Tree Pivot-Minors and Linear Rank-Width*

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Abstract

Tree-width and its linear variant path-width play a central role for the graph minor relation. In particular, Robertson and Seymour (1983) proved that for every tree T, the class of graphs that do not contain T as a minor has bounded path-width. For the pivot-minor relation, rank-width and linear rank-width take over the role of tree-width and path-width. As such, it is natural to examine if, for every tree T, the class of graphs that do not contain T as a pivot-minor has bounded linear rank-width. We first prove that this statement is false whenever T is a tree that is not a caterpillar. We conjecture that the statement is true if T is a caterpillar. We are also able to give partial confirmation of this conjecture by proving:

- for every tree T, the class of T-pivot-minor-free distance-hereditary graphs has bounded linear rank-width if and only if T is a caterpillar;
- for every caterpillar T on at most four vertices, the class of T-pivot-minor-free graphs has bounded linear rank-width.

To prove our second result, we only need to consider $T = P_4$ and $T = K_{1,3}$, but we follow a general strategy: first we show that the class of T-pivot-minor-free graphs is contained in some class of (H_1, H_2)-free graphs, which we then show to have bounded linear rank-width. In particular, we prove that the class of $(K_3, S_{1,2,2})$-free graphs has bounded linear rank-width, which strengthens a known result that this graph class has bounded rank-width.

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1 Introduction

In order to increase our understanding of graph classes, it is natural to consider some notion of "width" and to research what properties graph classes of bounded width may have. We say that a graph class has *bounded* width (for some specific width parameter) if there exists a constant c such that the width of every graph in the class is at most c. In particular, this type of structural research has been done in the context of graph containment problems, where the aim is to determine whether a graph H appears as a "pattern" inside some other graph G. Here, a pattern is defined by specifying a set of graph operations that may be used to obtain H from G. For instance, a graph G contains a graph H as a *minor* if H can be obtained from G via a sequence of vertex deletions, edge deletions and edge contractions.

Tree-width and its *linear* variant, path-width, are the best-known graph width parameters due to their relevance for graph minor theory [47]. Rank-width is another well-known parameter, introduced by Oum and Seymour [43]. The rank-width of a graph G expresses the minimum width k of a tree-like structure obtained by recursively splitting the vertex set of G in such a way that each cut induces a matrix of rank at most k (see Section 2 for a formal definition). Rank-width is more general than tree-width in the sense that every graph class of bounded tree-width has bounded rank-width, but there are classes for which the reverse does not hold, for example, the class of all complete graphs [17].

The notion of rank-width has important algorithmic implications, as many NP-complete decision problems are known to be polynomial-time solvable not only for graph classes of bounded tree-width, but also for graph classes of bounded rank-width; see [16, 25, 28, 32, 44] for a number of meta-theorems capturing such decision problems. Rank-width is equivalent to clique-width [43], another important and well-studied width parameter. Linear rank-width is a linearized variant of rank-width, known to be equivalent to linear clique-width (see, for example, [42]) and to be closely related to the trellis-width of linear codes [31]. We formally define the notions of rank-width and linear rank-width in Section 2.

The problem of determining whether a given graph has linear rank-width at most k for some given integer k is NP-complete (this follows from a result of Kashyap [31]). On the positive side, Jeong, Kim, and Oum [29] gave an FPT algorithm for deciding whether a graph has linear rank-width at most k. Ganian [26] and Adler, Farley, and Proskurowski [1] characterized the graphs of linear rank-width at most 1. Recently, Nešetřil et al. [39] showed that every class of bounded linear rank-width is linearly χ-bounded. However, our knowledge on linear rank-width, the topic of this paper, is still limited.

Motivation

To increase our understanding of rank-width and linear rank-width, we may want to verify if classical results for tree-width and path-width stay valid when we replace tree-width with rank-width and path-width with linear rank-width. The following two structural results, related to path-width and tree-width, form the core of the Graph Minor Structure Theorem. Here, a graph G is *H-minor-free* for some graph H if G does not contain H as a minor.

Theorem 1.1 (Robertson and Seymour [45]). *For every tree T, the class of T-minor-free graphs has bounded path-width.*

Theorem 1.2 (Roberson and Seymour [46]). *For every planar graph H, the class of H-minor-free graphs has bounded tree-width.*

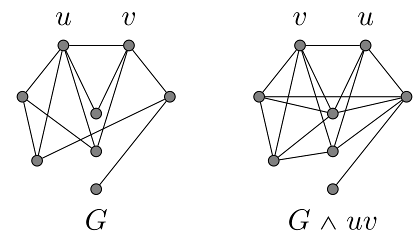

Figure 1: A graph before and after pivoting an edge (the example is taken from [20]).

It is known that edge deletions and contractions may increase the rank-width and linear rank-width [15]. Hence, working with minors is not a suitable approach for understanding rank-width and linear rank-width. Therefore, Oum [40] proposed the notions of vertex-minors and pivot-minors, two closely related notions, which were called ℓ-reductions and p-reductions, respectively, in [11]. Taking vertex-minors or pivot-minors does *not* increase the rank-width or linear rank-width of a graph [40].

To define the notions of a vertex-minor and a pivot-minor, we need some terminology. The *local complementation* at a vertex u in a graph G replaces every edge of the subgraph induced by the neighbours of u with a non-edge, and vice versa. The resulting graph is denoted by $G*u$. An *edge pivot* is the operation that takes an edge uv, first applies a local complementation at u, then at v, and then at u again. We denote the resulting graph $G \wedge uv = G*u*v*u$. It is known that $G*u*v*u = G*v*u*v$ [40], and thus $G \wedge uv = G \wedge vu$. An alternative definition of the edge pivot operation is as follows. Let S_u be the set of neighbours of u that are non-adjacent to v and let S_v be the set of neighbours of v that are non-adjacent to u, whereas we denote the set of common neighbours of u and v by S_{uv}. We replace every edge between any two vertices in distinct sets from $\{S_u \backslash \{v\}, S_v \backslash \{u\}, S_{uv}\}$ by a non-edge and vice versa. Afterwards, we delete every edge between u and S_u and add every edge between u and S_v. We also delete every edge between v and S_v and add every edge between v and S_u. We refer to Figure 1 for an example.

A graph H is a *vertex-minor* of a graph G if H can be obtained from G by a sequence of local complementations and vertex deletions. A graph H is a *pivot-minor* of a graph G if H can be obtained from G by a sequence of edge pivots and vertex deletions. Hence H is a vertex-minor of G if H is a pivot-minor of G, but the reverse is not necessarily true. A graph is *H-vertex-minor-free* if it contains no vertex-minor isomorphic to H, and similarly, a graph is *H-pivot-minor-free* if it contains no pivot-minor isomorphic to H.

It is natural to ask whether parallel statements of Theorems 1.1 and 1.2 exist for rank-width or linear rank-width in terms of vertex-minors or pivot-minors. Below we discuss the state-of-the-art for this research direction.

Related Work

A *circle graph* is the intersection graph of chords on a circle, and it is known that the class of circle graphs is closed under taking vertex-minors. Bouchet [11] characterized circle graphs in terms of three forbidden vertex-minors. Oum [40] showed that the class of circle graphs has unbounded rank-width, and asked, as an analogue to Theorem 1.2 for the vertex-minor relation, whether for every circle graph H, the class of H-vertex-minor-free graphs has bounded rank-width. Recently, Geelen et al. [27] gave an affirmative answer to this question.

Theorem 1.3 (Geelen, Kwon, McCarty, and Wollan [27])**.** *For every circle graph H, the class of H-vertex-minor-free graphs has bounded rank-width.*

Every pivot-minor of a graph is also a vertex-minor. Hence, for every graph H, the class of H-vertex-minor-free graphs is contained in the class of H-pivot-minor-free graphs. This leads to the question whether we can strengthen Theorem 1.3 by replacing the vertex-minor relation with the pivot-minor relation. However, this is not the case. In order to see this, we first observe that bipartite graphs are closed under taking pivot-minors [40]. Hence, no bipartite graph contains a non-bipartite circle graph as a pivot-minor. Now consider the class $\mathcal{G}$ of $n \times n$ grids, which has unbounded rank-width. As $\mathcal{G}$ is a subclass of bipartite graphs, $\mathcal{G}$ is H-pivot-minor-free for every non-bipartite circle graph H (such as, for example, $H = K_3$). Hence, for every non-bipartite graph H, the class of H-pivot-minor-free graphs has unbounded rank-width. This means we can only hope to strengthen Theorem 1.3 by considering bipartite circle graphs H, and Oum [41] conjectured the following analogue to Theorem 1.2 for the pivot-minor relation:

Conjecture 1 (Oum [41])**.** *For every bipartite circle graph H, the class of H-pivot-minor-free graphs has bounded rank-width.*

So far, Conjecture 1 has been verified for bipartite graphs [40], circle graphs [41], and line graphs [41]. If Conjecture 1 holds for all graphs, then this would imply both Theorems 1.2 and 1.3 [40].

We now turn to linear rank-width, for which Kanté and Kwon [30] conjectured the following analogue to Theorem 1.1 for the vertex-minor relation:

Conjecture 2 (Kanté and Kwon [30])**.** *For every tree T, the class of T-vertex-minor-free graphs has bounded linear rank-width.*

So far, Conjecture 2 has been verified on every class of graphs whose prime graphs, with respect to split decompositions, have bounded linear rank-width [30]. For example, prime distance-hereditary graphs have at most three vertices, and therefore, for every tree T, the class of T-vertex-minor-free distance-hereditary graphs has bounded linear rank-width. Moreover, Conjecture 2 holds for every path T [33].

Our Focus and Results

We focus on the remaining analogue, namely the analogue to Theorem 1.1 for the pivot-minor relation. We first prove that we cannot hope for a result that holds for every tree T. A *caterpillar* is a tree that contains a path P, such that every vertex not on P has a neighbour in P.

Theorem 1.4. *If T is a tree that is not a caterpillar, then the class of T-pivot-minor-free distance-hereditary graphs has unbounded linear rank-width.*

Due to Theorem 1.4, we conjecture the following:

Conjecture 3. *For every* caterpillar *T, the class of T-pivot-minor-free graphs has bounded linear rank-width.*

In contrast, the aforementioned result of Kwon et al. [33] confirming Conjecture 2 if T is a path implies that Conjecture 2 *has* been confirmed if T is a caterpillar: every caterpillar T is a pivot-minor of some path P [34, Theorem 4.6] and consequently, if T is a caterpillar, then the class of T-vertex-minor-free graphs is contained in the class of P-vertex-minor-free graphs.

By the fact that every caterpillar T is a pivot-minor of some path P and the fact that every path P is a caterpillar by definition, we can also formulate Conjecture 3 as follows:

Conjecture 3 (alternative formulation). *For every* path P*, the class of* P*-pivot-minor-free graphs has bounded linear rank-width.*

We make two contributions to Conjecture 3. We first show, in Section 4, that Conjecture 3 holds for distance-hereditary graphs.

Theorem 1.5. *Let* $n \geqslant 3$ *be an integer. Every* P_n*-pivot-minor-free distance-hereditary graph has linear rank-width at most* $2n - 5$*.*

Theorems 1.4 and 1.5, together with the fact that every caterpillar is a pivot-minor of some path, yields the following dichotomy.

Corollary 1.6. *For every tree* T*, the class of* T*-pivot-minor-free distance-hereditary graphs has bounded linear rank-width if and only if* T *is a caterpillar.*

If a graph G is P_4-pivot-minor-free, then G has no induced subgraph isomorphic to P_4. This implies that G is distance-hereditary. Hence, Theorem 1.5 has the following consequence:

Corollary 1.7. *Every* P_4*-pivot-minor-free graph has linear rank-width at most* 3*.*

Below we give a short alternative proof of Corollary 1.7 (without an explicit bound) after introducing a more general strategy. A graph G is H*-free* if G does not contain the graph H as an induced subgraph, and G is $(H_1, \ldots, H_p)$*-free* for some set of graphs $\{H_1, \ldots, H_p\}$ if G is H_i-free for every $i \in \{1, \ldots, p\}$. We can now try to obtain for a caterpillar T, a constant bound on the linear rank-width of a T-pivot-minor-free graph by adapting the following *general* strategy:

Step 1. Show that the class of T-pivot-minor-free graphs is a subclass of a class of (H_1, H_2)-free graphs for some graphs H_1 and H_2.

Step 2. Show that this class of (H_1, H_2)-free graphs has bounded linear rank-width.

An advantage of this strategy is that it will lead to a stronger result that forms the start of a *systematic* study into boundedness of linear rank-width of (H_1, H_2)-free graphs. This would address Open Problem 7.5 in [22], which asks for such a result. We refer to Section 6 for a further discussion on this.

To illustrate our general strategy for the case where $H = P_4$, we can do as follows. In Step 1, we observe that every P_4-pivot-minor graph is (P_4, dart)-free (see Figure 2 for an illustration of the dart). In Step 2, we use a result of Brignall, Korpelainen, and Vatter [14], who showed that a class of (P_4, H)-free graphs has bounded linear rank-width if and only if H is a threshold graph. Hence, as the dart is a threshold graph, the class of (P_4, dart)-free graphs, and thus the class of P_4-pivot-minor-free graphs, has bounded linear rank-width.

Whether P_n-pivot-minor-free graphs have bounded linear rank-width for $n \geqslant 5$ remains a challenging open question. In the remainder, we focus on the other tree T on four vertices besides the P_4, which is the *claw* $K_{1,3}$ (the 4-vertex star). We will prove the following result.

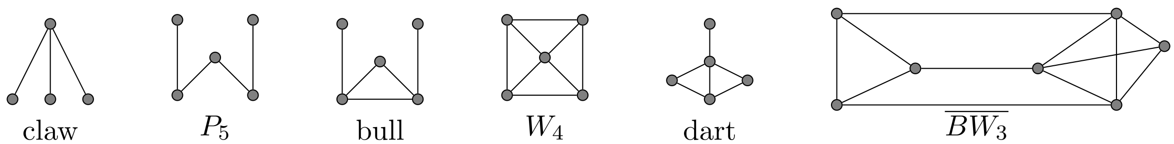

Figure 2: The graphs claw, P_5, bull, W_4, dart, and $\overline{BW_3}$.

Theorem 1.8. *Every claw-pivot-minor-free graph has linear rank-width at most* 59.

As a consequence, we have verified (the original formulation of) Conjecture 3 for every caterpillar T on at most four vertices. Since every tree on at most four vertices is a caterpillar we have in fact shown that if T is a tree on at most four vertices, then the class of T-pivot-minor-free graphs has bounded linear rank-width.

We first explain how we perform Step 1. In our previous paper [20], we proved that a graph is claw-pivot-minor-free if and only if it is $(\text{bull}, \text{claw}, P_5, W_4, \overline{BW_3})$-free; see Figure 2 for pictures of these forbidden induced graphs. It is readily seen that to prove boundedness of linear rank-width of some graph class $\mathcal{G}$ one may restrict to connected graphs in $\mathcal{G}$. In [20] we showed that a graph G is $(\text{bull}, \text{claw}, P_5)$-free if and only if every component of G is $3P_1$-free (the graph $3P_1$ consists of three isolated vertices). Hence, we derive the following result, in which we specify the graphs H_1 and H_2 of Step 1 as $H_1 = 3P_1$ and $H_2 = W_4$.

Theorem 1.9 (Dabrowski et al. [20])**.** *Let G be a connected graph. Then G is claw-pivot-minor-free if and only if G is $(3P_1, W_4, \overline{BW_3})$-free. In particular, the class of connected claw-pivot-minor-free graphs belongs to the class of $(3P_1, W_4)$-free graphs.*

As Step 2, we must prove that $(3P_1, W_4)$-free graphs have bounded linear rank-width. With an eye on a future classification of boundedness of linear rank-width for (H_1, H_2)-free graphs, we aim to prove boundedness for classes of (H_1, H_2)-free graphs that are as large as possible. For integers $1 \leqslant i \leqslant j \leqslant k$, let $S_{i,j,k}$ denote the *subdivided claw*, which is the graph obtained from the claw by subdividing its three edges $i-1$ times, $j-1$ times and $k-1$ times, respectively; see also Figure 3 and note that $S_{1,1,1} = K_{1,3}$. The *complement* of a graph G is the graph $\overline{G}$ with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv \mid u, v \in V(G) \text{ with } u \neq v \text{ and } uv \notin E(G)\}$. The graph K_3 denotes the triangle. As $3P_1 = \overline{K_3}$ and $W_4 = \overline{P_1 + 2P_2}$ is an induced subgraph of the complement of $\overline{S_{1,2,2}}$, the class of $(3P_1, W_4)$-free graphs is contained in the class of $(\overline{K_3}, \overline{S_{1,2,2}})$-free graphs. We prove the following result in Section 5:

Theorem 1.10. *Every $(K_3, S_{1,2,2})$-free graph has linear rank-width at most* 58.

As observed in Lemma 2.2 in Section 2, complementing a graph may increase its linear rank-width by at most 1. Hence, Theorems 1.9 and 1.10 and this observation imply Theorem 1.8.

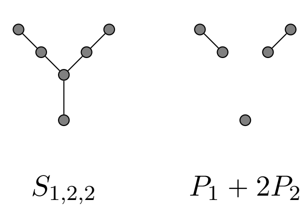

Figure 3: The graphs $S_{1,2,2}$ and $P_1 + 2P_2$; note that $P_1 + 2P_2$ is an induced subgraph of $S_{1,2,2}$.

Dabrowski et al. [19] proved that the class of $(K_3, S_{1,2,2})$-free graphs has bounded rank-width. As every class of bounded linear rank-width has bounded rank-width, but the reverse is not necessarily true, Theorem 1.10 is a strengthening of their result. Moreover, Theorem 1.10 is tight in the sense that even the class of $S_{1,2,3}$-free bipartite graphs is known to have unbounded linear rank-width [4].

It remains to prove Theorems 1.4, 1.5 and 1.10, which we do in Sections 3, 4 and 5, respectively. We discuss future research in Section 6.

2 Preliminaries

In this paper, all graphs have no loops and no multiple edges. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. For $S \subseteq V(G)$, let $G[S] = (S, \{uv \mid uv \in E, u, v \in S\})$ denote the subgraph of G induced by S. For convenience, we write $G[v_1, v_2, \ldots, v_m]$ for $G[\{v_1, v_2, \ldots, v_m\}]$. A graph H is an *induced subgraph* of G if $H = G[S]$ for some $S \subseteq V(G)$. For a vertex $v \in V(G)$, we let $G - v$ be the graph obtained from G by removing v. For a set $S \subseteq V(G)$, we let $G - S$ be the graph obtained from G by removing all vertices in S. For an edge $e \in E(G)$, we let $G - e$ be the graph obtained from G by removing e. For a set $F \subseteq E(G)$, we let $G - F$ be the graph obtained from G by removing all edges in F.

The set of neighbours of a vertex v in a graph G is denoted by $N_G(v)$. The size of $N_G(v)$ is the *degree* of v. For a set $A \subseteq V(G)$, we let $N_G(A)$ denote the set of all vertices in $V(G)\backslash A$ that have a neighbour in A. Two vertices v and w in G are *twins* if $N_G(v)\backslash\{w\} = N_G(w)\backslash\{v\}$. We say that twins v and w are *false twins* if v is not adjacent to w. An edge e of a connected graph G is a *cut edge* if $G - e$ is disconnected.

Let A and B be two disjoint vertex subsets of a graph G. We let $G \times (A, B)$ be the graph obtained from G by taking a *bipartite complementation* between A and B, that is, by replacing each edge between a vertex of A and a vertex of B by a non-edge, and vice versa. We say that A is *complete* to B if a is adjacent to b for every $a \in A$ and every $b \in B$, whereas A is *anti-complete* to B if a is not adjacent to b for every $a \in A$ and every $b \in B$. If A is complete or anti-complete to B, then A is *trivial* to B. If A consists of one vertex v, then we may say that v is complete or anti-complete to B.

A set F of edges is a *matching* in a graph G if no two edges in F have a common end-vertex. A set S of vertices in a graph G is an *independent set* if no two vertices in S are adjacent, whereas S is a *clique* if every pair of vertices in S is adjacent. The complete graph K_n is the graph on n vertices that form a clique. The complete bipartite graph $K_{n,m}$ is the bipartite graph with a bipartition (A, B) such that $|A| = n$, $|B| = m$, and A is complete to B. The graph W_n is the graph on $n + 1$ vertices that is obtained from a cycle on n vertices by adding one vertex that is made adjacent to all vertices in the cycle. The *length* of a path is the number of edges in the path.

For two graphs G and H on disjoint vertex sets, we let $G + H$ be the disjoint union of G and H, which has $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H)$. We let pG denote the disjoint union of p copies of G. The *subdivision* of an edge uv in a graph removes the edge uv and introduces a new vertex w that is made adjacent (only) to u and v.

A graph H is the 1-*subdivision* of a graph G if H is obtained from G by subdividing each edge of G exactly once. A graph G is *distance-hereditary* if for every connected induced subgraph H of G and every two vertices v, w in H, the distance between v and w in H is the same as the distance in G.

Let G be a graph with vertices $x_1, \ldots, x_n$. Let $A = A_G$ denote the *adjacency matrix* of G, that is, entry $A_G(i,j) = 1$ if x_i is adjacent to x_j and $A_G(i,j) = 0$ if x_i is not adjacent to x_j. For a subset $X \subseteq V(G)$, the matrix $A[X, V(G)\backslash X]$ is the $|X| \times |V(G)\backslash X|$ submatrix of A restricted to the rows of X and the columns of $V(G)\backslash X$. The *cut-rank function* of G is the function $\text{cutrk}_G : 2^{V(G)} \to \mathbb{N}$ such that for each $X \subseteq V(G)$,

$$\text{cutrk}_G(X) := \text{rank}(A_G[X, V(G)\backslash X]),$$

where we compute the rank *over the binary field*. A *linear ordering* of G is a permutation of the vertices of G. The *width* of a linear ordering $(x_1, \ldots, x_n)$ of G is defined as $\max_{1 \leqslant i \leqslant n}\{\text{cutrk}_G(\{x_1, \ldots, x_i\})\}$. The *linear rank-width* $\text{lrw}(G)$ of G is the minimum width over all linear orderings of G.

Let X and Y be two disjoint subsets of vertices of a graph. For an ordering $(x_1, \ldots, x_n)$ of the vertices of X and an ordering $(y_1, \ldots, y_m)$ of the vertices of Y, we define the ordering

$$(x_1, \ldots, x_n) \oplus (y_1, \ldots, y_m) := (x_1, \ldots, x_n, y_1, \ldots, y_m).$$

The cut-rank function is invariant under taking local complementation. This implies that the linear rank-width of a graph does not increase when taking its vertex-minor.

Lemma 2.1 (Bouchet [10]; See Oum [40])**.** *If G is obtained from H by a sequence of local complementations, then* $\text{cutrk}_G(X) = \text{cutrk}_H(X)$ *for all $X \subseteq V(G)$. So, if G is a vertex-minor of H, then* $\text{lrw}(G) \leqslant \text{lrw}(H)$.

We need three structural lemmas on linear rank-width. Recall that the complement of a graph G is the graph $\overline{G}$ with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv \mid u, v \in V(G) \text{ with } u \neq v \text{ and } uv \notin E(G)\}$.

Lemma 2.2. *If G has linear rank-width k, then $\overline{G}$ has linear rank-width at most $k+1$.*

Proof. Let H be the graph obtained from G by adding a vertex a complete to $V(G)$. Observe that $\overline{G} = (H * a)[V(G)]$. Then, $\text{lrw}(\overline{G}) \leqslant \text{lrw}(H * a) = \text{lrw}(H) \leqslant \text{lrw}(G) + 1$. Here, the second step follows from Lemma 2.1, and the third step follows from the fact that adding one vertex to a graph may increase the linear rank-width by at most one. □

Lemma 2.3. *Let G be a graph and A and B be two disjoint vertex subsets of G. If G has linear rank-width k, then $G \times (A, B)$ has linear rank-width at most $k+2$.*

Proof. Let H be the graph obtained from G by adding two adjacent vertices a and b such that

- $N_H(a) \cap V(G) = A$ and $N_H(b) \cap V(G) = B$.

Observe that $G \times (A, B) = (H \wedge ab)[V(G)]$. Since adding two vertices may increase the linear rank-width by at most two, we have $\text{lrw}(H \wedge ab) = \text{lrw}(H) \leqslant \text{lrw}(G) + 2$. Therefore, we have $\text{lrw}(G \times (A, B)) \leqslant \text{lrw}(G) + 2$. □

Let I be a set of pairwise twins in a graph G. We define $G//I$ as the graph obtained from G by removing all the vertices of I except one vertex.

Lemma 2.4. *Let G be a graph and $I_1, I_2, \ldots, I_m$ be pairwise disjoint subsets of $V(G)$ such that each I_i is a set of pairwise twins in G. Then* $\text{lrw}(G) \leqslant \text{lrw}(G//I_1//I_2//\cdots//I_m) + 1$.

Proof. Let $H := G//I_1//I_2//\cdots//I_m$. For each $j \in \{1,\dots,m\}$, let w_j be the vertex kept from I_j in H. Let $I := \bigcup_{j\in\{1,\dots,m\}} I_j$. Suppose that L_H is a linear ordering of H with the optimal width. We obtain a linear ordering L_G of G from L_H by replacing each $w_j \in \{w_1,\dots,w_m\}$ with any linear ordering of I_j. Let $L_G := (v_1, v_2, \dots, v_n)$. We claim that L_G has width at most $\operatorname{lrw}(H)+1$. Let $i \in \{1,\dots,n\}$, and let $L := \{v_j \mid 1 \leqslant j \leqslant i\}$ and $R := V(G)\backslash L$. It suffices to show that $\operatorname{rank}(A(G)[L,R]) \leqslant \operatorname{lrw}(H)+1$. We will use the fact that

(∗) if two rows of a matrix M are the same, then the matrix obtained from M by removing one of these rows has the same rank as M, and the same argument holds for columns.

We observe that at most one set of $I_1, I_2, \dots, I_m$ may have a vertex in both L and R. Let $J_1 := \{i \in \{1,2,\dots,m\} \mid I_i \cap L \neq \varnothing\}$ and $J_2 := \{i \in \{1,2,\dots,m\} \mid I_i \cap R \neq \varnothing\}$, and let $W_1 := \{w_i \mid i \in J_1\}$ and $W_2 := \{w_i \mid i \in J_2\}$. First assume that no set of $I_1, I_2, \dots, I_m$ has a vertex in both L and R. By (∗), we have $\operatorname{rank}\big(A(G)[L,R]\big) = \operatorname{rank}\big(A(G)[(L\backslash I)\cup W_1, (R\backslash I)\cup W_2]\big)$. As the partition $((L\backslash I)\cup W_1, (R\backslash I)\cup W_2)$ is considered when computing the width of L_H, we have $\operatorname{rank}(A(G)[L,R]) \leqslant \operatorname{lrw}(H)$. Therefore, we may assume that there is a set I_p of $I_1,\dots,I_m$ having a vertex in both L and R. Let $x \in I_p \cap L$ and $y \in I_p \cap R$. By (∗), we have

$$\begin{aligned}&\operatorname{rank}\big(A(G)[L,R]\big)\\ =&\operatorname{rank}\big(A(G)[(L\backslash I)\cup(W_1\backslash\{w_p\})\cup\{x\}, (R\backslash I)\cup(W_2\backslash\{w_p\})\cup\{y\}]\big).\end{aligned}$$

Note that one of the partitions $((L\backslash I)\cup W_1, (R\backslash I)\cup(W_2\backslash\{w_p\}))$ and $((L\backslash I)\cup(W_1\backslash\{w_p\}), (R\backslash I)\cup W_2)$ is considered when computing the width of L_H. Since $x, y, w_p \in I_p$, the matrix

$$A(G)[(L\backslash I)\cup(W_1\backslash\{w_p\})\cup\{x\}, (R\backslash I)\cup(W_2\backslash\{w_p\})\cup\{y\}]$$

can be obtained from $A(G)[(L\backslash I)\cup W_1, (R\backslash I)\cup(W_2\backslash\{w_p\})]$ by adding one column corresponding to y, and it also can be obtained from $A(G)[(L\backslash I)\cup(W_1\backslash\{w_p\}), (R\backslash I)\cup W_2]$ by adding one row corresponding to x. This implies that $\operatorname{rank}(A(G)[L,R]) \leqslant \operatorname{lrw}(H)+1$. We conclude that $\operatorname{lrw}(G) \leqslant \operatorname{lrw}(H)+1$. □

A *path decomposition* of a graph G is an ordered family $\mathcal{B} = \{B_1,\dots,B_r\}$ of subsets of $V(G)$ satisfying the following:

1. For every $v \in V(G)$ there exists a $t \in \{1,\dots,r\}$ with $v \in B_t$.
2. For every $uv \in E(G)$ there exists a $t \in \{1,\dots,r\}$ with $\{u,v\} \subseteq B_t$.
3. For every $v \in V(G)$, the set $\{t \in \{1,\dots,r\} \mid v \in B_t\}$ consists of consecutive integers.

The *width* of a path decomposition $(P,\mathcal{B})$ is defined as $\max\{|B_t| \mid t \in \{1,\dots,r\}\} - 1$. The *path-width* of G is the minimum width among all path decompositions of G.

We finish this section by proving that every tree with linear rank-width 1 is a caterpillar (recall that a caterpillar is a tree that contains a path P, such that every vertex not on P has a neighbour in P).

Theorem 2.5 (Adler and Kanté [2]). *For every tree T, the linear rank-width of a tree T is equal to the path-width of T.*

Theorem 2.6 (Takahashi, Ueno, and Kajitani [48]). *Let G be a tree and k be a positive integer. Then G has path-width at most k if and only if for every vertex v, $G - v$ has at most two connected components with path-width exactly k and all other connected components of $G - v$ have path-width less than k.*

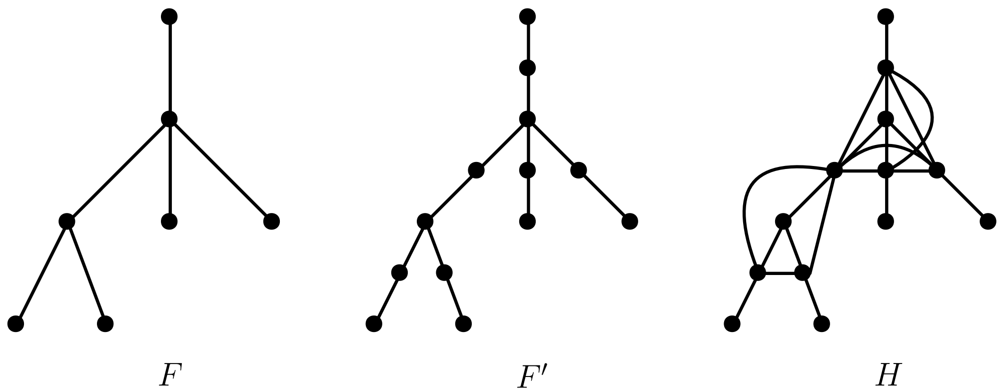

Figure 4: A tree F, the tree F' that is obtained from F by subdividing each edge of F once, and the graph $H \in \mathcal{C}$ that is obtained from F' after applying a local complementation at every vertex of degree at least 3 in F'.

Lemma 2.7. *A tree has linear rank-width at most* 1 *if and only if it is a caterpillar.*

Proof. Let T be a tree. First suppose that T is not a caterpillar. Then T contains $S_{2,2,2}$ as an induced subgraph. Thus, T contains a vertex v such that $T - v$ contains at least three connected components each containing an edge. So, $T - v$ contains three connected components having path-width at least 1. By Theorem 2.6, T has path-width at least 2, and by Theorem 2.5, T has linear rank-width at least 2.

Now suppose that T is a caterpillar. We prove by induction on $|V(T)|$ that T has path-width at most 1. We may assume that T has at least two vertices. Note that for every vertex v, $T - v$ has at most two connected components having an edge, which are still caterpillars, and all the other connected components are isolated vertices. So, $T - v$ has at most two connected components having path-width 1 by induction, and all the other connected components have path-width 0. Thus, by Theorem 2.6, T has path-width at most 1. This proves the claim. We now apply Theorem 2.5 to conclude that every caterpillar has linear rank-width at most 1. □

3 The Proof of Theorem 1.4

In this section, we prove Theorem 1.4, which states that for every tree T that is not a caterpillar, the class of T-pivot-minor-free graphs has unbounded linear rank-width.

Let $\mathcal{C}$ be the class of graphs that can be obtained from the 1-subdivision of a tree by applying a local complementation at every vertex of degree at least 3. We give an example of a graph in $\mathcal{C}$ in Figure 4. Our proof of Theorem 1.4 consists of the following parts:

1. we show that $\mathcal{C}$ has unbounded linear rank-width;
2. we show that every graph in $\mathcal{C}$ is a distance-hereditary graph with some additional properties needed to prove the third step; and
3. we show that every graph in $\mathcal{C}$ is T-pivot-minor-free whenever T is a tree that is not a caterpillar.

We start with the following lemma that proves the first part.

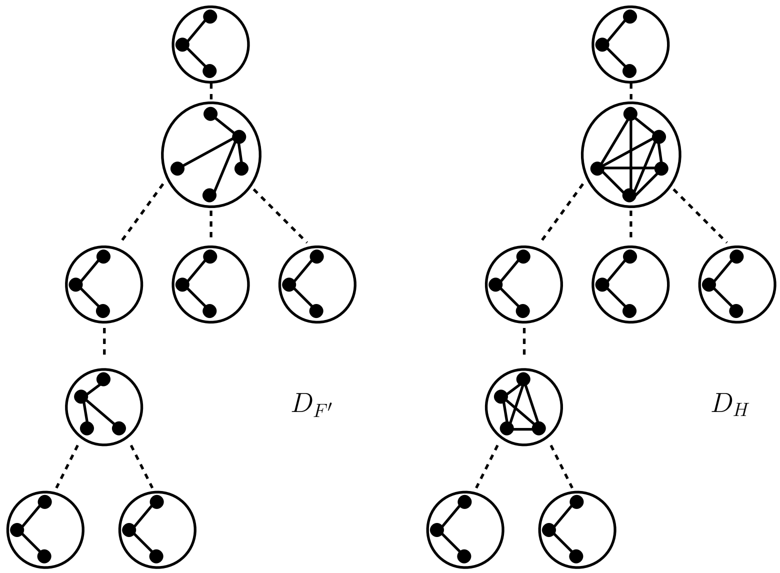

Figure 5: For the graphs F' and H in Figure 4, the decompositions $D_{F'}$ and D_H are the canonical split decompositions of F' and H, respectively. Dashed edges denote marked edges and each circle denotes a bag. Note that by the definition of a split, every bag contains at least three vertices.

Lemma 3.1. *The class $\mathcal{C}$ has unbounded linear rank-width.*

Proof. Adler and Kanté [2] proved that trees have unbounded linear rank-width. Note that the 1-subdivision H of a graph G contains G as a vertex-minor: for every subdivided vertex in H, perform a local complementation and remove it; this yields G. So, by Lemma 2.1, the class of 1-subdivisions of trees also has unbounded linear rank-width. As local complementations do not change the linear rank-width of a graph by Lemma 2.1, this means that $\mathcal{C}$ has unbounded linear rank-width. □

We will now prove that $\mathcal{C}$ is a subclass of the class of distance-hereditary graphs with some additional useful properties. In order to do this, we need the notion of a canonical split decomposition of a graph [18], which we define below.

A vertex partition (X, Y) of a connected graph G is a *split* of G if $|X| \geqslant 2, |Y| \geqslant 2$, and $N_G(Y)$ is complete to $N_G(X)$. A connected graph G on at least five vertices is *prime* if it has no split. A connected graph D with a distinguished set of edges $M(D)$ is a *marked graph* if $M(D)$ is a matching and each edge in $M(D)$ is a cut edge. An edge in $M(D)$ is a *marked edge*, and every other edge of D is an *unmarked edge*. A vertex incident with a marked edge is a *marked vertex*, and every other vertex of D is an *unmarked vertex*. Each connected component of $D - M(D)$ is a *bag* of D. If a marked edge e is incident with a vertex of a bag B, we say that B is *incident* with e. A bag B_1 of D is a neighbour bag of a bag B_2 of D if there is a marked edge incident with both B_1 and B_2. The *decomposition tree* of D is the graph obtained from D by contracting each bag into a vertex.

If G has a split (X, Y), we construct a marked graph D on the vertex set $V(G) \cup \{x_1, y_1\}$ for some new vertices x_1 and y_1 such that

- for every two distinct vertices x, y with $\{x, y\} \subseteq X$ or $\{x, y\} \subseteq Y$, the property $xy \in E(G)$

holds if and only if $xy \in E(D)$,

- $x_1 y_1$ is a new marked edge,
- X is anti-complete to Y,
- x_1 is complete to $N_G(Y)$ (with only unmarked edges) and has no neighbours in $V(G)\backslash N_G(Y)$,
- y_1 is complete to $N_G(X)$ (with only unmarked edges) and has no neighbours in $V(G)\backslash N_G(X)$.

The graph D is a *simple decomposition* of G. To obtain a split decomposition, we will recursively take a simple decomposition of a bag, and when we take a simple decomposition of a bag, all the marked vertices remain marked vertices. That is, a *split decomposition* of a connected graph G is a marked graph D defined inductively to be either G or a marked graph obtained from a split decomposition D' of G by replacing some bag B of D' by the bags of a simple decomposition B' of B and keeping all the marked edges between vertices of $V(B)$ and vertices of $V(D')\backslash V(B)$. We give an example of a split decomposition in Figure 5.

For a marked edge xy of a marked graph D, the *recomposition of D along xy* is the marked graph $(D \wedge xy) - \{x, y\}$, where when we pivot xy, we add unmarked edges between $N_D(x)\backslash\{y\}$ and $N_D(y)\backslash\{x\}$. This operation can be seen as merging two adjacent bags B_1 and B_2 into one bag B where the union of B_1 and B_2 was a simple decomposition of B. It is not hard to see that if D is a split decomposition of G, then G can be obtained from D by recomposing along all the marked edges.

A split decomposition D is *canonical* if each bag of D is either a prime graph, a star, or a complete graph, and for every marked edge xy in D, recomposing xy results in a split decomposition having a bag that is neither a prime graph, a star, nor a complete graph. We note that the split decompositions in Figure 5 are canonical. We say that a bag is a *star bag* if it is a star and a *complete* bag if it is a complete graph.

Theorem 3.2 (Cunningham and Edmonds [18]). *Every connected graph has a unique canonical split decomposition, up to isomorphism.*

Bouchet [9] described how split decompositions change under applying local complementations. A vertex v in a split decomposition D *represents* an unmarked vertex x (or is a *representative* of x) if either $v = x$ or there is a path of even length from v to x in D starting with a marked edge such that marked edges and unmarked edges appear alternately in the path. Observe that for every vertex v in a split decomposition D of a graph G, there exists a vertex $x \in V(G)$ such that v represents x. Two unmarked vertices x and y are *linked* in D if there is a path from x to y in D such that unmarked edges and marked edges appear alternately in the path.

A *local complementation* at an unmarked vertex x in a split decomposition D, denoted by $D*x$, is the operation that replaces each bag B containing a representative w of x with $B*w$.

Lemma 3.3 (Bouchet [9]). *Let D be the canonical split decomposition of a connected graph G. If x is an unmarked vertex of D, then $D * x$ is the canonical split decomposition of $G * x$.*

Let x and y be linked unmarked vertices in a split decomposition D, and let P be the path in D linking x and y where unmarked edges and marked edges appear alternately. Observe that such a path is unique. The *pivoting on xy of D*, denoted by $D \wedge xy$, is the split decomposition obtained as follows: for each bag B containing an unmarked edge vw of P, we replace B with $B \wedge vw$.

Lemma 3.4 (Adler, Kanté, and Kwon [3])**.** *Let D be the canonical split decomposition of a connected graph G. If $xy \in E(G)$, then $D \wedge xy$ is the canonical split decomposition of $G \wedge xy$.*

An *internal* edge of a tree is an edge that is not incident with a vertex of degree 1. We also need the following result due to Bouchet [9].

Lemma 3.5 (Bouchet [9])**.** *A canonical split decomposition of a tree T can be constructed by replacing each internal edge of T by a path of length 3, the middle edge of the path being the marked edge.*

A bag of a split decomposition is a *branching bag* if it is incident with at least three marked edges. Let $\mathcal{M}$ be the set of all distance-hereditary graphs in which each connected component admits a canonical split decomposition with the property that every branching bag is a complete bag. We will prove that $\mathcal{C}$ is a subclass of $\mathcal{M}$, so every graph of $\mathcal{C}$ is a distance-hereditary graph with the additional property that every branching bag in its canonical split decomposition is a complete bag. In order to do this we need one more lemma.

Lemma 3.6 (Bouchet [9])**.** *A connected graph is distance-hereditary if and only if every bag of its canonical split decomposition is either a star or a complete graph.*

Lemma 3.7. *$\mathcal{C}$ is a subclass of $\mathcal{M}$.*

Proof. Let $G \in \mathcal{C}$. By the definition of $\mathcal{C}$, we find that G is obtained from the 1-subdivision T of some tree by performing local complementations at vertices of degree at least 3. By Lemma 3.5, we can first construct the canonical split decomposition of T. By Lemma 3.3 we can then perform local complementations at corresponding unmarked vertices in the canonical split decomposition of T to obtain the canonical split decomposition of G (we refer again to Figure 5 for an example). By construction, every bag of the canonical split decomposition of G is a star or a complete bag. Hence, by Lemma 3.6, we find that G is a distance-hereditary graph. From our construction we also note that every branching bag is complete. We conclude that $G \in \mathcal{M}$. □

We will now prove that for every tree T that is not a caterpillar, $\mathcal{M}$ is T-pivot-minor-free. In order to do this we first show that $\mathcal{M}$ is closed under taking pivot-minors.

Lemma 3.8. *The class $\mathcal{M}$ is closed under taking pivot-minors.*

Proof. Let $G \in \mathcal{M}$. It is sufficient to show the following:

(1) for $v \in V(G)$, $G - v$ is in $\mathcal{M}$,

(2) for an edge $wz \in E(G)$, $G \wedge wz$ is in $\mathcal{M}$.

We may assume that G is connected. Let D be the canonical split decomposition of G.

We first show (2). Let wz be an edge of G. By Lemma 3.4, $D \wedge wz$ is the canonical split decomposition of $G \wedge wz$. Let P be the path in D linking w and z where unmarked edges and marked edges appear alternately. By the definition of pivoting in a canonical split decomposition, we obtain $D \wedge wz$ from D as follows: for each bag B containing an unmarked edge xy of P, we replace B with $B \wedge xy$. It is easy to observe that if B is a complete bag, then $B \wedge xy$ is again a complete bag, and if B is a star bag, then $B \wedge xy$ is again a star bag. Therefore, $G \wedge wz$ is again contained in $\mathcal{M}$.

It remains to prove (1). Suppose that $v \in V(G)$ and let B be the bag containing v. We may assume that D has at least two bags, otherwise the statement follows immediately.

Case 1. B is a complete bag.
If $|V(B)| \geqslant 4$, then after removing v in G, we find that B is still a complete bag of size at least 3. So, $D - v$ is a canonical split decomposition of $G - v$. If $|V(B)| = 3$, then $B - v$ is merged with one of the neighbour bags of B. This process does not change the type of the neighbour bag. It is possible that the two neighbour bags B_1 and B_2 of B in D are star bags, and they can be merged after $B - v$ is merged with a neighbour bag. In this case, each of B_1 and B_2 has at most two neighbour bags, and after merging B_1 and B_2, it is again a star bag that has at most two neighbour bags. Thus, $G - v$ is in $\mathcal{M}$ again.

Case 2. B is a star bag and v is a leaf of B.
If $|V(B)| \geqslant 4$, then $D - v$ is the canonical split decomposition of $G - v$. If $|V(B)| = 3$, then $B - v$ is merged with one of the neighbour bags of B. This process does not change the type of the neighbour bag. There might be two cases where the two neighbour bags B_1 and B_2 of B in D are merged after $B - v$ is merged with a neighbour bag. If B_1 and B_2 are complete bags, then the merged bag becomes a complete bag. If B_1 and B_2 are star bags, then each of them has at most two neighbour bags in D, and after merging, the new bag is again a star bag that has at most two neighbour bags. So, $G - v$ is in $\mathcal{M}$.

Case 3. B is star bag and v is the center of B.
Then v is a cut vertex of G; that is, $G - v$ is disconnected. Furthermore, each component of $G - v$ either consists of a single vertex, or it admits a split decomposition obtained from a connected component of $D - V(B)$ by removing a leaf of a star bag or a vertex in a complete bag. Thus, each component of $G - v$ is in $\mathcal{M}$, and thus, $G - v$ is also in $\mathcal{M}$. This concludes the proof of the lemma. □

We also need a known characterization of graphs of linear rank-width at most 1 in terms of their canonical split decompositions.

Lemma 3.9 (Kanté and Kwon [30])**.** *Let G be a connected graph with canonical split decomposition D. Then G has linear rank-width at most 1 if and only if G is distance-hereditary and the decomposition tree of D is a path.*

We are now ready to prove Theorem 1.4.

Theorem 1.4 (restated). *If T is a tree that is not a caterpillar, then the class of T-pivot-minor-free distance-hereditary graphs has unbounded linear rank-width.*

Proof. Let T be a tree that is not a caterpillar. As $\mathcal{C} \subseteq \mathcal{M}$ by Lemma 3.7 and $\mathcal{C}$ has unbounded linear rank-width by Lemma 3.1, it follows that $\mathcal{M}$ has unbounded linear rank-width. Moreover, $\mathcal{M}$ is a subclass of the class of distance-hereditary graphs. Hence, to prove the theorem it remains to show that every graph in $\mathcal{M}$ is T-pivot-minor-free.

Suppose, for contradiction, that T is a pivot-minor of some graph $H \in \mathcal{M}$. As $\mathcal{M}$ is closed under pivot-minors by Lemma 3.8, we find that $T \in \mathcal{M}$. Let L_T be a canonical split decomposition of T. Since T is a tree, L_T has no complete bags. So, by the definition of

$\mathcal{M}$, L_T has no branching bags, and thus, the decomposition tree of L_T is a path. Since T is distance-hereditary and the decomposition tree of L_T is a path, T has linear rank-width at most 1 by Lemma 3.9. Thus, by Lemma 2.7, T is a caterpillar, a contradiction. □

4 The Proof of Theorem 1.5

In this section, we prove Theorem 1.5, which states that P_n-pivot-minor-free distance-hereditary graphs have bounded linear rank-width.

To prove Theorem 1.5, we use the canonical split decomposition of a distance-hereditary graph, discussed in Section 3. A sequence $B_1, B_2, \ldots, B_n$ of distinct bags in a canonical split decomposition is a *path of bags* if for each $i \in \{1, \ldots, n-1\}$, B_{i+1} is a neighbour bag of B_i. As we will explain in the proof of Theorem 1.5, it follows from the definition of a canonical split decomposition, that at least half of the bags in a path of bags are star bags. By applying some pivot operations, we can extract a long path as a pivot-minor in this case. So, we may assume that the decomposition tree of the canonical split decomposition of a given graph has no long path. We use the following result which relates the path-width of a decomposition tree and the linear rank-width of the graph to conclude the theorem.

Proposition 4.1 (Kanté and Kwon [30])**.** *Let D be the canonical split decomposition of a connected distance-hereditary graph G, and let T_D be the decomposition tree of D. Then $\frac{1}{2}\operatorname{pw}(T_D) \leqslant \operatorname{lrw}(G) \leqslant \operatorname{pw}(T_D) + 1$.*

We also use a tight version of Theorem 1.1. We will use the fact that if a graph contains a minor isomorphic to P_n, then it also contains a subgraph isomorphic to P_n.

Theorem 4.2 (Bienstock, Robertson, Seymour, and Thomas [6])**.** *For every tree T on n vertices, the class of T-minor-free graphs has path-width at most $n-2$.*

We are now ready to prove Theorem 1.5.

Theorem 1.5 (restated). *Let $n \geqslant 3$ be an integer. Every P_n-pivot-minor-free distance-hereditary graph has linear rank-width at most $2n-5$.*

Proof. Let G be a distance-hereditary graph having no pivot-minor isomorphic to P_n. We will show that G has linear rank-width at most $2n-5$. We may assume that G is connected. Let D be the canonical split decomposition of G and let T_D be its decomposition tree.

We claim that D has no path with $2n-4$ bags. Suppose that such a path of bags $B_1, B_2, \ldots, B_{2n-4}$ exists. As no two complete bags are neighbour bags in a canonical split decomposition, at most $n-2$ bags in the sequence are complete bags. Thus, there are at least $n-2$ star bags. Let $B_{i_1}, B_{i_2}, \ldots, B_{i_t}$ be the sequence of all star bags in $B_1, B_2, \ldots, B_{2n-4}$ where $1 \leqslant i_1 < i_2 < \cdots < i_t \leqslant 2n-4$. For convenience, we assign $B_0 = B_{2n-3} = \varnothing$.

We claim that for every $k \in \{1, \ldots, t\}$, there is a graph G_k pivot-equivalent to G with a canonical split decomposition D_k such that

(∗) for every $k' \in \{1, \ldots, k\}$, the bag $D_k[V(B_{i_{k'}})]$ is a star bag whose center has no neighbour in $V(B_{i_{k'}-1}) \cup V(B_{i_{k'}+1})$.

For $k = 1$, suppose that D does not satisfy the property (∗). Choose a vertex v' in B_{i_1} that has no neighbour in $V(B_{i_1-1}) \cup V(B_{i_1+1})$. Such a vertex exists, as each bag has at least three vertices. Let v be the vertex of G represented by v' (recall that for every marked vertex, there is a vertex of G that it represents). We further choose a vertex w of G represented by the center w' of B_{i_1}. Since $v'w'$ is an edge of B_{i_1}, v is linked to w in D, and therefore, v is adjacent to w in G.

Note that for a star H with a leaf b and center a, $H \wedge ab$ is the star with vertex set $V(H)$ whose center is b. Thus, in $D \wedge vw$, v' becomes the center of $B_{i_1} \wedge v'w'$, which has no neighbour in $V(B_{i_1-1}) \cup V(B_{i_1+1})$. Thus, $G_1 = G \wedge vw$ and $D_1 = D \wedge vw$ satisfy $(*)$.

Now, assume that $k > 1$ and the property $(*)$ is satisfied for $k-1$. If the center of $D_{k-1}[V(B_{i_k})]$ has no neighbour in $V(B_{i_k-1}) \cup V(B_{i_k+1})$, then $G_k = G_{k-1}$ and $D_k = D_{k-1}$ satisfy $(*)$. So, we may assume that the center of $D_{k-1}[V(B_{i_k})]$ has a neighbour in $V(B_{i_k-1}) \cup V(B_{i_k+1})$. We distinguish two cases.

Case 1. The center of $D_{k-1}[V(B_{i_k})]$ has a neighbour in $V(B_{i_k-1})$.

Note that if $D_{k-1}[V(B_{i_k-1})]$ is a star bag, then by the inductive hypothesis, its center has no neighbour in $V(B_{i_k})$. But this is not possible by the definition of a canonical split decomposition. Thus, $D_{k-1}[V(B_{i_k-1})]$ is a complete bag. We choose a vertex v in G_{k-1} represented by a vertex in $D_{k-1}[V(B_{i_k-1})]$ having no neighbour in $V(B_{i_k-2}) \cup V(B_{i_k})$ and choose a vertex w in G_{k-1} represented by a vertex in $D_{k-1}[V(B_{i_k})]$ having no neighbour in $V(B_{i_k-1}) \cup V(B_{i_k+1})$. Observe that v is adjacent to w in G_{k-1}, because the center of $D_{k-1}[V(B_{i_k})]$ has a neighbour in $V(B_{i_k-1})$. Thus, in $D_{k-1} \wedge vw$, the bag induced by $V(B_{i_k})$ is a star bag whose center has no neighbour in $V(B_{i_k-1}) \cup V(B_{i_k+1})$. As the bags on $V(B_{i_1}), \ldots, V(B_{i_{k-1}})$ are not changed by this pivot operation, $G_k = G_{k-1} \wedge vw$ and $D_k = D_{k-1} \wedge vw$ satisfy $(*)$.

Case 2. The center of $D_{k-1}[V(B_{i_k})]$ has a neighbour in $V(B_{i_k+1})$.

We choose a vertex v in G_{k-1} represented by a vertex in $D_{k-1}[V(B_{i_k+1})]$ having no neighbour in $V(B_{i_k}) \cup V(B_{i_k+2})$ and choose a vertex w in G_{k-1} represented by a vertex in $D_{k-1}[V(B_{i_k})]$ having no neighbour in $V(B_{i_k-1}) \cup V(B_{i_k+1})$ and linked to v in D_{k-1}. Such a vertex w exists, because the center of $D_{k-1}[V(B_{i_k})]$ has a neighbour in $V(B_{i_k+1})$, and thus, if $D_{k-1}[V(B_{i_k+1})]$ is a star, then its center has a neighbour in $V(B_{i_k})$. This implies that v is adjacent to w in G_{k-1}. In $D_{k-1} \wedge vw$, the bag induced by $V(B_{i_k})$ is a star bag whose center has no neighbour in $V(B_{i_k-1}) \cup V(B_{i_k+1})$. As the bags on $V(B_{i_1}), \ldots, V(B_{i_{k-1}})$ are not changed by this pivot operation, $G_k = G_{k-1} \wedge vw$ and $D_k = D_{k-1} \wedge vw$ satisfy $(*)$.

Hence, we have found that the claim holds.

Now, in D_t, let v_j be a vertex of G_t represented by the center of $D_t[V(B_{i_j})]$ for each $j \in \{1, \ldots, t\}$, and let v_0 be a vertex of G_t represented by a leaf of $D_t[V(B_{i_1})]$ which has no neighbour in $V(B_{i_1+1})$, and let v_{t+1} be a vertex of G_t represented by a leaf of $D_t[V(B_{i_t})]$ which has no neighbour in $V(B_{i_t-1})$. It is not difficult to check that $v_0v_1v_2 \cdots v_tv_{t+1}$ is an induced path of G_t on $t+2 \geqslant n$ vertices. This contradicts the assumption that G has no pivot-minor isomorphic to P_n. We conclude that D has no path with $2n-4$ bags $B_1, B_2, \ldots, B_{2n-4}$.

The above means that the decomposition tree T_D has no path on $2n-4$ vertices. By Theorem 4.2, T_D has path-width at most $2n-6$. By Proposition 4.1, G has linear rank-width at most $2n-5$. □

5 The Proof of Theorem 1.10

In this section, we prove Theorem 1.10, which states that the class of $(K_3, S_{1,2,2})$-free graphs has linear rank-width at most 58. We prove the following statements in this order:

1. bipartite $2P_2$-free graphs, which form a subclass of bipartite $(P_1 + 2P_2)$-free graphs, have linear rank-width at most 1;
2. bipartite $(P_1 + 2P_2)$-free graphs, which form a subclass of bipartite $S_{1,2,2}$-free graphs, have linear rank-width at most 3;
3. bipartite $S_{1,2,2}$-free graphs have linear rank-width at most 3;
4. non-bipartite $(K_3, C_5, S_{1,2,2})$-free graphs have linear rank-width at most 3; and
5. $(K_3, S_{1,2,2})$-free graphs with an induced C_5 have linear rank-width at most 58.

Note that Statements 3–5 cover all cases for proving Theorem 1.10. So, we first consider bipartite $2P_2$-free graphs.

Lemma 5.1. *Every bipartite $2P_2$-free graph has linear rank-width at most* 1.

Proof. Let G be a bipartite $2P_2$-free graph with bipartition (A, B). It is well known [49] that a bipartite graph with bipartition (X_1, X_2) is $2P_2$-free if and only if it is a bipartite chain graph, that is, for each $i \in \{1, 2\}$, the neighbourhoods of the vertices in X_i can be ordered linearly with respect to the inclusion relation. We may assume that G is connected. Hence, as G is $2P_2$-free, we can define a sequence $A_1, A_2, \ldots, A_m$ of pairwise vertex-disjoint subsets of A such that

- $A_1 \cup A_2 \cup \cdots \cup A_m = A$,
- each A_i is a maximal set of pairwise twins in G,
- for integers $i, j \in \{1, \ldots, m\}$ with $i < j$, $N_G(A_i) \subsetneq N_G(A_j)$.

If $m = 1$, then G is complete bipartite. In this case, we take a linear ordering L_1 of A_1 and a linear ordering L_2 of $V(G) \backslash A_1$ arbitrarily. It is not hard to see that $L_1 \oplus L_2$ is a linear ordering of width at most 1. Hence G has linear rank-width at most 1.

Now suppose that $m \geqslant 2$. In this case, G is not complete bipartite. Notice that for each $i \in \{2, \ldots, m\}$, there is a vertex $v \in B$ that has a neighbour in A_i but does not have a neighbour in $A_1 \cup \cdots \cup A_{i-1}$; otherwise, vertices in $A_{i-1} \cup A_i$ have the same neighbourhood in B, which contradicts the maximality of A_i. For each $i \in \{2, \ldots, m\}$, let $B_i := N_G(A_i) \backslash N_G(A_{i-1})$, and let $B_1 := N_G(A_1)$. Since G is connected, we have $B = B_1 \cup B_2 \cup \cdots \cup B_m$.

For each $i \in \{1, \ldots, m\}$, let L_i^A be an ordering of A_i and L_i^B be an ordering of B_i. It is not difficult to check that the linear ordering $L_1^B \oplus L_1^A \oplus L_2^B \oplus L_2^A \oplus \cdots \oplus L_m^B \oplus L_m^A$ has width at most 1. □

We now consider bipartite $(P_1 + 2P_2)$-free graphs and show the following lemma.

Lemma 5.2. *Every bipartite $(P_1 + 2P_2)$-free graph has linear rank-width at most* 3.

Proof. Let G be a bipartite $(P_1 + 2P_2)$-free graph with bipartition (A, B). Then $G \times (A, B)$ is P_5-free. We may assume without loss of generality that $G \times (A, B)$ is connected. Then, as $G \times (A, B)$ is also bipartite, $G \times (A, B)$ is readily seen to be $2P_2$-free. By Lemma 5.1, $G \times (A, B)$ has linear rank-width at most 1. By Lemma 2.3, $G = G \times (A, B) \times (A, B)$ has linear rank-width at most 3. □

We now consider $S_{1,2,2}$-free bipartite graphs and need two results by Lozin [35].

Lemma 5.3 (Lozin [35])**.** *Every connected bipartite $(S_{1,2,2}, P_7)$-free graph is $(P_1 + 2P_2)$-free.*

Lemma 5.4 (Lozin [35])**.** *Let G be a bipartite $S_{1,2,2}$-free graph with no twins. If G contains an induced P_7, then G is $K_{1,3}$-free (and thus G has maximum degree at most 2).*

Proposition 5.5. *Every bipartite $S_{1,2,2}$-free graph has linear rank-width at most* 3.

Proof. Let G be a bipartite $S_{1,2,2}$-free graph with bipartition (A, B). We may assume that G is connected. If G is P_7-free, then by Lemma 5.3, G is $(P_1 + 2P_2)$-free, and by Lemma 5.2, G has linear rank-width at most 3. Thus, we may assume that G contains an induced subgraph isomorphic to P_7. Note that G is not a complete bipartite graph.

Let $I_1, I_2, \ldots, I_m$ be the vertex partition of G such that each I_i is a maximal set of pairwise twins in G. Since G is connected and bipartite, each I_i is contained in one of A or B. Let $G_1 := G//I_1//I_2//\cdots//I_m$. Note that G_1 is also connected. We claim that G_1 has no twins. Note that G_1 is not an edge, because G is not a complete bipartite graph. So, G_1 has at least three vertices. Moreover, G_1 still has an induced subgraph isomorphic to P_7, as P_7 has no twins.

Suppose for contradiction that G_1 has two twins v_1 and v_2, and assume that v_1 and v_2 were identified from I_{i_1} and I_{i_2} for some i_1 and i_2, respectively. Since each v_i has a neighbour and G_1 has at least three vertices, v_1 and v_2 are in the same part of the bipartition, and thus I_{i_1} and I_{i_2} are contained in the same part of the bipartition of G. Thus I_{i_1} and I_{i_2} have the same neighbourhoods in G, contradicting the fact that they are maximal sets of pairwise twins in G. So, G_1 has no twins. Then G_1 has linear rank-width at most 2 because by Lemma 5.4 every vertex has degree at most 2. By Lemma 2.4, we conclude that G has linear rank-width at most 3. ☐

We now consider $(K_3, C_5, S_{1,2,3})$-free graphs and need the following result as a lemma.

Lemma 5.6 (Dabrowski, Dross, and Paulusma [19])**.** *Let G be a connected $(K_3, C_5, S_{1,2,3})$-free graph that does not contain a pair of false twins. Then G is either bipartite or an induced cycle.*

Proposition 5.7. *Every non-bipartite $(K_3, C_5, S_{1,2,2})$-free graph has linear rank-width at most* 3.

Proof. Let G be a connected non-bipartite $(K_3, C_5, S_{1,2,2})$-free graph. By Lemma 5.6, G is a graph obtained from an induced cycle $C = c_1c_2\cdots c_kc_1$ by adding false twins. For each $i \in \{1, \ldots, k\}$, let U_i be the maximal set of false twins containing c_i in G. As $G//U_1//U_2//\cdots//U_k$ is isomorphic to C and C has linear rank-width at most 2, by Lemma 2.4, we find that G has linear rank-width at most 3. ☐

We now consider $(K_3, S_{1,2,2})$-free graphs that contain an induced C_5. We first introduce some additional terminology and lemmas. A graph is 3-*partite* if its vertex set can be partitioned into three independent sets. We need the following known result.

Theorem 5.8 (Alecu et al. [5])**.** *Let G be a 3-partite graph on n vertices with vertex partition (V_1, V_2, V_3) such that*

(a) for every $a \in V_1$, $b \in V_2$, $c \in V_3$, $G[a, b, c]$ is isomorphic to neither K_3 nor $3P_1$,

(b) $G[V_1 \cup V_2]$*,* $G[V_1 \cup V_3]$*, and* $G[V_2 \cup V_3]$ *are* $2P_2$*-free.*

Then $V(G)$ *admits a linear ordering* $x_1, \ldots, x_n$ *and a labelling* $\ell : V(G) \to \{a, b, c\}$ *such that* $x_i x_j \in E(G)$ *for* $i < j$ *if and only if* $(\ell(x_i), \ell(x_j)) \in \{(a, b), (b, c), (c, a)\}$*.*

The following lemma follows from Theorem 5.8 after observing that each cut of the linear ordering $x_1, \ldots, x_n$ has cut-rank at most 3, because it has at most three different rows.

Lemma 5.9. *Let* G *be a* 3*-partite graph with vertex partition* (V_1, V_2, V_3) *such that*

(a) for every $a \in V_1$*,* $b \in V_2$*,* $c \in V_3$*,* $G[a, b, c]$ *is isomorphic to neither* K_3 *nor* $3P_1$*,*

(b) $G[V_1 \cup V_2]$*,* $G[V_1 \cup V_3]$*, and* $G[V_2 \cup V_3]$ *are* $2P_2$*-free.*

Then G *has linear rank-width at most* 3*.*

Let G be a graph and V_1, V_2, V_3 be three pairwise disjoint independent sets of G. We denote the subgraph of G induced by $V_1 \cup V_2 \cup V_3$ as $G[V_1, V_2, V_3]$. Moreover, if $G[V_1, V_2, V_3]$ satisfies conditions (a) and (b) in Lemma 5.9, then we say that $G[V_1, V_2, V_3]$ is *nice*.

We are now ready to prove the following result. We note that Brandstädt, Mahfud and Mosca [12] gave an alternative proof of the result from [19], which shows that $(K_3, S_{1,2,2})$-free graphs have bounded rank-width. Some parts of the proof of our result below are similar to parts of the proof of [12]. As we need to use slightly different arguments, we have chosen to keep our proof self-contained. However, we explicitly indicate whenever there is overlap between our arguments and the ones used in [12].

Proposition 5.10. *Every* $(K_3, S_{1,2,2})$*-free graph that contains an induced* C_5 *has linear rank-width at most* 58*.*

Proof. Let G be a $(K_3, S_{1,2,2})$-free graph that contains an induced subgraph C isomorphic to C_5. We may assume without loss of generality that G is connected. We write $C = c_1c_2c_3c_4c_5c_1$ and interpret subscripts modulo 5. Let $U := V(G) \backslash V(C)$.

Since G is K_3-free, every vertex in U has either no neighbours in C or exactly one neighbour in C or exactly two neighbours, which are not consecutive in C.

We claim that every vertex of U has a neighbour in C. This can be seen as follows. Suppose, for contradiction, that U contains a vertex that has no neighbour in C. As G is connected, this means that U contains two vertices w_1 and w_2, such that w_1 has a neighbour in C and w_2 is adjacent to w_1 but w_2 has no neighbour in C. If w_1 has exactly one neighbour c_i in C, then $G[c_i, c_{i+1}, c_{i-1}, c_{i-2}, w_1, w_2]$ is isomorphic to $S_{1,2,2}$. If w_1 has two neighbours c_{i-1}, c_{i+1} in C, then $G[w_1, w_2, c_{i-1}, c_i, c_{i-2}, c_{i-3}]$ is isomorphic to $S_{1,2,2}$. However, both cases are not possible, as G is $S_{1,2,2}$-free. Hence, we conclude that every vertex in U has a neighbour in C.

Consequently, we can partition U into ten parts $\{V_1, \ldots, V_5, W_1, \ldots, W_5\}$ such that for each $i \in \{1, \ldots, 5\}$,

- V_i is the set of vertices whose unique neighbour in C is c_i, and
- W_i is the set of vertices that are adjacent to c_{i-1} and c_{i+1}.

Each set in $\{V_1, \ldots, V_5, W_1, \ldots, W_5\}$ is an independent set as G is K_3-free. We verify basic relations between these parts. Let $i \in \{1, \ldots, 5\}$.

(1) V_i is complete to $V_{i-1} \cup V_{i+1}$, and anti-complete to $V_{i-2} \cup V_{i+2}$.

Suppose, for contradiction, that there are $a \in V_i$ and $b \in V_{i+1}$ that are not adjacent. Then $G[c_i, a, c_{i+1}, b, c_{i-1}, c_{i-2}]$ is isomorphic to $S_{1,2,2}$, a contradiction. Thus, there are no such vertices. This implies that V_i is complete to V_{i+1}, and by symmetry also complete to V_{i-1}. Suppose that there are $a \in V_i$ and $b \in V_{i+2}$ that are adjacent. Then $G[c_i, c_{i+1}, a, b, c_{i-1}, c_{i-2}]$ is isomorphic to $S_{1,2,2}$, a contradiction. Hence, V_i is anti-complete to V_{i+2} and by symmetry also anti-complete to V_{i-2}. ◇

(2) W_i is anti-complete to $W_{i-2} \cup W_{i+2}$.

This is because G is K_3-free. ◇

(3) W_i is complete to V_i, and anti-complete to $V_{i-1} \cup V_{i+1}$.

Suppose that there are vertices $a \in W_i$ and $b \in V_i$ that are not adjacent to each other. Then $G[c_{i-1}, a, c_i, b, c_{i-2}, c_{i-3}]$ is isomorphic to $S_{1,2,2}$, a contradiction. This implies that W_i is complete to V_i. As G is K_3-free, W_i is anti-complete to $V_{i-1} \cup V_{i+1}$. ◇

(4) $G[V_i \cup W_{i+2}]$, $G[V_i \cup W_{i-2}]$, $G[W_i \cup W_{i+1}]$ are $2P_2$-free.

Suppose that there are $a_1, a_2 \in V_i$ and $b_1, b_2 \in W_{i+2}$ such that $a_1b_1, a_2b_2 \in E(G)$ and $a_1b_2, a_2b_1 \notin E(G)$. Then $G[c_i, c_{i-1}, a_1, b_1, a_2, b_2]$ is isomorphic to $S_{1,2,2}$, a contradiction. So, $G[V_i \cup W_{i+2}]$ is $2P_2$-free, and by symmetry $G[V_i \cup W_{i-2}]$ is $2P_2$-free. Suppose that there are $a_1, a_2 \in W_i$ and $b_1, b_2 \in W_{i+1}$ such that $a_1b_1, a_2b_2 \in E(G)$ and $a_1b_2, a_2b_1 \notin E(G)$. Then $G[c_{i-1}, c_{i-2}, a_1, b_1, a_2, b_2]$ is isomorphic to $S_{1,2,2}$, a contradiction. ◇

(5) For $v \in V_i, w \in W_{i+2}, z \in W_{i-2}$, $\{v, w, z\}$ is not an independent set.

Suppose that such v, w, z forming an independent set exist. Then $G[c_i, v, c_{i+1}, w, c_{i-1}, z]$ is isomorphic to $S_{1,2,2}$, a contradiction. ◇

By Claims (4) and (5) and the fact that G is K_3-free, we deduce that for each $i \in \{1, \ldots, 5\}$, $G[V_i, W_{i-2}, W_{i+2}]$ is a nice 3-partite graph. Moreover, V_i is complete to V_{i-1}, V_{i+1}, W_i and anti-complete to $V_{i+2}, V_{i+3}, W_{i-1}, W_{i+1}$, whereas W_i is anti-complete to W_{i-2}, W_{i+2}. By doing bipartite complementations between V_i and $V_{i+1} \cup W_i$ for each $i \in \{1, \ldots, 5\}$, we may assume that each edge not incident to a vertex of C belongs to $G[V_i, W_{i+2}, W_{i-2}]$ for some $i \in \{1, \ldots, 5\}$.

Our goal is now to compute a refined set of pairwise non-intersecting 3-partite graphs by doing a small number of bipartite complementations such that each new 3-partite graph satisfies conditions (a) and (b) of Lemma 5.9.

We first observe that each $G[V_i, W_{i-2}, W_{i+2}]$ intersects only $G[V_{i+1}, W_{i-1}, W_{i-2}]$ and $G[V_{i-1}, W_{i+2}, W_{i+1}]$. We now aim, for each $i \in \{1, \ldots, 5\}$, to split V_i, W_{i-2} and W_{i+2} in such a way that we can construct the desired non-intersecting 3-partite graphs after some bipartite complementations. We will use the same construction as in [12], but the way we use the different sets differs from [12].

For each $i \in \{1, \ldots, 5\}$, let us define the following partition of W_i:

$$\begin{aligned} W_i^- &= \{x \in W_i \mid x \text{ has a non-neighbour in } W_{i+1}\}, \\ W_i^+ &= \{x \in W_i \backslash W_i^- \mid x \text{ has a non-neighbour in } W_{i-1}\}, \\ W_i^* &= W_i \backslash (W_i^- \cup W_i^+). \end{aligned}$$

By definition, $W_i^+ \cup W_i^*$ is complete to W_{i+1} and W_i^- is complete to W_{i+1}^*. We claim that W_i^- is also complete to W_{i+1}^-. Suppose that a vertex $x \in W_i^-$ has a non-neighbour y in W_{i+1}^-.

Then, by definition y has a non-neighbour z in W_{i+2}. Therefore, $G[c_{i+1}, z, x, c_{i-1}, c_{i+2}, y]$ is isomorphic to $S_{1,2,2}$ because x is not adjacent to z by (2), a contradiction. Thus, W_i^- is complete to W_{i+1}^-.

We now show some relationships between the V_i's and W_i's, which were also proven in [12], but we add the proofs for completeness. Let $i \in \{1, \ldots, 5\}$.

(a) V_i is anti-complete to W_{i+2}^+ and W_{i-2}^-.

Suppose, for contradiction, that a vertex $x \in V_i$ has a neighbour $z \in W_{i+2}^+ \cup W_{i-2}^-$. First suppose that $z \in W_{i+2}^+$. Then, by definition, z has a non-neighbour $y \in W_{i+1}$. However, now $G[c_i, c_{i-1}, x, z, y, c_{i+2}]$ is isomorphic to $S_{1,2,2}$, a contradiction. Now suppose that $z \in W_{i-2}^-$. Then, by definition, z has a non-neighbour $y \in W_{i-1}$. However, now $G[c_i, c_{i+1}, y, c_{i-2}, x, z]$ is isomorphic to $S_{1,2,2}$, another contradiction. $\diamond$

(b) If a vertex $x \in V_i$ has a neighbour in W_{i+2}^* (resp. W_{i-2}^*), then x is complete to W_{i+2}^- and anti-complete to W_{i-2} (resp. complete to W_{i-2}^+ and anti-complete to W_{i+2}).

Suppose $x \in V_i$ has a neighbour y in W_{i+2}^*. Because W_{i+2}^* is complete to $W_{i+3} = W_{i-2}$, we find that x is anti-complete to W_{i-2}. Suppose that x has a non-neighbour z in W_{i+2}^-. By definition, there is a vertex $w \in W_{i-2}$ not adjacent to z, but adjacent to y. Then, $G[y, x, w, c_{i-1}, c_{i+1}, z]$ is isomorphic to $S_{1,2,2}$, a contradiction.

Now suppose that $x \in V_i$ has a neighbour y in W_{i-2}^*. Because by definition W_{i-2}^* is complete to $W_{i-3} = W_{i+2}$, we find that x is anti-complete to W_{i+2}. Suppose that x has a non-neighbour z in W_{i-2}^+. By definition, z has a non-neighbour w in W_{i+2}, which is adjacent to y. Again, $G[y, x, w, c_{i+1}, c_{i-1}, z]$ is isomorphic to $S_{1,2,2}$, a contradiction. $\diamond$

For each $i \in \{1, \ldots, 5\}$, let

$$\begin{aligned} V_i^1 &= \{x \in V_i \mid x \text{ has a neighbour in } W_{i+2}^*\}, \\ V_i^2 &= \{x \in V_i \mid x \text{ has a neighbour in } W_{i-2}^*\}, \\ V_i^3 &= \{x \in V_i \mid x \text{ has no neighbour in } W_{i+2}^* \cup W_{i-2}^*\}. \end{aligned}$$

From (a), we know that for each $x \in V_i$, in the graph $G[V_i \cup W_{i-2} \cup W_{i+2}]$, x can have neighbours in only $W_{i+2}^- \cup W_{i+2}^*$ and in $W_{i-2}^+ \cup W_{i-2}^*$. From (b), every vertex in V_i^1 is complete to W_{i+2}^- and anti-complete to W_{i-2}, and every vertex in V_i^2 is complete to W_{i-2}^+ and anti-complete to W_{i+2}. Thus, V_i^1, V_i^2, V_i^3 are disjoint sets.

The common neighbours between a vertex $x \in V_i$ and $y \in V_{i+1}$ are in W_{i-2}^* and the common neighbours between $x \in V_i$ and $y \in V_{i-1}$ are in W_{i+2}^*. For each $i \in \{1, \ldots, 5\}$, we define the following 3-partite graphs:

$$\begin{aligned} G_i^1 &= G[V_i^1, V_{i-1}^2, W_{i+2}^*], \\ G_i^2 &= G[V_i^3, W_{i+2}^-, W_{i-2}^+]. \end{aligned}$$

Now, we do some bipartite complementations. First we do bipartite complementations between $W_i^+ \cup W_i^*$ and W_{i+1}, and between W_i^- and $W_{i+1}^- \cup W_{i+1}^*$ for each i. The resulting graph has the property that the edges between W_i and W_{i+1} are always between W_i^- and W_{i+1}^+. Secondly, we do bipartite complementations between V_i^1 and W_{i+2}^-, and V_i^2 and W_{i-2}^+ for each i. Lastly, we remove C. Observe that the edges of the remaining graph are all contained in one G_i^j for some $i \in \{1, \ldots, 5\}$ and $j \in \{1, 2\}$, and by definition the G_i^j's are pairwise disjoint.

Because all of the graphs G_i^j are disjoint, the linear rank-width of the resulting graph is the maximum linear rank-width of its connected components. Since each connected component is a nice 3-partite graph, by Lemma 5.9, we conclude that the linear rank-width of the resulting graph is at most 3. We will now count the number of times we applied bipartite complementations. For each $i \in \{1, \ldots, 5\}$, we did one bipartite complementation to keep only the edges between V_i and $W_{i-2} \cup W_{i+2}$, and then two bipartite complementations to remove the edges between W_i and W_{i+1} apart from those between W_i^- and W_{i+1}^+, and finally two bipartite complementations to remove the edges between V_i^1 and W_{i+2}^-, and V_i^2 and W_{i-2}^+, that is, in total five bipartite complementations, resulting in a total of 25 bipartite complementations. So, by Lemma 2.3, the graph $G - V(C)$ has linear rank-width at most $3 + 2 * 25 = 53$. Because $|V(C)| = 5$, we conclude that the linear rank-width of G is at most 58. □

We are now ready to prove Theorem 1.10.

Theorem 1.10 (restated). *Every $(K_3, S_{1,2,2})$-free graph has linear rank-width at most* 58.

Proof. Let G be a $(K_3, S_{1,2,2})$-free graph. We may assume that G is connected. If G is bipartite, we use Proposition 5.5. If G is non-bipartite but C_5-free, we use Proposition 5.7. In the remaining case, we use Proposition 5.10. □

6 Concluding Remarks

In this paper we researched the relationship between pivot-minors and boundedness of linear rank-width. We first proved that for every tree T that is not a caterpillar, the class of T-pivot-minor-free graphs has unbounded linear rank-width. We then posed Conjecture 3, which states that an affirmative answer can be found whenever T *is* a caterpillar. We were only able to give an affirmative answer to this conjecture that holds for every caterpillar T, if the class of T-pivot-minor-free graphs is, in addition, also distance-hereditary. We also proved that the class of $K_{1,3}$-pivot-minor-free graphs has bounded linear rank-width. As a next step for proving Conjecture 3, it seems natural to consider the case where $T = K_{1,r}$ for $r \geqslant 4$. We also proved Conjecture 3 for $P = P_4$. Since Conjecture 3 is equivalent to the alternative conjecture that for every path P, the class of P-pivot-minor-free graphs has bounded linear rank-width, the case where $P = P_5$ is another interesting open case.

For obtaining our results (in particular, the case where $T = K_{1,3}$) we followed a general strategy consisting of two steps. We believe this strategy is also useful for making further progress towards Conjecture 3. However, Step 1 of the strategy requires us to find a hereditary graph class (class of graphs that can be characterized by a set of forbidden induced subgraphs) that contains the class of T-pivot-minor-free graphs under consideration. In general, finding an appropriate hereditary graph class is a challenging task.

The fact that $K_{1,3}$-pivot-minor-free graphs have bounded linear rank-width follows from a stronger result that we showed, namely that $(K_3, S_{1,2,2})$-free graphs have bounded linear rank-width. Showing this stronger result will be useful for a systematic study on the boundedness of linear rank-width of (H_1, H_2)-free graphs. Such a classification already exists for H-free graphs, as observed in [22]: for a graph H, the class of H-free graphs has bounded linear rank-width if and only if H is a subgraph of P_3 not isomorphic to $3P_1$. We note that similar classifications also exist for other width parameters: for the tree-width of (H_1, H_2)-free graphs [7], which was later generalized to a classification for tree-width of $\mathcal{H}$-free graphs, where $\mathcal{H}$ is a finite set of graphs [37], rank-width of H-free graphs (see [24]), rank-width of H-free bipartite graphs [23, 36, 38], and up to five non-equivalent open cases, rank-width of (H_1, H_2)-free graphs (see [8] or [22]), and for the mim-width of H-free graphs [13], whereas there is still an infinite number of open cases left for the mim-width of (H_1, H_2)-free graphs [13].

We leave a systematic study into boundedness of linear rank-width of (H_1, H_2)-free graphs for future research. Here, we only collect known results. The class of (H_1, H_2)-free graphs has bounded linear rank-width if

- one of H_1 and H_2 is a subgraph of P_3 that is not isomorphic to $3P_1$ [22],
- $(H_1, H_2) = (K_3, S_{1,2,2})$ and $(3P_1, \overline{S_{1,2,2}})$ (Theorem 1.10)
- $(H_1, H_2) = (P_4, F)$ where F is a threshold graph [14].

The class of (H_1, H_2)-free graphs has unbounded linear rank-width if

- $(H_1, H_2) = (K_3, S_{1,2,3})$ or $(3P_1, \overline{S_{1,2,3}})$ [4],
- $(H_1, H_2) = (P_4, F)$ where F is not a threshold graph [14],
- all known cases where (H_1, H_2)-free graphs have unbounded rank-width.

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