LOOP-ERASED RANDOM WALK AS A SPIN SYSTEM OBSERVABLE

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ABSTRACT. The determination of the Hausdorff dimension of the scaling limit of loop-erased random walk is closely related to the study of the one-point function of loop-erased random walk, i.e., the probability a loop-erased random walk passes through a given vertex. Recent work in the theoretical physics literature has investigated the Hausdorff dimension of loop-erased random walk in three dimensions by applying field theory techniques to study spin systems that heuristically encode the one-point function of loop-erased random walk. Inspired by this, we introduce two different spin systems whose correlation functions can be rigorously shown to encode the one-point function of loop-erased random walk.

1. INTRODUCTION

Loop-erased random walk is, informally speaking, the probability measure on self-avoiding walks that results from removing the loops from simple random walk in chronological order. We give a precise description in Section 2.2 below. Loop-erased random walk is a fundamental probabilistic object with connections to spanning trees and the uniform spanning forest [26, 20], amongst other topics. In two dimensions, loop-erased random walk has SLE_2 as a scaling limit [15], while in four and higher dimensions it scales to Brownian motion [14]. It is possible to prove the scaling limit of loop-erased random walk exists in three dimensions [12], but many open questions remain, see [16, 3] and references therein.

The preceding results have been used to determine the fractal (Hausdorff) dimension $\dim_H(\mathcal{K}_d)$ of the scaling limit $\mathcal{K}_d$ of d-dimensional loop-erased random walk when $d \neq 3$: $\dim_H(\mathcal{K}_2) = \frac{5}{4}$ and $\dim_H(\mathcal{K}_d) = 2$ for $d \geq 4$. Shiraishi [22] has given a characterization of $\dim_H(\mathcal{K}_3)$, but the numerical value is not known rigorously. These results are all based on probabilistic tools.

Loop-erased random walk is also of interest within theoretical physics. An interesting recent development has been the use of non-rigorous field theory techniques for the determination of the Hausdorff dimension $\dim_H(\mathcal{K}_d)$ [8, 25]. While there is a long history of the interplay between field theories and random walks [23, 7, 6], geometric properties of loop-erased random walk are less obviously connected to a field theory due to 'erasure' in its definition.

In this note we describe two rigorous spin system representations of loop-erased random walk. Both representations translate the problem of the determination of the Hausdorff dimension for loop-erased random walk into a discrete spin system problem. This can be viewed as a mathematical justification for the starting point of the non-rigorous field theory steps contained in [25], albeit for somewhat different spin systems

than the one considered in [25]. Our proofs use a combination of Grassmann integration and Viennot's combinatorial theory of heaps of pieces [1, 24], and pass through intermediary representations of our spin systems in terms of a graphical loop models.

We defer a precise descriptions of our spin systems to Section 2 below, but remark here that they both contain two bosonic components and four fermionic components, and as a result a weight of $2-4 = -2$ for each loop. This is natural, as various "$O(-2)$" models have been connected with simple random walk in the past in physics [2, 4, 9, 11] and combinatorics and probability [24, 10, 18]. As highlighted in [25], the extra components of a spin system involved in writing $-2 = 2 - 4$ enables one to capture loop-erased random walk statistics in addition to simple random walk statistics.

To give a flavour of our results, let $B_R(0)$ denote the ball of radius R about the origin in $\mathbb{Z}^d$, and let $\partial B_R(0)$ denote the vertex boundary of the ball. For three distinct vertices a, b and c let $U_{m^2}(a, b, c)$ denote either of the three-point correlation functions introduced below in Section 2 on $B_R(0)$ with vertex weights $m^2 > 0$ on $\partial B_R(0)$.

Theorem 1. *Let* $\mathsf{LE}(\omega)$ *denote the loop-erased random walk trajectory that results from simple random walk started from* $0 \in B_R(0)$ *and stopped at* $\partial B_R(0)$*. Then for any vertex* $b \in B_{R-1}(0) \setminus \{0\}$,

$$(1.1) \qquad \mathbb{P}\left[b \in \mathsf{LE}(\omega)\right] = \lim_{m^2 \to \infty} \sum_{c \in \partial B_R(0)} U_{m^2}(0, b, c).$$

The left-hand side in (1.1) is the one-point function for loop-erased random walk. The scaling behaviour of the one-point function encodes $\dim_{\mathrm{H}}(\mathcal{K}_d)$, see [17] for a precise statement when $d = 3$. Theorem 1 is thus our promised translation of the problem of determining $\dim_{\mathrm{H}}(\mathcal{K}_d)$ into a spin system problem.

A non-rigorous analysis of a spin system similar to the one of this paper has been used to estimate $\dim_{\mathrm{H}}(\mathcal{K}_3) \approx 1.62$ [25], which agrees with extensive simulations [27]. Rigorously establishing a similar result would be extremely interesting. Another very interesting direction would be to investigate if there are further connections between our spin systems and SLE_2 in two dimensions. We comment on some further future directions below in Section 2.3.

2. Precise formulation of the result

Our main result holds for any finite connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The precise formulation relies on the use of Grassmann algebras and Grassmann integration. The reader unfamiliar with this subject can consult, e.g., [1, Section 2].

2.1. **The spin systems.** We first introduce the spin systems used in our representation. This requires some notation. Let $\mathcal{V}$ be a finite set. Consider the Grassmann

algebra whose generators are $\{\xi_x^{(i)}, \eta_x^{(i)}\}_{x\in\mathcal{V}, i=1,\ldots,4}$. That is, these variables are all pairwise anticommuting. We set $\phi_x = \eta_x^{(3)}\xi_x^{(3)}$ and $\psi_x = \eta_x^{(4)}\xi_x^{(4)}$; these are commuting elements of the algebra. Note that $\phi_x^2 = \psi_x^2 = 0$ for all x.

The spins of our spin systems are six-tuples σ_x for $x \in \mathcal{V}$. For notational reasons that will become clear in what follows, we write the six-tuples as triples of pairs $\sigma_x^{(i)} = (u_x^{(i)}, v_x^{(i)})$, for $i = 1, 2, 3$. The $u^{(i)}$ and $v^{(i)}$ are given by $(u_x^{(i)}, v_x^{(i)}) = (\eta_x^{(i)}, \xi_x^{(i)})$ for $i = 1, 2$ and $(u_x^{(3)}, v_x^{(3)}) = (\phi_x, \psi_x)$. We define a product

$$(2.1) \qquad \sigma_x \cdot \sigma_y \equiv \sigma_x^{(1)} \cdot \sigma_y^{(1)} + \sigma_x^{(2)} \cdot \sigma_y^{(2)} + \sigma_x^{(3)} \cdot \sigma_y^{(3)},$$

where

$$(2.2) \qquad \sigma_x^{(i)} \cdot \sigma_y^{(i)} \equiv u_x^{(i)} v_y^{(i)} + u_y^{(i)} v_x^{(i)}.$$

Let $\beta = (\beta_{xy})_{x,y\in\mathcal{V}}$ be a collection of non-negative and symmetric edge weights, i.e., $\beta_{xy} = \beta_{yx} \geq 0$, and $\beta_{xx} = 0$ for all x. Further, let $(m_x^2)_{x\in\mathcal{V}}$ be a collection of non-negative vertex weights, and define $r_x \equiv m_x^2 + \sum_y \beta_{xy}$.

2.1.1. *Symmetric action.* For notational convenience we introduce the so-called τ-field on pairs of vertices $x \neq y$, cf. [5]:

$$(2.3) \qquad \tau_{xy}^{(i)} \equiv \beta_{xy}\sigma_x^{(i)} \cdot \sigma_y^{(i)}, \qquad i = 1, 2, 3,$$

and we also introduce the shorthand

$$(2.4) \qquad (\nabla \cdot \tau)_x^2 \equiv \sum_{i=1}^{3} \left(\sum_{y\in\mathcal{V}} \tau_{xy}^{(i)} \right)^2.$$

Define an action S by

$$(2.5) \quad S \equiv \frac{1}{2} \sum_{x\in\mathcal{V}} \left(r_x \sigma_x \cdot \sigma_x + \sum_y (\tau_{xy}^{(3)} - \tau_{xy}^{(1)} - \tau_{xy}^{(2)}) - \frac{1}{2}((\nabla \cdot \tau)_x^2)^2 + \frac{1}{12}((\nabla \cdot \tau)_x^2)^3 \right).$$

Note that S is an even (i.e., commuting) element of the Grassmann algebra, so its exponential is well-defined. We define the partition function Z by

$$(2.6) \qquad Z \equiv \int \prod_{x\in\mathcal{V}} \left(\prod_{i=1}^{4} \partial_{\xi_x^{(i)}} \partial_{\eta_x^{(i)}} \right) e^S.$$

For F an element of the Grassmann algebra we define a normalized expectation by

$$(2.7) \qquad \langle F \rangle \equiv \frac{1}{Z} \int \prod_{x\in\mathcal{V}} \left(\partial_{\xi_x^{(i)}} \partial_{\eta_x^{(i)}} \right) F e^S.$$

This is a rational function of the edge and vertex variables, and in our cases of interest it will be clear that the evaluation of this rational function is finite.

Our main observable of interest will be, for distinct $a, b, c \in \mathcal{V}$,

$$(2.8) \qquad U(a, b, c) \equiv C_{bc} \langle u_c^{(2)} v_b^{(2)} u_b^{(3)} v_b^{(3)} u_b^{(1)} v_a^{(1)} \rangle, \quad C_{bc} \equiv m_c^2 (m_b^2 + \sum_{y\in\mathcal{V}} \beta_{by})^2.$$

Remark. *The products $\phi_x\psi_y$ and $\psi_x\phi_y$ in (2.2) can be replaced with the symmetric expression $\phi_x\phi_y + \psi_x\psi_y$ on bipartite graphs by exchanging the roles of ϕ and ψ on one bipartition. On non-bipartite graphs there is also natural symmetrization procedure, but the notation of (2.2) will be more convenient for what follows.*

Remark. *We have formulated the action (2.5) in a combinatorially convenient manner. A more conventional form from the viewpoint of field theory can be obtained by algebraic manipulations, see [21].*

2.1.2. *Chiral action.* For our second representation we use a variant of the term $\nabla \cdot \tau$ from Section 2.1.1. Define

$$(2.9) \qquad \tau'_x \equiv \sum_{i=1}^{3}\sum_{y\in\mathcal{V}}\beta_{xy}v_y^{(i)}u_x^{(i)},$$

and define the action

$$(2.10) \qquad S' \equiv \frac{1}{2}\sum_x\left(r_x\sigma_x\cdot\sigma_x + 2\tau'_x - (\tau'_x)^2 + \frac{2}{3}(\tau'_x)^3\right).$$

The observable of interest for the action S' is, for $a, b, c \in \mathcal{V}$ distinct,

$$(2.11) \qquad U'(a,b,c) \equiv C'_{bc}\langle u_c^{(2)}v_b^{(2)}u_b^{(1)}v_a^{(1)}\rangle', \qquad C'_{bc} \equiv m_c^2(m_b^2 + \sum_{y\in\mathcal{V}}\beta_{by}),$$

where $\langle\cdot\rangle'$ is the normalized expectation defined by replacing S by S' in (2.6)–(2.7).

2.2. **Loop-erased random walk.** Let $\beta = (\beta_{xy})_{x,y\in\mathcal{V}}$ and $(m_x^2)_{x\in\mathcal{V}}$ be as Section 2.1. The set of edges $\mathcal{E}$ with strictly positive weights induces a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. We will assume that the graph $\mathcal{G}$ is connected, and that $m_x^2 > 0$ for some $x \in \mathcal{V}$.

Let $\blacktriangle \notin \mathcal{V}$ be an additional 'cemetery' vertex. We define a discrete-time Markov chain X with state space $\mathcal{V} \cup \{\blacktriangle\}$, $\blacktriangle$ an absorbing state, by setting

$$(2.12) \qquad \mathbb{P}\left[X_{n+1} = y \mid X_n = x\right] = \begin{cases} \frac{\beta_{xy}}{m_x^2 + \sum_y \beta_{xy}} & y \neq \blacktriangle, \\ \frac{m_x^2}{m_x^2 + \sum_y \beta_{xy}} & y = \blacktriangle. \end{cases}$$

Henceforth we will simply refer to X as a simple random walk, and we view the walk as a sequence of nearest-neighbour vertices of $\mathcal{G}$.

To formally define loop-erasure, some definitions are needed. Given a *length $k \geq 1$* and a finite walk $\omega = (\omega_1, \omega_2, \dots, \omega_k)$ we let $|\omega| = k$. A walk is *self-avoiding* if $\omega_i = \omega_j$ implies $i = j$. A *rooted and directed cycle* is a walk with $\omega_1 = \omega_{|\omega|}$ such that $(\omega_1, \dots, \omega_{|\omega|-1})$ is a non-empty self-avoiding walk. Sometimes this is called a self-avoiding polygon, but note our definition includes self-avoiding polygons of length three that use the same edge twice. A *directed cycle* is an equivalence class of rooted and oriented cycles, the equivalence being under cyclic shifts.

Given a walk ω, suppose K is the first index such that $(\omega_1, \dots, \omega_K)$ is not a self-avoiding walk. If K is finite, let $K' < K$ be the unique index such that $(\omega_{K'}, \dots, \omega_K)$ is a rooted oriented cycle. Define $L(\omega) = (\omega_1, \dots, \omega_{K'}, \omega_{K+1}, \dots, \omega_{|\omega|})$ if K is finite, and $L(\omega) = \omega$ otherwise. Note $L(\omega)$ is a walk. For any finite walk ω we define the *loop-erasure* $\mathsf{LE}(\omega)$ *of* ω to be the walk that results from iteratively applying L. This

operation stabilizes after finitely many steps since each application of L to a walk that is not self-avoiding reduces the length of the walk by at least one.

Let $h_\blacktriangle$ be the (almost surely finite) hitting time of $\blacktriangle$, and let $h_\blacktriangle^- = h_\blacktriangle - 1$. *Loop-erased random walk* is the law of $\mathsf{LE}((X_n)_{n<h_\blacktriangle})$, i.e., the law the loop-erasure of X considered up until the time of the first jump to $\blacktriangle$.

2.3. **Main results.** Our first result concerns the action S. Recall that we assume throughout the paper that $\mathcal{G}$ is a finite graph.

Theorem 2. *Assume $\mathcal{G}$ is connected. For three distinct vertices $a, b, c \in \mathcal{V}$,*

$$(2.13) \quad U(a, b, c) = \mathbb{P}_a\left[b \in \mathsf{LE}((X_n)_{n<h_\blacktriangle}) \text{ and } X_{h_\blacktriangle^-} = c\right] \cdot (1 + O(m_c^{-2})) \quad \text{as } m_c^2 \to \infty.$$

We obtain a similar result for the action S', but without any need to take $m_c^2 \to \infty$.

Theorem 3. *Assume $\mathcal{G}$ is connected. For three distinct vertices $a, b, c, \in \mathcal{V}$ such that $m_c^2 > 0$,*

$$(2.14) \qquad U'(a, b, c) = \mathbb{P}_a\left[b \in \mathsf{LE}((X_n)_{n<h_\blacktriangle}) \text{ and } X_{h_\blacktriangle^-} = c\right].$$

Theorem 1 follows from the preceding results, since taking the killing rate m_x^2 to infinity for $x \in A \subset \mathcal{V}$ results in a random walk stopped on the set A. We give more details below after a brief discussion of these theorems.

By considering spin systems with additional components it seems possible to construct observables that encode the multipoint functions of loop-erased random walk. Variations on our formulas that replace the 'bosonic' variables $u^{(3)}$ and $v^{(3)}$ with standard bosons also appear possible. We remark that bosonic variables with square 0 such as $u^{(3)}$ and $v^{(3)}$ can be viewed as a way to implement a Nienhuis-type action [19] on general graphs.

We briefly compare the actions S and S'. The spin system defined by the action S is closer to what is studied in [25] than the spin system defined by the action S', as S manifestly inherits the symmetries of the products in (2.2). On the other hand, the exact identity of Theorem 3 arises in part from the asymmetry in the action S'. From the point of view studying $\dim_H(\mathcal{K}_d)$ one would presumably like as simple of an action and observable as possible. Based on the non-rigorous methods and results of [25] one is lead to conjecture that a modification of the action S that removes the six-body term (S_6 below) would lead to the same behaviour. A justification of this conjecture would be quite interesting. Whether the six-body term in S' is similarly negligible is not clear to us.

Proof of Theorem 1. We consider the action S; for S' the argument is very similar. Let $z \in \partial B_R(0)$, and let (X_n) be a random walk on $B_R(0)$. It suffices to prove that

$$(2.15) \qquad \lim_{m^2 \to \infty} \mathbb{P}_{0,m^2}\left[y \in \mathsf{LE}((X_n)_{n<h_\blacktriangle}) \text{ and } X_{h_\blacktriangle^-} = z\right] = \mathbb{P}_0\left[y \in \mathsf{LE}((X_n)_{n \leq h_z})\right],$$

where the left-hand side is the probability for a random walk killed on the boundary with rate m^2 and the right-hand side is the probability for a random walk without killing. This can be seen by coupling the random walks together until they first hit the

boundary. The probability that y is in the loop-erasure of the stopped process is then precisely the probability that y is in the loop-erasure of the killed process, conditionally on the killed process being killed at the first visit to the boundary.

Since the probability the killed random walk is killed at its first visit to the boundary tends to one as $m^2 \to \infty$, the claim follows from Theorem 2.   ∎

The remainder of the paper is organized as follows. In Section 3 we prove Theorem 2. Theorem 3 has a similar but somewhat simpler proof, which is given in Section 4.

3. Proof of Theorem 2

A *coloured graph* is a graph whose edges are each assigned a non-empty subset of the colours $\{1, 2, 3\}$. We do not permit graphs to have self-loops (i.e., edges $\{x, x\}$), though we will make use of self-loops in the proofs below. A connected component is *monochromatic* if the edges of the component are all assigned exactly one colour. Given a coloured graph G, a vertex x, and a colour i, we say that the *coloured vertex* (x, i) is present if there is vertex y such that the edge $\{x, y\}$ is coloured i in G.

We remark that it is possible to view coloured graphs as multigraphs in which each repeated edge is assigned a distinct colour.

3.1. **Graphical representation of the partition function.** For a coloured graph G whose connected components are either self-avoiding walks or monochromatic cycles, define

$$(3.1) \qquad w_0(G) \equiv (-1)^{F(G)} \prod_{xy \in G} \beta_{xy} \prod_{x, i \notin V(G)} r_x,$$

where $F(G)$ is the number of cycles in G coloured either 1 or 2, the first product is over all edges in G, and the last product is over all coloured vertices not in G.

Proposition 4. *The partition function* (2.6) *associated to* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *is given by*

$$(3.2) \qquad Z = \sum_G w_0(G),$$

where the sum is over directed coloured subgraphs G of $\mathcal{G}$ whose connected components are monochromatic directed cycles.

The proof will make use of terms S_0, S_1, S_4 and S_6 defined by

$$(3.3) \qquad S_0(i; x) \equiv r_x u_x^{(i)} v_x^{(i)},$$

$$(3.4) \qquad S_1(i; x, y) \equiv \beta_{xy} v_y^{(i)} u_x^{(i)},$$

$$(3.5) \qquad S_4(i, j; x, \boldsymbol{y}) \equiv \left(\prod_{k=1}^{4} \beta_{xy_k} \right) v_{y_1}^{(i)} u_x^{(i)} v_x^{(i)} u_{y_2}^{(i)} v_{y_3}^{(j)} u_x^{(j)} v_x^{(j)} u_{y_4}^{(j)},$$

$$(3.6) \qquad S_6(x, \boldsymbol{y}) \equiv \left(\prod_{k=1}^{6} \beta_{xy_k} \right) v_{y_1}^{(1)} u_x^{(1)} v_x^{(1)} u_{y_2}^{(1)} v_{y_3}^{(2)} u_x^{(2)} v_x^{(2)} u_{y_4}^{(2)} v_{y_5}^{(3)} u_x^{(3)} v_x^{(3)} u_{y_6}^{(3)},$$

where $\boldsymbol{y}$ represents four- and six-tuples of vertices in $\mathcal{V}$ in the last two displays, respectively, while i and j range over $\{1, 2, 3\}$. In what follows it will be contextually clear when $\boldsymbol{y}$ is a four- or six-tuple.

Lemma 5. *The action can be rewritten as*

$$(3.7) \qquad S = \sum_{x,i} S_0(i; x) + \sum_{x,y,i} S_1(i; x, y) - \sum_{x,\boldsymbol{y},i,j} S_4(i, j; x, \boldsymbol{y}) + 2 \sum_{x,\boldsymbol{y}} S_6(x, \boldsymbol{y}).$$

Proof. This is an algebraic reformulation of (2.5). ∎

Lemma 6. *The exponential of the action can be rewritten as*

$$(3.8) \quad e^S = \prod_{x,i}(1 + S_0(i; x)) \prod_{x,y,i}(1 + S_1(i; x, y)) \prod_{x,\boldsymbol{y},i,j}(1 - S_4(i, j; x, \boldsymbol{y})) \prod_{x,\boldsymbol{y}}(1 + 2S_6(x, \boldsymbol{y})).$$

Proof. Each summand $\tilde{S}$ in (3.7) is nilpotent. In particular, $\exp(\tilde{S}) = 1 + \tilde{S}$. Since each $\tilde{S}$ is even the exponential of the sum is the product of the exponentials. ∎

Proof of Proposition 4. We represent the terms in the expansion of the products in (3.8) as coloured graphs with self-loops by viewing each monomial $v_y^{(i)} u_x^{(i)}$ as a directed edge with colour i from x to y. If $x = y$ this is a self-loop. Explicitly,

- terms $S_0(i; x)$ give self-loops (x, x) of colour i,
- terms $S_1(i; x, y)$ give directed edges (x, y) of colour i,
- terms $S_4(i, j; x, \boldsymbol{y})$ give directed edges (x, y_1) and (y_2, x) of colour i and (x, y_3) and (y_4, x) of colour j,
- terms $S_6(x, \boldsymbol{y})$ give directed edges (x, y_1) and (y_2, x) of colour 1, (x, y_3) and (y_4, x) of colour 2, and (x, y_5), (y_6, x) of colour 3.

By definition, $\int e^S$ is the coefficient of the top degree term of e^S. By the above, this coefficient is a sum over coloured directed graphs G that use every vertex of $\mathcal{G}$ with every possible colour. Note that such a graph G can arise in multiple ways, i.e., from different choices of terms in the expansions of the products in (3.8). For any such choice the weight of the graph has a factor r_x for each self-loop at x; a factor β_{xy} for each coloured directed edge (x, y); a numerical factor determined by the coefficients of the S_i; and a sign determined by the re-ordering of the Grassmann variables. The last two considerations depend on the choices of terms in the expansions of the products in (3.8).

The next step is to characterize the graphs G that give a non-zero contribution. It suffices to characterize the contributing graphs up to self-loops, whose existence is implied by the fact that only top degree terms contribute to the integral. In what follows we implicitly discuss only edges that are not self-loops. We first prove that all graphs with non-zero weight are unions of disjoint cycles. It suffices to establish that

(i) the coloured in- and out-degree of each vertex in G is the same in any graph with non-zero weight,
(ii) if the in-degree of some vertex in G is two, then the weight is zero,
(iii) if the in-degree of some vertex in G is three, then the weight is zero.

The first claim is immediate since self-loops add coloured in- and out-degree equally.

For the second claim, fix a graph G and a vertex x of in- and out-degree two. Let $\boldsymbol{y}$ be the 4-tuple of vertices to which x connects. Two of the edges must be of colour i, and two of colour j. There are two ways (fixing all other choices) to create the graph G that differ in how the monomials associated to edges containing x are chosen. The possibilities are $S_1(i; x, y_1)S_1(i; x, y_2)S_1(j; x, y_3)S_1(j; x, y_4)$ or $-S_4(i, j; x, \boldsymbol{y})$, which sum to zero.

For the third claim, we argue as above. There are now five possibilities for how the edges connecting x to the vertices in the six-tuple $\boldsymbol{y}$ could be chosen:

- a product of six $S_1(i; x, y)$,
- a product of one $-S_4(i, j; x, \boldsymbol{y}')$ with two $S_1(k; x, y)$ for $y \in \boldsymbol{y} \setminus \boldsymbol{y}'$ and $k = \{1, 2, 3\} \setminus \{i, j\}$,
- a factor $S_6(x, \boldsymbol{y})$.

The first option contributes weight 1, the last weight $+2$, while the middle three contribute weight -1. Thus these possibilities sum to zero.

To complete the proof of the proposition, we note that each self-loop has weight r_x, and there is one such factor for each vertex x and each colour i that is not contained in some cycle. Thus all that remains is to justify the sign $(-1)^{F(G)}$. Each directed cycle $(\omega_1, \dots, \omega_k)$ is associated to a product $S_1(i; \omega_1, \omega_2)S_1(i; \omega_2, \omega_3)\dots S_1(i; \omega_k, \omega_1)$, and each individual S_i term commutes, i.e., letting $\omega_{k+1} = \omega_1$,

$$(3.9) \quad S_1(i; \omega_k, \omega_{k+1})\dots S_1(i; \omega_1, \omega_2) = \left(\prod_{i=1}^{|\omega|} \beta_{\omega_i}\beta_{\omega_{i+1}}\right) v_{\omega_{k+1}}^{(i)} u_{\omega_k}^{(i)} v_{\omega_k}^{(i)} u_{\omega_{k-1}}^{(i)} \dots v_{\omega_2}^{(i)} u_{\omega_1}^{(i)}.$$

Moving the final factor of $u_{\omega_1}^{(i)}$ to the front contributes a factor -1 if and only if the colour $i \in \{1, 2\}$ as in this case $u_{\omega_1}^{(i)}$ is odd, while $u_{\omega_1}^{(3)}$ is even. ∎

An immediate consequence of this coloured graph representation is the following colourless variant. For a directed cycle C define

$$(3.10) \qquad w(C) \equiv \prod_{xy \in C} \frac{\beta_{xy}}{r_x} = \prod_{xy \in C} \frac{\beta_{xy}}{m_x^2 + \sum_s \beta_{xs}},$$

and let $\mathcal{L}$ denote the collection of graphs whose connected components are all directed cycles. Equivalently, will may think of an element $L \in \mathcal{L}$ as a set of disjoint directed cycles.

Corollary 7. *The partition function Z can be expressed as*

$$(3.11) \qquad Z = \left(\prod_{s \in \mathcal{V}} r_s^3\right) \sum_{L \in \mathcal{L}} (-1)^{|L|} \prod_{C \in L} w(C).$$

Proof. This follows from Proposition 4 by summing over the possible colours of each component, which yields a weight of $1 - 1 - 1 = -1$ for each directed cycle. The vertex weights follow since every vertex in a cycle is also in exactly two coloured self-loops, and every vertex not in a cycle is in exactly three coloured self-loops. ∎

3.2. **Computing $U(a, b, c)$, I.** In this section we compute a graphical formula for the numerator of (2.8), i.e., for the Grassmann integral of

$$(3.12) \qquad u_c^{(2)} v_b^{(2)} u_b^{(3)} v_b^{(3)} u_b^{(1)} v_a^{(1)} e^S,$$

when $a, b, c \in \mathcal{V}$ are all distinct. Towards this goal, we introduce two weighted sums of coloured graphs, where the weight $w_0(G)$ is given by (3.1):

$$(3.13) \qquad \Gamma(a, b, c) \equiv \sum_G \frac{w_0(G)}{r_b}, \qquad \Theta(a, b, c) \equiv \sum_G \sum_C \frac{w_0(G \cup C)}{r_b}.$$

We will see in the following that the main contribution comes from Γ, and Θ will be an error term. The sum defining $\Gamma(a, b, c)$ is a sum over coloured subgraphs G which are unions of a coloured self-avoiding walk γ and coloured directed cycles subject to the following conditions, in which $V(\gamma)$ denotes the vertices contained in γ:

- (i) γ is a self-avoiding walk from a to c that passes through b, of colour 1 from a to b, and of colour 2 from b to c,
- (ii) the directed cycles are pairwise disjoint,
- (iii) directed cycles of colour 1 do not intersect $V(\gamma) \setminus \{c\}$,
- (iv) directed cycles of colour 2 do not intersect $V(\gamma) \setminus \{a\}$,
- (v) directed cycles of colour 3 do not intersect $V(\gamma) \setminus \{a, c\}$, and contain at most one of a and c.

The sum defining Θ is a sum over coloured directed cycles C of colour 3 that contain both a and c and coloured graphs G such that the coloured graph $C \cup G$ satisfies (i)–(iv) above and

- (v') directed cycles of colour 3 do not intersect $V(\gamma) \setminus \{a, c\}$.

Proposition 8. *For distinct $a, b, c \in \mathcal{V}$,*

$$(3.14) \qquad \int u_c^{(2)} v_b^{(2)} u_b^{(3)} v_b^{(3)} u_b^{(1)} v_a^{(1)} e^S = \Gamma(a, b, c) + \Theta(a, b, c).$$

Proof of Proposition 8. Recall e^S has the product form given by Lemma 6. Our first step is to expand this product and see which terms combine with the prefactor in (3.12) to give a top-degree term. As in the proof of Proposition 4 we characterize the contributing coloured graphs up to self-loops. After doing this we will justify the claimed weight.

We first claim that, for each vertex $y \notin \{a, b, c\}$,

- (i) the coloured in- and out-degrees of y are the same in any coloured graph G with non-zero weight, and
- (ii) if the total in-degree of y is two or three, then the weight is zero.

The proof of these claims is exactly as in the proof of Proposition 4 (this proof only used the combinatorics of the terms that arise from expanding e^S).

We next consider the vertices a, b, c:

- (i) a must have colour 1 out-degree one and in-degree zero; c must have colour 2 in-degree one and out-degree zero; b must have colour 1 in-degree one, colour 2 out-degree one, and colour 3 in- and out-degree zero.

(ii) Excepting the above points, the coloured in- and out-degrees of a and c must be equal,

(iii) a cannot have out-degree three, and c cannot have in-degree three.

The factor $u_c^{(2)}v_b^{(2)}u_b^{(3)}v_b^{(3)}u_b^{(1)}v_a^{(1)}$ enforces the first claim: for example, a must have colour 1 out-degree one since the only terms in e^S that contain $u_a^{(1)}$ but not $v_a^{(1)}$ are of the form $S_1(1; a, \cdot)$. The claims for b and c follow similarly. The second claim follows since self-loops contribute in- and out-degree one. The last claim follows by arguing as in the proof of Proposition 4; we give the argument for a (it is similar for c). If a had out-degree three, then it must have in- and out-degree one for colours 2 and 3. Let $\boldsymbol{y}$ be the four-tuple of vertices to which a is connected, by colour 2 to y_1, y_2, and colour 3 to y_3, y_4. There are two ways this can occur: $S_1(2; a, y_1)S_1(2; a, y_2)S_1(3; a, y_3)S_1(3; a, y_4)$ or $-S_4(2, 3; a, \boldsymbol{y})$, and these sum to zero.

The preceding facts establish the claimed coloured graph structure. All that remains is to justify the weight. This is nearly as in the proof of Proposition 4: each self-loop at $y \in \mathcal{V}$ contributes a factor r_y, and each cycle coloured 1 or 2 contributes a factor -1. Note that there is never a self-loop at b of colour 3 due to the factor $u_b^{(3)}v_b^{(3)}$. Up to the sign arising from the edges in γ, this gives the weight $r_b^{-1}w_0(G)$. All that remains is to show that the self-avoiding walk γ from a to c does not contribute a sign. If $k^\star$ is the index such that $\omega_{k^\star} = b$, then this follows from observing that the weight of the walk has the form

$$(3.15) \qquad \underbrace{u_{\omega_k}^{(2)}v_{\omega_1}^{(1)}}_{\text{I}} v_{\omega_k}^{(2)}u_{\omega_{k-1}}^{(2)}v_{\omega_{k-1}}^{(2)}u_{\omega_{k-2}}^{(2)} \cdots v_{\omega_{k^\star+1}}^{(2)}u_{\omega_{k^\star}}^{(2)} \underbrace{v_{\omega_{k^\star}}^{(2)}u_{\omega_{k^\star}}^{(1)}}_{\text{II}} v_{\omega_{k^\star}}^{(1)}u_{\omega_{k^\star-1}}^{(1)} \cdots v_{\omega_2}^{(1)}u_{\omega_1}^{(1)},$$

where the factors labelled I and II come from the prefactor in the integrand. Moving the initial factor of $v_{\omega_1}^{(1)}$ through to the end proves the claim. ∎

Given a self-avoiding walk γ, let $\mathcal{L}_\gamma$ be the collection of sets of pairwise disjoint oriented cycles that are all vertex-disjoint from γ. Further, define

$$(3.16) \qquad w(\gamma) \equiv \prod_{i=1}^{|\omega|-1} \frac{\beta_{\omega_i \omega_{i+1}}}{r_{\omega_i}} = \prod_{i=1}^{|\omega|-1} \frac{\beta_{\omega_i \omega_{i+1}}}{m_{\omega_i}^2 + \sum_{y \in \mathcal{V}} \beta_{\omega_i y}}.$$

Proposition 9. *For any three distinct vertices a, b, c,*

$$(3.17) \qquad \Gamma(a, b, c) = \frac{\prod_x r_x^3}{r_c r_b^2} \sum_{\gamma \in \mathsf{SAW}(a,c)} \sum_{L \in \mathcal{L}_\gamma} (-1)^{|L|} w(\gamma) 1_{b \in \gamma} \prod_{C \in L} w(C),$$

where $\mathsf{SAW}(a, c)$ denotes the set of self-avoiding walks from a to c.

Proof. The identity follows from the definition of Γ by summing over the possible colourings. The self-avoiding walk must be vertex-disjoint from all directed cycles since any cycle intersecting exactly one endpoint can only have two possible colours, one of which is 3, and hence has total weight $1 - 1 = 0$.

What remains is to explain the prefactor. Recall the definition (3.1) of w_0. The first product $\prod_{xy \in G} \beta_{xy}$ contains exactly those β_{xy} that appear in $w(\gamma)$ and in $\prod_C w(C)$. We are left with treating the r factors, that is, for each vertex x we should count the numbers of colours i for which (x, i) is not present in G.

For $x \notin G$ there is a contribution of r_x^3. If $x \in G \setminus \{a, b, c\}$, it has a single incoming edge and a single outgoing edge, both of the same colour, leaving two colours that are not present. This is a contribution of r_x^2. Since the weight w (either $w(\gamma)$ if $x \in \gamma$ or $w(C)$ if $x \in C$) contains a factor r_x^{-1} we are left with the prefactor r_x^3.

The vertex a has exactly one outgoing edge of colour 1, leading to a contribution of r_a^2. Since $w(\gamma)$ contains a factor r_a^{-1} we are left with the prefactor r_a^3. The vertex b has adjacent edges of colours 1 and 2 but not of colour 3. Hence only $(b, 3)$ is not present in G, and the corresponding factor is r_b. Since $w(\gamma)$ contains a factor r_b^{-1} and in the definition (3.13) of Γ there is an additional r_b^{-1}, we are left with the prefactor r_b. Finally, for the vertex c, both $(c, 1)$ and $(c, 3)$ are not present in G, yielding a factor r_c^2. Since γ does not take any steps from c, there is no additional factor coming from $w(\gamma)$, and we are left with r_c^2. ∎

3.3. Heaps of pieces and loop-erased walk.

An important combinatorial structure, closely related to loop erased random walks, is a *heap of cycles*. This was initially studied in [24] as an application of Viennot's more general theory of heaps of pieces. We will briefly recall the definitions and results that we need; an enjoyable introduction to the theory can be found in [13].

Definition 10. *A* labeled heap of cycles *on $\mathcal{G}$ is a partially ordered set $(\mathcal{X}, \preceq)$, each of whose elements $\alpha \in \mathcal{X}$ is assigned a cycle C_α, such that for every $\alpha, \beta \in \mathcal{X}$:*

(i) *if C_α intersects C_β then α and β are comparable; and*
(ii) *if $\alpha \preceq \beta$ and β covers α ($\alpha \preceq \gamma \preceq \beta \Rightarrow \gamma \in \{\alpha, \beta\}$) then C_α intersects C_β.*

Two labeled heaps of cycles are isomorphic if they are isomorphic as ordered sets, with an isomorphism which respects the cycle assignment. A heap of cycles *is an equivalence class (under isomorphism) of labeled heaps of cycles.*

In the sequel we will be slightly informal and will refer to the cycles in a heap of cycles, as opposed to speaking of the labels of the elements of a labelled heap of cycles. We will be interested in weighted heaps of cycles. To this end, let $\{q_e\}_{e \in \mathcal{E}}$ be a set of formal commuting variables. Define, for each subgraph $G = (V(G), E(G))$ of $\mathcal{G}$, a weight $q(G) = \prod_{e \in E(G)} q_e$. For a heap of cycles L we define $q(L) = \prod_{C \in L} q(C)$.

We now come to the connection between heaps of cycles and loop-erased walk. Consider a walk ω from x to z. By performing the loop-erasure process, we obtain a self-avoiding path $\gamma = \mathsf{LE}(\omega)$, and a set of erased cycles. In order to reconstruct ω from γ and this set of cycles, however, we need some extra information on the order in which the cycles were erased. The essence of [24, Proposition 6.3], due to Viennot, is that this additional information is, in fact, the heap of cycles structure. Formally, for $V \subset \mathcal{V}$, let $\mathcal{H}_V$ denote the set of heaps of cycles whose maximal elements intersect V.

Theorem 11. *Fix a self-avoiding walk γ. There is a weight-preserving bijection between $\mathcal{H}_\gamma$ and the set of paths whose loop erasure is γ, weighted by their loops. In particular,*

$$(3.18) \qquad \sum_{\omega \,:\, \mathsf{LE}(\omega) = \gamma} q(\omega) = q(\gamma) \sum_{L \in \mathcal{H}_\gamma} q(L).$$

This theorem was stated without proof as [24, Proposition 6.3]; we remark that proofs can be found in [18] or [10, Appendix A].

Recall that $\mathcal{L}$ denotes the set of all collection of non-intersecting cycles. Equivalently, $\mathcal{L}$ is the set of heaps of cycles with empty order; sometimes these are called *trivial* heaps. For any set $V \subset \mathcal{V}$ of vertices let $\mathcal{L}_V$ denote that set of trivial heaps of cycles in $\mathcal{L}$ that do not contain any cycle intersecting V. We will also make use of the following general identity, which is the basic connection between the graph-theoretic objects of the previous section and walks.

Proposition 12 (Proposition 5.3 of [24]). *For every $V \subset \mathcal{V}$, as formal series in $\{q_e\}$,*

$$(3.19) \qquad \frac{\sum_{L \in \mathcal{L}_V} (-1)^{|L|} q(L)}{\sum_{L \in \mathcal{L}} (-1)^{|L|} q(L)} = \sum_{L \in \mathcal{H}_V} q(L).$$

We remark that while the preceding proposition is stated without proof [24, Proposition 5.3], the proof follows from the methods therein.

3.4. **Computing $U(a, b, c)$, II.** Let $\sum_{\omega : a \to c}$ denote the sum over all walks beginning at a and ending at c. Recall from (2.8) that $C_{bc} = m_c^2 r_b^2$.

Proposition 13. *For any three distinct vertices $a, b, c \in \mathcal{V}$,*

$$(3.20) \qquad \frac{C_{bc} \Gamma(a, b, c)}{Z} = \sum_{\omega : a \to c} 1_{b \in \mathsf{LE}(\omega)} \mathbb{P}_a[(X_n)_{n < h_\blacktriangle} = \omega].$$

Proof. By Proposition 9 and Corollary 7, setting $q_{xy} = \beta_{xy} r_x^{-1}$,

$$(3.21) \qquad \frac{\Gamma(a, b, c)}{Z} = r_b^{-2} r_c^{-1} \sum_{\gamma \in \mathsf{SAW}(a,c)} q(\gamma) 1_{b \in \gamma} \frac{\sum_{L \in \mathcal{L}_\gamma} (-1)^{|L|} q(L)}{\sum_{L \in \mathcal{L}} (-1)^{|L|} q(L)}$$

$$(3.22) \qquad = r_b^{-2} r_c^{-1} \sum_{\gamma \in \mathsf{SAW}(a,c)} q(\gamma) 1_{b \in \gamma} \sum_{L \in \mathcal{H}_\gamma} q(L);$$

using Proposition 12 for the last equality. We can now apply Theorem 11, obtaining

$$(3.23) \qquad \frac{\Gamma(a, b, c)}{Z} = r_b^{-2} r_c^{-1} \sum_{\omega : a \to c} 1_{b \in \mathsf{LE}(\omega)} q(\omega)$$

as formal power series. The result holds in the sense of convergent power series when interpreting q_{xy} numerically as $\beta_{xy} r_x^{-1}$, as after multiplying by C_{bc} we recognize the right-hand side to be the right-hand side of (3.20) since $m_c^2 r_c^{-1}$ is the probability of a jump from c to $\blacktriangle$. $\blacksquare$

Next we show that the term Θ is negligible compared to Γ as $m_c^2 \to \infty$.

Proposition 14. *For any three distinct vertices $a, b, c \in \mathcal{V}$*

$$(3.24) \qquad \left| \frac{\Theta(a, b, c)}{Z} \right| \leq \frac{\Gamma(a, b, c)}{Z} \cdot \mathbb{P}_a[X_{h_\blacktriangle} = c] \cdot \frac{\sum_{v \in \mathcal{V}} \beta_{cv}}{m_c^2} \max_{v \in \mathcal{V}} \mathbb{E}_v[h_\blacktriangle^-].$$

Proof. In each term of Θ we identify three coloured self-avoiding walks γ_i. The first goes from a to c passing through b, with colour 1 from a to b and colour 2 from b to c. The second goes from a to c and is of colour 3, and the third goes from c to a and is of colour 3. Moreover, the γ_i intersect only at a and c. Let $V(\gamma_i)$ denote the vertices in γ_i, and let $V = V(\gamma_1) \cup V(\gamma_2) \cup V(\gamma_3)$. Then, summing over the colouring of the other edges as in the proof of Proposition 9,

$$(3.25) \qquad \Theta(a,b,c) = \frac{\prod_{x\in\mathcal{V}} r_x^3}{r_c r_b^2} \sideset{}{'}\sum_{\gamma_1,\gamma_2,\gamma_3} \sum_{L\in\mathcal{L}_V} q(\gamma_1)q(\gamma_2)q(\gamma_3)(-1)^{|L|}q(L),$$

with $q_{xy} = \beta_{xy} r_y^{-1}$. The notation $\sum'_{\gamma_1,\gamma_2,\gamma_3}$ means that the self-avoiding walks γ_i intersect only at a and c. The $(-1)^{|L|}$ factor comes from the fact that each loop in L is counted with weight -1 twice (for colouring 1 and 2) and once with weight $+1$ (for colouring 3).

By Proposition 12 and Corollary 7, as in the proof of Proposition 13,

$$(3.26) \qquad \frac{\Theta(a,b,c)}{Z} = \frac{1}{r_c r_b^2} \sideset{}{'}\sum_{\gamma_1,\gamma_2,\gamma_3} q(\gamma_1)q(\gamma_2)q(\gamma_3) \sum_{L\in\mathcal{H}_V} q(L).$$

Controlling this term requires a small digression. Let L_1 and L_2 be two heaps of cycles. The *superposition* $L_1 \odot L_2$ *of* L_2 *on* L_1 is the heap of cycles that results from putting L_2 on top of L_1. More formally, the elements of $L_1 \odot L_2$ are the disjoint union of the elements of L_1 and those of L_2, the cycle assignment remains the same, and the order is the transitive closure of the following order: fix elements α_1 and α_2 such that C_{α_1} intersects C_{α_2}. If both elements belong to L_1 (respectively, L_2) they keep the order they had in L_1 (respectively, L_2). If, on the other hand, $\alpha_1 \in L_1$ and $\alpha_2 \in L_2$, then $\alpha_1 < \alpha_2$ (and vice versa). Superposition is an associative binary operation, and q is a homomorphism for this algebraic structure, i.e., $q(L_1 \odot L_2) = q(L_1)q(L_2)$, see [24].

For any two sets V_1, V_2, the superposition $L = L_1 \odot L_2$ of a heap of cycles $L_2 \in \mathcal{H}_{V_1}$ on a heap of cycles $L_1 \in \mathcal{H}_{V_2}$ is in $\mathcal{H}_V$ for $V = V_1 \cup V_2$. Moreover, this map is surjective. That is, any heap of cycles $L \in \mathcal{H}_V$ can be expressed as $L = L_1 \odot L_2$ for some $L_1 \in \mathcal{H}_{V_1}$ and $L_2 \in \mathcal{H}_{V_2}$. To see this, given a heap in $\mathcal{H}_V$, define L_1 to consist of the cycles in L that intersect V_1 or are smaller than a cycle intersecting V_1, let L_2 will consist of the other elements of L. It is then straightforward to verify that $L_1 \in \mathcal{H}_{V_1}$, $L_2 \in \mathcal{H}_{V_2}$, and $L = L_1 \odot L_2$.

We now return to estimating $\frac{\Theta(a,b,c)}{Z}$. We continue to write q_{xy} in place of $\beta_{xy} r_x^{-1}$ for brevity, but to make estimates we are considering series as analytical objects. Since the weights are positive, (3.26) and the surjectivity of superposition implies

$$(3.27) \qquad \left| \frac{\Theta(a,b,c)}{Z} \right| \le \frac{1}{r_c r_b^2} \sideset{}{'}\sum_{\gamma_1,\gamma_2,\gamma_3} q(\gamma_1)q(\gamma_2)q(\gamma_3) \sum_{L_1\in\mathcal{H}_{\gamma_1}} \sum_{L_2\in\mathcal{H}_{\gamma_2}} \sum_{L_3\in\mathcal{H}_{\gamma_3}} q(L_1)q(L_2)q(L_3),$$

where the right-hand side may be infinite.

We now bound this expression by replacing $\sum'$ by $\sum$, i.e., by relaxing the constraint that the γ_i are mutually self-avoiding. This results in a product of three terms – a self-avoiding walk γ_1 from a to c passing through b, and two self-avoiding walks γ_2 and

γ_3, one from a to c and the other from c to a. The first term is, by following the proof of Corollary 13,

$$(3.28) \qquad \frac{1}{r_c r_b^2} \sum_{\gamma_1} q(\gamma_1) \sum_{L_1 \in \mathcal{H}_{\gamma_1}} q(L_1) = \frac{\Gamma(a, b, c)}{Z}.$$

The second term, $\sum_{\gamma_2} \sum_{L_2 \in \mathcal{H}_{\gamma_2}} q(\gamma_2) q(L_2)$, can be analyzed in the same fashion. By Theorem 11,

$$(3.29) \qquad \sum_{\gamma: \, a \to c} \sum_{L \in \mathcal{L}_\gamma} q(\gamma) q(L) = \sum_{\omega: \, a \to c} q(\omega) = \frac{m_c^2 + \sum_{v \in \mathcal{V}} \beta_{cv}}{m_c^2} \, \mathbb{P}_a \left[X_{h_{\blacktriangle}^-} = c \right].$$

The third term, $\sum_{\gamma_3} \sum_{L_3 \in \mathcal{H}_{\gamma_3}} q(\gamma_3) q(L_3)$, counts all walks from c to a by Theorem 11. The first step, to some x such that $\beta_{cx} > 0$, has weight $\frac{\beta_{cx}}{r_c}$, so

$$(3.30)$$
$$\sum_{\gamma: \, c \to a} \sum_{L \in \mathcal{H}_\gamma} q(\gamma) q(L) = \sum_x \frac{\beta_{cx}}{r_c} \sum_{\omega: \, x \to a} q(\omega) = \sum_x \frac{\beta_{cx}}{r_c} \sum_{k \in \mathbb{N}} \sum_{\substack{\omega: \, x \to a \\ |\omega| = k}} \mathbb{P}_x[(X_n)_{n \leq k} = \omega].$$

The internal sum on the right-hand side is $\mathbb{P}_x[h_{\blacktriangle} > k]$, and summing this over k yields $\mathbb{E}_x[h_{\blacktriangle}^-]$. Optimizing the upper bound over x gives

$$(3.31) \qquad \sum_{\gamma_3} \sum_{L_3 \in \mathcal{H}_{\gamma_3}} q(\gamma_3) q(L_3) \leq \frac{\sum_{x \in \mathcal{V}} \beta_{cx}}{m_c^2 + \sum_{v \in \mathcal{V}} \beta_{cv}} \, \max_{x \in \mathcal{V}} \mathbb{E}_s[h_{\blacktriangle}].$$

Taking the product of all three terms yields (3.24). ∎

3.5. Proof of Theorem 2.

Proof of Theorem 2. This follows from Proposition 8 combined with Propositions 13 and 14. ∎

4. Proof of Theorem 3

In this section we prove Theorem 3; since the argument are rather similar to those of the preceding section we will be a little brief in some places.

Lemma 15. *The exponential of the action S' from (2.10) is given by*

$$(4.1) \qquad e^{S'} = \prod_{x,i} \left(1 + r_x u_x^{(i)} v_x^{(i)} \right) \prod_x \left(1 + \sum_{i,y} \beta_{xy} v_y^{(i)} u_x^{(i)} \right).$$

Proof. Using $\log(1 + x) = x - x^2/2 + x^3/3 + \dots$, the logarithm of (4.1) is
$$(4.2)$$
$$\sum_{x,i} r_x u_x^{(i)} v_x^{(i)} + \sum_x \sum_{y,i} \beta_{xy} v_y^{(i)} u_x^{(i)} - \frac{1}{2} \sum_x \left(\sum_{y,i} \beta_{xy} v_y^{(i)} u_x^{(i)} \right)^2 + \frac{1}{3} \sum_x \left(\sum_{y,i} \beta_{xy} v_y^{(i)} u_x^{(i)} \right)^3,$$

since any higher order terms include a factor of $(u_x^{(i)})^2 = 0$ for some i. This establishes the result. ∎

Proposition 16. *Let $Z' \equiv \int e^{S'}$. Then $Z = Z'$.*

Proof. We represent the terms in the expansion of the products in (4.1) as coloured subgraphs of $\mathcal{G}$ with self-loops by viewing each monomial $v_y^{(i)} u_x^{(i)}$ as a directed edge with colour i from x to y, and each monomial $u_x^{(i)} v_x^{(i)}$ as a self-loop of colour i. We recall that $\beta_{xx} = 0$ for all x.

By definition, $\int e^S$ is the coefficient of the top degree term of e^S. This coefficient is a sum over coloured directed graphs G with self-loops such that the coloured in- and out-degree of every vertex is one for all colours.

We first consider which graphs G without self-loops that result from expanding the second product in (4.1) can make a non-zero contribution. We claim that a non-zero contribution can only arise if for each vertex x, x has either in- and out-degree zero, or if x has colour in- and out-degree one for exactly one colour.

To see this, note that a self-loop of colour i at x contributes in- and out-degree one of colour i at x. Hence if G had a vertex with unequal coloured in- and out-degrees, the term cannot be top-degree. Moreover, the form of the product in (4.1) ensures each vertex has out-degree at most one. This proves the claim, and it follows that the contributing G are unions of pairwise disjoint monochromatic directed cycles.

To conclude the proof, we note that the coefficient in front of a term represented by such a graph G is equal $w_0(G)$, just as in Proposition 4. $\blacksquare$

We next consider the numerator of $U'(a, b, c)$ as defined in (2.11). Let

$$(4.3) \qquad \Gamma'(a, b, c) \equiv \sum_G w_0(G).$$

where the sum defining Γ' is a sum over coloured subgraphs G which are unions of a coloured self-avoiding walk γ and coloured directed cycles subject to the following conditions, in which $V(\gamma)$ denotes the vertices contained in γ:

    (i) γ is a self avoiding walk from a to c that passes through b, of colour 1 from a to b, and of colour 2 from b to c,

    (ii) the directed cycles are pairwise disjoint,

    (iii) directed cycles of colour 1 do not intersect $V(\gamma) \setminus \{c\}$,

    (iv) directed cycles of colour 2 do not intersect $V(\gamma)$,

    (v) directed cycles of colour 3 do not intersect $V(\gamma) \setminus \{c\}$.

Proposition 17. *For distinct $a, b, c \in \mathcal{V}$,*

$$(4.4) \qquad \int u_c^{(2)} v_b^{(2)} u_b^{(1)} v_a^{(1)} e^{S'} = \Gamma'(a, b, c).$$

Proof. We will expand the product (4.1), and see which terms combine with the prefactor in (4.4) to give a top-degree term. As in the proof of Proposition 16, we will first characterize the subgraphs of $\mathcal{G}$ corresponding to these terms.

Assume that G is such a graph. We first consider a vertex $x \notin \{a, b, c\}$. Then, just as in the proof of Proposition 16, either x has in- and out-degree zero, or x has colour in- and out-degree one for exactly one colour.

Next we consider the vertices $\{a, b, c\}$, beginning with a. Due to the term $v_a^{(1)}$ in (4.4), there cannot be a self-loop of colour 1 at a. Therefore, since G corresponds to a

top-degree term, $u_a^{(1)}$ must appear in the expansion of the second term in (4.1). That is, a has an outgoing edge of colour 1, and no other outgoing edges. Moreover, the terms $u_a^{(2)}$ and $u_a^{(3)}$ must appear as self-loops. Thus G contains a single edge adjacent to a, which is outgoing of colour 1.

For b, the term $v_b^{(2)}$ implies b has a single outgoing edge, of colour 1. The term $u_b^{(1)}$ implies b has an incoming edge of colour 2. Since $u_b^{(3)}$ can only appear as a self loop, we conclude that b has in- and out- degree zero of colour 3.

For c, the term $u_c^{(2)}$ implies c has in-degree one and out-degree zero for colour 2. An argument as in the proof of Proposition 16 shows c can either have (i) no adjacent edges of colours 1 and 3, or (ii) in- and out-degree one exactly one of these colours.

These observations imply that the graph indeed has the structure given in the definition of Γ'. The weight can be calculated as in the proof of Proposition 8. ∎

Proposition 18. *For any three distinct vertices $a, b, c \in \mathcal{V}$, $\Gamma'(a, b, c) = r_b^{-1}\Gamma(a, b, c)$, i.e.,*

$$(4.5) \qquad \Gamma'(a, b, c) = \frac{\prod_x r_x^3}{r_c r_b} \sum_{\gamma \in \mathsf{SAW}(a,c)} \sum_{L \in \mathcal{L}_\gamma} (-1)^{|L|} w(\gamma) 1_{b \in \gamma} \prod_{C \in L} w(C).$$

Proof. As is the proof of Proposition 9, we sum the possible colourings in (4.3). Note that graphs that intersect γ at the vertex c appear in (4.3) have total weight 0, since each such graph appears once when the cycle intersecting c is of colour 1 and once when it has colour 3, and these two coloured graph have opposite weights.

The vertex factor is calculated in the same manner as in Proposition 9. ∎

Proof of Theorem 3. By the definition (2.11) of $U'(a, b, c)$ and Propositions 16–18,

$$U'(a, b, c) = \frac{C'_{bc}\Gamma'(a, b, c)}{Z'} = \frac{C_{bc}\Gamma(a, b, c)}{Z},$$

and hence the Theorem follows by applying Proposition 13. ∎

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