The Boundary Element Method Applied to the Solution of the Diffusion-Wave Problem

by

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Abstract

A Boundary Element Method formulation is developed for the solution of the two-dimensional diffusion-wave problem, which is governed by a partial differential equation presenting a time fractional derivative of order α, with $1.0 < \alpha <\leq 2.0$. In the proposed formulation, the fractional derivative is transferred to the Laplacian through the Riemann-Liouville integro-differential operator; then, the basic integral equation of the method is obtained through the Weighted Residual Method, with the fundamental solution of the Laplace equation as the weighting function. In the final expression, the presence of additional terms containing the history contribution of the boundary variables constitutes the main difference between the proposed formulation and the standard one. The proposed formulation, however, works well for $1.5 \leq \alpha < 2.0$, producing results with good agreement with the analytical solutions and with the Finite Difference ones.

1. Introduction

The diffusion-wave problem is governed by a fractional differential equation in which the time derivative of the classical wave propagation problem is substituted by an integro-differential operator that interpolates the integer order derivatives. In other words: the diffusion-wave problem is governed by a partial differential equation with a time derivative of order α, in which $1 < \alpha < 2$. When $\alpha = 2$ one obtains the classical wave propagation problem, that can be treated as a particular case of the more general problem represented by the diffusion-wave equation. From the above, it is readily seen that the diffusion-wave problem belongs to the domain of the fractional calculus. Although the name *fractional calculus* became a somewhat of a misnomer, see Miller and Ross [1], as generalized operators can include rational or irrational, positive or negative, real or complex orders, it was kept due to the tradition in the textbooks, e.g. Ortigueira [2], and also in the present work. The fractional time derivative is represented by the Caputo derivative, see Ortigueira [2]. In what concerns the diffusion-wave equation, it has been employed to the solution of viscoelastic problems, e.g. Mainardi and Paradisi [3,4]. A great deal of attention has been given lately to the study of fractional differential equations, as many physical processes are more accurately modelled by these equations. The reader will find a lot of applications of the fractional differential equations in the works by El-Saka and Ahmed [5], Dalir and Bashour [6], Sun et al. [7].

Among the applications listed by Sun et al. [7] one can find, among others, those related to physics, signal and image processing, mechanics and dynamic systems, biology, and environmental science.

This work is concerned with the development of a BEM formulation for the solution of the two-dimensional diffusion-wave problem. The use of the Laplace equation fundamental solution, called from now on simply Laplace fundamental solution, generates a D-BEM type formulation, which for the problem to be solved can be named DDW-BEM, with the first D meaning domain, and DW meaning diffusion-wave: note that the name diffusion-wave implicitly implies that the problem belongs to the fractional calculus domain. In the absence of a proper fundamental solution, this is the most straightforward approach to be followed. Another BEM formulation, based on a quite similar approach, for the solution of the anomalous diffusion problem, in which $0 < \alpha < 1$, was recently developed by the authors: the promising results in Carrer et al. [8], allied to the previous authors' experience in dealing with these kinds of problems, encouraged the development of the present formulation. After the choice of the fundamental solution and, consequently, of the type of the formulation to be developed, the fractional derivative is transferred from the time to the space variables. This is accomplished through the application of the Riemann-Liouville integro-differential operator to all terms of the differential equation, see Ortigueira [2]. In the next step, familiar to all BEM researchers, the basic integral equation is obtained through the Weighted Residual Method, see Brebbia et al. [9]. The presence of additional terms containing the history contribution of the boundary variables, in the final expression of the BEM integral equation, constitutes the main difference between the proposed formulation and the standard one. This was already expected, as while the operators in the standard calculus are local, fractional operators are non-local, which means that the next state of a given system depends not only on its current but also on its previous states, that is, depends on the history. Naturally, the solution of this kind of problem is more time-consuming than the solution of the classical one. If the order of the time derivative is assumed to be equal to 2, the classical wave propagation equation arises.

The Finite Difference Method (FDM) has been widely used for the solution of fractional partial differential equations, and most of the FDM works are also concerned with the solution of the anomalous diffusion equation, for which $0 < \alpha < 1$: Murillo and Youste [10] presented explicit FDM for fractional diffusion and diffusion-wave equations; Youste and Acedo [11], Murio [12], presented, respectively, explicit and implicit FDM for fractional diffusion equation. The works by Meerschaert and Tadjeran [13], Li and Li [14], deal with space fractional advection-diffusion equation. The Finite Element Method (FEM) formulations appear in small number. The works by Deng [15], Roop [16], and Huang et al. [17], must be cited. In relation to the BEM, the works by Katsikadelis [18] and Dehghan and Safarpoor [19] must be cited. Katsikadelis [18] presented a method based on the concept of the analog equation for the solution of differential equations that contain Caputo fractional time derivatives of order α, with $0 < \alpha \le 1$, and β, with $1 < \beta \le 2$, representing fractional type damping and inertia forces, respectively. Dehghan and Safarpoor [19] presented the solution of the time-fractional modified anomalous subdiffusion equation, that contains two Riemann-Liouville fractional derivative operators of orders $(1 - \alpha)$ and $(1 - \beta)$, with $1 < \alpha,\beta < 1$, and the solution of the time-fractional convection–diffusion equation with constant coefficients, that contains the Caputo fractional derivative operator of order α, with $1.0 < \alpha \leq 2.0$.

After this brief exposition, it is possible to state now that the main novelty of this work is the DDW-BEM formulation itself. It is important to mention that some features of DDW-BEM turn it attractive and advantageous when compared with other numerical formulations; for instance: *i*) the simplicity of the fundamental solution and, consequently, of the matrices coefficients generated by the boundary and domain integrations, *ii*) the suitability for the solution of non-linear problems, *iii*) the computation of the boundary variable $q = \mathrm{d}u/\mathrm{d}n$ with the same level of accuracy of the field variable u (see, regarding this feature, the comparison between BEM and FDM results for q in Figure 9). Regardless these positive features, the authors think that it is important to provide the researches with as many numerical tools as possible for the solution of the most diverse kinds of problems.

For brevity, the numerical results from the DDW-BEM formulation, as there is no room for doubt, will be called simply BEM results throughout the text. To obtain these results, linear elements and triangular linear cells were employed, respectively, for the boundary and for the domain approximations. Linear elements and triangular linear cells were adopted in other works by the authors, see references [8], [20]; therefore, taking advantage of the authors' experience they were adopted in the formulation developed here. The Houbolt method, see Houbolt [21], was employed here for the time approximation. A Finite Difference Method (FDM) formulation was also developed to play the role of an auxiliary formulation and BEM and FDM results are compared with the analytical solutions in all the examples, showing a good agreement between them. It is important to mention that the application of both the DDW-BEM and FDM formulations is limited to the range $1.5 \leq \alpha \leq 2.0$. This limitation resembles that of the formulation developed for the solution of anomalous diffusion problems, where the restriction was limited to the interval $0.5 \leq \alpha \leq 1.0$, see Carrer et al. [8]. The development of an alternative BEM formulation, capable of overcoming this limitation and being able to solve problems covering all the range $1.0 \leq \alpha \leq 2$ is a very important and challenging task.

A strong dependency between the time-step length and the order of the time derivative seems to be a common feature regarding the numerical solution of problems presenting fractional derivatives, no matter the method adopted. A simple rule can be stated as: for smaller values of α, much smaller must be the selected values of Δt. This statement is confirmed in all the examples.

2. Basic concepts and the DDW-BEM formulation

2.1. Fractional Derivatives

For a given function, say *f*, Caputo and Riemann-Liouville fractional derivatives are defined, respectively, as:

$$\frac{\partial_C^\alpha f}{\partial t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{1}{(t-\tau)^{1+\alpha-n}} \frac{\partial^n f(\tau)}{\partial \tau^n} d\tau \tag{1}$$

and

$$\frac{\partial_{RL}^\alpha f}{\partial t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{1+\alpha-n}} d\tau \tag{2}$$

In equations (1) and (2), $n-1 < \alpha < n$, and $\Gamma(\ldots)$ is the Gamma function. Note that, from the definitions given by Equations (1) and (2), both Caputo and Riemann-Liouville are integro-differential operators in which the sequence integration-differentiation is inverted. When the initial conditions are null, both definitions coincide, see Ortigueira [2].

If $n = 1$, then $0 < \alpha < 1$, and one has the fractional diffusion equation, also called sub-diffusion equation or anomalous diffusion equation, used to describe diffusion processes when the diffusion is anomalous. If $n = 2$, one has the fractional diffusion-wave equation, used to describe viscoelastic problems, and $1 < \alpha < 2$.

2.2. The Diffusion-wave Problem

The governing equation for the diffusion-wave problem, for 2D problems, is written as:

$$\frac{1}{c^2}\frac{\partial_C^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad , \qquad 1 < \alpha < 2 \tag{3}$$

where c is the wave propagation velocity. Here, x, y and t are the space-time variables, and $u = u(x,y,t)$ is the field variable, which is assumed to be a causal function of time, i.e., vanishing for $t < 0$. In the next equations, a point of coordinates (x,y) is represented by X, that is, $X = (x,y)$; consequently, one has $u = u(X,t)$.

Before proceeding, a brief comment, regarding Equation (3), seems to be necessary, as one of the oldest applications of the fractional calculus is viscoelasticity: Equation (3) governs the propagation of stress waves in viscoelastic solids, which exhibit a power law creep of degree p with $0 < p < 1$, provided that $1 < \alpha < 2$ and $\alpha = 2 - p$, see Mainardi and Paradisi [3].

If the boundary Γ of the domain Ω is represented as: $\Gamma = \Gamma_u \cup \Gamma_q$, then the boundary conditions can be defined as follows:

$$\text{Dirichlet boundary condition: } u(X,t) = \hat{u}(X,t), \text{ over } \Gamma_u \tag{4}$$

and

$$\text{Neumann boundary condition: } \frac{du(X,t)}{dn(X)} = q(X,t) = \hat{q}(X,t), \text{ over } \Gamma_q \tag{5}$$

The initial conditions are:

$$u(X,0) = u_0(X) \tag{6}$$

and

$$\frac{\partial u(X,t)}{\partial t}|_{t=0}=\dot{u}_0(X) \tag{7}$$

To replace the fractional time derivative in the left-hand-side of Equation (3) by an integer derivative of order 2, the following property of the Riemann-Liouville operator:

$$\frac{\partial_{RL}^{2-\alpha}}{\partial t^{2-\alpha}}\left(\frac{\partial_C^{\alpha} u(X,t)}{\partial t^{\alpha}}\right)=\frac{\partial^2 u(X,t)}{\partial t^2} \tag{8}$$

is employed in both sides of Equation (3). The resulting equation reads:

$$\frac{1}{c^2}\frac{\partial^2 u(X,t)}{\partial t^2}=\frac{\partial_{RL}^{2-\alpha}}{\partial t^{2-\alpha}}\left(\frac{\partial^2 u(X,t)}{\partial x^2}+\frac{\partial^2 u(X,t)}{\partial y^2}\right)=\frac{1}{\Gamma(\alpha)}\frac{d^2}{dt^2}\int_0^t\frac{1}{(t-\tau)^{1-\alpha}}\left(\frac{\partial^2 u(X,\tau)}{\partial x^2}+\frac{\partial^2 u(X,\tau)}{\partial y^2}\right)d\tau \tag{9}$$

Then, as mentioned at the Introduction, Equation (9) is the equation to be solved by the BEM, or rather, by the DDW-BEM formulation.

3. The Boundary Element Method

The basic BEM integral equation was obtained through the Weighting Residuals Method (WRM), with the Laplace fundamental solution, $w = w(\xi, X)$, playing the role of the weighting function for the domain residuals. The weighting functions for the residuals at Γ_u and Γ_q are chosen in such a way that approximations to the boundary conditions are avoided. In the weighting functions, $\xi = (\xi_x, \xi_y)$ is called the source point and $X = (x,y)$, the field point. As the steps required to arrive at the DDW-BEM integral equation are quite similar to those presented by Carrer et al. [8], and that a comprehensive discussion concerning this matter can be found there, the DDW-BEM integral equation will be presented directly, without any loss to the interested reader. It reads:

$$
\begin{aligned}
&c(\xi)u_{n+1}(\xi) = \int_\Gamma q_{n+1} w d\Gamma - \int_\Gamma u_{n+1} Q d\Gamma - \\
&\frac{\Gamma(\alpha)}{c^2(\alpha-1)\Delta t^\alpha}\int_\Omega \ddot{u}_{n+1} w d\Omega - \sum_{j=0}^{n-1} B_{(n+1)(j+1)}\left[c(\xi)u_{n+1}(\xi) + \int_\Gamma u_{j+1} Q d\Gamma - \int_\Gamma q_{j+1} w d\Gamma\right]
\end{aligned}
\tag{10}
$$

To shorten Equation (10), the following notation is employed:

$$
\begin{aligned}
u_{n+1} &= u(X, t_{n+1}) \\
q_{n+1} &= q(X, t_{n+1})
\end{aligned}
\tag{11}
$$

with $t_{n+1}=(n+1)\Delta t$, Δt is the time-step,

$$
Q = Q(\xi, X) = \frac{dw(\xi, X)}{dn(X)} = \frac{\partial w}{\partial x} n_x(X) + \frac{\partial w}{\partial y} n_y(X)
\tag{12}
$$

and

$$
B_{(n+1)(j+1)} = \frac{1}{(n+1-j)^{2-\alpha}} - \frac{1}{(n-j)^{2-\alpha}}
\tag{13}
$$

Note that the coefficient $c(\xi)$ in the term between brackets in Equation (10) is the same that appears in the standard BEM formulation.

The non-local behaviour of the fractional operators is responsible for the presence, in Equation (10), of the summation symbol, that is, the computation of the variables *u* and *q* at a given time depends not only on their current values, but also on all their previous values; in other words, it depends on the history.

The presence of a domain integral in Equation (10), whose integrand contains the second order time derivative of the variable of interest, requires the domain discretization.

Finally, Houbolt method, see Houbolt [21], is used to approximate $\ddot{u}_{n+1}$ in the domain integral on the left-hand-side in Equation (10). The Houbold method comes from the differentiation at time t_{n+1} of the cubic Lagrange interpolation of $u = u(t)$ from time $t_{n-2} = (n-2)\Delta t$ to the time $t_{n+1} = (n+1)\Delta t$. It is given by:

$$\ddot{u}_{n+1} = \frac{1}{\Delta t^2}\left(2u_{n+1} - 5u_n + 4u_{n-2} - u_{n-2}\right) \tag{14}$$

At the beginning of the analysis, in other words, at the beginning of the time-marching process, $n = 0$; consequently, a good start of the analysis requires appropriate approximations for u_{-2} and u_{-1}. The expressions below:

$$u_{-2} = u_0 - 2\Delta t\dot{u}_0 \quad \text{and} \quad u_{-1} = u_0 - \Delta t\dot{u}_0 \tag{15}$$

were proposed by Carrer et al. [20] and proved to be useful in the solution of problems presenting non-null initial conditions. Indeed, expressions (15) were employed in the first and in the third examples.

Equation (14) is substituted in Equation (10), and all the boundary and the domain integrations are carried out in the discretized version of Equation (10), obtained after discretizing the boundary discretization with linear elements, and the domain with triangular linear cells. The resulting matrix form of Equation (10), valid for $n \geq 1$, is written as:

$$\begin{bmatrix} \boldsymbol{H}^{bb} & \boldsymbol{0} \\ \boldsymbol{H}^{db} & \boldsymbol{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{u}_{n+1}^{b} \\ \boldsymbol{u}_{n+1}^{d} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{G}^{bb} \\ \boldsymbol{G}^{db} \end{bmatrix} \left\{ \boldsymbol{q}_{n+1}^{b} \right\} -$$

$$\sum_{j=0}^{n-1} B_{(n+1)(j+1)} \left(\begin{bmatrix} \boldsymbol{H}^{bb} & \boldsymbol{0} \\ \boldsymbol{H}^{db} & \boldsymbol{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{u}_{j+1}^{b} \\ \boldsymbol{u}_{j+1}^{d} \end{Bmatrix} - \begin{bmatrix} \boldsymbol{G}^{bb} \\ \boldsymbol{G}^{db} \end{bmatrix} \left\{ \boldsymbol{q}_{j+1}^{b} \right\} \right) + \tag{16}$$

$$\frac{\Gamma(\alpha)}{c^2(\alpha-1)\Delta t^{\alpha}} \begin{bmatrix} \boldsymbol{M}^{bb} & \boldsymbol{M}^{bd} \\ \boldsymbol{M}^{db} & \boldsymbol{M}^{dd} \end{bmatrix} \begin{Bmatrix} 2\boldsymbol{u}_{n+1}^{b} - 5\boldsymbol{u}_{n}^{b} + 4\boldsymbol{u}_{n-1}^{b} - \boldsymbol{u}_{n-2}^{b} \\ 2\boldsymbol{u}_{n+1}^{d} - 5\boldsymbol{u}_{n}^{d} + 4\boldsymbol{u}_{n-1}^{d} - \boldsymbol{u}_{n-2}^{d} \end{Bmatrix}$$

For $n = 0$, Equation (16) is rewritten, without the summation symbol, as:

$$\begin{bmatrix} \boldsymbol{H}^{bb} & \boldsymbol{0} \\ \boldsymbol{H}^{db} & \boldsymbol{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{u}_{n+1}^{b} \\ \boldsymbol{u}_{n+1}^{d} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{G}^{bb} \\ \boldsymbol{G}^{db} \end{bmatrix} \left\{ \boldsymbol{q}_{n+1}^{b} \right\} - \tag{17}$$

$$\frac{\Gamma(\alpha)}{c^2(\alpha-1)\Delta t^{\alpha}} \begin{bmatrix} \boldsymbol{M}^{bb} & \boldsymbol{M}^{bd} \\ \boldsymbol{M}^{db} & \boldsymbol{M}^{dd} \end{bmatrix} \begin{Bmatrix} 2\boldsymbol{u}_{n+1}^{b} - 5\boldsymbol{u}_{n}^{b} + 4\boldsymbol{u}_{n-1}^{b} - \boldsymbol{u}_{n-2}^{b} \\ 2\boldsymbol{u}_{n+1}^{d} - 5\boldsymbol{u}_{n}^{d} + 4\boldsymbol{u}_{n-1}^{d} - \boldsymbol{u}_{n-2}^{d} \end{Bmatrix}$$

In the matrices in Equations (16) and (17), the double superscripts are interpreted as follows: the first indicates the position of the source point and the second, the position of the field point, whereas the superscripts *b* and *d* in the vectors correspond to boundary and domain variables. The identity matrix, $\boldsymbol{I}$, comes from $c(\xi) = 1$.

The solution of this kind of problem is naturally more time consuming than the solution of the classical one: the presence of the summation symbol in Equation (16) confirms this assertion. However, due to the use of the Laplace fundamental solution the matrices employed in the computation of the history contribution are assembled only once, at the beginning of the analysis, and are the same as those of the standard formulation; this feature makes, almost certainly, the DDW-BEM formulation less time consuming than a possible formulation to be developed and based on the use of a time-dependent fundamental solution.

The fundamental solution is given by:

$$w = w(\xi, X) = -\frac{1}{2\pi} \ln r \tag{18}$$

where r is the distance between field and source points. As mentioned before, its simplicity makes the DDW-BEM formulation quite attractive and versatile.

4. Examples

For simplicity the DDW-BEM results will be referred to as BEM results. In the three examples, the results for u are presented as functions of time for selected boundary nodes or internal points, and as functions of the spatial coordinates for chosen values of time. In the second example, the results for q are also presented as functions of time. In the third example, a discussion concerning the convergence analysis was also included. The BEM results are always compared with the analytical solutions, for $\alpha = 2.0$, 1.8, and 1.5, and with the results from an FDM formulation, whose development is included in the Appendix.

As already mentioned in the Introduction, the presence of additional terms containing the history contribution of the boundary variables, in the final expression of the BEM integral equation, turns the solution of the diffusion-wave equation more time-consuming than the solution of the classical wave propagation problem. For this reason, the CPU time for the analyses carried out with $\alpha = 1.8$ and $\alpha = 1.5$ are compared with the CPU time for $\alpha = 2.0$, when no history contribution computation is required. Note that the CPU time, as presented here, is related only with the time-marching process; the time for the matrices assemblage is not included, as it is the same, regardless the value of α.

The BEM meshes are described by the number of linear elements, n_Γ, and by the number of triangular linear cells, n_Ω.

The BEM and FDM results we obtained from computer programs developed by the authors, written in the language Fortran 90.

Finally, it is assumed that a consistent system of units has been adopted.

4.1. Bar with sinusoidal initial condition

This one-dimensional problem, which can be interpreted as the transverse vibration of a string in the interval $0 \leq x \leq \pi$, is treated here as a two-dimensional problem, which is defined over the region $0 \leq x \leq \pi$ and $0 \leq y \leq \pi/2$, with the boundary conditions below:

$$\hat{u}(0, y, t) = \hat{u}(\pi, y, t) = 0 \tag{19}$$

$$\hat{q}(x, 0, t) = \hat{q}(x, \pi/2, t) = 0 \tag{20}$$

Note that Neumann boundary conditions, adopted at $y = 0$ and $y = \pi/2$, aim at simulating the one-dimensional problem.

The initial conditions are:

$$u_0(x, y) = \sin x \tag{21}$$

$$\dot{u}_0(x, y) = 0 \tag{22}$$

For $c = 1$, the analytical solution is given by, see Murillo and Yuste [10], Ray [22]:

$$u(x,t) = E_\alpha\left(-t^\alpha\right)\sin x \tag{23}$$

In equation (23), $E_\alpha(\ldots)$ is the Mittag-Leffler function, see Ortigueira [2], defined according to:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)} \tag{24}$$

where $z \in C$, α is the fractional order of the derivative, and $\Gamma(\ldots)$ is the gamma function. For $\alpha = 2$:

$$E_2\left(-z^2\right) = \cos z \tag{25}$$

and Equation (23) falls in the well-known analytical solution of the wave propagation problem.

The BEM mesh is depicted in Figure 1; it has $n_\Gamma = 48$ and $n_\Omega = 256$ triangular linear cells. In the FDM one-dimensional analyses, $\Delta x = \pi/16$.

It is useful to define a dimensionless parameter, say β, to provide a guess of the time step length, as follows:

$$\beta = \frac{c\Delta t}{l} \tag{26}$$

where l is the length of the smallest boundary element. Carrer et al. [8] suggested that $\beta = 1/3$ is a good choice for the time step for the solution of the standard wave propagation problem using D-BEM formulations. From the mesh in Figure 1, one has $l = \pi/16$ and, consequently, $\Delta t = 0.065$. The BEM results depicted in Figure 2, for $u(\pi/2,\pi/4,t)$, were obtained with $\Delta t = 0.05$, a value smaller than the suggested one, for which $\beta = 0.255$. The BEM results for $\alpha = 1.8$, depicted in Figure 3, show that the convergence to the analytical solution requires the choice of very small values for the time step. Starting with $\Delta t = 0.05$, then adopting $\Delta t = 0.025$, good results were achieved only with $\Delta t = 0.00625$, a value 8 times smaller than that chosen for $\alpha = 2.0$. For $\alpha = 1.5$, the convergence to the analytical solution is slower than that for $\alpha = 1.8$, and much smaller values of the time step are required, as can be seen in Figure 4. A simple conclusion, from the previous figures, is immediate: as the parameter α becomes smaller, extremely small values of Δt will be required. In terms of the variable β, taking the smaller values of Δt in Figures 3 and 4, for $\alpha = 1.8$ one has $\beta = 0.032$, and for $\alpha = 1.5$, $\beta = 0.0064$. The results for $u(x,\pi/4,3.0)$ and $u(x,\pi/4,6.0)$,

$0 \le x \le \pi$, are depicted, respectively, in Figures 5 and 6. As an overall conclusion, the BEM and FDM results, depicted in Figures 2 – 4, can be considered acceptable. FDM analyses were carried out with the same Δt. However, both formulations failed on producing results for $\alpha < 1.5$.

In what follows, here and in the next examples, the variable t_{CPU}^{α} corresponds the CPU time for the analysis carried out with a prescribed value of the parameter α. Table 1 presents a comparison between $t_{CPU}^{1.8}$, $t_{CPU}^{1.5}$, and $t_{CPU}^{2.0}$, with $t_{CPU}^{2.0}$ assumed as the reference value. For completeness, $t_{CPU}^{2.0}$ = 0.347 sec. Figure 7 provides a good representation of the increase in the computer time as the value of α becomes smaller. A similar pattern was observed in the other examples and, for this reason, only Figure 7 was displayed.

4.2. Bar with an axial load

The second example can be interpreted as the longitudinal vibration of a bar with an axial load. Null initial conditions are assumed; the boundary conditions are:

$$\hat{u}(0,y,t)=0 \tag{27}$$

$$\hat{q}(L,y,t)=\frac{P}{EA} \tag{28}$$

where P is the applied load, E is the longitudinal elasticity modulus and A is the cross-section area.

As this is also a one-dimensional problem, it is simulated by adopting the boundary conditions below at $y = 0$ and $y = L/2$:

$$\hat{q}(x,0,t)=\hat{q}(x,L/2,t)=0 \tag{29}$$

The analytical solution reads:

$$u(x,t)=P\left(x-\frac{8L}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)E_\alpha\left(-\lambda_n^2c^2t^\alpha\right)\sin(\lambda_n x)\right) \tag{30}$$

with:

$$\lambda_n=\frac{n\pi}{2L} \tag{31}$$

The analyses were carried out with the same mesh already used in the previous example, but now $L = 12$. For $\alpha = 2.0$, the results for $u(L,L/2,t)$ and $u(L/2,L/2,t)$ can be seen in Figure 8, and for $q(0,L/2,t)$, in Figure 9. The time step was selected by taking $\beta = 1/3$; consequently, $\Delta t = 0.25$. The presence of time discontinuities for q makes this example a very difficult one, and in order to obtain accurate numerical results, not only must these discontinuities be well represented but also the numerical damping in u should be avoided. For these kinds of problems, the time-domain BEM formulations, usually denoted as TD-BEM, produce very accurate results, with small numerical damping and with the causality well represented. TD-BEM formulations for bi-dimensional problems can be found, for instance, in Mansur [23]. In the TD-BEM formulations for two-dimensional problems, as no boundary approximations are required, very accurate results can be found and used for comparative purposes. Concerning this matter, the interested reader is referred to Carrer and Costa [24].

The results for $\alpha = 1.8$ were obtained with $\Delta t = 0.125$, are presented in Figures 10 and 11. For $\alpha = 1.5$, a much smaller time step was required: in this case, $\Delta t = 0.03125$; see the results in Figures 12 and 13. The same values of Δt were employed in the one-dimensional FDM analyses.

For $\alpha = 2.0$, BEM results for $u(L/2,L/2,t)$ are in better agreement with the analytical solution than FDM results, although some damping is noticed in both numerical results: see Figure 8. In what concerns $q(0,L/2,t)$, see Figure 9, the analytical solution is poorly represented by the FDM results. This happens because $q(0,L/2,t)$ is computed by employing a forward finite difference formula in a post-processing stage: this feature of the FDM formulation brings a strong disadvantage in the computation of q. The BEM results, on the other hand, are in quite good agreement with the analytical solution, presenting some oscillations only in the vicinity of the discontinuity points. Note that the oscillations around the discontinuity points in the analytical solution, due to the Gibbs phenomenon, have been removed.

As the parameter α becomes smaller, some damping is introduced, see Figures 10 – 13; this damping seems to be responsible for the better agreement of the FDM results with the BEM results and also with the analytical solutions. Also, note the absence of discontinuities for q in Figures 11 and 13, and the improvement of the FDM results when compared with those from Figure 9.

Figures 14 and 15 depict the results for $t = 18$ and $t = 24$. The presence of numerical damping is noticeable when $\alpha = 2.0$, in Figure 14 at $x = 6.0$ and in Figure 15 at $x = 12.0$. For $\alpha = 1.8$ the results show a good agreement, whereas for $\alpha = 1.5$, as already expected, the agreement is not so good. In these last six figures, the analytical solution is given by Equation (30). However, due to the increasing values of the Mittag-Leffler function arguments in Equation (30), its use is not so straightforward, and care should be taken when computing the analytical solution. As an overall conclusion, the BEM results are in quite good agreement with the analytical solution.

Regarding the CPU time, Table 2 presents a comparison between $t_{CPU}^{1.8}$, $t_{CPU}^{1.5}$, and $t_{CPU}^{2.0}$. Once more, $t_{CPU}^{2.0}$ = 0.347 sec. is assumed as the reference value.

4.3. Square Membrane with Initial Condition

In this example, the transverse vibration of a square membrane defined in the region $0 \leq x,y \leq L$ is analysed. The boundary and the initial conditions are:

$$u(0,y,t) = u(L,y,t) = u(x,0,t) = u(x,L,t) = 0 \tag{32}$$

$$u(x,y,0) = \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{L}\right) \tag{33}$$

$$\dot{u}(x,y,0) = 0 \tag{34}$$

Here, c = 1.0, and L = 10.0.

A study concerning the convergence of the numerical results to the analytical solution is carried out here through the computation of the relative L_2 error norm, E_2, computed as follows:

$$E_2 = \frac{\left\| u_{analytical} - u_{BEM} \right\|_{L^2(\Omega)}}{\left\| u_{analytical} \right\|_{L^2(\Omega)}} \tag{35}$$

In Equation (35) $u_{analytical}$ is given by:

$$u_{analytical} = u(x,y,t) = E_\alpha\left(-\frac{2\pi^2}{L^2}c^2 t^\alpha\right)\sin\left(\frac{\pi x}{L}\right)\ \sin\left(\frac{\pi y}{L}\right) \tag{36}$$

where the Mittag-Leffler function $E_\alpha(...)$ was already defined by Equation (24). As before, for α = 2.0, one has the analytical solution for the wave propagation problem:

$$u(x,y,t) = \cos\left(\frac{\sqrt{2}\pi ct}{L}\right)\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{L}\right) \tag{37}$$

In Equation (35), the BEM results, u_{BEM}, come from the analyses carried out with three different meshes, presenting an increasing refinement: in mesh 1, $n_\Gamma = 40$ and $n_\Omega = 200$; in mesh 2, $n_\Gamma = 80$ and $n_\Omega = 800$ and, in mesh 3, $n_\Gamma = 160$ and $n_\Omega = 3200$. Mesh 3 is depicted in Figure 16, and Table 3 shows the time-steps, selected according to the mesh and the order of the time derivative.

Figures 17 – 19 present the results for the E_2 norm at $t = 3.0$, 6.0, 9.0 as functions of both the mesh refinement and the parameter α. One can notice that the results for $\alpha = 1.8$ and $\alpha = 1.5$ present errors minor than those obtained for $\alpha = 2.0$, at $t = 3.0$, for the three meshes. As the time advances, the same behavior is observed at $t = 6.0$, but now only with meshes 1 and 2. At $t = 9.0$, the results for $\alpha = 2.0$ present the minor errors, although the errors computed at this time are greater than the errors computed at $t = 6.0$.

Figure 20 presents the BEM results at $u(L/2,L/2,t)$ obtained with the more refined mesh. Finally, Figures 21 – 23 present the BEM and FDM results at t = 6, for $\alpha = 2.0$, 1.8, 1.5. The left-hand-side of these last three figures present the BEM results for all the domain, whereas in the right-hand-side BEM and FDM results, as functions of the x coordinate with $y = 5.0$, are compared with the analytical solution. FDM analyses were carried out with the same time-steps employed by the BEM analyses. Also, a mesh similar to mesh 3 was employed, FDM analysis being carried out with 1681 nodes. BEM results always present a good agreement with the analytical solution but, for $\alpha = 1.5$ good results were provided by the FDM only with the use of a smaller time-step. On the right-hand-side of Figure 23 one can observe the convergence of the FDM results to the analytical solution as the time-step is reduced from $\Delta t = 0.005$ to $\Delta t = 0.00125$.

Before presenting the CPU time, a slight modification is introduced in the previous notation: now $t_{CPU}^{\alpha,mesh}$ represents the CPU time for the analysis carried out with a prescribed value of the parameter α and with one of the three meshes, already numbered as 1,2 and 3. The reference value is $t_{CPU}^{2.0,1}$ = 0.094 sec. From Table 4, which presents a comparison between the CPU times, the strong dependence of Δt with respect to the order α of the derivative becomes clear. In the concluding remarks from the work by Yuste and Acedo [11], one can read: “the number of steps needed to reach even moderate times would become prohibitively large”. Although the reference [11] deals with the fractional, or anomalous, diffusion equation, the same conclusion applies for the diffusion-wave problems analyzed in this work.

Conclusions

The solution of fractional differential equations is a matter that has received a great deal of attention in recent years due to their various applications. In many cases, fractional order-models are more adequate than integer-order ones, because fractional derivatives and integrals enable the description of the memory properties of various materials and processes. Bearing this in mind, beside the challenge that every new piece of research brings about, a BEM formulation for the solution of the equation that governs the so-called diffusion-wave problem is proposed, developed, and validated through the analyses of some examples, in which the BEM results are compared with the analytical solutions and also with the results from a FDM formulation. In this equation the order of the time derivative, designated here by the Greek letter α, varies between one and two. This is also the main contribution of this work. In the proposed formulation, the fractional Caputo derivative is substituted by an ordinary second order time derivative by using the Riemann-Liouville operator, which means that the differential equation is rewritten with the Riemann-Liouville fractional derivative applied at the Laplacian of the differential equation. The reasoning, at the beginning of the work, was that proceeding in this way could be advantageous for the development of the formulation. The resulting formulation is of the type known as D-BEM, named here, for general purposes, as DDW-BEM, with DW meaning diffusion-wave. Note that the standard D-BEM formulation can be looked upon now as a particular case of the DDW-BEM, in which the history contribution is not taken into account. The computation of the history contribution was already expected, as while the operators in the standard calculus are local, in the fractional calculus they are non-local: this means that the next state of a given system depends not only on its current but also on its previous states, that is, depends on the history. Another feature, common to other numerical methods, is a strong dependence of the time step on the order of the time derivative, that is, the reduction in the order of the time derivative must be followed by a corresponding reduction in the time step length, in such a way that very small time steps are required for the solution of even simple problems. The DDW-BEM formulation was able to produce accurate results only for time derivatives of order greater than or equal to 1.5. In spite of this drawback, the good agreement observed between the BEM results, analytical solutions, and FDM results when $1.5 \leq \alpha < 2.0$, on the other hand, evidences the potential of the formulation. This assertion becomes clear from the third example for $\alpha = 1.5$, where BEM results present a better agreement with the analytical solution than the FDM results, which were obtained by employing a much smaller time-step. Further developments are also encouraged; among them, the most important seems to be the development of a BEM formulation capable of overcoming the limitation shown by the DDW-BEM formulation presented here; in other words, the development of a BEM formulation capable of solving problems covering all the range $1.0 < \alpha < 2.0$ is a very important and challenging task. Another possible development is concerned with the assumption of linear time variation for the variables u and q in the computation of the Riemann-Liouville fractional derivative. Besides, the approximation of the second order time derivative is always a matter that still deserves attention, and the use of the Newmark method appears as a possible alternative to the Houbolt method. The use of variable time-steps could be a good attempt to reduce the CPU time, the time-step length increasing for larger values of the time. The application of the present formulation for scattered data problems also deserves attention. Finally, to conclude, the solution of problems governed by differential equations with space- and time-fractional derivatives is a matter that deserves attention; the authors' opinion is that an approach similar to that presented in this work can be used to the solution to such kind of problems.

Appendix

The Finite Difference Method for the Diffusion-Wave Equation

For the development of a FDM formulation, Equation (9) is rewritten below:

$$\frac{1}{c^2}\frac{\partial^2 u(X,t)}{\partial t^2} = \frac{1}{\Gamma(\alpha)}\frac{d^2}{dt^2}\int_0^t \frac{1}{(t-\tau)^{1-\alpha}}\left(\frac{\partial^2 u(X,\tau)}{\partial x^2}+\frac{\partial^2 u(X,\tau)}{\partial y^2}\right)d\tau \tag{A1}$$

To shorten the notation, the second order space derivatives are represented as:

$$\frac{d^2 u}{dx^2} = u_{xx} \text{ and } \frac{d^2 u}{dy^2} = u_{yy} \tag{A2}$$

The term containing the derivative on the right-hand-side on equation (A1) is rewritten as:

$$\frac{d^2}{dt^2}\int_0^t \frac{u_{xx}(X,\tau)+u_{yy}(X,\tau)}{(t-\tau)^{1-\alpha}}d\tau = \frac{d}{dt}\left(\frac{d}{dt}\int_0^t \frac{u_{xx}(X,\tau)+u_{yy}(X,\tau)}{(t-\tau)^{1-\alpha}}d\tau\right) \tag{A3}$$

Then, the Leibnitz rule is applied to the term between parentheses, as follows:

$$\frac{d}{dt}\int_0^t \frac{u_{xx}(X,\tau)+u_{yy}(X,\tau)}{(t-\tau)^{1-\alpha}}d\tau = \int_0^t \frac{d}{dt}\left(\frac{u_{xx}(X,\tau)+u_{yy}(X,\tau)}{(t-\tau)^{1-\alpha}}\right)d\tau + \frac{u_{xx}(X,\tau)+u_{yy}(X,\tau)}{(t-\tau)^{1-\alpha}}\frac{dt}{dt}\Big|_{\tau=t} \tag{A4}$$

As $1 < \alpha < 2$, the second term on the right-hand-side on equation (A4) vanishes, and one can write:

$$\frac{d}{dt}\int_0^t \frac{u_{xx}(X,\tau)+u_{yy}(X,\tau)}{(t-\tau)^{1-\alpha}}d\tau = (\alpha-1)\int_0^t \frac{u_{xx}(X,\tau)+u_{yy}(X,\tau)}{(t-\tau)^{2-\alpha}}d\tau \tag{A5}$$

After substituting (A5) in (A3), and the resulting expression in equation (A1), one has:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=\frac{(\alpha-1)}{\Gamma(\alpha)}\frac{d}{dt}\int_0^t \frac{u_{xx}(X,\tau)+u_{yy}(X,\tau)}{(t-\tau)^{2-\alpha}}d\tau \tag{A6}$$

Now, by assuming the time is divided in time-steps of the same length Δt, the application of the forward finite difference formula to equation (A6) enables one to write:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=\frac{(\alpha-1)}{\Gamma(\alpha)}\frac{1}{\Delta t}\left(\int_0^{t+\Delta t}\frac{u_{xx}(X,\tau)+u_{yy}(X,\tau)}{(t+\Delta t-\tau)^{2-\alpha}}d\tau-\int_0^t \frac{u_{xx}(X,\tau)+u_{yy}(X,\tau)}{(t-\tau)^{2-\alpha}}d\tau\right) \tag{A7}$$

If, in equation (A7), $t = t_n = n\Delta t$, $n = 0, 1, 2, \ldots$, the first and the second integrals on the right-hand-side of equation (A7) can be evaluated numerically by the backward rectangular rule, thus generating the following expressions:

$$\int_0^{n\Delta t+\Delta t}\frac{u_{xx}(X,\tau)+u_{yy}(X,\tau)}{(t-\tau)^{2-\alpha}}d\tau=\sum_{k=0}^{n}\frac{u_{xx}(X,k\Delta t)+u_{yy}(X,k\Delta t)}{(n+1-k)^{2-\alpha}}\Delta t^{\alpha-1} \tag{A8}$$

And

$$\int_0^{n\Delta t}\frac{u_{xx}(X,\tau)+u_{yy}(X,\tau)}{(t-\tau)^{2-\alpha}}d\tau=\sum_{k=0}^{n-1}\frac{u_{xx}(X,k\Delta t)+u_{yy}(X,k\Delta t)}{(n-k)^{2-\alpha}}\Delta t^{\alpha-1} \tag{A9}$$

Substituting (A8) and (A9) in (A7), the following expression arises:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=\frac{(\alpha-1)}{\Gamma(\alpha)}\Delta t^{\alpha-2}\left(\sum_{k=0}^{n}\frac{u_{xx}(X,k\Delta t)+u_{yy}(X,k\Delta t)}{(n+1-k)^{2-\alpha}}-\sum_{k=0}^{n-1}\frac{u_{xx}(X,k\Delta t)+u_{yy}(X,k\Delta t)}{(n-k)^{2-\alpha}}\right) \tag{A10}$$

By introducing the following notation:

$$u_{xx}(X, k\Delta t) = u_{xx,k} \quad \text{and} \quad u_{yy}(X, k\Delta t) = u_{yy,k} \tag{A11}$$

Equation (A10) can be rewritten as:

$$\frac{\partial^2 u}{\partial t^2} = \frac{c^2(\alpha - 1)}{\Gamma(\alpha)} \Delta t^{\alpha-2} \left(u_{xx,n} + u_{yy,n} + \sum_{j=0}^{n-1} B_{nj} \left(u_{xx,j} + u_{yy,j} \right) \right) \tag{A12}$$

where:

$$B_{nk} = \frac{1}{(n+1-k)^{2-\alpha}} - \frac{1}{(n-k)^{2-\alpha}} \tag{A13}$$

The time and space second order derivatives that appear in Equation (A12) are substituted by their finite difference formulae computed at $t = \underline{t_n}$, and at $x = x_i$ and $y = y_i$; these formulae are written below:

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_{n+1}^{i,j} - 2u_n^{i,j} + u_{n-1}^{i,j}}{\Delta t^2} \tag{A14}$$

$$u_{xx,n} = \frac{u_n^{i+1,j} - 2u_n^{i,j} + u_n^{i-1,j}}{\Delta x^2} \tag{A15}$$

$$u_{yy,n} = \frac{u_n^{i,j+1} - 2u_n^{i,j} + u_n^{i,j-1}}{\Delta x^2} \tag{A16}$$

After substituting (A14) – (A16) into (A12), one finally obtains the finite difference expression of Equation (A1); it reads, for n > 0:

$$u_{n+1}^{i,j} = 2u_n^{i,j} - u_{n-1}^{i,j} + \frac{c^2(\alpha-1)}{\Gamma(\alpha)}\frac{\Delta t^\alpha}{\Delta x^2}\left(u_n^{i+1,j} - 2u_n^{i,j} + u_n^{i-1,j} + \sum_{k=0}^{n-1} B_{nk}\left(u_k^{i+1,j} - 2u_k^{i,j} + u_k^{i-1,j}\right)\right)$$

$$+ \frac{c^2(\alpha-1)}{\Gamma(\alpha)}\frac{\Delta t^\alpha}{\Delta y^2}\left(u_n^{i,j+1} - 2u_n^{i,j} + u_n^{i,j-1} + \sum_{k=0}^{n-1} B_{nk}\left(u_k^{i,j+1} - 2u_k^{i,j} + u_k^{i,j-1}\right)\right) \qquad \text{(A17)}$$

When n = 0, one has:

$$u_1^{i,j} = u_0^{i,j} + \frac{1}{2}\frac{c^2(\alpha-1)\Delta t^\alpha}{\Gamma(\alpha)}\left(\frac{u_0^{i+1,j} - 2u_0^{i,j} + u_0^{i-1,j}}{\Delta x^2} + \frac{u_0^{i,j+1} - 2u_0^{i,j} + u_0^{i,j-1}}{\Delta y^2}\right) + \Delta t\dot{u}_0^{i,j} \qquad \text{(A18)}$$

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Tables

Table 1. Bar with sinusoidal initial condition: CPU time

α	$t_{CPU}^{\alpha} / t_{CPU}^{2.0}$
1.8	1375.0
1.5	34457.0

Table 2. Bar with an axial load: CPU time

α	$t_{CPU}^{\alpha} / t_{CPU}^{2.0}$
1.8	163.0
1.5	2706.0

Table 3. Time steps employed in the square membrane analyses

α	Δt_{mesh1}	Δt_{mesh2}	Δt_{mesh3}
2.0	0.30	0.15	0.075
1.8	0.10	0.05	0.025
1.5	0.05	0.01	0.005

Table 4. Square Membrane with Initial Condition: CPU time

α	$t_{CPU}^{\alpha,1} / t_{CPU}^{2.0,1}$	$t_{CPU}^{\alpha,2} / t_{CPU}^{2.0,1}$	$t_{CPU}^{\alpha,3} / t_{CPU}^{2.0,1}$
2.0	1.0	18.0	920.0
1.8	51.0	2830.0	297665.0
1.5	195.0	71930.0	7428268619.0

Caption to the Figures

Figure 1. Rectangular mesh with 48 boundary elements and 256 cells.

Figure 2. Bar with sinusoidal initial condition: BEM and FDM results compared with the analytical solution for $u(\pi/2,\pi/4,t)$ with $\alpha = 2.0$.

Figure 3. Bar with sinusoidal initial condition: BEM and FDM results compared with the analytical solution for $u(\pi/2,\pi/4,t)$ with $\alpha = 1.8$.

Figure 4. Bar with sinusoidal initial condition: BEM and FDM results compared with the analytical solution for $u(\pi/2,\pi/4,t)$ with $\alpha = 1.5$.

Figure 5. Bar with sinusoidal initial condition: BEM and FDM results compared with the analytical solution for $u(x,\pi/4,3.0)$ with $\alpha = 2.0$, $\alpha = 1.8$, and $\alpha = 1.5$.

Figure 6. Bar with sinusoidal initial condition: BEM and FDM results compared with the analytical solution for $u(x,\pi/4,6.0)$ with $\alpha = 2.0$, $\alpha = 1.8$, and $\alpha = 1.5$.

Figure 7. Bar with sinusoidal initial condition: BEM CPU time for $\alpha = 2.0$, $\alpha = 1.8$, and $\alpha = 1.5$.

Figure 8. Bar with axial load: BEM and FDM results compared with the analytical solution for $u(L,L/2,t)$ and $u(L/2,L/2,t)$ with $\alpha = 2.0$.

Figure 9. Bar with axial load: BEM and FDM results compared with the analytical solution for $q(0,L/2,t)$ with $\alpha = 2.0$.

Figure 10. Bar with axial load: BEM and FDM results compared with the analytical solution for $u(L,L/2,t)$ and $u(L/2,L/2,t)$ with $\alpha = 1.8$.

Figure 11. Bar with axial load: BEM and FDM results compared with the analytical solution for $q(0,L/2,t)$ with $\alpha = 1.8$.

Figure 12. Bar with axial load: BEM and FDM results compared with the analytical solution for $u(L,L/2,t)$ and $u(L/2,L/2,t)$ with $\alpha = 1.5$.

Figure 13. Bar with axial load: BEM and FDM results compared with the analytical solution for $q(0,L/2,t)$ with $\alpha = 1.5$.

Figure 14. Bar with axial load: BEM and FDM results compared with the analytical solution for $u(x,L/,18.0)$ with α = 2.0, α = 1.8, and α = 1.5.

Figure 15. Bar with axial load: BEM and FDM results compared with the analytical solution for $u(x,L/,24.0)$ with α = 2.0, α = 1.8, and α = 1.5.

Figure 16. Square mesh with 160 boundary elements and 3200 cells.

Figure 17. Square Membrane with Initial Condition: BEM results convergence for α = 2.0, 1.8, 1.5 at t = 3.0.

Figure 18. Square Membrane with Initial Condition: BEM results convergence for α = 2.0, 1.8, 1.5 at t = 6.0.

Figure 19. Square Membrane with Initial Condition: BEM results convergence for α = 2.0, 1.8, 1.5 at t = 9.0.

Figure 20. Square Membrane with Initial Condition: BEM and FDM results, obtained with mesh 3, compared with the analytical solution for $u(L/2,L/2,t)$ with α = 2.0, α = 1.8, and α = 1.5.

Figure 21. Square Membrane with Initial Condition, α = 2.0: BEM results for $u(x,y,6.0)$ (left) and BEM and FDM results compared with the analytical solution for $u(x,L/2,6.0)$ (right).

Figure 22. Square Membrane with Initial Condition, α = 1.8: BEM results for $u(x,y,6.0)$ (left) and BEM and FDM results compared with the analytical solution for $u(x,L/2,6.0)$ (right).

Figure 22. Square Membrane with Initial Condition, α = 1.5: BEM results for $u(x,y,6.0)$ (left) and BEM and FDM results compared with the analytical solution for $u(x,L/2,6.0)$ (right).

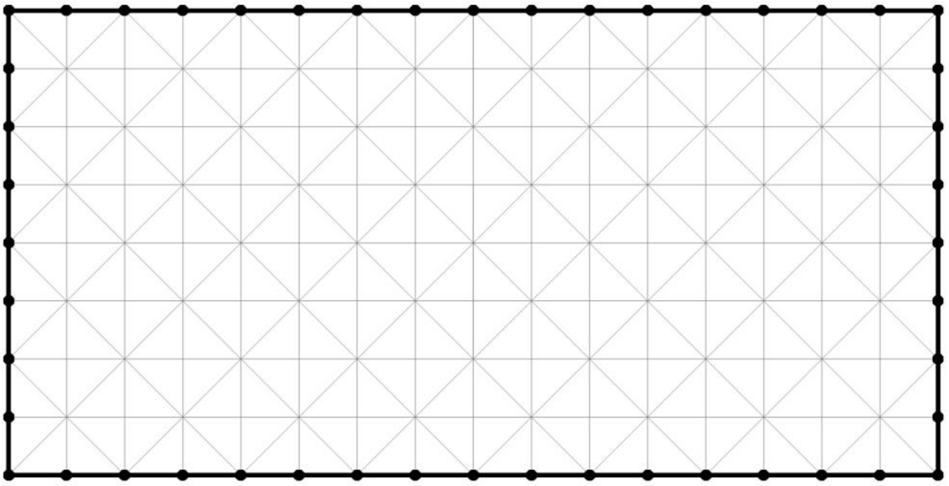


FIGURE 1

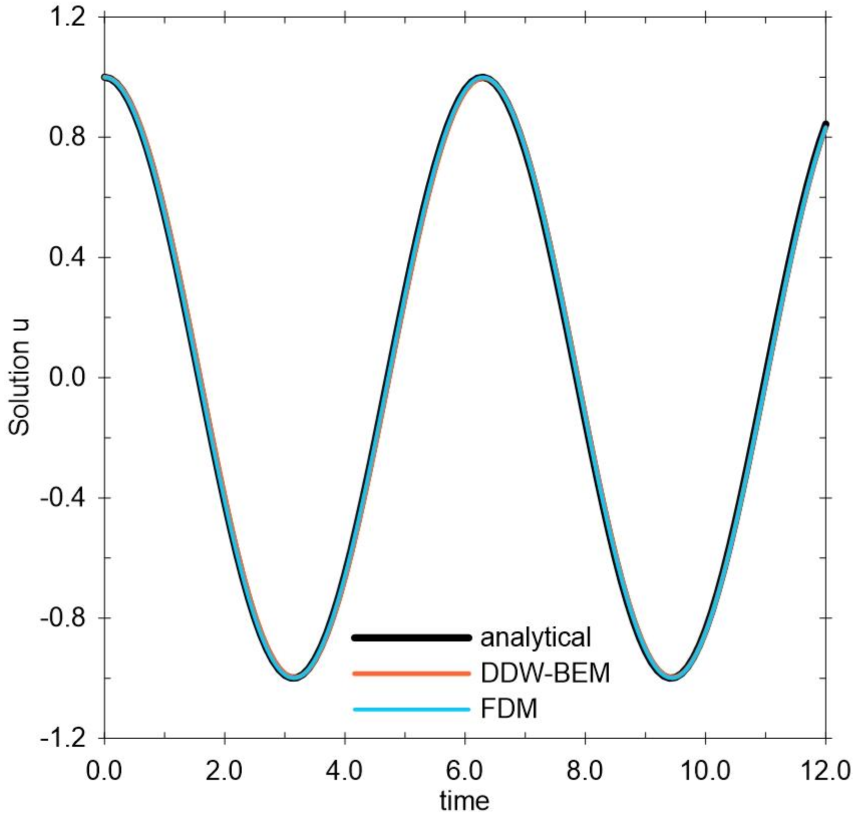
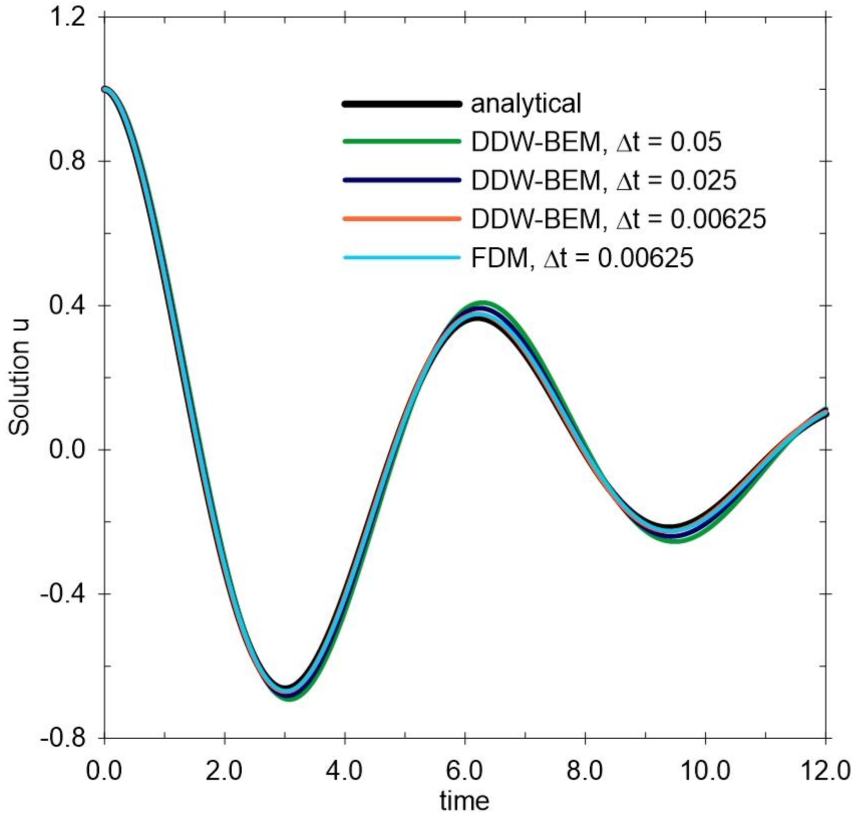
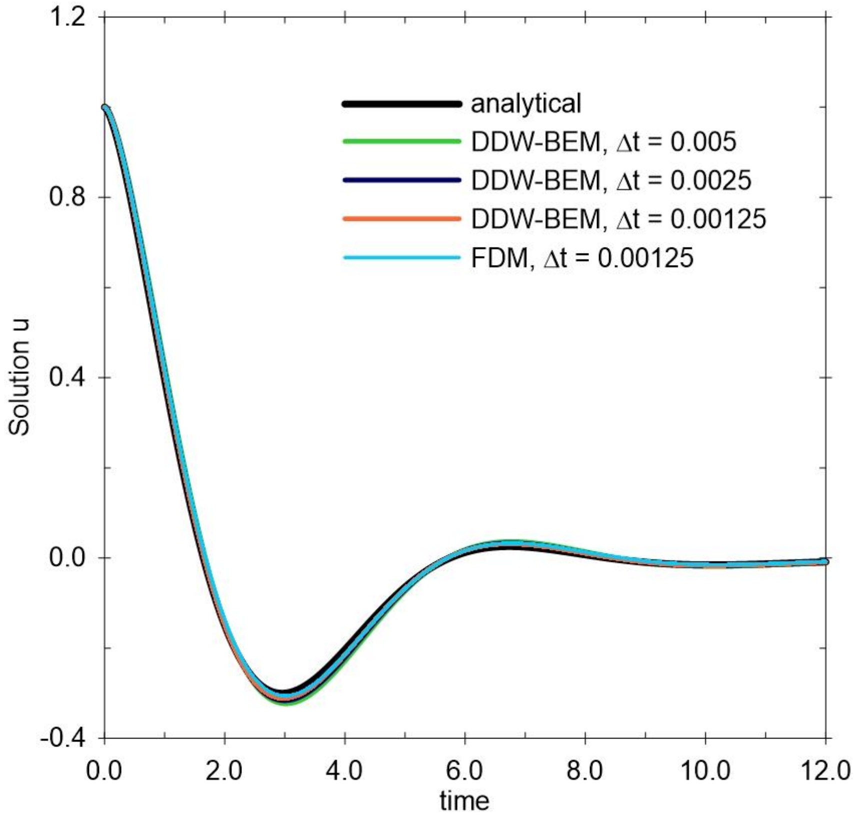

1.2
0.8
0.4
0.0
-0.4
-0.8
-1.2
Solution u
0.0
2.0
4.0
6.0
8.0
10.0
12.0
time
analytical
DDW-BEM
FDM

FIGURE 2



analytical
DDW-BEM, Δt = 0.05
DDW-BEM, Δt = 0.025
DDW-BEM, Δt = 0.00625
FDM, Δt = 0.00625
Solution u
time
1.2
0.8
0.4
0.0
-0.4
-0.8
0.0
2.0
4.0
6.0
8.0
10.0
12.0

FIGURE 3



1.2
0.8
0.4
0.0
-0.4
Solution u
0.0
2.0
4.0
6.0
8.0
10.0
12.0
time
analytical
DDW-BEM, ∆t = 0.005
DDW-BEM, ∆t = 0.0025
DDW-BEM, ∆t = 0.00125
FDM, ∆t = 0.00125

FIGURE 4

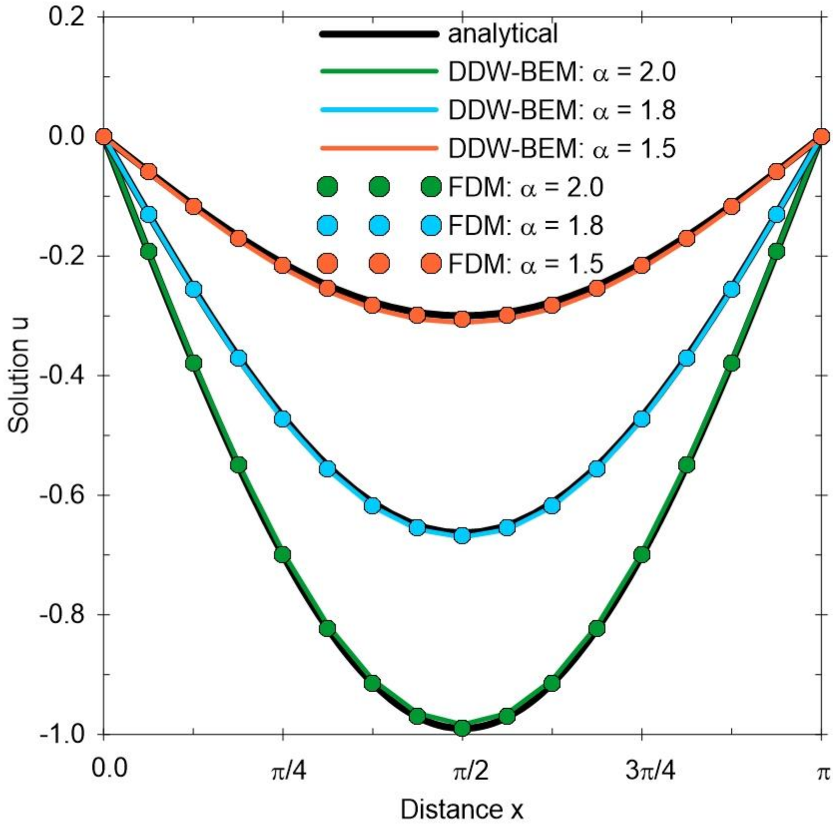
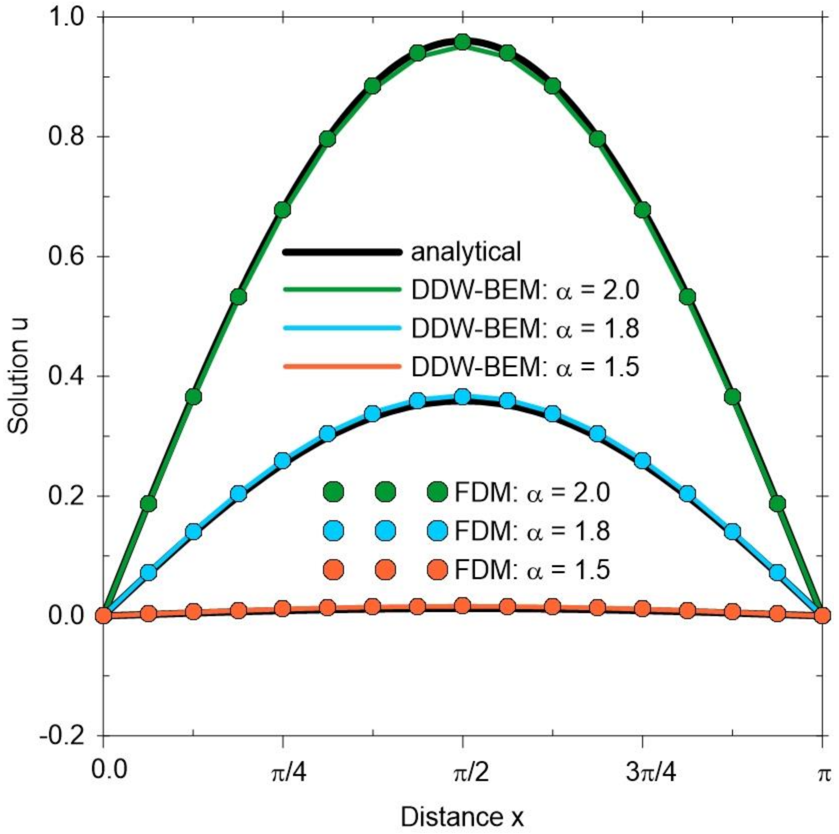
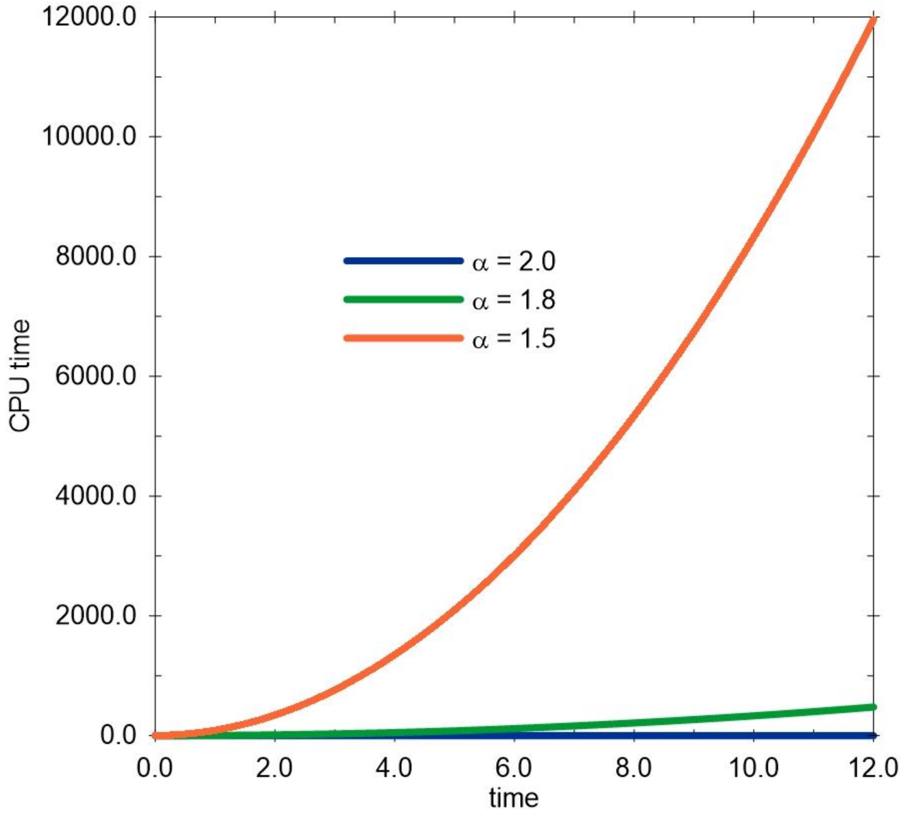

analytical
DDW-BEM: α = 2.0
DDW-BEM: α = 1.8
DDW-BEM: α = 1.5
FDM: α = 2.0
FDM: α = 1.8
FDM: α = 1.5
0.2
0.0
-0.2
-0.4
-0.6
-0.8
-1.0
Solution u
0.0
π/4
π/2
3π/4
π
Distance x

FIGURE 5



1.0
0.8
0.6
0.4
0.2
0.0
-0.2
Solution u
0.0
π/4
π/2
3π/4
π
Distance x
analytical
DDW-BEM: α = 2.0
DDW-BEM: α = 1.8
DDW-BEM: α = 1.5
FDM: α = 2.0
FDM: α = 1.8
FDM: α = 1.5

FIGURE 6



12000.0
10000.0
8000.0
6000.0
4000.0
2000.0
0.0
CPU time
0.0
2.0
4.0
6.0
8.0
10.0
12.0
time
α = 2.0
α = 1.8
α = 1.5

FIGURE 7

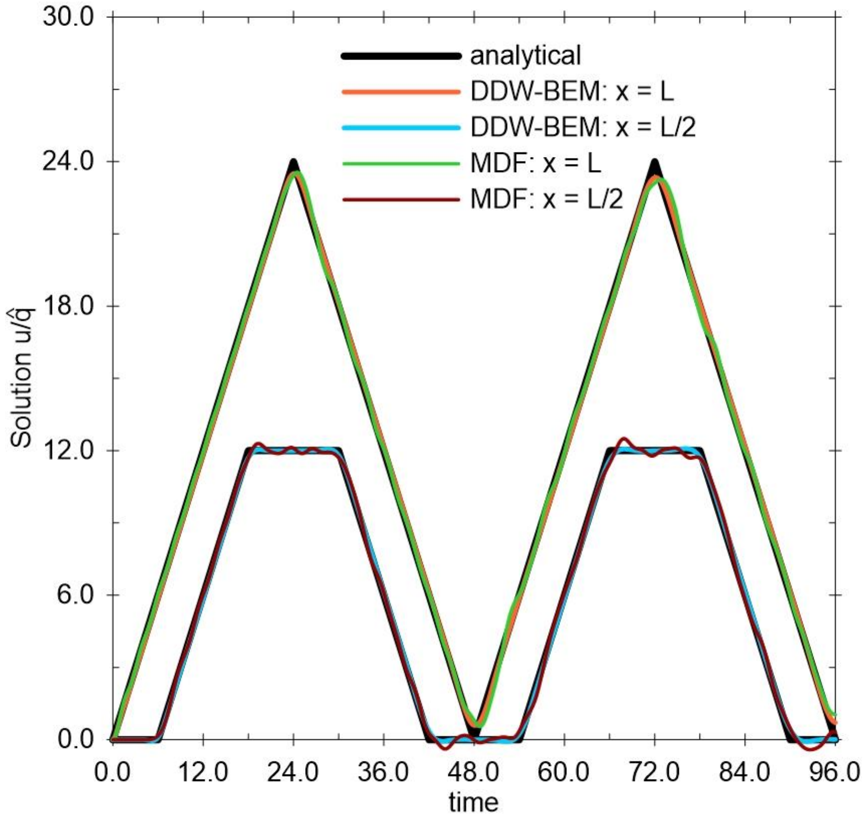

30.0
24.0
18.0
12.0
6.0
0.0
Solution u/q̂
0.0
12.0
24.0
36.0
48.0
60.0
72.0
84.0
96.0
time
analytical
DDW-BEM: x = L
DDW-BEM: x = L/2
MDF: x = L
MDF: x = L/2

FIGURE 8

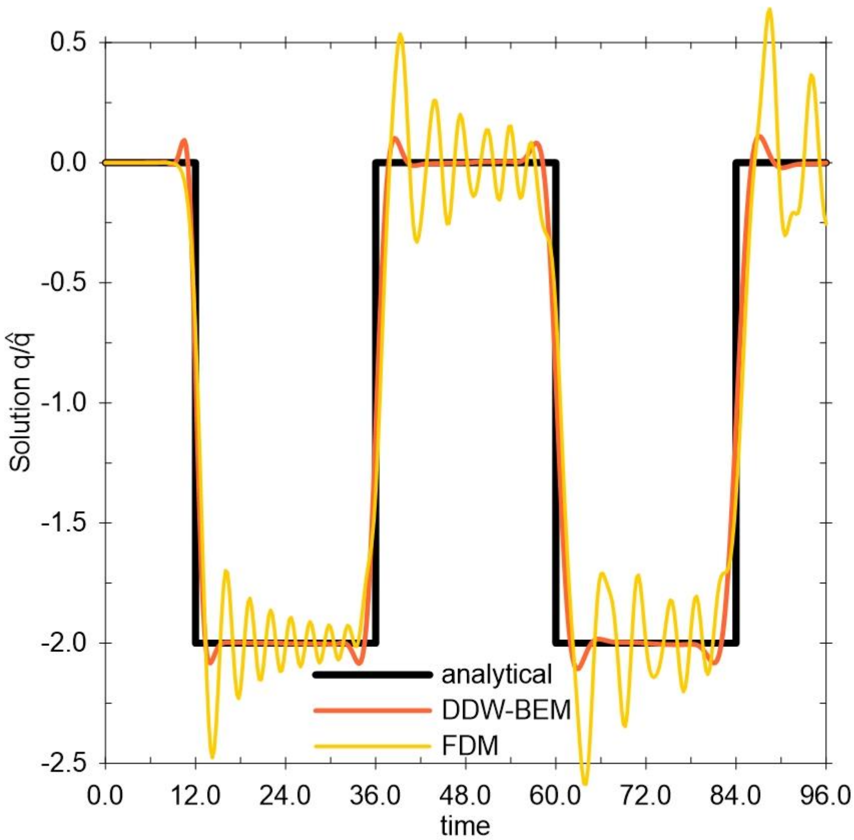
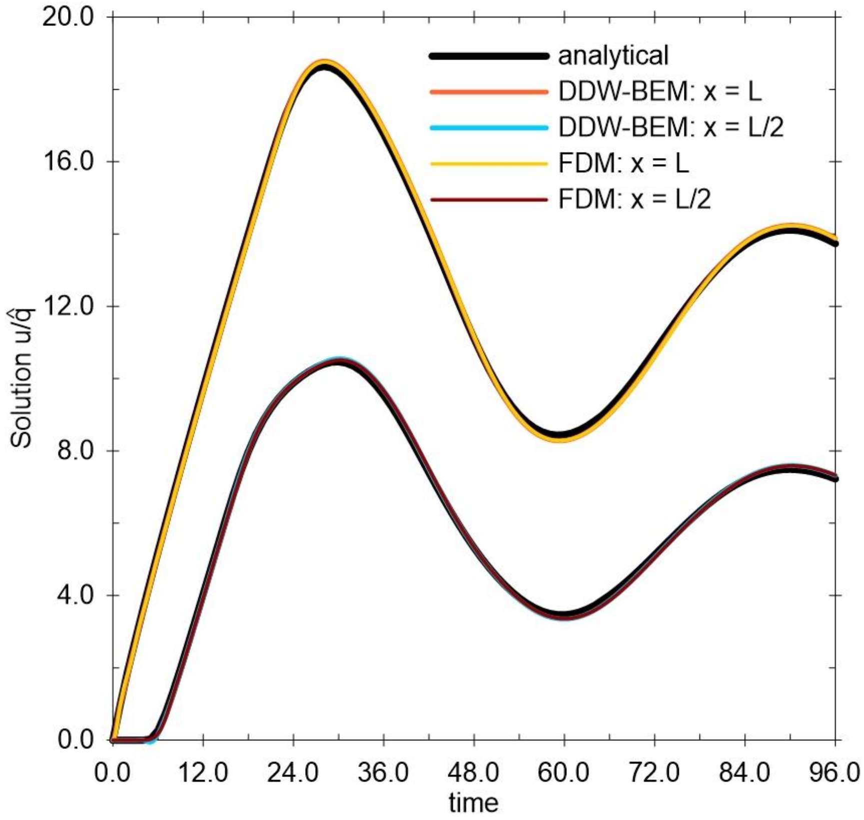
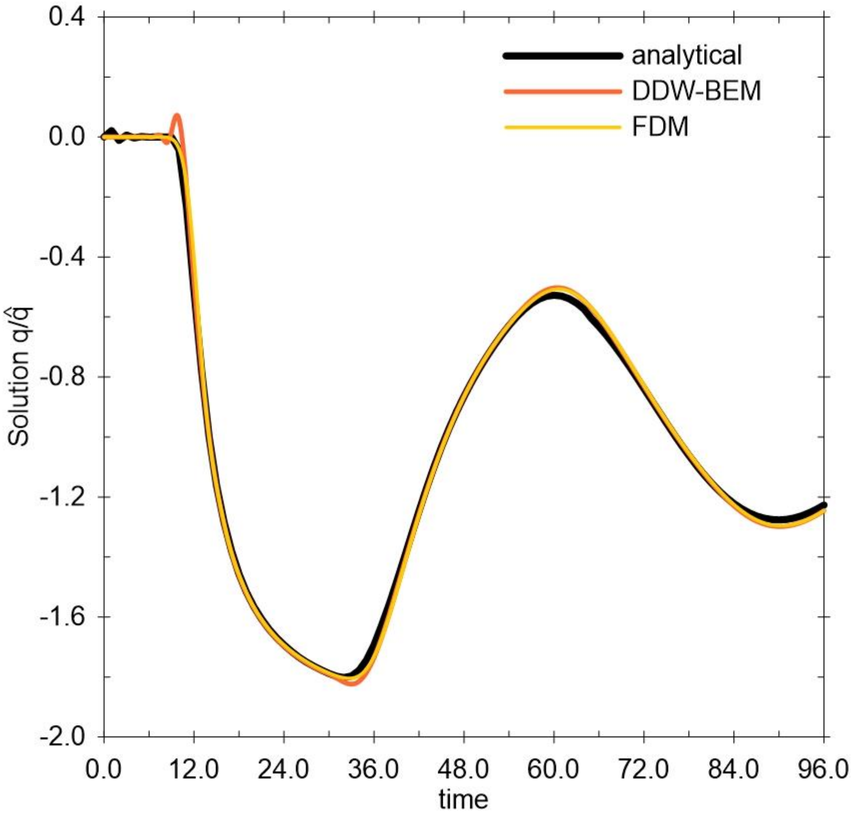

0.5
0.0
-0.5
-1.0
-1.5
-2.0
-2.5
Solution q/q̂
0.0
12.0
24.0
36.0
48.0
60.0
72.0
84.0
96.0
time
analytical
DDW-BEM
FDM

FIGURE 9



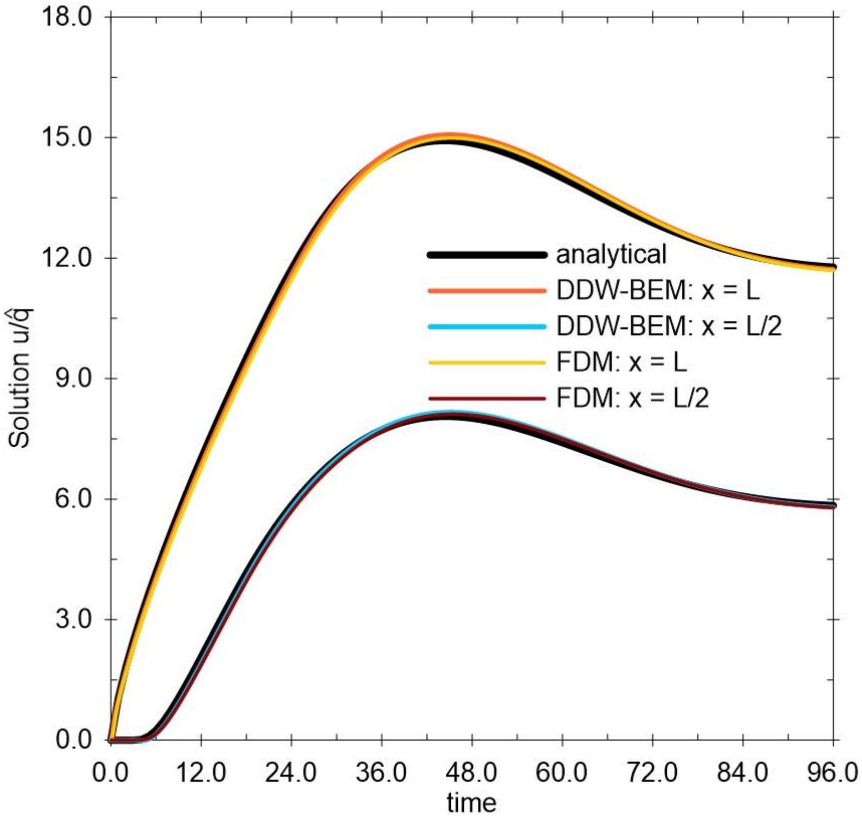
analytical
DDW-BEM: x = L
DDW-BEM: x = L/2
FDM: x = L
FDM: x = L/2
Solution u/q̂
time
20.0
16.0
12.0
8.0
4.0
0.0
0.0
12.0
24.0
36.0
48.0
60.0
72.0
84.0
96.0

FIGURE 10



0.4
0.0
-0.4
-0.8
-1.2
-1.6
-2.0
Solution q/q̂
0.0
12.0
24.0
36.0
48.0
60.0
72.0
84.0
96.0
time
analytical
DDW-BEM
FDM

FIGURE 11



18.0
15.0
12.0
9.0
6.0
3.0
0.0
Solution u/q̂
0.0
12.0
24.0
36.0
48.0
60.0
72.0
84.0
96.0
time
analytical
DDW-BEM: x = L
DDW-BEM: x = L/2
FDM: x = L
FDM: x = L/2

FIGURE 12

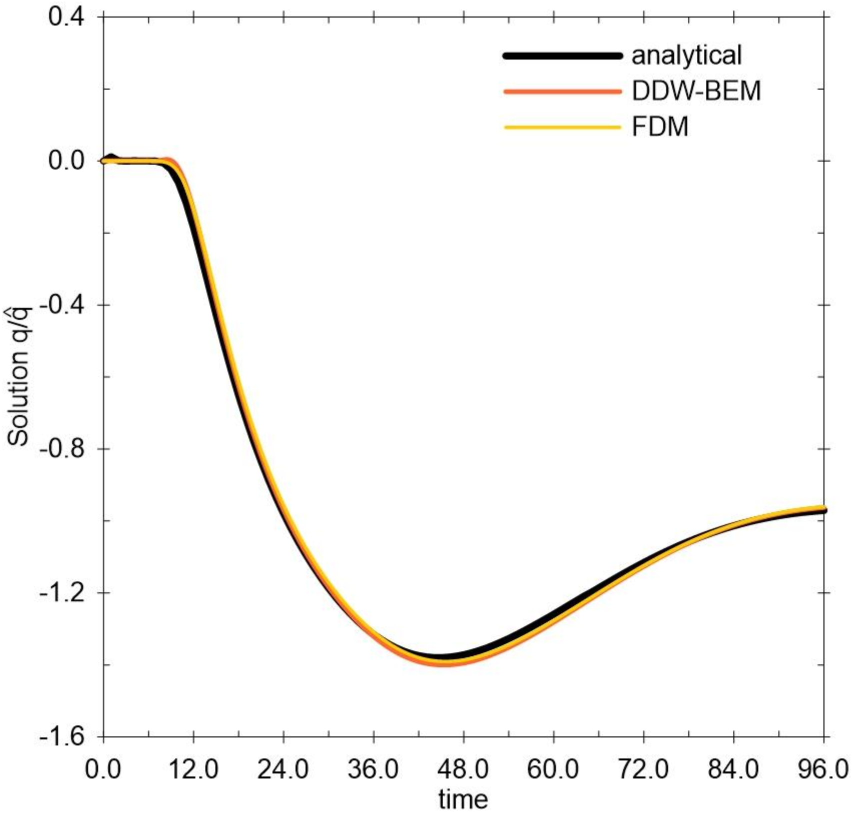
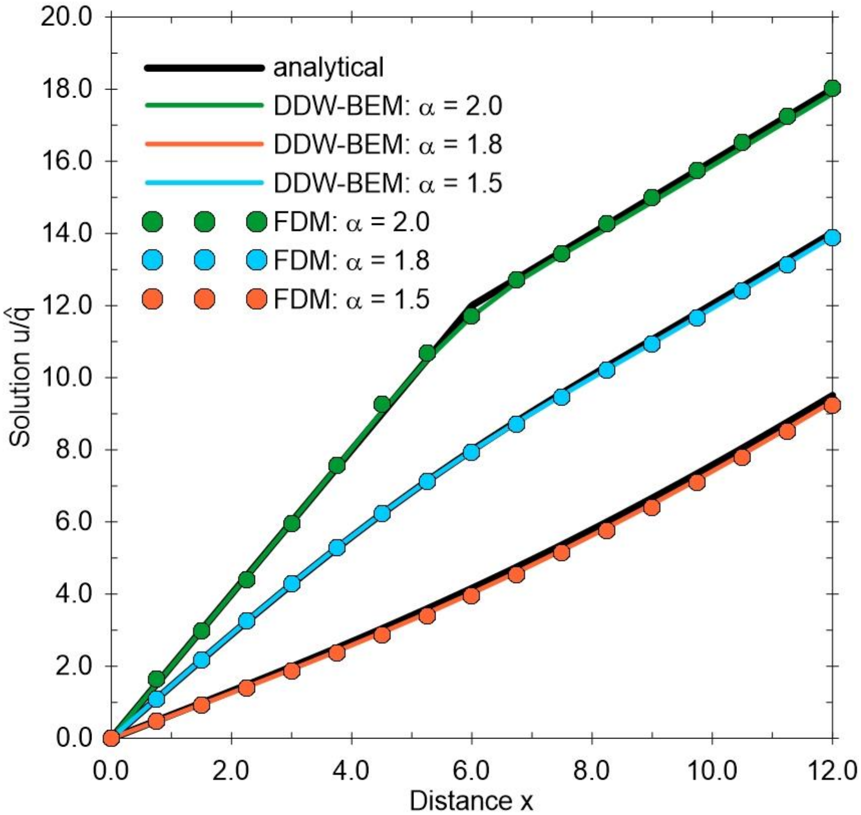
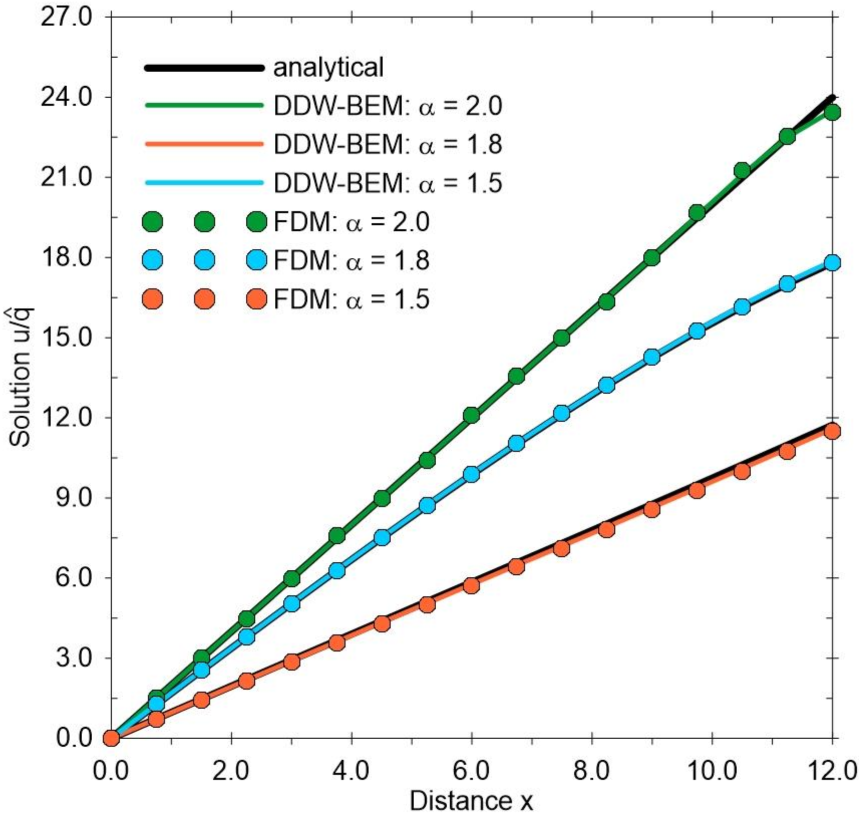

0.4
0.0
-0.4
-0.8
-1.2
-1.6
Solution q/q̂
0.0
12.0
24.0
36.0
48.0
60.0
72.0
84.0
96.0
time
analytical
DDW-BEM
FDM

FIGURE 13



20.0
18.0
16.0
14.0
12.0
10.0
8.0
6.0
4.0
2.0
0.0
0.0
2.0
4.0
6.0
8.0
10.0
12.0
Distance x
Solution u/q̂
analytical
DDW-BEM: α = 2.0
DDW-BEM: α = 1.8
DDW-BEM: α = 1.5
FDM: α = 2.0
FDM: α = 1.8
FDM: α = 1.5

FIGURE 14



27.0
24.0
21.0
18.0
15.0
12.0
9.0
6.0
3.0
0.0
0.0
2.0
4.0
6.0
8.0
10.0
12.0
Distance x
Solution u/q̂
analytical
DDW-BEM: α = 2.0
DDW-BEM: α = 1.8
DDW-BEM: α = 1.5
FDM: α = 2.0
FDM: α = 1.8
FDM: α = 1.5

FIGURE 15

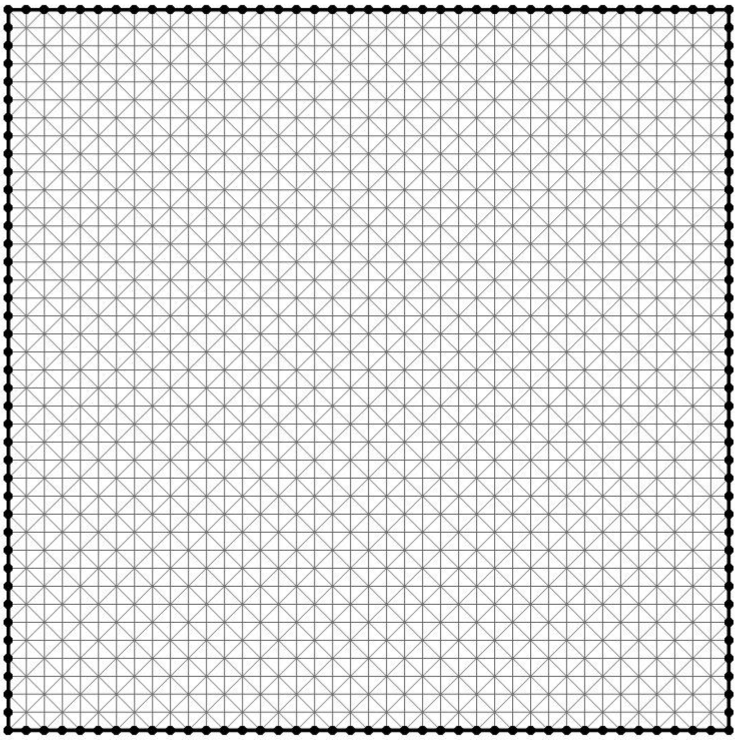
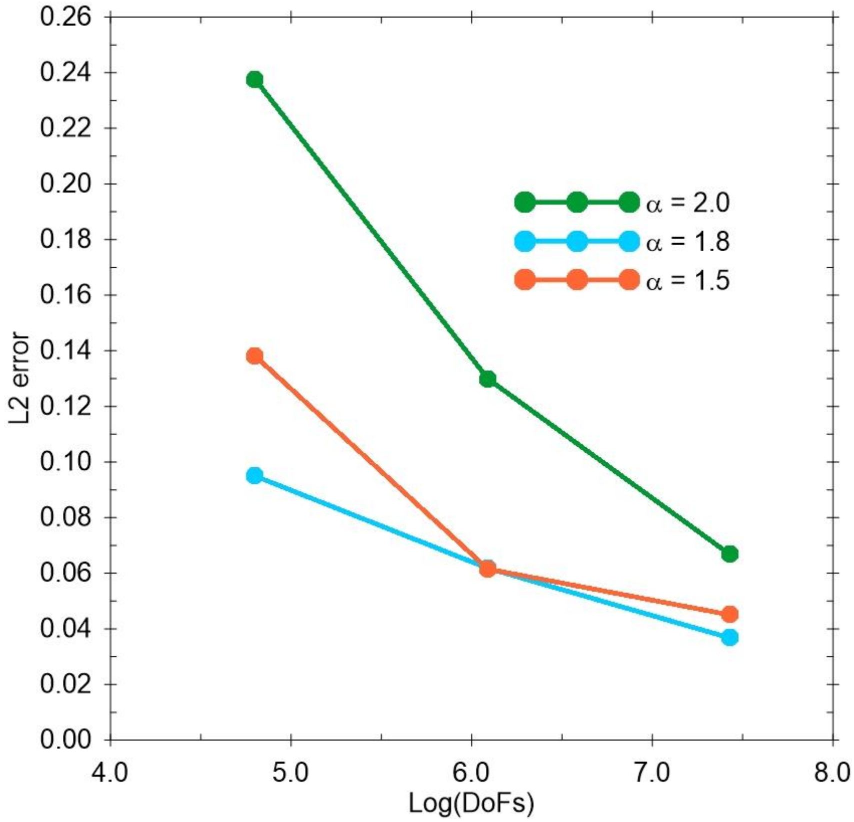


FIGURE 16



0.26
0.24
0.22
0.20
0.18
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
L2 error
4.0
5.0
6.0
7.0
8.0
Log(DoFs)
α = 2.0
α = 1.8
α = 1.5

FIGURE 17

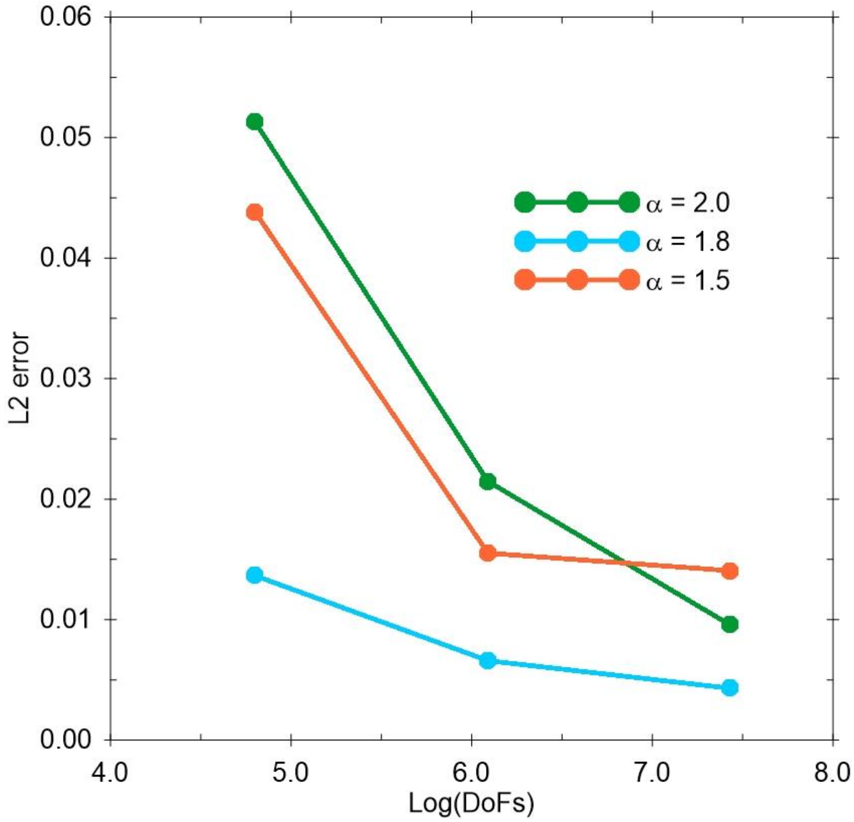

0.06
0.05
0.04
0.03
0.02
0.01
0.00
L2 error
4.0
5.0
6.0
7.0
8.0
Log(DoFs)
α = 2.0
α = 1.8
α = 1.5

FIGURE 18

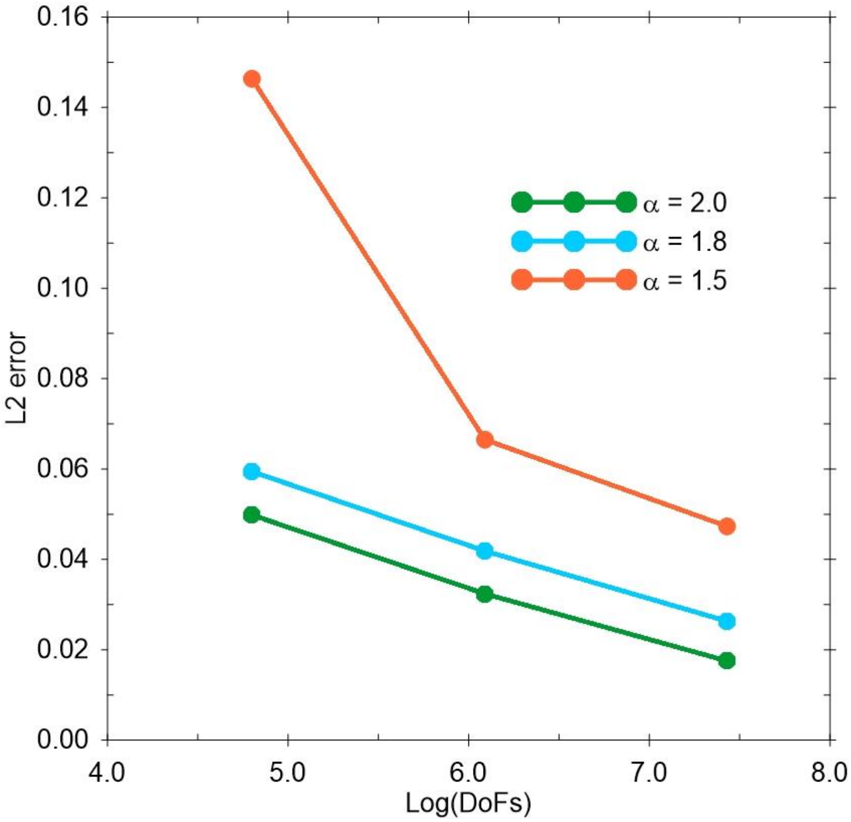

0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
L2 error
4.0
5.0
6.0
7.0
8.0
Log(DoFs)
α = 2.0
α = 1.8
α = 1.5

FIGURE 19

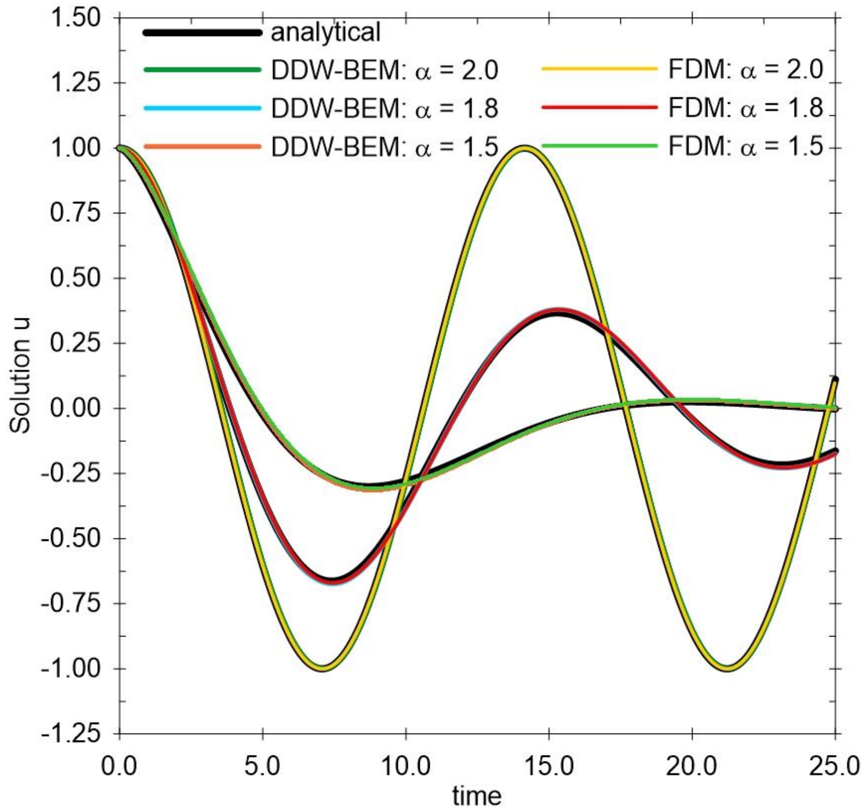

1.50
1.25
1.00
0.75
0.50
0.25
0.00
-0.25
-0.50
-0.75
-1.00
-1.25
Solution u
0.0
5.0
10.0
15.0
20.0
25.0
time
analytical
DDW-BEM: α = 2.0
DDW-BEM: α = 1.8
DDW-BEM: α = 1.5
FDM: α = 2.0
FDM: α = 1.8
FDM: α = 1.5

FIGURE 20

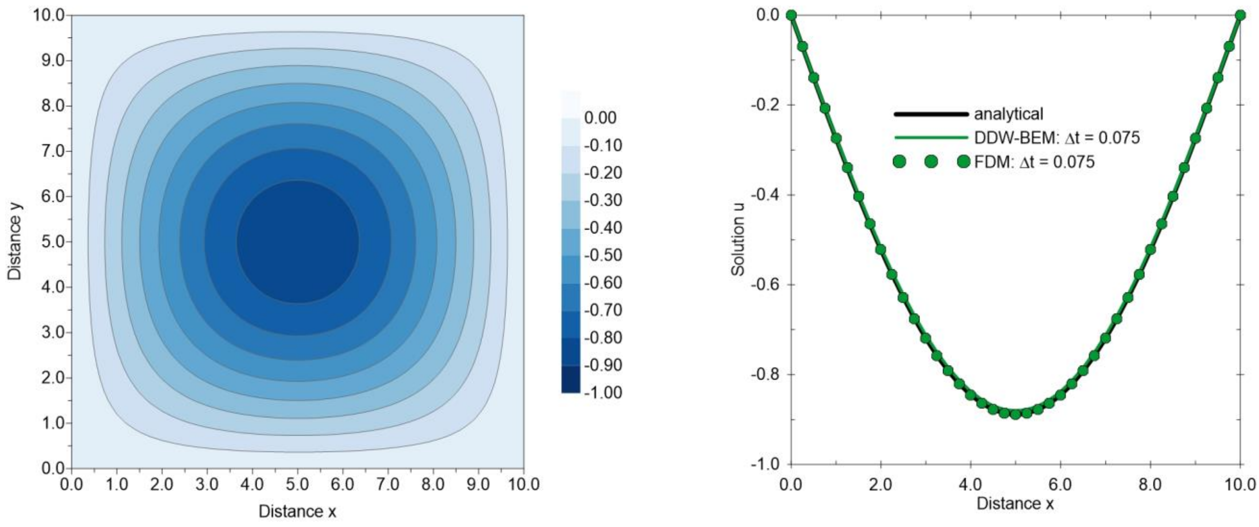
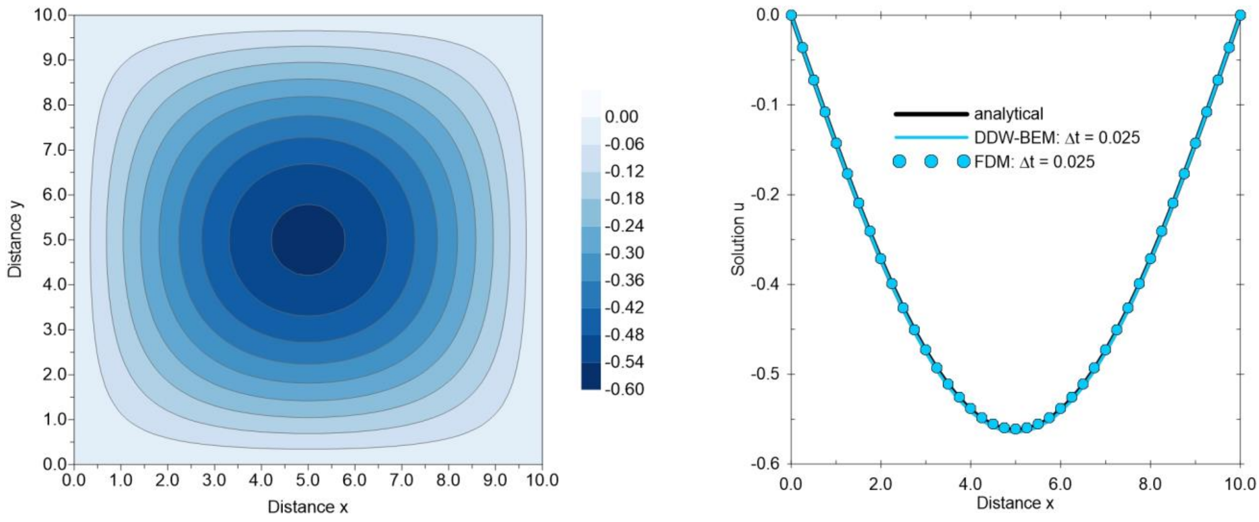
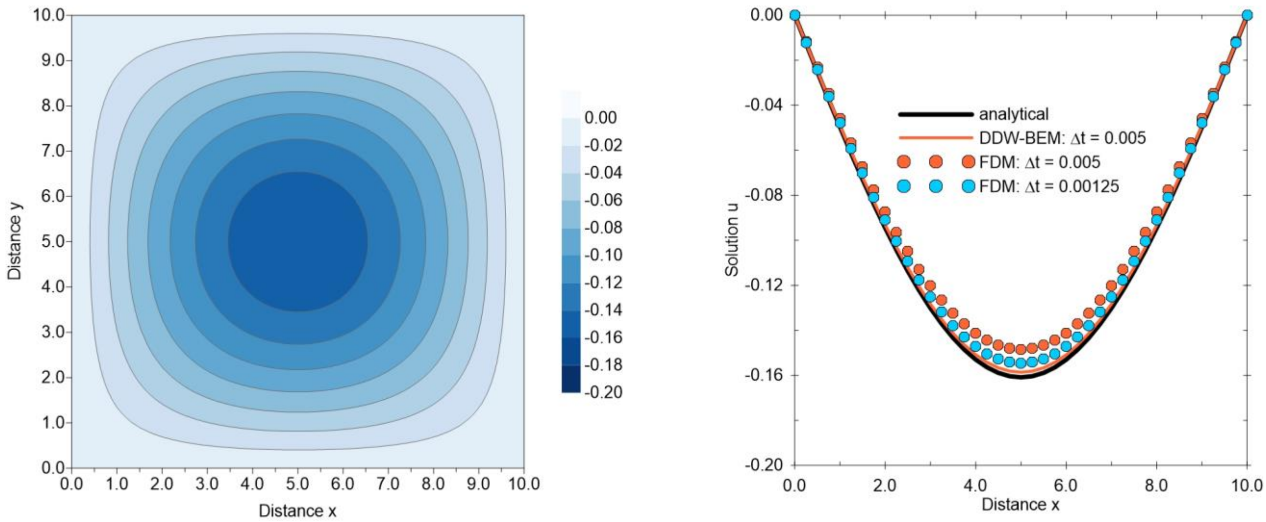

10.0
9.0
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0.0
Distance y
0.0 1.0 2.0 3.0 4.0 5.0 6.0 7.0 8.0 9.0 10.0
Distance x
0.00
-0.10
-0.20
-0.30
-0.40
-0.50
-0.60
-0.70
-0.80
-0.90
-1.00
0.0
-0.2
-0.4
-0.6
-0.8
-1.0
Solution u
analytical
DDW-BEM: Δt = 0.075
FDM: Δt = 0.075
0.0 2.0 4.0 6.0 8.0 10.0
Distance x

FIGURE 21



10.0
9.0
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0.0
Distance y
0.0 1.0 2.0 3.0 4.0 5.0 6.0 7.0 8.0 9.0 10.0
Distance x
0.00
-0.06
-0.12
-0.18
-0.24
-0.30
-0.36
-0.42
-0.48
-0.54
-0.60
0.0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
Solution u
0.0 2.0 4.0 6.0 8.0 10.0
Distance x
analytical
DDW-BEM: Δt = 0.025
FDM: Δt = 0.025

FIGURE 22



10.0
9.0
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0.0
Distance y
0.0 1.0 2.0 3.0 4.0 5.0 6.0 7.0 8.0 9.0 10.0
Distance x
0.00
-0.02
-0.04
-0.06
-0.08
-0.10
-0.12
-0.14
-0.16
-0.18
-0.20
0.00
-0.04
-0.08
-0.12
-0.16
-0.20
Solution u
analytical
DDW-BEM: Δt = 0.005
FDM: Δt = 0.005
FDM: Δt = 0.00125
0.0 2.0 4.0 6.0 8.0 10.0
Distance x

FIGURE 23