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A note on eigenvalue bounds for non-compact manifolds

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Abstract

In this article we prove upper bounds for the Laplace eigenvalues λ_k below the essential spectrum for strictly negatively curved Cartan–Hadamard manifolds. Our bound is given in terms of k^2 and specific geometric data of the manifold. This applies also to the particular case of non-compact manifolds whose sectional curvature tends to $-\infty$, where no essential spectrum is present due to a theorem of Donnelly/Li. The result stands in clear contrast to Laplacians on graphs where such a bound fails to be true in general.

KEYWORDS

Cheeger inequality, eigenvalues, Laplacian, negative curvature, Riemannian manifold

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1 | INTRODUCTION

In 1979 Donnelly and Li [5] proved a criterion for discrete spectrum of the Laplacian on Riemannian manifolds in terms of decreasing sectional curvature. This complemented a result by Weyl for Schrödinger operators with increasing potential.

In particular, let M be a complete Riemannian manifold and let Δ be the Laplacian. We denote by K_r the supremum of the sectional curvatures at points outside of $B_r(x_0)$, the ball of radius r about some arbitrary base point x_0, that is

$$K_r := \sup\big\{K(\sigma) \mid \sigma \subset T_pM \text{ two-dimensional subspace, } p \in M \backslash B_r(x_0)\big\}. \tag{1.1}$$

Then the theorem of Donnelly/Li reads as follows.

Theorem 1.1 (Donnelly/Li). *Let M be a complete simply connected negatively curved Riemannian manifold. If $K_r \to -\infty$ as $r \to \infty$, then Δ has purely discrete spectrum.*

In this note we give an upper bound on the eigenvalues λ_k (listed with increasing order and counting multiplicities) in terms of k^2 and specific geometric data of the manifold. While this bound is a classical result in the case of compact manifolds, it stands in clear contrast to case of Laplacians on graphs. Indeed, for graphs any asymptotics of eigenvalues can occur, see e.g. [2].

Our result is based on so-called improved Cheeger inequalities which were introduced in the setting of finite graphs in [7]. A dimension-free version of these improved Cheeger inequalities in the manifold setting was derived in [8] to prove an eigenvalue ratio result for closed weighted manifolds of nonnegative Bakry–Émery curvature. In this article, we discuss an application in the case of negative curvature: we use an adaption of the improved Cheeger inequalities for general non-closed manifolds (Theorem 2.1 below) to derive the following result on eigenvalues below the essential spectrum for strictly negatively curved Cartan–Hadamard manifolds:

Theorem 1.2. *Let M be a complete simply connected Riemannian manifold with strictly negative curvature, that is $K_0 < 0$ (with K_r defined in (1.1)). Then, we have for the L^2-eigenvalues $\lambda_0 < \lambda_1 \leq \dots$ of the Laplacian below the essential spectrum*

$$\lambda_k \leq \frac{128\mu^2}{|K_0|(\dim(M)-1)^2} k^2, \quad k \geq 1,$$

where

$$\mu = \inf_{r,s>0, x\in M} \frac{\mathrm{vol}(B_{r+s}(x) \setminus B_s(x))}{r^2 \mathrm{vol}(B_s(x))}.$$

Remark 1.3. Using the result of Cheng [4], one can obtain a different upper bound as follows. For a ball $B_r(x) \subset M$ with lower Ricci curvature bound larger than $(n-1)R$ with $R < 0$, and $n = \dim(M)$, Cheng obtains for the Dirichlet eigenvalues of this ball

$$\lambda_k(B_r(x)) \leq \frac{n^2}{4}|R| + \frac{(1+\pi^2)(1+2^{4n})}{r^2} k^2$$

for odd dimensions and an estimate with somewhat better constants for the even-dimensional case and all $k \geq 0$, see [4, Corollary 2.3] and [3, Theorem 7, Chapter III]. (Note that Cheng proves this result for closed manifolds but his arguments work also without modification in the case of the compact manifold $B_r(x)$ with Dirichlet boundary conditions. Note also that under the assumptions of Theorem 1.2, we have $R \leq K_0$.) By domain monotonicity, [3, Corollary 1, Chapter I], we have for all eigenvalues $\lambda_k(M)$ of M below the essential spectrum

$$\lambda_k(M) \leq \lambda_k(B_r(x)).$$

This yields an upper estimate with different geometric constants.

2 | PROOF OF OUR MAIN RESULT

We introduce the following notation. For a Riemannian manifold M let vol be its volume measure and d the Riemannian distance. For a Borel set $A \subseteq M$ the boundary measure $\mathrm{vol}^+(A)$ is defined as

$$\mathrm{vol}^+(A) = \liminf_{r\to 0} \frac{\mathrm{vol}(O_r(A)) - \mathrm{vol}(A)}{r},$$

where $O_r(A) = \{x \in M \mid d(x,a) \leq r \text{ for some } a \in A\}$. If A has positive volume and finite boundary measure, we let

$$\phi(A) = \frac{\mathrm{vol}^+(A)}{\mathrm{vol}(A)}$$

and $\phi(A) = \infty$ otherwise. The Cheeger constant of a non-compact Riemannian manifold M is defined as (see [3, p. 95])

$$h = h(M) = \inf_{A \subseteq M} \phi(A).$$

We deduce our main result, Theorem 1.2 above, from the following result for general manifolds which was shown in the setting of closed manifolds, [8, Theorem 1.6]. The basic idea of the proof is an extension of the methods of [7, Lemma 4, Proposition 2] developed for finite graphs to prove the so-called improved Cheeger inequalities.

Theorem 2.1. *Let M be a complete Riemannian manifold. Then, we have for the L^2-eigenvalues $\lambda_0 \leq \lambda_1 \leq \dots$ of the Laplacian below the essential spectrum*

$$h^2 \lambda_k \leq 128 k^2 \lambda_0^2, \quad k \geq 1.$$

The proof of this theorem is based on an estimate which was proven for compact manifolds in [8, Theorem 3.1]. Although the proof carries over we recall the proof here for the convenience of the reader. To this end let $f \geq 0$ be a function on M that is supported on a set of positive measure and define

$$\phi(f) = \inf_{t\geq 0} \phi\big(M_f(t)\big),$$

where $M_f(t) = \{x \in M \mid f(x) > t\}$ is the level set of f for $t \in \mathbb{R}$. Furthermore, we denote the L^p norm by $\|\cdot\|_p$ for $p \in [1,\infty]$. The following proposition is the essential ingredient in the proof of Theorem 2.1.

Proposition 2.2 (Non-compact version of Theorem 3.1 [8]). *Let M be a complete Riemannian manifold with L^2-eigenvalues*

$$\lambda_0 \leq \lambda_1 \leq \dots$$

of the Laplacian below the essential spectrum and let $f \geq 0$ be a bounded Lipshitz function in $L^2(M)$. Then,

$$\phi(f) \leq 8\sqrt{2}\frac{k}{\sqrt{\lambda_k}}\frac{\||\nabla f|\|_2^2}{\|f\|_2^2}, \quad k \geq 1.$$

Proof. Here we sketch the core arguments of the proof. For more details we refer the reader to [8]. We assume $|\nabla f| \in L^2(M)$ since otherwise the asserted inequality is trivial.

For a finite set $\theta \subset \mathbb{R}$, let $\psi_\theta : \mathbb{R} \to \mathbb{R}$ be defined by

$$\psi_\theta(s) = \arg\min_{t\in\theta} |s-t|,$$

$\eta_\theta : \mathbb{R} \to \mathbb{R}$ be defined by

$$\eta_\theta(s) = |s - \psi_\theta(s)|,$$

and $\eta_{\theta,f} : M \to [0,\infty)$

$$\eta_{\theta,f} = \eta_\theta \circ f = |f - \psi_\theta \circ f|$$

be the difference of f and its approximation $\psi_\theta \circ f$. Note that we have $0 \leq \eta_{\theta,f} \leq f$.

Now, fix $k \in \mathbb{N}$ for the rest of the proof and let $t_0 = 0$. Assume $t_0 < t_1 < \cdots < t_{j-1}$ are given. If there is $t \geq t_{j-1}$ such that

$$\left\|\eta_{\{t_{j-1},t\},f} 1_{f^{-1}((t_{j-1},t])}\right\|_2^2 = \frac{1}{k\lambda_k}\||\nabla f|\|_2^2 =: C_0, \tag{2.1}$$

then let t_j be the smallest such $t \geq t_{j-1}$. Otherwise, let $t_j = \|f\|_\infty$. Observe that

$$f_j = \eta_{\{t_{j-1},t_j\},f} 1_{f^{-1}((t_{j-1},t_j])}, \quad j \geq 1,$$

are positive disjointly supported Lipshitz functions which are trivial whenever $t_j = \|f\|_\infty$. Moreover, $f_j \in L^2$ since $0 \leq f_j \leq f$ and $f \in L^2(M)$. Furthermore, by the reverse triangle inequality we have $\left|f_j(x) - f_j(y)\right| \leq |f(x) - f(y)|$, $x, y \in M$. Therefore, as the supports of the f_j are disjoint, we obtain

$$\sum_{j=1}^{\infty} \left|\nabla f_j\right|^2 \leq |\nabla f|^2$$

and therefore, $\left|\nabla f_j\right| \in L^2(M)$ whenever $|\nabla f| \in L^2(M)$. By completeness of the Riemannian manifold, the Laplacian is essentially selfadjoint. Thus, the f_j's are included in the form domain of the Laplacian since $f_j, \left|\nabla f_j\right| \in L^2(M)$. We show the following claim.

Claim: $t_{2k} = \|f\|_\infty$.

In the case $t_{2k} < \|f\|_\infty$, we infer by the arguments above and by the fact that in this case $\|f_j\|_2^2 = C_0$ for all $j = 1, \dots, 2k$

$$\sum_{j=1}^{2k} \frac{\| |\nabla f_j| \|_2^2}{\|f_j\|_2^2} \le \frac{1}{C_0} \| |\nabla f| \|_2^2 = k\lambda_k.$$

By the assumption $t_{2k} < \|f\|_\infty$, the functions f_j are non-zero and therefore non-constant. Thus, there exist at least $k+1$ of the f_j's such that

$$\frac{\| |\nabla f_j| \|_2^2}{\|f_j\|_2^2} \le \lambda_k.$$

Hence, the inequality above for $k+1$ orthogonal functions stands in contradiction the Min-Max-Principle and the claim is proven.

So let $\theta = \{0 = t_0 < t_1 \le \dots \le t_{2k} = \|f\|_\infty\}$. By (2.1) and what we have shown above, we obtain

$$\|f - \psi_\theta \circ f\|_2^2 = \sum_{j=1}^{2k} \left\| \eta_{\{t_j, t\}, f} 1_{f^{-1}((t_j, t])} \right\|_2^2 \le \frac{2}{\lambda_k} \| |\nabla f| \|_2^2. \tag{2.2}$$

In order to estimate the L^2 norm of $f - \psi_\theta \circ f$ from below, we observe that the function $h : M \to \mathbb{R}$

$$h(x) = \int_0^{f(x)} \eta_\theta(t)\, dt$$

has the same level sets as f and therefore,

$$\phi(f) = \phi(h) \le \frac{\| |\nabla h| \|_1}{\|h\|_1},$$

where the last inequality follows from the area formula and the co-area inequality [1, Lemma 3.2] (for more details see [8, Lemma 2.4]). Firstly, we find by the fundamental theorem of calculus and the chain rule and secondly by the Cauchy–Schwarz inequality and $\eta_{\theta,f} = f - \psi_\theta \circ f$ that

$$\| |\nabla h| \|_1 = \| |\nabla f| (\eta_\theta \circ f) \|_1 \le \| |\nabla f| \|_2 \| f - \psi_\theta \circ f \|_2.$$

Thirdly, is it elementary to estimate

$$h \ge \frac{1}{8k} f^2$$

by choosing $t_j \le f(x) \le t_{j+1}$ for $x \in M$ and estimatingh

$$h(x) \ge \frac{1}{4} \left(\sum_{i=0}^{j-1} \left(t_{i+1} - t_i\right)^2 + \left(f(x) - t_j\right)^2 \right) \ge \frac{1}{8k} \left(\sum_{i=0}^{j-1} \left(t_{i+1} - t_i\right) + \left(f(x) - t_j\right) \right)^2 = \frac{1}{8k} f(x)^2.$$

These considerations together with (2.2) yield

$$\phi(f) \le \frac{\| |\nabla h| \|_1}{\|h\|_1} \le 8k \frac{\| |\nabla f| \|_2 \| f - \psi_\theta \circ f \|_2}{\|f\|_2^2} \le 8\sqrt{2} \frac{k}{\sqrt{\lambda_k}} \frac{\| |\nabla f| \|_2^2}{\|f\|_2^2},$$

which finishes the proof. □

With the help of this proposition we are now in the position to prove Theorem 2.1.

Proof of Theorem 2.1. We observe that for any n we have

$$\phi(f) \leq \phi(f \wedge n),$$

where $f \wedge n = \min\{f, n\}$. Moreover, by the proposition above we have

$$\phi(f \wedge n) \leq 8\sqrt{2}\frac{k}{\sqrt{\lambda_k}} \frac{\int_M |\nabla f \wedge n|^2 \,\mathrm{dvol}}{\int_M |f \wedge n|^2 \,\mathrm{dvol}}.$$

Since $\phi(f) \leq \phi(f \wedge n)$, $|\nabla(f \wedge n)| \leq |\nabla f|$ and $\int_M |f \wedge n|^2 \,\mathrm{dvol} \to \int_M |f|^2 \,\mathrm{dvol} = 1$, $n \to \infty$, we conclude

$$\phi(f) \leq 8\sqrt{2}\frac{k}{\sqrt{\lambda_k}} \frac{\int_M |\nabla f|^2 \,\mathrm{dvol}}{\int_M |f|^2 \,\mathrm{dvol}}.$$

We choose f to be an eigenfunction to λ_0. Then, f is a Lipshitz function in $L^2(M)$ with a definite sign which can be chosen to be positive. Then, by the definition of the Cheeger constant and the proposition above, we have

$$h\sqrt{\lambda_k} \leq 8\sqrt{2}k\lambda_0,$$

which finishes the proof. □

Finally, we present the proof of our main result, Theorem 1.2 in the Introduction.

Proof of Theorem 1.2. Let us first derive $h^2 \geq (\dim(M) - 1)^2|K_0|$: In the definition of the Cheeger constant, we can restrict ourselves to sets A with smooth boundary. Let $A \subset M$ be such a set, let $\hat{x} \in M$ be a point with positive distance to A, and let $d_{\hat{x}} : M \to [0, \infty)$ be the distance function to $\hat{x}$. Then $d_{\hat{x}}$ is a smooth function on A (since the exponential map $\exp_{\hat{x}} : T_{\hat{x}}M \to M$ is a diffeomorphism). By the Laplacian Comparison Theorem (see, e.g., [6, (3)]), we have

$$\Delta_M d_{\hat{x}}(x) \geq (\dim(M) - 1)\sqrt{-K_0} \coth\big(\sqrt{-K_0} d_{\hat{x}}(x)\big).$$

This implies that $\Delta_M d_{\hat{x}}(x) \geq (\dim(M) - 1)\sqrt{|K_0|}$ for all $x \in A$ and, therefore, on the one hand,

$$\int_A \Delta_M d_{\hat{x}} \,\mathrm{dvol} \geq (\dim(M) - 1)\sqrt{|K_0|}\,\mathrm{vol}(A),$$

and, on the other hand, using the Gauß Divergence Theorem,

$$\int_A \Delta_M d_{\hat{x}} \,\mathrm{dvol} = \int_{\partial A} \langle \mathrm{grad}\, d_{\hat{x}}, \nu\rangle \,\mathrm{dvol}_{\partial A} \leq \mathrm{vol}^+(A),$$

where ν is the outward unit normal vector of ∂A. Combining both inequalities leads to the proof of the above estimate of the Cheeger constant h.

Furthermore, let $\eta = (1 - d(\cdot, B_s(x))/r)_+$. Then,

$$\lambda_0 \leq \frac{\int_M |\nabla\eta|^2 \,\mathrm{dvol}}{\int_M |\eta|^2 \,\mathrm{dvol}} = \frac{\mathrm{vol}(B_{r+s}(x) \setminus B_s(x))}{r^2(\mathrm{vol}(B_s(x)) + \int_{(B_{r+s}(x))\setminus B_s(x)} (r - d(y, B_s(x))^2 \,\mathrm{dvol}(y))}$$

$$\leq \frac{\mathrm{vol}(B_{r+s}(x) \setminus B_s(x))}{r^2 \mathrm{vol}(B_s(x))}.$$

Hence, combining this with Proposition 2.1 we conclude the statement of the theorem. □

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