CLASSIFICATION OF NON-FREE KLEINIAN GROUPS GENERATED BY TWO PARABOLIC TRANSFORMATIONS

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ABSTRACT. We give a full proof to Agol's announcement on the classification of non-free Kleinian groups generated by two parabolic transformations.

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1. INTRODUCTION

Motivated by knot theory, Riley studied Kleinian groups generated by two parabolic transformations (see [51, 52, 53, 54, 55]). In particular, the construction of the complete hyperbolic structure on the figure-eight knot complement [52] inspired Thurston to establish the uniformisation theorem of Haken manifolds. The space of

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marked subgroups of $\mathrm{PSL}(2,\mathbb{C})$ generated by two non-commuting parabolic transformations is parametrised by a non-zero complex number. There is an open set, $\mathcal{R}$, called the *Riley slice of Schottky space*, of Kleinian groups of this type that are free and discrete, and for which the quotient of the domain of discontinuity is a four times punctured sphere. For every group in $\mathcal{R}$, the Klein manifold (the quotient of union of the hyperbolic space and the domain of discontinuity) is homeomorphic to the complement of the 2-strand trivial tangle. Keen and Series [30] studied the Riley slice by applying their theory of pleating rays, and it was supplemented by Komori and Series [33]. Motivated by knot theory, Akiyoshi, Sakuma, Wada and Yamashita [6] studied the combinatorial structures of the Ford domains, by extending Jorgensen's work [29] on punctured torus groups, which leads to a natural tessellation of $\mathcal{R}$ (see Figure 0.2b in [6]). Ohshika and Miyachi [46] proved that the closure of $\mathcal{R}$ is equal to the space of marked Kleinian groups with two parabolic generators which are free and discrete. Building on his joint work [25], [27] and [38] with Gehring, Hinkkanen and Marshall, respectively, Martin [37] identified the exterior of $\mathcal{R}$ as the Julia set of a certain semigroup of polynomials and proved a "supergroup density theorem" for groups in the exterior of $\mathcal{R}$. The problem to detect freeness and non-freeness of (not necessarily discrete) groups generated by two non-commuting parabolic transformations has attracted attention of various researchers (see [35, 24, 63, 31] and references therein).

In this paper, we are interested in Kleinian groups that are in the complement of the closure of $\mathcal{R}$, namely the groups that are discrete but not free. The essential simple loops on the boundary of the complement of the 2-strand trivial tangle, which are not null homotopic in the ambient space, are parametrised by a slope r in $\mathbb{Q}/2\mathbb{Z}$. The Heckoid groups, introduced by Riley [54] and formulated by Lee and Sakuma [34] following Agol [2], are Kleinian groups with two parabolic generators in which the element corresponding to the curve α_r of slope r has finite order. The most extreme case is the group $G(r)$ where this element is the identity, in which case, the quotient of hyperbolic space by this group is the complement of a 2-bridge knot or link.

In [1, Theorem 4.3], Adams proved that a non-free and torsion-free Kleinian group Γ is generated by two parabolic transformations if and only if the quotient hyperbolic manifold $\mathbb{H}^3/\Gamma$ is homeomorphic to the complement of a 2-bridge link $K(r)$ which is not a torus link. (We regard a knot as a one-component link.) This refines the result of Boileau and Zimmermann [11, Corollary 3.3] that a link in S^3 is a 2-bridge link if and only if its link group is generated by two meridians.

In 2002, Agol [2] announced the following classification theorem of non-free Kleinian groups generated by two parabolic transformations, which generalises Adams' result. The main purpose of this paper is to give a full proof to this theorem.

Theorem 1.1. *A non-free Kleinian group Γ is generated by two non-commuting parabolic elements if and only if one of the following holds.*

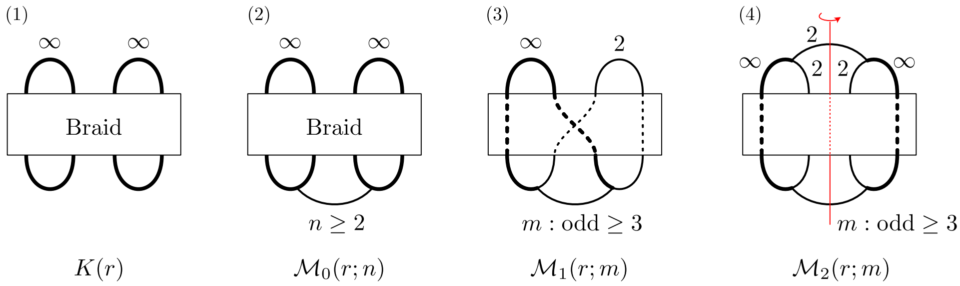

FIGURE 1. Weighted graphs representing 2-bridge links and Heckoid orbifolds, where the thick edges with weight ∞ correspond to parabolic loci and thin edges with integral weights represent the singular set. See Definition 3.4 for the precise description of the weighted graphs.

(1) Γ *is conjugate to the hyperbolic* 2*-bridge link group,* $G(r)$*, for some rational number* $r = q/p$*, where* p *and* q *are coprime integers such that* $q \not\equiv \pm 1 \pmod{p}$*.*
(2) Γ *is conjugate to the Heckoid group,* $G(r;n)$*, for some* $r \in \mathbb{Q}$ *and some* $n \in \frac{1}{2}\mathbb{N}_{\geq 3}$*.*

In the remainder of the introduction, we explain the meaning of the theorem more precisely.

Recall that a 2*-bridge link* is a knot or a two-component link which is represented by a diagram in the x-y plane that has two maximal points and two minimal points with respect to the height function determined by the y-coordinate. We may assume that the two maximal points and the two minimal points, respectively, have the same y-coordinates. Such a diagram gives a plait (or plat) representation of the 2-bridge link consisting of two upper bridges, two lower bridges, and a 4-strand braid connecting the upper and lower bridges (see Figure 1(1)). The 2-bridge links are parametrized by the set $\mathbb{Q}\cup\{\infty\}$, and the 2-bridge link corresponding to $r \in \mathbb{Q}\cup\{\infty\}$ is denoted by $K(r)$ and is called the 2*-bridge link of slope* r (see Section 2 for the precise definition). If $r = \infty$ then $K(r)$ is the 2-component trivial link, and if $r \in \mathbb{Z}$ then $K(r)$ is the trivial knot. If $r = q/p \in \mathbb{Q}$, where p and q are coprime integers, then $K(q/p)$ is hyperbolic, i.e., $S^3 - K(r)$ admits a complete hyperbolic structure of finite volume, if and only if $q \not\equiv \pm 1 \pmod{p}$. In this case, there is a torsion-free Kleinian group Γ, unique up to conjugation, such that $\mathbb{H}^3/\Gamma$ is homeomorphic to the link complement $S^3 - K(r)$ as oriented manifold. We denote the Kleinian group Γ by $G(r)$, and call it the *hyperbolic* 2*-bridge link group of slope* r.

The Heckoid groups were first introduced by Riley [54] as an analogy of the classical Hecke groups considered by Hecke [26]. The topological structure of their quotient orbifolds was worked out by Lee and Sakuma [34], following the description by Agol [2]. Specifically, they showed that the Heckoid groups are the orbifold fundamental groups of the Heckoid orbifolds illustrated in Figure 1(2)-(4). (See [7, 10, 20] for basic terminologies and facts concerning orbifolds.) These figures illustrate weighted graphs (S^3, Σ, w) whose explicit descriptions are given by Definition 3.4. For each weighted graph (S^3, Σ, w) in the figure, let (M_0, P) be the pair of a compact 3-orbifold M_0 and a compact 2-suborbifold P of ∂M_0 determined by the rules described below. Let Σ_∞ be the subgraph of Σ consisting of the edges with weight ∞, and let Σ_s be the subgraph of Σ consisting of the edges with integral weight.

(1) The underlying space $|M_0|$ of the orbifold M_0 is the complement of an open regular neighbourhood of the subgraph Σ_∞.
(2) The singular set of M_0 is $\Sigma_0 := \Sigma_s \cap |M_0|$, where the index of each edge of the singular set is given by the weight $w(e)$ of the corresponding edge e of Σ_s.
(3) For an edge e of Σ_∞, let P be the 2-suborbifold of ∂M_0 defined as follows.
   (a) In Figure 1(2), P consists of two annuli in ∂M_0 whose cores, respectively, are meridians of the two edges of Σ_∞.
   (b) In Figure 1(3), P consists of an annulus in ∂M_0 whose core is a meridian of the single edge of Σ_∞.
   (c) In Figure 1(4), P consists of two copies of the annular orbifold $D^2(2,2)$ (the 2-orbifold with underlying space the disc and with two cone points of index 2) in ∂M_0 each of which is bounded by a meridian of an edge of Σ_∞.

By [34, Lemmas 6.3 and 6.6], the orbifold pair (M_0, P) is a Haken pared orbifold (see Definition 3.1 or [10, Definition 8.3.7]) and admits a unique complete hyperbolic structure, which is geometrically finite (see Section 3 or [34, Proposition 6.7]). Namely there is a geometrically finite Kleinian group Γ, unique up to conjugation, such that $M := \mathbb{H}^3/\Gamma$ is isomorphic to the interior of the compact orbifold M_0, such that P represents the parabolic locus. The pair (M_0, P) is also regarded as a relative compactification of the pair consisting of a non-cuspidal part of M and its boundary (see Section 3).

We denote the pared orbifold $\mathcal{M} := (M_0, P)$ by $\mathcal{M}_0(r; n)$, $\mathcal{M}_1(r; m)$, or $\mathcal{M}_2(r; m)$ according as it is described by the weighted graph in Figure 1(2), (3), or (4). We also denote the Kleinian group Γ by $\pi_1(\mathcal{M})$.

Then the assertion (2) of the main Theorem 1.1 is equivalent to the following assertion (2')

(2') Γ *is conjugate to the Kleinian group* $\pi_1(\mathcal{M})$ *for some pared orbifold* $\mathcal{M} = \mathcal{M}_0(r; n)$, $\mathcal{M}_1(r; m)$, *or* $\mathcal{M}_2(r; m)$ *in Definition 3.4.*

Agol [2] also announced the following classification of parabolic generating pairs of the groups in Theorem 1.1, which refines and extends Adams' results that every hyperbolic 2-bridge link group has only finitely many parabolic generating pairs

[1, Corollary 4.1] and that the figure-eight knot group has precisely two parabolic generating pairs up to equivalence [1, Corollary 4.6].

Theorem 1.2. *(1) If* Γ *is a hyperbolic* 2*-bridge link group, then it has precisely two parabolic generating pairs, up to equivalence.*

(2) If Γ *is a Heckoid group, then it has a unique parabolic generating pair, up to equivalence.*

Here, by a *parabolic generating pair* of a Kleinian group Γ, we mean an unordered pair $\{\alpha, \beta\}$ of parabolic transformations α and β that generate Γ. Two parabolic generating pairs $\{\alpha, \beta\}$ and $\{\alpha', \beta'\}$ are said to be *equivalent* if $\{\alpha', \beta'\}$ is equal to $\{\alpha^{\epsilon_1}, \beta^{\epsilon_2}\}$ for some $\epsilon_1, \epsilon_2 \in \{\pm 1\}$ up to simultaneous conjugacy. In the companion [4] of this paper by Shunsuke Aimi, Donghi Lee, Shunsuke Sakai and the fourth author, an alternative proof of the theorem is given.

Theorems 1.1 and 1.2 are beautifully illustrated by a figure produced by Yasushi Yamashita upon request of Caroline Series, which is to be included in her article [59] in preparation. The figure is produced by using the results announced in [6, Section 3 of Preface]. (See also Figure 0.2b in [6], which was also produced by Yamashita.) For further properties of Heckoid groups, please see the article [5] in preparation.

This paper is organised as follows. In Section 2, we recall basic facts concerning 2-bridge links. In Section 3, we give the precise definitions of the Heckoid orbifolds and Heckoid groups. In Section 4, we give the classification of dihedral orbifolds, i.e., good orbifolds with dihedral orbifold fundamental groups (Theorem 4.1), which holds a key to the proof of the main theorem. In Section 5, we prove the relative tameness theorem for hyperbolic orbifolds (Theorem 5.1), following Bowditch's proof of the tameness theorem for hyperbolic orbifolds ([15]). This theorem is used in the treatment of geometrically infinite two parabolic generator non-free Kleinian groups. In fact, it turns out there are no such groups. In Section 6, we introduce a convenient method for describing pared orbifolds (Convention 6.1) and the concept of an orbifold surgery (Definition 6.3), and then prove a simple but useful lemma for orbifold surgeries (Lemma 6.4). In Section 7, we follow Adams [1], and recall basic facts concerning two parabolic generator Kleinian groups, in particular an estimate of the length of parabolic generators with respect to the maximal cusp (Lemma 7.1). In Section 8, we give an outline of the proof of the main theorem. Sections 9, 10, and 11 are devoted to the proof of the main theorem. In the appendix, which consists of Sections 12 and 13, we give the classification of geometric dihedral orbifolds that is necessary for the proof Theorem 4.1.

Throughout this paper, we use the following notation.

Notation 1.3. (1) For an orbifold $\mathcal{O}$, the symbol $\pi_1(\mathcal{O})$ denotes the orbifold fundamental group of $\mathcal{O}$, $H_1(\mathcal{O})$ denotes the abelianisation of $\pi_1(\mathcal{O})$, and $H_1(\mathcal{O}; \mathbb{Z}_2)$ denotes $H_1(\mathcal{O}) \otimes \mathbb{Z}_2$.

(2) For a natural number n, $\mathbb{Z}_n$ denotes the cyclic group (or the ring) $\mathbb{Z}/n\mathbb{Z}$ of order n, and $(\mathbb{Z}_n)^\times$ denotes the unit group of the ring $\mathbb{Z}/n\mathbb{Z}$.

(3) By a *dihedral group*, we mean a group generated by two elements of order 2. Thus it is isomorphic to the group $D_n := \langle a, b \,|\, a^2, b^2, (ab)^n \rangle$ for some $n \in \mathbb{N} \cup \{\infty\}$. Note that D_n has order $2n$ or ∞ according to whether $n \in \mathbb{N}$ or $n = \infty$. Note also that the order 2 cyclic group D_1 is also regarded as a dihedral group.

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2. Basic facts concerning 2-bridge links

In this section, we recall basic facts concerning 2-bridge links, which we use in the definitions of the Heckoid orbifolds and the Heckoid groups. The description of 2-bridge links given in this section is a mixture of those in [14, 58].

Let $\mathcal{J}$ be the group of isometries of the Euclidean plane $\mathbb{R}^2$ generated by the π-rotations around the points in $\mathbb{Z}^2$. Set $(\boldsymbol{S}^2, \boldsymbol{P}^0) = (\mathbb{R}^2, \mathbb{Z}^2)/\mathcal{J}$ and call it the *Conway sphere*. Then $\boldsymbol{P}^0$ consists of four points in the 2-sphere $\boldsymbol{S}^2$. Let $\check{\boldsymbol{S}}^2 := \boldsymbol{S}^2 - \boldsymbol{P}^0$ be the complementary 4-times punctured sphere. For each $s \in \mathbb{Q} \cup \{\infty\}$, let α_s as be the simple loop in $\check{\boldsymbol{S}}^2$ obtained as the projection of a line in $\mathbb{R}^2 - \mathbb{Z}^2$ of slope s. Then α_s is *essential* in $\check{\boldsymbol{S}}^2$, i.e., it does not bound a disc nor a once-punctured disc in $\check{\boldsymbol{S}}^2$. Conversely, any essential simple loop in $\check{\boldsymbol{S}}^2$ is isotopic to α_s for a unique $s \in \mathbb{Q} \cup \{\infty\}$: we call s the *slope* of the essential loop. For each $s \in \mathbb{Q} \cup \{\infty\}$, let δ_s be the pair of mutually disjoint arcs in $\boldsymbol{S}^2$ with $\partial\delta_s = \boldsymbol{P}^0$, obtained as the image of the union of the lines in $\mathbb{R}^2$ which intersect $\mathbb{Z}^2$. Note that the union $\delta_{0/1} \cup \delta_{1/0}$ is a circle in $\boldsymbol{S}^2$ containing $\boldsymbol{P}^0$, which divides $\boldsymbol{S}^2$ into two discs $\boldsymbol{S}^2_+ := pr([0,1] \times [0,1])$ and $\boldsymbol{S}^2_- := pr([1,2] \times [0,1])$, where $pr : \mathbb{R}^2 \to \boldsymbol{S}^2$ is the projection.

Let $\boldsymbol{B}^3 := \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2+z^2 \le 2\}$ be the round 3-ball in $\mathbb{R}^3 \subset \mathbb{R}^3 \cup \{\infty\} \cong S^3$, whose boundary contains the set P^0 consisting of the four marked points

$$\mathrm{SW} := (-1,-1,0), \quad \mathrm{SE} := (1,-1,0), \quad \mathrm{NE} := (1,1,0), \quad \mathrm{NW} := (-1,1,0).$$

Fix a homeomorphism $\theta : (\boldsymbol{S}^2, \boldsymbol{P}^0) \to (\partial\boldsymbol{B}^3, P^0)$ satisfying the following conditions (see Figure 2).

(1) θ maps the quadruple $(pr(0,0), pr(1,0), pr(1,1), pr(0,1))$ to the quadruple $(\mathrm{SW}, \mathrm{SE}, \mathrm{NE}, \mathrm{NW})$.

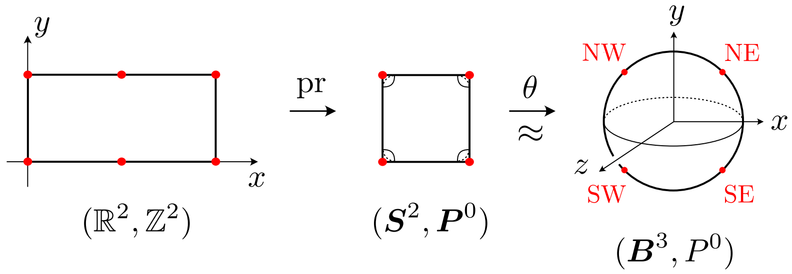


FIGURE 2. Conway sphere $(\boldsymbol{S}^2, \boldsymbol{P}^0) = (\mathbb{R}^2, \mathbb{Z}^2)/\mathcal{J}$ and the homeomorphism $\theta : (\boldsymbol{S}^2, \boldsymbol{P}^0) \to (\partial \boldsymbol{B}^3, \boldsymbol{P}^0)$

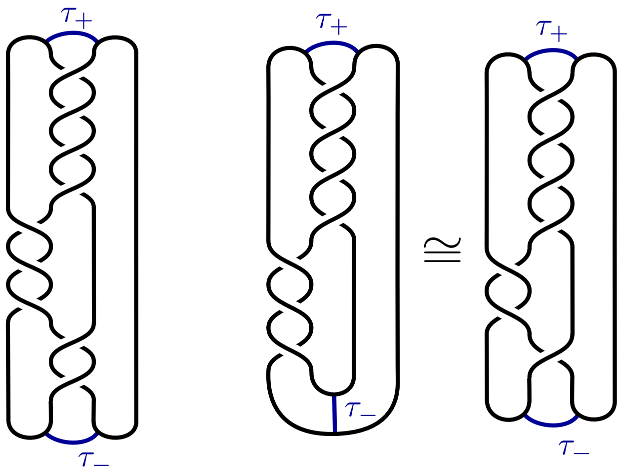
(2) θ maps the circle $\delta_{0/1} \cup \delta_{1/0}$ to the equatorial circle $\partial \boldsymbol{B}^3 \cap (\mathbb{R}^2 \times \{0\})$, and maps the hemispheres $\boldsymbol{S}^2_+$ and $\boldsymbol{S}^2_-$ onto the hemispheres $\partial \boldsymbol{B}^3 \cap (\mathbb{R}^2 \times \mathbb{R}_{\geq 0})$ to $\partial \boldsymbol{B}^3 \cap (\mathbb{R}^2 \times \mathbb{R}_{\leq 0})$, respectively.

(3) θ is equivariant with respect to the natural $(\mathbb{Z}_2)^2$-actions on $(\boldsymbol{S}^2, \boldsymbol{P}^0)$ and $(\partial \boldsymbol{B}^3, P^0)$. Here the natural $(\mathbb{Z}_2)^2$-action on $(\boldsymbol{S}^2, \boldsymbol{P}^0)$ is that which lifts to the group of isometries of the Euclidean plane $\mathbb{R}^2$ generated by the π-rotations around the points in $(\frac{1}{2}\mathbb{Z})^2$, and the natural $(\mathbb{Z}_2)^2$-action on $(\partial \boldsymbol{B}^3, P^0)$ is that generated by the π-rotations about the coordinate axes of $\mathbb{R}^3$.

We identify $(\partial \boldsymbol{B}^3, P^0)$ with $(\boldsymbol{S}^2, \boldsymbol{P}^0)$ through the homeomorphism θ. Thus for $s \in \mathbb{Q} \cup \{\infty\}$, α_s is regarded as an essential simple loop in $\partial \boldsymbol{B}^3 - P^0$, and δ_s is regarded as a union of two disjoint arcs in $\partial \boldsymbol{B}^3$ such that $\partial \delta_s = P^0$. Moreover, we can choose α_s and δ_s so that they are $(\mathbb{Z}_2)^2$-invariant.

For a rational number $r = q/p \in \mathbb{Q} \cup \{\infty\}$, let $t(r)$ be a pair of arcs properly embedded in $\boldsymbol{B}^3$ such that $t(r) \cap \partial \boldsymbol{B}^3 = \partial t(r) = P^0$, which is obtained from δ_r by pushing its interior into $\operatorname{int} \boldsymbol{B}^3$. The pair $(\boldsymbol{B}^3, t(r))$ is called the *rational tangle of slope* r. We may assume $t(r)$ is invariant by the natural $(\mathbb{Z}_2)^2$-action on $\boldsymbol{B}^3$. In particular, the x-axis intersects $t(r)$ transversely in two points: Let τ_r be the subarc of the x-axis they bound, and call it the *core tunnel* of $(\boldsymbol{B}^3, t(r))$ (see Figure 4). Two meridional circles of $t(r)$ near $\partial \tau_r$ together with a subarc of τ_r forms a graph in $\boldsymbol{B}^3 - t(r)$ homeomorphic to a pair of eyeglasses. This determines a *canonical generating meridian pair* of the rank 2 free group $\pi_1(\boldsymbol{B}^3 - t(r)) \cong \pi_1(\check{\boldsymbol{S}}^2)/\langle\langle \alpha_r \rangle\rangle$.

By gluing the boundaries of the rational tangles $(\boldsymbol{B}^3, t(\infty))$ and $(\boldsymbol{B}^3, t(r))$ by the identity map, we obtain a link in the 3-sphere: we denote it by $(S^3, K(r))$, and call it the 2-*bridge link of slope* $r = q/p$. The number of components, $|K(r)|$, of $K(r)$ is one or two (i.e., $K(r)$ is a knot or a two-component link) according to whether the denominator p is odd or even. The images of the core tunnels τ_∞ and τ_r in $(S^3, K(r))$ are called the *upper tunnel* and the *lower tunnel* of $K(r)$, respectively. We denote them by τ_+ and τ_-, respectively. The canonical generating meridian



$r = \dfrac{7}{29} = [4, 3, 2]$ $\qquad$ $r = \dfrac{3}{13} = [4, 3]$

FIGURE 3. 2-bridge link diagram

pairs of $\pi_1(\boldsymbol{B}^3 - t(\infty))$ and $\pi_1(\boldsymbol{B}^3 - t(r))$ descend to generating meridian pairs of the link group $\pi_1(S^3 - K(r)) \cong \pi_1(\check{\boldsymbol{S}}^2)/\langle\langle \alpha_\infty, \alpha_r \rangle\rangle$: we call them the *upper meridian pair* and the *lower meridian pair*, respectively.

When we need to care about the orientation of the ambient 3-sphere S^3, we regard $(S^3, K(r))$ as being obtained from $(-\boldsymbol{B}^3, t(\infty))$ and $(\boldsymbol{B}^3, t(r))$, where $\boldsymbol{B}^3$ inherits the standard orientation of $\mathbb{R}^3$. In other words, we identify the ambient 3-sphere S^3 with the one-point compactification $\mathbb{R}^3 \cup \{\infty\}$ of $\mathbb{R}^3$, in such a way that the $\boldsymbol{B}^3$ containing $t(r)$ is identified with the original round ball $\boldsymbol{B}^3$ via the identity map, whereas the $\boldsymbol{B}^3$ containing $t(\infty)$ is identified with $\mathrm{cl}(\mathbb{R}^3 \cup \{\infty\} - \boldsymbol{B}^3)$ via the inversion ι in $\partial \boldsymbol{B}^3$. Thus $K(r) = t(r) \cup \iota(t(\infty)) \subset \mathbb{R}^3 \cup \{\infty\} = S^3$. Under this orientation convention, a regular projection is read from the continued fraction expansion

$$r = [a_1, a_2, \cdots, a_n] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n}}},$$

in such a way that a_i corresponds to the a_i right-hand or left-hand half-twists according to whether i is odd or even (see Figure 3).

The natural $(\mathbb{Z}_2)^2$-actions on $(\boldsymbol{B}^3, t(\infty))$ and $(\boldsymbol{B}^3, t(r))$ can be glued to produce a $(\mathbb{Z}_2)^2$-action on $(S^3, K(r))$. Let f and h be the generators of the action whose restrictions to $(\boldsymbol{B}^3, t(\infty)))$ are the π-rotations about the y-axis and x-axis, respectively (see Figure 4). We call f, h, and fh, respectively, the *vertical involution*, the

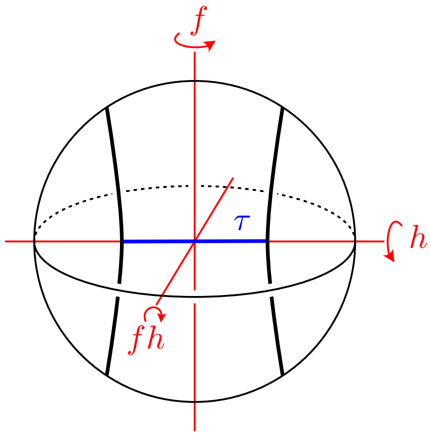


FIGURE 4. Natural $(\mathbb{Z}_2)^2$-actions on $(\boldsymbol{B}^3, t(\infty))$ consisting of the vertical involution f, the horizontal involution h, and the planar involution fh

horizontal involution, and the *planar involution* of $K(r)$. They are characterized by the following properties.

(1) $\mathrm{Fix}(h)$ contains τ_+, whereas each of $\mathrm{Fix}(f)$ and $\mathrm{Fix}(fh)$ intersects τ_+ transversely in a single point.
(2) The horizontal simple loop α_0 in $\partial(\boldsymbol{B}^3 - t(\infty))$ is mapped by f to itself preserving orientation, and it is mapped by fh to itself reversing orientation.

If the rational number $r = q/p$ satisfies the congruence $q^2 \equiv 1 \pmod p$, then $K(r)$ admits an additional orientation-preserving symmetry which interchanges $(\boldsymbol{B}^3, t(\infty))$ and $(\boldsymbol{B}^3, t(r))$. For a description of such symmetries, see e.g. [4, Sections 4 and 6], [56, Section 3].

We finally recall the classification theorem for 2-bridge links due to Schubert [60] (cf. [17, Chapter 12]).

Proposition 2.1. *For two rational numbers $r = q/p$ and $r' = q'/p'$, with p and p' positive, the following holds.*

(1) There is an orientation-preserving auto-homeomorphism φ of S^3 which maps $K(r)$ to $K(r')$ if and only if $p = p'$ and either $q \equiv q' \pmod p$ or $qq' \equiv 1 \pmod p$. Moreover the following hold.

(a) If $p = p'$ and $q \equiv q' \pmod p$, then there there is an orientation-preserving auto-homeomorphism φ of S^3 which maps $(K(r), \tau_+, \tau_-)$ to $(K(r'), \tau_+, \tau_-)$ and respects the $(\mathbb{Z}_2)^2$-action. Moreover, the conjugate of the vertical involution of $K(r)$ by φ is either the vertical or planar involution of $K(r')$, according to whether $q' \equiv q \pmod{2p}$ or $q' \equiv q + p \pmod{2p}$.

(b) *If $p = p'$ and $qq' \equiv 1 \pmod{p}$, then there there is an orientation-preserving auto-homeomorphism of S^3 which maps $(K(r), \tau_+, \tau_-)$ to $(K(r'), \tau_-, \tau_+)$ which respects the $(\mathbb{Z}_2)^2$-action.*

(2) There is an orientation-reversing auto-homeomorphism φ of S^3 which maps $K(r)$ to $K(r')$ if and only if $p = p'$ and either $q \equiv -q' \pmod{p}$ or $qq' \equiv -1 \pmod{p}$.

3. Heckoid orbifolds and Heckoid groups

In this section, we recall the definition of Heckoid orbifolds and Heckoid groups given by [34, Section 3].

Consider the quotient orbifold $(\boldsymbol{B}^3 - t(\infty))/(\mathbb{Z}_2)^2$, where $(\mathbb{Z}_2)^2$ is the natural action illustrated in Figure 4. Note that its boundary is identified with $\check{\boldsymbol{S}}^2/(\mathbb{Z}_2)^2 \cong S^2(2,2,2,\infty)$, which is the quotient of $\mathbb{R}^2 - \mathbb{Z}^2$ by the group generated by the π-rotations around the points in $(\frac{1}{2}\mathbb{Z})^2$. Note that $\pi_1(\check{\boldsymbol{S}}^2)$ is identified with a normal subgroup of $\pi_1(\check{\boldsymbol{S}}^2/(\mathbb{Z}_2)^2)$ of index 4. For each $s \in \mathbb{Q} \cup \{\infty\}$, let β_s be the simple loop in $\check{\boldsymbol{S}}^2/(\mathbb{Z}_2)^2$ obtained as the projection of a line in $\mathbb{R}^2 - (\frac{1}{2}\mathbb{Z})^2$ of slope s. The simple loop α_s in $\check{\boldsymbol{S}}^2$ doubly covers β_s, and so we have $\alpha_s = \beta_s^2$ as conjugacy classes in $\pi_1(\check{\boldsymbol{S}}^2/(\mathbb{Z}_2)^2)$.

For $r \in \mathbb{Q}$ and $m \in \mathbb{N}_{\geq 3}$, consider the 3-orbifold $\boldsymbol{B}(\infty; 2) := \mathrm{cl}(\boldsymbol{B}^3 - N(t_\infty))/(\mathbb{Z}_2)^2$, attach a 2-handle orbifold $D^2(m) \times I$ to it along the simple loop β_r. Since β_r divides $\check{\boldsymbol{S}}^2/(\mathbb{Z}_2)^2 \cong S^2(2,2,2,\infty)$ into $D^2(2,2)$ and $D^2(2,\infty)$, the resulting 3-orbifold has a spherical boundary $S^2(2,2,m) \cong S^2/D_m$, where D_m is the dihedral group of order $2m$ (cf. Notation 1.3(3)). Cap this spherical boundary with the 3-handle orbifold $B^3(2,2,m) \cong B^3/D_m$, and denote the resulting 3-orbifold by $\mathcal{H}(r;m)$. (Though this orbifold was denoted by $\mathcal{O}(r;m)$ in [34], we employ this symbol, because we use the symbol $\mathcal{O}$ to mainly denote spherical dihedral orbifolds.) Then we have

$$\pi_1(\mathcal{H}(r;m)) \cong \pi_1(S^2(2,2,2,\infty))/\langle\langle \beta_\infty^2, \beta_r^m \rangle\rangle.$$

Let P be the annular orbifold $\mathrm{fr}\, N(t_\infty)/(\mathbb{Z}_2)^2 \cong D^2(2,2)$ on $\partial\mathcal{H}(r;m)$, and continue to denote the orbifold pair $(\mathcal{H}(r;m), P)$ by the symbol $\mathcal{H}(r;m)$.

In [34, Section 6], it is proved that the orbifold pair $\mathcal{H}(r;m)$ is a pared 3-orbifold (see [10, Definition 8.3.7]).

Definition 3.1. An orbifold pair (M_0, P) is a *pared* 3*-orbifold* if it satisfies the following conditions

(1) M_0 is a compact, orientable, irreducible 3-orbifold which is very good (i.e., M_0 has a finite manifold cover).
(2) $P \subset \partial M_0$ is a disjoint union of incompressible toric and annular 2-suborbifolds.
(3) Every rank 2 free abelian subgroup of $\pi_1(M_0)$ is conjugate to a subgroup of some $\pi_1(P_i)$, where $P_i \subset P$ is a connected component.

(4) Any properly embedded annular 2-suborbifold $(A, \partial A)$ of (M_0, P) whose boundary rests on essential loops in P is parallel to P.

It is also observed in [34, Section 6] that $\mathcal{H}(r;m) = (\mathcal{H}(r;m), P)$ is a Haken pared orbifold (see [10, Definitions 8.0.1 and 8.3.7]). Hence, by the hyperbolization theorem of Haken pared orbifolds [10, Theorem 8.3.9], the pared orbifold $\mathcal{H}(r;m)$ admits a geometrically finite complete hyperbolic structure, namely, the interior of the orbifold $\mathcal{H}(r;m)$ admits a geometrically finite complete hyperbolic structure such that P represents the parabolic locus (see Section 5 for definitions).

Moreover, such a hyperbolic structure is unique, because the ends of the non-cuspidal part of $\mathcal{H}(r;m)$ are isomorphic to (a turnover) $\times [0, \infty)$, which are quasi-isometrically rigid, and every orbifold homeomorphism between two geometrically finite structures preserving the parabolicity in both directions is isotopic to a quasi-isometry, as can be seen by the same argument as Marden's theorem [36]. We denote the unique (up to conjugation) Kleinian group that uniformises the pared orbifold $\mathcal{H}(r;m)$ by the symbol $\pi_1(\mathcal{H}(r;m))$.

Now the Heckoid groups and the Heckoid orbifolds are defined as follows [34, p.242 and Definition 3.2].

Definition 3.2. For $r \in \mathbb{Q}$ and $n = \frac{m}{2} \in \frac{1}{2}\mathbb{N}_{\geq 3}$, the *Heckoid group $G(r;n)$ of slope r and index n* is the Kleinian group that is obtained as the image of the natural homomorphism

$$\psi : \pi_1(\mathrm{cl}(\boldsymbol{B}^3 - N(t_\infty))) \to \pi_1(\mathrm{cl}(\boldsymbol{B}^3 - N(t_\infty))/(\mathbb{Z}_2)^2) \to \pi_1(\mathcal{H}(r;m)) < \mathrm{PSL}(2, \mathbb{C}).$$

The *Heckoid orbifold $\mathcal{S}(r;n)$ of slope r and index n* is the pared orbifold, that is obtained as the covering of the pared orbifold $\mathcal{H}(r;m)$ associated with the subgroup $G(r;n) < \pi_1(\mathcal{H}(r;m))$. We also denote the Kleinian group $G(r;n)$ by $\pi_1(\mathcal{S}(r;n))$.

Then we have the following proposition. (The main Theorem 1.1 implies that the converse to the first assertion of the proposition holds.)

Proposition 3.3. *For any $r \in \mathbb{Q}$ and $n = \frac{m}{2} \in \frac{1}{2}\mathbb{N}_{\geq 3}$, the Heckoid group is a (non-free) Kleinian group with nontrivial torsion which is generated by two non-commuting parabolic transformations. Moreover, the image of the conjugacy class of the simple loop α_r in $G(r;n)$ is an elliptic transformation of rotation angle $\frac{2\pi}{n} = \frac{4\pi}{m}$.*

Proof. Let $\{x, y\}$ be the canonical generating meridian pair of the rank 2 free group $\pi_1(\mathrm{cl}(\boldsymbol{B}^3 - N(t_\infty)))$ (see Section 2). Then $G(r;n)$ is generated by the image $\{\psi(x), \psi(y)\} \subset \pi_1(\mathcal{H}(r;m))$. Since $\pi_1(\mathcal{H}(r;m))$ is the Kleinian group which uniformises the pared orbifold $\mathcal{H}(r;m)$, the generating pair of $G(r;n)$ consists of non-commuting parabolic transformations. Since $\alpha_r = \beta_r^2$ and since β_r is a meridian of the singular set of $\mathcal{H}(r;n)$ of index $2n = m$, it follows that $\psi(\alpha_r)$ is an elliptic transformation of rotation angle $\frac{2\pi}{n} = \frac{4\pi}{m}$. □

Next, we recall the topological description of the Heckoid orbifolds. In Definition 3.2, the Heckoid orbifold $\mathcal{S}(r;n)$ is defined as a covering of the pared orbifold $\mathcal{H}(r;m)$. Their explicit topological description is given by [34, Propositions 5.2 and 5.3], which says that the Heckoid orbifold $\mathcal{S}(r;n)$ is isomorphic to one of the orbifold pairs depicted in Figure 1, that is specified by the following formula.

$$\mathcal{S}(r;n) \cong \begin{cases} \mathcal{M}_0(r;n) & \text{if } n \in \mathbb{N}_{\geq 2}, \\ \mathcal{M}_1(\hat{r};m) & \text{if } n = m/2 \text{ for some odd } m > 2 \text{ and if } p \text{ is odd}, \\ \mathcal{M}_2(\hat{r};m) & \text{if } n = m/2 \text{ for some odd } m > 2 \text{ and if } p \text{ is even}, \end{cases}$$

where $\hat{r}$ is defined from $r = q/p$ by the following rule.

$$\hat{r} = \begin{cases} \frac{q/2}{p} & \text{if } p \text{ is odd and } q \text{ is even}, \\ \frac{(p+q)/2}{p} & \text{if } p \text{ is odd and } q \text{ is odd}, \\ \frac{q}{p/2} & \text{if } p \text{ is even}. \end{cases}$$

Thus the following precise definition of the orbifold pairs in Figure 1 gives an explicit topological picture of the Heckoid orbifold $\mathcal{S}(r;n)$.

Definition 3.4. (1) For $r \in \mathbb{Q}$ and for a positive integer $n \geq 2$, $\mathcal{M}_0(r;n)$ denotes the orbifold pair determined by the weighted graph $(S^3, K(r) \cup \tau_-, w_0)$, where w_0 is given by

$$w_0(K(r)) = \infty, \quad w_0(\tau_-) = n.$$

(2) For $r = q/p \in \mathbb{Q}$ with p odd and an odd integer $m \geq 3$, $\mathcal{M}_1(r;m)$ denotes the orbifold pair determined by the weighted graph $(S^3, K(r) \cup \tau_-, w_1)$, where w_1 is given by the following rule. Let J_1 and J_2 be the edges of the graph $K(r) \cup \tau_-$ distinct from τ_-. Then

$$w_1(J_1) = \infty, \quad w_1(J_2) = 2, \quad w_1(\tau_-) = m.$$

(3) For $r = q/p \in \mathbb{Q}$ and an odd integer $m \geq 3$, $\mathcal{M}_2(r;m)$ denotes the orbifold pair determined by the weighted graph $(S^3, K(r) \cup \tau_+ \cup \tau_-, w_2)$, where w_2 is given by the following rule. Let J_1 and J_2 be unions of two mutually disjoint edges of the graph $K(r) \cup \tau_+ \cup \tau_-$ distinct from $\tau_\pm$. Moreover, if p is even, then both J_1 and J_2 are preserved by the vertical involution f of $K(r)$. (Thus f interchanges the two components of each of J_1 and J_2.) Then

$$w_2(J_1) = \infty, \quad w_2(J_2) = 2, \quad w_2(\tau_+) = 2, \quad w_2(\tau_-) = m.$$

In Definition 3.4(3), the 'identity' $w_2(J_1) = \infty$ means that $w_2(e) = \infty$ for each edge e contained in J_1. Similarly, $w_2(J_2) = 2$ means that $w_2(e) = 2$ for each edge e contained in J_2. We employ this kind of convention throughout the paper.

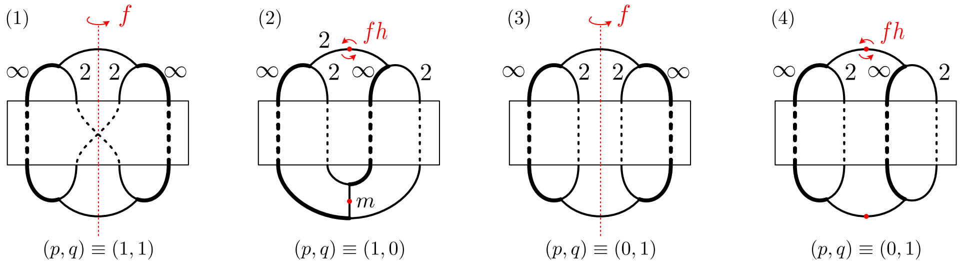


FIGURE 5. The first two figures (1) and (2) illustrate Remark 3.5(2), and the last two figures (3) and (4) illustrate Remark 3.5(3). See also [4, Figures in Section 7].

Remark 3.5. (1) Because of the $(\mathbb{Z}_2)^2$-symmetry of 2-bridge links, the choice of the edges J_1 and J_2 in (2) and (3) does not affect the isomorphism class of the resulting orbifolds (see [34, Remark 5.4]).

(2) Suppose p is odd. Then, in the definition of $\mathcal{M}_2(r;m)$, the disjointness condition of J_1 and J_2 determines the pair (J_1, J_2) up to the horizontal involution h of $(S^3, K(r) \cup \tau_+ \cup \tau_-)$. Moreover, according to whether q is odd or even, both J_1 and J_2 are preserved by f or fh, respectively (see Figure 5(1),(2)).

(3) Suppose p is even. Then, in the definition of $\mathcal{M}_2(r;m)$, the condition that both J_1 and J_2 are preserved by f is not essential in the following sense. Let $J_{i,1}$ and $J_{i,2}$ be the components of J_i for $i = 1, 2$, such that $J_{1,1} \cap J_{2,1} = \emptyset$ and $J_{1,2} \cap J_{2,2} = \emptyset$. Set $J_1' = J_{1,1} \cup J_{2,1}$ and $J_2' = J_{1,2} \cup J_{2,2}$. Then J_1' and J_2' are unions of two mutually disjoint edges of the graph $K(r) \cup \tau_+ \cup \tau_-$ distinct from $\tau_\pm$, such that both J_1' and J_2' are preserved by the planar involution fh, instead of the vertical involution f (see Figure 5(3),(4)). Let w_2' be the weight function on the graph $K(r) \cup \tau_+ \cup \tau_-$ defined by

$$w_2'(J_1') = \infty, \quad w_2'(J_2') = 2, \quad w_2'(\tau_+) = 2, \quad w_2'(\tau_-) = m.$$

Then $(S^3, K(r) \cup \tau_+ \cup \tau_-, w_2')$ represents the orbifold $\mathcal{M}_2(r';m)$, where $r' = (p+q)/p$ for $r = q/p$. This follows from the fact that there is a homeomorphism from $(S^3, K(r) \cup \tau_+ \cup \tau_-)$ to $(S^3, K(r') \cup \tau_+ \cup \tau_-)$ sending $(\tau_\pm, J_1, J_2)$ to $(\tau_\pm, J_1', J_2')$ (see Proposition 2.1(1a)).

4. Classification of dihedral orbifolds

In this section, we give a classification of the dihedral orbifolds, which plays a key role in the proof of the main theorem. We refer to [8, 9, 20] for standard terminologies for orbifolds.

By using the the orbifold theorem, the geometrisation theorem of compact orientable 3-manifolds, and the classification of geometric dihedral orbifolds (see Appendix), we obtain the following classification of good orbifolds with dihedral orbifold fundamental groups.

Theorem 4.1. *Let $\mathcal{O}$ be a compact orientable 3-orbifold with nonempty singular set satisfying the following conditions.*

(i) *$\mathcal{O}$ does not contain a bad 2-suborbifold.*
(ii) *Any component of $\partial\mathcal{O}$ is not spherical.*
(iii) *$\pi_1(\mathcal{O})$ is a dihedral group.*

Then $\mathcal{O}$ is isomorphic to one of the following orbifolds.

(1) The spherical dihedral orbifold $\mathcal{O}(r; d_+, d_-)$ represented by the weighted graph $(S^3, K(r) \cup \tau_+ \cup \tau_-, w)$ for some $r \in \mathbb{Q}$ and coprime positive integers d_+ and d_-, where w is given by the following rule (see Figure 6).

$$w(K(r)) = 2, \quad w(\tau_+) = d_+, \quad w(\tau_-) = d_-.$$

(2) The $S^2 \times \mathbb{R}$ orbifold $\mathcal{O}(\infty)$ represented by the weighted graph $(S^3, K(\infty), w)$, where w takes the value 2 at each component of the 2-bridge link $K(\infty)$ of slope ∞, i.e. the 2-component trivial link.

(3) The $S^2 \times \mathbb{R}$ orbifold $\mathcal{O}(\mathbb{RP}^3, O)$ represented by the weighted graph $(\mathbb{RP}^3, O, w)$, where O is the trivial knot in the projective 3-space $\mathbb{RP}^3$ with $w(O) = 2$.

(4) The orbifold $D^2(2, 2) \times I$.

Remark 4.2. For the orbifold $\mathcal{O}(r; d_+, d_-)$, if $d_+ = 1$ (resp. $d_- = 1$), then τ_+ (resp. τ_-) does not belong to the singular set (cf. Convention 6.2(1)). In particular, $\mathcal{O}(r) := \mathcal{O}(r; 1, 1)$ is the π-orbifold associated with the 2-bridge link $K(r)$ in the sense of [11], i.e. the orbifold with underlying space S^3 and with singular set $K(r)$, whose index is 2. In Adam's classification of torsion-free Kleinian groups generated by two parabolic transformations [1, Theorem 4.3], the π-orbifolds $\mathcal{O}(r)$ played a key role, whereas the orbifolds $\mathcal{O}(r; d_+, d_-)$ play the corresponding key role in this paper.

Proof. Let $\mathcal{O}$ be a 3-orbifold satisfying the three conditions. We first treat the case where $\mathcal{O}$ is irreducible, i.e., any spherical 2-suborbifold of $\mathcal{O}$ bounds a *discal 3-suborbifold* (a quotient of a 3-ball by a finite orthogonal group). We can observe that $\mathcal{O}$ is topologically atoroidal as follows. Suppose on the contrary that $\mathcal{O}$ contains an essential toric suborbifold F. Then the inclusion map induces an injective homomorphism from $\pi_1(F)$ into $\pi_1(\mathcal{O})$, as explained below. Since $\mathcal{O}$ does not contain a bad 2-suborbifold by the condition (i), $\mathcal{O}$ is very good, by [8, Corollary 1.3]. Thus by applying the equivariant loop theorem to the group action, $\pi_1(F)$ embeds into $\pi_1(\mathcal{O})$ (see [9, Corollary 3.20]). This contradicts the fact that the dihedral group $\pi_1(\mathcal{O})$ does not contain $\mathbb{Z}^2$.

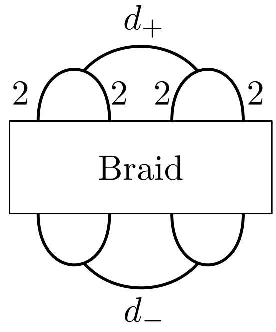


FIGURE 6. The spherical dihedral orbifold $\mathcal{O}(r; d_+, d_-)$

Hence, by the orbifold theorem [8, Corollary 1.2], $\mathcal{O}$ is geometric, i.e., either $\operatorname{int}\mathcal{O}$ admits one of Thurston's geometries or $\mathcal{O}$ is a discal 3-orbifold. The latter possibility does not happen by the assumption (ii), and so $\operatorname{int}\mathcal{O}$ admits one of Thurston's geometry. If the geometry is S^3, then by Proposition 12.2, $\mathcal{O}$ is isomorphic to the orbifold $\mathcal{O}(r; d_+, d_-)$ in (1). If the geometry is $S^2 \times \mathbb{R}$, then by Proposition 13.1, $\mathcal{O}$ is isomorphic to the orbifold $\mathcal{O}(\infty)$ in (2) or the orbifold $\mathcal{O}(\mathbb{RP}^3, O)$ in (3). (But this does not happen, because these orbifolds are reducible whereas we currently assume that $\mathcal{O}$ is irreducible.) If the geometry is one of the remaining 6 geometries, then by Proposition 13.2, $\mathcal{O}$ is isomorphic to the orbifold $D^2(2,2) \times I$ in (4).

Next, we treat the case when $\mathcal{O}$ is reducible. Note that $\mathcal{O}$ does not contain a non-separating spherical 2-suborbifold, because $H_1(\mathcal{O})$ is finite. Thus we do not need to worry about the paradoxical problems concerning spherical splitting of 3-orbifolds pointed out by Petronio [50]. By [50, Theorems 0.1], there is a finite system of spherical 2-suborbifolds $\mathcal{S}$ such that (a) no component of $\mathcal{O} - \mathcal{S}$ is *punctured discal* (a discal 3-orbifold minus regular neighbourhoods of a finite set) and (b) all *prime factors* of $\mathcal{O}$ (the orbifolds obtained from the components of $\mathcal{O} - \mathcal{S}$ by capping the boundary components with discal orbifolds) are irreducible. It should be noted that some prime component may be a manifold, i.e., its branching locus is empty. By Perelman's geometrisation theorem of compact orientable 3-manifolds [47, 48, 49] (see also [7, 18, 32, 44, 45]) and the the geometrisation theorem of compact orientable 3-orbifolds (see e.g. [9, Theorem 3.27]), each prime component of $\mathcal{O}$ admits a canonical decomposition into geometric pieces by a family of essential toric 2-orbifolds. In particular, each prime factor has a nontrivial orbifold fundamental group. Since the only nontrivial free product decomposition of a dihedral group is the decomposition of the infinite dihedral group D_∞ into the free product $\mathbb{Z}_2 * \mathbb{Z}_2$, $\mathcal{O}$ is the connected sum (along a 2-sphere with empty branching set) of two irreducible 3-orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$, such that $\pi_1(\mathcal{O}_i) \cong \mathbb{Z}_2$. Since $\mathcal{O}_i$ is geometric, $\mathcal{O}_i$ is isomorphic to (a) the discal 3-orbifold $B^3/\mathbb{Z}_2$, (b) the orbifold (S^3, O, w), where O is a trivial knot and $w(O) = 2$, or (c) $\mathbb{RP}^3$. By condition (ii), $\mathcal{O}_i$ cannot be a discal orbifold. Since

$\mathcal{O} = \mathcal{O}_1 \# \mathcal{O}_2$ has nonempty ramification locus, at least one of $\mathcal{O}_i$ is not isomorphic to $\mathbb{RP}^3$. Hence, $\mathcal{O}$ is isomorphic to the orbifold $(S^3, O, w) \# (S^3, O, w) \cong \mathcal{O}(\infty)$ in (2) or the orbifold $(S^3, O, w) \# \mathbb{RP}^3 \cong \mathcal{O}(\mathbb{RP}^3, O)$ in (3). □

Remark 4.3. By considering the image of $\mathcal{O}(r; d_+, d_-)$ by a π-rotation about a horizontal axis in Figure 6, we can interchange the role of d_+ and d_-. To be precise, we can see from Proposition 2.1(1b) that $\mathcal{O}(q/p; d_+, d_-) \cong \mathcal{O}(q'/p; d_-, d_+)$ if $qq' \equiv 1 \pmod p$.

5. Relative tameness theorem for hyperbolic orbifolds

We first recall basic terminology for hyperbolic orbifolds, following [9, Chapter 6]. Let Γ be a finitely generated Kleinian group and $M = \mathbb{H}^3/\Gamma$ the quotient hyperbolic orbifold. For a real number $\epsilon > 0$, the ϵ-thin part $M_{(0,\epsilon]}$ of M is the set of all points $x \in M$ such that $d(\tilde{x}, \gamma\tilde{x}) \le \epsilon$ for some lift $\tilde{x}$ of x to $\mathbb{H}^3$ and some $\gamma \in \Gamma$ of order $> 1/\epsilon$ (including ∞). By the Margulis Lemma, there is a constant $\mu > 0$, such that for any real number $\epsilon \in (0, \mu]$, each component X of $M_{(0,\epsilon]}$ is either a Margulis tube or a cuspidal end. Here a *Margulis tube* is a compact quotient of the r-neighbourhood of a geodesic in $\mathbb{H}^3$ by an elementary subgroup of Γ which preserves the geodesic, and a *cuspidal end* is the quotient of a horoball in $\mathbb{H}^3$ by an elementary parabolic subgroup of Γ which preserves the horoball.

Topologically, a cuspidal end is a product $F \times [0, +\infty)$, where F is a Euclidean 2-orbifold. Thus we have the following possibilities for F.

(1) F is the open annulus $S^1 \times \mathbb{R}$ or $S^2(2, 2, \infty)$, the quotient of $S^1 \times \mathbb{R}$ by an involution.
(2) F is the torus T^2 or $S^2(2, 2, 2, 2)$, the quotient of T^2 by an involution.
(3) F is $S^2(2, 3, 6)$, $S^2(2, 4, 4)$ or $S^2(3, 3, 3)$, the quotient of T^2 by a finite cyclic group action of order 6, 4 or 3, respectively.

A cusp $F \times [0, +\infty)$ is said to be *rigid* if $F \cong S^2(2, 3, 6)$, $S^2(2, 4, 4)$ or $S^2(3, 3, 3)$. Otherwise it is said to be *flexible*. It is well-known that a cusp $F \times [0, +\infty)$ is rigid if and only if the holonomy representation of the orbifold fundamental group $\pi_1(F \times [0, +\infty)$ admits no nontrivial deformation (see [39, Proposition 1]).

Let $M^{\mathrm{cusp}}_{(0,\epsilon]}$ be the union of the cuspidal ends of $M_{(0,\epsilon]}$, and let $M_0 := M - \mathrm{int}\, M^{\mathrm{cusp}}_{(0,\epsilon]}$ be the non-cuspidal part of M. Then $P := \partial M_0$ is a disjoint union of euclidean 2-orbifolds, and is called the *parabolic locus* of M_0. Note that $M \cong \mathrm{int}\, M_0$ and that P consists of (*closed*) toric orbifolds (closed 2-orbifolds obtained as quotients of the 2-dimensional torus) and *open* annular orbifolds (open 2-orbifolds obtained as quotients of the open annulus $S^1 \times \mathbb{R}$).

The following theorem is an orbifold version of (the relative version of) the tameness theorem established by Agol [2] and Calegari-Gabai [19] (see also Soma [62] and Bowditch [15]).

Theorem 5.1. *Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-orbifold with finitely generated orbifold fundamental group Γ. Then there is a compact 3-orbifold $\bar{M}_0$ and a compact suborbifold $\bar{P}$ of $\partial \bar{M}_0$, such that (i) $\operatorname{int} \bar{M}_0 = \operatorname{int} M_0 \cong M$ and (ii) the interior of $\bar{P}$ in $\partial \bar{M}_0$ is equal to $P = \partial M_0$.*

Proof. We give a proof following the arguments of Bowditch [15, Section 6.6] (cf. [2, Lemma 14.3]). By Selberg's lemma, M admits a finite regular manifold cover, namely there is a complete hyperbolic manifold N and a finite group G of orientation-preserving isometries of N such that $N/G \cong M$. The inverse image, N_0, of M_0 in N forms a G-invariant non-cuspidal part of N, and we have $N_0/G \cong M_0$. By the relative version of the tameness theorem [19, Theorem 7.3] (cf. [15, Section 6]), there is a compact 3-manifold $\bar{N}_0$ and a compact submanifold $\bar{Q}$ of $\partial \bar{N}_0$, such that (i) $\operatorname{int} \bar{N}_0 = \operatorname{int} N_0$ and (ii) the interior of $\bar{Q}$ in $\partial \bar{N}_0$ is equal to ∂N_0. Let $D(N_0)$ and be the double of N_0 along ∂N_0. Then the action of G on N_0 extends to an action on $D(N_0)$, and $D(N_0)/G$ is isomorphic to the double, $D(M_0)$, of M_0 along ∂M_0. Consider the double, $D(\bar{N}_0)$, of $\bar{N}_0$ along $\bar{Q}$. Then $D(\bar{N}_0)$ is a compact manifold with interior $D(N_0)$. By [42, Theorem 8.5], the action of G on $D(N_0)$ extends to an action on $D(\bar{N}_0)$, and $\operatorname{int}(D(\bar{N}_0)/G) = D(N_0)/G$ is identified with $D(M_0)$. Let $\bar{M}_0$ be the closure in $D(\bar{N}_0)/G$ of one of the two copies of M_0 in $D(M_0) \subset D(\bar{N}_0)/G$, and let $\bar{P}$ be the image of $\bar{Q} \subset D(\bar{N}_0)$ in $D(\bar{N}_0)/G$. Then the pair $(\bar{M}_0, \bar{P})$ satisfies the desired conditions. □

The above theorem together with the following theorem enables us to reduce the treatment of geometrically infinite case to that of geometrically finite case.

Theorem 5.2. *Under the setting of Theorem 5.1, $(\bar{M}_0, \bar{P})$ is a pared orbifold. Moreover, the pared orbifold $(\bar{M}_0, \bar{P})$ admits a geometrically finite complete hyperbolic structure. Namely, there is a geometrically finite Kleinian group Γ' such that (i) the orbifold $\mathbb{H}^3/\Gamma'$ is isomorphic to the orbifold $\operatorname{int} \bar{M}_0 \cong M$ and (ii) P is the parabolic locus of Γ'.*

Proof. The first assertion that $(\bar{M}_0, \bar{P})$ is a pared orbifold can be proved as in the proof of [43, Corollary 6.10 in Chapter V]. So we prove the second assertion that the pared orbifold $(\bar{M}_0, \bar{P})$ admits a geometrically finite hyperbolic structure. If the orbifold $\bar{M}_0$ is Haken in the sense of [10, Definition 8.0.1] then it follows from [10, Theorem 8.3.9] that the pared orbifold $(\bar{M}_0, \bar{P})$ admits a geometrically finite hyperbolic structure, as desired. So we may assume the orbifold $\bar{M}_0$ is non-Haken, i.e., either it contains no essential 2-suborbifold or it contains an essential turnover. In the first case, ∂M_0 consists only of turnovers by [9, Proposition 9.4]. This implies that every end of $M \cong \operatorname{int} \bar{M}_0$ has a neighbourhood isomorphic to the product of (a turnover) $\times [0, \infty)$. Since a hyperbolic turnover is always realised by a totally geodesic surface, each end has a neighbourhood containing no closed geodesics. Thus every end of the hyperbolic orbifold M is geometrically finite and rigid. Thus M admits a unique complete hyperbolic structure, and it is geometrically finite. In the latter case, by the turnover splitting theorem [9, Theorem 4.8], $\bar{M}_0$ admits a decomposition by a finite disjoint family of essential hyperbolic turnovers into Haken orbifolds and small orbifolds. By the orbifold theorem, each piece admits a geometrically finite hyperbolic structure, respecting the parabolic locus. By gluing these hyperbolic structures along the totally geodesic hyperbolic turnovers, we obtain a geometrically finite hyperbolic structure on $(\bar{M}_0, \bar{P})$. □

Remark 5.3. In [2], Agol suggested to prove the last assertion of Theorem 5.2 by using a relative version of the work of Feighn and Mess [23, Theorem 2] which proves the existence of a compact core of an orbifold $M = \mathbb{H}^3/\Gamma$ with a finitely generated orbifold fundamental group Γ. Such a relative version is proved by Matsuzaki [39, Lemma 2] under the assumption that Γ is indecomposable (over finite cyclic groups and with respect to the parabolic subgroups) in the sense of [39, Definition in p.26]. But we are not sure if non-free two-parabolic generator Kleinian groups satisfy this property. Though Theorem 5.1, which is proved by using the deep tameness theorem, of course, guarantees the existence of a relative core of complete hyperbolic orbifolds with finitely generated fundamental groups, we are not sure if more 'elementary' proof is possible.

6. Orbifold surgery

In this section, we introduce a convenient method for representing pared orbifolds by weighted graphs, generalising the convention in the introduction (Convention 6.1). Then we introduce the concept of an orbifold surgery (Definition 6.3), which is a key ingredient of the proof of the main theorem, and prove a basic Lemma 6.4 for the orbifold surgery. At the end of this section, we also state another basic Lemma 6.5 concerning the $\mathbb{Z}_2$-homology of an orbifold, which is repeatedly used in the proof of the main theorem.

Convention 6.1. Consider a triple (W, Σ, w), where W is a compact oriented 3-manifold, Σ is a finite trivalent graph properly embedded in W, and w is a function on the edge set of Σ which takes value in $\mathbb{N}_{\geq 2} \cup \{\infty\}$. Here, a loop component of Σ is regarded as a single edge, $\Sigma \cap \partial W$ is the set of degree 1 vertices of Σ, and all other vertices have degree 3. For each edge e of Σ, its value $w(e)$ by w is called the *weight* of the edge. We call the triple (W, Σ, w) a *weighted graph* and call w the *weight function* of the weighted graph. Let Σ_∞ be the subgraph of Σ consisting of the edges with weight ∞, and let Σ_s be the subgraph of Σ consisting of the edges with integral weight.

We regard each component, F, of ∂W as a 2-orbifold as follows: the underlying space is the complement of an open regular neighbourhood of $F \cap \Sigma_\infty$ in F, and the singular set is $F \cap \Sigma_s$, where the index of a singular point is given by the weight of the corresponding edge of Σ_s. We assume that the following condition (SC) is satisfied.

(SC) For any sphere component S of ∂W, the corresponding 2-orbifold is not a bad orbifold, a spherical orbifold, a discal orbifold, nor an annulus. Namely, (i) $|S \cap W| \geq 3$ and (ii) if $|S \cap W| = 3$ then $\sum_{i=1}^{3} \frac{1}{w(e_i)} \leq 1$, where e_i $(i = 1, 2, 3)$ are the (germs of) edges of Σ which have an endpoint in F.

A trivalent vertex v of Σ is said to be *spherical*, *euclidean* or *hyperbolic* according to whether $\sum_{i=1}^{3} \frac{1}{w(e_i)}$ is bigger than, equal to, or smaller than 1, where e_i $(i = 1, 2, 3)$ are the (germs of) edges incident on v. Let V_E (resp. V_H) be the set of the euclidean (resp. hyperbolic) vertices.

Let M_0 be the complement of an open regular neighbourhood of $\Sigma_\infty \cup V_E \cup V_H$ in M. Then M_0 has the structure of an orbifold, with singular set $\Sigma_0 := M_0 \cap \Sigma_s$, where the indices of the edges of Σ_0 are given by w.

For each edge e of Σ_∞, let $m_e \subset \partial M_0$ be a meridian loop of e, let P_∞ be the disjoint union of the regular neighbourhoods in ∂M_0 of m_e, where e runs over the edges of Σ_∞. The condition (SC) implies that each component of $\mathrm{cl}(\partial M_0 - P_\infty)$ is either a euclidean or hyperbolic 2-orbifold. Let P be the union of P_∞ and the euclidean components of $\mathrm{cl}(\partial M_0 - P_\infty)$. Then P is a disjoint union of euclidean 2-orbifolds.

We call (M_0, P) the *orbifold pair determined by the weighted graph* (M, Σ, w).

Convention 6.2. It is sometimes convenient to employ the following slight extension of Convention 6.1.

(1) We allow w to have an edge e with $w(e) = 1$. In this case, we consider the weighted graph (W, Σ', w'), where Σ' is the subgraph of Σ consisting of those edges with $w(e) \neq 1$ and w' is the restriction of w to Σ'. If Σ' is also trivalent graph properly embedded in W and the condition (SC) is satisfied, then we define the orbifold pair determined by (W, Σ, w) to be that determined by (W, Σ', w').

(2) We allow a quadrivalent vertex, v, such that the four edge germs incident on it have index 2. In this case, v represents a parabolic locus, $P(v)$, isomorphic to $S^2(2, 2, 2, 2)$.

A key ingredient of the proof of the main theorem is an orbifold surgery.

Definition 6.3. Let (M_0, P) be a pared orbifold, represented by a weighted graph (W, Σ, w) satisfying the condition (SC). By replacing the weight function w with another weight function w' (which also takes value in $\mathbb{N}_{\geq 2} \cup \{\infty\}$), we obtain another weighted graph (W, Σ, w'). This fails to satisfy the condition (SC) only when some sphere component S of the topological boundary ∂W determines a spherical 2-orbifold with three singular points. In this case, we cap all such sphere boundaries of W with a cone over $(S, S \cap \Sigma)$ to obtain a new weighted graph, which we call the *augmentation* of (W, Σ, w'). It satisfies the condition (SC), and determines an orbifold pair (N_0, Q). We call the 3-orbifold $\mathcal{O} := N_0$ the orbifold obtained from (M_0, P) by the *orbifold surgery* determined by the replacement of the weight function w with the new weight function w'.

The following simple lemma is used repeatedly in the proof of the main theorem.

Lemma 6.4. *Let (M_0, P) be a pared orbifold, and let $\mathcal{O}$ be the orbifold obtained from (M_0, P) by an orbifold surgery. Then $\mathcal{O}$ does not contain a bad 2-suborbifold and $\partial\mathcal{O}$ does not contain a spherical component. In particular, $\mathcal{O}$ is very good.*

Proof. Let (W, Σ, w) be a weighted graph representing the pared orbifold (M_0, P), and let w' be the weight function on Σ that gives the orbifold $\mathcal{O} = N_0$, where (N_0, Q) is the orbifold pair that is represented by the augmentation of (W, Σ, w'). Assume to the contrary that N_0 contains a bad 2-suborbifold, S, which is either a teardrop $S^2(n)$ or a spindle $S^2(m, n)$ for some integers $m > n \geq 2$. Since the underlying space $|S|$ is disjoint from the vertex set of the singular set, $\Sigma(N_0)$, of N_0, we may assume $|S|$ is a submanifold of W transversal to Σ. Then it determines a suborbifold, S^*, of M_0, such that $|S^*| = |S| \cap |M_0|$. The singular set of S^* is equal to $|S^*| \cap \Sigma_s$, where Σ_s is the subgraph of Σ consisting of the edges of integral w-weight, and the index of each singular point is given by the w-weight of the corresponding edge of Σ_s.

First, suppose that $S \cong S^2(n)$ for $n \geq 2$. Let e be the edge of Σ such that $|S| \cap e$ is the singular point of S. If e is an edge of Σ_s, then S^* is isomorphic to the teardrop $S^2(w(e))$, which contradicts the fact that M_0 is good. If e is an edge of Σ_∞, then S^* is a disc whose boundary is an essential simple loop on P. This contradicts the fact that P is incompressible in M_0.

Next, suppose that $S \cong S^2(m, n)$ for $m > n \geq 2$. Let e_1 and e_2 be the edges of Σ corresponding to the singular point of S of index m and n, respectively. Then $w'(e_1) = m \neq n = w'(e_2)$, and so e_1 and e_2 are distinct. If both e_1 and e_2 are contained in Σ_s, then $S^* \cong S^2(m^*, n^*)$ for some $m^*, n^* \geq 2$. Since M_0 does not contain a bad 2-suborbifold, m^* and n^* must be equal, and hence S^* is an spherical suborbifold of M_0. Since M_0 is irreducible, S^* bounds a discal 3-orbifold. This implies e_1 and e_2 determine the same edge of $\Sigma(N_0)$. By the condition (SC), this in turn implies $e_1 = e_2$, a contradiction. If exactly one of e_1 and e_2 is contained in Σ_s, then S^* is a discal orbifold whose boundary is an essential simple loop on P. This contradicts the assumption that P is incompressible in M_0. If none of e_1 and e_2 is contained in Σ_s, then S^* is an annulus whose boundary consists of a pair of essential simple loops on P. Thus S^* is parallel to P by Definition 3.1(4), and so $e_1 = e_2$, a contradiction.

Thus we have proved that $\mathcal{O} = N_0$ does not contain a bad 2-suborbifold. The assertion that $\partial\mathcal{O}$ does not contain a spherical orbifold follows from the fact that $\mathcal{O} = N_0$ is represented by the augmentation of (W, Σ, w'). The assertion that $\mathcal{O}$ is very good follows from [8, Corollary 1.3], which is a consequence of the orbifold theorem. □

Another key tool for the proof of the main theorem is the homology with $\mathbb{Z}_2$ coefficient. Under Notation 1.3, we have the following lemma, which can be easily deduced from the definition of $H_1(\mathcal{O}; \mathbb{Z}_2)$ and the Alexander duality.

Lemma 6.5. *Suppose an orbifold $\mathcal{O}$ is represented by a weighted graph (S^3, Σ, w) in S^3. Let $\Sigma_{\rm even}$ be the subgraph of Σ spanned by the edges of even weight. Then $H_1(\mathcal{O}; \mathbb{Z}_2)$ is determined by $H_1(\Sigma_{\rm even}; \mathbb{Z}_2)$. To be precise, we have the following natural isomorphisms.*

$$H_1(\mathcal{O}; \mathbb{Z}_2) \cong H_1(S^3 - \Sigma_{\rm even}; \mathbb{Z}_2) \cong H^1(\Sigma_{\rm even}; \mathbb{Z}_2) \cong \mathrm{Hom}(H_1(\Sigma_{\rm even}; \mathbb{Z}_2), \mathbb{Z}_2)$$

In particular, the following hold.

(1) $H_1(\mathcal{O}; \mathbb{Z}_2)$ is generated by the meridians of edges of $\Sigma_{\rm even}$.
(2) The meridian of an edge of Σ of odd degree represents the trivial element of $H_1(\mathcal{O}; \mathbb{Z}_2)$.
(3) Let e_i $(i = 1, 2, 3)$ be edges of Σ incident on a vertex of Σ, and suppose that $w(e_1)$ is odd and $w(e_2)$ and $w(e_3)$ are even. Then the meridians of e_2 and e_3 represent the same element of $H_1(\mathcal{O}; \mathbb{Z}_2)$.

7. Canonical horoball pairs for Kleinian groups generated by two parabolic transformations

Throughout Sections 7 $\sim$ 11, $\Gamma = \langle \alpha, \beta \rangle$ denotes a non-elementary Kleinian group generated by two parabolic transformations α and β, and $M = \mathbb{H}^3/\Gamma$ denotes the quotient hyperbolic 3-orbifold. Let η be the geodesic joining the parabolic fixed points of α and β, and let h be the π-rotation around η. Then we have

$$(h\alpha h^{-1}, h\beta h^{-1}) = (\alpha^{-1}, \beta^{-1}).$$

We call h the *inverting elliptic element* for the parabolic generating pair $\{\alpha, \beta\}$ of the Kleinian group Γ. As shown in [64, Section 5.4], we can find a geodesic intersecting η orthogonally, such that the π-rotation, f, around it satisfies the following identity.

$$(f\alpha f^{-1}, f\beta f^{-1}) = (\beta, \alpha).$$

We call f the *exchanging elliptic element* for the parabolic generating pair $\{\alpha, \beta\}$ of the Kleinian group Γ. It should be noted that fh is the exchanging elliptic element for the parabolic generating pair $\{\alpha, \beta^{-1}\}$ of Γ.

By abuse of notation, we denote the isometries of M induced by f and h by the same symbols f and h, respectively. Each of them is either the identity map or a (nontrivial) involution of M, i.e., its order is 1 or 2. We call the isometries f and h, the *exchanging involution* and the *inverting involution* of M associated with the parabolic generating pair $\{\alpha, \beta\}$. It should be noted that if Γ is isomorphic to a hyperbolic 2-bridge link group $G(K(r))$ and $\{\alpha, \beta\}$ is the upper-meridian pair, then the involutions f and h on $M \cong S^3 - K(r)$ are the restrictions of the vertical and horizontal involutions of $K(r)$ (see Figure 4). This is the reason why we use the symbols f and h with two different meanings.

Let $\hat{\Gamma} := \langle \Gamma, f \rangle$ be the group generated by Γ and the exchanging elliptic element f associated with the parabolic generating pair $\{\alpha, \beta\}$ of Γ. Then $\hat{\Gamma}$ is a Kleinian group which is either equal to Γ or a $\mathbb{Z}_2$-extension of Γ according to whether f

belongs to Γ or not. Let $\hat{M} := \mathbb{H}^3/\hat{\Gamma}$ be the quotient hyperbolic orbifold, and let $\hat{C}_{\alpha,\beta}$ be the maximal cusp of $\hat{M}$ corresponding to the conjugacy class of $\hat{\Gamma}$ containing both α and $\beta = f\alpha f^{-1}$. Then the inverse image $p^{-1}(\hat{C}_{\alpha,\beta})$ of $\hat{C}_{\alpha,\beta}$ by the projection $p : \mathbb{H}^3 \to \hat{M}$ is a union of horoballs with disjoint interiors but whose boundaries have nonempty tangential intersections. We call it the *canonical horoball system* associated with the parabolic generating pair $\{\alpha, \beta\}$ of Γ. If a parabolic element γ of Γ stabilises a member of the canonical horoball system, we denote the horoball by H_γ. We denote the translation length of γ on the horosphere ∂H_γ by the symbol $|\gamma| = |\gamma|_{\partial H_\gamma}$, and call it the *length of* γ *in the canonical horosphere.* We call the pair (H_α, H_β) the *canonical horoball pair* for the parabolic generating pair $\{\alpha, \beta\}$ of the Kleinian group Γ.

Note that the definition of $|\gamma|$ depends on the parabolic generating pair $\{\alpha, \beta\}$, because the exchanging elliptic element f is involved in the definition. However, it actually depends only on the pair $\{\mathrm{Fix}(\alpha), \mathrm{Fix}(\beta)\}$, because any orientation-preserving isometry, which exchanges $\mathrm{Fix}(\alpha)$ and $\mathrm{Fix}(\beta)$, also exchanges the members H_α and H_β of the canonical horoball pair associated with $\{\alpha, \beta\}$. (Otherwise, the product of f and an unexpected involution, which exchanges $\mathrm{Fix}(\alpha)$ and $\mathrm{Fix}(\beta)$ but does not exchange H_α and H_β, gives a loxodromic transformation which fixes the parabolic fixed points $\mathrm{Fix}(\alpha)$ and $\mathrm{Fix}(\beta)$. This contradicts the assumption that Γ is discrete.)

The following lemmas are proved by Adams [1, Lemma 3.1, Theorem 3.2, and p.197] (see also Brenner [16]). Since they holds a key to the proof of the main theorem and since we described the setting in a slightly different way, we include the proof.

Lemma 7.1. *Under the above setting, the following hold.*

(1) For any parabolic element $\gamma \in \Gamma$ which stabilises a member of the canonical horoball system, we have $|\gamma| \geq 1$.
(2) $1 \leq |\alpha| = |\beta|$.
(3) If Γ is non-free then $|\alpha| = |\beta| < 2$.

Proof. (1) We may assume ∂H_γ is the horosphere $\mathbb{C} \times \{1\}$ in the upper half space model $\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_+$. Then some other member, H_g, of the canonical horoball system touches ∂H_γ and hence has Euclidean diameter 1. Since $\gamma(H_g) = H_{\gamma g \gamma^{-1}}$ is also a member of the canonical horoball system, H_g and $\gamma(H_g)$ have disjoint interiors. Hence we have $|\gamma| \geq 1$.

(2) Since α and β are conjugate in $\hat{\Gamma}$, $|\alpha|$ and $|\beta|$ are equal. Moreover, $|\alpha| = |\beta|$ is ≥ 1 by (1).

(3) We refer the proof to [1, Theorem 3.2] and Brenner [16]. ☐

Lemma 7.2. *Both α and β are primitive in Γ.*

Proof. If Γ is a free, then the assertion follows from the fact that any member of a free-generating system of a free group is primitive. So, we may assume Γ is non-free. Suppose on the contrary that one of the two elements, say α, is imprimitive, namely there is an element $\alpha_0 \in \Gamma$ and an integer $n \geq 2$ such that $\alpha = \alpha_0^n$. Then $|\alpha| = n|\alpha_0| \geq n \geq 2$ by Lemma 7.1(1). But, this contradicts Lemma 7.1(3). □

8. Outline of the proof of Theorem 1.1

We now state an outline of the proof of Theorem 1.1. Since the if part is clear (cf. Proposition 3.3), we prove the only if part. To this end, we summarise the setting of Theorem 1.1.

Assumption 8.1. Let $\Gamma = \langle \alpha, \beta \rangle$ be a non-free Kleinian group generated by two non-commuting parabolic transformations α and β, and let $M = \mathbb{H}^3/\Gamma$ be the quotient hyperbolic orbifold. Let M_0 be the non-cuspidal part of M, and $P = \partial M_0$ the parabolic locus. By Theorem 5.1, (M_0, P) admits a relative compactification $(\bar{M}_0, \bar{P})$, which is a pared orbifold by Theorem 5.2. The pared orbifold $(\bar{M}_0, \bar{P})$ can be represented by a weighted graph (W, Σ, w), where W is a compact 3-manifold, Σ is a trivalent graph properly embedded in W, and w is a weight function on the edge set of Σ (see Convention 6.1). *We abuse notation to denote the (compact) pared orbifold $(\bar{M}_0, \bar{P})$ by (M_0, P).* We denote the components of $\bar{P}$, which is now denoted by P, corresponding to the cusps C_α and C_β by P_α and P_β, respectively.

Outline of the proof of Theorem 1.1. Under Assumption 8.1, the proof is divided into the following two cases.

Case 1. $P_\alpha \cong P_\beta$ is a flexible cusp (Section 9 for generic case and Section 11 for exceptional case).

Case 2. $P_\alpha \cong P_\beta$ is a rigid cusp (Section 10).

In both cases, the first task is to find an orbifold surgery that yields an orbifold $\mathcal{O}$ with dihedral orbifold fundamental group.

In Case 1, this can be generically done by using Lemma 7.2. In fact, if $P_\alpha \cong P_\beta$ is a flexible cusp, then Lemma 7.2 implies that each of the parabolic elements α and β can be represented by simple loops of P_α and P_β, respectively. Generically, these simple loops are disjoint, and such an surgery obviously exists. This generic case is treated in Section 9.

However, there is an exceptional case where $P_\alpha = P_\beta \cong S^2(2, 2, 2, 2)$ and the simple loops representing α and β intersect nontrivially (Lemma 9.1). In this case, the exchanging elliptic element f does not belong to Γ, and we need to consider the $\mathbb{Z}_2$-extension $\hat{\Gamma} := \langle \Gamma, f \rangle$ of Γ and consider the corresponding pared orbifold $(\hat{M}_0, \hat{P}) := (M_0, P)/f$, where $\hat{P}_{\alpha\beta}$ is isomorphic to the rigid cusp $S^2(2, 4, 4)$. The treatment of this case is deferred to Section 11, after the treatment of the rigid cusp Case 2 in Section 10, described below.

In Case 2, if $P_\alpha \cong P_\beta$ is isomorphic to either $S^2(2,4,4)$ or $S^2(2,3,6)$, the dihedral surgery can be found by using an estimate of the shortest, second shortest, and third shortest lengths of parabolic elements on the maximal rigid cusp, which in turn is based on Lemma 7.1. If $P_\alpha \cong P_\beta$ is isomorphic to $S^2(3,3,3)$, the inverting parabolic element h does not belong to Γ, and we consider the $\mathbb{Z}_2$-extension $\Gamma_h := \langle \Gamma, h\rangle$ and the corresponding pared orbifold $(M_{h,0}, P_h) := (M_0, P)/h$. The images of P_α and P_β in this quotient are isomorphic to $S^2(2,3,6)$, and this case can be treated by using arguments in the case where $P_\alpha \cong P_\beta \cong S^2(2,3,6)$.

After finding an orbifold surgery that yields an orbifold $\mathcal{O}$ with dihedral orbifold fundamental group, we can appeal to the classification Theorem 4.1 of the dihedral orbifolds, because Lemma 6.4 guarantees that the orbifold $\mathcal{O}$ satisfies the three conditions in Theorem 4.1. So, $\mathcal{O}$ belongs to the list in the theorem. The original pared orbifold (M_0, P) is obtained from the dihedral orbifold $\mathcal{O}$ by inverse surgery operations. Through case-by-case arguments, by using the homology with $\mathbb{Z}_2$-coefficients, a result concerning the symmetries of the spherical dihedral orbifold (Corollary 12.7), and a 'surgery trick' (the last paragraph in Case 1 in Section 10 and Case 1 in Section 11), we prove the following.

(1) If $P_\alpha \cong P_\beta$ is a flexible cusp, then, in the generic case, the pared orbifold (M_0, P) is isomorphic to either a hyperbolic 2-bridge link exterior or a Heckoid orbifold (Section 9): in the exceptional case, we encounter a contradiction (Section 11).

(2) If $P_\alpha \cong P_\beta$ is a rigid cusp, then we encounter a contradiction (Section 10).

This ends an outline of the proof of the main Theorem 1.1. □

9. Proof of Theorem 1.1 - flexible cusp: generic case -

Under Assumption 8.1, suppose that $P_\alpha \cong P_\beta$ is a flexible cusp. Then the 2-orbifold $P_\alpha \cong P_\beta$ is isomorphic to the torus T^2, the pillowcase $S^2(2,2,2,2)$, the annulus A^2, or $D^2(2,2)$. The following fact is the starting point of this section.

Lemma 9.1. *Under the above setting, α and β are represented by simple loops on P_α and P_β, respectively. Moreover, if $P_\alpha = P_\beta$, then one of the following holds.*

(1) The parabolic elements α and β are represented by the same (possibly oppositely oriented) simple loop.

(2) $P_\alpha = P_\beta \cong S^2(2,2,2,2)$, $f \notin \Gamma$, and $P_\alpha/f = P_\beta/f \cong S^2(2,4,4)$, where the first f is the exchanging elliptic element associated with $\{\alpha, \beta\}$ and the last two f's denote the involution on (M_0, P) induced by the exchanging elliptic element f (see Figure 11).

Proof. The first assertion directly follows from Lemma 7.2, because any primitive parabolic element in the orbifold fundamental group of the 2-dimensional orbifold T^2, $S^2(2,2,2,2)$, A^2, or $D^2(2,2)$ is represented by a simple loop on the 2-orbifold. For the proof of the second assertion, suppose that $P_\alpha = P_\beta$. If the exchanging elliptic element f belongs to Γ, then β is conjugate to α in Γ, and so they are represented by the same simple loop. Thus we may suppose $f \notin \Gamma$. Then f descends to a nontrivial orientation-preserving involution on M, which we continue to denote by f, on the flexible cusp P_α. By the classification of orientation-preserving involutions on flexible cusps, we can observe that either (a) the involution f on M_0 preserves or reverses the homotopy class of each essential simple loop on P_α, or (b) $P_\alpha \cong S^2(2,2,2,2)$ and $P_\alpha/f \cong S^2(2,4,4)$. In the first case, α and $\beta^{\pm 1}$ are represented by the same simple loop, and so we obtain the desired conclusion. $\square$

In this section, we treat the case where either $P_\alpha \neq P_\beta$ or $P_\alpha = P_\beta$ and the conclusion (1) in Lemma 9.1 holds. Thus we assume the following condition in the remainder of this section. The other case is treated in Section 11.

Assumption 9.2. Under Assumption 8.1, we further assume that (a) $P_\alpha \cong P_\beta$ is a flexible cusp and that (b) either $P_\alpha \neq P_\beta$ or $P_\alpha = P_\beta$ and the conclusion (1) in Lemma 9.1 holds. It should be noted that either α and β are represented by disjoint simple loops or they are represented by the same (possibly oppositely oriented) simple loop.

Under this assumption, we can apply an orbifold surgery on (M_0, P) to the pared orbifold (M_0, P) to obtain a dihedral orbifold, $\mathcal{O}$, as follows. Note that Assumption 9.2 implies that the pared orbifold (M_0, P) is represented by a weighted graph $(W, \Sigma, \tilde{w})$, such that there are (possibly identical) edges e_α and e_β of Σ whose meridians represent α and β, respectively. Let w be a weight function on Σ which is identical with $\tilde{w}$, except that $w(e_\alpha) = w(e_\beta) = 2$. Then the orbifold $\mathcal{O}$ represented by the augmentation of the weighted graph (W, Σ, w) is a result of an "order 2" orbifold surgery on (M_0, P), and $\pi_1(\mathcal{O})$ is dihedral, as shown below.

Note that there is a natural epimorphism from $\Gamma = \pi_1(M_0)$ to $\pi_1(\mathcal{O})$, and the images of α and β in $\pi_1(\mathcal{O})$ have order ≤ 2. Moreover, the images of α and β have the same order, because (a) if $f \in \Gamma$ then α and β are conjugate in Γ and so in $\pi_1(\mathcal{O})$, and (b) if $f \notin \Gamma$ then f descends to an involution on $\mathcal{O}$ which interchanges the images of α and β. So $\pi_1(\mathcal{O})$ is either the trivial group or a dihedral group. Since $\mathcal{O}$ is very good by Lemma 6.4 and since $\mathcal{O}$ has nonempty singular set, $\pi_1(\mathcal{O})$ is nontrivial and so isomorphic to a dihedral group.

Thus $\mathcal{O}$ satisfies the three conditions in Theorem 4.1 and so $\mathcal{O}$ belongs to the list in the theorem. We have the following lemma.

Lemma 9.3. *The orbifold $\mathcal{O}$ is isomorphic to the spherical dihedral orbifold $\mathcal{O}(r; d_+, d_-)$ for some $r \in \mathbb{Q}$ and coprime positive integers d_+ and d_-.*

Proof. We show that the possibilities (2), (3) and (4) in Theorem 4.1 cannot happen. Suppose (2) happens. Then we can see that one of the following holds, by recalling the fact that $\mathcal{O}$ is obtained from the pared orbifold (M_0, P) an order 2 orbifold surgery.

(i) M_0 is the exterior of the two-component trivial link, $P = \partial M_0$, and the singular set of M_0 is empty.
(ii) The underlying space of M_0 is the solid torus (the exterior of a trivial knot), $P = \partial M_0$, and the singular set is a trivial knot in the solid torus with index 2.

In each case, (M_0, P) is reducible, a contradiction.

By the same reasoning, we can see that (3) cannot happen.

If (4) happens, then as in the above, we can see that one of the following holds, where $(B^3, t_1 \cup t_2)$ is a two-strand trivial tangle.

(i) $(M_0, P) \cong (\mathrm{cl}(B^3 - N(t_1 \cup t_2)), \mathrm{fr}\, N(t_1 \cup t_2))$ and the singular set of M_0 is empty.
(ii) $(M_0, P) \cong (\mathrm{cl}(B^3 - N(t_1)), \mathrm{fr}\, N(t_1))$ and the singular set of M_0 is t_2 with index 2.

In the first case, $\Gamma = \pi_1(M_0)$ is a rank 2 free group, which contradicts the assumption that Γ is non-free. In the second case, note that $H_1(M_0)$, the abelianization of the orbifold fundamental group $\pi_1(M_0)$, is $\mathbb{Z} \oplus \mathbb{Z}_2$. On the other hand, both α and β are represented by the core loop of the annulus $P = \mathrm{fr}\, N(t_1)$, and the pair $\{\alpha, \beta\}$ cannot generate $H_1(M_0)$, a contradiction. □

By Lemma 9.3, the original orbifold (M_0, P) is recovered from $\mathcal{O} = \mathcal{O}(r; d_+, d_-)$ by applying the inverse orbifold surgery operation. This leads us to the following proposition.

Proposition 9.4. *Under the notation in Lemma 9.3, the following hold, if necessary by replacing $r = q/p$ with q'/p where $q' = q + p$ or $qq' \equiv 1 \pmod p$.*

(1) If $|K(r)| = 1$, then one of the following holds.
  (i) *$d_+ = d_- = 1$ and $(M_0, P) \cong (E(K(r)), \partial E(K(r)))$, where $q \not\equiv \pm 1 \pmod p$. Here $E(K(r))$ denotes the exterior of $K(r)$, i.e. the complement of an open regular neighbourhood of $K(r)$.*
  (ii) *$d_+ = 1$, $d_- \geq 2$, and $(M_0, P) \cong \mathcal{M}_0(r; d_-)$.*
  (iii) *$d_+ = 1$, d_- is an odd integer ≥ 3, and $(M_0, P) \cong \mathcal{M}_1(r; d_-)$.*
  (iv) *$d_+ = 2$, d_- is an odd integer ≥ 3, and $(M_0, P) \cong \mathcal{M}_2(r; d_-)$.*

(2) If $|K(r)| = 2$, then one of the following holds.
  (i) *$d_+ = d_- = 1$ and $(M_0, P) \cong (E(K(r)), \partial E(K(r)))$, where $q \not\equiv \pm 1 \pmod p$.*
  (ii) *$d_+ = 1$, $d_- \geq 2$, and $(M_0, P) \cong \mathcal{M}_0(r; d_-)$.*
  (iii) *$d_+ = 2$, d_- is an odd integer ≥ 3, and $(M_0, P) \cong \mathcal{M}_2(r; d_-)$.*

Proof. Recall that $\mathcal{O} = \mathcal{O}(r; d_+, d_-)$ is represented by the weighted graph $(S^3, K(r) \cup \tau_+ \cup \tau_-, w)$ for some $r \in \mathbb{Q}$ and for some coprime positive integers d_+ and d_-, and w is given by the following rule (see Figure 6):

$$w(K(r)) = 2, \quad w(\tau_+) = d_+, \quad w(\tau_-) = d_-$$

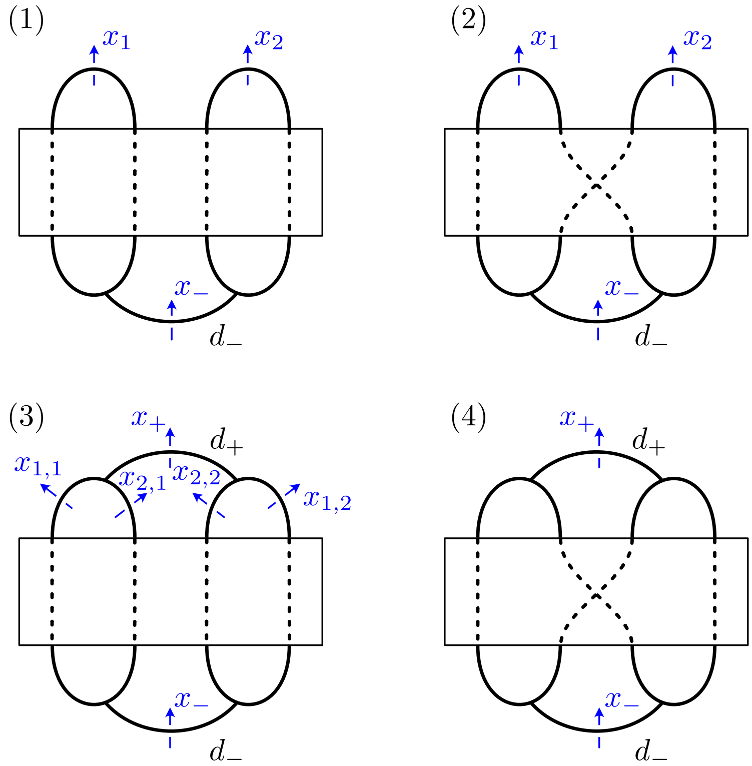


FIGURE 7. $H_1(\mathcal{O};\mathbb{Z}_2) \cong H_1(S^3 - \Sigma_{\text{even}};\mathbb{Z}_2)$, where Σ_{even} is the subgraph of $\Sigma = K(r) \cup \tau_+ \cup \tau_-$ spanned by the edges of even weight.

Then (M_0, P) is represented by the weighted graph $(S^3, K(r) \cup \tau_+ \cup \tau_-, \tilde{w})$, where $\tilde{w}$ is obtained from w by replacing the label 2 of the edges e_α and e_β, which correspond to P_α and P_β respectively, with the label ∞. By Remark 4.3, we may assume $1 \le d_+ \le d_-$, if necessary by replacing $r = q/p$ with q'/p where $qq' \equiv 1 \pmod{p}$.

Case 1. $d_+ = d_- = 1$. Then $\Sigma(\mathcal{O})$ is the 2-bridge link $K(r)$. Thus $(M_0, P) \cong (S^3, K(r), \tilde{w})$, where either (a) $\tilde{w}(K(r)) = \infty$ or (b) $K(r)$ is a 2-component link $K_1 \cup K_2$ and $(\tilde{w}(K_1), \tilde{w}(K_2)) = (\infty, 2)$. In the first case, $(M_0, P) \cong (E(K(r)), \partial E(K(r)))$, and so $\mathbb{H}^3/\Gamma$ is the hyperbolic 2-bridge link complement, $S^3 - K(r)$: in particular, $q \not\equiv \pm 1 \pmod{p}$. In the second case, both α and β are meridians of the component K_1, which contradicts the fact that $H_1(\mathcal{O};\mathbb{Z}_2) \cong (\mathbb{Z}_2)^2$.

Case 2. $d_+ = 1 < d_-$.

Subcase 2.1. $|K(r)| = 2$ (see Figure 7(1)). Then the edge set of $\Sigma(\mathcal{O})$ consists of τ_- and the two components K_1, K_2 of $K(r)$. Let x_-, x_1 and x_2 be the meridians of τ_-, K_1 and K_2, respectively. By Lemma 6.5, $H_1(\mathcal{O};\mathbb{Z}_2) \cong (\mathbb{Z}_2)^2$ is freely generated by $\{x_1, x_2\}$, and moreover we have $x_- = 0$. Since $H_1(\mathcal{O};\mathbb{Z}_2)$ is generated by (the images of) α and β, we may assume $e_\alpha = K_1$ and $e_\beta = K_2$. Thus (M_0, P) is represented by $(S^3, K(r) \cup \tau_-, \tilde{w})$, where $\tilde{w}(K_1) = \tilde{w}(K_2) = \infty$ and $\tilde{w}(\tau_-) = d_-$. Hence $(M_0, P) \cong \mathcal{M}_0(r; d_-)$.

Subcase 2.2. $|K(r)| = 1$ (see Figure 7(2)). Then the edge set of $\Sigma(\mathcal{O})$ consists of τ_- and the two subarcs J_1 and J_2 of $K(r)$ bounded by $K(r) \cap \tau_-$. Let x_-, x_1 and x_2 be the meridians of τ_-, J_1 and J_2, respectively.

Suppose first that d_- is odd. Then we see by Lemma 6.5 that $x_- = 0$ in $H_1(\mathcal{O}; \mathbb{Z}_2) \cong \mathbb{Z}_2$ and that $H_1(\mathcal{O}; \mathbb{Z}_2) \cong \mathbb{Z}_2$ is generated by $x_1 = x_2$. Hence one of the following holds.

(1) $\{e_\alpha, e_\beta\} = \{J_1, J_2\}$ and so (M_0, P) is represented by $(S^3, K(r) \cup \tau_-, \tilde{w})$, where $\tilde{w}(J_1) = \tilde{w}(J_2) = \infty$ and $\tilde{w}(\tau_-) = d_-$. Hence $(M_0, P) \cong \mathcal{M}_0(r; d_-)$.

(2) $e_\alpha = e_\beta = J_i$ for $i = 1$ or 2. By the symmetry of $\mathcal{O}$, we may assume $i = 1$ and so (M_0, P) is represented by $(S^3, K(r) \cup \tau_-, \tilde{w})$, where $\tilde{w}(J_1) = \infty$, $\tilde{w}(J_2) = 2$ and $\tilde{w}(\tau_-) = d_-$. Hence $(M_0, P) \cong \mathcal{M}_1(r; d_-)$.

Suppose next that d_- is even. Then $x_1 + x_2 + x_- = 0$ in $H_1(\mathcal{O}; \mathbb{Z}_2) \cong (\mathbb{Z}_2)^2$. Since $H_1(\mathcal{O}; \mathbb{Z}_2)$ is generated by α and β, we have $e_\alpha \neq e_\beta$. This implies that the exchanging elliptic element f for $\{\alpha, \beta\}$ does not belong to Γ, and f descends to an involution on $\mathcal{O}$ interchanging e_α with e_β. We now use Corollary 12.7 on the symmetry of the orbifold $\mathcal{O}(r; d_+, d_-)$. We first consider the generic case where $p \neq 1$ (i.e., $K(r)$ is a nontrivial knot) or $d_- > 2$. (Recall the current assumption $d_+ = 1$.) Then, by Corollary 12.7(1), any orientation-preserving involution of $\mathcal{O}$ preserves τ_-. So, e_α and e_β are different from $\tau_\pm$, and therefore $\{e_\alpha, e_\beta\} = \{J_1, J_2\}$. Hence, as in the previous case, we can conclude $(M_0, P) \cong \mathcal{M}_0(r; d_-)$. In the exceptional case where $p = 1$ and $d_- = 2$, The orbifold $\mathcal{O} \cong \mathcal{O}(0/1; 1, 2)$ has the 3-fold cyclic symmetry as illustrated in Figure 15. Thus, if necessary after applying this symmetry, we may assume $\{e_\alpha, e_\beta\} = \{J_1, J_2\}$. Hence we have $(M_0, P) \cong \mathcal{M}_0(r; d_-) \cong \mathcal{M}_0(0/1; 2)$.

Since we repeatedly use the above argument in the remainder of the proof of Proposition 9.4, we state an expanded version of the argument as a lemma.

Lemma 9.5. *Under the setting of Proposition 9.4, suppose* $(d_+, d_-) \neq (1, 1)$ *and* $H_1(\mathcal{O}; \mathbb{Z}_2) \cong (\mathbb{Z}_2)^2$. *Then* $e_\alpha \neq e_\beta$, *and the exchanging elliptic element* f *does not belong to* Γ *and it descends to an orientation-preserving involution of* $\mathcal{O} = \mathcal{O}(r; d_+, d_-)$ *interchanging* e_α *and* e_β. *Moreover, the following hold.*

(1) Except when $p = 1$ *and* $\{d_+, d_-\} = \{1, 2\}$, e_α *and* e_β *are different from* $\tau_\pm$.

(2) If $d_+, d_- \geq 2$, *then the inverting elliptic element* h *belongs to* Γ.

Proof. We have only to prove (2). If h does not belong to Γ, then it descends to an orientation-preserving involution of $\mathcal{O}(r; d_+, d_-)$ which preserves both e_α and e_β. However, if $d_+, d_- \geq 2$, then by Corollary 12.7(2), no orientation-preserving involution of $\mathcal{O}(r; d_+, d_-)$ preserves an edge of the singular set different from $\tau_\pm$. This contradicts the assertion (1). □

Case 3. $2 \le d_+ \le d_-$. Since d_+ and d_- are coprime, we see $2 \le d_+ < d_-$ and one of d_+ and d_- is odd.

Subcase 3.1. $|K(r)| = 2$ (see Figure 7(3)). Let K_1 and K_2 be the components of $K(r)$, and let $J_{i,j}$ $(1 \le i, j \le 2)$ be the edges of $\Sigma(\mathcal{O})$ such that $K_j = J_{1,j} \cup J_{2,j}$ for $j = 1, 2$ and that the vertical involution of $K(r)$ interchanges $J_{i,1}$ and $J_{i,2}$ for $i = 1, 2$. Let $x_\pm$ and $x_{i,j}$ be the meridians of $\tau_\pm$ and $J_{i,j}$, respectively. Then by using Lemma 6.5 and the fact that one of d_+ and d_- is odd, we see that $H_1(\mathcal{O}; \mathbb{Z}_2) \cong (\mathbb{Z}_2)^2$ is freely generated by $x_1 := x_{1,1} = x_{2,1}$ and $x_2 := x_{1,2} = x_{2,2}$: moreover we have $x_\pm = 0$. Hence, we may assume $e_\alpha \subset K_1$ and $e_\beta \subset K_2$. Since the horizontal involution of $K(r)$ interchanges $J_{1,j}$ and $J_{2,j}$ $(j = 1, 2)$, we may assume $e_\alpha = J_{1,1} \subset K_1$ and $e_\beta = J_{i,2} \subset K_2$ for some $i = 1$ or 2. By Lemma 9.5(2), we have $h \in \Gamma$, and so $P_\alpha \cong P_\beta$ is homeomorphic to $D^2(2, 2)$ or $S^2(2, 2, 2, 2)$. Since $2 \le d_+ < d_-$, we must have $d_+ = 2$. If $i = 1$, i.e. $e_\beta = J_{1,2}$, then $\tilde{w}$ is given by

$$\tilde{w}(J_{1,1}) = \tilde{w}(J_{1,2}) = \infty, \quad \tilde{w}(J_{1,2}) = \tilde{w}(J_{2,2}) = 2, \quad \tilde{w}(\tau_+) = 2, \quad \tilde{w}(\tau_-) = d_-.$$

Since the vertical involution of $K(r)$ preserves $J_1 := J_{1,1} \cup J_{1,2}$, we see that (M_0, P) is isomorphic to $\mathcal{M}_2(r; d_-)$. If $i = 2$, i.e. $e_\beta = J_{2,2}$, then the planar involution of $K(r)$ preserves $J_1 := J_{1,1} \cup J_{2,2}$. Hence, we see by Remark 3.5 that (M_0, P) is isomorphic to $\mathcal{M}_2(r'; d_-)$, where $r' = (p + q)/p$.

Subcase 3.2. $|K(r)| = 1$ (see Figure 7(4)). Suppose first that one of d_+ and d_- is even. Then $H_1(\mathcal{O}; \mathbb{Z}_2) \cong (\mathbb{Z}_2)^2$ by Lemma 6.5. Hence, by Lemma 9.5(2), both e_α and e_β are contained in $K(r)$, and $h \in \Gamma$. In particular, $P_\alpha \cong P_\beta \cong D^2(2, 2)$ or $S^2(2, 2, 2, 2)$. Let e_i $(1 \le i \le 4)$ be the edges of the singular set of $\mathcal{O}$ contained in the knot $K(r)$ in this cyclic order. We also assume that $\partial\tau_+ = (e_1 \cap e_2) \cup (e_3 \cap e_4)$ and $\partial\tau_- = (e_2 \cap e_3) \cup (e_4 \cap e_1)$. Since the $(\mathbb{Z}_2)^2$-symmetry of $\mathcal{O}(r; d_+, d_-)$ acts transitively on the edge set $\{e_i\}_{1 \le i \le 4}$ (see Figure 14), we may assume $e_1 = e_\alpha$ and so $\tilde{w}(e_1) = \infty$. Since e_α joins τ_+ with τ_- and since $d_\pm$ are coprime integers such that $2 \le d_+ \le d_-$, the condition that $P_\alpha \cong D^2(2, 2)$ or $S^2(2, 2, 2, 2)$ implies that $d_+ = 2$ and $d_- \ge 3$. This in turn implies that $P_\alpha \cong P_\beta \cong D^2(2, 2)$. Since $\partial D^2(2, 2)$ is isotopic to the simple loop α in ∂M_0, we must have $\tilde{w}(e_2) = 2$. Thus e_β is equal to e_3 or e_4. However, if $e_\beta = e_4$ then e_α, e_β, and the odd index edge τ_- share a vertex, it follows from Lemma 6.5(3) that the meridian α of e_1 and the meridian β of e_4 represent the same element of $H_1(\mathcal{O}; \mathbb{Z}_2) \cong (\mathbb{Z}_2)^2$, a contradiction. Hence $e_\beta = e_3$. Set $J_1 = e_1 \cup e_3$ and $J_2 = e_2 \cup e_4$. Then J_1 and J_2 are disjoint, $K(r) = J_1 \cup J_2$ and the following hold.

$$\tilde{w}(J_1) = \infty,\ \tilde{w}(J_2) = 2,\ \tilde{w}(\tau_+) = 2,\ \tilde{w}(\tau_-) = d_-$$

Hence we have $(M_0, P) \cong \mathcal{M}_2(r; d_-)$ (cf. Remark 3.5(2)).

Suppose finally that both d_+ and d_- are odd. Then, by Lemma 6.5, the meridians $x_\pm$ of $\tau_\pm$ represent the trivial element of $H_1(\mathcal{O}; \mathbb{Z}_2) \cong \mathbb{Z}_2$, and hence both e_α and $e_\beta = f(e_\alpha)$ are contained in $K(r)$. On the other hand, since $d_\pm > 2$, we have $P_\alpha \cong P_\beta$ is homeomorphic to an annulus, and hence the inverting elliptic element h

descends to an involution of $\mathcal{O}$ which preserves each of the two mutually different edges e_α and e_β and restricts to an orientation-reversing involution on each of the edges. But, such an involution does not exist by Corollary 12.7(2), a contradiction.

This completes the proof of Proposition 9.4. □

10. Proof of Theorem 1.1 - rigid cusp case -

Under Assumption 8.1, suppose that $P_\alpha \cong P_\beta$ is a rigid cusp. Thus the 2-orbifold $P_\alpha \cong P_\beta$ is isomorphic to $S^2(p,q,r)$ where $(p,q,r) = (2,4,4)$, $(2,3,6)$, or $(3,3,3)$.

Let $G < \Gamma$ be the orbifold fundamental group $\pi_1(P_\alpha)$, and let Λ be the subgroup of G consisting of parabolic transformations. We may assume that (a) G stabilises the ideal point ∞ of the upper-half space model of $\mathbb{H}^3$, and (b) the boundary ∂H_α of the canonical horoball H_α is identified with the horosphere $\mathbb{C} \times \{1\} \subset \mathbb{H}^3$. For each element $g \in \Lambda$, let $|g|$ be the length of g in the canonical horosphere (see Section 7), namely $|g| = |g|_{\partial H_\alpha}$, the translation length of g in ∂H_α, and simply call it the *length* of g. Let $L_1(\Lambda) > 0$ be the minimum of the lengths of nontrivial elements of Λ. More generally, for each $n \in \mathbb{N}$, let $L_n(\Lambda)$ be the n-th shortest length of nontrivial elements of Λ.

Case 1. $P_\alpha \cong S^2(2,4,4)$. Then $G \cong \langle a,b,c \mid a^2,\ b^4,\ c^4, abc\rangle$, and Λ is the rank 2 free abelian group with free basis $\{b^2a, c^2a\}$. We may assume the action of G on the horosphere $\partial H_\alpha = \mathbb{C} \times 1 \cong \mathbb{C}$ is given by the following rule. There is a positive real ℓ such that a is the π rotation about 0, and b and c are the $\pi/2$ rotations about ℓ and ℓi, respectively. We can easily observe the following.

- (i) The shortest length $L_1(\Lambda)$ is equal to 2ℓ, and it is attained precisely by the conjugates of b^2a in G. (Note that $c^2a = (b^{-1}a^{-1})^2a = b^{-1}a^{-1}b^{-1} = b^3ab^{-1} = b(b^2a)b^{-1}$ is conjugate to b^2a.)
- (ii) The second shortest length $L_2(\Lambda)$ is equal to $2\sqrt{2}\ell$, and it is attained precisely by the conjugates of b^2ac^2a in G.
- (iii) The third shortest length $L_3(\Lambda)$ is equal to 4ℓ, and it is attained precisely by the conjugates of $(b^2a)^2$ in G.

By Lemma 7.1(1), $2\ell = L_1(\Lambda) \geq 1$, and so $\ell \geq \frac{1}{2}$. Since Γ is non-free, Lemma 7.1(3) implies that the length $|\alpha|$ of the parabolic element $\alpha \in \Lambda$ is less than 2. Since $L_3(\Lambda) = 4\ell \geq 2$, $|\alpha|$ is equal to either $L_1(\Lambda)$ or $L_2(\Lambda)$. By using this fact, we obtain the following lemma.

Lemma 10.1. *The parabolic element $\alpha \in \Lambda$ is conjugate to b^2a or b^2ac^2a in G. Moreover the following hold.*

- *(1) If α is conjugate to b^2a, then the images of α by the natural epimorphisms from $G \cong \pi_1(S^2(2,4,4))$ to $\pi_1(S^2(2,2,2))$, $\pi_1(S^2(2,2,4))$, and $\pi_1(S^2(2,4,2))$ have order 2.*
- *(2) If α is conjugate to b^2ac^2a, then the images of α by the natural epimorphisms from $G \cong \pi_1(S^2(2,4,4))$ to $\pi_1(S^2(2,2,2))$, $\pi_1(S^2(2,2,4))$, and $\pi_1(S^2(2,4,2))$*

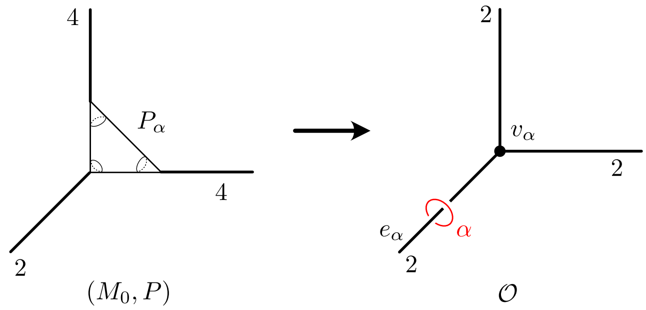


FIGURE 8. Orbifold surgery on the rigid cusp $S^2(2,4,4)$: The parabolic locus P_α of the pared orbifold (M_0, P) shrinks into the vertex v_α of the singular set of the orbifold $\mathcal{O}$. By Lemma 10.2, the homology class $[\alpha] \in H_1(\mathcal{O};\mathbb{Z}_2)$ determined by the parabolic element $\alpha \in \Gamma$ is represented by the meridian of the edge e_α of the singular set $\Sigma(\mathcal{O})$ incident on v_α whose index in the original orbifold M_0 is 2.

have order 1*,* 2 *and* 2*, respectively. Moreover, the* $\mathbb{Z}_2$*-homology class of* α *vanishes.*

Proof. The assertion in the first line follows from the observations preceding the lemma. The assertions (1) and (2) can be checked easily, by using the fact that b^2a is conjugate to c^2a in G. □

Now let $\mathcal{O}$ be the orbifold obtained from the pared orbifold (M_0, P) by the orbifold surgery as illustrated in Figure 8. Namely, for each index 4 edge of the singular set which has an endpoint in P_α or P_β, we replace the index 4 with the index 2, and then cap all resulting spherical boundary components with discal 3-orbifolds. Then each of P_α and P_β shrinks into a vertex of $\mathcal{O}$ with link $S^2(2,2,2)$, which we denote by v_α and v_β, respectively. We denote by e_α (resp. e_β) the edge of the singular set $\Sigma(\mathcal{O})$ incident on v_α (resp. v_β) whose index in the original orbifold M_0 is 2.

Lemma 10.2. *The orbifold* $\mathcal{O}$ *is isomorphic to a spherical dihedral orbifold* $\mathcal{O}(r; d_+, d_-)$ *for some* $r \in \mathbb{Q}$ *and coprime positive integers* d_+ *and* d_-*. Moreover,* $\alpha \in \Lambda$ *is conjugate to* b^2a *in* G*, and the homology class* $[\alpha] \in H_1(\mathcal{O};\mathbb{Z}_2)$ *determined by* α *is equal to the meridian of the edge* e_α*. Similarly, the homology class* $[\beta] \in H_1(\mathcal{O};\mathbb{Z}_2)$ *is equal to the meridian of the edge* e_β*.*

Proof. By Lemma 10.1, α is conjugate to b^2a or b^2ac^2a in $G = \pi_1(S^2(2,4,4))$, its image in $\pi_1(S^2(2,2,2))$ has order 2 or 1 accordingly. Hence the image of α in $\pi_1(\mathcal{O})$ has order ≤ 2. Moreover, the images of α and β have the same order, because (a) if the exchanging involution f belongs to Γ then α and β are conjugate in Γ and so in $\pi_1(\mathcal{O})$, and (b) if $f \notin \Gamma$ then f descends to an involution on $\mathcal{O}$ which interchanges the images of α and β. Hence $\pi_1(\mathcal{O})$ is either the trivial group or a dihedral group. Since $\mathcal{O}$ is very good by Lemma 6.4 and since it has a singular point with link $S^2(2,2,2)$, $\pi_1(\mathcal{O})$ is a noncyclic dihedral group. Hence, by Theorem 4.1, $\mathcal{O}$ is isomorphic to a spherical dihedral orbifold $\mathcal{O}(r; d_+, d_-)$.

We prove the remaining assertions. If $\alpha \in \Gamma$ is conjugate to b^2ac^2a, then it descends to the trivial element of $\pi_1(S^2(2,2,2))$, and so it represents the trivial element of $\pi_1(\mathcal{O})$. This contradicts the fact that $\pi_1(\mathcal{O})$ is a dihedral group generated by the images of α and β. Hence α is conjugate to b^2a. This implies that the $\mathbb{Z}_2$-homology class $[\alpha] \in H_1(\mathcal{O}; \mathbb{Z}_2)$ is equal to that represented by the element a, and so it is the meridian of the edge e_α. The existence of the exchanging elliptic element f implies the corresponding assertion for $[\beta]$. □

Lemma 10.3. *The pared orbifold (M_0, P) is represented by the weighted graph $(S^3, K(r) \cup \tau_+ \cup \tau_-, \tilde{w})$ for some $r \in \mathbb{Q}$, where $\tilde{w}$ is determined by the following rule (see Figure 9):*

$$\tilde{w}(J_1) = 2, \quad \tilde{w}(J_2) = 4, \quad \tilde{w}(\tau_+) = 4, \quad \tilde{w}(\tau_-) = m,$$

for some odd integer $m \geq 3$, where J_1 and J_2 are unions of two mutually disjoint edges of the graph $K(r) \cup \tau_+ \cup \tau_-$ distinct from $\tau_\pm$, such that $K(r) = J_1 \cup J_2$. Moreover, P_α and P_β correspond to distinct endpoints of $\partial\tau_+$.

Proof. By Lemma 10.2, $\mathcal{O}$ is represented by the weighted graph $(S^3, K(r) \cup \tau_+ \cup \tau_-, w)$ for some $r \in \mathbb{Q}$, where w is given by the rule

$$w(K(r)) = 2, \quad w(\tau_+) = d_+, \quad w(\tau_-) = d_-$$

for some coprime positive integers d_+ and d_-. Since $\mathcal{O}$ is obtained from (M_0, P) by an orbifold surgery, there is a weight function $\tilde{w}$ on the graph $K(r) \cup \tau_+ \cup \tau_-$ such that the pared orbifold (M_0, P) is represented by the weighted graph $(S^3, K(r) \cup \tau_+ \cup \tau_-, \tilde{w})$. By Remark 4.3, we may assume d_- is odd, if necessary by replacing $r = q/p$ with q'/p where $qq' \equiv 1 \pmod{p}$. Hence, we see $H_1(\mathcal{O}; \mathbb{Z}_2) \cong (\mathbb{Z}_2)^2$ by Lemma 6.5. Since $H_1(\mathcal{O}; \mathbb{Z}_2)$ is generated by $[\alpha]$ and $[\beta]$, which are the meridians of the edges e_α and e_β, respectively (see Lemma 10.2), we have $e_\alpha \neq e_\beta$.

Since the links of v_α and v_β are isomorphic to $S^2(2,2,2)$, we see $w(\tau_+) = 2$ and $\{v_\alpha, v_\beta\} \subset \partial\tau_+$. Since e_α (resp. e_β) is the unique edge of the trivalent graph $K(r) \cup \tau_+ \cup \tau_-$ incident on the vertex v_α (resp. v_β) with $\tilde{w}$-weight 2, and since $e_\alpha \neq e_\beta$, we see that v_α and v_β are distinct endpoints of τ_+. (If $v_\alpha = v_\beta$, then its 'link' in M_0 is of the form $S^2(2,2,*) \not\cong S^2(2,4,4)$.) Hence P_α and P_β correspond to distinct endpoints of $\partial\tau_+$.

We observe that e_α and e_β are not equal to τ_+. If, say e_α was equal to τ_+, then it is incident on $v_\beta \in \partial\tau_+$. Since $\tilde{w}(e_\alpha) = 2$, this implies we have $e_\alpha = e_\beta$, a contradiction. This observation implies that both e_α and e_β are contained in $K(r)$.

We next observe that $d_- \geq 3$. If $d_- = 1$, then the endpoints of e_α and e_β are all contained in $\partial\tau_+ = \{v_\alpha, v_\beta\}$. This together with the previous observation implies that the 'links' of v_α and v_β in M_0 are isomorphic to $S^2(2,2,4)$, a contradiction.

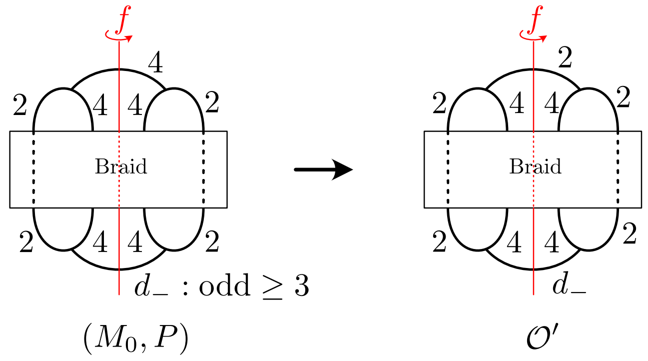


FIGURE 9. The possible pared orbifold (M_0, P) in Lemma 10.3 and the orbifold $\mathcal{O}'$ obtained by the orbifold surgery. In this figure, we apply further normalisation so that J_1 and J_2 are invariant by the vertical involution f (cf. Remark 3.5(3)).

We now show that e_α and e_β are disjoint. If they are not disjoint, then they share an endpoint of τ_-, which has odd weight d_-. This implies that the meridians of e_α and e_β represent an identical element of $H_1(\mathcal{O}; \mathbb{Z}_2)$ (see Lemma 6.5(3)), and so $[\alpha] = [\beta]$, a contradiction.

Set $J_1 := e_\alpha \cup e_\beta$ and let $J_2 := \mathrm{cl}(K(r) - J_1)$. Then J_1 and J_2 satisfy the desired conclusion with $m = d_-$. □

We show that the situation described in Lemma 10.3 cannot happen. To this end, we perform another orbifold surgery on (M_0, P) which replaces the weight 4 of τ_+ with 2. To be precise, we consider the orbifold $\mathcal{O}'$ represented by the weighted graph $(S^3, K(r) \cup \tau_+ \cup \tau_-, \tilde{w}')$ for some $r \in \mathbb{Q}$, where $\tilde{w}'$ is given by the following rule.

$$\tilde{w}'(J_1) = 2, \quad \tilde{w}'(J_2) = 4, \quad \tilde{w}'(\tau_+) = 2, \quad \tilde{w}'(\tau_-) = m$$

Note that $P_\alpha \cong S^2(2,4,4)$ shrinks into a singular point of $\mathcal{O}'$ with link $S^2(2,2,4)$ or $S^2(2,4,2)$. Since α is conjugate to b^2a in $\pi_1(S^2(2,4,4)) < \pi_1(M_0) = \Gamma$, we see by Lemma 10.1(1) that the image of α in $\pi_1(\mathcal{O}')$ has order ≤ 2. The same argument can be applied to β and we see that the image of β in $\pi_1(\mathcal{O}')$ also has order ≤ 2. Since $\mathcal{O}'$ is very good by Lemma 6.4 and since the singular set of $\mathcal{O}'$ contains a trivalent vertex, $\pi_1(\mathcal{O}')$ is a noncyclic dihedral group. Since the singular set of $\mathcal{O}'$ contains four trivalent vertices, Theorem 4.1 implies that $\mathcal{O}'$ must be isomorphic to a spherical dihedral orbifold $\mathcal{O}(r'; d'_+, d'_-)$ with $d'_+, d'_- \geq 2$. In particular, the singular set $\Sigma(\mathcal{O}')$ of $\mathcal{O}'$ must contain precisely four or five edges with index 2. This contradicts the fact that $\Sigma(\mathcal{O}')$ contains precisely three edges of index 2 (see Figure 9).

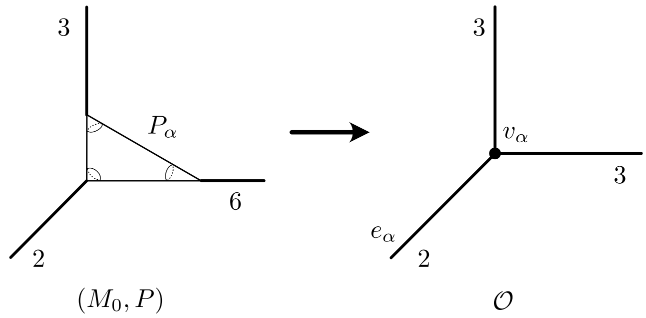


FIGURE 10. Orbifold surgery on the rigid cusp $S^2(2,3,6)$

Case 2. $P_\alpha \cong S^2(2,3,6)$. Then $G \cong \langle a,b,c \mid a^2,\ b^3,\ c^6,\ abc\rangle$, and Λ is the rank 2 free abelian group with free basis $\{ac^3, c(ac^3)c^{-1} = cac^2\}$. We may assume the action of G on the horosphere $\partial H_\alpha = \mathbb{C} \times 1 \cong \mathbb{C}$ is given by the following rule. There is a positive real ℓ such that a is the π rotation about $\sqrt{3}\ell$, b is the $2\pi/3$ rotation about $2\ell e^{\pi i/6} = \sqrt{3}\ell + \ell i$, and c is the $\pi/3$ rotations about 0. The action of the generators of Λ is given by

$$ac^3(z) = z + 2\sqrt{3}\ell, \quad cac^2(z) = z + 2\sqrt{3}\ell e^{\pi i/3}.$$

We can easily observe the following.

(i) $L_1(\Lambda) = 2\sqrt{3}\ell$, and it is attained precisely by the conjugates of ac^3 in G.
(ii) $L_2(\Lambda) = 6\ell$, and it is attained precisely by the conjugates of $(ac^3)(cac^2) = ac^4ac^2$ in G.
(iii) $L_3(\Lambda) = 4\sqrt{3}\ell$, and it is attained precisely by the conjugates of $(ac^3)^2$ in G.

By Lemma 7.1(1), $2\sqrt{3}\ell = L_1(\Lambda) \geq 1$, and so $\ell \geq \frac{1}{2\sqrt{3}}$. Since Γ is non-free, Lemma 7.1(3) implies that the length $|\alpha|$ of the parabolic element $\alpha \in \Lambda$ is less than 2. Hence we obtain the following.

Lemma 10.4. *The parabolic element $\alpha \in \Lambda$ is conjugate to ac^3 or ac^4ac^2 in G.*

Now let $\mathcal{O}$ be the orbifold obtained from the pared orbifold (M_0, P) by the orbifold surgery as illustrated in Figure 10. Namely, for each edge of the singular set which has the index 6 cone point of P_α or P_β as an endpoint, we replace the weight 6 with the new weight 3, and then cap all resulting spherical boundary components with discal 3-orbifolds. Then P_α and P_β shrink into singular points, v_α and v_β, of $\mathcal{O}$ with link $S^2(2,3,3)$.

Lemma 10.5. *The image of α by the natural epimorphism from $\pi_1(S^2(2,3,6))$ to $\pi_1(S^2(2,3,3))$ has order 2.*

Proof. By Lemma 10.4, α is conjugate to either ac^3 or ac^4ac^2 in $G = \pi_1(S^2(2,3,6))$. Moreover, the images of ac^3 and ac^4ac^2 in $\pi_1(S^2(2,3,3)) \cong \langle a,b,c \mid a^2,\ b^3,\ c^3,\ abc\rangle$

have order 2. This is obvious for ac^3, and the assertion for ac^4ac^2 is verified as follows. In $\pi_1(S^2(2,3,3))$, we have $1 = b^3 = (ac^{-1})^3$ and so $ac^{-1} = (ac^{-1})^{-2} = (ca)^2$. Hence the image of ac^4ac^2 in $\pi_1(S^2(2,3,3))$ is equal to $acac^{-1} = ac(ca)^2 = ac^2aca = (ca)^{-1}a(ca)$. Thus it is conjugate to a, and so has order 2, as desired. □

By the above lemma, the image of α in $\pi_1(\mathcal{O})$ has order ≤ 2. The existence of the exchanging elliptic element f implies that the images of α and β in $\pi_1(\mathcal{O})$ have the same order. Thus $\pi_1(\mathcal{O})$ is either a dihedral group or the trivial group. Since $\mathcal{O}$ is very good by Lemma 6.4 and since the singular set of $\mathcal{O}$ contains a trivalent vertex, $\pi_1(\mathcal{O})$ is a noncyclic dihedral group. Hence, Theorem 4.1 implies that $\mathcal{O}$ must be isomorphic to a spherical dihedral orbifold $\mathcal{O}(r; d_+, d_-)$ with $(d_+, d_-) \neq (1,1)$. However, the orbifold $\mathcal{O}(r; d_+, d_-)$ does not contain a singular point with link $S^2(2,3,3)$, a contradiction.

Case 3. $P_\alpha \cong S^2(3,3,3)$. Then the inverting elliptic element h does not belong to Γ, and the group, Γ_h, obtained from Γ by adding h is a $\mathbb{Z}_2$-extension of Γ. Consider the hyperbolic orbifold $M_h := \mathbb{H}^3/\Gamma_h$. Then M_h is the quotient of $M = \mathbb{H}^3/\Gamma$ by the isometric involution induced by h, which we continue to denote by h. Set $(M_{h,0}, P_h) := (M_0/h, P/h)$, $P_{h,\alpha} := P_\alpha/h$ and $P_{h,\beta} := P_\beta/h$. Then $P_{h,\alpha} \cong P_{h,\beta}$ is isomorphic to $S^2(2,3,6)$. Thus $G_h := \pi_1(P_{h,\alpha}) \cong \langle a, b, c \mid a^2,\ b^3,\ c^6,\ abc\rangle$. Since the subgroup Γ of Γ_h generated by α and β is non-free, we see by the arguments in Case 2 that α is conjugate to ac^3 or ac^4ac^2 in G_h.

Let $\mathcal{O}_h$ be the orbifold obtained from the pared orbifold $(M_{h,0}, P_h)$ by the orbifold surgery as illustrated in Figure 10 at both $P_{h,\alpha}$ and $P_{h,\beta}$. Then $P_{h,\alpha}$ and $P_{h,\beta}$ shrink into singular points, $v_{h,\alpha}$ and $v_{h,\beta}$, of $\mathcal{O}_h$ with link $S^2(2,3,3)$. The images of α and β in $\pi_1(\mathcal{O}_h)$ have the same order ≤ 2, and so the subgroup of $\pi_1(\mathcal{O}_h)$ they generate is either a dihedral group or the trivial group. This subgroup has index ≤ 2 in $\pi_1(\mathcal{O}_h)$, because $\Gamma = \langle \alpha, \beta\rangle$ has index 2 in Γ_h. Hence the group $\pi_1(\mathcal{O}_h)$ is a trivial group, a dihedral group, $\mathbb{Z}_2$ (the $\mathbb{Z}_2$-extension of the trivial group) or a $\mathbb{Z}_2$-extension of a dihedral group.

Since $\mathcal{O}_h$ is very good by Lemma 6.4 and $\mathcal{O}_h$ contains a singular point with link $S^2(2,3,3)$, $\pi_1(\mathcal{O})$ is either a noncyclic dihedral group or a $\mathbb{Z}_2$-extension of a noncyclic dihedral group. Hence Theorem 4.1 implies that $\mathcal{O}_h$ is isomorphic to (a) a spherical dihedral orbifold $\mathcal{O}(r; d_+, d_-)$ or (b) the quotient of $\mathcal{O}(r; d_+, d_-)$ by an isometric involution, where $(d_+, d_-) \neq (1,1)$. Since $\mathcal{O}(r; d_+, d_-)$ does not have a singular point with link $S^2(2,3,3)$, (a) cannot happen, and so we may assume (b) holds. Since $\pi_1(S^2(2,3,3))$ does not have an index 2 subgroup, the link of an inverse image of the singular point $v_{h,\alpha}$ in the double cover $\mathcal{O}(r; d_+, d_-)$ of $\mathcal{O}_h$ is also isomorphic to $S^2(2,3,3)$. But, this is impossible. Hence P_α cannot be isomorphic to $S^2(3,3,3)$.

Thus we have proved that $P_\alpha \cong P_\beta$ cannot be a rigid cusp.

11. Proof of Theorem 1.1 - flexible cusp: exceptional case -

In this section, we treat the case where the following assumption is satisfied, and prove that this assumption is never satisfied.

Assumption 11.1. Under Assumption 8.1, we further assume that $P_\alpha = P_\beta$ and it is a flexible cusp $S^2(2,2,2,2)$ and that the conclusion (2) in Lemma 9.1 holds. Namely, $f \notin \Gamma$, and $P_\alpha/f = P_\beta/f \cong S^2(2,4,4)$ (see Figure 11).

Let $\hat{\Gamma} := \langle \Gamma, f\rangle$ be the group generated by Γ and f. Let $\hat{M} := \mathbb{H}^3/\hat{\Gamma}$ be the quotient hyperbolic orbifold. Let $\hat{M}_0$ be the non-cuspidal part of $\hat{M}$, and $\hat{P} = \partial \hat{M}_0$ the parabolic locus. By abuse of notation, we denote the pared orbifold obtained as the relative compactification of $(\hat{M}_0, \hat{P})$ by the same symbol $(\hat{M}_0, \hat{P})$. We denote the component of the compact euclidean 2-orbifold $\hat{P}$ corresponding to the conjugacy class containing α and $\beta = f\alpha f^{-1}$ by $\hat{P}_{\alpha\beta}$. Thus $\hat{P}_{\alpha\beta} \cong P_\alpha/f = P_\beta/f \cong S^2(2,4,4)$ and $(M_0, P)/f \cong (\hat{M}_0, \hat{P})$, where f denotes the involution on the pared orbifold (M_0, P) induced by the exchanging involution f. In particular, M_0 is the double orbifold covering of $\hat{M}_0$, associated with the homomorphism $\xi : \pi_1(\hat{M}_0) = \Gamma \to \mathbb{Z}_2$ such that $\xi(\alpha) = \xi(\beta) = 0$ and $\xi(f) = 1$. We denote the homomorphism $H_1(\hat{M}_0;\mathbb{Z}_2) \to \mathbb{Z}_2$ induced by ξ by the same symbol.

Note that $\pi_1(\hat{P}_{\alpha\beta}) \cong \pi_1(S^2(2,4,4)) \cong \langle a, b, c \mid a^2,\ b^4,\ c^4, abc\rangle$. As in Case 1 in Section 10, we identify $\pi_1(\hat{P}_{\alpha\beta})$ with the stabiliser $\mathrm{Stab}_{\hat{\Gamma}}(\mathrm{Fix}(\alpha))$. Then the proof of Lemma 10.1 also works in this setting, because $\{\alpha, \beta\}$ generates the non-free subgroup Γ of the Kleinian group $\hat{\Gamma}$, and we have the following lemma.

Lemma 11.2. *The parabolic element α is conjugate to b^2a or b^2ac^2a in $\mathrm{Stab}_{\hat{\Gamma}}(\mathrm{Fix}(\alpha)) \cong \pi_1(S^2(2,4,4))$, and of course, the assertions (1) and (2) in Lemma 10.1 also hold.*

Let e_i and $\hat{e}_i$ $(i = 1, 2, 3)$ be the edges of the singular sets $\Sigma(M_0)$ and $\Sigma(\hat{M}_0)$ as illustrated in Figure 11. Thus e_2 and e_3 are contained in the fixed point set of the involution f on M_0, and $\hat{e}_i$ is the image of e_i by the covering projection $M_0 \to \hat{M}_0$. (Note that it can happen that some of them are identical, though their germs near the parabolic locus are different.) Then the following holds.

Lemma 11.3. *The homomorphism $\xi : H_1(\hat{M}_0;\mathbb{Z}_2) \to \mathbb{Z}_2$, that determines the double orbifold covering $M_0 \to \hat{M}_0$, satisfies*

$$\xi(m_1) = 0, \quad \xi(m_2) = \xi(m_3) = 1,$$

where m_i denotes the meridian of the edge $\hat{e}_i$. Moreover, the homology class $[f] \in H_1(\hat{M}_0;\mathbb{Z}_2)$ determined by $f \in \hat{\Gamma}$ is equal to either m_2 or m_3.

Proof. The formula for ξ follows from the fact that the fixed point set of the involution f on M_0 contains e_2 and e_3, which project to $\hat{e}_2$ and $\hat{e}_3$, respectively. It is also obvious that $\xi([f]) = 1$. So, if $H_1(\hat{M}_0;\mathbb{Z}_2) \cong \mathbb{Z}_2$, we have $[f] = m_2 = m_3$. Suppose

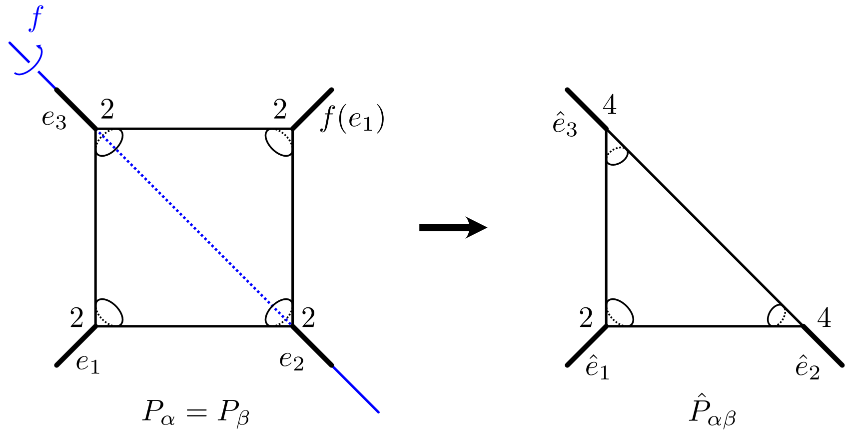


FIGURE 11. Assumption 11.1 assumes that $f \notin \Gamma$ descends to an involution, f, on $P_\alpha = P_\beta \cong S^2(2,2,2,2)$ such that $P_\alpha/f = P_\beta/f \cong S^2(2,4,4)$.

that $H_1(\hat{M}_0;\mathbb{Z}_2) \cong (\mathbb{Z}_2)^2$. Then, since $H_1(\hat{M}_0;\mathbb{Z}_2)$ is generated by $[f]$ and $[\alpha]$, we see $[\alpha] \neq 0$. So, α is conjugate to b^2a by Lemma 11.2. (Otherwise α is conjugate to b^2ac^2a and so $[\alpha] = 0$.) Therefore $[\alpha] = [a] = m_1$ is contained in $\mathrm{Ker}(\xi) \cong \mathbb{Z}_2$. Thus $\mathrm{Ker}(\xi)$ is generated by m_1. Since $\xi([f]) = \xi(m_2)$, it follows that $[f]$ is equal to either m_2 or $m_1 + m_2 = m_3$. □

Let $\hat{\mathcal{O}}$ be the orbifold obtained from the pared orbifold $(\hat{M}_0, \hat{P}_{\alpha\beta})$ by the orbifold surgery that replaces the index 4 of the edges $\hat{e}_2$ and $\hat{e}_3$ with the index 2. Then $\hat{P}_{\alpha\beta}$ shrinks into a singular point, $v_{\alpha\beta}$, with link $S^2(2,2,2)$, and the image of α in $\pi_1(\hat{\mathcal{O}})$ has order ≤ 2 by Lemma 11.2. Since $\hat{\Gamma}$ is generated by f and α, $\pi_1(\hat{\mathcal{O}})$ is either trivial, $\mathbb{Z}_2 = D_1$ or a noncyclic dihedral group. By using Lemma 6.4, Theorem 4.1 and the fact that $\hat{\mathcal{O}}$ has a singular point with link $S^2(2,2,2)$, we see that $\hat{\mathcal{O}}$ is isomorphic to a spherical dihedral orbifold $\mathcal{O}(r; d_+, d_-)$ with noncyclic dihedral orbifold fundamental group. Moreover, we may assume that $d_+ = 2$ and that $v_{\alpha\beta}$ is an endpoint of τ_+. By Lemma 6.5, we have $H_1(\hat{\mathcal{O}};\mathbb{Z}_2) \cong (\mathbb{Z}_2)^2$.

Lemma 11.4. *Under the above setting, $|K(r)| = 1$ and so the edges $\hat{e}_i$ $(i = 1, 2, 3)$ are all distinct.*

Proof. We first observe that α cannot be conjugate to b^2ac^2a. In fact, if α was conjugate to b^2ac^2a, then its image in $\pi_1(\hat{\mathcal{O}})$ is trivial by Lemma 11.2 (cf. Lemma 10.1(2)), and so $\pi_1(\hat{\mathcal{O}})$ is generated by the image of f. This contradicts the fact that $\pi_1(\hat{\mathcal{O}})$ is a noncyclic dihedral group. This observation together with Lemma 11.2 implies that α is conjugate to b^2a and so $[\alpha] = [a] = m_1 \in \mathrm{Ker}(\xi)$. Moreover,

$[f] = m_2$ or m_3 by Lemma 11.3. Hence $H_1(\hat{\mathcal{O}}; \mathbb{Z}_2) \cong (\mathbb{Z}_2)^2$ is generated by the meridians of the three edges $\hat{e}_i$ $(i = 1, 2, 3)$ incident on the vertex $v_{\alpha\beta} \in \partial\tau_+$.

Now suppose on the contrary that $|K(r)| = 2$. Then we see, by using Lemma 6.5, that the meridian of τ_+ represents the trivial element of $H_1(\hat{\mathcal{O}}; \mathbb{Z}_2)$ and the meridians of the remaining two edges incident on $v_{\alpha\beta} \in \partial\tau_+$ represent the identical element of $H_1(\hat{\mathcal{O}}; \mathbb{Z}_2)$. This contradicts the fact that $H_1(\hat{\mathcal{O}}; \mathbb{Z}_2) \cong (\mathbb{Z}_2)^2$. Hence $|K(r)| = 1$. This implies that the the edges $\hat{e}_i$ $(i = 1, 2, 3)$ incident on $v_{\alpha\beta} \in \partial\tau_+$ are all distinct, as desired. □

Recall that the weights of the edges $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ of $\Sigma(\hat{M}_0)$ are $2, 4, 4$. Since $\hat{e}_2 \neq \hat{e}_3$ by Lemma 11.4, we can apply the orbifold surgery on $(\hat{M}_0, \hat{P})$ of "type $(2, 4, 4) \to (2, 2, 4)$", namely we can replace the index 4 of the edge $\hat{e}_2$ of the singular set $\Sigma(\hat{M}_0)$ with the index 2, and leave the other indices, including the index 4 of $\hat{e}_3$, unchanged. We denote the resulting orbifold by $\hat{\mathcal{O}}_{(2,2,4)}$. By Lemma 11.2, α has order at most 2 in $\pi_1(\hat{\mathcal{O}}_{(2,2,4)})$. Hence, by using Lemma 6.4, Theorem 4.1, and the fact that $\hat{\mathcal{O}}_{(2,2,4)}$ has a singular point with link $S^2(2, 2, 4)$, we see that $\hat{\mathcal{O}}_{(2,2,4)}$ is isomorphic to a spherical dihedral orbifold $\mathcal{O}(r; d_+, d_-)$ with noncyclic dihedral orbifold fundamental group. Moreover, we may assume $d_+ = 4$ and that the parabolic locus $P_{\alpha\beta}$ degenerates into a singular point, $v_{\alpha\beta}$, which is an endpoint of τ_+. It should be noted that the edge $\hat{e}_3$ of $\Sigma(\hat{M}_0)$ corresponds to τ_+. (Here, we reset the notation, and the symbols $\mathcal{O}(r; d_+, d_-)$ and $v_{\alpha\beta}$ now represent objects different from those they had represented in the paragraph preceding Lemma 11.4.)

Case 1. $d_- \geq 3$. We apply the orbifold surgery on $(\hat{M}_0, \hat{P})$ of "type $(2, 4, 4) \to (2, 4, 2)$", namely we replace the index 4 of the edge $\hat{e}_3 = \tau_+$ of the singular set $\Sigma(\hat{M}_0)$ with the index 2, and leave the other indices, including the index 4 of $\hat{e}_2$, unchanged. (This is possible by Lemma 11.4.) We denote the resulting orbifold by $\hat{\mathcal{O}}_{(2,4,2)}$. By Lemma 11.2, α has order at most 2 in $\pi_1(\hat{\mathcal{O}}_{(2,4,2)})$. Hence, again by using Lemma 6.4, Theorem 4.1, and the fact that $\hat{\mathcal{O}}_{(2,4,2)}$ has a singular point with link $S^2(2, 4, 2)$, we see that $\hat{\mathcal{O}}_{(2,4,2)}$ is isomorphic to a spherical dihedral orbifold with noncyclic dihedral orbifold fundamental group. Note that the edges $\hat{e}_2$ and τ_- of $\Sigma(\hat{\mathcal{O}}_{(2,4,2)})$, which have indices 4 and $d_- \geq 3$, respectively, share a common endpoint (see Figure 12). But this cannot happen in any spherical dihedral orbifold, a contradiction.

Case 2. $d_- = 1$. Then $(\hat{M}_0, \hat{P})$ is represented by a weighted graph $(S^3, K(r) \cup \tau_+, \hat{w})$, such that $K(r) = \hat{e}_1 \cup \hat{e}_2$ is a knot, $\hat{e}_3 = \tau_+$, and

$$\hat{w}(\tau_+) = 4, \quad \hat{w}(\hat{e}_1) = 2, \quad \hat{w}(\hat{e}_2) = 4.$$

Recall that the subset $\hat{e}_2 \cup \hat{e}_3 = \hat{e}_2 \cup \hat{\tau}_+$ of $\Sigma(\hat{M}_0)$ are the images of the fixed point set of the involution f on M_0. This implies that the map $|M_0| \to |\hat{M}_0|$ induced by the orbifold covering $M_0 \to \hat{M}_0$ is the double branched covering branched over

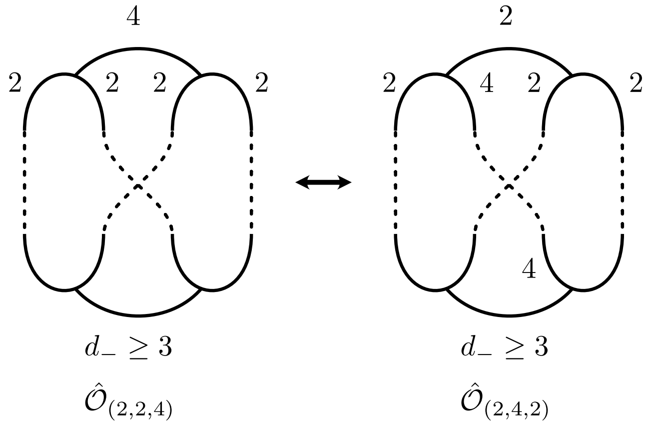


FIGURE 12. Since $\hat{\mathcal{O}}_{(2,2,4)} \cong \mathcal{O}(r; d_+, d_-)$ with $\hat{e}_3 = \tau_+$ is as in the left figure, $\hat{\mathcal{O}}_{(2,4,2)}$ is as in the right figure. The latter orbifold has a singular point with link $S^2(2, 4, d_-)$ with $d_- \geq 3$, and so it cannot be a spherical dihedral orbifold.

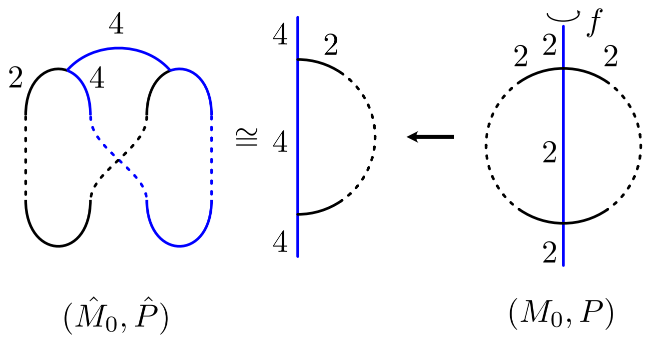


FIGURE 13. Since the orbifold covering $M_0 \to \hat{M}_0$ induces the double branched covering $|M_0| \to |\hat{M}_0|$ branched over $\hat{e}_2 \cup \hat{e}_3 = \hat{e}_2 \cup \hat{\tau}_+$, the orbifold (M_0, P) is as illustrated in the right figure.

$\hat{e}_2 \cup \hat{e}_3$. Hence (M_0, P) is represented by the weighted graph illustrated in Figure 13. Here, we assume the extended Convention 6.2, and the two 4-valent vertices represent parabolic loci isomorphic to $S^2(2, 2, 2, 2)$. Hence we see by Lemma 6.4 that $H_1(M_0; \mathbb{Z}) \cong (\mathbb{Z}_2)^3$, a contradiction.

Thus we have proved that the situation in Assumption 11.1 cannot occur. This completes the proof of the main Theorem 1.1.

12. Appendix 1: Spherical orbifolds with dihedral orbifold fundamental groups

In this appendix, we classify the orientable spherical 3-orbifolds with dihedral orbifold fundamental groups (Proposition 12.2), and determine the (orientation-preserving) isometry groups of these orbifolds (Propositions 12.5 and 12.6). Proposition 12.2 is used in the proof of Theorem 4.1, and Corollary 12.7 is used in Section 9. Propositions 12.5 and 12.6 are used in the companion [4] of this paper. The classification of the spherical dihedral orbifolds is implicitly contained in Dunber's work [22], which classifies the Seifert fibered orbifolds. The isometry groups of the dihedral spherical orbifolds obtained as the π-orbifolds associated with 2-bridge links are calculated by [57, 28]. Moreover, in the recent papers [40, 41], Mecchia and Seppi classified the Seifert fibered spherical 3-orbifolds and calculated the isometry groups of such orbifolds. Since every spherical dihedral orbifold is Seifert fibered, the results in this section are implicitly contained in [40, 41]. However, we give a self-contained proof, because it is not a simple task to translate their results into the form we need.

We first recall basic facts concerning the 3-dimensional spherical geometry following [61, 57]. Let $\mathcal{H}$ be the quaternion skew field. We use the symbol q to denote a generic quaternion

$$q = a + bi + cj + dk \in \mathcal{H} \quad (a, b, c, d \in \mathbb{R}).$$

(We believe this does not cause any confusion, even though q is also used to denote the numerator of a rational number $r = q/p$.) For each $q \in \mathcal{H}$, $\bar{q} = a - bi - cj - dk$ denotes its conjugate, $\Re(q) = a$ denotes its real part, and $|q|$ denotes its norm $\sqrt{q\bar{q}} = \sqrt{a^2 + b^2 + c^2 + d^2}$. We identify S^n ($n = 1, 2, 3$) with the following subspaces of $\mathcal{H}$.

$$S^3 := \{q \in \mathcal{H} \,|\, |q| = 1\}$$
$$S^2 := \{q \in \mathcal{H} \,|\, |q| = 1, \quad \Re(q) = 0\}$$
$$S^1 := \{z \in \mathbb{C} \subset \mathcal{H} \,|\, |z| = 1\}$$

The norm $|\cdot|$ induces a Euclidean metric on $\mathcal{H}$, and S^n ($n = 1, 2, 3$) are endowed with the induced metrics. The subspaces S^3 and S^1 form a Lie group with respect to the restriction of the product in $\mathcal{H}$. The group S^3 acts on itself by conjugation leaving S^2 invariant. This gives an epimorphism $\psi : S^3 \to \mathrm{Isom}^+(S^2)$, with $\ker \psi = \langle -1 \rangle$, defined by

$$\psi(q)(x) = qx\bar{q} \quad (q \in S^3,\ x \in S^2).$$

If $q = \cos\theta + q_0 \sin\theta$ with $q_0 \in S^2$, then $\psi(q)$ is the rotation of S^2, by angle 2θ, with fixed points $\pm q_0$.

For a positive integer n, any cyclic subgroup of order n (resp. any dihedral subgroup of order $2n$) of $\mathrm{Isom}^+(S^2)$ is conjugate to the subgroup $\mathbb{Z}_n := \psi(\mathbb{Z}_n^*)$

(resp. $\mathbb{D}_n := \psi(\mathbb{D}_n^*)$), where $\mathbb{Z}_n^* := \langle \omega \rangle$ and $\mathbb{D}_n^* := \langle \omega,\ j\rangle$ with $\omega = \exp(\pi i/n)$. Note that these groups are contained in the subgroup $\mathbb{D}_S := \langle S^1, j\rangle = S^1 \sqcup S^1 j$ of S^3. Then the following hold (see, e.g. [57, Proposition 2.6]).

Lemma 12.1. (1) *If $n \geq 2$, then the normaliser $N(\mathbb{Z}_n^*)$ of $\mathbb{Z}_n^*$ in S^3 is equal to $\mathbb{D}_S$.*

(2) *If $n \geq 3$, then the normaliser $N(\mathbb{D}_n^*)$ of $\mathbb{D}_n^*$ in S^3 is equal to $\mathbb{D}_{2n}^*$. If $n = 2$, then $N(\mathbb{D}_n^*)$ is equal to the binary octahedral group $O^* = \psi^{-1}(O)$, where $O < \mathrm{Isom}^+(S^2)$ is the octahedral group obtained as the subgroup of $\mathrm{Isom}^+(S^2)$ preserving the regular octahedron in the 3-dimensional Euclidean subspace $\langle i, j, k\rangle$ of $\mathcal{H}$ spanned by the 6 vertices $\{\pm i, \pm j, \pm k\}$.*

Let $\phi : S^3 \times S^3 \to \mathrm{Isom}^+(S^3)$ be the homomorphism defined by

$$\phi(q_1, q_2)(q) = q_1 q q_2^{-1}.$$

Then ϕ is an ephimorphism with $\mathrm{Ker}\,\phi = \langle(-1, -1)\rangle \cong \mathbb{Z}_2$.

We occasionally identify $S^3 \subset \mathcal{H}$ with the unit sphere

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$$

in $\mathbb{C}^2$ by the correspondence $q = z_1 + z_2 j \leftrightarrow (z_1, z_2)$. Let $\mathrm{L} : S^1 \times S^1 \to \mathrm{Isom}^+(S^3)$ be the injective homomorphism defined by

$$\mathrm{L}(\omega_1, \omega_2)(z_1, z_2) = (\omega_1 z_1, \omega_2 z_2).$$

When $\omega_\ell = \exp(2\pi i \frac{k_\ell}{n_\ell})$ $(\ell = 1, 2)$, where $\frac{k_\ell}{n_\ell}$ is a rational number, we write

$$\mathrm{L}(\omega_1, \omega_2) = \mathrm{L}(\frac{k_1}{n_1}, \frac{k_2}{n_2}), \tag{1}$$

because its restriction to the circles $S^3 \cap (\mathbb{C} \times \{0\})$ and $S^3 \cap (\{0\} \times \mathbb{C})$ are the '$\frac{k_1}{n_1}$-rotation' and '$\frac{k_2}{n_2}$-rotation', respectively. Though the symbol $\mathrm{L}(\cdot, \cdot)$ is used in two different ways, we believe this does not cause any confusion, because its meaning is clearly understood from the context according to whether $\cdot$ is a unit complex or a rational number.

Observe that

$$\phi(\eta_1, \eta_2) = \mathrm{L}(\eta_1 \bar{\eta}_2, \eta_1 \eta_2) \quad ((\eta_1, \eta_2) \in S^1 \times S^1).$$

In particular, we have

$$\phi(S^1 \times S^1) = \mathrm{L}(S^1 \times S^1) < \mathrm{Isom}^+(S^3). \tag{2}$$

Consider the isometries $J := \phi(j, j)$, $J_1 := \phi(1, j)$ and $J_2 := \phi(j, 1)$, which acts on $S^3 \subset \mathbb{C}^2$ as follows.

$$J(z_1, z_2) = (\bar{z}_1, \bar{z}_2),\ J_1(z_1, z_2) = (z_2, -z_1),\ J_2(z_1, z_2) = (-\bar{z}_2, \bar{z}_1)$$

Observe $J = J_1 J_2$ and that

$$\phi(\mathbb{D}_S \times \mathbb{D}_S) = \langle \mathrm{L}(S^1 \times S^1), J, J_1\rangle, \quad \langle J, J_1\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2. \tag{3}$$

In fact, $\phi(\mathbb{D}_S \times \mathbb{D}_S)$ is the split extension of $\mathrm{L}(S^1 \times S^1)$ by $\langle J, J_1\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, where the action of $\langle J, J_1\rangle$ on $\mathrm{L}(S^1 \times S^1)$ by conjugation is given by the following formula.

$$J\mathrm{L}(\omega_1, \omega_2)J^{-1} = \mathrm{L}(\bar{\omega}_1, \bar{\omega}_2), \quad J_1\mathrm{L}(\omega_1, \omega_2)J_1^{-1} = \mathrm{L}(\omega_2, \omega_1) \tag{4}$$

The following proposition gives a classification of the orientable spherical 3-orbifolds with dihedral orbifold fundamental groups.

Proposition 12.2. *Let $\mathcal{O}$ be an oriented spherical 3-orbifold. Then $\pi_1(\mathcal{O})$ is isomorphic to a dihedral group, if and only if $\mathcal{O}$ is isomorphic to the orbifold, $\mathcal{O}(r; d_1, d_2)$, represented by the weighted graph $(S^3, K(r) \cup \tau_+ \cup \tau_-, w)$ in Figure 6 for some $r \in \mathbb{Q}$ and coprime positive integers d_1 and d_2, where w is given by the following rule.*

$$w(K(r)) = 2, \quad w(\tau_+) = d_1, \quad w(\tau_-) = d_2.$$

In fact, $\mathcal{O}(r; d_1, d_2)$ with $r = q/p$ is isomorphic to S^3/Γ, where Γ is the subgroup of $\mathrm{Isom}^+(S^3)$ given by

$$\Gamma = \left\langle \mathrm{L}(\frac{k_1}{pd_2}, \frac{k_2}{pd_1}),\ J \right\rangle \cong D_n \quad \text{with } n = pd_1d_2 \tag{5}$$

for some integers k_1 and k_2 such that

$$\gcd(pd_2, k_1) = 1, \quad \gcd(pd_1, k_2) = 1, \quad k_2 \equiv qk_1 \pmod{p}. \tag{6}$$

Moreover, the spherical structure of $\mathcal{O}(r; d_1, d_2)$ is unique, i.e., if Γ' is a subgroup of $\mathrm{Isom}^+(S^3)$ such that S^3/Γ' is isomorphic to $\mathcal{O}(r; d_1, d_2)$ as oriented orbifolds, then Γ' is conjugate to the subgroup Γ defined by (5).

Proof. We first prove the only if part of the first assertion. Let Γ be a subgroup of $\mathrm{Isom}^+(S^3)$ isomorphic to the dihedral group D_n, and let f and h be the elements of Γ such that

$$\Gamma \cong \langle f, h \mid f^n = 1, h^2 = 1, hfh^{-1} = f^{-1}\rangle.$$

(Though the symbols Γ, f and h are used in different meanings in the previous sections, we believe this does not cause any confusion.) We show that S^3/Γ is isomorphic to some $\mathcal{O}(q/p; d_1, d_2)$, such that $n = pd_1d_2$.

Claim 12.3. *After taking conjugation in $\mathrm{Isom}^+(S^3)$, we may assume $f = \mathrm{L}(\frac{k_1}{pd_2}, \frac{k_2}{pd_1})$, where p, d_1, d_2, k_1, and k_2 are positive integers such that $\gcd(d_1, d_2) = 1$, $\gcd(pd_2, k_1) = 1$, $\gcd(pd_1, k_2) = 1$, and $n = pd_1d_2$.*

Proof of Claim 12.3. Since any element of S^3 is conjugate to an element in S^1, we may assume, by taking conjugation, that $f \in \phi(S^1 \times S^1) = \mathrm{L}(S^1 \times S^1)$ (see (2)). Since f has order n, we may assume $f = \mathrm{L}(\frac{k_1'}{n}, \frac{k_2'}{n})$ for some integers k_1' and k_2' such that $\gcd(n, k_1', k_2') = 1$. For $\ell = 1, 2$, set $d_\ell = \gcd(n, k_\ell')$, $n_\ell = \frac{n}{d_\ell}$ and $k_\ell = \frac{k_\ell'}{d_\ell}$, so that $f = \mathrm{L}(\frac{k_1'}{n}, \frac{k_2'}{n}) = \mathrm{L}(\frac{k_1}{n_1}, \frac{k_2}{n_2})$, where $\gcd(k_1, n_1) = \gcd(k_2, n_2) = 1$. Note also that $\gcd(d_1, d_2) = \gcd(n, k_1', k_2') = 1$. Set $p = \gcd(n_1, n_2)$ and $n_\ell' = \frac{n_\ell}{p}$ $(\ell = 1, 2)$. Then

$n = \mathrm{lcm}(n_1, n_2) = pn_1'n_2'$. Thus $n_1 d_1 = n = pn_1'n_2' = n_1 n_2'$ and so $d_1 = n_2'$. Similarly we have $d_2 = n_1'$. Hence we have $n = pd_1d_2$ and $f = \mathrm{L}(\frac{k_1}{n_1}, \frac{k_2}{n_2}) = \mathrm{L}(\frac{k_1}{pd_2}, \frac{k_2}{pd_1})$. $\square$

Now consider the subgroup $\langle f^p \rangle \cong \mathbb{Z}_{d_1 d_2}$ generated by $f^p = \mathrm{L}(\frac{k_1}{d_2}, \frac{k_2}{d_1})$. Since $\gcd(d_1, d_2) = 1$, we have

$$\langle f^p \rangle \cong \langle f^{pd_2} \rangle \times \langle f^{pd_1} \rangle \cong \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2}.$$

Note that

$$\langle f^{pd_2} \rangle = \langle \mathrm{L}(0, \frac{k_2 d_2}{d_1}) \rangle = \langle \mathrm{L}(0, \frac{1}{d_1}) \rangle, \quad \langle f^{pd_1} \rangle = \langle \mathrm{L}(\frac{k_1 d_1}{d_2}, 0) \rangle = \langle \mathrm{L}(\frac{1}{d_2}, 0) \rangle.$$

Hence we have

$$\langle f^p \rangle = \langle \mathrm{L}(0, \frac{1}{d_1}) \rangle \times \langle \mathrm{L}(\frac{1}{d_2}, 0) \rangle.$$

Thus $S^3/\langle f^p \rangle$ is the orbifold with underlying space S^3 and with singular set the Hopf link, where one component has index d_1 and the other component has index d_2. To give a precise description of this orbifold, identify S^3 with the join $S^1 * S^1$, by the correspondence $(tz_1, \sqrt{1-t^2}z_2) \leftrightarrow tz_1 + (1-t)z_2$. Thus the first and second factor circles of $S^1 * S^1$ correspond to the circles $S^1 \times \{0\}$ and $\{0\} \times S^1$ in $S^3 \subset \mathbb{C}^2$, respectively. For $\omega \in S^1$, let $\mathrm{L}(\omega)$ be the isometry of S^1 defined by $\mathrm{L}(\omega)(z) = \omega z$ ($z \in S^1$). Then the isometry $\mathrm{L}(\omega_1, \omega_2)$ is identified with the self-homeomorphism $\mathrm{L}(\omega_1) * \mathrm{L}(\omega_2)$ of $S^1 * S^1$, defined by

$$(\mathrm{L}(\omega_1) * \mathrm{L}(\omega_2))(tz_1 + (1-t)z_2) = t\omega_1 z_1 + (1-t)\omega_2 z_2.$$

Under the above convention, the orbifold $S^3/\langle f^p \rangle$ is described as follows. The underlying space of the orbifold is given by

$$|S^3/\langle f^p \rangle| \cong \left(S^1/\mathrm{L}(\frac{1}{d_2})\right) * \left(S^1/\mathrm{L}(\frac{1}{d_1})\right) \cong S^1 * S^1 \cong S^3,$$

and the singular set is the union of the two circles which gives the join structure of S^3, where the first factor circle (which corresponds to $S^1/\mathrm{L}(\frac{1}{d_2})$) has index d_1 and the second factor circle (which corresponds to $S^1/\mathrm{L}(\frac{1}{d_1})$) has index d_2. Here, $\mathrm{L}(\frac{1}{d_\ell})$ denotes $\mathrm{L}(e^{\frac{2\pi i}{d_\ell}})$ as in (1).

The isometry f descends to the periodic isomorphism of the orbifold $S^3/\langle f^p \rangle \cong S^1 * S^1$ given by $\mathrm{L}(\frac{k_1}{p}) * \mathrm{L}(\frac{k_2}{p})$, because the periodic map $\mathrm{L}(\frac{k_1}{pd_2})$ (resp. $\mathrm{L}(\frac{k_2}{pd_1})$) on S^1 descends to the periodic map $\mathrm{L}(\frac{k_1}{p})$ (resp. $\mathrm{L}(\frac{k_2}{p})$) on the circle $S^1/\mathrm{L}(\frac{1}{d_2})$ (resp. $S^1/\mathrm{L}(\frac{1}{d_1})$). Note that $\langle \mathrm{L}(\frac{k_1}{p}) * \mathrm{L}(\frac{k_2}{p}) \rangle = \langle \mathrm{L}(\frac{1}{p}) * \mathrm{L}(\frac{q}{p}) \rangle$, with $q \equiv k_1^{-1} k_2 \pmod p$, where k_1^{-1} is the inverse of k_1 in the multiplicative group $(\mathbb{Z}_p)^\times$ (cf. Notation 1.3(2)). Hence we see that the orbifold $S^3/\langle f \rangle$ is isomorphic to the orbifold, $\mathcal{O}(L(p,q), d_1, d_2)$, with underlying space the lens space, $L(p,q)$, and with singular set the union of the core circles of the standard genus 1 Heegaard splitting of $L(p,q)$ with indices d_1 and d_2, respectively. (Though the notation $L(p,q)$ looks similar to the notation $\mathrm{L}(\cdot,\cdot)$ in (1), we believe there is no fear of confusion.)

Since $h^2=1$ and $hfh^{-1}=f^{-1}$, we see by using Lemma 12.1(1) that $h=\phi(q_1,q_2)$ for some $(q_1,q_2)\in S^1j\times S^1j$. Since any element of S^1j is conjugate to j by an element of S^1, we may assume $h=\phi(j,j)=J$, and so $h(z_1,z_2)=(\bar{z}_1,\bar{z}_2)$. This implies that the involution h of S^3 descends to the hyper-elliptic involution of $|S^3/\langle f\rangle|\cong L(p,q)$. Recall that (i) the quotient map determined by the hyper-elliptic involution gives the double branched covering of S^3 branched over the 2-bridge link $K(q/p)$ and that (ii) the core circles of the genus 1 Heegaard splitting project to the upper and lower tunnels, respectively. Hence, the quotient S^3/Γ, with $\Gamma=\langle f,h\rangle\cong D_n$, is isomorphic to the orbifold $\mathcal{O}(q/p;d_1,d_2)$. This completes the proof of the only if part of the first assertion. The proof also shows that the group Γ is given by the formula (5) for some integers k_1 and k_2 satisfying the condition (6).

The if part of the first assertion follows from the above argument and the following claim.

Claim 12.4. *For any rational number $r=q/p$ and a pair of coprime integers (d_1,d_2), there is a pair (k_1,k_2) of integers which satisfies the condition (6).*

Proof of Claim 12.4. Consider the homomorphism

$$\Psi:(\mathbb{Z}_{pd_2})^\times\times(\mathbb{Z}_{pd_1})^\times\to(\mathbb{Z}_p)^\times\times(\mathbb{Z}_p)^\times\to(\mathbb{Z}_p)^\times,$$

where the first homomorphism is the product of the natural projections and the second homomorphism maps $(k_1,k_2)\in(\mathbb{Z}_p)^\times\times(\mathbb{Z}_p)^\times$ to $k_1^{-1}k_2\in(\mathbb{Z}_p)^\times$. Then both of the two homomorphisms are surjective and so is their composition Ψ. Regard the numerator q of the rational number $r=q/p$ as an element of $(\mathbb{Z}_p)^\times$, and let (k_1,k_2) be a pair of integers which projects to an element in the inverse image $\Psi^{-1}(q)$. Then (k_1,k_2) satisfies the condition (6). □

Finally we prove the uniqueness of the spherical structure on the orbifold $\mathcal{O}(q/p;d_1,d_2)$. The preceding arguments show that the triple $(q/p,d_1,d_2)\in\mathbb{Q}\times\mathbb{N}\times\mathbb{N}$ uniquely determines a dihedral subgroup $\Gamma<\mathrm{Isom}^+(S^3)$, up to conjugation, such that S^3/Γ is isomorphic to $\mathcal{O}(q/p;d_1,d_2)$ as oriented orbifolds. Thus we have only to show that there are no unexpected orientation-preserving topological isomorphism between two orbifolds, $\mathcal{O}(q/p;d_1,d_2)=S^3/\Gamma$ and $\mathcal{O}(q'/p';d_1',d_2')=S^3/\Gamma'$.

Assume that $\mathcal{O}(q/p;d_1,d_2)$ and $\mathcal{O}(q'/p';d_1',d_2')$ are isomorphic as oriented orbifolds. Then $pd_1d_2=p'd_1'd_2'$ and $\{d_1,d_2\}=\{d_1',d_2'\}$, because they have isomorphic orbifold fundamental groups and the same index sets of the singular sets. In particular we have $p=p'$.

Suppose first that $n:=pd_1d_2\geq 3$. Then D_n has the unique cyclic subgroup of index 2, and so each of $\mathcal{O}(q/p;d_1,d_2)$ and $\mathcal{O}(q'/p';d_1',d_2')$ has the unique double orbifold covering with cyclic orbifold fundamental group. The underlying spaces of the covering orbifolds are the lens spaces $L(p,q)$ and $L(p',q')$, respectively. Hence, by the classification of lens spaces, we have $p = p'$ and either $q \equiv q' \pmod p$ or $qq' \equiv 1 \pmod p$. Moreover, by using the uniqueness of the genus one Heegaard splittings (see [12, 13]), we see that $\mathcal{O}(q/p; d_1, d_2)$ and $\mathcal{O}(q'/p'; d'_1, d'_2)$ are isomorphic as oriented orbifolds if and only if one of the following conditions hold.

(1) $p = p'$, $q \equiv q' \pmod p$, and $(d_1, d_2) = (d'_1, d'_2)$.
(2) $p = p'$, $qq' \equiv 1 \pmod p$, and $(d_1, d_2) = (d'_2, d'_1)$.

In both cases, we can see that the subgroups Γ and Γ' are conjugate in $\mathrm{Isom}^+(S^3)$.

In the exceptional case when $n := pd_1d_2 = 2$, we have either (i) $p = p' = 1$ and $\{d_1, d_2\} = \{d'_1, d'_2\} = \{1, 2\}$ or (ii) $p = p' = 2$ and $d_1 = d_2 = d'_1 = d'_2 = 1$. We can easily see that the subgroups Γ and Γ' are conjugate in $\mathrm{Isom}^+(S^3)$.

This completes the uniqueness of the spherical structure. □

Next, we calculate the (orientation-preserving) isometry group of the dihedral spherical 3-orbifold $\mathcal{O}(q/p; d_1, d_2)$. If $(d_1, d_2) = (1, 1)$, then $\mathcal{O}(q/p; d_1, d_2)$ is the π-orbifold, $\mathcal{O}(q/p)$, associated with the 2-bridge link $K(q/p)$ (cf. [11]) and its isometry group is calculated by [57, Theorem 4.1] and [28, Corollary 3.2.11]. (There are errors in [57, Theorem 4.1] for the special case when $p = 1, 2$. There are also misprints for the generic case in the statement of Theorem 4.1, though the correct result can be found in the tables in [57, p.184].)

Proposition 12.5. *The orientation-preserving isometry group of the spherical orbifold $\mathcal{O}(q/p) := \mathcal{O}(q/p; 1, 1)$ is described as follows.*

(1) If $q \not\equiv \pm 1 \pmod p$, then the following holds.

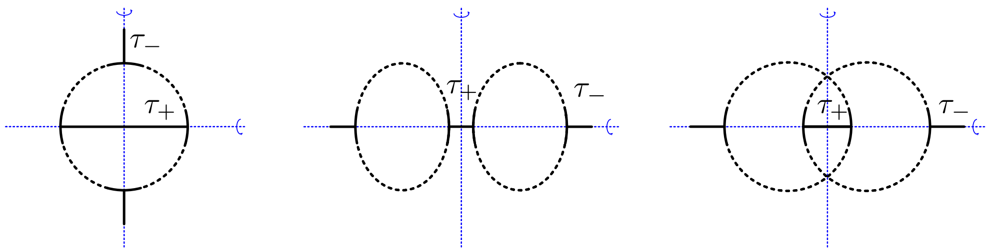
$$\mathrm{Isom}^+(\mathcal{O}(q/p)) \cong \begin{cases} (\mathbb{Z}_2)^2 & \text{if } q^2 \not\equiv 1 \pmod p \\ D_4 & \text{if } p \text{ is odd and } q^2 \equiv 1 \pmod p \\ & \text{or if } p \text{ is even and } q^2 \equiv p+1 \pmod{2p} \\ (\mathbb{Z}_2)^3 & \text{if } p \text{ is even and } q^2 \equiv 1 \pmod{2p} \end{cases}$$

(2) If $q \equiv \pm 1 \pmod p$, then then the following holds.

$$\mathrm{Isom}^+(\mathcal{O}(q/p)) \cong \begin{cases} S^1 \rtimes \mathbb{Z}_2 & \text{if } p \text{ is odd and } \geq 3 \\ S^1 \rtimes (\mathbb{Z}_2)^2 & \text{if } p \text{ is even and } \geq 4 \\ (S^1 \times S^1) \rtimes (\mathbb{Z}_2)^2 & \text{if } p = 2 \\ (S^1 \times S^1) \rtimes \mathbb{Z}_2 & \text{if } p = 1 \end{cases}$$

In the remainder of this section, we treat the remaining case $(d_1, d_2) \neq (1, 1)$. In the very special case, when $p = 1$ and $\{d_1, d_2\} = \{1, 2\}$, we call $\mathcal{O}(0/1; 1, 2)$ the *trivial θ-orbifold*, because its singular set is the trivial θ-curve in S^3. Then we have the following proposition.

Proposition 12.6. *The orientation-preserving isometry group of the spherical dihedral orbifold $\mathcal{O}(q/p; d_1, d_2)$ with $(d_1, d_2) \neq (1, 1)$ is described as follows.*



$|K(r)| = 1$ $\quad$ $|K(r)| = 2$, $\mathrm{lk}(K_1, K_2) \equiv 0 \pmod 2$ $\quad$ $|K(r)| = 2$, $\mathrm{lk}(K_1, K_2) \equiv 1 \pmod 2$

FIGURE 14. $\mathrm{Isom}^+(\mathcal{O}(q/p; d_1, d_2)) \cong (\mathbb{Z}_2)^2$ if $(d_1, d_2) \neq (1, 1)$ and $\mathcal{O}(q/p; d_1, d_2)) \not\cong \mathcal{O}(0/1; 1, 2)$.

(1) $\mathrm{Isom}^+(\mathcal{O}(q/p; d_1, d_2)) \cong (\mathbb{Z}_2)^2$, *except when* $\mathcal{O}(q/p; d_1, d_2)$ *is isomorphic to the trivial* θ*-orbifold* $\mathcal{O}(0/1; 1, 2)$*, i.e. except when* $p = 1$ *and* $\{d_1, d_2\} = \{1, 2\}$.

(2) For the the trivial θ*-orbifold* $\mathcal{O}(0/1; 1, 2)$*, we have* $\mathrm{Isom}^+(\mathcal{O}(0/1; 1, 2)) \cong D_3 \times \mathbb{Z}_2$.

Before proving the proposition, we give the following consequence of the proposition (and the orbifold theorem), which is used in the proof of the main theorem.

Corollary 12.7. *Consider a spherical orbifold* $\mathcal{O}(q/p; d_1, d_2)$ *with* $(d_1, d_2) \neq (1, 1)$*, and let* g *be an orientation-preserving involution of the orbifold. Then the following hold.*

(1) Except when $p = 1$ *and* $\{d_1, d_2\} = \{1, 2\}$*, (i.e. except when* $\mathcal{O}(q/p; d_1, d_2)$ *is isomorphic to the trivial* θ*-orbifold* $\mathcal{O}(0/1; 1, 2)$*),* g *stabilises the edges* τ_+ *and* τ_- *of the singular set (when it is contained in the singular set).*

(2) If $d_1, d_2 \geq 2$*, then* g *does not stabilise any edge of the singular set different from* $\tau_\pm$*.*

Proof of Corollary 12.7. By the orbifold theorem, we may assume g is an isometry of the spherical orbifold. (This is proved by applying the orbifold theorem to the finite group action on the universal cover S^3 of $\mathcal{O}(q/p; d_1, d_2)$ generated by a lift of g and the covering transformation group.) On the other hand, the action of $\mathrm{Isom}^+(\mathcal{O}(q/p; d_1, d_2))$ in the generic case is as illustrated in Figure 14. (See also [4, Figure 6-8], and replace the weights ∞ with 2, then we obtain the desired visualisation, besides the exceptional case.) The exceptional case where the orbifold is the trivial θ-orbifold is illustrated in Figure 15. The assertion (1) is now obvious from Figure 14. The assertion (2) also follows from the figure by noting that $K(r)$ consists of four edges if $d_1, d_2 \geq 2$ (otherwise, $K(r)$ consists of two edges). $\square$

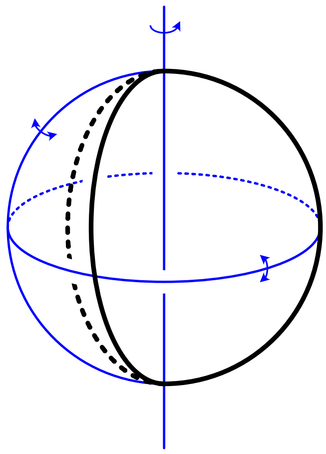


FIGURE 15. $\mathrm{Isom}^+(\mathcal{O}(0/1;1,2)) \cong D_3 \times \mathbb{Z}_2$. The singular set of the trivial θ-orbifold $\mathcal{O}(0/1;1,2)$ is the standardly embedded θ-graph in $S^2 \subset S^3$, consisting of three geodesics joining the north and south poles, which are permuted by the $2\pi/3$-rotation around the earth axis. The orientation-preserving isometry group is visualised as the product of the dihedral group D_3 generated by the π-rotations about the three great circles containing the singular edges and cyclic group $\mathbb{Z}_2$ generated by the π-rotation about the equator.

The proof of Proposition 12.6 presented below is parallel to that of [57, Theorem 4.1]. Consider the dihedral spherical 3-orbifold $\mathcal{O}(q/p;d_1,d_2)$ with $(d_1,d_2) \neq (1,1)$. By Proposition 12.2, the orbifold fundamental group $\pi_1(\mathcal{O}(q/p;d_1,d_2))$ is identified with the subgroup

$$\Gamma = \left\langle \mathrm{L}(\frac{k_1}{pd_2}, \frac{k_2}{pd_1}),\ J \right\rangle = \langle \mathrm{L}(\omega_1,\omega_2),\ J\rangle < \mathrm{Isom}^+(S^3)$$

where $\omega_1 = \exp(2\pi i \frac{k_1}{pd_2})$, $\omega_2 = \exp(2\pi i \frac{k_2}{pd_1})$ for some integers k_1 and k_2 satisfying the condition (6). Pick $(\eta_1,\eta_2) \in S^1 \times S^1$ such that $(\omega_1,\omega_2) = (\eta_1\bar{\eta}_2, \eta_1\eta_2)$. Then $\Gamma = \langle \phi(\eta_1,\eta_2), \phi(j,j)\rangle$. Set

$$\tilde{\Gamma} := \phi^{-1}(\Gamma) = \langle (\eta_1,\eta_2),(j,j)\rangle < S^3 \times S^3.$$

Then $\mathrm{Isom}^+\,\mathcal{O}(q/p;d_1,d_2) \cong N(\tilde{\Gamma})/\tilde{\Gamma}$, where $N(\tilde{\Gamma})$ is the normaliser of $\tilde{\Gamma}$ in $S^3 \times S^3$.

For $\ell = 1,2$, set $\tilde{\Gamma}_\ell = \mathrm{pr}_\ell(\tilde{\Gamma})$, where $\mathrm{pr}_\ell : S^3 \times S^3 \to S^3$ is the projection to the ℓ-th factor. Then $\tilde{\Gamma}_\ell = \langle \eta_\ell, j\rangle = \mathbb{D}^*_{m_\ell}$ for some positive integer m_ℓ. Then the following lemma is obvious from the definition of $\mathbb{D}^*_{m_\ell}$, where $o(\cdot)$ denotes the order of a group element.

Lemma 12.8. *(1)* $o(\eta_\ell^2) = m_\ell$.

(2) If m_ℓ is even, then $o(\eta_\ell) = 2m_\ell$. If m_ℓ is odd, then $o(\eta_\ell)$ is either m_ℓ or $2m_\ell$.

Note that the orientation-reversing isometry $c : S^3 \to S^3$, defined by $c(q) = \bar{q}$, acts on $\mathrm{Isom}^+(S^3)$ by conjugation, as follows:

$$c\phi(q_1, q_2)c^{-1} = \phi(q_2, q_1)$$

Hence, we assume $m_1 \le m_2$ without loss of generality.

Lemma 12.9. *(1) $2 \le m_1 \le m_2$.*

(2) If $m_1 = 2$, then $m_2 = 2m_2'$ for some odd integer m_2', and $\{d_1, d_2\} = \{1, 2\}$. Moreover, $m_1 = m_2 = 2$ if and only if $p = 1$.

Proof. (1) Suppose on the contrary that $m_1 = 1$. Then $\eta_1 = \pm 1$, and so $(\omega_1, \omega_2) = \pm(\bar{\eta}_2, \eta_2)$. This implies $pd_2 = o(\omega_1) = o(\omega_2) = pd_1$ and therefore $d_1 = d_2$. Since $\gcd(d_1, d_2) = 1$, we have $d_1 = d_2 = 1$, a contradiction.

(2) Suppose $m_1 = 2$. Then $\eta_1 = \pm i$, and so $(\omega_1, \omega_2) = \pm(i\bar{\eta}_2, i\eta_2)$. By using Lemma 12.8, we can verify the following, from which the assertion (2) follows.

(i) If m_2 is odd, then $(o(\omega_1), o(\omega_2)) = (4m_2, 4m_2)$ and so $d_1 = d_2 = 1$ as in (1), a contradiction.

(ii) If $m_2 = 2m_2'$ for some odd integer m_2', then $\omega_1^{m_2'} = -\omega_2^{m_2'} = \pm 1$ and so $\{pd_2, pd_1\} = \{o(\omega_1), o(\omega_2)\} = \{m_2', 2m_2'\}$. Hence we have $p = m_2'$ and $\{d_1, d_2\} = \{1, 2\}$.

(iii) If $m_2 = 4m_2'$ for some integer m_2', then $\omega_1^{2m_2'} = -\omega_2^{2m_2'} = \pm i$ and so $o(\omega_1) = o(\omega_2) = 8m_2'$. Hence $d_1 = d_2 = 1$, a contradiction.

□

Lemma 12.10. *Except for the special case where $p = 1$ and $\{d_1, d_2\} = \{1, 2\}$, namely except when $\mathcal{O}(q/p; d_1, d_2)$ is the trivial θ-orbifold, we have*

$$N(\tilde{\Gamma}) < \mathbb{D}^*_{2m_1} \times \mathbb{D}^*_{2m_2} < \mathbb{D}_S \times \mathbb{D}_S.$$

Proof. If $m_1 \ge 3$, then Lemma 12.1 implies $N(\tilde{\Gamma}_\ell) = \mathbb{D}^*_{2m_\ell}$ for each $\ell = 1, 2$ (because $m_2 \ge m_1$ by assumption), and hence we have $N(\tilde{\Gamma}) < N(\tilde{\Gamma}_1) \times N(\tilde{\Gamma}_2) < \mathbb{D}^*_{2m_1} \times \mathbb{D}^*_{2m_2}$.

Since $m_2 \ge m_1 \ge 2$ by Lemma 12.9(1), we have only to treat the case where $m_1 = 2$. Since we exclude the case where $p = 1$ and $\{d_1, d_2\} = \{1, 2\}$, Lemma 12.9(2) implies $m_2 \ge 3$, and so $N(\tilde{\Gamma}_2) = \mathbb{D}^*_{2m_2}$. On the other hand, since $m_1 = 2$, we see by Lemma 12.1 that $N(\tilde{\Gamma}_1) = O^*$. Hence $N(\tilde{\Gamma}) < O^* \times \mathbb{D}^*_{2m_2}$.

Now observe that the decomposition $\mathbb{D}_S = S^1 \sqcup S^1 j$ induces the decomposition of $\tilde{\Gamma} < \mathbb{D}_S \times \mathbb{D}_S$ into the following two non-empty subsets.

$$\tilde{\Gamma}^{(1)} := \tilde{\Gamma} \cap (S^1 \times S^1) \quad \text{and} \quad \tilde{\Gamma}^{(j)} := \tilde{\Gamma} \cap (S^1 j \times S^1 j).$$

Note that $\mathrm{pr}_1(\tilde{\Gamma}^{(1)}) = \langle i \rangle = \{\pm 1, \pm i\}$ and $\mathrm{pr}_1(\tilde{\Gamma}^{(j)}) = \langle i \rangle j = \{\pm j, \pm k\}$. Pick an arbitrary element $(q_1, q_2) \in N(\tilde{\Gamma})$. Then $q_2 \in \mathbb{D}^*_{2m_2} < \mathbb{D}_S$, and so the inner-automorphism of S^3 determined by q_2 preserves the subgroup $S^1 < \mathbb{D}_S$. Thus the inner-automorphism of $S^3 \times S^3$ determined by (q_1, q_2) preserves the subset $\tilde{\Gamma}^{(1)}$ of $\tilde{\Gamma} = \tilde{\Gamma}^{(1)} \sqcup \tilde{\Gamma}^{(j)}$. Hence the inner-automorphism of S^3 determined by q_1 preserves the subgroup $\mathrm{pr}_1(\tilde{\Gamma}^{(1)}) = \langle i \rangle$, and so it preserves the subset $\{\pm i\}$, i.e., $q_1 i \bar{q}_1 = \pm i$. By the description of O^* in Lemma 12.1, this implies that $q_1 \in \mathbb{D}_2^* < \mathbb{D}_{2m_1}^*$. Hence $(q_1, q_2) \in \mathbb{D}_{2m_1}^* \times \mathbb{D}_{2m_2}^*$, as desired. □

Lemma 12.11. *The normaliser $N(\Gamma)$ of Γ in $\mathrm{Isom}^+ S^3$ is contained in $\langle \mathrm{L}(S^1 \times S^1), J \rangle$.*

Proof. By the formula (3) and Lemma 12.10, we have

$$N(\Gamma) = \phi(N(\tilde{\Gamma})) < \phi(\mathbb{D}_S \times \mathbb{D}_S) = \langle \mathrm{L}(S^1 \times S^1), J, J_1 \rangle.$$

Since $J \in \Gamma$, we have only to show that $J_1 \notin N(\Gamma)$. To this end, recall that $\Gamma = \langle \mathrm{L}(\frac{k_1}{pd_2}, \frac{k_2}{pd_1}), J \rangle$. Now suppose on the contrary that $J_1 \in N(\Gamma)$. Then the conjugation by J_1 preserves the subgroup $\langle \mathrm{L}(\frac{k_1}{pd_2}, \frac{k_2}{pd_1}) \rangle$ and we have $J_1 \mathrm{L}(\frac{k_1}{pd_2}, \frac{k_2}{pd_1}) J_1^{-1} = \mathrm{L}(\frac{k_2}{pd_1}, \frac{k_1}{pd_2})$ by (4). Thus we have $d_1 = d_2$ and so $d_1 = d_2 = 1$, a contradiction. Hence $J_1 \notin N(\Gamma)$ as desired. □

Lemma 12.12. *Except when $\mathcal{O}(q/p; d_1, d_2)$ is the trivial θ-orbifold, we have the following.*

$$\begin{aligned} N(\Gamma) &= \left\langle \mathrm{L}(\frac{k_1}{2pd_2}, \frac{k_2}{2pd_1}),\ \mathrm{L}(\frac{1}{2}, 0),\ \mathrm{L}(0, \frac{1}{2}),\ J \right\rangle \\ &\cong \left\langle \mathrm{L}(\frac{k_1}{2pd_2}, \frac{k_2}{2pd_1}),\ \mathrm{L}(\frac{1}{2}, 0),\ \mathrm{L}(0, \frac{1}{2}) \right\rangle \rtimes \langle J \rangle \end{aligned}$$

Proof. Recall that $\Gamma = \langle \mathrm{L}(\omega_1, \omega_2), J \rangle$ where $\omega_1 = \exp(2\pi i \frac{k_1}{pd_2})$ and $\omega_2 = \exp(2\pi i \frac{k_2}{pd_1})$. Set $\sqrt{\omega_1} = \exp(\pi i \frac{k_1}{pd_2})$ and $\sqrt{\omega_2} = \exp(\pi i \frac{k_2}{pd_1})$. Suppose an element $\mathrm{L}(\zeta_1, \zeta_2) \in \mathrm{L}(S^1 \times S^1)$ belongs to $N(\Gamma)$. Then $\mathrm{L}(\zeta_1^2, \zeta_2^2) J = \mathrm{L}(\zeta_1, \zeta_2) J \mathrm{L}(\zeta_1, \zeta_2)^{-1} \in \Gamma$, and hence (ζ_1, ζ_2) belongs to the subgroup $\langle (\sqrt{\omega_1}, \sqrt{\omega_2}),\ (-1, 1),\ (1, -1) \rangle$.

Conversely, the image by L of any element in the above subgroup belongs to $N(\Gamma)$. Hence

$$\begin{aligned} N(\Gamma) \cap \mathrm{L}(S^1 \times S^1) &= \langle \mathrm{L}(\sqrt{\omega_1}, \sqrt{\omega_2}),\ \mathrm{L}(-1, 1),\ \mathrm{L}(1, -1) \rangle \\ &= \left\langle \mathrm{L}(\frac{k_1}{2pd_2}, \frac{k_2}{2pd_1}),\ \mathrm{L}(\frac{1}{2}, 0),\ \mathrm{L}(0, \frac{1}{2}) \right\rangle. \end{aligned}$$

(Recall the abuse of notation given by (1).) This together with Lemma 12.11 implies the desired result. □

Proof of Proposition 12.6. Consider the spherical dihedral orbifold $\mathcal{O}(q/p; d_1, d_2)$ with $(d_1, d_2) \neq (1, 1)$. We first treat the generic case where $\mathcal{O}(q/p; d_1, d_2)$ is not the trivial θ-orbifold $\mathcal{O}(0/1; 1, 2)$. Then, by using Lemma 12.12 and the fact that $J \in \Gamma$, $\mathrm{Isom}^+ \mathcal{O}(q/p; d_1, d_2) \cong N(\Gamma)/\Gamma$ is isomorphic to the quotient of the group

$N := \langle \mathrm{L}(\frac{k_1}{2pd_2}, \frac{k_2}{2pd_1}), \mathrm{L}(\frac{1}{2}, 0), \mathrm{L}(0, \frac{1}{2})\rangle$ by its subgroup $G := \langle \mathrm{L}(\frac{k_1}{pd_2}, \frac{k_2}{pd_1})\rangle$. Set $a = \mathrm{L}(\frac{k_1}{2pd_2}, \frac{k_2}{2pd_1})$, $b_1 = \mathrm{L}(\frac{1}{2}, 0)$ and $b_2 = \mathrm{L}(0, \frac{1}{2})$. It should be noted that $\langle b_1, b_2\rangle \cong (\mathbb{Z}_2)^2$ and that the subset $\{b_1, b_2, b_1 b_2\}$ is equal to the set of all order 2 elements of $\mathrm{L}(S^1 \times S^1)$.

Note that the order of $L(\frac{k_1}{2pd_2})$ is equal to $2pd_2$ or pd_2 according to whether k_1 is odd or even. Similarly, the order of $L(\frac{k_2}{2pd_1})$ is equal to $2pd_1$ or pd_1 according to whether k_2 is odd or even. Thus the order of $a = \mathrm{L}(\frac{k_1}{2pd_2}, \frac{k_2}{2pd_1})$ is $2pd_1d_2$ or pd_1d_2, where the latter happens if and only if both k_1 and k_2 are even.

Case 1. $o(a) = 2pd_1d_2$. Then the element $a^{pd_1d_2} \in L(S^1 \times S^1)$ has order 2. Hence it is equal to one of the elements of $\{b_1, b_2, b_1b_2\}$. Thus $\langle a\rangle \cap \langle b_1, b_2\rangle \cong \mathbb{Z}_2$. This implies that $N \cong \langle a \,|\, a^{2pd_1d_2}\rangle \oplus \langle b_\ell \,|\, b_\ell^2\rangle$ for some $\ell \in \{1, 2\}$. Since G corresponds to the subgroup $\langle a^2\rangle$, we have

$$\mathrm{Isom}^+ \mathcal{O}(q/p; d_1, d_2) \cong N/\langle a^2\rangle \cong \langle a \,|\, a^2\rangle \oplus \langle b_\ell \,|\, b_\ell^2\rangle \cong (\mathbb{Z}_2)^2.$$

Case 2. $o(a) = pd_1d_2$. Since $o(a^2) = o(\mathrm{L}(\frac{k_1}{pd_2}, \frac{k_2}{pd_1})) = pd_1d_2$, we have $o(a) = o(a^2)$, and so $o(a) = pd_1d_2$ is odd. Thus $\langle a\rangle \cap \langle b_1, b_2\rangle = \{1\}$, and therefore $N \cong \langle a \,|\, a^{pd_1d_2}\rangle \oplus \langle b_1 \,|\, b_1^2\rangle \oplus \langle b_2 \,|\, b_2^2\rangle$. Hence, we have

$$\mathrm{Isom}^+ \mathcal{O}(q/p; d_1, d_2) \cong N/\langle a^2\rangle \cong N/\langle a\rangle \cong \langle b_1 \,|\, b_1^2\rangle \oplus \langle b_2 \,|\, b_2^2\rangle \cong (\mathbb{Z}_2)^2.$$

This completes the proof of Proposition 12.6 in the generic case.

In the exceptional case, where $\mathcal{O}(q/p; d_1, d_2)$ is the trivial θ-orbifold $\mathcal{O}(0/1; 1, 2)$, we may assume

$$\tilde{\Gamma} = \langle (i, i), (j, j)\rangle = \{\pm(1, 1), \pm(i, i), \pm(j, j), \pm(k, k)\}.$$

Then, by using Lemma 12.1, we can see that $N(\tilde{\Gamma}) = \{(q, q) \,|\, q \in O^*\} \rtimes \langle J_1\rangle$. Hence we have

$$\mathrm{Isom}^+ \mathcal{O}(0/1; 1, 2) \cong (O^*/(\mathbb{Z}_2)) \rtimes \mathbb{Z}_2 \cong D_3 \times \mathbb{Z}_2.$$

□

13. Appendix 2: Non-spherical geometric orbifolds with dihedral orbifold fundamental groups

In this section, we classify the non-spherical geometric orbifolds with dihedral orbifold fundamental groups (Propositions 13.1 and 13.2). These results are used in the proof of Theorem 4.1.

We first deal with the dihedral orbifolds with $S^2 \times \mathbb{R}$ geometry.

Proposition 13.1. *Let $\mathcal{O}$ be a compact orientable $S^2 \times \mathbb{R}$ orbifold with nonempty singular set which satisfies the following conditions.*

(i) *No component of $\partial\mathcal{O}$ is spherical.*
(ii) *$\pi_1(\mathcal{O})$ is a dihedral group.*

Then $\mathcal{O}$ is isomorphic to one of the following orbifolds.

(1) $\mathcal{O}(\infty)$, *the orbifold represented by the weighted graph* $(S^3, K(\infty), w)$, *where* w *takes the value* 2 *at each component of the* 2*-bridge link* $K(\infty)$ *of slope* ∞, *i.e. the* 2*-component trivial link.*

(2) $\mathcal{O}(\mathbb{RP}^3, O)$, *the orbifold with underlying space* $\mathbb{RP}^3$ *whose singular set is the trivial knot (i.e., the boundary of an embedded disc in* $\mathbb{RP}^3$*) with index* 2.

Proof. By the assumption that $\mathcal{O}$ has the geometry $S^2 \times \mathbb{R}$, we have $\pi_1(\mathcal{O}) < \mathrm{Isom}(S^2 \times \mathbb{R}) \cong \mathrm{Isom}(S^2) \times \mathrm{Isom}(\mathbb{R})$ and $\mathrm{int}\, \mathcal{O} \cong (S^2 \times \mathbb{R})/\pi_1(\mathcal{O})$. By the condition (ii), $\pi_1(\mathcal{O}) \cong D_n$ for some $n \in \mathbb{N} \cup \{\infty\}$.

If $n \in \mathbb{N}$, then the action of the finite dihedral group $\pi_1(\mathcal{O})$ on $S^2 \times \mathbb{R}$ extends to an action on the compact 3-manifold $S^2 \times [-\infty, \infty]$, where $[-\infty, \infty] \cong I$ is a compactification of $\mathbb{R}$, and $\mathcal{O}$ is identified with $S^2 \times [-\infty, \infty]/\pi_1(\mathcal{O})$. Thus $\mathcal{O}$ has a spherical boundary component, which contradicts the condition (i). So $n = \infty$ and $\pi_1(\mathcal{O}) \cong \langle f, h \mid h^2,\ hfh = f^{-1} \rangle$. Since the action of $\pi_1(\mathcal{O})$ on $S^2 \times \mathbb{R}$ is properly discontinuous, $f \in \mathrm{Isom}(S^2 \times \mathbb{R})$ is the product of a (possibly trivial) rotation of S^2 and a nontrivial translation of $\mathbb{R}$. Thus the orbifold $\mathcal{O}(f) := (S^2 \times \mathbb{R})/\langle f \rangle$ is homeomorphic to the manifold $S^2 \times S^1$. The isometry h descends to a fiber-preserving involution of $\mathcal{O}(f) \cong S^2 \times S^1$ which acts on the second factor as a reflection. Thus $\mathcal{O} = \mathcal{O}(f)/h$ is the quotient of $S^2 \times [0, 1]$ by an equivalence relation $(x, 0) \sim (\gamma_0(x), 0)$ and $(x, 1) \sim (\gamma_1(x), 1)$ where γ_0 and γ_1 are orientation-reversing involutions of S^2. Thus γ_i is conjugate to either the reflection in a great circle or the antipodal map. According to the combination (reflection, reflection), (reflection, antipodal map), or (antipodal map, antipodal map), $\mathcal{O}$ is isomorphic to $\mathcal{O}(\infty)$, $\mathcal{O}(\mathbb{RP}^3, O)$, or $\mathbb{RP}^3 \# \mathbb{RP}^3$. The last case cannot happen because $\mathcal{O}$ has the empty singular set. □

The following proposition deals with the dihedral orbifolds with the remaining 6 geometries.

Proposition 13.2. *Let* $\mathcal{O}$ *be a compact orientable* 3*-orbifold with nonempty singular set which has one of the* 6 *geometries different from* S^3 *and* $S^2 \times \mathbb{R}$ *and satisfies the following conditions.*

(i) $\pi_1(\mathcal{O})$ *is a dihedral group.*

(ii) *No component of* $\partial \mathcal{O}$ *is spherical.*

Then $\mathcal{O}$ *is isomorphic to* $D^2(2, 2) \times I$.

Proof. Let X be the geometry which $\mathcal{O}$ possesses. Then X is $\mathbb{H}^3$, $\mathbb{E}^3$, $\widetilde{SL_2}(\mathbb{R})$, *Nil* or *Sol*, and $\mathrm{int}\, \mathcal{O}$ is isomorphic to X/Γ for some discrete subgroup $\Gamma \cong \pi_1(\mathcal{O})$ of $\mathrm{Isom}(X)$. Note that the underlying topological space of X is homeomorphic to $\mathbb{R}^3$. The proof is divided into two cases according to whether $\pi_1(\mathcal{O})$ is finite or infinite.

Case 1. Suppose that $\pi_1(\mathcal{O})$ is a finite dihedral group D_n. Then, as will be shown below, the action of D_n on X has a global fixed point x. Then the exponential map from T_xX, the tangent space to X at x, to X is a D_n-equivariant homeomorphism. This implies that $\partial\mathcal{O} \cong S^2(2,2,n)$, contradicting the condition (ii).

The existence of a global fixed point can be proved as follows. For the constant curvature case $X = \mathbb{H}^3$ or $\mathbb{E}^3$, this is well-known. We shall first deal with the case where X is Nil, $\mathbb{H}^2 \times \mathbb{R}$, or $\widetilde{\mathrm{SL}_2\mathbb{R}}$. Recall that there is an exact sequence

$$1 \to \mathrm{Isom}(\mathbb{R}) \to \mathrm{Isom}(X) \to \mathrm{Isom}(E) \to 1,$$

where E is the Euclidean plane $\mathbb{E}^2$ when X is Nil and the hyperbolic plane $\mathbb{H}^2$ when X is $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{\mathrm{SL}_2\mathbb{R}}$. We also note that the projection $\mathrm{Isom}(X) \to \mathrm{Isom}(E)$ above is induced by a fibration $p : X \to E$. Let $\bar{D}_n$ be the image of D_n in $\mathrm{Isom}(E)$ and K the kernel in D_∞ of the projection to $\bar{D}_n$. Then the action of $\bar{D}_n$ on E has a global fixed point y, and the action of K on the fibre $p^{-1}(y)$ has a global fixed point since both of them are finite. Thus D_n has a global fixed point on X when X is Nil, $\mathbb{H}^2 \times \mathbb{R}$, or $\widetilde{\mathrm{SL}_2\mathbb{R}}$.

We shall now show the same property when $X = Sol$. In this case, there is an exact sequence

$$1 \to \mathrm{Isom}(\mathbb{E}^2) \to \mathrm{Isom}(Sol) \to \mathrm{Isom}(\mathbb{R}) \to 1,$$

and the projection $\mathrm{Isom}(Sol) \to \mathrm{Isom}(\mathbb{R})$ is induced by a fibration $q : Sol \to \mathbb{R}$. Let $\bar{D}_n$ be the projection of D_n in $\mathrm{Isom}(\mathbb{R})$. Then $\bar{D}_n$ is either trivial or $\mathbb{Z}_2$ generated by a reflection on $\mathbb{R}$. In either case, it fixes a point y on $\mathbb{R}$. In the former case, $D_n \cong \langle g, h \mid g^2, h^2,\ (gh)^n \rangle$ acts on the fibre $q^{-1}(y)$ by Euclidean isometries in such a way that g and h correspond to reflections, and hence D_n fixes a point on the fibre. In the latter case, the kernel K of the projection $D_n \to \bar{D}_n$ is isomorphic to $\mathbb{Z}_n$, and fixes a point on the fibre in the same way. Thus we have shown that D_n has a fixed point also in the case when $X = Sol$.

Case 2. Suppose $\pi_1(\mathcal{O})$ is the infinite dihedral group $D_\infty \cong \langle g, h \mid g^2, h^2 \rangle$. First we shall consider the case when X has constant curvature. Then g and h are order 2 elliptic transformations, and hence fix pointwise axes a_g and a_h respectively. They do not meet each other since otherwise the action fixes their intersection and cannot be faithful and discrete. Let ℓ be the common perpendicular to a_g and a_h if it exists. (This does not exist when $X = \mathbb{H}^3$ and a_g touches a_h at infinity. This exceptional case will be considered later.) Let Π_g be the totally geodesic plane containing a_h and perpendicular to ℓ. We define Π_h in the same way. Then the region cobounded by Π_g and Π_h constitutes a fundamental domain of the action of D_∞. Suppose now that $X = \mathbb{H}^3$ and a_g touches a_h at infinity. Then there is a totally geodesic plane H containing both a_g and a_h and it is preserved by D_∞. We then let Π_g and Π_h be totally geodesic planes containing a_g and a_h respectively, which are perpendicular to H. Then a fundamental domain is cobounded by Π_g and Π_h again. Therefore, in either case, we can see $\mathrm{int}\,\mathcal{O} \cong \mathrm{int}\,D^2(2,2) \times \mathbb{R}$ and so $\mathcal{O} \cong D^2(2,2) \times I$.

Next, we shall consider the case when X is Nil or $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{\mathrm{SL}_2\mathbb{R}}$. As before, let $\bar{D}_\infty$ be the projection of D_∞ to $\mathrm{Isom}(E)$, and K the kernel of the projection. We first observe that the images $\bar{g}$ and $\bar{h}$ of g and h in $\bar{D}_\infty$ are nontrivial. In fact, if say $\bar{g}$ is trivial, then g acts on $\mathbb{R}$ as a nontrivial order 2 isometry. Thus g acts on $\mathbb{R}$ by a reflection, and so the image $\bar{g}$ must be an orientation-reversing isometry on E, which contradicts our assumption that $\bar{g}$ is trivial.

Since g and h have order 2, their images $\bar{g}$ and $\bar{h}$ in $\bar{D}_\infty$ also have order 2. We first deal with the case where both of them are orientation-preserving, i.e. π-rotations. Let y_g and y_h be the centres of the π-rotations $\bar{g}$ and $\bar{h}$, respectively. Then g and h are π-rotations about the geodesics $p^{-1}(y_g)$ and $p^{-1}(y_h)$, respectively. Since the action of D_∞ is faithful, we have $y_g \neq y_h$. Now consider the geodesic line ℓ in E containing y_g and y_h, and let ℓ_g and ℓ_h be the lines which intersects ℓ perpendicularly at y_g and y_h, respectively. Then ℓ_g and ℓ_h are disjoint, and they cobound a region R in E. We see that $p^{-1}(R)$ is a fundamental region of the action of D_∞, and we have $\mathrm{int}\,\mathcal{O} \cong \mathrm{int}\,D^2(2,2) \times \mathbb{R}$.

We next treat the case where both of $\bar{g}$ and $\bar{h}$ are orientation-reversing, i.e. reflections. Let a_g and a_h be the axes of the reflections $\bar{g}$ and $\bar{h}$, respectively. Then g and h are the 'symmetries' with respect to the geodesics $\tilde{a}_g$ and $\tilde{a}_h$, respectively, where $\tilde{a}_g$ and $\tilde{a}_h$ are lifts of a_g and a_h, respectively. If a_g and a_h are disjoint, they cobound a region R in E, and $p^{-1}(R)$ is a fundamental region of the action of D_∞, and we have $\mathrm{int}\,\mathcal{O} \cong \mathrm{int}\,D^2(2,2) \times \mathbb{R}$. If a_g and a_h meet each other at a point $y \in E$. Then D_∞ acts effectively and discretely on the fiber $p^{-1}(y)$, and so the axes $\tilde{a}_g$ and $\tilde{a}_h$ intersect $p^{-1}(y)$ perpendicularly at distinct points, z_g and z_h, respectively. Let P_g and P_h be the ruled surfaces in X obtained as the unions of the geodesics which intersect $p^{-1}(y)$ perpendicularly at z_g and z_h, respectively. Then P_g and P_h are disjoint planes in X, and the domain they cobound is a fundamental domain of D_∞, and we can see $\mathrm{int}\,\mathcal{O} \cong \mathrm{int}\,D^2(2,2) \times \mathbb{R}$.

We now treat the case where one of $\bar{g}$ and $\bar{h}$ is orientation-preserving and the other is orientation-reversing. We may assume $\bar{g}$ is orientation-preserving and $\bar{h}$ is orientation-reversing. Let y_g be the center of the π-rotation $\bar{g}$, and let a_h be the axis of the reflection $\bar{h}$. If y_g belongs to a_h, then the axes of the π-rotations of g and h intersect, and the action of D_∞ cannot be discrete and faithful. So y_g is not contained in a_g. Let ℓ be a geodesic line in E which passes through y and is disjoint from a_h. Let R be the region in E bounded by a_h and ℓ. Then $p^{-1}(R)$ is a fundamental region of the action of D_∞, and we have $\mathrm{int}\,\mathcal{O} \cong \mathrm{int}\,D^2(2,2) \times \mathbb{R}$.

Finally, suppose that $X = Sol$. Then the projection $\bar{D}_\infty$ of D_∞ to $\mathrm{Isom}(\mathbb{R})$ is either trivial or $\mathbb{Z}_2$ or D_∞ itself. In the case when $\bar{D}_\infty$ is trivial, the generators g and h act on each fibre by π-rotations, and their fixed points must differ. Thus $\mathrm{int}\,\mathcal{O} \cong X/D_\infty$ is a bundle over $\mathbb{R}$ with fiber $\mathbb{E}^2/D_\infty \cong \mathrm{int}\,D^2(2,2)$. So we have $\mathrm{int}\,\mathcal{O} \cong \mathrm{int}\,D^2(2,2) \times \mathbb{R}$. In the case when $\bar{D}_\infty$ is $\mathbb{Z}_2$, the action of $\bar{D}_\infty$ is a reflection with respect to a point x. We set $P = q^{-1}(x)$. Then the g and h act on P by reflections, and by the same argument as in the previous paragraph, we have a homeomorphism $\operatorname{int}\mathcal{O} \cong D^2(2,2) \times \mathbb{R}$. Finally, suppose that $\bar{D}_\infty = D_\infty$. Then g and h fix points x_g and x_h on $\mathbb{R}$ respectively, and they differ. We consider fibres $\Pi_g = q^{-1}(x_g)$ and $\Pi_h = q^{-1}(x_h)$. The elements act on Π_g and Π_h as reflections with axes $a_g \subset \Pi_g$ and $a_h \subset \Pi_h$. Then the region cobounded by Π_g and Π_h constitutes a fundamental region for the action of D_∞, and we see that $\operatorname{int}\mathcal{O} \cong \operatorname{int} D^2(2,2) \times \mathbb{R}$. □

References

[1] C. Adams, *Hyperbolic* 3*-manifolds with two generators*, Comm. Anal. Geom. **4** (1996), 181–206.

[2] I. Agol, *The classification of non-free* 2*-parabolic generator Kleinian groups*, Slides of talks given at Austin AMS Meeting and Budapest Bolyai conference, July 2002, Budapest, Hungary.

[3] I. Agol, *Tameness of hyperbolic 3-manifolds*, arXiv:math/0405568.

[4] S. Aimi, D. Lee, S. Sakai, and M. Sakuma, *Classification of parabolic generating pairs of Kleinian groups with two parabolic generators*, arXiv:2001.11662 [math.GT].

[5] H. Akiyoshi, J. Parker, and M. Sakuma, *On Hecke groups and Heckoid groups*, in preparation.

[6] H. Akiyoshi, M. Sakuma, M. Wada, and Y. Yamashita, *Punctured torus groups and* 2*-bridge knot groups (I)*, Lecture Notes in Mathematics **1909**, Springer, Berlin, 2007.

[7] L. Bessiéres, G. Besson, S. Maillot, M. Boileau, J. Porti, *Geometrisation of 3-manifolds*, EMS Tracts in Mathematics, **13**. European Mathematical Society (EMS), Zürich, 2010. x+237 pp.

[8] M. Boileau, B. Leeb, and J. Porti, *Geometrization of 3-dimensional orbifolds*, Ann. of Math. **162** (2005), 195–290.

[9] M. Boileau, S, Maillot, and J. Porti, *Three-dimensional orbifolds and their geometric structures*, Panoramas et Synthèses [Panoramas and Syntheses], **15**, Société Mathématique de France, Paris, 2003. viii+167 pp.

[10] M. Boileau and J. Porti, *Geometrization of* 3*-orbifolds of cyclic type*, Appendix A by Michael Heusener and Porti, Astérisque No. 272 (2001).

[11] M. Boileau and B. Zimmermann, *The* π*-orbifold group of a link*, Math. Z. **200** (1989), 187–208.

[12] F. Bonahon, *Difféotopies des espaces lenticulaires*, Topology **22** (1983), 305–314.

[13] F. Bonahon and J. P. Otal, *Scindements de Heegaard des espaces lenticulaires*, C. R. Acad. Sci. Paris Sér. I Math. **294** (1982), 585–587.

[14] F. Bonahon and L. Siebenmann, *New geometric splittings of classical knots, and the classification and symmetries of arborescent knots*, `http://www-bcf.usc.edu/~fbonahon/Research/Publications.html`

[15] B. H. Bowditch, *Notes on tameness*, Enseign. Math. **56** (2010), 229–285.

[16] J. L. Brenner, *Quelques groupes libres de matrices*, C. R. Acad. Sci. Paris **241** (1955), 1689–1691.

[17] G. Burde, H. Zieschang and M. Heusener, *Knots. Third, fully revised and extended edition*, De Gruyter Studies in Mathematics, **5**. De Gruyter, Berlin, 2014. xiv+417.

[18] H.-D. Cao and X.-P. Zhu, *A complete proof of the Poincaré and geometrization conjectures —application of the Hamilton-Perelman theory of the Ricci flow*, Asian J. Math. **10** (2006), 165–492: *Erratum*, ibid, 663.

[19] D. Calegari and D. Gabai, *Shrinkwrapping and the taming of hyperbolic 3-manifolds*, J. Amer. Math. Soc. **19** (2006), 385–446.

[20] D. Cooper, C. Hodgson and S. Kerckhoff, *Three-dimensional orbifolds and cone-manifolds* MSJ Memoirs **5**, Mathematical Society of Japan, Tokyo, 2000. x+170 pp.

[21] W. D. Dunbar, *Hierarchies for 3-orbifolds*, Topology Appl. **29** (1988), 267–283.

[22] W. D. Dunbar, *Geometric orbifolds*, Rev. Mat. Univ. Complut. Madrid **1** (1988), 67–99.
[23] M. Feighn and G. Mess, *Conjugacy classes of finite subgroups of Kleinian groups*, Amer. J. Math. **113** (1991), 179–188.
[24] J. Gilman, *The structure of two-parabolic space: parabolic dust and iteration*, Geom. Dedicata **131** (2008), 27–48.
[25] F. W. Gehring and G. J. Martin, *Commutators, collars and the geometry of Möbius groups*, J. Anal. Math. **63** (1994), 175–219.
[26] E. Hecke, *Über die Bestimmung Dirichletscher Reihen durch ihre Funktionalgleichung*, Math. Ann. **112** (1936), 664–699.
[27] A. Hinkkanen and G. J. Martin, *The dynamics of semigroups of rational functions. I*, Proc. London Math. Soc. **73** (1996), 358–384.
[28] T. L. Jeevanjee, *Isometries of orbifolds double-covered by lens spaces*, J. Knot Theory Ramifications **12** (2003), 819–832.
[29] T. Jorgensen, *On pairs of once-punctured tori*, Kleinian groups and hyperbolic 3-manifolds (Warwick, 2001), 183–207, London Math. Soc. Lecture Note Ser., **299**, Cambridge Univ. Press, Cambridge, 2003.
[30] L. Keen and C. Series, *The Riley slice of Schottky space*, Proc. London Math. Soc. **69** (1994), 72–90.
[31] S. Kim and T. Koberda, *Non-freeness of groups generated by two parabolic elements*, arXiv:1901.06375.
[32] B. Kleiner and J. Lott, *Notes on Perelman's papers*, Geom. Topol. **12** (2008), 2587–2855.
[33] Y. Komori and C. Series, *The Riley slice revisited*, The Epstein birthday schrift, 303–316, Geom. Topol. Monogr., **1**, Geom. Topol. Publ., Coventry, 1998.
[34] D. Lee and M. Sakuma, *Epimorphisms from* 2*-bridge link groups onto Heckoid groups* (I), Hiroshima Math. J. **43** (2013), 239–264.
[35] R. C. Lyndon and J. L. Ullman, *Groups generated by two parabolic linear fractional transformations*, Canadian J. Math. **21** (1969), 1388–1403.
[36] A. Marden, *The Geometry of Finitely Generated Kleinian Groups*, Ann. Math. **99** (1974), 383–462.
[37] G. J. Martin, *Nondiscrete parabolic characters of the free group F_2: supergroup density and Nielsen classes in the complement of the Riley slice*, preprint.
[38] G. J. Martin and T. H. Marshall, *Polynomial Trace Identities in $SL(2,\mathbb{C})$, Quaternion Algebras, and Two-generator Kleinian Groups*, arXiv 1911.11643.
[39] K. Matsuzaki, *Structural stability of Kleinian groups*, Michigan Math. J. **44** (1997), 21–36.
[40] M. Mecchia and A. Seppi, *Fibered spherical* 3*-orbifolds*, Rev. Mat. Iberoam. **31** (2015), 811–840.
[41] M. Mecchia and A. Seppi, *Isometry groups and mapping class groups of spherical 3-orbifolds*, arXiv:1607.06281.
[42] W.H. Meeks and P. Scott, *Finite group actions on 3-manifolds*, Invent. Math. **86** (1986), 287–346.
[43] J.W. Morgan and H. Bass, *The Smith conjecture*, Pure Appl. Math., **112**, Academic Press, Orlando, FL, 1984.
[44] J. Morgan and G. Tian, *Ricci Flow and the Poincaré Conjecture*, Clay Mathematics Monographs, **3**. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2007. xlii+521 pp.
[45] J. Morgan and G. Tian, *The geometrization conjecture*, Clay Mathematics Monographs, **5**. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2014. x+291 pp.
[46] K. Ohshika and H. Miyachi, *Uniform models for the closure of the Riley slice*, In the tradition of Ahlfors-Bers. V, 249–306, Contemp. Math., **510**, Amer. Math. Soc., Providence, RI, 2010.

[47] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math/0211159 [math.DG].
[48] G. Perelman, *Ricci flow with surgery on three-manifolds*, arXiv:math/0303109 [math.DG].
[49] G. Perelman, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, arXiv:math/0307245 [math.DG].
[50] C. Petronio, *Spherical splitting of 3-orbifolds*, Math. Proc. Cambridge Philos. Soc. **142** (2007), 269–287.
[51] R. Riley, *Parabolic representations of knot groups. I*, Proc. London Math. Soc. **24** (1972), 217–242.
[52] R. Riley, *A quadratic parabolic group*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 281–288.
[53] R. Riley, *Parabolic representations of knot groups. II*, Proc. London Math. Soc. **31** (1975), 495–512.
[54] R. Riley, *Algebra for Heckoid groups*, Trans. Amer. Math. Soc. **334** (1992), 389–409.
[55] R. Riley, *Groups generated by two parabolics A and $B(w)$ for w in the first quadrant*, Output of an computer experiment.
[56] M. Sakuma, *On strongly invertible knots*, Algebraic and topological theories (Kinosaki, 1984), 176–196, Kinokuniya, Tokyo, 1986.
[57] M. Sakuma, *The geometries of spherical Montesinos links*, Kobe J. Math. **7** (1990), 167–190.
[58] M. Sakuma and J. Weeks, *Examples of canonical decompositions of hyperbolic link complements*, Japan. J. Math. (N.S.) **21** (1995), 393–439.
[59] C. Series, *All about the Riley slice*, in preparation.
[60] H. Schubert, *Knoten mit zwei Brücken*, Math. Z. **65** (1956), 133–170.
[61] P. Scott, *The geometries of* 3*-manifolds*, Bull. London Math. Soc. **15** (1983), 401–487.
[62] T. Soma, *Existence of ruled wrappings in hyperbolic 3-manifolds*, Geom. Topol. **10** (2006), 1173–1184.
[63] E. C. Tan and S. P. Tan, *Quadratic Diophantine equations and two generator Möbius groups*, J. Austral. Math. Soc. Ser. A **61** (1996), 360–368.
[64] W. Thurston, *The geometry and topology of three-manifolds*, available from http://library.msri.org/books/gt3m/

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