1\bar{2}\bar{1}\bar{3}\bar{2}\bar{1}\bar{3}\bar{2}(\bar{1}\bar{3}131)2312312\bar{1}3(123)^2\bar{2}\bar{1}2 \\
&\stackrel{\text{br}_3(2,3)}{=} (\bar{3}\bar{2}\bar{1})^2\bar{3}1\bar{2}\bar{1}\bar{3}\bar{2}\bar{1}(\bar{3}\bar{2}\bar{3}\bar{2}\bar{3})12312\bar{1}3(123)^2\bar{2}\bar{1}2 \\
&\stackrel{\text{br}_4(1,2)}{=} (\bar{3}\bar{2}\bar{1})^2\bar{3}1\bar{2}\bar{1}\bar{3}(\bar{2}\bar{1}212)312\bar{1}3(123)^2\bar{2}\bar{1}2 \\
&\stackrel{\text{br}_3(1,3)}{=} \bar{2}(2\bar{3}\bar{2})\bar{1}\bar{3}\bar{2}(\bar{1}\bar{3}1)\bar{2}(\bar{1}\bar{3}1)2(\bar{1}\bar{3}1)231(2\bar{3}\bar{2})\bar{1}2 \\
&\stackrel{\text{br}_3(2,3)}{=} \bar{2}(\bar{3}\bar{2}\bar{3})\bar{1}(\bar{3}\bar{2}\bar{3})\bar{1}(\bar{3}\bar{2}\bar{3})\bar{1}(\bar{3}\bar{2}\bar{3})1(\bar{3}\bar{2}\bar{3})1(\bar{3}\bar{2}\bar{3})1(\bar{3}\bar{2}\bar{3})\bar{1}2.
\end{aligned}$$

The last line is the identity using $\text{br}_7(R_1, R_3^{-1}R_2R_3)$.

Now consider (4). Recall that

$$(17) \quad Q_{1/2}R_1Q_{1/2}^{-1} = R_1R_2R_1^{-1}, \quad Q_{1/2}R_2Q_{1/2}^{-1} = R_3, \quad Q_{1/2}R_1Q_{1/2}^{-1} = R_3^{-1}R_1R_3.$$

First use $Q_{1/2}^2 = R_1R_2R_3$ to write $Q_{1/2}^{-3}R_1R_2R_3 = Q_{1/2}^{-1}$. Then use $Q_{1/2}^{-1}R_3Q_{1/2} = R_2$ and finally $Q_{1/2}^{-1}R_1R_2R_1R_2^{-1}R_1^{-1}Q_{1/2} = R_1R_3^{-1}R_2R_3R_1^{-1}$:

$$\begin{aligned} (Q_{1/2}^{-3}123\bar{2}\bar{1}Q_{1/2}^{-3}123\bar{2}\bar{1})(Q_{1/2}^312\bar{3}\bar{2}\bar{1}Q_{1/2}^312\bar{3}\bar{2}\bar{1}) &= \bar{Q}_{1/2}\bar{2}\bar{1}\bar{Q}_{1/2}3Q_{1/2}12Q_{1/2}12\bar{3}\bar{2}\bar{1} \\ &= \bar{Q}_{1/2}\bar{2}\bar{1}212Q_{1/2}12\bar{3}\bar{2}\bar{1} \\ &= \bar{Q}_{1/2}121\bar{2}\bar{1}Q_{1/2}12\bar{3}\bar{2}\bar{1} \\ &= 1\bar{3}23\bar{1}12\bar{3}\bar{2}\bar{1}. \end{aligned}$$

The last line is the identity using $\text{br}_3(R_2, R_3)$.

Part (5)

$$\begin{aligned} (P^{-1}1P^{-2}1)(P^2\bar{1}P^2\bar{1}) &= P^{-2}1\bar{3}23\bar{1}P^2\bar{1} \\ &= \bar{3}\bar{1}31\bar{3}\bar{1}. \end{aligned}$$

The last line is the identity using $\text{br}_3(R_1, R_3)$.

□

Since we know the order of P , Q or the other symmetry in each case, we combine Lemmas 5.5 and 5.6 to obtain the following corollary.

Corollary 5.7. (1) *In σ_5 : The natural presentation implies*

- $\text{br}_2(P^5, R_2R_3^{-1}R_2^{-1}R_1R_2R_3R_2^{-1})$;
- $\text{br}_6(R_2, R_3^{-1}R_2^{-1}R_1^{-1}R_2R_3R_2^{-1}R_1R_2R_3)$.

(2) *In $\mathbf{E}_2$: Let R_1 , R_2 and R_3 satisfy the braid relations in the natural presentation for $\mathbf{E}_2$ groups, and let S have the conjugation relations of equation (16). Then we have*

- $\text{br}_2(SR_1R_2R_3, R_2^{-1}R_1R_2)$, in particular, $\text{br}_2((R_1R_2R_3)^3, R_2^{-1}R_1R_2)$;
- $\text{br}_6(R_1R_2R_1^{-1}, R_3)$.

(3) *In $\bar{\mathbf{H}}_1$: The natural presentation implies*

- $\text{br}_2((R_1R_2R_3)^3, R_2^{-1}R_1R_2R_3R_2^{-1}R_1^{-1}R_2)$;
- $\text{br}_{14}(R_2^{-1}R_1R_2, R_3^{-1}R_1R_2R_1^{-1}R_3)$.

(4) *In $\mathbf{H}_2$: Let R_1 , R_2 and R_3 satisfy the braid relations in the natural presentation for $\mathbf{H}_2$ groups and the conjugation relations in equation (17). Then we have*

- $\text{br}_2(Q_{1/2}^3, R_2^{-1}R_1R_2)$, in particular $\text{br}_2((R_1R_2R_3)^3, R_2^{-1}R_1R_2)$;
- $\text{br}_{10}(R_1R_2R_3R_2^{-1}R_1^{-1}, R_2)$.

Proof. In each case, we use Lemma 5.5 with $C = R_1R_2R_3$.

(1) We set $A = R_2$ and $B = P^{-5}$. We have $P^{-5}R_2P^5 = R_3^{-1}R_2^{-1}R_1^{-1}R_2R_3R_2^{-1}R_1R_2R_3$ and so

$$2\bar{3}\bar{2}123\bar{2} = (123)(\bar{3}\bar{2}\bar{1}2\bar{3}\bar{2}123)\bar{2} = C(BA^{-1}B^{-1})A^{-1}.$$

This proves the first part. For the second, observe P has order 30 and so P^5 has order 6.

- (2) We write Γ for the group generated by R_1, R_2 and R_3 and relations given by the natural presentation. There is a unique $\sigma \in \text{Aut}(\Gamma)$ that satisfies

$$\sigma(R_1) = R_1, \sigma_1(R_2) = R_3, \sigma_1(R_3) = R_3^{-1}R_2R_3,$$

and $\sigma^3 = \text{Id}$. Now consider the corresponding morphism $\varphi : \mathbb{Z}/3\mathbb{Z} \rightarrow \text{Aut}(\Gamma)$, where $\varphi(k) = \sigma^k$, and write $\Gamma' = \Gamma \rtimes_{\varphi} \mathbb{Z}/3\mathbb{Z}$ for the corresponding semi-direct product.

Finally, we set $R'_j = (R_j, 0)$ and $S' = (0, 1) \in \Gamma'$, and consider $A = R'_1 R'_2 R_1'^{-1}$, $B = R_3'^{-1} R_2'^{-1} R_1'^{-1} S'^{-1}$. Then we have

$$BAB^{-1} = (S' R'_1 R'_2 R_3')^{-1} R'_1 R'_2 R_1'^{-1} (S' R'_1 R'_2 R_3') = R'_3$$

and so

$$R_2'^{-1} R'_1 R'_2 = (R'_1 R'_2 R_3') (R_3'^{-1}) R'_1 R_2'^{-1} R_1'^{-1} = C(BAB^{-1})A^{-1}.$$

Since S' and $R'_1 R'_2 R_3'$ commute and $R'_1 R'_2 R_3'$ has order 6, we see that $S' R'_1 R'_2 R_3'$ has order 6. Hence part (2) of Lemma 5.5 implies $\text{br}_6(R'_1 R'_2 R_1'^{-1}, R_3')$, which gives $\text{br}_6(R_1 R_2 R_1^{-1}, R_3)$ after projecting onto the first factor of the semi-direct product.

Also, part (1) of Lemma 5.5 gives $\text{br}_2(S' R'_1 R'_2 R_3', R_2'^{-1} R'_1 R'_2)$, i.e. $S' R'_1 R'_2 R_3'$ commutes with $R_2'^{-1} R'_1 R'_2$. Now S' has order 3 and $R_1 R_2 R_3$ has order 6, so $(S' R'_1 R'_2 R_3')^3 = (R'_1 R'_2 R_3')^3$, so $(R'_1 R'_2 R_3')^3$ also commutes with $R_2'^{-1} R'_1 R'_2$, hence the first part.

- (3) We set $A = R_2^{-1} R_1 R_3$ and $B = Q^{-3}$. We have $Q^{-3} R_2^{-1} R_1 R_3 Q^3 = R_3^{-1} R_1 R_2 R_1^{-1} R_3$. This means that

$$\bar{2}123\bar{2}\bar{1}2 = (123)(\bar{3}1\bar{2}\bar{1}3)(\bar{2}\bar{1}2) = C(BA^{-1}B^{-1})A^{-1}.$$

This proves the first part. For the second part, observe that Q has order 42 and so Q^3 has order 14.

- (4) The main difficulty is similar to the one in case (2), and it comes from the fact that the symmetry $Q_{1/2}$ is not in the group generated by R_1, R_2 and R_3 . Once again, we construct the symmetry inside a semi-direct product constructed group-theoretically from the group given by the natural presentation.

Start by noting that equation (17) defines an automorphism α of the group F defined by the $\mathbf{H}_2$ natural presentation. To see that the map $R_1 \mapsto R_1 R_2 R_1^{-1}$, $R_2 \mapsto R_3$, $R_3 \mapsto R_3^{-1} R_1 R_3$ induces a homomorphism $\alpha : F \rightarrow F$, one needs to verify that $\text{br}_3(3, \bar{3}13)$, $\text{br}_3(\bar{3}13, 12\bar{1})$, $\text{br}_5(12\bar{1}, 3)$, $\text{br}_5(12\bar{1}, \bar{3}13 \cdot 3 \cdot \bar{3}13)$. The only non-trivial verification is the fact that $\text{br}_5(1, \bar{3}23)$ implies $\text{br}_5(3, 12\bar{1})$, and this follows from an easy computation using $\text{br}_3(1, 3)$ and $\text{br}_3(2, 3)$.

One easily verifies that $\alpha : F \rightarrow F$ is bijective, so it is an automorphism of F . Explicit computation shows that α^2 is given by the conjugation in F by $R_1 R_2 R_3$, so α has order 30, and α^{15} has order 2. We denote by $\varphi : \mathbb{Z}/2\mathbb{Z} \rightarrow \text{Aut}(F)$ the corresponding morphism, and $F' = F \rtimes_{\varphi} \mathbb{Z}/2\mathbb{Z}$.

In F' , we have a symmetry playing the same role as $Q_{1/2}$, so from this point on we ignore the fact that $Q_{1/2}$ is not in F .

We set $A = R_3$ and $B = Q_{1/2}^{-3}$. Moreover, $Q_{1/2}^{-3}R_3Q_{1/2}^3 = R_3^{-1}R_1R_2R_1^{-1}R_3$ and so

$$\bar{2}12 = (123)(\bar{3}1\bar{2}\bar{1}3)(\bar{3}) = C(BA^{-1}B^{-1})A^{-1}.$$

This proves the first part. Finally, $Q_{1/2}$ has order 30 and so $Q_{1/2}^3$ has order 10, which proves the second part. $\square$

We need to treat σ_1 and σ_{10} separately, since they do not fit into the general framework described above.

Proposition 5.8. *For σ_1 : If $JR_iJ^{-1} = R_{i+1}$, $br_6(R_2, R_3)$ and P^8 then the braid relations $br_4(R_1, R_3^{-1}R_2R_3)$, $br_3(R_1, R_3^{-1}R_2^{-1}R_3R_2R_3)$ and $br_3(R_1, R_2R_3R_2R_3^{-1}R_2^{-1})$ are equivalent.*

Proof. Since $P = R_1J$ has order 8 we see that $R_1R_2R_3 = P^3 = P^{-5} = R_3^{-1}R_2^{-1}R_1^{-1}R_3^{-1}R_2^{-1}J$. Thus

$$\begin{aligned} 1(23\bar{2})1(23\bar{2})\bar{1}(2\bar{3}\bar{2})\bar{1}(2\bar{3}\bar{2}) &= (\bar{3}\bar{2}\bar{1}\bar{3}\bar{2}J)\bar{2}123\bar{2}\bar{1}2(\bar{J}23123)(2\bar{3}\bar{2}) \\ &= \bar{3}\bar{2}\bar{1}\bar{3}\bar{2}\bar{3}231\bar{3}\bar{2}\bar{3}231(232\bar{3}\bar{2}) \\ &= (\bar{3}\bar{2})(\bar{1}(\bar{3}\bar{2}\bar{3}23)1(\bar{3}\bar{2}\bar{3}23)1(\bar{3}\bar{2}\bar{3}23))(23). \end{aligned}$$

Hence $br_4(R_1, R_2R_3R_2^{-1})$ and $br_3(R_1, R_3^{-1}R_2^{-1}R_3R_2R_3)$ are equivalent. Similarly

$$\begin{aligned} 1(232\bar{3}\bar{2})1(23\bar{2}\bar{3}\bar{2})\bar{1}(23\bar{2}\bar{3}\bar{2}) &= (\bar{3}\bar{2}\bar{1}\bar{3}\bar{2}J)2\bar{3}\bar{2}123\bar{2}(\bar{J}23123)23\bar{2}\bar{3}\bar{2} \\ &= \bar{3}\bar{2}\bar{1}\bar{3}\bar{2}\bar{3}\bar{1}\bar{3}\bar{2}\bar{3}1\bar{3}\bar{2}\bar{3}1(232\bar{3}\bar{2}\bar{3}\bar{2}) \\ &= (\bar{3}\bar{2})(\bar{1}(\bar{3}\bar{2}\bar{3})\bar{1}(\bar{3}\bar{2}\bar{3})1(\bar{3}\bar{2}\bar{3})1(\bar{3}\bar{2}\bar{3}))(23). \end{aligned}$$

Hence $br_3(R_1, R_2R_3R_2R_3^{-1}R_2^{-1})$ and $br_4(R_1, R_3^{-1}R_2R_3)$ are equivalent. Finally, since

$$P^2R_1P^{-2} = R_1(R_2R_3R_2^{-1})R_1^{-1}, \quad P^2(R_3^{-1}R_2R_3)P^{-2} = R_1$$

we see that $br_4(R_1, R_2R_3R_2^{-1})$ and $br_4(R_1, R_3^{-1}R_2R_3)$ are equivalent. $\square$

Proposition 5.9. *For σ_{10} : If $JR_iJ^{-1} = R_{i+1}$, $br_5(R_2, R_3)$ and P^5 then*

- $br_2(R_1, R_3^{-1}R_2^{-1}R_3R_2R_3)$;
- $br_3(R_1, R_3^{-1}R_2R_3)$.

Proof. Since $P = R_1J$ has order 5, we have

$$R_1^{-1}R_3^{-1}R_2^{-1} = J(R_1J)^{-3}J^{-1} = J(R_1J)^2J^{-1} = R_2R_3J^{-1}$$

and so

$$\begin{aligned} \bar{1}(\bar{3}\bar{2}\bar{3}23)1(\bar{3}\bar{2}\bar{3}23) &= (23\bar{J})\bar{3}(J\bar{3}\bar{2})\bar{3}\bar{2}\bar{3}23 \\ &= 23\bar{2}\bar{3}\bar{2}\bar{3}\bar{2}\bar{3}23. \end{aligned}$$

The last line is the identity since $br_5(R_2, R_3)$. Similarly

$$\begin{aligned} \bar{1}(\bar{3}\bar{2}\bar{3})\bar{1}(\bar{3}\bar{2}\bar{3})1(\bar{3}\bar{2}\bar{3}) &= (23\bar{J})3\bar{1}\bar{3}(J\bar{3}\bar{2})\bar{3}\bar{2}\bar{3} \\ &= 23\bar{2}\bar{3}\bar{2}\bar{3}\bar{2}\bar{3}\bar{2}\bar{3}. \end{aligned}$$

Again, the last line is the identity since $\text{br}_5(R_2, R_3)$. $\square$

6. COMMENSURABILITY INVARIANTS

6.1. Adjoint trace fields. The basic commensurability invariant we will use is the adjoint trace field, i.e. the field generated by the traces of $\text{Ad}(\gamma)$, for all group elements γ . Because our groups preserve a Hermitian form, this is simply the field generated by $|\text{Tr} \gamma|^2$, where γ runs over all group elements. It is well-known that this field is finitely generated, and is a commensurability invariant (see section 2.5 of [22], Proposition (12.2.1) of [6] or [33]).

In order to compute these fields explicitly, we first use an upper bound given by Proutsevitch's trace formula, i.e. Theorems 4 and 10 of [35] (see also section 17.2 of [22]). A convenient formulation is given in Corollary 5.9 of [29], which gives the following:

Proposition 6.1. *For every triangle group generated by three reflections of order p , the field of traces $\mathbb{Q}(\text{TrAd } \Gamma)$ is totally real, contained in $\mathbb{Q}(|\rho|^2, |\sigma|^2, |\tau|^2, \rho\sigma\tau, \bar{\rho}\bar{\sigma}\bar{\tau}, a)$, where $a = e^{2\pi i/p}$.*

We have phrased this in terms of non-symmetric triangle groups, but the symmetric case is of course a special case obtained by taking $\rho = \sigma = \tau$.

In fact, for all groups we consider, the upper bound in Proposition 6.1 can be simplified. Indeed, $|\rho|^2, |\sigma|^2, |\tau|^2$ are all rational except for the Thompson groups of type $\mathbf{H}_2$, where $|\sigma|^2 = (3 + \sqrt{5})/2$, but the latter is then contained in $\mathbb{Q}(\rho\sigma\tau)$. Moreover, all number fields we consider are Galois, so the complex conjugate $\bar{\rho}\bar{\sigma}\bar{\tau}$ is also contained in $\mathbb{Q}(\rho\sigma\tau)$. We then get the following result.

Proposition 6.2. *If Γ is any of the sporadic, Thompson or Mostow lattices, the field $\mathbb{Q}(\text{TrAd } \Gamma)$ is contained in $\mathbb{Q}(\rho\sigma\tau, a)$.*

This gives an upper bound for the adjoint trace field, which is given by the real subfield of $\mathbb{Q}(\rho\sigma\tau, a)$. In fact we check that this upper bound is sharp. In order to do this, one needs to compute a few explicit traces, see the formulas in section 3.2. We also use

$$\text{Tr}(R_3 R_2 R_1) = 3 - |\rho|^2 - |\sigma|^2 - |\tau|^2 - u^3 \bar{\rho}\bar{\sigma}\bar{\tau},$$

since $|\text{Tr}(R_3 R_2 R_1)|^2$ often gives a generator of the adjoint trace field. In particular, since $\text{Tr}(R_1) = u^2 + 2\bar{u}$ where $a = u^3$, we get:

Proposition 6.3. *Let $\mu = 3 - |\rho|^2 - |\sigma|^2 - |\tau|^2$. The field $\mathbb{Q}(|a + 2|^2, |\mu - \bar{a}\rho\sigma\tau|^2)$ is contained in $\mathbb{Q}(\text{TrAd } \Gamma)$.*

The values of $|\text{Tr}(R_3 R_2 R_1)|^2$ for all sporadic and Thompson triangle lattices are given in Tables 6.1 and 6.2. Note that $|\mu - \bar{a}\rho\sigma\tau|^2$ generates the adjoint trace field for most groups, i.e. all but $\mathcal{T}(5, \mathbf{H}_2)$. For that group, $|a + 2|^2$ gives a generator.

From these values, it is a bit cumbersome (but not really difficult) to check that for each of the lattices we consider, the lower bound given by Proposition 6.3 has the same degree as the upper bound given by the real subfield of $\mathbb{Q}(\rho\sigma\tau, a)$, see Proposition 6.2.

τ	p	$ \text{Tr}(R_3 R_2 R_1) ^2$
σ_1	3	$3(11 + 2\sqrt{6})$
	4	$3(21 + 4\sqrt{2})$
	6	$3(31 + 2\sqrt{6})$
$\overline{\sigma}_4$	3	$(19 + 3\sqrt{21})/2$
	4	$17 + 3\sqrt{7}$
	5	λ
	6	$(49 + 3\sqrt{21})/2$
	8	$(34 + 15\sqrt{2} + 3\sqrt{14})/2$
	12	$(34 + 15\sqrt{3} + 3\sqrt{7})/2$
σ_5	2	$(7 + 3\sqrt{5})/2$
	3	$17 + 3\sqrt{5}$
	4	$(24 + 9\sqrt{3} + 3\sqrt{15})/2$
σ_{10}	3	$4(3 + \sqrt{5})$
	4	$(45 + 17\sqrt{5})/2$
	5	$(57 + 23\sqrt{5})/2$
	10	$39 + 16\sqrt{5}$

TABLE 6.1. Values of traces giving a generator for the adjoint trace field. In the table, λ is a generator for $\mathbb{Q}(\sqrt{14}\sqrt{5} + \sqrt{5})$, as can be seen from the fact that $\sqrt{14}\sqrt{5} + \sqrt{5} = (50\lambda^3 - 2040\lambda^2 + 18414\lambda - 18538)/2403$.

T	p	$ \text{Tr}(R_3 R_2 R_1) ^2$	$ \text{Tr}(R_1) ^2$
S₂	3	$3 + \sqrt{5}$	
	4	$(6 + \sqrt{3} + \sqrt{15})/2$	
	5	$-2\alpha_{15} + \alpha_{15}^2 + \alpha_{15}^3$	
E₂	4	$8 + 4\sqrt{3}$	
	6	16	
	12	$8 + 4\sqrt{3}$	
$\overline{\mathbf{H}}_1$	2	$2 + 2\cos(2\pi/7)$	
	7	$(1 + 2\cos(2\pi/7))^2$	
H₂	2	$(5 + \sqrt{5})/2$	
	3	$-1 - 5\alpha_{15} + 2(\alpha_{15}^2 + \alpha_{15}^3)$	
	5	$6 + 2\sqrt{5}$	
	10	$5 + 2\sqrt{5}$	
$\overline{\mathbf{H}}_2$	5	1	$4 + \sqrt{5}$

TABLE 6.2. Values of traces giving a generator for the adjoint trace field, for Thompson lattices. In the table, α_n stands for $2\cos(2\pi/n)$. We list $|\text{Tr}(R_1)|^2$ only if the first column does not already generate the adjoint trace field.

The adjoint trace fields for sporadic, Thompson and Mostow lattices are listed in the appendix. Those in the Appendix (section A.9) can be obtained in section 17.3 of Mostow's original paper [22], or more efficiently by converting the groups into hypergeometric monodromy groups using equation (21) and applying Lemma (12.5) of [6].

6.2. Signature spectrum and non-arithmeticity index. Let Γ be a lattice of $\mathrm{PU}(2, 1)$, and assume $k = \mathbb{Q}(\mathrm{TrAd} \Gamma)$ is a totally real number field. It follows from the discussion in section 12 in [6] that, up to complex conjugation, Γ is contained in a unique k -group whose real points give a group isomorphic to $\mathrm{PU}(2, 1)$. Moreover, k is the smallest possible number field with that property.

For the lattices considered in this paper, the k -structure is obvious, since they are all contained in the integer points $\mathrm{SU}(H, \mathcal{O}_{\mathbb{L}})$ of groups the form $G = \mathrm{SU}(H)$, where H is a Hermitian matrix with entries in $\mathbb{L}$, where $\mathbb{L}$ is a CM-field with maximal totally real subfield given by k . In particular, the automorphisms of $\mathbb{L}$ commute with complex conjugation, they all preserve k , and they come in complex conjugate pairs $\{\varphi_1, \overline{\varphi}_1, \dots, \varphi_r, \overline{\varphi}_r\}$, where φ_i and $\overline{\varphi}_i$ induce the same automorphism of k , but for $i \neq j$, φ_i and φ_j have different restrictions to k . Note also that the restriction of the automorphisms φ_j to k give all the automorphisms of k .

For every automorphism φ of k which is the restriction of some φ_j as above, the Galois conjugate group G^φ is given by $\mathrm{SU}(H^{\varphi_j})$, where H^{φ_j} is obtained from H by applying φ_j to every entry of H . Note that this is only well-defined up to complex conjugation.

The following arithmeticity criterion is well known (we often refer to this statement as the Mostow/Vinberg arithmeticity criterion), see section 4 of [22] or section 12 of [6].

Proposition 6.4. *Γ is arithmetic if and only if for every non-trivial automorphism φ of k , G^φ preserves a definite Hermitian form.*

This suggests a way to measure how far a given lattice is from being arithmetic, see the following definition.

Definition 6.5. *Let Γ be as above.*

- (1) *The signature spectrum of Γ is the set of signatures (p_i, q_i) of the Hermitian form preserved by G^{φ_i} , where φ_i ranges over all automorphisms of $k = \mathbb{Q}(\mathrm{TrAd} \Gamma)$.*
- (2) *The non-arithmeticity index of Γ is the number of non-trivial automorphisms φ of k such that G^φ preserves an indefinite Hermitian form.*

Note that the signature spectrum is not completely well-defined, since $\mathrm{SU}(H) = \mathrm{SU}(\lambda H)$ for any real number $\lambda \neq 0$ (in particular one could take $\lambda < 0$), but this is really the only ambiguity. Observe also that the signature spectrum clearly determines the non-arithmeticity index.

Now the key observation is that a lattice in $\mathrm{SU}(2, 1)$ acts irreducibly on $\mathbb{C}^3$, so it preserves a unique Hermitian form (up to scaling). This is of course also true for the Galois conjugates of a given lattice. Since a subgroup of finite index in a lattice is also a lattice, we get that the non-arithmeticity index is a commensurability invariant. For future reference, we summarize this discussion in the statement Proposition 6.6.

Proposition 6.6. *Let Γ be as above, and let $\Gamma' \subset \Gamma$ be a subgroup of finite index. Then Γ and Γ' have the same signature spectrum, and the same non-arithmeticity index.*

6.3. Commensurators. In order to refine the partition into commensurability classes, it is also useful to consider properties of the commensurator proved by Margulis. Recall that the commensurator of Γ in G is the group $C_G(\Gamma)$ of elements $g \in G$ such that $\Gamma \cap g\Gamma g^{-1}$ has finite index in both Γ and $g\Gamma g^{-1}$. The following result follows from Theorem IX.1.13 in [20].

Theorem 6.7. *Let Γ be a non-arithmetic lattice in $G = \mathrm{PU}(2, 1)$. Then Γ has finite index in $C_G(\Gamma)$, in particular $C_G(\Gamma)$ is a lattice.*

In section 7.2, we will use the following reformulation of Theorem 6.7 (obtained from the latter by taking Γ to be the common commensurator of Γ_1 and Γ_2).

Proposition 6.8. *Suppose Γ_1 and Γ_2 are commensurable non-arithmetic lattices in $\mathrm{PU}(2, 1)$. Then there exists a lattice Γ and a $g \in \mathrm{PU}(2, 1)$ such that Γ_1 and $g\Gamma_2 g^{-1}$ are both finite index subgroups of Γ .*

7. COMMENSURABILITY RELATIONS

The goal of this section is to prove Theorem 1.4. The detailed statement, giving explicit representatives for each commensurability class, is given in the form of a table, see Table A.2. The end result is that among 2-dimensional non-arithmetic Deligne-Mostow, sporadic and Thompson lattices, there are precisely 22 commensurability classes.

7.1. Some isomorphisms between triangle groups.

7.1.1. Non-rigid non-equilateral triangle groups.

Proposition 7.1. (1) *For every p , the group $\mathcal{T}(p, \mathbf{S}_1)$ is conjugate to the sporadic triangle group $\mathcal{S}(p, \bar{\sigma}_4)$.*

(2) *For every p , the group $\mathcal{T}(p, \mathbf{E}_1)$ is conjugate to the sporadic triangle group $\mathcal{S}(p, \sigma_1)$.*

Proof: We start with the proof of part (1).

We write R_1, R_2, R_3 for standard generators of a sporadic triangle group for $\bar{\sigma}_4$. Recall that this is characterized up to conjugation by $\mathrm{Tr} R_1 J = \bar{\sigma}_4$, and that this implies that $R_1 J$ has order 7,

$$(18) \quad (R_i R_j)^2 = (R_j R_i)^2$$

i.e. $\mathrm{br}(R_i, R_j) = 4$ (for $i \neq j$), and

$$(19) \quad R_1(R_2 R_3 R_2^{-1})R_1 = (R_2 R_3 R_2^{-1})R_1(R_2 R_3 R_2^{-1})$$

$$(20) \quad R_1(R_3^{-1} R_3 R_2)R_1 = (R_3^{-1} R_3 R_2)R_1(R_3^{-1} R_3 R_2)$$

i.e. $\mathrm{br}(R_1, R_2 R_3 R_2^{-1}) = \mathrm{br}(R_1, R_3^{-1} R_3 R_2) = 3$.

Now consider the group elements $M_1 = R_3^{-1} R_2 R_3$, $M_2 = R_2 R_3 R_2^{-1}$ and $M_3 = R_1$. These three matrices generate the sporadic group, since

$$R_1 = M_3, \quad R_2 = M_2^{-1} M_1 M_2, \quad R_3 = M_1 M_2 M_1^{-1}.$$

We claim that the three isometries M_1, M_2, M_3 (can be simultaneously conjugated to) generate an $\mathbf{S}_1$ group. Let $\mathbf{m}_1 = R_3^{-1}\mathbf{n}_2$, $\mathbf{m}_2 = R_2\mathbf{n}_3$ and $\mathbf{m}_3 = \mathbf{n}_1$ be polar vectors to the mirrors of R_1, R_2 and R_3 . Then the parameters associated with $\langle M_1, M_2, M_3 \rangle$ are

$$\begin{aligned}\rho' &= (u^2 - \bar{u}) \frac{\langle \mathbf{m}_2, \mathbf{m}_1 \rangle}{\|\mathbf{m}_2\| \|\mathbf{m}_1\|} = -\bar{u}^2 \bar{\sigma}_4, \\ \sigma' &= (u^2 - \bar{u}) \frac{\langle \mathbf{m}_3, \mathbf{m}_2 \rangle}{\|\mathbf{m}_3\| \|\mathbf{m}_2\|} = \tau' = (u^2 - \bar{u}) \frac{\langle \mathbf{m}_1, \mathbf{m}_3 \rangle}{\|\mathbf{m}_1\| \|\mathbf{m}_3\|} = u(\bar{\sigma}_4 - \sigma_4^2).\end{aligned}$$

Since $\bar{\sigma}_4 - \sigma_4^2 = 1$ we see that $|\rho'| = \sqrt{2}$, $|\sigma'| = |\tau'| = 1$ and $\rho'\sigma'\tau' = -\bar{\sigma}_4 = (1 + i\sqrt{7})/2$. These are the same parameters as for $\mathbf{S}_1$. Therefore the two groups are conjugate.

The proof of part (2) is similar. In that case the sporadic group is defined by $\tau = -1 + i\sqrt{2}$, $(R_1 R_2 R_3)$ has order 8 and

$$\text{br}(R_j, R_k) = 6, \text{ br}(R_1, R_2 R_3 R_2^{-1}) = 4, \text{ br}(R_1, R_2 R_3 R_2 R_3^{-1} R_2^{-1}) = 3.$$

Explicit generators of type $\mathbf{E}_1$ are given by

$$M_1 = R_2 R_3 R_2 R_3^{-1} R_2^{-1}, \quad M_2 = R_1 R_3^{-1} R_2^{-1} R_3 R_2 R_3 R_1^{-1}, \quad M_3 = R_1,$$

and these generate the same group because

$$R_1 = M_3, \quad R_2 = (M_3^{-1} M_2^{-1} M_3) M_1 (M_3^{-1} M_2 M_3), \quad R_3 = M_1 (M_3^{-1} M_2 M_3) M_1^{-1}.$$

Setting $\mathbf{m}_1 = R_2 R_3 \mathbf{n}_2$, $\mathbf{m}_2 = R_1 R_3^{-1} R_2^{-1} \mathbf{n}_3$ and $\mathbf{m}_3 = \mathbf{n}_1$ we have

$$\rho' = 4\bar{\tau}^2 - 11\tau + \tau^4 = i\sqrt{2}, \quad \sigma' = -\bar{u}(2\tau - \bar{\tau}^2) = \bar{u}, \quad \tau' = -u(2\tau - \bar{\tau}^2) = u.$$

□

The Thompson groups with $p = 2$ were shown to be commensurable to explicit Mostow groups in [43]. In a similar vein, we have the following.

Proposition 7.2. (1) *The group $\mathcal{T}(7, \bar{\mathbf{H}}_1)$ is conjugate to the Mostow group $\Gamma(7, 9/14)$.*
 (2) *The group $\mathcal{T}(5, \bar{\mathbf{H}}_2)$ is conjugate to the Mostow group $\Gamma(5, 7/10)$.*

Proof:

- (1) In the group $\Gamma(7, 9/14)$, one verifies that $M_1 = R_1$, $M_2 = (R_2 R_1^{-1} R_2)^{-1} R_1 (R_2 R_1^{-1} R_2)$ and $M_3 = R_3$ are conjugate to standard generators for $\mathcal{T}(7, \bar{\mathbf{H}}_1)$. Writing $\mathbf{m}_1 = \mathbf{n}_1$, $\mathbf{m}_2 = (R_2 R_1^{-1} R_2)^{-1} \mathbf{n}_1$, $\mathbf{m}_3 = \mathbf{n}_3$ and $\tau^3 = 1$, we find

$$\rho' = e^{6\pi i/7}(-1 - i\sqrt{7})/2, \quad \sigma' = e^{6\pi i/7} \bar{\tau}, \quad \tau' = \tau.$$

Since $\tau^3 = 1$ this means

$$|\rho'| = \sqrt{2}, \quad |\sigma'| = |\tau'| = 1, \quad \rho'\sigma'\tau' = e^{-2\pi i/7}(-1 - i\sqrt{7})/2$$

as required.

One can check that

$$R_2 = M_3(M_2 M_3 M_2^{-1} M_1)^{-3} M_1^{-1} M_2 M_1,$$

which shows that M_1, M_2, M_3 generate the same group as R_1, R_2, R_3 .

- (2) In the group $\Gamma(5, 7/10)$, one verifies that $M_1 = R_1$, $M_2 = R_2^{-1}R_3R_2$ and $M_3 = R_2$ are conjugate to standard generators for $\mathcal{T}(5, \bar{\mathbf{H}}_2)$. Indeed, writing $\mathbf{m}_1 = \mathbf{n}_1$, $\mathbf{m}_2 = R_2^{-1}\mathbf{n}_3$, $\mathbf{m}_3 = \mathbf{n}_2$ and arguing as above:

$$\rho' = -\bar{u}\bar{\tau} - u^2\tau^2, \quad \sigma' = -u^3\bar{\tau}, \quad \tau' = -u\bar{\tau}.$$

In $\Gamma(5, 7/10)$ we have $u = e^{2\pi i/15}$ and $\tau = -e^{-i\pi/3}$. Hence $|\rho'| = 2\cos(\pi/5)$, $|\sigma'| = |\tau'| = 1$ and $\rho'\sigma'\tau' = -e^{2\pi i/5} - e^{4\pi i/5}$.

Moreover, R_1 , R_2 and R_3 generate the corresponding Mostow group (the clearly generate the subgroup generated by R_1 , R_2 and R_3 , and that subgroup is equal to the group generated by R_1 and J , because 3 does not divide the order of R_1J , which is 4, see [37] for instance).

□

7.1.2. Rigid non-equilateral triangle groups. In this section we explain some relations of rigid triangle groups (Thompson groups with parameters $\mathbf{S}_2$, $\mathbf{S}_3$, $\mathbf{S}_4$ or $\mathbf{E}_3$, see Table 3.4) with other triangle groups.

Proposition 7.3. *For every $p = 5, 6, 7, 8, 9, 10, 12, 18$, $\mathcal{T}(p, \mathbf{S}_3)$ is the Livné group with parameter p .*

Proof: This follows from changes of parameters as in [17]. More specifically, in $\mathcal{T}(p, \mathbf{S}_3)$, the complex reflections R_1 , $R_1R_2R_1^{-1}$, R_3 are generators that pairwise have braid length 3. This allows us to identify as Mostow groups, see section A.9 for more details. □

Proposition 7.4. *The lattices $\mathcal{T}(p, \mathbf{S}_4)$, $p = 4, 5, 6, 8, 12$ are isomorphic to arithmetic Mostow groups.*

Proof: In the Mostow group generated by R_1 and J , the elements $J(R_1R_2)^{-1}$ and R_1R_3J are complex reflections, with known angle (see [22], or [28]).

Note also that $J(R_1R_2)^{-1}$ commutes with R_2 , since

$$JR_2^{-1}R_1^{-1} \cdot R_2 \cdot R_1R_2J^{-1} \cdot R_2^{-1} = JR_1J^{-1}R_2^{-1} = Id,$$

where we have used the braid relation $\text{br}(R_1, R_2) = 3$. Similarly, one checks that R_1R_3J commutes with R_3 .

Also, we have

$$(R_1JR_2^{-1}R_1^{-1})^2 = R_1(JR_2^{-1})^2R_1^{-1} = (JR_2^{-1})^2.$$

This implies that the braid length $\text{br}(R_1, J(R_1R_2)^{-1})$ is either 2 or 4, but one easily checks that these two complex reflections do not commute, so $\text{br}(R_1, J(R_1R_2)^{-1}) = 4$.

Above, we have used the fact that R_1 commutes with $(JR_2^{-1})^2$, which is true since

$$[R_1, (JR_2^{-1})^2] = R_1JR_2^{-1}J^{-1}J^{-1}R_2^{-1}R_1^{-1}R_2JJR_2J^{-1} = R_1R_3^{-1}R_1^{-1}R_3^{-1}R_1R_3 = Id.$$

In the five Mostow groups listed in Table 7.1, the corresponding elements (either $J(R_1R_2)^{-1}$ or $(R_1R_3J)^{-1}$ depending on the order of generators) have the same order as R_j , and the elements in the second column are (2, 3, 4) triangle group generators. □

Mostow group	(2,3,4)-generators
$\Gamma(4, 1/4)$	$R_1, J(R_1 R_2)^{-1}, R_2$
$\Gamma(5, 1/10)$	$R_1, J(R_1 R_2)^{-1}, R_2$
$\Gamma(6, 0)$	$R_1, J(R_1 R_2)^{-1}, R_2$
$\Gamma(8, 1/8)$	$R_1, (R_1 R_3 J)^{-1}, R_3$
$\Gamma(12, 1/4)$	$R_1, (R_1 R_3 J)^{-1}, R_3$

TABLE 7.1. We write these 5 Mostow groups as (2,3,4)-triangle groups, by considering the triple of reflections in the second column.

Proposition 7.5. *For every $p > 2$, $\mathcal{T}(p, \mathbf{S}_5)$ is isomorphic to the group $\mathcal{S}(p, \sigma_{10})$.*

Proof: This follows from two successive changes of generators as in [17]. One checks that the 2, 3, 5; 5 triangle groups are the same as 3, 5, 5; 2 triangle groups, which are the same as 5, 5, 5; 3 triangle groups. The latter correspond to sporadic σ_{10} groups. $\square$

For the special case $p = 10$ in Proposition 7.5, we have an extra isomorphism.

Proposition 7.6. *The group $\mathcal{T}(10, \mathbf{S}_5)$ is isomorphic to $\mathcal{T}(10, \mathbf{H}_2)$ (and also to $\mathcal{S}(10, \sigma_{10})$).*

Proof: In the group $\mathcal{T}(10, \mathbf{H}_2)$, one considers $M = ((R_1 R_2 R_3)^2 R_2^{-1} R_1 R_2)^{-3}$, which is a complex reflection with angle $\pi/5$.

One checks by explicit computation that the matrices R_2, M, R_3 generate a (2, 3, 5)-triangle group, i.e. the group $\mathcal{T}(10, \mathbf{S}_5)$. $\square$

7.2. Determination of the number of commensurability classes. In this section, we summarize the current lower bound on the number of commensurability classes of non-arithmetic lattices in $\mathrm{PU}(2, 1)$. A lot of this can be done by using only rough commensurability invariants, i.e. cocompactness, adjoint trace fields, and non-arithmeticity index (see section 6.2).

The table for Mostow and Deligne-Mostow groups show that there are at most 13 commensurability classes of Deligne-Mostow lattices in $\mathrm{PU}(2, 1)$. As mentioned above, the results in [7], [18] and [21] imply that there are in fact precisely 9 commensurability classes there.

7.2.1. Cocompact groups. Among the non-arithmetic Thompson lattices, the groups $\mathcal{T}(5, \mathbf{S}_2)$ and $\mathcal{T}(3, \mathbf{H}_2)$ cannot be commensurable to any Deligne-Mostow lattice nor to any sporadic group, but in principle they could be commensurable with each other. We will now exclude that possibility:

Proposition 7.7. *The groups $\Gamma_1 = \mathcal{T}(5, \mathbf{S}_2)$ and $\Gamma_2 = \mathcal{T}(3, \mathbf{H}_2)$ are not commensurable.*

We give an argument that relies on the following volume estimate for lattices containing complex reflections of large order.

Proposition 7.8. *Let Γ be a discrete subgroup of $\mathrm{PU}(2, 1)$ containing a complex reflection A of order $n \geq 7$. Let m_A denote the mirror of A , and let Γ_A denote the stabilizer of m_A in Γ . Then*

$$\mathrm{Vol}(\Gamma \backslash \mathbf{H}_{\mathbb{C}}^2) \geq \frac{\pi(1 - 2 \sin \frac{\pi}{n})}{2n \sin \frac{\pi}{n}} \mathrm{Vol}(m_A / \Gamma_A).$$

Moreover, if there is no $g \in \Gamma$ such that $g(m_A)$ is orthogonal to m_A , then

$$\mathrm{Vol}(\Gamma \backslash \mathbf{H}_{\mathbb{C}}^2) \geq \frac{\pi(1 - 2 \sin \frac{\pi}{n})}{n \sin \frac{\pi}{n}} \mathrm{Vol}(m_A / \Gamma_A).$$

Proof. Normalize in the Siegel domain so that $m_A = \{(\zeta, v, u) \in \mathbf{H}_{\mathbb{C}}^2 : \zeta = 0\}$. Then, a point $(\zeta_1, v_1, u_1) \in \mathbf{H}_{\mathbb{C}}^2$ a distance δ from L_A satisfies

$$\cosh^2 \left(\frac{\delta}{2} \right) = \frac{|\zeta_1|^2 + u_1}{u_1}.$$

In other words $|\zeta_1|^2 = u_1(\cosh(\delta) - 1)/2$. Let $N(\delta)$ be the δ -neighborhood of m_A . Then

$$\begin{aligned} \mathrm{Vol}(N(\delta) / \Gamma_A) &= \frac{1}{n} \int_u \int_v \int_x \int_y \frac{4}{u^3} du dv dx dy \\ &= \frac{1}{n} \int_u \int_v \frac{4\pi |\zeta_1|^2}{u^3} du dv \\ &= \frac{2\pi(\cosh \delta - 1)}{n} \int_u \int_v \frac{1}{u^2} du dv \\ &= \frac{2\pi(\cosh \delta - 1)}{n} \mathrm{Vol}(m_A / \Gamma_A) \end{aligned}$$

Using Theorem 5.2 of [16], we see that if there is no $g \in \Gamma$ so that $g(m_A)$ is orthogonal to m_A and if $\cosh(\delta) \geq \frac{1}{2 \sin(\pi/n)}$ then $N(\delta)$ does not intersect its images under elements of $\Gamma - \Gamma_A$. For such a δ we have

$$\begin{aligned} \mathrm{Vol}(\Gamma \backslash \mathbf{H}_{\mathbb{C}}^2) &\geq \mathrm{Vol}(N(\delta) / \Gamma_A) \\ &\geq \frac{\pi(1 - 2 \sin \frac{\pi}{n})}{n \sin \frac{\pi}{n}} \mathrm{Vol}(m_A / \Gamma_A). \end{aligned}$$

□

Proof: (of Proposition 7.7) We argue by contradiction, let us assume they are commensurable. Then by Proposition 6.8 we may assume that both of them are contained in a common lattice Γ .

Recall that Γ_1 has Euler characteristic $133/300$, and Γ_2 has Euler characteristic $26/75$. Let us denote by d_j the index of Γ_j in Γ . Since $133/300d_1 = 26/75d_2$, and 133 and 26 are relatively prime, we must have $d_1 = 133d'_1$ and $d_2 = 26d'_2$ for some integers d'_1, d'_2 . In other words, the Euler characteristic of Γ is of the form $1/300d$ for some integer d .

We denote by $R_j^{(k)}$ the j -th standard generator of Γ_k . Consider the $\mathbb{C}$ -Fuchsian subgroup F_1 of Γ_1 , generated by $R_1^{(1)}$ and $R_2^{(1)}$. Since these two reflections braid with length 5, F_1 is central extension of a $(2, 5, 5)$ -triangle group, with center generated by a complex reflection $(R_1^{(1)} R_1^{(2)})^2$ of order 10 (see Proposition 2.5).

The commensurator Γ contains a possibly larger Fuchsian subgroup $F \supset F_1$, which is a central extension of either a $(2, 5, 5)$ -triangle group or a $(2, 4, 5)$ -triangle group, with center of order n , where n is a multiple of 10.

Suppose first that there is no $g \in \Gamma$ such that $g(m_1)$ is orthogonal to m_1 . Then by Proposition 7.8,

$$\text{Vol}(\Gamma \setminus \mathbf{H}_{\mathbb{C}}^2) \geq V\pi \frac{1 - 2 \sin \frac{\pi}{n}}{n \cdot \sin \frac{\pi}{n}} \geq V\pi \frac{1 - 2 \sin \frac{\pi}{10}}{10 \cdot \sin \frac{\pi}{10}},$$

where $V = \pi/10$ is the co-area of the $(2, 4, 5)$ -triangle group.

It follows that

$$\chi(\Gamma \setminus \mathbf{H}_{\mathbb{C}}^2) = \frac{3}{8\pi^2} \text{Vol}(\Gamma \setminus \mathbf{H}_{\mathbb{C}}^2) \geq \frac{3}{400} \frac{\sqrt{5} - 1}{2} > \frac{1}{216},$$

which is impossible, since Γ has Euler characteristic $1/300d_1 < 1/216$.

Hence we assume there exists a $g \in \Gamma$ such that $g(m_1)$ is orthogonal to m_1 . In that case, gRg^{-1} gives an element of order 10 acting on m_F as a rotation of order 10, where $R = (R_1^{(1)} R_1^{(2)})^2$ generates the center (i.e. pointwise stabilizer) of m_F .

Now F is a central extension of either a $(2, 4, 5)$ or a $(2, 5, 5)$ -triangle group, but these triangle groups contain no element of order 10, which is a contradiction. $\square$

7.2.2. Non-cocompact groups. Among the non-cocompact lattices we constructed, there are three pairs of non-arithmetic lattices with the same rough commensurability invariants; the following result shows that these pairs are actually in different commensurability classes.

Proposition 7.9. (1) *The groups $\mathcal{S}(3, \sigma_1)$ and $\mathcal{S}(6, \sigma_1)$ are not commensurable.*
 (2) *The groups $\mathcal{S}(4, \sigma_5)$ and $\mathcal{T}(4, \mathbf{S}_2)$ are not commensurable.*
 (3) *The groups $\Gamma(6, 1/6)$ and $\mathcal{T}(4, \mathbf{E}_2)$ are not commensurable.*

Our proof of Proposition 7.9 relies on studying the cusps of the groups $\mathcal{S}(3, \sigma_1)$, $\mathcal{S}(4, \sigma_5)$ and $\Gamma(6, 1/6)$. An alternative proof of part (3) follows from work of Kappes and Möller [18], because of Proposition 7.10.

Proposition 7.10. *The group $\mathcal{T}(4, \mathbf{E}_2)$ is (conjugate to) a subgroup of index 3 in the Deligne-Mostow group $\Gamma_{\mu, \Sigma}$ for $\mu = (3, 3, 5, 6, 7)/12$, $\Sigma = \mathbb{Z}_2$*

Proof: A presentation for Deligne-Mostow groups without three-fold symmetry is given by Pasquinelli in [31]. The group $\Gamma_{\mu, \Sigma}$ is $(3, 4, 4)$ in her notation. The most convenient presentation is the alternative one given at the end of Section 4.1 of [31], which for this group is

$$\left\langle A, B, R : \begin{array}{l} A^4, B^3, R^4, (BRA)^{24}, (ARBR)^6, \\ \text{br}_4(B, R), \text{br}_2((BRA)^{-2}, R), \text{br}_2(A, B) \end{array} \right\rangle.$$

One then easily verifies that writing $R_1 = A$, $R_2 = R$, $R_3 = BRB^{-1}$ gives a group isomorphic to $\mathcal{T}(4, \mathbf{E}_2)$. This is done by showing each of these two presentations implies the other. Furthermore, one can extend this isomorphism by writing $S = B$ where S is the extra symmetry from Remark 3.4 (3). Using the fact that B has order 3, the relation $\text{br}_4(B, R)$ is equivalent to $\text{br}_3(R, BRB^{-1})$. Then using this relation, the relation $\text{br}_2((BRA)^{-2}, R)$ is equivalent to $\text{br}_4(A, R)$. Together with $\text{br}_2(A, B)$ this immediately gives $\text{br}_4(A, BRB^{-1})$ and $\text{br}_4(A, R(BRB^{-1})R^{-1})$. Moreover, B commutes with $ARBR$ and so $(ARBR)^6$ is equivalent to $(AR(BRB^{-1}))^6$. Finally, using $\text{br}_4(A, R)$, $\text{br}_4(B, R)$ and $\text{br}_2(A, B)$ we have $(BRA)^{24} = (BRAR)^{24}(\bar{A}BR\bar{B}AR)^{-12}$. Thus $(BRA)^{24}$ is equivalent to $(\bar{A}BR\bar{B}AR)^{12}$, where we also use $(BRAR)^6$. $\square$

We remark that exactly the same argument shows that $\mathcal{T}(3, \mathbf{E}_2)$ is (conjugate to) a subgroup of index 3 in the Deligne-Mostow group $\Gamma_{\mu, \Sigma}$ for $\mu = (1, 1, 3, 3, 4)/6$, and $\Sigma = \mathbb{Z}_2$ acting by swapping just the factors $1/6$.

For each of the groups $\mathcal{S}(3, \sigma_1)$, $\mathcal{S}(4, \sigma_5)$ and $\Gamma(6, 1/6)$, the mirrors m_1 and m_2 of R_1 and R_2 intersect in a point p_{12} on $\partial\mathbf{H}_{\mathbb{C}}^2$. This means that the group $\langle R_1, R_2 \rangle$ is a parabolic group. We give an upper bound on the largest cusp neighborhood that is precisely invariant under this parabolic group. We discuss volume bounds for cusp groups in a more general context, since we believe that these results could have wider applications.

We begin by considering the general structure of parabolic groups Γ_{∞} generated by two complex reflections A and B both with order p . The center $Z(\Gamma_{\infty})$ of $\Gamma_{\infty} = \langle A, B \rangle$ is generated by a vertical translation, which we denote by T . We need to consider the following three cases (see Proposition 2.5).

- (1) $p = 3$: In this case, A and B braid with length 6, Γ_{∞} is a central extension of the rotation subgroup of a $(3, 3, 3)$ triangle group and $T = (AB)^3$.
- (2) $p = 4$: In this case, A and B braid with length 4, Γ_{∞} is a central extension of the rotation subgroup of a $(2, 4, 4)$ triangle group and $T = (AB)^2$.
- (3) $p = 6$: In this case, A and B braid with length 3, Γ_{∞} is a central extension of the rotation subgroup of a $(2, 3, 6)$ triangle group and $T = (AB)^3$.

We will actually use some more detailed information about Γ_{∞} . This information can be deduced from the presentations of Heisenberg lattices in Section 7.1 Dekimpe [5]. Instead of giving details of this, we give a geometrical proof instead. To that end, let Λ_{∞} denote the subgroup of Γ_{∞} consisting of Heisenberg translations. Note that Λ_{∞} is a central extension of the translation subgroup of the corresponding triangle group.

Since $Z(\Gamma_{\infty})$ is a group of Heisenberg translations, it is contained in Λ_{∞} , and in fact $Z(\Gamma_{\infty}) = Z(\Lambda_{\infty})$. Moreover, the commutator subgroup of Λ_{∞} is a finite index subgroup of $Z(\Gamma_{\infty})$. Our next goal is to determine that index in each of the three cases.

Lemma 7.11. *Let $\Gamma_{\infty} = \langle A, B \rangle$ be as above. Then, the Heisenberg lattice Λ_{∞} has index p in Γ_{∞} and is generated by $A^{-1}B$, AB^{-1} and T . The commutator subgroup of Λ_{∞} is all of $Z(\Gamma_{\infty})$ when $p = 3$ or 6 and has index 2 in $Z(\Gamma_{\infty})$ when $p = 4$.*

Proof: This lemma could be deduced from Dekimpe [5] as indicated above.

Let Γ_* denote the rotation subgroup of one of the above Euclidean triangle groups, and let Λ_* be its translation subgroup. In each case $\langle A^{-1}B, AB^{-1} \rangle$ projects to Λ_* and the index of Λ_* in Γ_* is p . This may easily be checked using Euclidean geometry, for example by normalizing the projection of A to be $z \mapsto e^{2\pi i/p}z$ and the projection of B to be $z \mapsto e^{2\pi i/p}z + 1$. This gives an obvious isomorphism between Λ_* and the discrete ring $\mathbb{Z}[e^{2\pi i/p}]$ (recall that $p = 3, 4$ or 6). Thus Λ_∞ has index p in Γ_∞ and is generated by $A^{-1}B$, AB^{-1} and T .

Each commutator $[C, D]$ in Λ_∞ is a vertical translation whose length is proportional to the area of the parallelogram spanned by the projections of C and D in $\mathbb{C}$; see page 446 of [26] for example. With the above normalization, it is clear that the parallelogram spanned by the projections of $A^{-1}B$ and AB^{-1} has the smallest area among any positive area parallelograms spanned by elements of Λ_* . Hence, the commutator subgroup of Λ_∞ is generated by $[A^{-1}B, AB^{-1}]$. It remains to write this commutator as a power of the generator T of $Z(\Gamma_\infty)$. We split this into three cases:

(1) $p = 3$: Since A and B have order 3, we see that

$$\begin{aligned} [A^{-1}B, AB^{-1}] &= (A^{-1}B)(AB^{-1})(B^{-1}A)(BA^{-1}) \\ &= A(ABABAB)A^{-1} = T. \end{aligned}$$

(2) $p = 4$: In this case A and B have order 4 and $(AB)^2 = (BA)^2$. Therefore

$$\begin{aligned} [A^{-1}B, AB^{-1}] &= (A^{-1}B)(AB^{-1})(B^{-1}A)(BA^{-1}) \\ &= A^2(ABAB)BABA^{-1} = A(AB)^4A^{-1} = T^2. \end{aligned}$$

(3) $p = 6$. In this case, A and B have order 6 and satisfy the classical braid relation. Therefore:

$$\begin{aligned} [A^{-1}B, AB^{-1}] &= (A^{-1}BA)B^{-1}B^{-1}(ABA^{-1}) \\ &= BAB^{-4}AB = (BAB)(BAB) = T. \end{aligned}$$

□

We now give a formula for the volumes of certain cusp neighborhoods associated to the group Γ_∞ . This follows the methods in [26].

Proposition 7.12. *Let Γ be a discrete subgroup of $\mathrm{SU}(2, 1)$ and let Γ_∞ be a parabolic subgroup of Γ fixing a point of $\partial\mathbf{H}_{\mathbb{C}}^2$. Let Λ_∞ be the maximal lattice of Heisenberg translations in Γ_∞ and let m be the index of Λ_∞ in Γ_∞ . Let T be a generator of $Z(\Gamma_\infty)$ and let q be the positive integer so that the shortest non-trivial commutator in Λ_∞ is T^q . Let C be any element of $\Gamma - \Gamma_\infty$. If B_∞ is any horoball that is precisely invariant under Γ_∞ in Γ , then*

$$\mathrm{Vol}(\Gamma_\infty \backslash B_\infty) \leq \frac{(3 - \mathrm{tr}(TCTC^{-1}))q}{2m}.$$

Proof. We construct a horoball B'_∞ that intersects its image under C , and so is not precisely invariant, and so that

$$\mathrm{Vol}(\Gamma_\infty \backslash B'_\infty) = \frac{(3 - \mathrm{tr}(TCTC^{-1}))q}{2m}.$$

Following the normalization in [26], we use the second Hermitian form (denoted by J_0 in [26]) and we suppose that Γ_∞ fixes q_∞ , which corresponds to $[1, 0, 0]^t$. Without loss of generality, suppose that $C(q_\infty) = q_o$ is the origin in Heisenberg coordinates, which corresponds to $[0, 0, 1]^t$. Let B'_∞ be a horoball based at q_∞ and consider its image $B'_o = C(B'_\infty)$ under C based at the point q_o . Suppose that the height of B'_∞ is chosen so that B'_∞ and B'_o are disjoint, but their boundaries intersect in a single point. If C has the form given in equation (1.3) of [26], this condition is precisely that the height of B'_∞ is $h = 2/|c|$. Again, following the normalization in [26], suppose that

$$T = \begin{pmatrix} 1 & 0 & it/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore, using the formula for $\text{Vol}(B'_\infty/\Gamma_\infty)$ given on page 446 of [26], we have:

$$\text{Vol}(\Gamma_\infty \backslash B'_\infty) = \frac{1}{2h^2} \cdot \frac{t^2 q}{m} = \frac{|c|^2 t^2 q}{8m}.$$

In order to express this in an invariant way, we want to write $|c|^2 t^2$ in terms of traces. Since we suppose that C sends q_∞ to q_o , this means that

$$CTC^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ |c|^2 it/2 & 0 & 1 \end{pmatrix}.$$

Hence $|c|^2 t^2/4 = 3 - \text{tr}(TCTC^{-1})$, which gives the result. $\square$

We want to apply this result in the case where Γ is one of $\mathcal{S}(3, \sigma_1)$, $\mathcal{S}(4, \sigma_5)$ or $\Gamma(6, 1/6)$; the parabolic subgroup is $\Gamma_\infty = \langle R_1, R_2 \rangle$ and the map C is J . We have already found the integers m (which is p in each case) and q needed to apply the theorem. It remains to find $\text{tr}(TCTC^{-1})$.

Lemma 7.13. *Suppose that R_1 , R_2 , u and τ are as given in Section 3.1. If $R_1 R_2$ is parabolic then $|\tau|^2 = 2 - u^3 - \bar{u}^3$.*

Proof. The trace of $R_1 R_2$ is $u(2 - |\tau|^2) + \bar{u}^2$ and the intersection of the mirrors of R_1 and R_2 , denoted p_{12} , corresponds to a $\bar{u}^2$ -eigenvector $\mathbf{p}_{12}$. In order for $R_1 R_2$ to be parabolic, it must have a repeated eigenvalue $\bar{u}^2$ whose eigenspace is spanned by $\mathbf{p}_{12}$. In particular, the trace of $R_1 R_2$ is $u^4 + 2\bar{u}^2$. The result follows by solving for $|\tau|^2$ in

$$u(2 - |\tau|^2) + \bar{u}^2 = \text{tr}(R_1 R_2) = u^4 + 2\bar{u}^2.$$

$\square$

We will be interested in three cases:

- (1) $p = 3$: This means $u^3 + \bar{u}^3 = -1$ and $|\tau|^2 = 3$.
- (2) $p = 4$: This means $u^3 + \bar{u}^3 = 0$ and $|\tau|^2 = 2$.
- (3) $p = 6$: This means $u^3 + \bar{u}^3 = 1$ and $|\tau|^2 = 1$.

Proposition 7.14. *The center of $\Gamma_\infty = \langle R_1, R_2 \rangle$ is generated by a scalar multiple of*

$$T_{12} = \begin{pmatrix} 1 & 0 & (2+u^6)((u^5-u^2)\bar{\tau}+u^2\tau^2) \\ 0 & 1 & -(2+\bar{u}^6)((\bar{u}^5-\bar{u}^2)\tau+\bar{u}^2\bar{\tau}^2) \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof. We perform the calculation in each of the three cases.

(1) When $p = 3$ the center is generated by $(R_1 R_2)^3$. Just using $|\tau|^2 = 3$, we obtain:

$$(R_1 R_2)^3 = \begin{pmatrix} u^3 & 0 & \bar{u}^6(1-u^3)((u^5-u^2)\bar{\tau}+u^2\tau^2) \\ 0 & u^3 & -u^3(1-\bar{u}^3)((\bar{u}^5-\bar{u}^2)\tau+\bar{u}^2\bar{\tau}^2) \\ 0 & 0 & \bar{u}^6 \end{pmatrix}.$$

Using $u^3 + \bar{u}^3 = -1$, we see that $u^3 = \bar{u}^6$ and $1-u^3 = 2+u^6$, and so $(R_1 R_2)^3 = u^3 T_{12}$.

(2) When $p = 4$ the center is generated by $(R_1 R_2)^2$. Just using $|\tau|^2 = 2$, we obtain:

$$(R_1 R_2)^2 = \begin{pmatrix} -u^2 & 0 & \bar{u}^4((u^5-u^2)\bar{\tau}+u^2\tau^2) \\ 0 & -u^2 & u^2((\bar{u}^5-\bar{u}^2)\tau+\bar{u}^2\bar{\tau}^2) \\ 0 & 0 & \bar{u}^4 \end{pmatrix}.$$

Using $u^3 + \bar{u}^3 = 0$, we see that $-u^2 = \bar{u}^4$ and $1 = 2+u^6$, and so $(R_1 R_2)^2 = -u^2 T_{12}$.

(3) When $p = 6$ the center is generated by $(R_1 R_2)^3$. Just using $|\tau|^2 = 1$, we obtain:

$$(R_1 R_2)^3 = \begin{pmatrix} -u^3 & 0 & \bar{u}^6(1+u^3)((u^5-u^2)\bar{\tau}+u^2\tau^2) \\ 0 & -u^3 & u^3(1+\bar{u}^3)((\bar{u}^5-\bar{u}^2)\tau+\bar{u}^2\bar{\tau}^2) \\ 0 & 0 & \bar{u}^6 \end{pmatrix}.$$

Using $u^3 + \bar{u}^3 = 1$, we see that $-u^3 = \bar{u}^6$ and $1+u^3 = 2+u^6$, and so $(R_1 R_2)^3 = -u^3 T_{12}$.

□

Corollary 7.15. *Let $\Gamma_\infty = \langle R_1, R_2 \rangle$. If B_∞ is any horoball that is precisely invariant under Γ_∞ in Γ then*

$$\text{Vol}(\Gamma_\infty \backslash B_\infty) \leq \frac{q}{2p} \left| (2+u^6)((u^5-u^2)\bar{\tau}+u^2\tau^2) \right|^2$$

where $q = 1$ when $p = 3$ or 6 and $q = 2$ when $p = 4$.

Proof. We apply Proposition 7.12 with $C = J$. This means that

$$J T_{12} J^{-1} = T_{23} = \begin{pmatrix} 1 & 0 & 0 \\ (2+u^6)((u^5-u^2)\bar{\tau}+u^2\tau^2) & 1 & 0 \\ -(2+\bar{u}^6)((\bar{u}^5-\bar{u}^2)\tau+\bar{u}^2\bar{\tau}^2) & 0 & 1 \end{pmatrix}.$$

Hence

$$\text{tr}(T_{12} T_{23}) = 3 - \left| (2+u^6)((u^5-u^2)\bar{\tau}+u^2\tau^2) \right|^2.$$

The result follows since we know $m = p$ and $q = 1$ for $p = 3, 6$ and $q = 2$ for $p = 4$. □

On the other hand, we have the following lower bound for $\text{Vol}(\Gamma_\infty \backslash B_\infty)$:

Proposition 7.16. *Let Γ_1 be a lattice in $\mathrm{SU}(2, 1)$. Suppose that Γ_1 is an index d subgroup of a lattice Γ in $\mathrm{SU}(2, 1)$. Suppose that $\Gamma_1 \backslash \mathbf{H}_{\mathbb{C}}^2$ has only one cusp and let Γ_1^∞ be the corresponding parabolic subgroup. If B_1 is the largest horoball that is precisely invariant under Γ_1^∞ in Γ_1 then*

$$\mathrm{Vol}(\Gamma_1^\infty \backslash B_1) \geq d/4.$$

Proof. Let Γ_∞ be a parabolic subgroup of Γ , which we may and will assume contains Γ_1^∞ . Since $\Gamma_1 \backslash \mathbf{H}_{\mathbb{C}}^2$ has only one cusp, the index of Γ_1^∞ in Γ_∞ is the same as the index of Γ_1 in Γ , namely d . From Theorem 4.1 of [26] we know that there is a horoball B so that $\mathrm{Vol}(\Gamma_\infty \backslash B) \geq 1/4$. Clearly a horoball precisely invariant under Γ_∞ in Γ is also precisely invariant under Γ_1^∞ in Γ_1 , and hence $B \subset B_1$. Since Γ_1^∞ has index d in Γ_∞ , the corresponding covering $\Gamma_1^\infty \backslash B \rightarrow \Gamma_\infty \backslash B$ has degree d . This implies that

$$\mathrm{Vol}(\Gamma_1^\infty \backslash B_1) \geq \mathrm{Vol}(\Gamma_1^\infty \backslash B) = d \mathrm{Vol}(\Gamma_\infty \backslash B) \geq d/4.$$

□

Combining Corollary 7.15 and Proposition 7.16 we get the following bound on the index d of a lattice containing Γ :

$$d \leq \frac{2q}{p} \left| (2 + u^6)((u^5 - u^2)\bar{\tau} + u^2\tau^2) \right|^2.$$

For the groups we are interested in, this bound is:

(1) $\mathcal{S}(3, \sigma_1)$:

$$d \leq \frac{2}{3} \left| (2 + u^6)((u^5 - u^2)\bar{\tau} + u^2\tau^2) \right|^2 = 6 + 2\sqrt{6} < 11.$$

(2) $\mathcal{S}(4, \sigma_5)$:

$$d \leq \left| (2 + u^6)((u^5 - u^2)\bar{\tau} + u^2\tau^2) \right|^2 = \frac{7 + \sqrt{5} + 3\sqrt{3} + \sqrt{15}}{2} < 10.$$

(3) $\Gamma(6, 1/6)$:

$$d \leq \frac{1}{3} \left| (2 + u^6)((u^5 - u^2)\bar{\tau} + u^2\tau^2) \right|^2 = 2 + \sqrt{3} < 4.$$

Proof. (Proposition 7.9)

(1) Suppose that the groups $\mathcal{S}(3, \sigma_1)$ and $\mathcal{S}(6, \sigma_1)$ are commensurable. Then by Proposition 6.8 we may assume that they are contained in a common lattice Γ , say with indices d_1 and d_2 respectively. The corresponding orbifold Euler characteristics are:

$$\chi(\mathcal{S}(3, \sigma_1) \backslash \mathbf{H}_{\mathbb{C}}^2) = 2/9, \quad \chi(\mathcal{S}(6, \sigma_1) \backslash \mathbf{H}_{\mathbb{C}}^2) = 43/72.$$

Therefore the orbifold Euler characteristic of the commensurator Γ is

$$\frac{2}{9d_1} = \frac{43}{72d_2}.$$

That is, $43d_1 = 16d_2$ and so $d_1 \geq 16$. This contradicts the above bound of 11 on the index of any lattice containing $\mathcal{S}(3, \sigma_1)$.

- (2) Suppose the groups $\mathcal{S}(4, \sigma_5)$ and $\mathcal{T}(4, \mathbf{S}_2)$ are commensurable. Their orbifold Euler characteristics are:

$$\chi(\mathcal{S}(4, \sigma_5) \backslash \mathbf{H}_{\mathbb{C}}^2) = 17/36, \quad \chi(\mathcal{T}(4, \mathbf{S}_2) \backslash \mathbf{H}_{\mathbb{C}}^2) = 1/3.$$

Arguing as before, the index d_1 of $\mathcal{S}(4, \sigma_5)$ in its commensurator must be at least 17, contradicting the above bound of 10.

- (3) Suppose the groups $\Gamma(6, 1/6)$ and $\mathcal{T}(4, \mathbf{E}_2)$ are commensurable. Their orbifold Euler characteristics are:

$$\chi(\Gamma(6, 1/6) \backslash \mathbf{H}_{\mathbb{C}}^2) = 11/144, \quad \chi(\mathcal{T}(4, \mathbf{E}_2) \backslash \mathbf{H}_{\mathbb{C}}^2) = 17/32.$$

As before, d_1 must be at least 11, contradicting the above bound of 4.

□

A. COMBINATORIAL DATA, COMMENSURABILITY INVARIANTS, PRESENTATIONS

For each of the lattice considered in this paper (where our algorithm produces a fundamental domain), we list

- (1) The type of the triangle group, i.e. six braid lengths together with the order of $Q = R_1 R_2 R_3$. We will use the notation $a, b, c; d, e, f; g$ for

$$\text{br}(2, 3), \text{br}(3, 1), \text{br}(1, 2); \text{br}(1, \bar{3}23), \text{br}(1, 23\bar{2}), \text{br}(3, 12\bar{1}); o(Q),$$

where $o(Q)$ means the order of Q .

- (2) The orbifold Euler characteristic, and basic commensurability invariants (adjoint trace field, cocompactness, arithmeticity and non-arithmeticity index);
- (3) The values of the order p of reflections such that the group is a lattice. Values in parentheses indicate that our algorithm fails to give a fundamental domain for that group (see section 5.4 for details of how the algorithm fails in each case);
- (4) The rough structure of the invariant shell, in the form of a list of side representatives. Recall that $[k] a; b, c$ stands for a pyramid with a k -gon as its base, which occurs when $\text{br}(b, c) = k$;
- (5) A presentation in terms of generators and relations (for the sake of brevity and clarity, we write the braid relation $(ab)^{n/2} = (ba)^{n/2}$ as $\text{br}_n(a, b)$). We give slightly more relations than in the relations present in the natural presentations (see sections 5.6), so that the reader can reconstruct the fundamental domain from the presentation. More specifically, we list every braid relation that corresponds to the apex of some side representative in the domain. If there is truncation of that apex for some values of p , then we also list a power relation next to the corresponding braid relation.
- (6) A list of conjugacy classes of vertex stabilizers;
- (7) A list of singular points in the quotient, with the type of (cyclic) singularity.

Dashed tables list groups that we know to be lattices, but where our algorithm does not produce a fundamental domain. If that is the case, and we know an alternative description for the group that makes the algorithm work, we give it in the commensurability invariant table.

A.1. Sporadic σ_1 .

Triangle group type: 6,6,6; 4,4,4; 8

Lattice for $p = 3, 4, 6$.

Commensurability invariants:

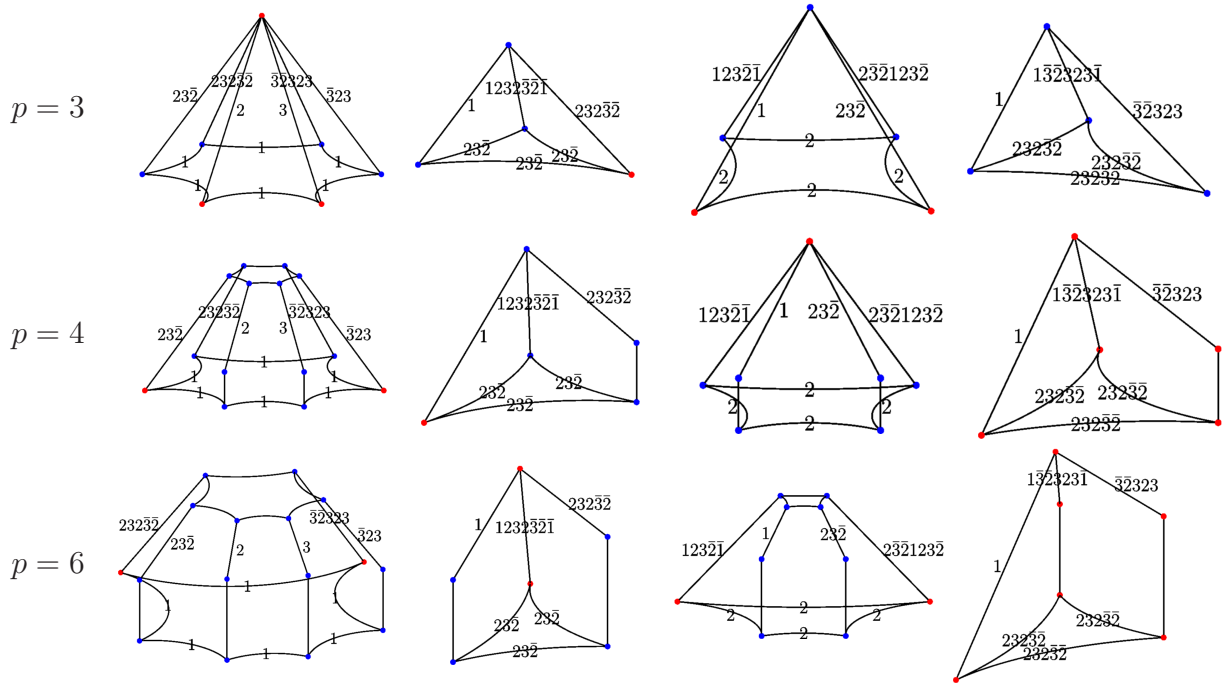
p	χ^{orb}	$\mathbb{Q}(\text{TrAd } \Gamma)$	CM field	C?	A?
3	2/9	$\mathbb{Q}(\sqrt{6})$	$\mathbb{Q}(i\sqrt{2}, i\sqrt{3})$	NC	NA(1)
4	7/16	$\mathbb{Q}(\sqrt{2})$	$\mathbb{Q}(i, \sqrt{2})$	NC	NA(1)
6	43/72	$\mathbb{Q}(\sqrt{6})$	$\mathbb{Q}(i\sqrt{2}, i\sqrt{3})$	NC	NA(1)

Presentations:

$$\left\langle R_1, R_2, R_3, J \mid R_1^p, J^3, (R_1 J)^8, R_3 = J R_2 J^{-1} = J^{-1} R_1 J, \right. \\ \left. \text{br}_6(R_1, R_2), (R_1 R_2)^{\frac{3p}{p-3}}, \text{br}_4(R_1, R_2 R_3 R_2^{-1}), (R_1 R_2 R_3 R_2^{-1})^{\frac{4p}{p-4}} \right. \\ \left. \text{br}_3(R_1, R_2 R_3 R_2 R_3^{-1} R_2^{-1}), \text{br}_3(R_1, R_3^{-1} R_2^{-1} R_3 R_2 R_3) \right\rangle$$

Combinatorics:

Triangle	$\#(P\text{-orb})$	Top trunc.	Top ideal
[6] 1; 2, 3	8	$p = 4, 6$	$p = 3$
[4] 2; 1, $23\bar{2}$	8	$p = 6$	$p = 4$
[3] $23\bar{2}$; 1, $23\bar{2}3\bar{2}$	8		$p = 6$
[3] $23\bar{2}3\bar{2}$; 1, $\bar{3}\bar{2}3\bar{2}3$	8		$p = 6$



Vertex stabilizers:

p	Vertex	Order	Nature	Vertex	Order	Nature
3	p_{12}	∞	Cusp			
	$p_{1,23\bar{2}}$	72	G_5			
	$p_{1,232\bar{3}\bar{2}}$	24	G_4	$p_{1,\bar{3}\bar{2}323}$	24	G_4
4	$p_{1,(12)^3}$	16	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$p_{1,(13)^3}$	16	$\mathbb{Z}_4 \times \mathbb{Z}_4$
	$p_{1,23\bar{2}}$	∞	Cusp			
	$p_{1,232\bar{3}\bar{2}}$	96	G_8	$p_{1,\bar{3}\bar{2}323}$	96	G_8
6	$p_{1,(12)^3}$	12	$\mathbb{Z}_6 \times \mathbb{Z}_2$	$p_{1,(13)^3}$	12	$\mathbb{Z}_6 \times \mathbb{Z}_2$
	$p_{1,(123\bar{2})^2}$	36	$\mathbb{Z}_6 \times \mathbb{Z}_6$	$p_{1,(1\bar{3}23)^2}$	36	$\mathbb{Z}_6 \times \mathbb{Z}_6$
	$p_{1,232\bar{3}\bar{2}}$	∞	Cusp	$p_{1,\bar{3}\bar{2}323}$	∞	Cusp

Singular points

p	Element	Type
3, 4, 6	J	$\frac{1}{3}(1, 2)$
	P	$\frac{1}{8}(1, 3)$
4, 6	$R_1 R_2$	$\frac{1}{3}(1, 1)$
6	$R_1 R_2 R_3 R_2^{-1}$	$\frac{1}{2}(1, 1) = A_1$

A.2. Sporadic $\bar{\sigma}_4$.

Triangle group type: 4,4,4; 3,3,3; 7

Lattice for $p = 3, 4, 5, 6, 8, 12$.

Commensurability invariants:

p	χ^{orb}	$\mathbb{Q}(\text{TrAd } \Gamma)$	CM field	C?	A?
3	2/63	$\mathbb{Q}(\sqrt{21})$	$\mathbb{Q}(i\sqrt{3}, i\sqrt{7})$	C	A
4	25/224	$\mathbb{Q}(\sqrt{7})$	$\mathbb{Q}(i, \sqrt{7})$	NC	NA(1)
5	47/280	$\mathbb{Q}(\sqrt{\frac{5+\sqrt{5}}{14}})$	$\mathbb{Q}(\zeta_5, i\sqrt{7})$	C	NA(2)
6	25/126	$\mathbb{Q}(\sqrt{21})$	$\mathbb{Q}(i\sqrt{3}, i\sqrt{7})$	NC	NA(1)
8	99/448	$\mathbb{Q}(\sqrt{2}, \sqrt{7})$	$\mathbb{Q}(\zeta_8, i\sqrt{7})$	C	NA(2)
12	221/1008	$\mathbb{Q}(\sqrt{3}, \sqrt{7})$	$\mathbb{Q}(\zeta_{12}, i\sqrt{7})$	C	NA(2)

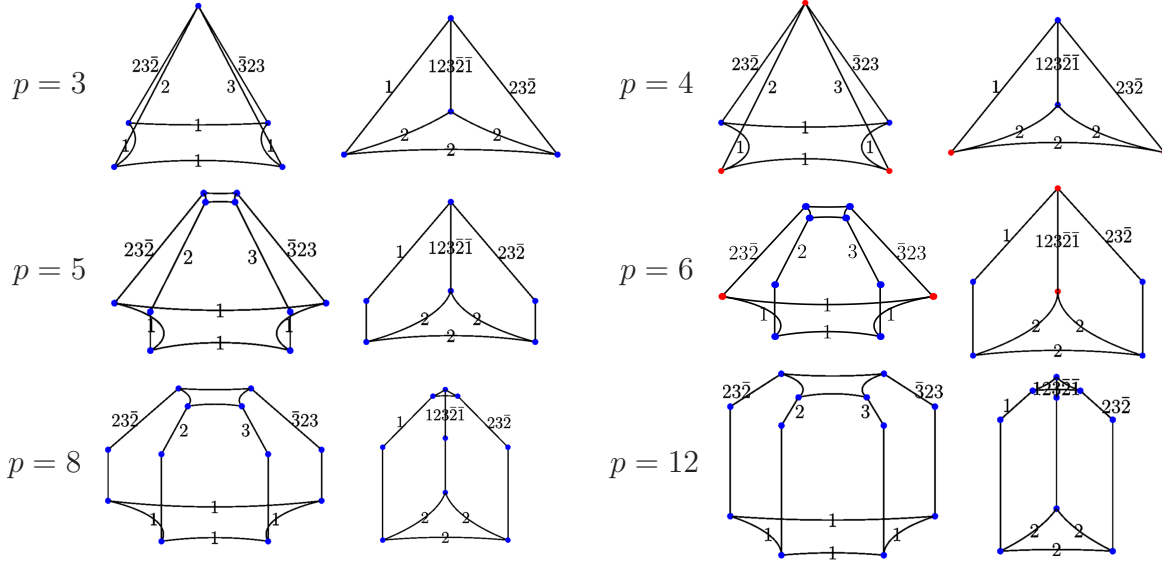
Presentations:

$$\left\langle R_1, R_2, R_3, J \mid R_1^p, J^3, (R_1 J)^7, R_3 = J R_2 J^{-1} = J^{-1} R_1 J, \right. \\ \left. \text{br}_4(R_1, R_2), (R_1 R_2)^{\frac{4p}{p-4}}, \text{br}_3(R_1, R_2 R_3 R_2^{-1}), (R_1 R_2 R_3 R_2^{-1})^{\frac{6p}{p-6}} \right\rangle$$

Combinatorics:

Triangle	$\#(P\text{-orb})$	Top trunc.	Top ideal
[4] 1; 2, 3	7	$p = 5, 6, 8, 12$	$p = 4$
[3] 2; 1, 23 $\bar{2}$	7	$p = 8, 12$	$p = 6$

Vertex stabilizers:



p	Vertex	Order	Nature	Vertex	Order	Nature
3	p_{12}	72	G_5			
	$p_{1,23\bar{2}}$	24	G_4			
4	p_{12}	∞	Cusp			
	$p_{1,23\bar{2}}$	96	G_8			
5	$p_{1,(12)^2}$	50	$\mathbb{Z}_5 \times \mathbb{Z}_{10}$	$p_{1,(13)^2}$	50	$\mathbb{Z}_5 \times \mathbb{Z}_{10}$
	$p_{1,23\bar{2}}$	600	G_{16}			
6	$p_{1,(12)^2}$	36	$\mathbb{Z}_6 \times \mathbb{Z}_6$	$p_{1,(13)^2}$	36	$\mathbb{Z}_6 \times \mathbb{Z}_6$
	$p_{1,23\bar{2}}$	∞	Cusp			
8	$p_{1,(12)^2}$	32	$\mathbb{Z}_8 \times \mathbb{Z}_4$	$p_{1,(13)^2}$	32	$\mathbb{Z}_8 \times \mathbb{Z}_4$
	$p_{1,(123\bar{2})^3}$	64	$\mathbb{Z}_8 \times \mathbb{Z}_8$			
12	$p_{1,(12)^2}$	36	$\mathbb{Z}_{12} \times \mathbb{Z}_3$	$p_{1,(13)^2}$	36	$\mathbb{Z}_{12} \times \mathbb{Z}_3$
	$p_{1,(123\bar{2})^3}$	48	$\mathbb{Z}_{12} \times \mathbb{Z}_4$			

Singular points:

p	Element	Type
3,4,5,6,8,12	J	$\frac{1}{3}(1, 2)$
3,4,5,6,8,12	P	$\frac{1}{7}(1, 3)$
3,4,5,6,8,12	$R_2 R_3 R_2^{-1} P^2$	A_1
5,6,8,12	$R_2 R_3$	A_1
8,12	$R_1 R_2 R_3 R_2^{-1}$	$\frac{1}{3}(1, 1)$
8,12	$R_1 R_2 R_3 R_2^{-1} R_1$	A_1

A.3. Sporadic σ_5 .

Triangle group type: 4,4,4; 5,5,5; 30

P^5 is a complex reflection

Lattice for $p = 2, 3, 4$.

Commensurability invariants:

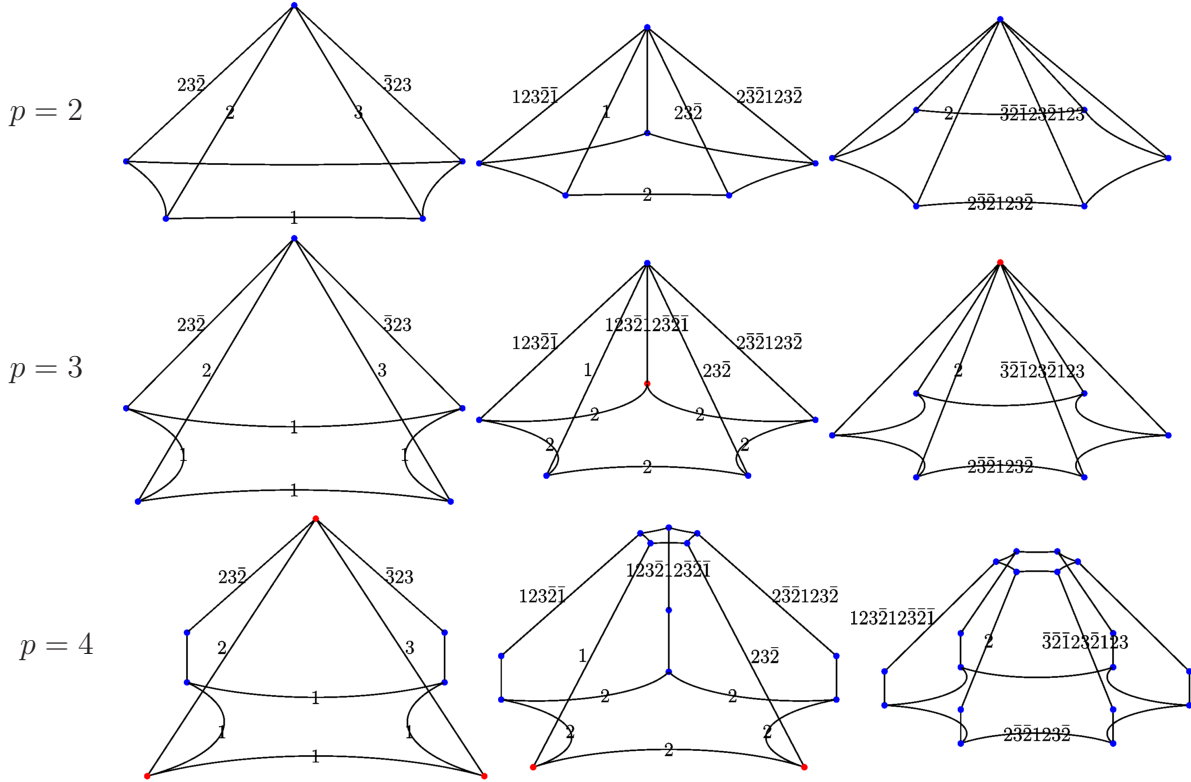
p	χ^{orb}	$\mathbb{Q}(\text{TrAd } \Gamma)$	CM field	C?	A?
2	1/45	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\sigma_5^3) = \mathbb{Q}(\omega \frac{\sqrt{5}-3i\sqrt{3}}{2})$	C	A
3	49/180	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\omega \frac{\sqrt{5}-3i\sqrt{3}}{2})$	NC	NA(1)
4	17/36	$\mathbb{Q}(\sqrt{3}, \sqrt{5})$	$\mathbb{Q}(i, \omega \frac{\sqrt{5}-3i\sqrt{3}}{2})$	NC	NA(3)

Presentations:

$$\left\langle R_1, R_2, R_3, J \mid R_1^p, J^3, (R_1 J)^{30}, R_3 = J R_2 J^{-1} = J^{-1} R_1 J, \text{br}_4(R_1, R_2), \text{br}_5(R_1, R_2 R_3 R_2^{-1}), \right. \\ \left. (R_1 R_2 R_3 R_2^{-1})^{\frac{10p}{3p-10}}, \text{br}_6(R_2, R_3^{-1} R_2^{-1} R_1^{-1} R_2 R_3 R_2^{-1} R_1 R_2 R_3), (R_2 \cdot R_3^{-1} R_2^{-1} R_1^{-1} R_2 R_3 R_2^{-1} R_1 R_2 R_3)^{\frac{3p}{p-3}} \right\rangle$$

Combinatorics:

Triangle	$\#(P\text{-orb})$	Top trunc.	Top ideal
[4] 1; 2, 3	30		$p = 4$
[5] 2; 1, $23\bar{2}$	30	$p = 4$	
[6] $2\bar{3}\bar{2}123\bar{2}$; 2, $\bar{3}\bar{2}\bar{1}23\bar{2}123$	5	$p = 4$	$p = 3$



Vertex stabilizers:

p	Vertex	Order	Nature
2	p_{12}	8	$G(4, 4, 2)$
	$p_{1,23\bar{2}}$	10	$G(5, 5, 2)$
	$p_{2,123\bar{2}123\bar{2}\bar{1}}$	72	$G(6, 1, 2)^{(*)}$
3	p_{12}	72	G_5
	$p_{1,23\bar{2}}$	360	G_{20}
	$p_{2,123\bar{2}123\bar{2}\bar{1}}$	∞	Cusp ^(*)
4	p_{12}	∞	Cusp
	$p_{1,(123\bar{2})^5}$	16	$\mathbb{Z}_4 \times \mathbb{Z}_4$
	$p_{2,(2 \cdot 123\bar{2}123\bar{2}\bar{1})^3}$	48	$\mathbb{Z}_{12} \times \mathbb{Z}_4^{(*)}$

Singular points:

p	Element	Type
2,3,4	J	$\frac{1}{3}(1, 2)$
2,3,4	P	$\frac{1}{5}(1, 2)$
4	123212321	A_1
4	$P^5\bar{2}$	A_1
4	$123\bar{2}$	$\frac{1}{5}(1, 3)$

A.4. Sporadic σ_{10} .

Triangle group type: 5,5,5; 3,3,3; 5

Lattice for $p = 3, 4, 5, 10$.

Commensurability invariants:

p	χ^{orb}	$\mathbb{Q}(\text{TrAd } \Gamma)$	CM field	C?	A?
3	1/45	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(i\sqrt{3}, \sqrt{5})$	C	A
4	3/32	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(i, \sqrt{5})$	C	A
5	1/8	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\zeta_5)$	C	A
10	13/100	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\zeta_5)$	C	A

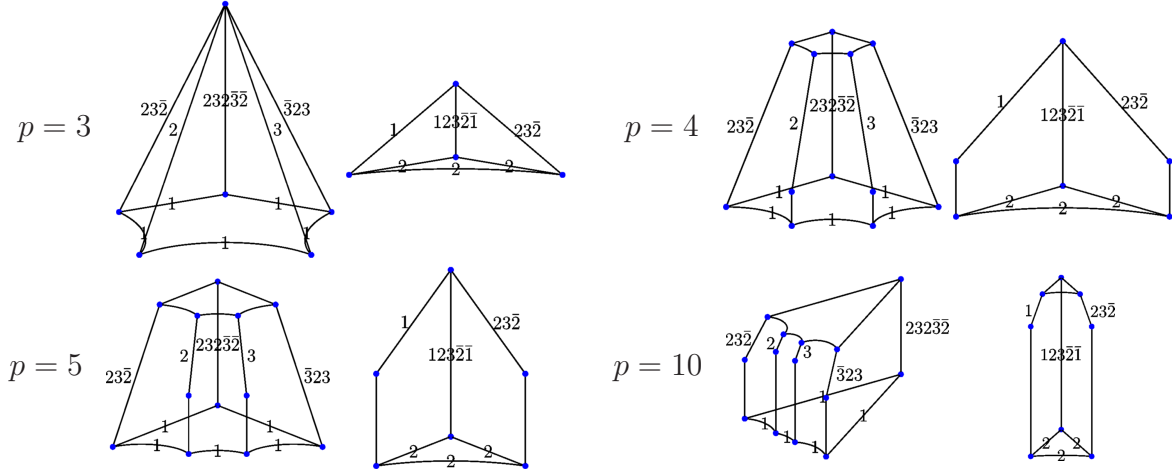
Presentations:

$$\left\langle R_1, R_2, R_3, J \mid R_1^p, J^3, (R_1 J)^5, R_3 = J R_2 J^{-1} = J^{-1} R_1 J, \right. \\ \left. \text{br}_5(R_1, R_2), (R_1 R_2)^{\frac{10p}{3p-10}}, \text{br}_3(R_1, R_2 R_3 R_2^{-1}), (R_1 R_2 R_3 R_2^{-1})^{\frac{6p}{p-6}} \right\rangle$$

Combinatorics:

Triangle	$\#(P\text{-orb})$	Top trunc.	Top ideal
[5] 1; 2, 3	5	$p = 4, 5, 10$	
[3] 2; 1, $23\bar{2}$	5	$p = 10$	

Vertex stabilizers:



p	Vertex	Order	Nature
3	p_{12}	360	G_{20}
	$p_{1,23\bar{2}}$	24	G_4
	$p_{1,\bar{3}\bar{2}323}$	9	$\mathbb{Z}_3 \times \mathbb{Z}_3$
4	$p_{1,(13)^5}$	16	$\mathbb{Z}_4 \times \mathbb{Z}_4$
	$p_{1,23\bar{2}}$	96	G_8
	$p_{1,\bar{3}\bar{2}323}$	16	$\mathbb{Z}_4 \times \mathbb{Z}_4$
5	$p_{1,(13)^5}$	10	$\mathbb{Z}_5 \times \mathbb{Z}_2$
	$p_{1,23\bar{2}}$	600	G_{16}
	$p_{1,\bar{3}\bar{2}323}$	25	$\mathbb{Z}_5 \times \mathbb{Z}_5$
10	$p_{1,(13)^5=id}$	10	$\mathbb{Z}_{10}$
	$p_{1,(123\bar{2})^3}$	50	$\mathbb{Z}_{10} \times \mathbb{Z}_5$
	$p_{1,\bar{3}\bar{2}323}$	100	$\mathbb{Z}_{10} \times \mathbb{Z}_{10}$

Singular points:

p	Element	Type
3,4,5	Q	$\frac{1}{5}(1, 2)$
	$R_2 Q^2$	A_1
4,5	$R_2 Q^4$	$\frac{1}{5}(1, 2)$
4,5	$23\bar{2}(Q\bar{2})^2$	A_1
5	$R_3 Q^4$	A_1

A.5. Thompson S_2 .

Triangle group type: 3,3,4; 5,5,5; 5

Lattice for $p = 3, 4, 5$.

Commensurability invariants:

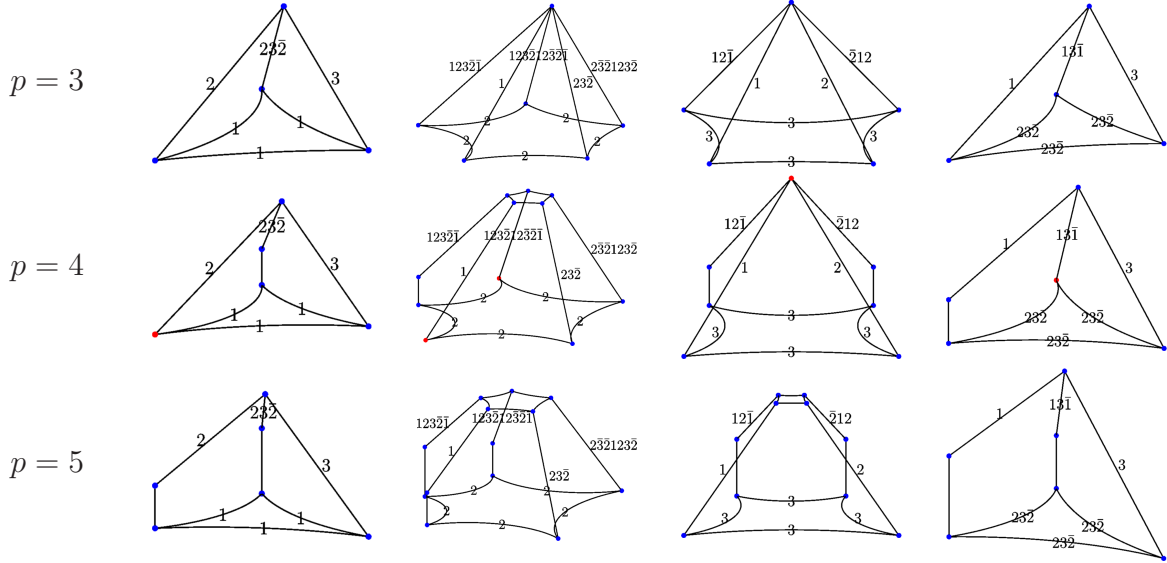
p	χ^{orb}	$\mathbb{Q}(\text{TrAd } \Gamma)$	CM field	C?	A?
3	2/15	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(i\sqrt{2}, \sqrt{5})$	C	A
4	1/3	$\mathbb{Q}(\sqrt{3}, \sqrt{5})$	$\mathbb{Q}(i, \sqrt{3}, \sqrt{5})$	NC	NA(3)
5	133/300	$\mathbb{Q}(\cos(2\pi/15))$	$\mathbb{Q}(\zeta_{15})$	C	NA(1)

Presentations:

$$\left\langle R_1, R_2, R_3 \mid R_1^p, R_2^p, R_3^p, (R_1 R_2 R_3)^5, \text{br}_3(R_1, R_3), \text{br}_3(R_2, R_3) \right. \\ \left. \text{br}_4(R_1, R_2), (R_1 R_2)^{\frac{4p}{p-4}}, \text{br}_5(R_1, R_2 R_3 R_2^{-1}), (R_1 R_2 R_3 R_2^{-1})^{\frac{10p}{3p-10}} \right\rangle$$

Combinatorics:

Triangle	$\#(P\text{-orb})$	Top trunc.	Top ideal
[3] 1; 2, 3	5		
[3] $23\bar{2}$; 1, 3	5		
[4] 3; 1, 2	5	$p = 5$	$p = 4$
[5] 2; 1, $23\bar{2}$	5	$p = 4, 5$	



Vertex stabilizers:

p	Vertex	Order	Nature	Vertex	Order	Nature
3	p_{13}	24	G_4	p_{23}	24	G_4
	p_{12}	72	G_5			
	$p_{1,23\bar{2}}$	360	G_{20}			
4	p_{13}	96	G_8	p_{23}	96	G_8
	p_{12}	∞	Cusp			
	$p_{1,(123\bar{2})^5}$	16	$\mathbb{Z}_4 \times \mathbb{Z}_4$			
5	p_{13}	600	G_{16}	p_{23}	600	G_{16}
	$p_{1,(12)^2}$	50	$\mathbb{Z}_5 \times \mathbb{Z}_{10}$			
	$p_{1,(123\bar{2})^5}$	10	$\mathbb{Z}_5 \times \mathbb{Z}_2$			

Singular points:

p	Element	Type
3,4,5	Q $R_2 Q^2$	$\frac{1}{5}(1, 2)$ A_1
4,5	$R_2 Q^4$	$\frac{1}{5}(1, 2)$
4,5	$23\bar{2}(Q\bar{2})^2$	A_1
5	$R_3 Q^4$	A_1

A.6. Thompson E_2 .

Triangle group type: 3,4,4; 4,4,6; 6
 Q^3 is a complex reflection

Lattice for $p = 3, 4, 6, (12)$.

Commensurability invariants:

p	χ^{orb}	$\mathbb{Q}(\text{TrAd } \Gamma)$	CM field	C?	A?
3	1/4	$\mathbb{Q}$	$\mathbb{Q}(i\sqrt{3})$	NC	A
4	17/32	$\mathbb{Q}(\sqrt{3})$	$\mathbb{Q}(\zeta_{12})$	NC	NA(1)
6	3/4	$\mathbb{Q}$	$\mathbb{Q}(i\sqrt{3})$	NC	A

p	χ^{orb}	$\mathbb{Q}(\text{TrAd } \Gamma)$	CM field	C?	A?	Alt.
12	1/4	$\mathbb{Q}(\sqrt{3})$	$\mathbb{Q}(\zeta_{12})$	C	A	?

Presentations ($p = 3, 4, 6$ only):

$$\left\langle R_1, R_2, R_3 \mid R_1^p, R_2^p, R_3^p, (R_1 R_2 R_3)^6, \text{br}_3(R_2, R_3), \text{br}_4(R_3, R_1), (R_1 R_3)^{\frac{4p}{p-4}}, \text{br}_4(R_1, R_2), \right. \\ \left. (R_1 R_2)^{\frac{4p}{p-4}}, \text{br}_4(R_1, R_2 R_3 R_2^{-1}), (R_1 R_2 R_3 R_2^{-1})^{\frac{4p}{p-4}}, \text{br}_6(R_3, R_1 R_2 R_1^{-1}), (R_3 R_1 R_2 R_1^{-1})^{\frac{3p}{p-3}} \right\rangle$$

Combinatorics:

Triangle	$\#(P\text{-orb})$	Top trunc.	Top ideal
[3] 1; 2, 3	6	$p = 12$	$p = 6$
[4] $23\bar{2}$; 1, 3	6	$p = 6, 12$	$p = 4$
[4] 3; 1, 2	6	$p = 6, 12$	$p = 4$
[4] 2; 1, $23\bar{2}$	6	$p = 6, 12$	$p = 4$
[6] $\bar{3}13$; $12\bar{1}$, 3	3	$p = 4, 6, 12$	$p = 3$

Vertex stabilizers:

Q^3 is a complex reflection

Lattice for $p = 2, (7)$.

Commensurability invariants:

p	χ^{orb}	$\mathbb{Q}(\text{TrAd } \Gamma)$	CM field	C?	A?
2	1/49	$\mathbb{Q}(\cos(2\pi/7))$	$\mathbb{Q}(\zeta_7)$	C	A

p	χ^{orb}	$\mathbb{Q}(\text{TrAd } \Gamma)$	CM field	C?	A?	Alt.
7	1/49	$\mathbb{Q}(\cos(2\pi/7))$	$\mathbb{Q}(\zeta_7)$	C	A	$\Gamma(7, 9/14)$

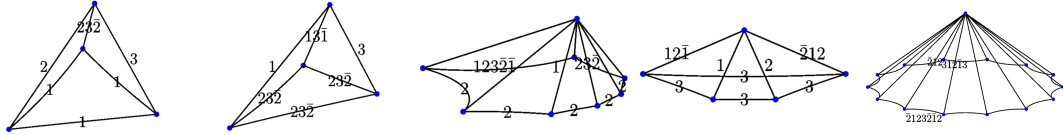
Presentations ($p = 2$ only):

$$\langle R_1, R_2, R_3 \mid R_1^2, R_2^2, R_3^2, (R_1 R_2 R_3)^{42}, (R_1 R_2 R_3 R_2^{-1})^7 \\ \text{br}_3(R_2, R_3), \text{br}_3(R_3, R_1), \text{br}_4(R_1, R_2) \rangle$$

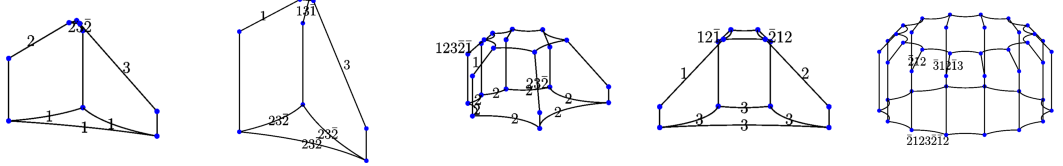
Combinatorics:

Triangle	$\#(P\text{-orb})$	Top trunc.	Top ideal
[3] 1; 2, 3	42	$p = 7$	
[3] $23\bar{2}$; 1, 3	42	$p = 7$	
[4] 3; 1, 2	42	$p = 7$	
[7] 2; 1, $23\bar{2}$	42	$p = 7$	
[14] $\bar{2}123\bar{2}\bar{1}2$; $\bar{2}12, \bar{3}12\bar{1}3$	3	$p = 7$	

$p = 2$



$p = 7$



Vertex stabilizers:

p	Vertex	Order	Nature	Vertex	Order	Nature
2	p_{13}	6	$G(3, 3, 2)$	p_{23}	6	$G(3, 3, 2)$
	p_{12}	8	$G(4, 4, 2)$			
	$p_{1,23\bar{2}}$	14	$G(7, 7, 2)$			
	$p_{2,123\bar{2}12\bar{3}\bar{2}\bar{1}}$	392	$G(14, 1, 2)^{(*)}$			

Singular points:

p	Element	Order	Eigenvalues	Type
2	Q	42	(ω, ζ_{42})	$\frac{1}{3}(1, 1)$

A.8. Thompson H_2 .

Triangle group type: 3,3,5; 5,5,5; 15

Q^3 is a complex reflection

Lattice for $p = 2, 3, 5, (10), (-5)$.

Commensurability invariants:

p	χ^{orb}	$\mathbb{Q}(\text{TrAd } \Gamma)$	CM field	C?	A?
2	1/100	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\zeta_5)$	C	A
3	26/75	$\mathbb{Q}(\cos(2\pi/15))$	$\mathbb{Q}(\zeta_{15})$	C	NA(1)
5	73/100	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\zeta_5)$	C	A

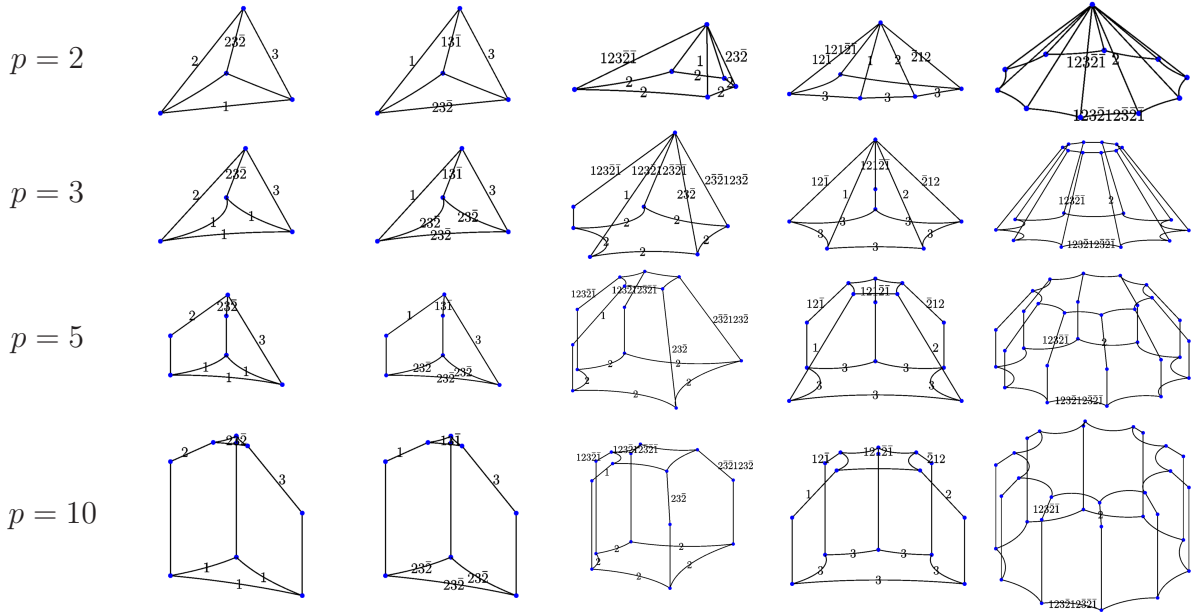
p	χ^{orb}	$\mathbb{Q}(\text{TrAd } \Gamma)$	CM field	C?	A?	Alt.
10	13/100	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\zeta_5)$	C	A	$\mathcal{S}(10, \sigma_{10})$
-5	1/200	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\zeta_5)$	C	A	$\Gamma(5, 7/10)$

Presentations ($p = 2, 3, 5$ only):

$$\left\langle R_1, R_2, R_3 \mid R_1^p, R_2^p, R_3^p, (R_1 R_2 R_3)^{15}, \text{br}_3(R_2, R_3), \text{br}_3(R_3, R_1), \right. \\ \left. \text{br}_5(R_1, R_2), (R_1 R_2)^{\frac{10p}{3p-10}}, \text{br}_5(R_1, R_2 R_3 R_2^{-1}), (R_1 R_2 R_3 R_2^{-1})^{\frac{10p}{3p-10}}, \right. \\ \left. \text{br}_{10}(R_3, R_1 R_2 R_1 R_2^{-1} R_1^{-1}), (R_3 R_1 R_2 R_1 R_2^{-1} R_1^{-1})^{\frac{5p}{2p-5}} \right\rangle$$

Combinatorics:

Triangle	$\#(P\text{-orb})$	Top trunc.	Top ideal
[3] 1; 2, 3	15	$p = 10$	
[3] $23\bar{2}$; 1, 3	15	$p = 10$	
[5] 2; 1, $23\bar{2}$	15	$p = 5, 10$	
[5] 3; 1, 2	15	$p = 5, 10$	
[10] $(123)^2 \bar{2} 1 2 (\bar{3} 2 \bar{1})^2$; $121\bar{2}\bar{1}$, 3	3	$p = 3, 5, 10$	



Vertex stabilizers:						
p	Vertex	Order	Nature	Vertex	Order	Nature
2	p_{13}	6	$G(3, 3, 2)$	p_{23}	6	$G(3, 3, 2)$
	$p_{1,23\bar{2}}$	10	$G(5, 5, 2)$	p_{12}	10	$G(5, 5, 2)$
	$p_{2,123\bar{2}\bar{1}}$	100	$G(10, 2, 2)^{(*)}$			
3	p_{13}	24	G_4	p_{23}	24	G_4
	$p_{1,23\bar{2}}$	360	G_{20}	p_{12}	360	G_{20}
	$p_{2,(2\cdot123\bar{2}\bar{1})^5}$	45	$\mathbb{Z}_3 \times \mathbb{Z}_{15}^{(*)}$	$p_{123\bar{2}\bar{1},(2\cdot123\bar{2}\bar{1})^5}$	45	$\mathbb{Z}_3 \times \mathbb{Z}_{15}^{(*)}$
5	p_{13}	600	G_{16}	p_{23}	600	G_{16}
	$p_{1,(123\bar{2})^5}$	10	$\mathbb{Z}_5 \times \mathbb{Z}_2$	$p_{2,(12)^5}$	10	$\mathbb{Z}_5 \times \mathbb{Z}_2$
	$p_{2,(\bar{3}\bar{2}\bar{1})^2\bar{2}\bar{1}2}$	25	$\mathbb{Z}_5 \times \mathbb{Z}_5$	$p_{123\bar{2}\bar{1},(\bar{3}\bar{2}\bar{1})^2\bar{2}\bar{1}2}$	25	$\mathbb{Z}_5 \times \mathbb{Z}_5$

Singular points

p	Element	Type
2,3,5	Q	$\frac{1}{3}(1, 1)$
5	$232(Q2)^2$	A_1
5	$2(Q\bar{3})^2$	A_1
5	$2Q^{-1}$	$\frac{1}{5}(1, 2)$
5	$3Q^{-1}$	$\frac{1}{5}(1, 2)$

A.9. Mostow triangle groups. In this section, we gather in the form of a table basic numerical invariants for the Mostow triangle group.

In order to obtain the results below, it is very useful to know that the group $\Gamma(p, t)$, generated by R_1 and J is always isomorphic to the hypergeometric monodromy group $\Gamma_{\mu, \Sigma}$ for exponents

$$(21) \quad \mu = \left(\frac{1}{2} - \frac{1}{p}, \frac{1}{2} - \frac{1}{p}, \frac{1}{2} - \frac{1}{p}, \frac{1}{4} + \frac{3}{2p} - \frac{t}{2}, \frac{1}{4} + \frac{3}{2p} + \frac{t}{2} \right),$$

and Σ corresponding to permutations of the first three weights. Moreover, if condition Σ -INT is satisfied but INT is *not* satisfied for the first three exponents (i.e. p is odd), then $\Gamma_{\mu, \Sigma}$ is the same as Γ_{μ} .

From the hypergeometric exponents, one can easily read off the adjoint trace field (which is the real subfield in the cyclotomic field $\mathbb{Q}(\zeta_d)$, where d is the least common denominator of the exponents), see Lemma (12.5) in [6].

Presentations for various of these groups have been given in several places, including [22], [12], [28], [45] for instance. A unified presentation for all Deligne-Mostow groups with three fold symmetry was given in [32]. It is straightforward to check that our presentation is equivalent to hers.

The non-arithmeticity index can be computed explicitly from the hypergeometric weights, since Proposition (12.7) of [6] gives a formula for the signature of Galois conjugates.

Finally, the volumes of Mostow lattices were tabulated by Sauter in [37]; note that Sauter lists volumes, but volumes are given by a universal constant ($8\pi^2/3$ if the holomorphic curvature is normalized to be -1) times the orbifold Euler characteristic. A lot of these volumes were also computed using different fundamental domains, see [28] for instance.

For some groups, namely $\Gamma(5, 1/2)$, $\Gamma(7, 3/14)$, $\Gamma(9, 1/18)$, the algorithm does not work quite as described in section 4.1.5, see section 5.4 for details.

Contrary to the previous sections of the Appendix, we do not list vertex stabilizers and singular points, since these were already described by Deligne and Mostow, see [7].

Triangle group type: $3, 3, 3; k, k, k; 2k$

Presentations:

$$\left\langle R_1, R_2, R_3, J \mid R_1^p, J^3, (R_1 J)^{2k}, JR_1 J^{-1} = R_2, JR_2 J^{-1} = R_3, \right. \\ \left. \text{br}_3(R_1, R_2), (R_1 R_2)^{\frac{6p}{p-6}}, (R_2 R_1 J)^{\frac{2kp}{(k-2)p-2k}} \right\rangle$$

Combinatorics:

Triangle	#(P-orbit)
[3] 1; 2, 3	$2k$
[k] 2; 1, 23 $\bar{2}$	2

Remark A.1. Note $2k$ stands for the order of $P = R_1 J$. The second type of faces should be omitted when $k = 2$, in which case 1 and 23 $\bar{2}$ commute, i.e. they braid with order 2. The latter groups correspond to the Livné family.

Commensurability invariants:

	p	t	$o(P)$	χ^{orb}	$\mathbb{Q}(\text{TrAd } \Gamma)$	CM field	C?	A?
$k = 2$	5	7/10	4	1/200	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\zeta_5)$	C	A
	6	2/3	4	1/72	$\mathbb{Q}$	$\mathbb{Q}(\zeta_3)$	NC	A
	7	9/14	4	1/49	$\mathbb{Q}(\cos(2\pi/7))$	$\mathbb{Q}(\zeta_7)$	C	A
	8	5/8	4	3/128	$\mathbb{Q}(\sqrt{2})$	$\mathbb{Q}(\zeta_8)$	C	A
	9	11/18	4	2/81	$\mathbb{Q}(\cos(2\pi/9))$	$\mathbb{Q}(\zeta_9)$	C	NA(1)
	10	3/5	4	1/40	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\zeta_5)$	C	A
	12	7/12	4	7/288	$\mathbb{Q}(\sqrt{3})$	$\mathbb{Q}(\zeta_{12})$	C	A
	18	5/9	4	13/648	$\mathbb{Q}(\cos(2\pi/9))$	$\mathbb{Q}(\zeta_9)$	C	A

	p	t	$o(P)$	χ^{orb}	$\mathbb{Q}(\text{TrAd } \Gamma)$	CM field	C?	A?
$k = 3$	4	5/12	6	1/72	$\mathbb{Q}(\sqrt{3})$	$\mathbb{Q}(\zeta_{12})$	C	A
	5	11/30	6	8/225	$\mathbb{Q}(\cos(2\pi/15))$	$\mathbb{Q}(\zeta_{15})$	C	A
	6	1/3	6	1/18	$\mathbb{Q}$	$\mathbb{Q}(\zeta_3)$	NC	A
	7	13/42	6	61/882	$\mathbb{Q}(\cos(2\pi/21))$	$\mathbb{Q}(\zeta_{21})$	C	NA(2)
	8	7/24	6	11/144	$\mathbb{Q}(\cos(2\pi/24))$	$\mathbb{Q}(\zeta_{24})$	C	NA(1)
	9	5/18	6	13/162	$\mathbb{Q}(\cos(2\pi/9))$	$\mathbb{Q}(\zeta_9)$	C	A
	10	4/15	6	37/450	$\mathbb{Q}(\cos(2\pi/15))$	$\mathbb{Q}(\zeta_{15})$	C	NA(2)
	12	1/4	6	1/12	$\mathbb{Q}(\sqrt{3})$	$\mathbb{Q}(\zeta_{12})$	C	A
	18	2/9	6	13/162	$\mathbb{Q}(\cos(2\pi/9))$	$\mathbb{Q}(\zeta_9)$	C	A

$k = 4$	p	t	$o(P)$	χ^{orb}	$\mathbb{Q}(\text{TrAd } \Gamma)$	CM field	C?	A?
	3	1/3	8	1/288	$\mathbb{Q}(\sqrt{3})$	$\mathbb{Q}(\zeta_{12})$	C	A
	4	1/4	8	1/32	$\mathbb{Q}$	$\mathbb{Q}(\zeta_4)$	NC	A
	5	1/5	8	23/400	$\mathbb{Q}(\cos(2\pi/20))$	$\mathbb{Q}(\zeta_{20})$	C	NA(1)
	6	1/6	8	11/144	$\mathbb{Q}(\sqrt{3})$	$\mathbb{Q}(\zeta_{12})$	NC	NA(1)
	8	1/8	8	3/32	$\mathbb{Q}(\sqrt{2})$	$\mathbb{Q}(\zeta_8)$	C	A
$k = 5$	12	1/12	8	7/72	$\mathbb{Q}(\sqrt{3})$	$\mathbb{Q}(\zeta_{12})$	C	A
	p	t	$o(P)$	χ^{orb}	$\mathbb{Q}(\text{TrAd } \Gamma)$	CM field	C?	A?
	3	7/30	10	2/225	$\mathbb{Q}(\cos(2\pi/15))$	$\mathbb{Q}(\zeta_{15})$	C	A
	4	3/20	10	33/800	$\mathbb{Q}(\cos(2\pi/20))$	$\mathbb{Q}(\zeta_{20})$	C	NA(2)
	5	1/10	10	13/200	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\zeta_5)$	C	A
$k = 6$	10	0	10	1/10	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\zeta_5)$	C	A
	p	t	$o(P)$	χ^{orb}	$\mathbb{Q}(\text{TrAd } \Gamma)$	CM field	C?	A?
	3	1/6	12	1/72	$\mathbb{Q}$	$\mathbb{Q}(\zeta_3)$	NC	A
	4	1/12	12	13/288	$\mathbb{Q}(\sqrt{3})$	$\mathbb{Q}(\zeta_{12})$	C	NA(1)
$k = 7$	6	0	12	1/12	$\mathbb{Q}$	$\mathbb{Q}(\zeta_3)$	NC	A
	p	t	$o(P)$	χ^{orb}	$\mathbb{Q}(\text{TrAd } \Gamma)$	CM field	C?	A?
	3	5/42	14	61/3528	$\mathbb{Q}(\cos(2\pi/21))$	$\mathbb{Q}(\zeta_{21})$	C	NA(2)
$k = 8$	p	t	$o(P)$	χ^{orb}	$\mathbb{Q}(\text{TrAd } \Gamma)$	CM field	C?	A?
	3	1/12	16	11/576	$\mathbb{Q}(\cos(2\pi/24))$	$\mathbb{Q}(\zeta_{24})$	C	NA(1)
	4	0	16	3/64	$\mathbb{Q}(\sqrt{2})$	$\mathbb{Q}(\zeta_8)$	C	A
$k = 9$	p	t	$o(P)$	χ^{orb}	$\mathbb{Q}(\text{TrAd } \Gamma)$	CM field	C?	A?
	3	1/18	18	13/648	$\mathbb{Q}(\cos(2\pi/9))$	$\mathbb{Q}(\zeta_9)$	C	A
$k = 10$	p	t	$o(P)$	χ^{orb}	$\mathbb{Q}(\text{TrAd } \Gamma)$	CM field	C?	A?
	3	1/30	20	37/1800	$\mathbb{Q}(\cos(2\pi/15))$	$\mathbb{Q}(\zeta_{15})$	C	NA(2)
$k = 12$	p	t	$o(P)$	χ^{orb}	$\mathbb{Q}(\text{TrAd } \Gamma)$	CM field	C?	A?
	3	0	24	1/48	$\mathbb{Q}(\sqrt{3})$	$\mathbb{Q}(\zeta_{12})$	C	A
$k = 5/2$	p	t	$o(P)$	χ^{orb}	$\mathbb{Q}(\text{TrAd } \Gamma)$	CM field	C?	A?
	5	1/2	10	1/200	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\zeta_5)$	C	A
$k = 7/2$	p	t	$o(P)$	χ^{orb}	$\mathbb{Q}(\text{TrAd } \Gamma)$	CM field	C?	A?
	7	3/14	14	1/49	$\mathbb{Q}(\cos(2\pi/7))$	$\mathbb{Q}(\zeta_7)$	C	A
$k = 9/2$	p	t	$o(P)$	χ^{orb}	$\mathbb{Q}(\text{TrAd } \Gamma)$	CM field	C?	A?
	9	1/18	18	2/81	$\mathbb{Q}(\cos(2\pi/9))$	$\mathbb{Q}(\zeta_9)$	C	NA(1)

A.10. Deligne-Mostow groups without 3-fold symmetry. In order to check that our lattices are not commensurable to any Deligne-Mostow lattice, we also need to consider the handful of groups in the Deligne-Mostow list whose hypergeometric exponents do not have a 3-fold symmetry. In Table A.1, we list such non-arithmetic groups, their orbifold Euler characteristics and the rough commensurability invariants.

It turns out that the group Γ_μ with $\mu = (4, 4, 5, 5, 6)/12$ is commensurable to the Mostow group $\Gamma(4, 1/12)$, and the group with $\mu = (6, 6, 9, 9, 10)/20$ is commensurable to $\Gamma(4, 3/20)$, see [7], [18] or [21].

μ	$\chi^{orb}(\Gamma_{\mu,\Sigma})$	$\mathbb{Q}(\text{TrAd } \Gamma)$	CM field	C?	A?
$(4, 4, 5, 5, 6)/12$	13/96	$\mathbb{Q}(\sqrt{3})$	$\mathbb{Q}(\zeta_{12})$	C	NA(1)
$(3, 3, 5, 6, 7)/12$	17/96	$\mathbb{Q}(\sqrt{3})$	$\mathbb{Q}(\zeta_{12})$	NC	NA(1)
$(6, 6, 9, 9, 10)/20$	99/800	$\mathbb{Q}(\cos(2\pi/20))$	$\mathbb{Q}(\zeta_{20})$	C	NA(2)

TABLE A.1. Invariants for non-arithmetic Deligne-Mostow groups $\Gamma_{\mu,\Sigma}$, such that μ has no 3-fold symmetry. Here Σ stands for the full symmetry group of μ .

A.11. Commensurability classes. In this section we summarize the analysis of commensurability classes of non-arithmetic lattices obtained in our paper, which brings to 22 the number of currently known non-arithmetic lattices in $\text{PU}(2, 1)$.

The result is given in Table A.2; groups in different large boxes are in distinct commensurability classes, either because they have different adjoint trace fields or because one is cocompact and the other is not. Within a large box, we separate groups by a solid line if they are known to be in distinct commensurability classes.

$\mathbb{Q}(\text{TrAd } \Gamma)$	C	NC
$\mathbb{Q}(\sqrt{2})$		$\mathcal{S}(4, \sigma_1)$
$\mathbb{Q}(\sqrt{3})$	$\Gamma(4, \frac{1}{12}), \frac{1}{12}(4, 4, 5, 5, 6)$	$\Gamma(6, \frac{1}{6}) \mid \frac{1}{12}(3, 3, 5, 6, 7), \mathcal{T}(4, \mathbf{E}_2)$
$\mathbb{Q}(\sqrt{5})$		$\mathcal{S}(3, \sigma_5)$
$\mathbb{Q}(\sqrt{6})$		$\mathcal{S}(3, \sigma_1) \mid \mathcal{S}(6, \sigma_1)$
$\mathbb{Q}(\sqrt{7})$		$\mathcal{S}(4, \overline{\sigma}_4)$
$\mathbb{Q}(\sqrt{21})$		$\mathcal{S}(6, \overline{\sigma}_4)$
$\mathbb{Q}(\sqrt{2}, \sqrt{3})$		$\Gamma(3, 1/12), \Gamma(8, \frac{7}{24})$
$\mathbb{Q}(\sqrt{2}, \sqrt{7})$	$\mathcal{S}(8, \overline{\sigma}_4)$	
$\mathbb{Q}(\sqrt{3}, \sqrt{5})$		$\mathcal{S}(4, \sigma_5) \mid \mathcal{T}(4, \mathbf{S}_2)$
$\mathbb{Q}(\sqrt{3}, \sqrt{7})$	$\mathcal{S}(12, \overline{\sigma}_4)$	
$\mathbb{Q}(\sqrt{\frac{5+\sqrt{5}}{2}})$	$\Gamma(4, \frac{3}{20}), \frac{1}{20}(6, 6, 9, 9, 10) \mid \Gamma(5, \frac{1}{5})$	
$\mathbb{Q}(\sqrt{\frac{5+\sqrt{5}}{14}})$	$\mathcal{S}(5, \overline{\sigma}_4)$	
$\mathbb{Q}(\cos \frac{2\pi}{9})$	$\Gamma(9, 11/18), \Gamma(9, 1/18)$	
$\mathbb{Q}(\cos \frac{2\pi}{15})$	$\Gamma(10, \frac{4}{15}), \Gamma(3, \frac{1}{30}) \mid \mathcal{T}(5, \mathbf{S}_2) \mid \mathcal{T}(3, \mathbf{H}_2)$	
$\mathbb{Q}(\cos \frac{2\pi}{21})$	$\Gamma(3, \frac{5}{42}), \Gamma(7, \frac{13}{42})$	

TABLE A.2. Commensurability classes of non-arithmetic lattice triangle groups.

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