

SU(2) channels the cancellation of K3 BPS states

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ABSTRACT: The conformal field theoretic elliptic genus, an invariant for $N = (2, 2)$ superconformal field theories, counts the BPS states in any such theory with signs, according to their bosonic or fermionic nature. For K3 theories, this invariant is the source of the Mathieu Moonshine phenomenon. There, the net number of $\frac{1}{4}$ -BPS states is positive for any conformal dimension above the massless threshold, but it may arise after cancellation of the contributions of an equal number of bosonic and fermionic BPS states present in non-generic theories, as is the case for the class of \mathbb{Z}_2 -orbifolds of toroidal SCFTs. Nevertheless, the space $\widehat{\mathcal{H}}$ of all BPS states that are generic to such orbifold theories provides a convenient framework to construct a particular generic space of states of K3 theories. We find a natural action of the group SU(2) on a subspace of $\widehat{\mathcal{H}}$ which is compatible with the cancellations of contributions from the corresponding non-generic states. In fact, we propose that this action channels those cancellations. As a by-product, we find a new subspace of the generic space of states in $\widehat{\mathcal{H}}$.

KEYWORDS: Conformal Field Models in String Theory, Conformal Field Theory, Extended Supersymmetry, Nonperturbative Effects

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1 Introduction

The BPS spectrum of quantum field theories with extended supersymmetry has long been recognised to yield crucial and in fact useful information. Indeed, it is the key to the construction of invariants which both allow extrapolations from weak to strong coupling and relate abstractly defined theories to geometry. K3 theories provide a rich family of examples. They are superconformal field theories with $N = (2, 2)$ worldsheet supersymmetry at central charges $(c, \bar{c}) = (6, 6)$, with spacetime supersymmetry and integer eigenvalues of the operators J_0 and \bar{J}_0 — the zero modes of the two $\mathfrak{u}(1)$ currents generating an affine subalgebra of the left and right $N = 2$ superconformal algebras — and without holomorphic BPS states at weight $\frac{1}{2}$. These requirements imply that every K3 theory has $N = (4, 4)$ worldsheet supersymmetry, and that its conformal field theoretic elliptic genus $\mathcal{E}(\tau, z)$ equals the complex elliptic genus of K3 surfaces [1]. The moduli space \mathcal{M}_{K3} of K3 theories has dimension 80 and possesses at least one connected component whose structure is well understood [2, 3] and which contains all \mathbb{Z}_2 -orbifold conformal field theories that

are obtained from toroidal CFTs [3]. This component is expected to parametrize all non linear sigma models on K3 surfaces. These, in turn, provide a rich environment in which to explore non perturbative effects.

More recently, the sporadic group Mathieu 24 (M_{24}) has made an intriguing appearance in the conformal field theoretic elliptic genus of K3 theories. First came a numerical observation by [4] quickly followed by further evidence from the calculation of twining genera [5–8], culminating in a proof that the numerology is truly a signature of M_{24} [9]. The prospect of having this sporadic group acting on a non linear sigma model on K3 was swiftly discarded in [10], prompting a finer inspection of the BPS spectrum of such theories, since $\mathcal{E}(\tau, z)$ counts $\frac{1}{2}$ - and $\frac{1}{4}$ -BPS states with signs, according to their bosonic or fermionic nature. In particular, this index encodes a net number of massive $\frac{1}{4}$ -BPS ground states, all of the same statistics. It is an invariant throughout \mathcal{M}_{K3} and therefore one is at liberty to explore its properties from any vantage point in that moduli space. It is known that for generic theories the net number is actually the total number of these $\frac{1}{4}$ -BPS states [11, 12], while non-generic theories, to which the \mathbb{Z}_2 -orbifold CFTs belong, typically possess a number of bosonic and fermionic $\frac{1}{4}$ -BPS states whose contributions cancel out in $\mathcal{E}(\tau, z)$. These states are therefore not encoded in $\mathcal{E}(\tau, z)$ and are not expected to be organised in non-trivial representations of M_{24} . Understanding the nature of these “excess” states in order to better grasp the role of M_{24} in relation to the elliptic genus of K3 is the object of the present work.

Although generic theories do not have such excess states and would therefore appear to be less complicated, no such theories are known explicitly. Hence they offer little scope for elucidating the M_{24} action on the BPS states counted by $\mathcal{E}(\tau, z)$ at present. In contrast, \mathbb{Z}_2 -orbifold CFTs provide an interesting laboratory, not the least because they all share the same spectrum of generic BPS states; in other words, the fine details of their underlying complex 2-tori are of no consequence for our analysis of this class of K3 theories. Moreover, these theories enjoy a wealth of beautiful mathematical structures, which are only beginning to show. Indeed, in this work we use a global $SU(2)$ action on a subspace of generic BPS states which we hope will provide a useful tool in the study of K3 theories beyond the known examples, and certainly beyond Mathieu Moonshine.

A few years ago, and guided by our symmetry surfing programme [13–15], we showed, in the framework of \mathbb{Z}_2 -orbifold CFTs, that a maximal subgroup of M_{24} called the *octad group* G acts naturally on a space of $\frac{1}{4}$ -BPS states at level one¹ whose dimension agrees with the massive $\frac{1}{4}$ -BPS contribution to the elliptic genus at level one [16]. Significantly, the octad group is not a subgroup of M_{23} and is the overarching group of all geometric symmetry groups G_i of different \mathbb{Z}_2 -orbifold CFTs on K3 [15]. These geometric symmetries are rooted in the construction of Kummer surfaces, obtained by minimally resolving the 16 singularities of the standard \mathbb{Z}_2 -quotient of a complex 2-torus $T_\Lambda = \mathbb{C}^2/\Lambda$ with Λ a rank 4 lattice. In the symmetry surfing programme, the ability to surf the \mathbb{Z}_2 -orbifold subvariety of \mathcal{M}_{K3} relies crucially on the existence, in the rank 16 Kummer lattice, of a “diagonal” direction invariant under the action of all geometric symmetry groups G_i . On the \mathbb{Z}_2 -orbifold CFT side, this is echoed by the existence of a “diagonal” exactly marginal

¹In the Ramond sector, states at level $n \in \mathbb{N}$ have conformal dimensions $(h, \bar{h}) = (\frac{1}{4} + n, \frac{1}{4})$.

state T^{diag} built on a twisted ground state $|\alpha_{\text{diag}}\rangle$ whose orthogonal complement in the 16-dimensional space of twisted ground states is a 15-dimensional subspace \mathcal{A} . We denote by H^\perp the space of all massive ground states in the Fock space over \mathcal{A} . H^\perp is graded and we write $H^\perp = \bigoplus_{n=1}^{\infty} H_n^\perp$ where H_n^\perp is the space of level n states in H^\perp , $n \in \mathbb{N}$. At level 1, it so happens that the net number of massive $\frac{1}{4}$ -BPS states counted by the elliptic genus corresponds to states solely built on twisted ground states in \mathcal{A} . Moreover, all the massive $\frac{1}{4}$ -BPS states built on $|\alpha_{\text{diag}}\rangle$ cancel in the counting against all massive $\frac{1}{4}$ -BPS states from the untwisted sector. This property helped us pinning down the octad group action on the remaining massive $\frac{1}{4}$ -BPS states at level 1, using Margolin’s construction of a 45-dimensional representation of M_{24} as a guide [17], but the situation is not typical at higher level. Yet the octad group continues to act on H^\perp , as elegantly demonstrated by Gaberdiel, Keller and Paul [18]. Moreover, they argue that G also acts on the space H^{rest} consisting of some $\frac{1}{4}$ -BPS states built on the twisted ground state $|\alpha_{\text{diag}}\rangle$ and some $\frac{1}{4}$ -BPS states from the untwisted sector such that the graded dimension of $H^{\text{rest}} \oplus H^\perp$ agrees precisely with the massive ground state contributions to $\mathcal{E}(\tau, z)$. Since the action of G arises from symmetry surfing, this beautifully supports the symmetry surfing programme.

Let us describe the $\frac{1}{4}$ -BPS states in a given K3 theory in more detail. Here and in the following, by this abbreviation we mean those BPS states which saturate the BPS bound for half of the antiholomorphic $N = 4$ worldsheet supersymmetries and which are massive with respect to the holomorphic $N = 4$ superconformal algebra. By \widehat{H}_n , $n \in \mathbb{N} \setminus \{0\}$, we denote the space of *all* $\frac{1}{4}$ -BPS *ground* states at level n in the given K3 theory. The findings of [18] imply that $\widehat{H}_n \cong H_n^\perp \oplus H_n^{\text{rest}} \oplus H_n^+$, where as above, the dimension of $H_n^\perp \oplus H_n^{\text{rest}}$ agrees with the massive ground state contributions at level n to $\mathcal{E}(\tau, z)$. Accordingly, H_n^+ accounts for “excess” states whose contributions to the conformal field theoretic elliptic genus cancel out. While H_n^\perp is well under control by the results of [18], the spaces $H_n^{\text{rest}} \oplus H_n^+$ have not yet released all their secrets. In particular, for $n > 1$ neither H_n^{rest} nor H_n^+ has been constructed explicitly so far in any K3 theory. In a very recent article [19], Keller and Zadeh have deformed \mathbb{Z}_2 -orbifold CFTs on K3 away from the orbifold point using second order conformal perturbation methods. They have shown that under a deformation by T^{diag} all the $\frac{1}{4}$ -BPS states in H_1^\perp remain $\frac{1}{4}$ -BPS, while those in H_1^+ cease to satisfy the bound once the initial theory is deformed away from the orbifold. Deforming away from the orbifold into a different direction lifts the states in a *different* space $\widetilde{H}_1^+ \cong H_1^+$, whilst $(\widetilde{H}_1^+)^\perp \subset H_1^+ \oplus H_1^\perp$ remains at the BPS bound. This fits very well in the overall picture of the elliptic genus providing information that remains unchanged regardless of the regime (perturbative or not) one is interested in, and regardless of the point in \mathcal{M}_{K3} one considers. It also fits nicely with the symmetry surfing predictions: depending on the direction of deformation, different symmetry groups remain unbroken. Hence different subspaces of $\widehat{H} = \bigoplus_{n=1}^{\infty} \widehat{H}_n$ remain stable under different deformations, but each of them is isomorphic, as a representation of the Virasoro- and $\mathfrak{u}(1)$ -current zero modes $L_0, \bar{L}_0, J_0, \bar{J}_0$, to the massive ground state contributions to the generic space of states \mathbb{H}_0 introduced in [12]. This in particular shows that the inclusion $H_n^\perp \oplus H_n^{\text{rest}} \hookrightarrow \widehat{H}$ of representations is not uniquely determined in general.

While the structure of H^\perp has been understood by the results of [16, 18], we now proceed to uncover more structure on the spaces $H_n^{\text{rest}} \oplus H_n^+$, with the hope of finding how \mathbb{Z}_2 -orbifold CFTs earmark excess BPS states. The results of [12] show that independently of the deformation, if deforming to a generic theory, then the *bosonic* excess states in \widehat{H}_n are precisely those states in \widehat{H}_n that transform in the vacuum representation of the antiholomorphic $N = 4$ superconformal algebra. Under deformation, the $N = 4$ representations built on these states combine with representations with *fermionic* $\frac{1}{4}$ -BPS ground states to form long representations away from the BPS bound. By the above, these excess fermionic ground states depend on the choice of deformation. To describe them for the deformation by T^{diag} , we construct a “geometric” action of the group $SU(2)$, henceforth denoted $SU(2)_{\text{geom}}$, under which the four free Dirac fermions of our theories and their superpartners transform as doublets and whose action commutes with that of the $N = 4$ superconformal algebra. All $\frac{1}{4}$ -BPS states in the graded space $H^{\text{rest}} \oplus H^+ = \bigoplus_{n=1}^{\infty} H_n^{\text{rest}} \oplus H_n^+$ transform under $SU(2)_{\text{geom}}$. We argue that for deformations in the diagonal direction T^{diag} , to be lifted to a long representation, both the fermionic and bosonic excess $\frac{1}{4}$ -BPS states in H^+ must transform in isomorphic representations of $SU(2)_{\text{geom}}$.

Before proceeding to the heart of our matter, we provide the context in which M_{24} emerged in K3 theories. This has the double aim of providing a quick overview for the reader who is not familiar with the subject, and of introducing some of the definitions and notations that will be used later in the paper.

Let $\tau, z \in \mathbb{C}$, with τ in the upper complex halfplane.² The genus one partition function $\mathcal{Z}^{N=(4,4)}$ of an $N = (4, 4)$ superconformal field theory at central charge $c = \bar{c} = 6$ is a modular covariant function

$$\mathcal{Z}^{N=(4,4)} = \frac{1}{2} \left\{ \mathcal{Z}^{\text{NS}} + \mathcal{Z}^{\text{R}} + \mathcal{Z}^{\widetilde{\text{NS}}} + \mathcal{Z}^{\widetilde{\text{R}}} \right\} \tag{1.1}$$

with

$$\begin{aligned} \mathcal{Z}^{\text{S}} &= \mathcal{Z}^{\text{S}}(\tau, z; \bar{\tau}, \bar{z}) := \text{Tr}_{\mathbb{H}^{\text{S}}} (y^{J_0} q^{L_0 - \frac{1}{4}} \bar{y}^{\bar{J}_0} \bar{q}^{\bar{L}_0 - \frac{1}{4}}), & \text{S} \in \{\text{NS}, \text{R}\}, \\ \mathcal{Z}^{\widetilde{\text{S}}} &= \mathcal{Z}^{\widetilde{\text{S}}}(\tau, z; \bar{\tau}, \bar{z}) := \text{Tr}_{\mathbb{H}^{\text{S}}} ((-1)^{J_0 - \bar{J}_0} y^{J_0} q^{L_0 - \frac{1}{4}} \bar{y}^{\bar{J}_0} \bar{q}^{\bar{L}_0 - \frac{1}{4}}) \end{aligned} \tag{1.2}$$

where the traces are taken over the subspaces \mathbb{H}^{NS} and \mathbb{H}^{R} of \mathbb{H} , the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ -graded complex vector space of all states in the superconformal field theory. The gradings split \mathbb{H} into the Neveu-Schwarz and Ramond sectors,³ each containing (worldsheet) bosons and fermions, that is

$$\mathbb{H} = \underbrace{\mathbb{H}_b^{\text{NS}} \oplus \mathbb{H}_f^{\text{NS}}}_{\mathbb{H}^{\text{NS}}} \oplus \underbrace{\mathbb{H}_b^{\text{R}} \oplus \mathbb{H}_f^{\text{R}}}_{\mathbb{H}^{\text{R}}}. \tag{1.3}$$

²By $\bar{\tau}, \bar{z} \in \mathbb{C}$ we denote the complex conjugates of $\tau, z \in \mathbb{C}$, which we occasionally include as arguments of a non-holomorphic function of τ, z . This is convenient whenever such a function allows a power series expansion in $q := \exp(2\pi i\tau), \bar{q} := \exp(-2\pi i\bar{\tau}), y := \exp(2\pi iz), \bar{y} := \exp(-2\pi i\bar{z})$, where q, \bar{q}, y, \bar{y} may also be viewed as independent formal variables.

³Referring to both left and right movers, as in a K3 theory, there exist no NSR or RNS states, by definition. Note that our description of K3 theories involves the internal CFT only.

Here, as representations of the $N = (4, 4)$ superconformal algebras, the spaces \mathbb{H}^{NS} and \mathbb{H}^{R} are isomorphically mapped to each other under spectral flow. The operators J_0 and \bar{J}_0 are the zero modes of the $\mathfrak{u}(1)$ currents which, together with the currents $J^\pm(z)$ and $\bar{J}^\pm(\bar{z})$, form two copies of the affine $\mathfrak{su}(2)$ subalgebra of the (small) $N = (4, 4)$ superconformal algebra. The partition function (1.1) may be expressed in terms of sesquilinear combinations of $N = 4$ unitary irreducible characters at central charges $c = 6$ and $\bar{c} = 6$. These characters are generating functions for short and long representations and were coined ‘massless’ and ‘massive’ respectively in [21] to signify that the corresponding representations have non-zero or zero Witten index respectively. We recall the expressions for these characters in appendix A.3, where we label the two massless characters in the Ramond sector as $\chi_0^{\text{R}}(\tau, z)$ and $\chi_{\frac{1}{2}}^{\text{R}}(\tau, z)$ while the Ramond massive characters are of the form $q^h \tilde{\chi}^{\text{R}}(\tau, z)$ with $\tilde{\chi}^{\text{R}} := \chi_{\frac{1}{2}}^{\text{R}} + 2\chi_0^{\text{R}}$ and where the conformal weight $h \in \mathbb{R}$ of the highest weight state is bounded below by $h > \frac{1}{4}$. The Ramond massless characters, on the other hand, have highest weight states whose conformal dimension saturates the bound $h = \frac{1}{4}$. For K3 theories, the partition function is of the form

$$\mathcal{Z}^{N=(4,4)}(\tau, z; \bar{\tau}, \bar{z}) = \frac{1}{2} \sum_{S \in \{\text{NS}, \text{R}, \tilde{\text{NS}}, \tilde{\text{R}}\}} \sum_{a,b} n_{ab} \chi_a^S(\tau, z) \overline{\chi_b^S(\tau, z)}, \quad n_{ab} \in \mathbb{N} \quad \forall a, b, \quad (1.4)$$

with a, b running over massless and massive $N = 4$ characters and with the term containing the vacuum of the theory having $n_{00} = 1$. The conformal field theoretic elliptic genus $\mathcal{E}(\tau, z)$ of a K3 theory is defined as the specialisation of $\mathcal{Z}^{\tilde{\text{R}}}$ where the antiholomorphic $\tilde{\text{R}}$ characters are projected to their Witten index value by setting $\bar{z} = 0$. We thus have

$$\mathcal{E}(\tau, z) := \text{Tr}_{\mathbb{H}^{\text{R}}} ((-1)^{J_0 - \bar{J}_0} y^{J_0} q^{L_0 - \frac{1}{4}} \bar{q}^{\bar{L}_0 - \frac{1}{4}}) = \sum_{a,b} n_{ab} \chi_a^{\tilde{\text{R}}}(\tau, z) \overline{\chi_b^{\tilde{\text{R}}}(\tau, 0)}. \quad (1.5)$$

As a consequence of the theory enjoying $N = (2, 2)$ worldsheet supersymmetry, \mathcal{E} is a holomorphic function of τ and z , and it counts (with opposite signs) the RR fermionic and bosonic states whose antiholomorphic signature is the Witten index of the massless representation they belong to. The first explicit calculation of this topological invariant was carried out within the framework of Gepner models and \mathbb{Z}_2 -orbifold CFTs in [1, (3.8)], where a spectral-flowed version of the conformal field theoretic elliptic genus (1.5) was used, namely $\Phi(\tau, z) := q^{1/4} y \mathcal{E}(\tau, z + \frac{\tau+1}{2})$. This was in order to make a direct parallel with the work of Witten [22]. The elliptic genus presented in [1, (5.10)-(5.12)] is the $z = 0$ specialisation of

$$\Phi(\tau, z) = 8 \left\{ \frac{\vartheta_2(\tau, z)^2}{\vartheta_4(\tau, 0)^2} - \frac{\vartheta_1(\tau, z)^2}{\vartheta_3(\tau, 0)^2} - \frac{\vartheta_4(\tau, z)^2}{\vartheta_2(\tau, 0)^2} \right\}. \quad (1.6)$$

With the help of (A.16a)–(A.16c) and (A.17), $\Phi(\tau, z)$ may be expressed as an infinite sum of irreducible $N = 4$ Neveu-Schwarz characters in the following way,

$$\Phi(\tau, z) = 20\chi_{\frac{1}{2}}^{\text{NS}}(\tau, z) - 2\chi_0^{\text{NS}}(\tau, z) + A(\tau) \tilde{\chi}^{\text{NS}}(\tau, z), \quad (1.7)$$

where

$$A(\tau) := 2 - 8q^{\frac{1}{8}} \eta(\tau) \sum_{i=2}^4 h_i(\tau)$$

and the functions $h_i(\tau)$ are the $\nu = 0$ specialisations of the functions $h_i(\tau, \nu)$ given in (A.8) and (A.9). $A(\tau)$ has Fourier expansion

$$A(\tau) = \sum_{n=1}^{\infty} A_n q^n = 2 \cdot (45q + 231q^2 + 770q^3 + 2277q^4 + \dots), \quad (1.8)$$

and $\frac{1}{2}(A(\tau) - 2)q^{-\frac{1}{8}} = h^{(2)}(\tau)$, the weakly holomorphic mock modular form on $SL(2, \mathbb{Z})$ presented in [23, (7.16)].

A list of the first 8 coefficients $A_n, n \in \{1, \dots, 8\}$, counting $\frac{1}{4}$ -BPS ground states, was given in [24], but the significance of these coefficients has only been realised since 2010,⁴ after the observation in [4] that they coincide with dimensions of representations of the sporadic group M_{24} . The existence of an infinite-dimensional M_{24} module underlying $A(\tau)$ was proven in [9]. Yet the role of M_{24} in the context of strings compactified on K3 surfaces remains a mystery, and this phenomenon has been named *Mathieu Moonshine*.

We have structured the remainder of this work as follows.

In section 2, we recall the ingredients from \mathbb{Z}_2 -orbifold CFTs relevant to our analysis. Moreover, we present an explicit construction of excess BPS states pertaining to $H_n^+, n \in \{1, 2\}$, including their decomposition into $SU(2)_{\text{geom}}$ multiplets. The information at level $n = 1$ was already provided in our work [16], but the full significance of $SU(2)_{\text{geom}}$ was not recognised then. The information at level $n = 2$ is new and requires a careful and detailed analysis of the data encoded in the partition function of \mathbb{Z}_2 -orbifold CFTs on K3. The states in H_2^{rest} are listed in appendix B.

Section 3 takes stock of the group theoretic information gleaned in the previous section, gives our rationale behind our construction of the $SU(2)_{\text{geom}} \times \overline{SU(2)}_{\text{geom}}$ action, and provides analytic expressions for untwisted and twisted partition functions that encode the $SU(2)_{\text{geom}}$ action.

A discussion and outlook is given in section 4.

Appendix A gathers helpful Jacobi theta function identities, as well as expressions for the $N = 4$ characters at central charge $c = 6$ involving Appell functions, whose definitions are also presented. This appendix also offers explanations for the analytic expressions appearing in section 3.

2 \mathbb{Z}_2 -orbifold CFTs

Since the elliptic genus \mathcal{E} is an invariant on the moduli space \mathcal{M}_{K3} of K3 theories, it encodes properties that all K3 theories share. In particular, apart from states in massless representations with respect to the holomorphic and the antiholomorphic $N = 4$ superconformal algebra, which will not be our concern here, \mathcal{E} counts a net number of $\frac{1}{4}$ -BPS states at each integer conformal weight strictly above threshold. By the results of [9], the corresponding contributions to \mathcal{E} agree with the graded character of a space

$$\mathcal{H}^{\text{BPS}} = \bigoplus_{n=1}^{\infty} (H_n \otimes \mathcal{H}_n^{N=4}), \quad (2.1)$$

⁴Moonshine, a clear, unaged whiskey, became legal in the US in 2010. We do not know whether this had any significance for the discovery of Mathieu Moonshine.

where $\mathcal{H}_n^{N=4}$ is an irreducible massive $N = 4$ representation at conformal weight n for each $n \in \mathbb{N} \setminus \{0\}$, and H_n is a finite dimensional representation of M_{24} and \bar{J}_0 . \mathcal{H}^{BPS} is the subspace of massive states in the generic space of states \mathbb{H}_0 introduced in [12]. Each H_n is an invariant of K3 theories, although the dimensions of the \hat{H}_n in the space

$$\hat{\mathcal{H}}^{\text{BPS}} = \bigoplus_{n=1}^{\infty} (\hat{H}_n \otimes \mathcal{H}_n^{N=4}) \tag{2.2}$$

of all massive $\frac{1}{4}$ -BPS states may vary from one K3 theory to another. In generic K3 theories, we have $\hat{H}_n \cong H_n$ for all n [11, 12], while in non-generic theories $\dim \hat{H}_n \geq \dim H_n$ for all n . When one deforms away from a non-generic theory, excess states, whose contributions to the elliptic genus cancel, are lifted into non-BPS representations off threshold. Although the results of [11, 12] imply that \mathcal{H}^{BPS} has a geometric description in terms of the chiral de Rham cohomology of K3, this space remains difficult to access. For any deformation of a non-generic theory, it is therefore valuable to gain insight on the subspace of excess states in $\hat{\mathcal{H}}^{\text{BPS}}$ whose contributions to \mathcal{E} cancel, and to identify the driver of such cancellations in non-generic yet accessible K3 theories.

2.1 Free fermions and bosons as building blocks

Our prototype of non-generic K3 theories is the class of \mathbb{Z}_2 -orbifold superconformal field theories, which we denote by

$$\mathcal{C} = \mathcal{T}/\mathbb{Z}_2 \quad \text{with } \mathcal{T} \text{ a toroidal SCFT at central charges } c = \bar{c} = 6. \tag{2.3}$$

The construction of \mathcal{C} is induced by the standard Kummer construction which minimally resolves the singularities of the \mathbb{Z}_2 -quotient of a complex 2-torus $T_\Lambda := \mathbb{C}^2/\Lambda$, with $\Lambda \subset \mathbb{C}^2$ a rank 4 lattice over \mathbb{Z} . Unlike generic K3 theories, these provide a framework to test the symmetry surfing idea explicitly and garner further clues for the construction of the putative VOA(s) associated with the M_{24} Moonshine module. To this effect, we restrict our attention to the symmetry groups G_i induced by geometric symmetries — including those stemming from shifts by half lattice vectors — of the underlying toroidal conformal field theories \mathcal{T} . As was detailed in [14–16], this is meaningful after the choice of a geometric interpretation for the theory \mathcal{T} on some torus T_Λ . The symmetry surfing programme also requires a choice of generators for the lattice $\Lambda \subset \mathbb{C}^2 \cong \mathbb{R}^4$. All in all, these choices induce an identification $\frac{1}{2}\Lambda/\Lambda \cong \mathbb{F}_2^4$, such that every geometric symmetry group G_i acts on the twisted ground states $T_{\vec{a}}$, $\vec{a} \in \mathbb{F}_2^4$, as permutation group by means of affine linear maps on the space of labels \mathbb{F}_2^4 .

The underlying toroidal CFT \mathcal{T} possesses two holomorphic free Dirac fermions $\chi_+^a(z)$ and their complex conjugates $\chi_-^a(z)$ with standard OPEs,

$$\chi_+^a(z) \chi_-^b(w) \sim \frac{1}{z-w} \delta^{ab}, \quad a, b \in \{1, 2\}. \tag{2.4}$$

Their bosonic superpartners $j_\pm^I(z)$ are built out of a set of four real holomorphic U(1)-currents $j^I(z)$, $I \in \{1, 2, 3, 4\}$, whose zero modes generate infinitesimal translations on the torus T_Λ . Here, $j^I(z)$ is the Noether current for the translation along the I^{th} coordinate

axis in the standard coordinate system that T_Λ inherits from $\mathbb{C}^2 \cong \mathbb{R}^4$ around each point. One has

$$j_\pm^1(z) = \frac{1}{\sqrt{2}}(j^1(z) \pm ij^2(z)), \quad j_\pm^2(z) = \frac{1}{\sqrt{2}}(j^3(z) \pm ij^4(z)), \quad (2.5)$$

with

$$j_+^a(z) j_-^b(w) \sim \frac{1}{(z-w)^2} \delta^{ab}, \quad a, b \in \{1, 2\}. \quad (2.6)$$

Under the action of the \mathbb{Z}_2 -orbifold group, the fields $\chi_\pm^a(z)$ and $j_\pm^a(z)$ flip sign, while the $N = 4$ SCA is invariant under this orbifold action, as follows from the following free field representation:

$$\begin{aligned} J^3 &= \frac{1}{2} \{:\chi_+^1 \chi_-^1: + :\chi_+^2 \chi_-^2:\} = \frac{1}{2} J, & J^\pm &= \pm : \chi_\pm^1 \chi_\pm^2: \\ G^\pm &= \sqrt{2} \{:\chi_\pm^1 j_\mp^1: + :\chi_\pm^2 j_\mp^2:\}, & G'^\pm &= \sqrt{2} \{:\chi_\mp^1 j_\mp^2: - :\chi_\mp^2 j_\mp^1:\}, \\ T &= \sum_{a=1}^2 :j_+^a j_-^a: + \frac{1}{2} \sum_{a=1}^2 \{:\partial \chi_+^a \chi_-^a: + :\partial \chi_-^a \chi_+^a:\}. \end{aligned} \quad (2.7)$$

The currents J^\pm and J^3 generate the $\mathfrak{su}(2)$ affine subalgebra of the $N = 4$ superconformal algebra, under which the Dirac fermions χ_\pm^a have charges $\pm \frac{1}{2}$, while their bosonic superpartners are uncharged, as is immediate from the form of the Cartan subalgebra current J^3 in (2.7). In contrast, the symmetry groups G_i also act linearly as subgroups of $SU(2)$ on χ_\pm^a and j_\pm^a . More precisely, χ_+^a and j_+^a transform as doublets $\mathbf{2}$ under this $SU(2)$, which will be referred to as ‘geometric’ $SU(2)_{\text{geom}}$ while χ_-^a and j_-^a transform as complex conjugate doublets $\bar{\mathbf{2}}$. In other words, if χ_+^a and j_+^a transform with

$$M = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta|^2 = 1, \quad (2.8)$$

then χ_-^a and j_-^a transform with \bar{M} . The action of $SU(2)_{\text{geom}}$ commutes with the $N = 4$ action, as can be inferred from the $SU(2)_{\text{geom}}$ invariance of the fields in (2.7).

In the antiholomorphic sector of the theory, the two Dirac fermions $\overline{\chi_+^a}(\bar{z})$ and their superpartners $\overline{j_+^a}(\bar{z})$ transform as doublets under a right-moving group $\overline{SU(2)}_{\text{geom}}$ whose action commutes with that of the antiholomorphic $N = 4$ superconformal algebra, while they are singlets under $SU(2)_{\text{geom}}$. Their complex conjugates $\overline{\chi_-^a}(\bar{z})$ and $\overline{j_-^a}(\bar{z})$ also transform as doublets under $\overline{SU(2)}_{\text{geom}}$ and as singlets under $SU(2)_{\text{geom}}$.

2.2 The Neveu-Schwarz partition function

With an eye to prepare the ground for future work on the VOA(s) expected to underlie the Mathieu Moonshine module, we choose to work in the Neveu-Schwarz sector. The \mathbb{Z}_2 -orbifold partition function in this sector is given by contributions from the two complex NS fermions and their bosonic superpartners (2.6), both untwisted and twisted by the \mathbb{Z}_2 action as in [1],

$$\mathcal{Z}^{\text{NS}} = \mathcal{Z}_{\text{untwisted}}^{\text{NS}} + \mathcal{Z}_{\text{twisted}}^{\text{NS}}. \quad (2.9)$$

The dependence on the moduli of the underlying toroidal theory becomes apparent in $\mathcal{Z}_{\text{untwisted}}^{\text{NS}}$, which depends on $\Gamma(\Lambda, B) \subset \mathbb{R}^4 \oplus \mathbb{R}^4$, the signature (4, 4) Narain lattice associated with the lattice Λ and the B -field of the underlying toroidal theory. Indeed, we have⁵

$$\mathcal{Z}_{\text{untwisted}}^{\text{NS}}(z, \bar{z}) = \frac{1}{2} \frac{1}{|\eta|^8} \left| \frac{\vartheta_3(z)}{\eta} \right|^4 \left\{ 1 + \sum_{\substack{(p_L:p_R) \in \Gamma(\Lambda, B) \\ (p_L:p_R) \neq (0,0)}} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2} \right\} + 8 \left| \frac{\vartheta_4(z)}{\vartheta_2} \right|^4 \quad (2.10)$$

and

$$\mathcal{Z}_{\text{twisted}}^{\text{NS}}(z, \bar{z}) = 8 \left| \frac{\vartheta_2(z)}{\vartheta_4} \right|^4 + 8 \left| \frac{\vartheta_1(z)}{\vartheta_3} \right|^4. \quad (2.11)$$

Different tori T_Λ lead to different Narain lattices, but the $\frac{1}{4}$ -BPS states emerging from non-zero momentum or winding are non-generic in the class of \mathbb{Z}_2 -orbifold CFTs (see, for example, [12, (3.7)-(3.8)] for the precise argument). The remaining spectrum of states is generic to \mathbb{Z}_2 -orbifold CFTs on K3, however, and is the object of our present analysis.

Restricting the graded trace that usually yields the Neveu-Schwarz partition function to the states with vanishing winding and momentum and the twisted sector thus yields

$$\mathcal{Z}^{\text{NS, generic}}(z, \bar{z}) = \frac{1}{2} \frac{1}{|\eta|^8} \left| \frac{\vartheta_3(z)}{\eta} \right|^4 + 8 \left| \frac{\vartheta_4(z)}{\vartheta_2} \right|^4 + \mathcal{Z}_{\text{twisted}}^{\text{NS}}(z, \bar{z}). \quad (2.12)$$

The BPS states we are interested in are Neveu-Schwarz states which under the antiholomorphic $N = 4$ superconformal algebra transform like elements of the chiral ring. In other words, we need to project to $\ker(2\bar{L}_0 - \bar{J}_0)$. These states are thus encoded in the conformal field theoretic elliptic genus of K3 spectral-flowed from the \tilde{R} sector to the NS sector, that is, in

$$\mathcal{E}^{\text{NS}}(\tau, z) := \text{Tr}_{\mathbb{H}^{\text{NS}}} \left((-1)^{\bar{J}_0} y^{J_0} q^{L_0 - \frac{1}{4}} \bar{q}^{\bar{L}_0 - \frac{\bar{J}_0}{2}} \right) = -q^{\frac{1}{4}} y \mathcal{E} \left(\tau, z + \frac{\tau + 1}{2} \right). \quad (2.13)$$

It may also be obtained from the generic Neveu-Schwarz partition function by inserting $\bar{z} = -\frac{\bar{\tau} + 1}{2}$, namely

$$\begin{aligned} \mathcal{Z}^{\text{NS, generic}} \left(z, \bar{z} = -\frac{\bar{\tau} + 1}{2} \right) &= \underbrace{8 \frac{\vartheta_4(z)^2}{\vartheta_2^2} \bar{q}^{-\frac{1}{4}}}_{\text{untwisted}} + \underbrace{8 \left\{ \frac{\vartheta_2(z)^2}{\vartheta_4^2} (-\bar{q}^{-\frac{1}{4}}) + \frac{\vartheta_1(z)^2}{\vartheta_3^2} \bar{q}^{-\frac{1}{4}} \right\}}_{\text{twisted}} \\ &= \mathcal{E}^{\text{NS}}(\tau, z) \bar{q}^{-\frac{1}{4}}. \end{aligned} \quad (2.14)$$

Note that our conventions ensure that the vacuum contributes to $\mathcal{E}^{\text{NS}}(\tau, z)$ with a positive sign, which we find natural, since it is bosonic.⁶

We now rewrite $\mathcal{E}^{\text{NS}}(\tau, z)$ in terms of $N = 4$ characters and Appell functions. To do so, we use the following notations introduced in [18, (B.2)–(B.4)]: by $U_{\ell=\frac{1}{2}}(z)$ we denote the

⁵The dependence on τ will often be understood but not explicitly referred to for easy reading of formulas.

⁶There is a global sign difference between equation (2.14) and [18, (2.5)], or equivalently, (1.6); we view this as a different choice of conventions, due to the fact that in [1, 18], the elliptic genus is expressed in the $\tilde{\text{NSR}}$ sector, while we work in the NSNS sector.

generating function for BPS states that transform as the vacuum under the antiholomorphic $N = 4$ superconformal algebra. To be invariant under the \mathbb{Z}_2 -orbifold action, there must be an even number of modes acting on the vacuum:

$$\begin{aligned}
 U_{\ell=\frac{1}{2}}(z) &:= \frac{1}{2}q^{-\frac{1}{4}} \left\{ \prod_{n=1}^{\infty} \frac{(1+q^{n-\frac{1}{2}}y)^2(1+q^{n-\frac{1}{2}}y^{-1})^2}{(1-q^n)^4} + \prod_{n=1}^{\infty} \frac{(1-q^{n-\frac{1}{2}}y)^2(1-q^{n-\frac{1}{2}}y^{-1})^2}{(1+q^n)^4} \right\} \\
 &= \frac{1}{2} \left\{ \frac{\vartheta_3(z)^2}{\eta^6} + 4 \frac{\vartheta_4(z)^2}{\vartheta_2^2} \right\} \\
 &= \chi_0^{\text{NS}}(z) + \sum_{n=1}^{\infty} f_n q^n \tilde{\chi}^{\text{NS}}(z). \tag{2.15}
 \end{aligned}$$

Similarly, $U_{\ell=0}(z)$ is the generating function for untwisted BPS states that transform as massless matter ground states under the antiholomorphic $N = 4$ superconformal algebra. These states are created from the vacuum by the action of a single mode of weight $\frac{1}{2}$ of an antiholomorphic Dirac fermion. Hence for such BPS states to be invariant under the \mathbb{Z}_2 action, there must be an odd number of holomorphic modes acting on the ground state:

$$\begin{aligned}
 U_{\ell=0}(z) &:= q^{-\frac{1}{4}} \left\{ \prod_{n=1}^{\infty} \frac{(1+q^{n-\frac{1}{2}}y)^2(1+q^{n-\frac{1}{2}}y^{-1})^2}{(1-q^n)^4} - \prod_{n=1}^{\infty} \frac{(1-q^{n-\frac{1}{2}}y)^2(1-q^{n-\frac{1}{2}}y^{-1})^2}{(1+q^n)^4} \right\} \\
 &= \frac{\vartheta_3(z)^2}{\eta^6} - 4 \frac{\vartheta_4(z)^2}{\vartheta_2^2} \\
 &= 4\chi_{\frac{1}{2}}^{\text{NS}}(z) + \sum_{n=1}^{\infty} g_n^{\text{inv}} q^n \tilde{\chi}^{\text{NS}}(z). \tag{2.16}
 \end{aligned}$$

In analogy with our notation (1.8), we introduce

$$f(\tau) := \sum_{n=1}^{\infty} f_n q^n \quad \text{and} \quad g^{\text{inv}}(\tau) := \sum_{n=1}^{\infty} g_n^{\text{inv}} q^n,$$

and using (A.16c) and (A.17) we obtain the following analytic expressions for these functions,

$$\begin{aligned}
 f(\tau) &= 2h_2(\tau) \eta(\tau) q^{\frac{1}{8}} + \frac{q^{\frac{1}{8}}}{2\eta(\tau)^3} - 1, \\
 g^{\text{inv}}(\tau) &= -4h_2(\tau) \eta(\tau) q^{\frac{1}{8}} + \frac{q^{\frac{1}{8}}}{\eta(\tau)^3}. \tag{2.17}
 \end{aligned}$$

Here, the function h_2 is a specialisation of a level one Appell function (see appendix A.2).

By (2.14), the contributions to $\mathcal{E}^{\text{NS}}(\tau, z)$ from the untwisted sector thus may be written as

$$8 \frac{\vartheta_4(z)^2}{\vartheta_2^2} = 2U_{\ell=\frac{1}{2}}(z) - U_{\ell=0}(z) \stackrel{(2.15), (2.16)}{=} 2\chi_0^{\text{NS}}(z) - 4\chi_{\frac{1}{2}}^{\text{NS}}(z) + (2f - g^{\text{inv}}) \tilde{\chi}^{\text{NS}}(z). \tag{2.18}$$

Proceeding in a similar fashion in the twisted sector, we use the function $T_{\ell=0}(z)$ introduced in [18, (B.7)–(B.8)] which gives the contributions to $\mathcal{E}^{\text{NS}}(\tau, z)$ from one twisted

sector:

$$\begin{aligned}
 T_{\ell=0}(z) &:= \frac{1}{2}q^{-\frac{1}{4}} \left\{ (y+2+y^{-1})q^{\frac{1}{2}} \prod_{n=1}^{\infty} \frac{(1+q^ny)^2(1+q^ny^{-1})^2}{(1-q^{n-\frac{1}{2}})^4} \right. \\
 &\quad \left. + (y-2+y^{-1})q^{\frac{1}{2}} \prod_{n=1}^{\infty} \frac{(1-q^ny)^2(1-q^ny^{-1})^2}{(1+q^{n-\frac{1}{2}})^4} \right\} \\
 &= \frac{1}{2} \left\{ \frac{\vartheta_2(z)^2}{\vartheta_4^2} - \frac{\vartheta_1(z)^2}{\vartheta_3^2} \right\} \\
 &= \chi_{\frac{1}{2}}^{\text{NS}}(z) + \sum_{n=1}^{\infty} g_n^{\text{tw}} q^n \tilde{\chi}^{\text{NS}}(z). \tag{2.19}
 \end{aligned}$$

We introduce the function

$$g^{\text{tw}}(\tau) := \sum_{n=1}^{\infty} g_n^{\text{tw}} q^n \stackrel{\text{(A.24)}}{=} -\frac{1}{2}(h_3(\tau) + h_4(\tau))\eta(\tau)q^{\frac{1}{8}}, \tag{2.20}$$

whose Fourier modes, alongside those of $f(\tau)$ and $g^{\text{inv}}(\tau)$, provide crucial data for our analysis. By (2.14) the contributions to $\mathcal{E}^{\text{NS}}(\tau, z)$ from the twisted sector thus read

$$8 \left\{ -\frac{\vartheta_2(z)^2}{\vartheta_4^2} + \frac{\vartheta_1(z)^2}{\vartheta_3^2} \right\} \stackrel{\text{(2.19)}}{=} -16T_{\ell=0}(z) \stackrel{\text{(2.19)}}{=} -16\chi_{\frac{1}{2}}^{\text{NS}}(z) - 16g^{\text{tw}}(\tau)\tilde{\chi}^{\text{NS}}(z), \tag{2.21}$$

where the factor 16 accounts for the number of linearly independent ground states in the twisted sector of the theory. Indeed, the twisted ground states are localised at the 16 singular points of the quotient T_{Λ}/\mathbb{Z}_2 . Altogether, (2.18) and (2.21) yield a decomposition of the conformal field theoretic elliptic genus according to

$$\mathcal{E}^{\text{NS}}(\tau, z) = 2\chi_0^{\text{NS}}(\tau, z) - 20\chi_{\frac{1}{2}}^{\text{NS}}(\tau, z) + (2f(\tau) - g^{\text{inv}}(\tau) - 16g^{\text{tw}}(\tau))\tilde{\chi}^{\text{NS}}(\tau, z) \tag{2.22}$$

and thus, by comparison with (1.7),

$$A(\tau) = -2f(\tau) + g^{\text{inv}}(\tau) + 16g^{\text{tw}}(\tau) = \sum_{n=1}^{\infty} A_n q^n. \tag{2.23}$$

With the decomposition (2.22) of the NS-elliptic genus \mathcal{E}^{NS} in hand, we will from now on continue to work in the Neveu-Schwarz sector, focussing on massive primary states that contribute to \mathcal{E}^{NS} . Note that in this sector, the level n accounted for by A_n agrees with the conformal weight.

2.3 Decomposition of the space $\widehat{\mathcal{H}}^{\text{BPS}}$ of massive $\frac{1}{4}$ -BPS states

As already mentioned in the introduction to this section, in non-generic K3 theories, the space of massive $\frac{1}{4}$ -BPS states $\widehat{\mathcal{H}}^{\text{BPS}}$ is larger at every level than the space \mathcal{H}^{BPS} of generic massive $\frac{1}{4}$ -BPS states, as the conformal field theoretic elliptic genus counts BPS states with signs. Ultimately, we wish to know to what extent one can identify the very states in the class of \mathbb{Z}_2 -orbifold CFTs, whose contributions cancel in the net count of \mathcal{E} under selected

deformations. The better we understand them, in particular the type of group action they may enjoy, the more we can hope to uncover the VOA structure(s) on the generic space of states that *does* contribute to the net count. Our guiding principle in this quest is symmetry surfing [14, 15], a programme we have developed over a period of years and that has passed a number of non-trivial tests, either through explicit calculations within \mathbb{Z}_2 -orbifold CFTs [16] or through a process of deformations away from the \mathbb{Z}_2 -orbifold point in two very interesting papers [18, 19].

In [16], symmetry surfing identifies a special one-dimensional subspace of the 16-dimensional space of twisted ground states in \mathbb{Z}_2 -orbifold conformal field theories on $K3$. Indeed in such theories, corresponding to the 16 fixed points of the standard \mathbb{Z}_2 action on T_Λ , there are 16 pairwise orthogonal twisted ground states $|\alpha_\beta\rangle$, labelled by $\beta \in \mathbb{F}_2^4$. By construction, the “diagonal” state

$$|\alpha_{\text{diag}}\rangle := \sum_{\beta \in \mathbb{F}_2^4} |\alpha_\beta\rangle \tag{2.24}$$

is invariant under all symmetries induced by geometric symmetries of the torus T_Λ , including shifts by elements of $\frac{1}{2}\Lambda$. It is thus invariant under the full overarching affine group

$$G := \text{Aff}(\mathbb{F}_2^4) = \mathbb{Z}_2^4 \times \text{GL}(\mathbb{F}_2^4) \stackrel{[20]}{\cong} \mathbb{Z}_2^4 \times A_8$$

which contains all the groups G_i of finite symplectic automorphisms on Kummer surfaces.

Conveniently, in [16] we found that at massive level one, the orthogonal complement of the Fock space built on $|\alpha_{\text{diag}}\rangle$ echoes the construction of a 45-dimensional representation of the group M_{24} by Margolin [17]. Inspired by his construction, we thus define $\mathcal{H}^\perp \subset \widehat{\mathcal{H}}^{\text{BPS}}$ as the Fock space built on the 15-dimensional orthogonal complement of $|\alpha_{\text{diag}}\rangle$ in the space of twisted ground states. This prompts the following ansatz,⁷ introduced similarly in [18],

$$\widehat{\mathcal{H}}^{\text{BPS}} = \mathcal{H}^\perp \oplus \mathcal{H}^{\text{rest}} \oplus \mathcal{H}^+ \tag{2.25}$$

with $\mathcal{H}^\perp = \bigoplus_{n=1}^\infty (H_n^\perp \otimes \mathcal{H}_n^{N=4})$, $\mathcal{H}^{\text{rest}} = \bigoplus_{n=1}^\infty (H_n^{\text{rest}} \otimes \mathcal{H}_n^{N=4})$, $\mathcal{H}^+ = \bigoplus_{n=1}^\infty (H_n^+ \otimes \mathcal{H}_n^{N=4})$, where as representations of \overline{J}_0 and the octad group G ,

$$H_n \cong H_n^\perp \oplus H_n^{\text{rest}} \quad \text{for all } n \in \mathbb{N}, n > 0, \tag{2.26}$$

and where H_n was defined in (2.1). Ultimately, (2.26) should extend to an isomorphism of representations of \overline{J}_0 and M_{24} . As was pointed out in [18], the ansatz (2.25) can be interpreted as identifying the massive contributions to the generic space of states as the subspace $\mathcal{H}^\perp \oplus \mathcal{H}^{\text{rest}}$ of $\widehat{\mathcal{H}}^{\text{BPS}}$ which remains at the BPS bound under deformations of our CFT by the exactly marginal deformation T_{diag} built on $|\alpha_{\text{diag}}\rangle$.

Table 1 summarises data for the first four conformal weights above threshold in terms of the Fourier coefficients of the generating functions $f(\tau)$, $g^{\text{inv}}(\tau)$ and $g^{\text{tw}}(\tau)$ for massive $\frac{1}{4}$ -BPS ground states contributing to the partition functions $U_{\ell=\frac{1}{2}}, U_{\ell=0}$ and $T_{\ell=0}$ respectively. By construction, we have $\dim H_n = A_n$ and $\dim H_n^\perp = 15g_n^{\text{tw}}$ and hence, by (2.23),

⁷All direct sums are understood as orthogonal direct sums.

level n	1	2	3	4	
A_n	90	462	1540	4554	net number of states in \mathcal{E}^{NS}
f_n	3	1	18	15	untwisted sector ($U_{\ell=\frac{1}{2}}$)
g_n^{inv}	0	16	8	72	untwisted sector ($U_{\ell=0}$)
g_n^{tw}	6	28	98	282	one twisted sector ($T_{\ell=0}$)
$A_n = 16g_n^{\text{tw}} + g_n^{\text{inv}} - 2f_n$	96-6	448+16-2	1568+8-36	4512+72-30	$\dim H_n(\mathcal{H}^{\text{BPS}})$
$g_n^{\text{tw}} + g_n^{\text{inv}} - 2f_n$	0	42	70	324	$\dim H_n^{\text{rest}}(\mathcal{H}^{\text{rest}})$
$15g_n^{\text{tw}}$	90	420	1470	4230	$\dim H_n^\perp(\mathcal{H}^\perp)$
$\hat{A}_n = 16g_n^{\text{tw}} + g_n^{\text{inv}} + 2f_n$	96+6	448+16+2	1568+8+36	4512+72+30	$\dim \hat{H}_n(\hat{\mathcal{H}}^{\text{BPS}})$
$g_n^{\text{tw}} + g_n^{\text{inv}} + 2f_n$	12	46	142	384	$\dim \hat{H}_n^{\text{rest}}$

Table 1. Data on the number of $\frac{1}{4}$ -BPS states emerging from different sectors of \mathbb{Z}_2 -orbifolds CFTs on K3.

$\dim H_n^{\text{rest}} = g_n^{\text{tw}} + g_n^{\text{inv}} - 2f_n$ for all $n \in \mathbb{N}$, $n > 0$. In other words, the excess states in H_n^+ , which are lifted from the BPS bound under a deformation by T_{diag} , belong to the twisted sector generated by $|\alpha_{\text{diag}}\rangle$ and the untwisted sector. While the results of [11, 12] imply that the holomorphic untwisted states accounted for by $2f_n$ all belong to H_n^+ , it is not possible at this stage to identify whether the remaining states in H_n^+ come from the diagonal twisted sector, the untwisted sector, or both. We will return to this point in section 3.

2.4 $\frac{1}{4}$ -BPS states at level one and two

To investigate the elusive properties of H_n^+ in general, we begin by studying the spaces $\hat{H}_n = H_n^\perp \oplus H_n^{\text{rest}} \oplus H_n^+$ at levels $n = 1$ and $n = 2$ more closely. We also introduce a consistent action of $\text{SU}(2)_{\text{geom}} \times \overline{\text{SU}(2)}_{\text{geom}}$ on $H_n^{\text{rest}} \oplus H_n^+$ at these levels. The rationale behind our construction will be explained in section 3 — based on the data collected at levels $n = 1$ and $n = 2$. Here and in the following, we denote the modes of the four free fermions and of their superpartners as $(\chi_\pm^k)_\ell$ and $(a_\pm^k)_m$ with ℓ and m either integers or half-integers in accordance with the boundary conditions imposed by the \mathbb{Z}_2 -orbifold construction. Following a wide-spread tradition, ℓ and m account for the *negative* contributions to the energy.

Level 1. At conformal weight $n = 1$, since $A_1 = 90 = 15g_1^{\text{tw}}$ and $g_1^{\text{inv}} = 0$, our ansatz is compatible with the claim that the contributions to \mathcal{E}^{NS} from the six-dimensional ($2f_1 = 6$) space of untwisted massive $\frac{1}{4}$ -BPS states cancels those from the six-dimensional ($g_1^{\text{tw}} = 6$) space of massive $\frac{1}{4}$ -BPS states in the diagonal twisted sector. This was already discussed in [16], where the geometric action of the group $\text{SU}(2)_{\text{geom}}$ was mentioned. We reproduce our results here in the Neveu-Schwarz sector, not the least because some interesting lessons can be drawn from this case.

To work in the Neveu-Schwarz sector of the \mathbb{Z}_2 -orbifold CFTs, we spectral flow from the Ramond sector and choose chiral-chiral ground states, i.e. states in the kernel of $(2L_0 - J_0)$ and $(2\bar{L}_0 - \bar{J}_0)$.

In the subspace of the untwisted sector accounted for by the partition function $U_{\ell=\frac{1}{2}}(z)$ (see (2.15)), this amounts to building states from the bosonic highest weight states Ω with conformal weights $(h, \bar{h}) = (0, 0)$ and $\check{\Omega} = (\bar{\chi}_+^1)_{-\frac{1}{2}}(\bar{\chi}_+^2)_{-\frac{1}{2}}\Omega$ with $(h, \bar{h}) = (0, 1)$ (recall that $J_0 = 2J_0^3$ in (2.7)). The six-dimensional space of states contributing to $2f_1 = 6$ is generated by the following $N = 4$ primaries:

$$\begin{aligned}
 & (\chi_+^1)_{-\frac{1}{2}}(\chi_-^2)_{-\frac{1}{2}}\Omega, \quad \{(\chi_+^1)_{-\frac{1}{2}}(\chi_-^1)_{-\frac{1}{2}} - (\chi_+^2)_{-\frac{1}{2}}(\chi_-^2)_{-\frac{1}{2}}\}\Omega, \quad (\chi_+^2)_{-\frac{1}{2}}(\chi_-^1)_{-\frac{1}{2}}\Omega, \\
 & (\chi_+^1)_{-\frac{1}{2}}(\chi_-^2)_{-\frac{1}{2}}\check{\Omega}, \quad \{(\chi_+^1)_{-\frac{1}{2}}(\chi_-^1)_{-\frac{1}{2}} - (\chi_+^2)_{-\frac{1}{2}}(\chi_-^2)_{-\frac{1}{2}}\}\check{\Omega}, \quad (\chi_+^2)_{-\frac{1}{2}}(\chi_-^1)_{-\frac{1}{2}}\check{\Omega}.
 \end{aligned} \tag{2.27}$$

The vacuum Ω is by definition invariant under $SU(2)_{\text{geom}} \times \overline{SU(2)}_{\text{geom}}$ and so is $\check{\Omega}$, since $\overline{\chi_+^a}$ with $a \in \{1, 2\}$ is a doublet under $\overline{SU(2)}_{\text{geom}}$ and a singlet under $SU(2)_{\text{geom}}$. Hence each row in (2.27) generates a $(3, 1)$ representation of the group $SU(2)_{\text{geom}} \times \overline{SU(2)}_{\text{geom}}$. In the diagonal twisted sector, the six dimensional space of states accounted for by $g_1^{\text{tw}} = 6$ is built on the $SU(2)_{\text{geom}} \times \overline{SU(2)}_{\text{geom}}$ invariant twisted ground state $|\alpha_{\text{diag}}\rangle$ and also transforms as the sum of two $(3, 1)$ representations of the $SU(2)_{\text{geom}} \times \overline{SU(2)}_{\text{geom}}$ action, generated by

$$\begin{aligned}
 & (a_+^1)_{-\frac{1}{2}}(\chi_-^2)_0|\alpha_{\text{diag}}\rangle, \quad \{(a_+^1)_{-\frac{1}{2}}(\chi_-^1)_0 - (a_+^2)_{-\frac{1}{2}}(\chi_-^2)_0\}|\alpha_{\text{diag}}\rangle, \quad (a_+^2)_{-\frac{1}{2}}(\chi_-^1)_0|\alpha_{\text{diag}}\rangle, \\
 & (a_-^1)_{-\frac{1}{2}}(\chi_-^1)_0|\alpha_{\text{diag}}\rangle, \quad \{(a_-^1)_{-\frac{1}{2}}(\chi_-^2)_0 + (a_-^2)_{-\frac{1}{2}}(\chi_-^1)_0\}|\alpha_{\text{diag}}\rangle, \quad (a_-^2)_{-\frac{1}{2}}(\chi_-^2)_0|\alpha_{\text{diag}}\rangle.
 \end{aligned} \tag{2.28}$$

So at level 1, the ‘‘excess’’ $\frac{1}{4}$ -BPS states in H^+ belong to a pair of isomorphic representations of $SU(2)_{\text{geom}} \times \overline{SU(2)}_{\text{geom}}$ of opposite fermion number. A detailed analysis of the fate of this 12-dimensional space of states when the K3 theories are deformed away from the \mathbb{Z}_2 -orbifolds has recently been carried out in [19]. The conclusion is that under the diagonal deformation T^{diag} , these states combine into non-BPS representations and thus cease to be accounted for by \mathcal{E}^{NS} ; in other words, deformations of a non-generic theory ‘lift’ a number of massive $\frac{1}{4}$ -BPS states, including all those contributing to $U_{\ell=\frac{1}{2}}(z)$. That this must happen under deformations to generic theories follows already from the analysis of [12].

Level 2. At conformal weight $n = 2$, in H_2^+ there is only a two-dimensional space of untwisted holomorphic excess states counted with one sign ($2f_2 = 2$). A priori we must find a two-dimensional subspace of the space $H_2^{\text{rest}} \oplus H_2^+$ of dimension $g_2^{\text{inv}} + g_2^{\text{tw}} = 44$, contributing to \mathcal{E}^{NS} with the opposite sign, in order to identify the subspace of H_2^+ matching the untwisted holomorphic excess states. The latter are singlets of $SU(2)_{\text{geom}} \times \overline{SU(2)}_{\text{geom}}$ given by

$$\begin{aligned}
 |s_{(2)}\rangle & := \left\{ \sum_{k=1}^2 (a_+^k)_{-1}(a_-^k)_{-1} - \sum_{k=1}^2 \{(\chi_+^k)_{-3/2}(\chi_-^k)_{-\frac{1}{2}} - (\chi_+^k)_{-\frac{1}{2}}(\chi_-^k)_{-3/2}\} \right. \\
 & \quad \left. + 2(\chi_+^1)_{-\frac{1}{2}}(\chi_-^1)_{-\frac{1}{2}}(\chi_+^2)_{-\frac{1}{2}}(\chi_-^2)_{-\frac{1}{2}} \right\} \Omega, \\
 |\check{s}_{(2)}\rangle & := \left\{ \sum_{k=1}^2 (a_+^k)_{-1}(a_-^k)_{-1} - \sum_{k=1}^2 \{(\chi_+^k)_{-3/2}(\chi_-^k)_{-\frac{1}{2}} - (\chi_+^k)_{-\frac{1}{2}}(\chi_-^k)_{-3/2}\} \right. \\
 & \quad \left. + 2(\chi_+^1)_{-\frac{1}{2}}(\chi_-^1)_{-\frac{1}{2}}(\chi_+^2)_{-\frac{1}{2}}(\chi_-^2)_{-\frac{1}{2}} \right\} \check{\Omega}.
 \end{aligned} \tag{2.29}$$

Remarkably, there are exactly two massive $\frac{1}{4}$ -BPS singlets with respect to the $SU(2)_{\text{geom}} \times \overline{SU(2)}_{\text{geom}}$ action in the diagonal twisted sector to match $|s_{(2)}\rangle$ and $|\check{s}_{(2)}\rangle$. These are also the only singlets under the action of the diagonal $SU(2)$ in $SU(2)_{\text{geom}} \times \overline{SU(2)}_{\text{geom}}$. There are none in the untwisted sector accounted for by $U_{\ell=0}$, a fact which serves as a justification in section 3 when we generalise our arguments. With the notation

$$A_{\epsilon\delta\rho}^{k\ell mn} := (a_{\epsilon}^k)_{-\frac{1}{2}} (a_{\delta}^{\ell})_{-\frac{1}{2}} (a_{\rho}^m)_{-\frac{1}{2}} (\chi_{-}^n)_0, \quad \epsilon, \delta, \rho \in \{+, -\}, \quad k, \ell, m, n \in \{1, 2\}, \quad \check{k} := 3 - k, \tag{2.30}$$

the two $SU(2)_{\text{geom}} \times \overline{SU(2)}_{\text{geom}}$ singlets in the diagonal twisted sector are given by

$$\begin{aligned} |\check{s}_{(2)}\rangle &:= \sum_{k=1}^2 \left\{ (a_{+}^k)_{-\frac{3}{2}} (\chi_{-}^k)_0 - 3(a_{+}^k)_{-\frac{1}{2}} (\chi_{-}^k)_{-1} - 3(-1)^k (a_{+}^k)_{-\frac{1}{2}} (\chi_{+}^{\check{k}})_{-1} (\chi_{-}^1)_0 (\chi_{-}^2)_0 \right. \\ &\quad \left. + 2(A_{++-}^{kkkk} + A_{++-}^{k\check{k}k\check{k}}) \right\} |\alpha_{\text{diag}}\rangle, \\ |\check{\check{s}}_{(2)}\rangle &:= \sum_{k=1}^2 (-1)^k \left\{ (a_{-}^k)_{-\frac{3}{2}} (\chi_{-}^{\check{k}})_0 - 3(a_{-}^k)_{-\frac{1}{2}} (\chi_{-}^{\check{k}})_{-1} + 3(-1)^k (a_{-}^k)_{-\frac{1}{2}} (\chi_{+}^k)_{-1} (\chi_{-}^1)_0 (\chi_{-}^2)_0 \right. \\ &\quad \left. + 2(A_{+--}^{k\check{k}k\check{k}} - A_{+--}^{k\check{k}k\check{k}}) \right\} |\alpha_{\text{diag}}\rangle. \end{aligned} \tag{2.31}$$

The remaining space of massive level two $\frac{1}{4}$ -BPS states in the diagonal twisted sector is 26-dimensional ($g_2^{\text{tw}} = 28$) and is presented in appendix B.1, while the 16-dimensional space of massive $\frac{1}{4}$ -BPS states in the untwisted sector accounted for by $U_{\ell=0}(z)$ is presented in appendix B.2. This detailed analysis supports the conjecture that the matching of excess $\frac{1}{4}$ -BPS states respects the $SU(2)_{\text{geom}}$ and the $\overline{SU(2)}_{\text{geom}}$ actions on these states, both in the untwisted and twisted sectors. We thus expect that the four states $|s_{(2)}\rangle, |\check{s}_{(2)}\rangle, |\tilde{s}_{(2)}\rangle, |\check{\check{s}}_{(2)}\rangle$ are lifted from the BPS bound under a deformation by T_{diag} . It would be interesting to confirm this prediction by conformal perturbation methods along the lines of [19]. In the next section we present general results corroborating our conjecture at low levels.

3 Geometric $SU(2)$ as a guiding principle

Let us now give a general definition of the $SU(2)_{\text{geom}} \times \overline{SU(2)}_{\text{geom}}$ action on the space $H^{\text{rest}} \oplus H^+$, along the lines indicated in section 2.1. We have defined this space as a space of massive ground states common to all \mathbb{Z}_2 -orbifold conformal field theories on K3, containing both twisted and untwisted contributions, mindful however that its decomposition into $H^{\text{rest}} \oplus H^+$ has not been carried out so far. All states in $H^{\text{rest}} \oplus H^+$ are elements of the Fock space representations obtained from the vacuum Ω and the diagonal twisted ground state $|\alpha_{\text{diag}}\rangle$ by the action of the modes of the free fermionic fields $\chi_{\pm}^a, a \in \{1, 2\}$, and their superpartners j_{\pm}^a , along with their antiholomorphic analogues. It thus suffices to state the action of $SU(2)_{\text{geom}} \times \overline{SU(2)}_{\text{geom}}$ on these fields, on the vacuum Ω and on $|\alpha_{\text{diag}}\rangle$, yielding an action on the entire Fock space built on these states. For each \mathbb{Z}_2 -orbifold conformal field theory on K3 we do this by using the left- and the right-moving action of the group $SU(2)$ which contains the linear part of the geometric symmetry group of our theory. In other words, we use the standard action of the group $SU(2)$ on \mathbb{C}^2 , which is a subgroup of

	$SU(2)_{\text{geom}}$	+1	-1
$\mathfrak{su}(2)$			
+1		χ_+^1	χ_+^2
-1		χ_-^2	χ_-^1

	$SU(2)_{\text{geom}}$	+1	-1
$\mathfrak{su}(2)$			
0		j_+^1	j_+^2
0		j_-^2	j_-^1

Table 2. Charges of free bosons and fermions under $SU(2)_{\text{geom}}$ and under the affine $\mathfrak{su}(2)$ subalgebra of the $N = 4$ superconformal algebra.

a global $SO(4)$ symmetry group of our $N = 4$ superconformal algebra, see for instance [25, §5.3]. Both Ω and $|\alpha_{\text{diag}}\rangle$ are invariant under every geometric symmetry group for any \mathbb{Z}_2 -orbifold conformal field theory on K3. We therefore choose both these states to be invariant under $SU(2)_{\text{geom}} \times \overline{SU(2)}_{\text{geom}}$, while the fields χ_{\pm}^a and j_{\pm}^a , $a \in \{1, 2\}$, transform as doublets under the action of $SU(2)_{\text{geom}}$ and trivially under $\overline{SU(2)}_{\text{geom}}$. Note that these fields also carry a $U(1)$ charge associated with the affine subalgebra $\mathfrak{su}(2)$ of the $N = 4$ superconformal algebra. Our conventions for the two sets of charges, which are summarised in table 2, inform on how to refine the holomorphic partition functions for the untwisted and twisted sectors so they encode the action of the group $SU(2)_{\text{geom}}$ on $\frac{1}{4}$ -BPS states. The charges of $\overline{\chi_{\pm}^a}$ and $\overline{j_{\pm}^a}$ under $\overline{SU(2)}_{\text{geom}}$ and $\overline{\mathfrak{su}(2)}$ are analogous. From our derivation of the $SU(2)_{\text{geom}} \times \overline{SU(2)}_{\text{geom}}$ action on $H^{\text{rest}} \oplus H^+$, we do not expect any meaningful extension to H^{\perp} . Indeed, although a well-defined action of $SU(2)$ exists on H^{\perp} , which is trivial on every twisted ground state, this action does not extend the action of the linear parts of our geometric symmetry groups to $SU(2)$, since these groups act non-trivially on H^{\perp} .

We introduce the complex variable $w := e^{2\pi i\nu}$, $\nu \in \mathbb{C}$, to track the $SU(2)_{\text{geom}}$ charges of $\frac{1}{4}$ -BPS states by refining the partition functions $U_{\ell=\frac{1}{2}}$, $U_{\ell=0}$, $T_{\ell=0}$ introduced in (2.15), (2.16) and (2.19) to

$$\begin{aligned}
 U_{\ell=\frac{1}{2}}(z, \nu) &:= -\frac{1}{2}(w^{-1} - 2 + w) \frac{\vartheta_3(z + \nu) \vartheta_3(z - \nu)}{\vartheta_1(\nu)^2} \\
 &\quad + \frac{1}{2}(w^{-1} + 2 + w) \frac{\vartheta_4(z + \nu) \vartheta_4(z - \nu)}{\vartheta_2(\nu)^2}, \tag{3.1a}
 \end{aligned}$$

$$\begin{aligned}
 U_{\ell=0}(z, \nu) &:= -(w^{-1} - 2 + w) \frac{\vartheta_3(z + \nu) \vartheta_3(z - \nu)}{\vartheta_1(\nu)^2} \\
 &\quad - (w^{-1} + 2 + w) \frac{\vartheta_4(z + \nu) \vartheta_4(z - \nu)}{\vartheta_2(\nu)^2}, \tag{3.1b}
 \end{aligned}$$

$$T_{\ell=0}(z, \nu) := \frac{1}{2} \frac{\vartheta_2(z + \nu) \vartheta_2(z - \nu)}{\vartheta_4(\nu)^2} - \frac{1}{2} \frac{\vartheta_1(z + \nu) \vartheta_1(z - \nu)}{\vartheta_3(\nu)^2}. \tag{3.1c}$$

Given the action of $\overline{SU(2)}_{\text{geom}}$ introduced above, the only states accounted for by the above partition functions which carry a non-trivial action of this group are those built on the \mathbb{Z}_2 -orbifold odd ground states $\tilde{\Omega}^1 := (\overline{\chi_+^1})_{-\frac{1}{2}} \Omega$ and $\tilde{\Omega}^2 := (\overline{\chi_+^2})_{-\frac{1}{2}} \Omega$. $U_{\ell=0}$ is the graded character for the space of \mathbb{Z}_2 -orbifold invariant states in the Fock space built on these two states, which transform as a doublet under $\overline{SU(2)}_{\text{geom}}$. To encode this action as well, we

therefore multiply $U_{\ell=0}(z, \nu)$ by $\frac{1}{2}(\bar{w} + \bar{w}^{-1})$. We treat \bar{w} as a formal variable separately from w , to keep the actions of $SU(2)_{\text{geom}}$ and $\overline{SU(2)}_{\text{geom}}$ apart, in the spirit of separating the action of the left- and the right-moving $N = 4$ superconformal algebras.

The action of the diagonal $SU(2)$ in $SU(2)_{\text{geom}} \times \overline{SU(2)}_{\text{geom}}$ is then captured by identifying \bar{w} as the complex conjugate of w . Indeed, by what was said above, for any given K3 theory the action of $SU(2)_{\text{geom}}$ and $\overline{SU(2)}_{\text{geom}}$ induces the action of the linear part of the geometric symmetry group G_i mentioned in section 2.3 on holomorphic and antiholomorphic fields, respectively. If G_i acts on the holomorphic fields by the representation ϱ , then it acts by the complex conjugate representation $\bar{\varrho}$ on the antiholomorphic partner fields. This is used, for example, in the construction of the corresponding partition functions, where the contributions from the antiholomorphic fields are simply obtained as the complex conjugates of the contributions coming from their holomorphic partners (see, for example, [1, (5.2)]). We remark that the partition functions $U_{\ell=\frac{1}{2}}(z, \nu)$, $\frac{1}{2}(\bar{w} + \bar{w}^{-1})U_{\ell=0}(z, \nu)$, $T_{\ell=0}(z, \nu)$ were used in [18, (C.5), (C.8), (C.10)] at $z = 0$ and at three specific values of w , with complex conjugates \bar{w} , which were interpreted as eigenvalues of the elements g of the linear parts of the geometric symmetry groups G_i . In that situation, $\bar{w} = w^{-1}$, and the expressions in [18, (C.5), (C.8), (C.10)] are invariant under $\bar{w} \leftrightarrow w$. Indeed, by construction, at these special values our partition functions are the characters of g . In [18], the latter play a crucial role in providing evidence for the symmetry surfing programme.

Let us first restrict our attention to the action of $SU(2)_{\text{geom}}$. Since the $SU(2)_{\text{geom}}$ action commutes with that of $N = 4$ and using (2.15), (2.16), (2.19), the refined partition functions enjoy a decomposition in $N = 4$ superconformal characters of the form

$$U_{\ell=\frac{1}{2}}(z, \nu) = \chi_0^{\text{NS}}(z) + f(\nu)\tilde{\chi}^{\text{NS}}(z), \tag{3.2a}$$

$$U_{\ell=0}(z, \nu) = 2(w + w^{-1})\chi_{\frac{1}{2}}^{\text{NS}}(z) + g^{\text{inv}}(\nu)\tilde{\chi}^{\text{NS}}(z), \tag{3.2b}$$

$$T_{\ell=0}(z, \nu) = \chi_{\frac{1}{2}}^{\text{NS}}(z) + g^{\text{tw}}(\nu)\tilde{\chi}^{\text{NS}}(z). \tag{3.2c}$$

As before, and by abuse of notation, $f(\nu)$ is our shorthand notation for $f(\tau, \nu)$, and $f(\tau) = f(\tau, \nu = 0)$, etc. By construction, $SU(2)_{\text{geom}}$ maps the spaces of massive ground states that contribute to $U_{\ell=\frac{1}{2}}$, $U_{\ell=0}$, $T_{\ell=0}$, respectively, to themselves, such that these three spaces decompose into direct sums of irreducible representations of $SU(2)_{\text{geom}}$. Therefore,

$$f(\nu) = \sum_{n=1}^{\infty} q^n \left(\sum_p f_{n,p} \chi_p^{\text{SU}(2)}(\nu) \right), \tag{3.3a}$$

$$g^{\text{inv}}(\nu) = \sum_{n=1}^{\infty} q^n \left(\sum_p g_{n,p}^{\text{inv}} \chi_p^{\text{SU}(2)}(\nu) \right), \tag{3.3b}$$

$$g^{\text{tw}}(\nu) = \sum_{n=1}^{\infty} q^n \left(\sum_p g_{n,p}^{\text{tw}} \chi_p^{\text{SU}(2)}(\nu) \right), \tag{3.3c}$$

with the $SU(2)$ character of the representation with isospin $p \in \frac{1}{2}\mathbb{N}$ given by

$$\chi_p^{\text{SU}(2)}(\nu) := \sum_{r=0}^{2p} e^{2\pi i(2p-2r)\nu} = \frac{\sin 2\pi(2p+1)\nu}{\sin 2\pi\nu} \tag{3.4}$$

and $f_{n,p}$, $g_{n,p}^{\text{inv}}$ and $g_{n,p}^{\text{tw}} \in \mathbb{N}$ the multiplicity of the $(2p + 1)$ -dimensional representation at level n in the two untwisted and the twisted diagonal sectors, respectively.

In fact, as explained in appendix A.5, using identities amongst Jacobi theta functions and Appell functions, one may rewrite (3.3a)–(3.3c) as

$$f(\tau, \nu) = -1 - \left\{ \frac{1}{2}(w - 2 + w^{-1}) h_1(\tau, \nu) - \frac{1}{2}(w + 2 + w^{-1}) h_2(\tau, \nu) \right\} \eta(\tau) q^{\frac{1}{8}}, \quad (3.5a)$$

$$g^{\text{inv}}(\tau, \nu) = -\{(w - 2 + w^{-1}) h_1(\tau, \nu) + (w + 2 + w^{-1}) h_2(\tau, \nu)\} \eta(\tau) q^{\frac{1}{8}}, \quad (3.5b)$$

$$g^{\text{tw}}(\tau, \nu) = -\frac{1}{2} \{h_3(\tau, \nu) + h_4(\tau, \nu)\} \eta(\tau) q^{\frac{1}{8}}. \quad (3.5c)$$

The three functions $f(\tau, \nu)$, $g^{\text{inv}}(\tau, \nu)$ and $g^{\text{tw}}(\tau, \nu)$ are the graded characters of certain subspaces of the space of massive $\frac{1}{4}$ -BPS ground states, whose decomposition into $H^{\text{rest}} \oplus H^+$ is at the heart of our investigation. Recall that we have defined the space H^+ as to contain pairs of ground states at opposite fermion numbers whose contributions to the elliptic genus \mathcal{E}^{NS} cancel. Upon deformation of our K3 theory by T^{diag} , each such pair is lifted to a common long $N = 4$ representation off the BPS bound. On the other hand, for every \mathbb{Z}_2 -orbifold CFT on K3, $\text{SU}(2)_{\text{geom}} \times \overline{\text{SU}(2)}_{\text{geom}}$ acts on $H^{\text{rest}} \oplus H^+$ as to restrict to the action of the linear part of the geometric symmetry group of the theory, which remains unbroken under deformations by T^{diag} . This, together with the symmetry surfing proposal, prompts us to postulate that the states in H^+ are paired up according to their transformation properties under $\text{SU}(2)_{\text{geom}} \times \overline{\text{SU}(2)}_{\text{geom}}$. In other words, we postulate that H^+ decomposes into pairs of isomorphic representations under $\text{SU}(2)_{\text{geom}}$ and $\overline{\text{SU}(2)}_{\text{geom}}$ with opposite fermion numbers.

To determine which subspaces of the spaces accounted for by $f(\nu)$, $g^{\text{inv}}(\nu)$ and $g^{\text{tw}}(\nu)$ may contribute to H^+ , we recall from the above that by the results of [12], all untwisted holomorphic states accounted for by $f(\nu)$ are non-generic and thus belong to H^+ . All of them are bosonic, while $g^{\text{inv}}(\nu)$ and $g^{\text{tw}}(\nu)$ account for fermionic states only. In other words, we must find $\text{SU}(2)_{\text{geom}} \times \overline{\text{SU}(2)}_{\text{geom}}$ representations matching those accounted for by $f(\nu)$ within the spaces whose graded characters are $g^{\text{inv}}(\nu)$ and $g^{\text{tw}}(\nu)$. Observe that by (A.11) and (3.5a)–(3.5c), $f(\nu)$ and $g^{\text{tw}}(\nu)$ are invariant under a shift of the variable ν by $\frac{1}{2}$, while $g^{\text{inv}}(\nu + \frac{1}{2}) = -g^{\text{inv}}(\nu)$. This shows that $g^{\text{inv}}(\nu)$ only accounts for representations of $\text{SU}(2)_{\text{geom}}$ with half-integer spin, while $f(\nu)$ and $g^{\text{tw}}(\nu)$ only account for representations with integer spin. Moreover, all states accounted for by $f(\nu)$ and $g^{\text{tw}}(\nu)$ transform trivially under $\overline{\text{SU}(2)}_{\text{geom}}$, in contrast to those accounted for by $g^{\text{inv}}(\nu)$. This, together with the evidence provided by the explicit calculations at levels $n = 1, 2$, prompts us to postulate that, at any level $n \in \mathbb{N}, n \geq 1$, only $\frac{1}{4}$ -BPS states belonging to the diagonal twisted sector can pair up with those occurring in the untwisted sector and accounted for by $f(\nu)$. In light of the decompositions (3.3a)–(3.3c) into characters of $\text{SU}(2)_{\text{geom}}$, this implies the claim that

$$\forall p, n \in \mathbb{N}: \quad g_{n,p}^{\text{tw}} - 2f_{n,p} \geq 0. \quad (3.6)$$

We have expanded the functions $f(\nu)$ and $g^{\text{tw}}(\nu)$ as q -power series and verified (3.6) up to $O(q^{101})$, supporting our postulates. We present the data up to $O(q^{16})$ in table 3 for reference. We hope to provide an analytic proof of (3.6) in the near future.

4 Discussion

The VOA(s) underlying Mathieu Moonshine remain elusive to the extent that even a consensus on whether or not to expect a link between Mathieu Moonshine and K3 theories has not been reached. The works of [11, 12] indicate that such a link could involve a *generic space of states* of K3 theories, in accordance with the ideas behind our symmetry surfing programme [13–16]. The present work is a contribution to the study of generic properties of K3 theories which we find interesting in their own right. This can be viewed as a preparation for a new attempt at the construction of a Mathieu Moonshine VOA on the generic space of states.

In [12], crucially building on the results of [11], it was shown that indeed there exists a generic space of states \mathbb{H}_0 for all K3 theories, roughly defined through the property that it embeds into the space of BPS states of every K3 theory as a representation of the holomorphic $N = 4$ superconformal algebra, extended by the zero modes \bar{L}_0, \bar{J}_0 of the Virasoro field and $\mathfrak{u}(1)$ current in the antiholomorphic $N = 4$ superconformal algebra. Although \mathbb{H}_0 can be modelled by the chiral de Rham cohomology of a K3 surface [11, 12], its detailed structure has not been studied so far. Approaching the space from the perspective of *non-generic* yet accessible K3 theories, we have, in the present work, studied more closely the structure of the $\frac{1}{4}$ -BPS states of \mathbb{Z}_2 -orbifold CFTs of toroidal theories. We propose a strategy to earmark the $\frac{1}{4}$ -BPS states that move off the BPS bound under the most symmetric deformation T^{diag} of \mathbb{Z}_2 -orbifold CFTs on K3, away from the \mathbb{Z}_2 -orbifold limit. Such states must come in pairs of opposite fermion numbers, such that their contributions cancel each other in the conformal field theoretic elliptic genus of K3, in order to be part of the same (long) non-BPS representation after deformation. In [16], where the first concrete study of BPS states contributing to the count at level $n = 1$ in the conformal field theoretic elliptic genus of K3 was undertaken, we identified a 15-dimensional space of twisted ground states carrying a Fock space representation \mathcal{H}^\perp which turns out to be generic in the above sense, that is, along deformations by T^{diag} . In the twisted sector, \mathcal{H}^\perp is the orthogonal complement of a ‘diagonal’ subspace from which T^{diag} arises. The findings were guided by the symmetry surfing programme [14, 15] and were inspired by Margolin’s construction of a 45-dimensional representation of M_{24} [17]. They highlighted the action of the octad group, a maximal subgroup of M_{24} , on the 45-dimensional subspace of \mathcal{H}^\perp at level 1. Evidence of the octad group action at all levels was provided in the work [18], fueling the symmetry surfing programme whose aim is to exhibit an M_{24} action on the generic space of states. More recently, Keller and Zadeh [19] deformed the \mathbb{Z}_2 -orbifold CFTs away from the Kummer point by a marginal operator and showed that if the deformation is taken in the diagonal direction T^{diag} , then indeed all the BPS states in the twisted sector of the original non-generic K3 theory that are orthogonal to \mathcal{H}^\perp move off the BPS bound under the deformation.

It remains that beyond level $n = 1$, we do not have total control on which BPS states move off the BPS bound under a given deformation. By the results of [12], we know that under deformations to generic K3 theories this happens for every untwisted massive state accounted for by the partition function $U_{\ell=\frac{1}{2}}$. Since each state that moves off the BPS bound pairs up with a state of opposite fermion number, to become part of the same long

non-BPS representation, one needs to identify the correct partners in either the untwisted subsector $U_{\ell=0}$ or the twisted sector. Following the results of [18], we know that under the deformation by T^{diag} , of the twisted states only those in the diagonal twisted sector can move off the BPS bound. We postulate that none of the untwisted states accounted for in $U_{\ell=0}$ do. At level $n = 1$ this trivially holds as the level one contribution to $U_{\ell=0}$ is zero. In fact, all states in the twisted diagonal sector move off the BPS bound under deformation by T^{diag} . This is a very special situation that does not persist at higher levels.

In order to identify potential states to pair up with the states accounted for in $U_{\ell=\frac{1}{2}}$ and move off the BPS bound under deformation by T^{diag} , we postulate compatibility with a geometric action of the group $SU(2)$, denoted $SU(2)_{\text{geom}}$ in this work. Indeed, as already pointed out in [16], the holomorphic Dirac fermion fields χ_{\pm}^a , $a \in \{1, 2\}$, and their superpartners, which are the building blocks of \mathbb{Z}_2 -orbifold CFTs, transform as doublets under $SU(2)_{\text{geom}}$. The group acts trivially on the vacuum and on the twisted diagonal ground state, and also on the antiholomorphic partners of the χ_{\pm}^a , $a \in \{1, 2\}$, and their superpartners. We have introduced refined partition functions that keep track of that group action on $\frac{1}{4}$ -BPS states in the untwisted and twisted diagonal sectors, and we were able to show that at level $n \leq 2$ only states stemming from the twisted diagonal sector carry representations of $SU(2)_{\text{geom}}$ that match those in the untwisted subsector accounted for by $U_{\ell=\frac{1}{2}}$. The $SU(2)_{\text{geom}}$ action on the $\frac{1}{4}$ -BPS states thus helps to identify sets of states in the diagonal twisted sector that may lift off the BPS bound under the deformation by T^{diag} . We note that a certain degree of indetermination remains, as the multiplicities of isospin p representations at any fixed level n in the diagonal twisted sector quickly exceed by far twice the multiplicities of isomorphic representations at the same level in the untwisted sector, as evidenced in table 3. Therefore, except for levels $n = 1, 2$, our postulate is not powerful enough to pin down the exact states that are generic along the deformation by T^{diag} .

As explained above, $SU(2)_{\text{geom}}$ in particular leaves the \mathbb{Z}_2 -orbifold odd ground states in the untwisted subsector accounted for by $U_{\ell=0}$ invariant. From the perspective of symmetry surfing, it is perhaps more natural to consider the action of a diagonal $SU(2)$ in $SU(2)_{\text{geom}} \times \overline{SU(2)_{\text{geom}}}$, where $\overline{SU(2)_{\text{geom}}}$ is the antiholomorphic analog of $SU(2)_{\text{geom}}$. In particular, under this diagonal action of $SU(2)$, the ground states in the untwisted sector accounted for by $U_{\ell=0}$ transform non-trivially. We argue that representations from this sector do not pair up with states accounted for by $U_{\ell=\frac{1}{2}}$ to form long representations off the BPS bound under any deformation. Indeed, if this were the case, then there should be a deformation in the underlying toroidal theory that would lift these states off the BPS bound. However, such states are generic to all toroidal theories and hence such deformations do not exist.

Altogether we expect the role of $SU(2)_{\text{geom}}$ to be helpful in understanding the generic space of states of K3 theories, independently of Mathieu Moonshine. An analysis of deformations beyond level $n = 1$ along the lines of those followed in [19] would certainly shed more light on the relevance of $SU(2)_{\text{geom}}$. The analysis of [19] already shows that any sufficiently small deformation away from the \mathbb{Z}_2 -orbifold conformal field theories on K3 reduces the space of $\frac{1}{4}$ -BPS states at massive level one to a generic space. Given the results of [12], the same must hold at arbitrary level. It would be interesting to understand the

structure of the underlying VOAs and their dependence on the details of the deformation. Indeed, is it possible that the dependence on the choice of deformation drops out entirely? Ultimately, this could answer some of the open questions of Mathieu Moonshine.

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A Modular and mock modular input

A.1 Jacobi theta functions

Let $q = e^{2\pi i\tau}$, $\tau \in \mathfrak{H}$ and $y = e^{2\pi iz}$, $z \in \mathbb{C}$. Our notations for the Jacobi theta functions are

$$\begin{aligned}
 \vartheta_1(z) &:= i \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}(n-\frac{1}{2})^2} y^{n-\frac{1}{2}} = iq^{\frac{1}{8}} y^{-\frac{1}{2}} \prod_{n=1}^{\infty} (1-q^n)(1-q^{n-1}y)(1-q^n y^{-1}), \\
 \vartheta_2(z) &:= \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n-\frac{1}{2})^2} y^{n-\frac{1}{2}} = q^{\frac{1}{8}} y^{-\frac{1}{2}} \prod_{n=1}^{\infty} (1-q^n)(1+q^{n-1}y)(1+q^n y^{-1}), \\
 \vartheta_3(z) &:= \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2} y^n = \prod_{n=1}^{\infty} (1-q^n)(1+q^{n-\frac{1}{2}}y)(1+q^{n-\frac{1}{2}}y^{-1}), \\
 \vartheta_4(z) &:= \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n^2} y^n = \prod_{n=1}^{\infty} (1-q^n)(1-q^{n-\frac{1}{2}}y)(1-q^{n-\frac{1}{2}}y^{-1}),
 \end{aligned} \tag{A.1}$$

with $\vartheta_i(0) := \vartheta_i$, $i \in \{2, 3, 4\}$ and $\vartheta_1(0) = 0$. All the theta function identities used in this paper can be found in [26]. In particular, for $z_1, z_2 \in \mathbb{C}$, the following addition formulae are useful, all of which can be proved by residue analysis,

$$\begin{aligned}
 \vartheta_1(z_1 + z_2)\vartheta_1(z_1 - z_2)\vartheta_4^2 &= \vartheta_3(z_1)^2\vartheta_2(z_2)^2 - \vartheta_2(z_1)^2\vartheta_3(z_2)^2 \\
 &= \vartheta_1(z_1)^2\vartheta_4(z_2)^2 - \vartheta_4(z_1)^2\vartheta_1(z_2)^2, \\
 \vartheta_4(z_1 + z_2)\vartheta_4(z_1 - z_2)\vartheta_2^2 &= \vartheta_4(z_1)^2\vartheta_2(z_2)^2 + \vartheta_3(z_1)^2\vartheta_1(z_2)^2 \\
 &= \vartheta_2(z_1)^2\vartheta_4(z_2)^2 + \vartheta_1(z_1)^2\vartheta_3(z_2)^2,
 \end{aligned} \tag{A.2}$$

and

$$\begin{aligned}
 \vartheta_2(z_1 \pm z_2)\vartheta_3(z_1 \mp z_2)\vartheta_2\vartheta_3 &= \vartheta_2(z_1)\vartheta_3(z_1)\vartheta_2(z_2)\vartheta_3(z_2) \mp \vartheta_1(z_1)\vartheta_4(z_1)\vartheta_1(z_2)\vartheta_4(z_2), \\
 \vartheta_2(z_1 \pm z_2)\vartheta_4(z_1 \mp z_2)\vartheta_2\vartheta_4 &= \vartheta_2(z_1)\vartheta_4(z_1)\vartheta_2(z_2)\vartheta_4(z_2) \mp \vartheta_1(z_1)\vartheta_3(z_1)\vartheta_1(z_2)\vartheta_3(z_2), \\
 \vartheta_3(z_1 \pm z_2)\vartheta_4(z_1 \mp z_2)\vartheta_3\vartheta_4 &= \vartheta_3(z_1)\vartheta_4(z_1)\vartheta_3(z_2)\vartheta_4(z_2) \mp \vartheta_1(z_1)\vartheta_2(z_1)\vartheta_1(z_2)\vartheta_2(z_2).
 \end{aligned} \tag{A.3}$$

The Dedekind η function is defined as

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \quad (\text{A.4})$$

and is related to the Jacobi theta functions by the identity

$$2\eta^3 = \vartheta_2 \vartheta_3 \vartheta_4. \quad (\text{A.5})$$

For future reference, we note

$$\begin{aligned} \vartheta_1^2\left(\frac{\tau+1}{2}\right) &= q^{-\frac{1}{4}} \vartheta_3^2(\tau), & \vartheta_2^2\left(\frac{\tau+1}{2}\right) &= -q^{-\frac{1}{4}} \vartheta_4^2(\tau), \\ \vartheta_3^2\left(\frac{\tau+1}{2}\right) &= 0, & \vartheta_4^2\left(\frac{\tau+1}{2}\right) &= q^{-\frac{1}{4}} \vartheta_2^2(\tau). \end{aligned} \quad (\text{A.6})$$

A.2 Appell functions

Let

$$\mathcal{K}_\ell(\tau, \nu, \mu) := \sum_{m \in \mathbb{Z}} \frac{q^{\frac{\ell m^2}{2}} w^{\ell m}}{1 - wxq^m}, \quad w = e^{2\pi i \nu}, \quad x = e^{2\pi i \mu} \quad (\text{A.7})$$

be the Appell function at level $\ell \in \mathbb{N}$, $\nu, \mu \in \mathbb{C}$, $\nu + \mu \notin \mathbb{Z}\tau + \mathbb{Z}$ [28–30]. Define

$$h_3(\tau, \nu) := \frac{1}{\eta(\tau)} \frac{1}{\vartheta_3(\tau, \nu)} q^{-\frac{1}{8}} \mathcal{K}_1\left(\tau, \nu, -\frac{\tau+1}{2}\right) = \frac{1}{\eta(\tau)} \frac{1}{\vartheta_3(\tau, \nu)} \sum_{m \in \mathbb{Z}} \frac{q^{\frac{m^2}{2} - \frac{1}{8}} w^m}{1 + wq^{m - \frac{1}{2}}}. \quad (\text{A.8})$$

By evaluating (A.8) at ν shifted by the three two-torsion points of an elliptic curve, one defines three new functions,

$$\begin{aligned} h_4(\tau, \nu) &:= h_3\left(\tau, \nu + \frac{1}{2}\right) = \frac{1}{\eta(\tau)} \frac{1}{\vartheta_4(\tau, \nu)} \sum_{m \in \mathbb{Z}} \frac{(-1)^m q^{\frac{m^2}{2} - \frac{1}{8}} w^m}{1 - wq^{m - \frac{1}{2}}}, \\ h_2(\tau, \nu) &:= h_3\left(\tau, \nu + \frac{\tau}{2}\right) = \frac{1}{\eta(\tau)} \frac{1}{\vartheta_2(\tau, \nu)} \sum_{m \in \mathbb{Z}} \frac{q^{\frac{m^2}{2} + \frac{m}{2}} w^{m + \frac{1}{2}}}{1 + wq^m}, \\ h_1(\tau, \nu) &:= h_3\left(\tau, \nu + \frac{\tau+1}{2}\right) = \frac{-i}{\eta(\tau)} \frac{1}{\vartheta_1(\tau, \nu)} \sum_{m \in \mathbb{Z}} \frac{(-1)^m q^{\frac{m^2}{2} + \frac{m}{2}} w^{m + \frac{1}{2}}}{1 - wq^m}. \end{aligned} \quad (\text{A.9})$$

The following properties are immediate:

$$h_i(-\nu) = h_i(\nu), \quad h_i(\nu + \tau) = h_i(\nu) \quad \forall i \in \{1, 2, 3, 4\} \quad (\text{A.10})$$

and

$$h_1\left(\nu + \frac{1}{2}\right) = h_2(\nu), \quad h_2\left(\nu + \frac{1}{2}\right) = h_1(\nu), \quad h_3\left(\nu + \frac{1}{2}\right) = h_4(\nu), \quad h_4\left(\nu + \frac{1}{2}\right) = h_3(\nu). \quad (\text{A.11})$$

Identities. Here and throughout the paper, $h_i := h_i(\nu = 0), i \in \{1, 2, 3, 4\}$. The following identities can be established by residue analysis,

$$h_3(\nu) = h_3 - \frac{\vartheta_1(\nu)^2}{\vartheta_3(\nu)^2} \frac{\eta^2}{\vartheta_3^2}, \quad h_4(\nu) = h_4 + \frac{\vartheta_1(\nu)^2}{\vartheta_4(\nu)^2} \frac{\eta^2}{\vartheta_4^2}, \quad h_2(\nu) = h_2 + \frac{\vartheta_1(\nu)^2}{\vartheta_2(\nu)^2} \frac{\eta^2}{\vartheta_2^2}. \quad (\text{A.12})$$

New ones are obtained by shifting ν by the three two-torsion points in (A.12),

$$\begin{aligned} h_4(\nu) &= h_3 - \frac{\vartheta_2(\nu)^2}{\vartheta_4(\nu)^2} \frac{\eta^2}{\vartheta_3^2}, & h_3(\nu) &= h_4 + \frac{\vartheta_2(\nu)^2}{\vartheta_3(\nu)^2} \frac{\eta^2}{\vartheta_4^2}, & h_1(\nu) &= h_2 + \frac{\vartheta_2(\nu)^2}{\vartheta_1(\nu)^2} \frac{\eta^2}{\vartheta_2^2}, \\ h_2(\nu) &= h_3 + \frac{\vartheta_4(\nu)^2}{\vartheta_2(\nu)^2} \frac{\eta^2}{\vartheta_3^2}, & h_1(\nu) &= h_4 + \frac{\vartheta_4(\nu)^2}{\vartheta_1(\nu)^2} \frac{\eta^2}{\vartheta_4^2}, & h_3(\nu) &= h_2 - \frac{\vartheta_4(\nu)^2}{\vartheta_3(\nu)^2} \frac{\eta^2}{\vartheta_2^2}, \\ h_1(\nu) &= h_3 + \frac{\vartheta_3(\nu)^2}{\vartheta_1(\nu)^2} \frac{\eta^2}{\vartheta_3^2}, & h_2(\nu) &= h_4 + \frac{\vartheta_3(\nu)^2}{\vartheta_2(\nu)^2} \frac{\eta^2}{\vartheta_4^2}, & h_4(\nu) &= h_2 - \frac{\vartheta_3(\nu)^2}{\vartheta_4(\nu)^2} \frac{\eta^2}{\vartheta_2^2}. \end{aligned} \quad (\text{A.13})$$

A.3 $N = 4$ characters at $c=6$

The characters for unitary, irreducible representations of the $N = 4$ superconformal algebra were first derived in [21]. When the central charge is $c = 6$, the representations fall into an infinite class of ‘long’ or ‘massive’ representations with Neveu-Schwarz characters of the form

$$q^h \tilde{\chi}^{\text{NS}}(\tau, z) = q^{h - \frac{1}{8}} \frac{\vartheta_3(\tau, z)^2}{\eta(\tau)^3}, \quad (\text{A.14})$$

with $h \in \mathbb{R}, h > 0$, the conformal weight of the highest weight state alongside two ‘short’ or ‘massless’ representations labelled by the $\mathfrak{su}(2)$ ‘spin’ $\ell \in \{0, \frac{1}{2}\}$ and the conformal weight $h = \ell$ of their highest weight states. The corresponding Neveu-Schwarz characters are expressible in a variety of ways. Here, we use their expressions in terms of the Appell functions $h_i(z)$ defined in (A.8) and (A.9). We have

$$\chi_{\frac{1}{2}}^{\text{NS}}(\tau, z) := \chi_{h=\ell=\frac{1}{2}}^{\text{NS}}(\tau, z) = h_3(\tau, z) \frac{\vartheta_3(\tau, z)^2}{\eta(\tau)^2} = h_3(\tau, z) \eta(\tau) q^{\frac{1}{8}} \tilde{\chi}^{\text{NS}}(\tau, z). \quad (\text{A.15})$$

This form of the characters was first presented in [27] and expresses the branching of $N = 4$ characters in an infinite sum of $N = 2$ characters at central charge $c = 6$. Inserting (A.12) and (A.13) into (A.15), one gets

$$\chi_{\frac{1}{2}}^{\text{NS}}(\tau, z) = -\frac{\vartheta_1(\tau, z)^2}{\vartheta_3(\tau)^2} + h_3(\tau) \eta(\tau) q^{\frac{1}{8}} \tilde{\chi}^{\text{NS}}(\tau, z) \quad (\text{A.16a})$$

$$= \frac{\vartheta_2(\tau, z)^2}{\vartheta_4(\tau)^2} + h_4(\tau) \eta(\tau) q^{\frac{1}{8}} \tilde{\chi}^{\text{NS}}(\tau, z) \quad (\text{A.16b})$$

$$= -\frac{\vartheta_4(\tau, z)^2}{\vartheta_2(\tau)^2} + h_2(\tau) \eta(\tau) q^{\frac{1}{8}} \tilde{\chi}^{\text{NS}}(\tau, z). \quad (\text{A.16c})$$

The second massless $N = 4$ character in the Neveu-Schwarz sector is given by

$$\chi_0^{\text{NS}}(\tau, z) := \chi_{h=\ell=0}^{\text{NS}}(\tau, z) = \tilde{\chi}^{\text{NS}}(\tau, z) - 2\chi_{\frac{1}{2}}^{\text{NS}}(\tau, z). \quad (\text{A.17})$$

Twisting by the fermion number operator one obtains

$$\begin{aligned}\tilde{\chi}^{\widetilde{\text{NS}}}(\tau, z) &= \tilde{\chi}^{\text{NS}}\left(\tau, z + \frac{1}{2}\right), \\ \chi_\ell^{\widetilde{\text{NS}}}(\tau, z) &= \chi_\ell^{\text{NS}}\left(\tau, z + \frac{1}{2}\right), \quad \ell \in \left\{0, \frac{1}{2}\right\}.\end{aligned}\tag{A.18}$$

Under spectral flow, the Neveu-Schwarz and Ramond characters flow to each other as

$$\begin{aligned}\tilde{\chi}^{\text{NS}}\left(\tau, z + \frac{\tau}{2}\right) &= q^{-\frac{1}{4}} y^{-1} \tilde{\chi}^{\text{R}}(\tau, z), & \tilde{\chi}^{\text{NS}}\left(\tau, z + \frac{\tau+1}{2}\right) &= -q^{-\frac{1}{4}} y^{-1} \tilde{\chi}^{\widetilde{\text{R}}}(\tau, z), \\ \chi_\ell^{\text{NS}}\left(\tau, z + \frac{\tau}{2}\right) &= q^{-\frac{1}{4}} y^{-1} \chi_{\frac{1}{2}-\ell}^{\text{R}}(\tau, z), & \chi_\ell^{\text{NS}}\left(\tau, z + \frac{\tau+1}{2}\right) &= -q^{-\frac{1}{4}} y^{-1} \chi_{\frac{1}{2}-\ell}^{\widetilde{\text{R}}}(\tau, z), \quad \ell \in \left\{0, \frac{1}{2}\right\}.\end{aligned}\tag{A.19}$$

Given (A.15) and (A.17), all $N = 4$ characters may be expressed in terms of Appell functions. In particular,

$$\begin{aligned}\chi_0^{\text{R}}(\tau, z) &= h_2(\tau, z) \frac{\vartheta_2(\tau, z)^2}{\eta(\tau)^2} = h_2(\tau, z) \eta(\tau) q^{\frac{1}{8}} \tilde{\chi}^{\text{R}}(\tau, z), \\ \chi_{\frac{1}{2}}^{\widetilde{\text{NS}}}(\tau, z) &= h_4(\tau, z) \frac{\vartheta_4(\tau, z)^2}{\eta(\tau)^2} = h_4(\tau, z) \eta(\tau) q^{\frac{1}{8}} \tilde{\chi}^{\widetilde{\text{NS}}}(\tau, z), \\ \chi_0^{\widetilde{\text{R}}}(\tau, z) &= h_1(\tau, z) \frac{\vartheta_1(\tau, z)^2}{\eta(\tau)^2} = h_1(\tau, z) \eta(\tau) q^{\frac{1}{8}} \tilde{\chi}^{\widetilde{\text{R}}}(\tau, z).\end{aligned}\tag{A.20}$$

The Witten index of the various representations is obtained by setting $z = 0$ in the $\widetilde{\text{R}}$ characters. The massive representations all have Witten index zero while by (A.19), (A.17) and (A.12),

$$\begin{aligned}\chi_{\frac{1}{2}}^{\widetilde{\text{R}}}(\tau, 0) &= -2 = -q^{\frac{1}{4}} \chi_0^{\text{NS}}\left(\tau, \frac{\tau+1}{2}\right), \\ \chi_0^{\widetilde{\text{R}}}(\tau, 0) &= 1 = -q^{\frac{1}{4}} \chi_{\frac{1}{2}}^{\text{NS}}\left(\tau, \frac{\tau+1}{2}\right).\end{aligned}\tag{A.21}$$

A.4 The functions $f(\tau)$, $g^{\text{inv}}(\tau)$ and $g^{\text{tw}}(\tau)$

The functions (2.17) and (2.20) can all be obtained by standard manipulations,

$$\begin{aligned}U_{\ell=\frac{1}{2}}(z) &\stackrel{(2.15)}{=} \frac{1}{2} \left\{ \frac{\vartheta_3(z)^2}{\eta^6} + 4 \frac{\vartheta_4(z)^2}{\vartheta_2^2} \right\} \\ &\stackrel{(A.14), (A.16c)}{=} \frac{1}{2\eta^3} q^{\frac{1}{8}} \tilde{\chi}^{\text{NS}}(z) + 2 \left\{ -\chi_{\frac{1}{2}}^{\text{NS}}(z) + h_2 \eta q^{\frac{1}{8}} \tilde{\chi}^{\text{NS}}(z) \right\} \\ &\stackrel{(A.17)}{=} \chi_0^{\text{NS}}(z) + \underbrace{\left\{ 2h_2 \eta q^{\frac{1}{8}} - 1 + \frac{1}{2\eta^3} q^{\frac{1}{8}} \right\}}_{=f(\tau)} \tilde{\chi}^{\text{NS}}(z),\end{aligned}\tag{A.22}$$

$$\begin{aligned}
 U_{\ell=0}(z) &\stackrel{(2.16)}{=} \frac{\vartheta_3(z)^2}{\eta^6} - 4 \frac{\vartheta_4(z)^2}{\vartheta_2^2} \\
 &\stackrel{(A.14), (A.16c)}{=} \frac{1}{\eta^3} q^{\frac{1}{8}} \tilde{\chi}^{\text{NS}}(z) - 4 \left\{ -\chi_{\frac{1}{2}}^{\text{NS}}(z) + h_2 \eta q^{\frac{1}{8}} \tilde{\chi}^{\text{NS}}(z) \right\} \\
 &= 4 \chi_{\frac{1}{2}}^{\text{NS}}(z) + \underbrace{\left\{ -4h_2 \eta q^{\frac{1}{8}} + \frac{1}{\eta^3} q^{\frac{1}{8}} \right\}}_{=g^{\text{inv}}(\tau)} \tilde{\chi}^{\text{NS}}(z)
 \end{aligned} \tag{A.23}$$

and

$$\begin{aligned}
 T_{\ell=0}(z) &\stackrel{(2.19)}{=} \frac{1}{2} \left\{ \frac{\vartheta_2(z)^2}{\vartheta_4^2} - \frac{\vartheta_1(z)^2}{\vartheta_3^2} \right\} \\
 &\stackrel{(A.16a), (A.16b)}{=} \frac{1}{2} \left\{ \chi_{\frac{1}{2}}^{\text{NS}}(z) - h_4 \eta q^{\frac{1}{8}} \tilde{\chi}^{\text{NS}}(z) \right\} + \frac{1}{2} \left\{ \chi_{\frac{1}{2}}^{\text{NS}}(z) - h_3 \eta q^{\frac{1}{8}} \tilde{\chi}^{\text{NS}}(z) \right\} \\
 &= \underbrace{\chi_{\frac{1}{2}}^{\text{NS}}(z) - \frac{1}{2} (h_3 + h_4) \eta q^{\frac{1}{8}} \tilde{\chi}^{\text{NS}}(z)}_{=g^{\text{tw}}(\tau)}.
 \end{aligned} \tag{A.24}$$

A.5 The functions $f(\tau, \nu)$, $g^{\text{inv}}(\tau, \nu)$ and $g^{\text{tw}}(\tau, \nu)$

From (3.2a)–(3.2c) and using (3.1a)–(3.1c) together with (A.16a)–(A.16c) as well as some theta function identities obtained from (A.3) and (A.5),

$$\begin{aligned}
 f(\nu) &= -1 + q^{\frac{1}{8}} \frac{\vartheta_2^2 \vartheta_3^2}{4\eta^3} \left\{ \frac{\vartheta_4(\nu)^2}{\vartheta_1(\nu)^2} + \frac{\vartheta_3(\nu)^2}{\vartheta_2(\nu)^2} \right\} - 2q^{\frac{1}{8}} \left\{ \frac{\vartheta_2^4}{4\eta^3} - h_3 \eta \right\} \\
 &\quad - (w + w^{-1}) q^{\frac{1}{8}} \frac{\vartheta_2^2 \vartheta_3^2}{8\eta^3} \left\{ \frac{\vartheta_4(\nu)^2}{\vartheta_1(\nu)^2} - \frac{\vartheta_3(\nu)^2}{\vartheta_2(\nu)^2} \right\},
 \end{aligned} \tag{A.25a}$$

$$\begin{aligned}
 g^{\text{inv}}(\nu) &= q^{\frac{1}{8}} \frac{\vartheta_2^2 \vartheta_3^2}{2\eta^3} \left\{ \frac{\vartheta_4(\nu)^2}{\vartheta_1(\nu)^2} - \frac{\vartheta_3(\nu)^2}{\vartheta_2(\nu)^2} \right\} + (w + w^{-1}) q^{\frac{1}{8}} \left\{ \frac{\vartheta_2^4}{2\eta^3} - 2\eta h_3 \right\} \\
 &\quad - (w + w^{-1}) q^{\frac{1}{8}} \frac{\vartheta_2^2 \vartheta_3^2}{4\eta^3} \left\{ \frac{\vartheta_4(\nu)^2}{\vartheta_1(\nu)^2} + \frac{\vartheta_3(\nu)^2}{\vartheta_2(\nu)^2} \right\},
 \end{aligned} \tag{A.25b}$$

$$g^{\text{tw}}(\nu) = q^{\frac{1}{8}} \left\{ \frac{\vartheta_2^4}{4\eta^3} - h_3 \eta \right\} - q^{\frac{1}{8}} \frac{\vartheta_2^2 \vartheta_3^2}{8\eta^3} \left\{ \frac{\vartheta_1(\nu)^2}{\vartheta_4(\nu)^2} + \frac{\vartheta_2(\nu)^2}{\vartheta_3(\nu)^2} \right\}. \tag{A.25c}$$

One now uses some of the relations (A.12) and (A.13) between Appell functions as well as the theta function identities (A.2) and (A.3) to obtain (3.5a)–(3.5c).

We also note that the functions $f(\nu)$, $g^{\text{inv}}(\nu)$ and $g^{\text{tw}}(\nu)$ may be written in terms of the massless characters of an $N = 4$ SCA at central charge $c = 6$. Taking advantage

of (A.17), (A.18), (A.19) and (A.20), one has

$$f(\nu) = -\frac{1}{2} \left[\frac{\chi_{\frac{1}{2}}^{\tilde{R}}(\nu)}{\tilde{\chi}^{\tilde{R}}(\nu)} + \frac{\chi_{\frac{1}{2}}^R(\nu)}{\tilde{\chi}^R(\nu)} \right] - \frac{1}{2}(w + w^{-1}) \left[\frac{\chi_0^{\tilde{R}}(\nu)}{\tilde{\chi}^{\tilde{R}}(\nu)} - \frac{\chi_0^R(\nu)}{\tilde{\chi}^R(\nu)} \right], \quad (\text{A.26})$$

$$g^{\text{inv}}(\nu) = - \left[\frac{\chi_{\frac{1}{2}}^{\tilde{R}}(\nu)}{\tilde{\chi}^{\tilde{R}}(\nu)} - \frac{\chi_{\frac{1}{2}}^R(\nu)}{\tilde{\chi}^R(\nu)} \right] - (w + w^{-1}) \left[\frac{\chi_0^{\tilde{R}}(\nu)}{\tilde{\chi}^{\tilde{R}}(\nu)} + \frac{\chi_0^R(\nu)}{\tilde{\chi}^R(\nu)} \right], \quad (\text{A.27})$$

$$g^{\text{tw}}(\nu) = -\frac{1}{2} \left[\frac{\chi_{\frac{1}{2}}^{\tilde{\text{NS}}}(\nu)}{\tilde{\chi}^{\tilde{\text{NS}}}(\nu)} + \frac{\chi_{\frac{1}{2}}^{\text{NS}}(\nu)}{\tilde{\chi}^{\text{NS}}(\nu)} \right]. \quad (\text{A.28})$$

A.6 Fourier coefficients of $f(\tau, \nu)$ and $g^{\text{tw}}(\tau, \nu)$

The data in table 3 is presented in support of our claim that under deformation of our \mathbb{Z}_2 -orbifold CFT on K3 by T^{diag} , only $\frac{1}{4}$ -BPS ground states stemming from the diagonal twisted sector pair up with those in the untwisted subsector accounted for by $U_{\ell=\frac{1}{2}}$ to form long representations off the BPS bound. For reference, we also present data in table 4 relating to the action of $\text{SU}(2)_{\text{geom}}$ on $\frac{1}{4}$ -BPS ground states from the untwisted subsector accounted for by $U_{\ell=0}$. We also note that for the action of the diagonal $\text{SU}(2)$ in $\text{SU}(2)_{\text{geom}} \times \overline{\text{SU}(2)}_{\text{geom}}$ (see section 3 and appendix B.2), the information in table 5 should be used. However, at level 1 (resp. 2), the two triplets (resp. the two singlets) from the untwisted subsector accounted for by $U_{\ell=\frac{1}{2}}$ only match the $\text{SU}(2)$ representation contributions of the $\frac{1}{4}$ -BPS ground states in the diagonal twisted sector: at level $n = 1$, there are just no states available in the untwisted subsector $U_{\ell=0}$, and there are no singlets in that sector at level $n = 2$.

$n \backslash p$	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
$1(g^{\text{tw}})$															3^2	$g_1^{\text{tw}} = 6$
$1(f)$														5^4	3	$f_1 = 3$
$2(g^{\text{tw}})$															3^2	$g_2^{\text{tw}} = 28$
$2(f)$															1	$f_2 = 1$
$3(g^{\text{tw}})$													7^6	5^6	3^8	$g_3^{\text{tw}} = 98$
$3(f)$													7^{10}	5^3	3	$f_3 = 18$
$4(g^{\text{tw}})$										9^8				5^{18}	3^{14}	$g_4^{\text{tw}} = 282$
$4(f)$													7^{30}	5	$f_4 = 15$	
$5(g^{\text{tw}})$										11^{10}	9^{14}		7^{30}	5^{34}	3^3	$g_5^{\text{tw}} = 728$
$5(f)$													7^5	5^3	3^5	$f_5 = 68$
$6(g^{\text{tw}})$									13^{12}	11^{18}	9^{42}		7^{60}	5^{76}	3^{58}	$g_6^{\text{tw}} = 1734$
$6(f)$													7^3	5^9	3^7	$f_6 = 89$
$7(g^{\text{tw}})$									15^{14}	13^{22}	11^{54}	9^{86}	7^{130}	5^{138}	3^{120}	$g_7^{\text{tw}} = 3864$
$7(f)$												9^7	7^9	5^{15}	3^{14}	$f_7 = 249$
$8(g^{\text{tw}})$									17^{16}	15^{26}	13^{66}	11^{112}	7^{240}	5^{272}	3^{204}	$g_8^{\text{tw}} = 8182$
$8(f)$												9^5	7^{18}	5^{22}	3^{22}	$f_8 = 358$
$9(g^{\text{tw}})$									19^{18}	17^{30}	15^{246}	13^{354}	7^{468}	5^{472}	3^{380}	$g_9^{\text{tw}} = 16618$
$9(f)$												11^9	7^{30}	5^{45}	3^{39}	$f_9 = 799$
$10(g^{\text{tw}})$									19^{34}	17^{90}	15^{164}	13^{304}	7^{824}	5^{856}	3^{634}	$g_{10}^{\text{tw}} = 32550$
$10(f)$												11^7	7^{50}	5^{67}	3^{60}	$f_{10} = 1236$
$11(g^{\text{tw}})$									19^{102}	17^{190}	15^{362}	13^{582}	7^{1482}	5^{1446}	3^{1104}	$g_{11}^{\text{tw}} = 61828$
$11(f)$												13^{11}	7^{95}	5^{112}	3^{97}	$f_{11} = 2419$
$12(g^{\text{tw}})$									19^{216}	17^{420}	15^{696}	13^{1140}	7^{2518}	5^{2474}	3^{1800}	$g_{12}^{\text{tw}} = 114352$
$12(f)$												13^9	7^{144}	5^{181}	3^{150}	$f_{12} = 3780$
$13(g^{\text{tw}})$									19^{478}	17^{810}	15^{1366}	13^{2054}	7^{4290}	5^{4054}	3^{3000}	$g_{13}^{\text{tw}} = 206526$
$13(f)$												15^{13}	7^{245}	5^{286}	3^{240}	$f_{13} = 6801$
$14(g^{\text{tw}})$									19^{924}	17^{1592}	15^{2470}	13^{3714}	7^{7050}	5^{6678}	3^{4778}	$g_{14}^{\text{tw}} = 365232$
$14(f)$												15^{11}	7^{386}	5^{436}	3^{359}	$f_{14} = 10659$
$15(g^{\text{tw}})$									19^{1818}	17^{2886}	15^{4476}	13^{6408}	7^{11572}	5^{10656}	3^{7690}	$g_{15}^{\text{tw}} = 633820$
$15(f)$												17^{15}	7^{608}	5^{694}	3^{551}	$f_{15} = 18137$

Table 3. $SU(2)_{\text{geom}}$ representations and multiplicities at level $n, 1 \leq n \leq 15$. An entry of type b^m at level n is understood as multiplicity $m = g_{n,p}^{\text{tw}}$ or $m = f_{n,p}$ of $SU(2)_{\text{geom}}$ representation of dimension $b = 2p + 1$.

$n \backslash p$	15/2	13/2	11/2	9/2	7/2	5/2	3/2	1/2	
1									$g_1^{\text{inv}} = 0$
2							4^4		$g_2^{\text{inv}} = 16$
3								2^4	$g_3^{\text{inv}} = 8$
4						6^8	4^4	2^4	$g_4^{\text{inv}} = 72$
5						6^4	4^{12}	2^4	$g_5^{\text{inv}} = 80$
6					8^{12}	6^{12}	4^{16}	2^{16}	$g_6^{\text{inv}} = 264$
7					8^8	6^{24}	4^{28}	2^{20}	$g_7^{\text{inv}} = 360$
8				10^{16}	8^{24}	6^{44}	4^{56}	2^{32}	$g_8^{\text{inv}} = 904$
9				10^{12}	8^{48}	6^{72}	4^{76}	2^{60}	$g_9^{\text{inv}} = 1360$
10			12^{20}	10^{36}	8^{84}	6^{132}	4^{140}	2^{92}	$g_{10}^{\text{inv}} = 2808$
11			12^{16}	10^{72}	8^{132}	6^{204}	4^{224}	2^{136}	$g_{11}^{\text{inv}} = 4360$
12		14^{24}	12^{48}	10^{132}	8^{252}	6^{344}	4^{348}	2^{236}	$g_{12}^{\text{inv}} = 8176$
13		14^{20}	12^{96}	10^{216}	8^{392}	6^{540}	4^{540}	2^{344}	$g_{13}^{\text{inv}} = 12816$
14	16^{28}	14^{60}	12^{180}	10^{400}	8^{656}	6^{864}	4^{860}	2^{524}	$g_{14}^{\text{inv}} = 22368$
15	16^{24}	14^{120}	12^{300}	10^{624}	8^{1044}	6^{1320}	4^{1272}	2^{812}	$g_{15}^{\text{inv}} = 34888$

Table 4. $SU(2)_{\text{geom}}$ representations and multiplicities at level $n, 1 \leq n \leq 15$. An entry of type b^m at level n is understood as multiplicity $m = g_{n,p}^{\text{inv}}$ of $SU(2)_{\text{geom}}$ representation of dimension $b = 2p + 1$.

B Quarter BPS states at level 2

B.1 Twisted sector

Note that $|\alpha_{\text{diag}}\rangle$ is invariant under $SU(2)_{\text{geom}} \times \overline{SU(2)}_{\text{geom}}$. There is a 28-dimensional space of massive $\frac{1}{4}$ -BPS states in the diagonal twisted sector at level 2 which accounts for g_2^{tw} and which transforms trivially under $\overline{SU(2)}_{\text{geom}}$. We have already presented two $SU(2)_{\text{geom}}$ -singlets in (2.31) which actually are the only singlets within the 28-dimensional space in question. The remaining 26-dimensional space transforms as two triplets and four quintuplets of $SU(2)_{\text{geom}}$. We use the notations introduced in (2.30) to give an explicit expression for these states.

The two triplets.

$$\{|t_1\rangle, |t_2\rangle, |t_3\rangle\} \quad \text{and} \quad \{|\tilde{t}_1\rangle, |\tilde{t}_2\rangle, |\tilde{t}_3\rangle\}, \quad (\text{B.1})$$

$n \backslash p$	8	7	6	5	4	3	2	1	0	
1										$g_1^{\text{inv}} = 0$
2							5^2	3^2		$g_2^{\text{inv}} = 16$
3								3^2	1^2	$g_3^{\text{inv}} = 8$
4						7^4	5^6	3^4	1^2	$g_4^{\text{inv}} = 72$
5						7^2	5^8	3^8	1^2	$g_5^{\text{inv}} = 80$
6					9^6	7^{12}	5^{14}	3^{16}	1^8	$g_6^{\text{inv}} = 264$
7					9^4	7^{16}	5^{26}	3^{24}	1^{10}	$g_7^{\text{inv}} = 360$
8				11^8	9^{20}	7^{34}	5^{50}	3^{44}	1^{16}	$g_8^{\text{inv}} = 904$
9				11^6	9^{30}	7^{60}	5^{74}	3^{68}	1^{30}	$g_9^{\text{inv}} = 1360$
10			13^{10}	11^{28}	9^{60}	7^{108}	5^{136}	3^{116}	1^{46}	$g_{10}^{\text{inv}} = 2808$
11			13^8	11^{44}	9^{102}	7^{168}	5^{214}	3^{180}	1^{68}	$g_{11}^{\text{inv}} = 4360$
12		15^{12}	13^{36}	11^{90}	9^{192}	7^{298}	5^{346}	3^{292}	1^{118}	$g_{12}^{\text{inv}} = 8176$
13		15^{10}	13^{58}	11^{156}	9^{304}	7^{466}	5^{540}	3^{442}	1^{172}	$g_{13}^{\text{inv}} = 12816$
14	17^{14}	15^{44}	13^{120}	11^{290}	9^{528}	7^{760}	5^{862}	3^{692}	1^{262}	$g_{14}^{\text{inv}} = 22368$
15	17^{12}	15^{72}	13^{210}	11^{462}	9^{834}	7^{1182}	5^{1296}	3^{1042}	1^{406}	$g_{15}^{\text{inv}} = 34888$

Table 5. $SU(2)_{\text{geom}} \times \overline{SU(2)}_{\text{geom}}$ representations and multiplicities at level n , $1 \leq n \leq 15$. An entry of type b^m at level n is understood as multiplicity $m = g_{n,p}^{\text{inv}}$ of the representation of dimension $b = 2p + 1$ of the diagonal $SU(2)$ in $SU(2)_{\text{geom}} \times \overline{SU(2)}_{\text{geom}}$.

with

$$\begin{aligned}
 |t_1\rangle &= \left((a_+^1)_{-\frac{3}{2}} (\chi_-^2)_0 + 3(a_+^1)_{-\frac{1}{2}} (\chi_-^2)_{-1} + 3(a_+^1)_{-\frac{1}{2}} (\chi_+^1)_{-1} (\chi_-^1)_0 (\chi_-^2)_0 - 6A_{++-}^{1112} \right. \\
 &\quad \left. - 6A_{++-}^{1222} \right) |\alpha_{\text{diag}}\rangle, \\
 |t_2\rangle &= \left(\sum_{k=1}^2 (-1)^k \left((a_+^k)_{-\frac{3}{2}} (\chi_-^k)_0 + 3(a_+^k)_{-\frac{1}{2}} (\chi_-^k)_{-1} \right) + 3 \sum_{k=1}^2 (a_+^k)_{-\frac{1}{2}} (\chi_+^{\check{k}})_{-1} (\chi_-^1)_0 (\chi_-^2)_0 \right. \\
 &\quad \left. - 6 \sum_{k=1}^2 (-1)^k (A_{++-}^{kkkk} + A_{++-}^{k\check{k}k\check{k}}) \right) |\alpha_{\text{diag}}\rangle, \\
 |t_3\rangle &= \left(- (a_+^2)_{-\frac{3}{2}} (\chi_-^1)_0 - 3(a_+^2)_{-\frac{1}{2}} (\chi_-^1)_{-1} + 3(a_+^2)_{-\frac{1}{2}} (\chi_+^2)_{-1} (\chi_-^1)_0 (\chi_-^2)_0 + 6A_{++-}^{2221} \right. \\
 &\quad \left. + 6A_{++-}^{2111} \right) |\alpha_{\text{diag}}\rangle, \tag{B.2}
 \end{aligned}$$

and

$$\begin{aligned}
 |\tilde{t}_1\rangle &= \left((a_-^2)_{-\frac{3}{2}} (\chi_-^2)_0 + 3(a_-^2)_{-\frac{1}{2}} (\chi_-^2)_{-1} + 3(a_-^2)_{-\frac{1}{2}} (\chi_+^1)_{-1} (\chi_-^1)_0 (\chi_-^2)_0 - 6A_{+--+}^{2222} \right. \\
 &\quad \left. - 6A_{+--+}^{1122} \right) |\alpha_{\text{diag}}\rangle, \\
 |\tilde{t}_2\rangle &= \left(\sum_{k=1}^2 \left((a_-^k)_{-\frac{3}{2}} (\chi_-^{\check{k}})_0 + 3(a_-^k)_{-\frac{1}{2}} (\chi_-^{\check{k}})_{-1} \right) - 3 \sum_{k=1}^2 (-1)^k (a_-^k)_{-\frac{1}{2}} (\chi_+^k)_{-1} (\chi_-^1)_0 (\chi_-^2)_0 \right. \\
 &\quad \left. - 6 \sum_{k=1}^2 (A_{+--+}^{k\check{k}k\check{k}} + A_{+--+}^{k\check{k}k\check{k}}) \right) |\alpha_{\text{diag}}\rangle, \\
 |\tilde{t}_3\rangle &= \left((a_-^1)_{-\frac{3}{2}} (\chi_-^1)_0 + 3(a_-^1)_{-\frac{1}{2}} (\chi_-^1)_{-1} - 3(a_-^1)_{-\frac{1}{2}} (\chi_+^2)_{-1} (\chi_-^1)_0 (\chi_-^2)_0 - 6A_{+--+}^{1111} \right. \\
 &\quad \left. - 6A_{+--+}^{2211} \right) |\alpha_{\text{diag}}\rangle, \tag{B.3}
 \end{aligned}$$

where $\check{k} := 3 - k$.

The four quintuplets.

$$\begin{aligned}
 &\{ A_{++++}^{1112} |\alpha_{\text{diag}}\rangle, (A_{++++}^{1111} - 3A_{++++}^{1122}) |\alpha_{\text{diag}}\rangle, (A_{++++}^{1222} - A_{++++}^{2111}) |\alpha_{\text{diag}}\rangle, \\
 &\quad (A_{++++}^{2222} - 3A_{++++}^{2211}) |\alpha_{\text{diag}}\rangle, A_{++++}^{2221} |\alpha_{\text{diag}}\rangle \}, \\
 &\{ A_{----}^{2222} |\alpha_{\text{diag}}\rangle, (A_{----}^{2221} + 3A_{----}^{1222}) |\alpha_{\text{diag}}\rangle, (A_{----}^{1122} + A_{----}^{2211}) |\alpha_{\text{diag}}\rangle, \\
 &\quad (A_{----}^{1112} + 3A_{----}^{2111}) |\alpha_{\text{diag}}\rangle, A_{----}^{1111} |\alpha_{\text{diag}}\rangle \}, \\
 &\{ A_{+++-}^{1122} |\alpha_{\text{diag}}\rangle, (A_{+++-}^{1112} + A_{+++-}^{1121} - 2A_{+++-}^{1222}) |\alpha_{\text{diag}}\rangle, \\
 &\quad (A_{+++-}^{1111} + A_{+++-}^{2222} - 2A_{+++-}^{2112} - 2A_{+++-}^{1221}) |\alpha_{\text{diag}}\rangle, \\
 &\quad (A_{+++-}^{2221} + A_{+++-}^{2212} - 2A_{+++-}^{2111}) |\alpha_{\text{diag}}\rangle, A_{+++-}^{2211} |\alpha_{\text{diag}}\rangle \}, \\
 &\{ A_{+---}^{1222} |\alpha_{\text{diag}}\rangle, (A_{+---}^{2222} - A_{+---}^{1221} - 2A_{+---}^{1122}) |\alpha_{\text{diag}}\rangle, \\
 &\quad (A_{+---}^{1112} - A_{+---}^{2221} + 2A_{+---}^{1211} - 2A_{+---}^{2122}) |\alpha_{\text{diag}}\rangle, \\
 &\quad (A_{+---}^{1111} - A_{+---}^{2112} - 2A_{+---}^{2211}) |\alpha_{\text{diag}}\rangle, A_{+---}^{2111} |\alpha_{\text{diag}}\rangle \}. \tag{B.4}
 \end{aligned}$$

B.2 Untwisted sector

Besides the two singlets (2.29), the untwisted sector at level 2 contains a 16-dimensional space of massive $\frac{1}{4}$ -BPS states contributing to g_2^{inv} . They are $N = 4$ primaries built as eight odd combinations of oscillator modes acting on the \mathbb{Z}_2 -orbifold odd ground states $\tilde{\Omega}^1 := (\bar{\chi}_+^1)_{-\frac{1}{2}} \Omega$ and $\tilde{\Omega}^2 := (\bar{\chi}_+^2)_{-\frac{1}{2}} \Omega$. These two states transform non-trivially under our geometric symmetry groups G_i . Indeed, as argued in section 3, they are invariant under the action of $\text{SU}(2)_{\text{geom}}$ and form a doublet under the action of $\overline{\text{SU}(2)}_{\text{geom}}$.

On the other hand, we find four quadruplet representations of $\text{SU}(2)_{\text{geom}}$ at level 2 in this sector. We introduce the notation

$$B_{\pm}^{k\ell m} := (a_{\pm}^k)_{-1} (\chi_+^{\ell})_{-\frac{1}{2}} (\chi_-^m)_{-\frac{1}{2}}, \quad k, \ell, m \in \{1, 2\}, \tag{B.5}$$

so that the 16-dimensional space is generated by the following 16 states:

$$\begin{aligned}
 &B_+^{112} \tilde{\Omega}^i, \quad (B_+^{122} + B_+^{212} - B_+^{111}) \tilde{\Omega}^i, \quad (B_+^{211} + B_+^{121} - B_+^{222}) \tilde{\Omega}^i, \quad B_+^{221} \tilde{\Omega}^i, \\
 &B_-^{212} \tilde{\Omega}^i, \quad (B_-^{112} + B_-^{211} - B_-^{222}) \tilde{\Omega}^i, \quad (B_-^{221} + B_-^{122} - B_-^{111}) \tilde{\Omega}^i, \quad B_-^{121} \tilde{\Omega}^i. \tag{B.6}
 \end{aligned}$$

where each row generates a $(4, 2)$ representation of $SU(2)_{\text{geom}} \times \overline{SU(2)}_{\text{geom}}$. Note that under the action of the diagonal subgroup $SU(2)$ of $SU(2)_{\text{geom}} \times \overline{SU(2)}_{\text{geom}}$, we obtain two copies of $\mathbf{4} \otimes \mathbf{2} \cong \mathbf{3} \oplus \mathbf{5}$, yielding no singlets altogether.

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