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Curvatures, graph products and Ricci flatness

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Abstract
In this paper, we compare Ollivier–Ricci curvature and Bakry–Émery curvature notions on combinatorial graphs and discuss connections to various types of Ricci flatness. We show that nonnegativity of Ollivier–Ricci curvature implies the nonnegativity of Bakry–Émery curvature under triangle-freeness and an additional in-degree condition. We also provide examples that both conditions of this result are necessary. We investigate relations to graph products and show that Ricci flatness is preserved under all natural products. While nonnegativity of both curvatures is preserved under Cartesian products, we show that in the case of strong products, non-negativity of Ollivier–Ricci curvature is only preserved for horizontal and vertical edges. We also prove that all distance-regular graphs of girth 4 attain their maximal possible curvature values.

KEYWORDS
curvature comparison, graph curvature, Ricci flat

1 | INTRODUCTION

1.1 | Motivation of the paper

Curvature is a fundamental notion in the setting of smooth Riemannian manifolds. There is no unique choice of an analogue of curvature in the setting of combinatorial graphs. Two possibilities are *Ollivier–Ricci curvature* and *Bakry–Émery curvature* which are both motivated by specific curvature properties of Riemannian manifolds. Ollivier–Ricci curvature, introduced in [16], is based on the observation that, in the case of positive/negative Ricci curvature, average

distances between corresponding points in two nearby small balls in Riemannian manifolds are smaller/larger than the distance between their centres. This fact is reinterpreted using the theory of Optimal Transportation of probability measures representing these balls. Bakry–Émery curvature, introduced in [1], is based on the so-called *curvature–dimension inequality* which reads for n-dimensional Riemannian manifolds (M, g) as follows:

$$\frac{1}{2}\Delta \|\operatorname{grad} f\|^2(x) \geq \langle \nabla f(x), \nabla \Delta f(x)\rangle + \frac{1}{n}(\Delta f(x))^2 + \operatorname{Ric}(\nabla f, \nabla f)(x) \tag{1}$$

for all $f \in C^\infty(M)$ and $x \in M$. Here, $\operatorname{Ric}(v, w)$ for tangent vectors v, w at x stands for the Ricci curvature of the manifold. This formula is a straightforward implication of Bochner's identity, a fundamental fact in Riemannian Geometry with many important consequences. Both curvature notions have been further discussed in the setting of graphs in several literatures (see, e.g., [14] for Ollivier–Ricci curvature and [12,15,18] for Bakry–Émery curvature). For the precise definitions of both notions in this paper, we refer to Section 2.

While there are many special cases in which these two discrete curvature notions are related, it is a challenging problem to develop a satisfactory general understanding of the agreements and differences of these two curvature notions.

One special family of graphs which have both nonnegative Ollivier–Ricci curvature and nonnegative Bakry–Émery curvature was introduced by Chung and Yau [6], namely, *Ricci flat graphs*. The notion of Ricci flatness was motivated by the structure of the d-dimensional grid $\mathbb{Z}^d$ (with vanishing Ollivier–Ricci and Bakry–Émery curvature) and the class of Ricci flat graphs contains all abelian Cayley graphs as a subset.

The motivation of this paper is to investigate various relations between these two curvature notions and the property of Ricci flatness with special focus on triangle-free graphs. We also present explicit examples of graphs related to our results. The curvatures of these examples were calculated numerically via the interactive web-application at https://www.mas.ncl.ac.uk/graph-curvature/. For more details about this very useful tool we refer the readers to [9].

1.2 | Statement of results

Let $G = (V, E)$ be a regular graph. Ollivier–Ricci curvature $\kappa_p(x, y)$ is defined on edges $\{x, y\} \in E$ and there is an idleness parameter $p \in [0, 1]$ involved. Lin, Lu and Yau introduced in [14] a modified notion of Ollivier–Ricci curvature, denoted by $\kappa_{\mathrm{LLY}}(x, y)$. Both notions are introduced in Definition 2.3. While it is known that $\kappa_0 \leq \kappa_{\mathrm{LLY}}$, our first result shows in Section 2.1 that positive κ_{LLY}-curvature implies the nonnegativity of κ_0-curvature:

Theorem 1.1. *Let $G = (V, E)$ be a regular graph. Then we have the following implication for all edges $\{x, y\} \in E$:*

$$\kappa_{\mathrm{LLY}}(x, y) > 0 \Rightarrow \kappa_0(x, y) \geq 0.$$

The Bakry–Émery curvature is defined on vertices and the inequality (1) involves a dimension parameter n. Since graphs do not have a well-defined dimension, a natural choice simplifying this inequality is $n = \infty$. The induced Bakry–Émery curvature value at a vertex x is then denoted by $\mathcal{K}_\infty(x)$ (see Definition 2.8).

Let us now turn to the above-mentioned notion of Ricci flatness. Ricci flatness is defined locally for individual vertices. In this paper we also introduce stronger types of Ricci flatness, namely, (R)-, (S)- and (RS)-Ricci flatness (see Definition 3.1). A fundamental consequence of Ricci flatness is that it implies both nonnegativity of Ollivier–Ricci and Bakry–Émery curvatures; the stronger property of (R)-Ricci flatness implies even strict positivity of these two curvatures (see Theorems 3.4 and 3.5).

Another basic property of Ricci flatness is that it is preserved under natural graph products (see Theorem 5.2). The graph products under consideration, namely, Cartesian product (involving horizontal and vertical edges), tensorial product (involving only diagonal edges), and the strong product (involving all three types of edges), are introduced in Definition 5.1. While Cartesian products preserve the nonnegativity of both Ollivier–Ricci curvature and Bakry–Émery curvature, in the case of strong products, nonnegative Ollivier–Ricci curvature is only preserved for horizontal and vertical edges (see Corollary 5.4).

We also consider the case of graphs which contain no triangles. In Section 4, we present our main result of this paper relating the two curvature notions. Ralli [17] gave an interesting criterion for curvature sign agreement of both curvature notions for triangle-free graphs which do not contain the complete bipartite graph $K_{2,3}$ as a subgraph. He mentions that the situation is much more unclear if one restricts to general triangle-free graphs. Our result requires triangle-freeness at a vertex x and the additional assumption that the in-degrees of vertices in the 2-sphere $S_2(x)$ are smaller or equal to 2. For $z \in S_2(x)$, the in-degree, denoted by $d_x^-(z)$, is the number of common neighbours of x and z. This assumption is weaker than nonexistence of $K_{2,3}$ as a subgraph.

Theorem 1.2. *Given a regular graph $G = (V, E)$, let $x \in V$ be a vertex not contained in a triangle and satisfying $d_x^-(z) \leq 2$ for all $z \in S_2(x)$. Then we have the following:*

(a) $\kappa_0(x, y) = 0$ *for all* $y \in S_1(x)$ *implies* $\mathcal{K}_\infty(x) \geq 0$.
(b) $\kappa_{\mathrm{LLY}}(x, y) = \frac{2}{d}$ *for all* $y \in S_1(x)$ *implies* $\mathcal{K}_\infty(x) = 2$.

It is an important remark here that $\kappa_0(x, y) = 0$, $\kappa_{\mathrm{LLY}}(x, y) = \frac{2}{d}$ and $\mathcal{K}_\infty(x) = 2$ are the maximum possible values of curvature for a vertex x not contained in a triangle. This curvature comparison result is proved by employing Ricci flatness, see Section 4. At the end of the section, we also provide examples to show that all conditions of the theorem are necessary.

In Section 6, we show that the curvatures of all *distance-regular graphs of girth* 4 and vertex degree d satisfy $\kappa_0 = 0$, $\kappa_{\mathrm{LLY}} = \frac{2}{d}$ and $\mathcal{K}_\infty = 2$ (see Theorem 6.2). In other words, all curvatures attain their maximal possible values for this interesting family of triangle-free graphs.

2 | CURVATURE NOTIONS

All graphs $G = (V, E)$ with vertex set V and edge set E in this paper are simple (i.e., without loops and multiple edges), undirected and connected, and we assume that the vertex degrees d_x of all vertices $x \in V$ are finite. Moreover, all our graphs are regular (i.e., $d_x = d$ for all $x \in V$) unless stated otherwise. Balls and spheres are denoted by

$$B_k(x) := \{z \in V : d(x, z) \leq k\},$$
$$S_k(x) := \{z \in V : d(x, z) = k\},$$

where $d : V \times V \to \mathbb{N} \cup \{0\}$ is the combinatorial distance function.

2.1 | Ollivier–Ricci curvature

We define the following probability distributions μ_x^p for any $x \in V, p \in [0, 1]$:

$$\mu_x^p(z) = \begin{cases} p & \text{if } z = x, \\ \dfrac{1 - p}{d_x} & \text{if } z \sim x, \\ 0 & \text{otherwise}. \end{cases}$$

Definition 2.1 (Transport plan and Wasserstein distance). Given $G = (V, E)$, let μ_1, μ_2 be two probability measures on V. A *transport plan* π transporting μ_1 to μ_2 is a function $\pi : V \times V \to [0, \infty)$ satisfying the following marginal constraints:

$$\mu_1(x) = \sum_{y \in V} \pi(x, y), \quad \mu_2(y) = \sum_{x \in V} \pi(x, y). \tag{2}$$

The *cost* of a transport plan π is given by

$$\text{cost}(\pi) = \sum_{y \in V} \sum_{x \in V} d(x, y) \pi(x, y).$$

The set of all transport plans satisfying (2) is denoted by $\Pi(\mu_1, \mu_2)$.

The *Wasserstein distance* $W_1(\mu_1, \mu_2)$ between μ_1 and μ_2 is then defined as

$$W_1(\mu_1, \mu_2) := \inf_\pi \text{cost}(\pi) = \inf_\pi \sum_{y \in V} \sum_{x \in V} d(x, y) \pi(x, y), \tag{3}$$

where the infimum runs over all transport plans $\pi \in \Pi(\mu_1, \mu_2)$.

Remark 2.2. Note that every $\pi \in \Pi(\mu_1, \mu_2)$ satisfies $\pi(x, y) = 0$ if $x \notin \text{supp}(\mu_1)$ or $y \notin \text{supp}(\mu_2)$. Therefore (3) can be rewritten as

$$W_1(\mu_1, \mu_2) = \inf_\pi \sum_{y \in \text{supp}(\mu_2)} \sum_{x \in \text{supp}(\mu_1)} d(x, y) \pi(x, y).$$

In other words, a transport plan π moves a mass distribution given by μ_1 into a mass distribution given by μ_2, and $W_1(\mu_1, \mu_2)$ is a measure for the minimal effort which is required for such a transition.

If μ_1 and μ_2 have finite supports, then there exists π which attains the infimum in (3). We call such π an *optimal transport plan* transporting μ_1 to μ_2.

Definition 2.3 (Ollivier–Ricci curvature). The *p-Ollivier–Ricci curvature* [16] on an edge $\{x, y\} \in E$ is

$$\kappa_p(x, y) = 1 - W_1\left(\mu_x^p, \mu_y^p\right),$$

where $p \in [0, 1]$ is called the *idleness* parameter.

The Ollivier–Ricci curvature introduced by Lin, Lu and Yau [14] is defined as

$$\kappa_{\mathrm{LLY}}(x, y) = \lim_{p \to 1} \frac{\kappa_p(x, y)}{1 - p}.$$

It was shown in [14, Lemma 2.1] that the function $p \mapsto \kappa_p(x, y)$ is concave, which implies

$$\kappa_p(x, y) \le \kappa_{\mathrm{LLY}}(x, y) \quad \text{for all } p \in [0, 1]. \tag{4}$$

Moreover, we have the following relation for edges $\{x, y\}$ with $d_x = d_y = d$ (see [3]):

$$\kappa_{\mathrm{LLY}}(x, y) = \frac{d + 1}{d} \kappa_{\frac{1}{d+1}}(x, y). \tag{5}$$

From the definition of the Wasserstein metric we can get an upper bound for W_1 by choosing a suitable transport plan. Using Kantorovich duality (see, e.g., [20, Ch. 5]), a fundamental concept in the optimal transport theory, we can approximate the opposite direction:

Theorem 2.4 (Kantorovich duality). *Given $G = (V, E)$, let μ_1, μ_2 be two probability measures on V. Then*

$$W_1(\mu_1, \mu_2) = \sup_{\substack{\phi: V \to \mathbb{R} \\ \phi \in 1\text{-Lip}}} \sum_{x \in V} \phi(x)(\mu_1(x) - \mu_2(x)),$$

where 1-Lip denotes the set of all 1-Lipschitz functions. If $\phi \in$ 1-Lip attains the supremum we call it an optimal Kantorovich potential *transporting μ_1 to μ_2.*

Note that both curvatures $\kappa_0(x, y)$ and $\kappa_{\mathrm{LLY}}(x, y)$ of an edge $\{x, y\}$ are already determined by the combinatorial structure of the induced subgraph $B_2(x)$. (In fact, by symmetry reasons, the combinatorial structure of the induced subgraph $B_2(x) \cap B_2(y)$ is sufficient.)

As the relation $\kappa_0 \le \kappa_{\mathrm{LLY}}$ is known from (4), now we will prove the surprising fact that strict positivity of κ_{LLY} implies the nonnegativity of κ_0 (as stated in Theorem 1.1 from Section 1).

Proof of Theorem 1.1. Let $G = (V, E)$ be d-regular. Using the relation (5), it suffices to prove

$$\kappa_{\frac{1}{d+1}}(x, y) > 0 \Rightarrow \kappa_0(x, y) \ge 0.$$

Let $\{x, y\} \in E$ be an edge with $\kappa_{\frac{1}{d+1}}(x, y) > 0$. We define the following sets:

$$\begin{aligned} T_{xy} &:= S_1(x) \cap S_1(y), \\ V_x &:= S_1(x) \backslash B_1(y), \\ V_y &:= S_1(y) \backslash B_1(x). \end{aligned}$$

In other words, T_{xy} is the set of common neighbours of x and y, V_x is the set of neighbours of x which have distance 2 to y and, similarly, V_y is the set of neighbours of y which have distance 2 to x.

We can choose an optimal transport plan $\pi_{\text{opt}} \in \Pi\left(\mu_x^{\frac{1}{d+1}}, \mu_y^{\frac{1}{d+1}}\right)$ with

(i) if $u \in T_{xy} \cup \{x\} \cup \{y\}$, then $\pi_{\text{opt}}(u, u) = \frac{1}{d+1}$,
(ii) if $u \in V_x$, then $\pi_{\text{opt}}(u, v) = \frac{1}{d+1}$ for exactly one $v \in V_y$ and 0 for others,
(iii) if $u \notin B_1(x)$, then $\pi_{\text{opt}}(u, v) = 0$ for $v \in V$.

The existence of an optimal transport plan satisfying (ii) (i.e., without splitting mass) follows from [4, Theorem 1.1] (see also [19, p. 5]). Moreover, this transport plan can be chosen to satisfy (i) by [3, Lemma 4.1]. Note that (iii) holds for any transport plan in $\Pi\left(\mu_x^{\frac{1}{d+1}}, \mu_y^{\frac{1}{d+1}}\right)$.

In other words, the optimal transport plan does not move the mass distributions at x, y or T_{xy}, and for the vertices in V_x it moves the mass distribution from one vertex completely to one vertex in V_y. Thus the optimal transport plan pairs the vertices at V_x and V_y. Let $u \in V_x$ and denote by $\tilde{u}$ the unique vertex in V_y for which $\pi_{\text{opt}}(u, \tilde{u}) = \frac{1}{d+1}$.

Let us then consider the Wasserstein distance. Using the optimal transport plan we can write

$$1 > 1 - \kappa_{\frac{1}{d+1}}(x, y) = W_1\left(\mu_x^{\frac{1}{d+1}}, \mu_y^{\frac{1}{d+1}}\right) = \frac{1}{d+1}\sum_{u \in V_x} d(u, \tilde{u}). \tag{6}$$

Note that $1 \le d(u_j, \tilde{u}_j) \le 3$ for all $u_j \in V_x$. Let

$$N_i := |\{u \in V_x : d(u, \tilde{u}) = i\}| \quad \text{for } i \in \{1, 2, 3\}.$$

It follows from (6) that $d + 1 > \sum_{u \in V_x} d(u, \tilde{u}) = N_1 + 2N_2 + 3N_3$, which implies

$$d \ge N_1 + 2N_2 + 3N_3. \tag{7}$$

Now we distinguish three cases.

Assume that $N_3 > 0$. Then there exists at least one vertex $w \in V_x$ satisfying $d(w, \tilde{w}) = 3$. Let π be a transport plan from μ_x^0 to μ_y^0 such that $\pi(w, x) = \frac{1}{d}$, $\pi(y, \tilde{w}) = \frac{1}{d}$ and $\pi(u, \tilde{u}) = \frac{1}{d}$ for all other pairs $(u, \tilde{u})$ on the support of π_{opt} except $(w, \tilde{w})$. Using this transport plan and (7), we have

$$W_1\left(\mu_x^0, \mu_y^0\right) \le \frac{1}{d}(2 + N_1 + 2N_2 + 3(N_3 - 1))$$
$$\le \frac{d-1}{d} < 1.$$

Thus $\kappa_0(x, y) > 0$.

Next, we assume $N_3 = 0$ and $N_2 > 0$. Then there exists at least one vertex $w \in V_x$ satisfying $d(w, \tilde{w}) = 2$, and we obtain, similarly as above,

$$W_1\left(\mu_x^0, \mu_y^0\right) \leq \frac{1}{d}(2 + N_1 + 2(N_2 - 1))$$

$$\leq \frac{1}{d}(N_1 + 2N_2 + 3N_3) \leq 1,$$

and therefore $\kappa_0(x, y) \geq 0$.

Finally, if $N_2 = N_3 = 0$, the optimal transport plan π_{opt} defines a perfect matching between the sets V_x and V_y, and therefore

$$W_1\left(\mu_x^0, \mu_y^0\right) \leq \frac{2 + (N_1 - 1)}{d} = \frac{N_1 + 1}{d} \leq 1,$$

since $N_1 = |V_x| \leq d - 1$, and again, $\kappa_0(x, y) \geq 0$, with equality if and only if $N_1 = d - 1$, which means $T_{xy} = \varnothing$. $\qquad\square$

Remark 2.5.

(a) The proof shows that $\kappa_{\mathrm{LLY}}(x, y) > 0$ implies $\kappa_0(x, y) > 0$ in the following cases:
 (i) $N_3 > 0$ or
 (ii) $N_3 = N_2 = 0$ and $\{x, y\}$ is contained in a triangle.
(b) The hypercubes Q^d satisfy $\kappa_{\mathrm{LLY}}(x, y) = \frac{2}{d} > 0$ and $\kappa_0(x, y) = 0$ for all edges $\{x, y\} \in E$.
(c) The triplex (see Figure 1a) satisfies $\kappa_{\mathrm{LLY}}(x, y) = 0$ and $\kappa_0(x, y) = -\frac{1}{3} < 0$ for all edges $\{x, y\} \in E$.
(d) The icosidodecahedral graph (see Figure 1b) satisfies $\kappa_{\mathrm{LLY}}(x, y) = 0$ and $\kappa_0(x, y) = 0$ for all edges $\{x, y\} \in E$. This implies that $\kappa_p(x, y) = 0$ for all $p \in [0, 1]$. Graphs with this property in all edges are called *bone-idle* (this notion was introduced in [3]).
(e) It was shown in [3] that $\kappa_{\mathrm{LLY}} \leq \kappa_0 + \frac{2}{d}$ and that $\kappa_{\mathrm{LLY}}, \kappa_0 \in \mathbb{Z}/d$. The result in the theorem further rules out the possibility that $\kappa_{\mathrm{LLY}} = \frac{1}{d}$ and $\kappa_0 = -\frac{1}{d}$ occur simultaneously.

The examples (b) and (c) show that the result in the theorem is sharp.

We finish this subsection with the following upper curvature bounds for κ_0 and κ_{LLY}:

Theorem 2.6 (see [13, Theorem 4] and [8, Proposition 2.7]). *Let $G = (V, E)$ be d-regular and $\{x, y\} \in E$. Then*

$$\kappa_0(x, y) \leq \frac{\#_\Delta(x, y)}{d}$$

and

$$\kappa_{\mathrm{LLY}}(x, y) \leq \frac{2 + \#_\Delta(x, y)}{d},$$

where $\#_\Delta(x, y)$ is the number of triangles containing $\{x, y\}$.

2.2 | Bakry–Émery curvature

This curvature notion was first introduced by Bakry and Émery in [1] and was applied on graphs in [12,15,18]. The definition of this curvature is based on the curvature–dimension inequality (1), which is equivalently rewritten as (8) with the help of the following Γ-calculus.

For any function $f : V \to \mathbb{R}$ and any vertex $x \in V$, the (nonnormalized) *Laplacian* Δ is defined via

$$\Delta f(x) := \sum_{y : y \sim x} (f(y) - f(x)).$$

Definition 2.7 (Γ and Γ_2 operators). Given $G = (V, E)$, we define for two functions $f, g : V \to \mathbb{R}$

$$2\Gamma(f, g) := \Delta(fg) - f\Delta g - g\Delta f;$$
$$2\Gamma_2(f, g) := \Delta\Gamma(f, g) - \Gamma(f, \Delta g) - \Gamma(\Delta f, g).$$

We write $\Gamma(f) := \Gamma(f, f)$ and $\Gamma_2(f, f) := \Gamma_2(f)$, for short.

Definition 2.8 (Bakry–Émery curvature). Given $G = (V, E)$, $\mathcal{K} \in \mathbb{R}$ and $\mathcal{N} \in (0, \infty]$. We say that a vertex $x \in V$ satisfies the *curvature–dimension inequality* $\mathrm{CD}(\mathcal{K}, \mathcal{N})$, if for any $f : V \to \mathbb{R}$, we have

$$\Gamma_2(f)(x) \geq \frac{1}{\mathcal{N}}(\Delta f(x))^2 + \mathcal{K}\Gamma(f)(x) \quad \text{for all } x \in V. \tag{8}$$

We call $\mathcal{K}$ a lower Ricci curvature bound of x, and $\mathcal{N}$ a dimension parameter. The graph $G = (V, E)$ satisfies $\mathrm{CD}(\mathcal{K}, \mathcal{N})$ (globally), if all its vertices satisfy $\mathrm{CD}(\mathcal{K}, \mathcal{N})$. At a vertex $x \in V$, let $\mathcal{K}(x, \mathcal{N})$ be the largest $\mathcal{K}$ such that (8) holds for all functions f at x for a given $\mathcal{N}$. We call $\mathcal{K}(x, \cdot)$ the *Bakry–Émery curvature function* of x and we define

$$\mathcal{K}_\infty(x) := \lim_{\mathcal{N} \to \infty} \mathcal{K}(x, \mathcal{N}).$$

In this paper, we will restrict our considerations to the curvature at ∞-dimension $\mathcal{K}_\infty : V \to \mathbb{R}$. Note that for the definition of $\mathcal{K}_\infty(x)$, the formula (8) simplifies to

$$\Gamma_2(f)(x) \geq \mathcal{K}\Gamma(f)(x) \quad \text{for all } x \in V.$$

The quadratic forms $\Gamma(\cdot, \cdot)(x)$ and $\Gamma_2(\cdot, \cdot)(x)$ can be represented by matrices $\Gamma(x)$ and $\Gamma_2(x)$ as follows:

$$\Gamma(f, g)(x) = \underline{f}\,\Gamma(x)\,\underline{g}^\top,$$
$$\Gamma_2(f, g)(x) = \underline{f}\,\Gamma_2(x)\,\underline{g}^\top,$$

where $\underline{f}, \underline{g}$ are the vector representations of f and g. The matrices $\Gamma(x), \Gamma_2(x)$ are symmetric with nonzero entries only in $B_1(x)$ and $B_2(x)$, respectively. So we can view them as local matrices by disregarding the vertices outside $B_2(x)$. For the explicit matrix entries of $\Gamma(x)$ and

$\Gamma_2(x)$ see [10, Subsections 2.2 and 2.3]. Note that these entries are already fully determined by the combinatorial structure of the *incomplete 2-ball around x*, denoted by $B_2^{\mathrm{inc}}(x)$, which is the induced subgraph of $B_2(x)$ with all edges within $S_2(x)$ removed.

We have the following general upper curvature bound similar to Theorem 2.6:

Theorem 2.9 (see [10, Corollary 3.3]). *Let* $G = (V, E)$ *be d-regular and* $x \in V$. *Then*

$$\mathcal{K}_\infty(x) \leq 2 + \frac{\#_\Delta(x)}{d},$$

where $\#_\Delta(x)$ *is the number of triangles containing x.*

Let us finally return to the examples from Section 2.1.

Remark 2.10. The examples in Remark 2.5 have the following Bakry–Émery and Ollivier–Ricci curvatures:

	$\kappa_0(x, y)$	$\kappa_{\mathrm{LLY}}(x, y)$	$\mathcal{K}_\infty(x)$
Hypercube Q^d	0	$\frac{2}{d}$	2
Triplex	$-\frac{1}{3}$	0	-1
Icosidodecahedral graph	0	0	$-\frac{3}{2}$

None of the regular graphs in the above table has curvature with opposite signs. We are not aware of any such examples and it would be interesting to find such graphs.

3 | RICCI FLAT GRAPHS

The notion of Ricci flat graphs was introduced in 1996 by Chung and Yau [6] in connection to a logarithmic Harnack inequality and is motivated by the structure of the *d*-dimensional grid $\mathbb{Z}^d$. Abelian Cayley graphs are prominent examples of Ricci flat graphs.

Definition 3.1. Let $G = (V, E)$ be a *d*-regular graph. We say that $x \in V$ is *Ricci flat* if there exist maps $\eta_i : B_1(x) \to V$ for $1 \leq i \leq d$ with the following properties:

(i) $\eta_i(u) \sim u$ for all $u \in B_1(x)$,
(ii) $\eta_i(u) \neq \eta_j(u)$ if $i \neq j$,
(iii) $\bigcup_j \eta_j(\eta_i x)) = \bigcup_j \eta_i(\eta_j x)$ for all i.

We also consider the following additional properties of the maps η_i:

(R) Reflexivity: $\eta_i^2(x) = x$ *for all i.*
(S) Symmetry: $\eta_j(\eta_i x) = \eta_i(\eta_j x)$ *for all i, j.*

If there exists a family of maps η_i for a given vertex $x \in V$ satisfying property (R) or property (S) in addition to (i)–(iii), we say that x is (R)-Ricci flat or (S)-Ricci flat, respectively. If there exists a family of maps η_i satisfying (i)–(iii) and (R) and (S) simultaneously, we say that x is (RS)-Ricci flat.

The d-dimensional grid $\mathbb{Z}^d$ is Ricci flat with the choices $\eta_i(x) = x + e_i$. The following lemma is a useful observation for the study of Ricci flatness of concrete examples.

Lemma 3.2. *Assume a family of maps $\eta_i \colon B_1(x) \to V$ satisfies (i)–(iii) of the above definition. Then each of these maps η_i is a bijective map between $B_1(x)$ and $B_1(\eta_i x)$.*

Proof. Assume that the family η_i satisfies (i)–(iii). It follows immediately from (i) and (ii) and regularity that

$$\bigcup_j \eta_j(u) = S_1(u) \quad \text{for all } u \in B_1(x).$$

This implies that (iii) is equivalent to

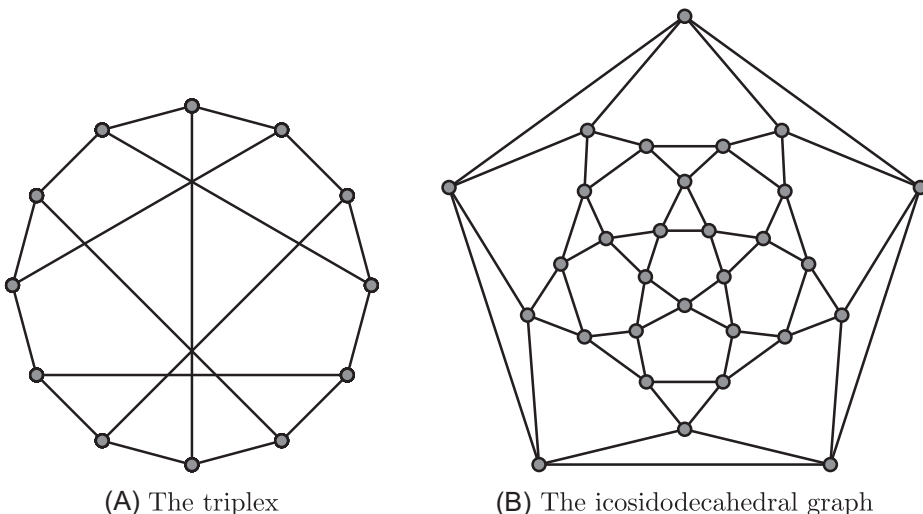$$S_1(\eta_i x) = \eta_i(S_1(x)) \quad \text{for all } i,$$

which, in turn, implies

$$B_1(\eta_i x) = S_1(\eta_i x) \cup \{\eta_i x\} = \eta_i(S_1(x)) \cup \eta_i(\{x\}) = \eta_i(B_1(x)). \tag{9}$$

Therefore, each map η_i must be injective, since

$$|\eta_i(B_1(x))| = |B_1(\eta_i x)| = |B_1(x)|.$$

Bijectivity from $B_1(x)$ to $B_1(\eta x)$ follows immediately from (9). $\square$

(A) The triplex   (B) The icosidodecahedral graph

FIGURE 1 Examples of graphs with $\kappa_{\mathrm{LLY}} = 0$. (A) The triplex and (B) the icosidodecahedral graph

Note that all Ricci flatness properties at a vertex x can be determined from the combinatorial structure of the incomplete 2-ball $B_2^{\text{inc}}(x)$ around x, which was introduced in Section 2.2.

Example 3.3. To help readers familiarize with the notion of Ricci flatness, we provide three examples of graphs and check whether each of them is Ricci flat.

(a) The incomplete 2-ball in Figure 2 with $S_1(x) = \{v_1, v_2, v_3\}$, $v_1 \sim v_2$ and $S_2(x) = \{v_4, v_5, v_6\}$, $v_4 \sim v_1$, $v_5 \sim v_2$, v_3 and $v_6 \sim v_3$ is not Ricci flat:

We show this by contradiction. Assume $\eta_i : B_1(x) \to V$ with properties (i)–(iii) exist. Without loss of generality, we can assume $\eta_i(x) = v_i$. Note that we must have $\eta_i(v_j) \in S_1(v_i) \cap S_1(v_j)$ for $1 \leq i, j \leq d$. This implies that we have the following choices for our maps η_j:

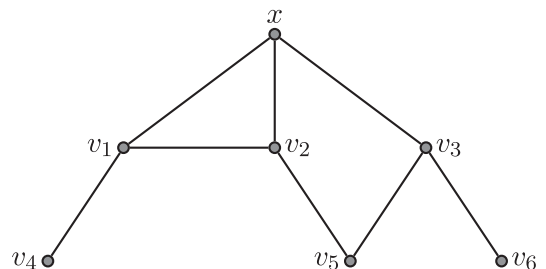|	x	v_1	v_2	v_3
η_1	v_1	x, v_2, v_4	x	x
η_2	v_2	x	x, v_1, v_5	x, v_5
η_3	v_3	x	x, v_5	x, v_5, v_6

Such a table can be presented concisely with the help of a $d \times d$ matrix A, namely, $A = (A_{ij})$ defined as follows: Let $S_1(x) = \{v_1, ..., v_d\}$ where $v_j := \eta_j(x)$, and $S_2(x) = : \{v_{d+1}, ..., v_t\}$ and, furthermore, $v_0 := x$. Then the entries $A_{ij} \in \{0, 1, ..., t\}$ of A are given via the relation

$$v_{A_{ij}} = \eta_i(v_j).$$

Then the table translates into the following possibilities for the entries of A:

$$\begin{pmatrix} 0, 2, 4 & 0 & 0 \\ 0 & 0, 1, 5 & 0, 5 \\ 0 & 0, 5 & 0, 5, 6 \end{pmatrix}.$$

The conditions (i)–(iii) require that all columns and rows of A have nonrepeating entries. Obviously, this is not possible in this case. Henceforth, we will use this matrix notation to simplify matters.

FIGURE 2 Graph that is not Ricci flat

(a) *The graph* $K_{3,3}$: Let $S_1(x) = \{v_1, v_2, v_3\}$ and $S_2(x) = \{v_4, v_5\}$ with $v_4, v_5 \sim v_1, v_2, v_3$. We have the following possibilities for the entries of the associated matrix A:

$$\begin{pmatrix} 0,4,5 & 0,4,5 & 0,4,5 \\ 0,4,5 & 0,4,5 & 0,4,5 \\ 0,4,5 & 0,4,5 & 0,4,5 \end{pmatrix}.$$

Note that (R)-Ricci flatness requires the existence of an associated matrix A with vanishing diagonal and (S)-Ricci flatness requires the existence of a symmetric matrix A. Therefore, x is (R)- and (S)-Ricci flat by the following matrix choices:

$$A_R = \begin{pmatrix} 0 & 4 & 5 \\ 5 & 0 & 4 \\ 4 & 5 & 0 \end{pmatrix}, \quad A_S = \begin{pmatrix} 0 & 4 & 5 \\ 4 & 5 & 0 \\ 5 & 0 & 4 \end{pmatrix}.$$

Note that x is not (RS)-Ricci flat since both properties (vanishing diagonal and symmetry) cannot be satisfied at the same time. In fact, the complete bipartite graphs $K_{d,d}$ are both (R)- and (S)-Ricci flats for all d, and (RS)-Ricci flat if and only if d is even (see details in the appendix of the arXiv version [7]).

(b) *Shrikhande graph*: Cayley graph $\mathbb{Z}_4 \times \mathbb{Z}_4$ with the generator set $\{\pm(0, 1), \pm(1, 0), \pm(1, 1)\}$. It is a strongly regular graph (see [5, p. 125]). The structure of the incomplete 2-ball $B_2^{\text{inc}}(x)$ around any vertex x is given in Figure 3. We have the following possibilities for the entries of the associated matrix A:

$$\begin{pmatrix} 0,2,6,7,12,15 & 0,7 & 0,2 & 0,12 & 0,6 & 0,15 \\ 0,7 & 0,1,3,7,8,13 & 0,8 & 0,3 & 0,13 & 0,1 \\ 0,2 & 0,8 & 0,2,4,8,9,14 & 0,9 & 0,4 & 0,14 \\ 0,12 & 0,3 & 0,9 & 0,3,5,9,10,12 & 0,10 & 0,5 \\ 0,6 & 0,13 & 0,4 & 0,10 & 0,4,6,10,11,13 & 0,11 \\ 0,15 & 0,1 & 0,14 & 0,5 & 0,11 & 0,1,5,11,14,15 \end{pmatrix}.$$
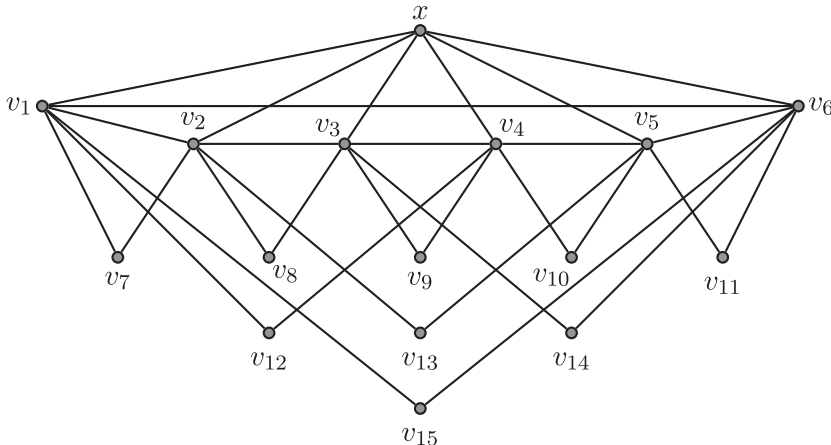

FIGURE 3 The incomplete 2-ball $B_2^{\text{inc}}(x)$ of the Shrikhande graph

Choosing 0 for diagonal entries fixes all other entries of the matrix. Moreover, this choice leads to a symmetric matrix, which shows that x is (RS)-Ricci flat.

3.1 | Ricci flatness and Ollivier–Ricci curvature

With regard to Ollivier–Ricci curvature we have the following general implications:

Theorem 3.4. *Let $G = (V, E)$ be d-regular.*

(a) If $x \in V$ is Ricci flat, then $\kappa_0(x, y) \geq 0$ for all edges $\{x, y\} \in E$.
(b) If $x \in V$ is (R)-Ricci flat, then $\kappa_{\text{LLY}}(x, y) \geq \frac{2}{d}$ for all edges $\{x, y\} \in E$.

Proof. For the proof of (a) we assume Ricci flatness at x with corresponding maps $\eta_i : B_1(x) \to V$. Let $y \in S_1(x)$. Recall that

$$S_1(x) = \{\eta_1(x), ..., \eta_d(x)\}.$$

Therefore, we have $y = \eta_i(x)$ for some $i \in \{1, ..., d\}$. We choose the following transport plan:

$$\pi(u, \eta_i(u)) = \frac{1}{d} \quad \text{for all } u \in S_1(x),$$

and $\pi(u, v) = 0$ for all other combinations. This implies

$$\sum_{v \in V} \pi(u, v) = \pi(u, \eta_i(u)) = \frac{1}{d} = \mu_x^0(u) \quad \text{for all } u \in S_1(x)$$

and (using Lemma 3.2)

$$\sum_{u \in V} \pi(u, v) = \pi\left(\eta_i^{-1}(v), v\right) = \frac{1}{d} = \mu_y^0(v) \quad \text{for all } v \in S_1(y),$$

which shows that $\pi \in \Pi\left(\mu_x^0, \mu_y^0\right)$. This leads to

$$W_1\left(\mu_x^0, \mu_y^0\right) \leq \text{cost}(\pi) = \sum_{u \in S_1(x)} \pi(u, \eta_i(u)) = 1,$$

which implies $\kappa_0(x, y) \geq 0$. We prove (b) similarly. Assume x is (R)-Ricci flat with corresponding maps η_i and $y = \eta_i(x)$. Note that we have $\eta_i(y) = x$ from reflexivity. This time, we choose the following transport plan $\pi \in \Pi\left(\mu_x^{1/(d+1)}, \mu_y^{1/(d+1)}\right)$:

$$\pi(u, \eta_i(u)) = \frac{1}{d+1} \quad \text{for all } u \in S_1(x)\backslash\{y\},$$

$\pi(x, x) = \pi(y, y) = \frac{1}{d+1}$, and $\pi(u, v) = 0$ for all other combinations. This leads to

$$W_1\left(\mu_x^{1/(d+1)}, \mu_y^{1/(d+1)}\right) \leq \text{cost}(\pi) = \sum_{u \in S_1(x) \setminus \{y\}} \pi(u, \eta_i(u)) = \frac{d-1}{d+1},$$

which implies $\varkappa_{1/(d+1)}(x, y) \geq \frac{2}{d+1}$ and

$$\varkappa_{\text{LLY}}(x, y) = \frac{d+1}{d} \varkappa_{1/(d+1)}(x, y) \geq \frac{2}{d}. \qquad \square$$

3.2 | Ricci flatness and Bakry–Émery curvature

With regard to Bakry–Émery curvature we have the following general implications:

Theorem 3.5. *Let $G = (V, E)$ be d-regular.*

(a) *If $x \in V$ is Ricci flat, then $\mathcal{K}_\infty(x) \geq 0$.*
(b) *If $x \in V$ is (R)-Ricci flat, then $\mathcal{K}_\infty(x) \geq 2$.*

Proof. The proof of statement (a) was already explained in [6,15]. This proof strategy can also be applied to prove statement (b). We present these proofs for the reader's convenience.

Recall from the definition that

$$2\Gamma_2(f, f)(x) = \Delta\Gamma(f, f)(x) - 2\Gamma(f, \Delta f)(x) \qquad (10)$$

and

$$2\Gamma(f, g)(x) = \Delta(fg)(x) - f(x)\Delta g(x) - g(x)\Delta f(x).$$

A useful identity to compute $\Gamma(f, g)$ is

$$2\Gamma(f, g)(x) = \sum_{y : y \sim x} (f(y) - f(x))(g(y) - g(x)).$$

Let us now consider the first term on the right-hand side (RHS) of (10) and use the identity $A^2 - B^2 = (A - B)^2 + 2B(A - B)$:

$$
\begin{aligned}
\Delta\Gamma(f, f)(x) &= \sum_{i=1}^{d} (\Gamma(f, f)(\eta_i x) - \Gamma(f, f)(x)) \\
&= \frac{1}{2} \sum_{i=1}^{d} \left[\sum_{j=1}^{d} (f(\eta_j \eta_i x) - f(\eta_i x))^2 - \sum_{j=1}^{d} (f(\eta_j x) - f(x))^2 \right] \\
&= \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} (f(\eta_j \eta_i x) - f(\eta_i x) - f(\eta_j x) + f(x))^2 \\
&\quad + \sum_{i=1}^{d} \sum_{j=1}^{d} (f(\eta_j x) - f(x))(f(\eta_j \eta_i x) - f(\eta_i x) - f(\eta_j x) + f(x)).
\end{aligned}
$$

On the other hand, we have for the second term on the RHS of (10), using Ricci flatness,

$$
\begin{aligned}
-2\Gamma(f, \Delta f)(x) &= -\sum_{j=1}^{d}(f(\eta_j x) - f(x))(\Delta f(\eta_j x) - \Delta f(x)) \\
&= -\sum_{j=1}^{d}\sum_{i=1}^{d}(f(\eta_j x) - f(x))(f(\eta_i \eta_j x) - f(\eta_j x) - f(\eta_i x) + f(x)) \\
&= -\sum_{j=1}^{d}\sum_{i=1}^{d}(f(\eta_j x) - f(x))(f(\eta_j \eta_i x) - f(\eta_i x) - f(\eta_j x) + f(x)).
\end{aligned}
$$

Adding both terms, we end up with

$$
2\Gamma_2(f, f)(x) = \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}(f(\eta_j \eta_i x) - f(\eta_i x) - f(\eta_j x) + f(x))^2 \geq 0,
$$

showing $\mathcal{K}_\infty(x) \geq 0$. Under the stronger condition of (R)-Ricci flatness, we can estimate $2\Gamma_2(f, f)(x)$ from below as follows:

$$
\begin{aligned}
2\Gamma_2(f, f)(x) &= \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}(f(\eta_j \eta_i x) - f(\eta_i x) - f(\eta_j x) + f(x))^2 \\
&\geq \frac{1}{2}\sum_{i=1}^{d}(f(\eta_i \eta_i x) - f(\eta_i x) - f(\eta_i x) + f(x))^2 \\
&= \frac{1}{2}\sum_{i=1}^{d}(2f(x) - 2f(\eta_i x))^2 = 4\Gamma(f, f)(x).
\end{aligned}
$$

This shows that $\Gamma_2(f, f)(x) \geq 2\Gamma(f, f)(x)$, which means that we have $\mathcal{K}_\infty(x) \geq 2$. $\qquad\square$

4 | TRIANGLE-FREE GRAPHS

In this section we focus on curvature comparison results for graphs without triangles. Our main result states that the nonnegativity of Ollivier–Ricci curvature implies the nonnegativity of Bakry–Émery curvature under a certain in-degree condition (see Corollary 1.2). This result is derived via Ricci flatness properties.

We start with particular upper curvature bounds in case of triangle-freeness:

Proposition 4.1. *Let $G = (V, E)$ be d-regular. Then we have the following upper curvature bounds:*

(i) $\kappa_0(x, y) \leq 0$ *for all edges $\{x, y\} \in E$ not contained in a triangle,*
(ii) $\kappa_{\mathrm{LLY}}(x, y) \leq \frac{2}{d}$ *for all edges $\{x, y\} \in E$ not contained in a triangle,*
(iii) $\mathcal{K}_\infty(x) \leq 2$ *for all $x \in V$ not contained in a triangle.*

Remark 4.2. Combining the proposition with the lower curvature bounds for Ricci flatness (Theorems 3.4 and 3.5), we obtain the following curvature equalities:

- If x is Ricci flat and the edge $\{x, y\} \in E$ is not contained in any triangle, then $\kappa_0(x, y) = 0$.
- If x is (R)-Ricci flat and the edge $\{x, y\} \in E$ is not contained in any triangle, then $\kappa_{\mathrm{LLY}}(x, y) = \frac{2}{d}$.
- If x is (R)-Ricci flat and not contained in any triangle, then $\mathcal{K}_\infty(x) = 2$.

Proof of Proposition 4.1. Although Statements (i) and (ii) are an implication from Theorem 2.6, we provide their proof here which presents a useful idea for the following remark.

Statement (i) follows from

$$
\begin{aligned}
W_1\left(\mu_x^0, \mu_y^0\right) &= \sum_{u \in S_1(x)} \sum_{v \in S_1(y)} d(u, v) \pi_{\mathrm{opt}}(u, v) \\
&\geq \sum_{u \in S_1(x)} \sum_{v \in S_1(y)} \pi_{\mathrm{opt}}(u, v) = 1,
\end{aligned}
\tag{11}
$$

since $S_1(x) \cap S_1(y) = \varnothing$. Here π_{opt} is an optimal transport plan in $\Pi\left(\mu_x^0, \mu_y^0\right)$.

For the proof of (ii), we only need to show

$$
\kappa_{\frac{1}{d+1}}(x, y) \leq \frac{2}{d+1}
$$

by (5). This follows from

$$
\begin{aligned}
W_1\left(\mu_x^{1/(d+1)}, \mu_y^{1/(d+1)}\right) &= \sum_{u \in B_1(x)} \sum_{v \in B_1(y)} d(u, v) \pi_{\mathrm{opt}}(u, v) \\
&\geq \left(\sum_{u \in B_1(x)} \sum_{v \in B_1(y)} \pi_{\mathrm{opt}}(u, v)\right) - \pi_{\mathrm{opt}}(x, x) - \pi_{\mathrm{opt}}(y, y) \\
&\geq 1 - \frac{2}{d+1},
\end{aligned}
\tag{12}
$$

since $B_1(x) \cap B_1(y) = \{x, y\}$ and $\pi_{\mathrm{opt}}(u, u) \leq \mu_x^{1/(d+1)}(u) \leq \frac{1}{d+1}$. Here π_{opt} is an optimal transport plan in $\Pi\left(\mu_x^{1/(d+1)}, \mu_y^{1/(d+1)}\right)$.

Statement (iii) is an implication from Theorem 2.9. □

Remark 4.3. Note that in Proposition 4.1, (ii) implies (i) by Theorem 1.1. Moreover, it follows from the above proof that sharpness of the bounds in (i) and (ii) has the following combinatorial interpretation in the triangle-free case:

(a) $\kappa_0(x, y) = 0$ is equivalent that there is a perfect matching between $S_1(x)$ and $S_1(y)$.
(b) $\kappa_{\mathrm{LLY}}(x, y) = \frac{2}{d}$ is equivalent that there is a perfect matching between $S_1(x) \backslash \{y\}$ and $S_1(y) \backslash \{x\}$.

A natural class of examples where all three upper bounds of Proposition 4.1 are attained is distance-regular graphs of girth 4 (see Section 6). To motivate our next result, let us focus on one particular example:

Example 4.4. Let $S_1(x) = \{v_1, ..., v_d\}$ and $S_2(x) = \{v_{ij} | 1 \leq i < j \leq d\}$ with $v_i, v_j \sim v_{ij}$. In fact this is the 2-ball of the d-dimensional hypercube Q^d and we have the following curvatures (see Remark 2.10):

$$\kappa_0(x, v_i) = 0, \quad \kappa_{\mathrm{LLY}}(x, v_i) = \frac{2}{d}, \quad \mathcal{K}_\infty(x) = 2.$$

We also like to mention that the vertex x in this example is (RS)-Ricci flat and that we have $d_x^-(z) = 2$ for all $z \in S_2(x)$.

Theorem 4.5. *Given a regular graph $G = (V, E)$, let $x \in V$ be a vertex not contained in a triangle and satisfying $d_x^-(z) \leq 2$ for all $z \in S_2(x)$. Then we have the following:*

(a) *$\kappa_0(x, y) = 0$ for all $y \in S_1(x)$ is equivalent to x being (S)-Ricci flat.*
(b) *$\kappa_{\mathrm{LLY}}(x, y) = \frac{2}{d}$ for all $y \in S_1(x)$ is equivalent to x being (RS)-Ricci flat.*

This result, together with Theorem 3.5, implies our main curvature comparison result in Theorem 1.2 from Section 1:

Proof of Theorem 1.2. Under the assumptions of Theorem 4.5, we first assume that $\kappa_0(x, y) = 0$ for all $y \in S_1(x)$. This implies that x is Ricci flat and, by Theorem 3.5(a), that $\mathcal{K}_\infty(x) \geq 0$.

Similarly, assuming $\kappa_{\mathrm{LLY}}(x, y) = \frac{2}{d}$ for all $y \in S_1(x)$, we know that x is (R)-Ricci flat, and Theorem 3.5(b) implies that $\mathcal{K}_\infty(x) \geq 2$. Since x is not contained in a triangle, this leads to $\mathcal{K}_\infty(x) = 2$ by Proposition 4.1(iii). □

Before we start with the proof of Theorem 4.5, let us introduce the following notion and discuss relations to existing results.

Definition 4.6. Let $G = (V, E)$ be a regular triangle-free graph and $x \in V$. We say that $y_1, y_2 \in S_1(x)$ are *linked* by $z \in S_2(x)$ if we have $y_1 \sim z \sim y_2$. We refer to z as a *link* of y_1 and y_2.

Ralli [17] investigated curvature implications for regular graphs without K_3 and $K_{3,2}$ as subgraphs. It is easy to check that this condition is equivalent to the following properties at all vertices x:

(i) x is not contained in a triangle,
(ii) $d_x^-(z) \leq 2$ for all $z \in S_2(x)$,
(iii) Any pair $y_1, y_2 \in S_1(x)$ has at most one link.

A consequence of his results is that conditions (i)–(iii) imply $\mathcal{K}_\infty(x) \leq 0$ or $\mathcal{K}_\infty(x) = 2$. Under these conditions, Ralli has the following equivalence:

$$\kappa_0(x, y) = 0 \quad \text{for all } y \in S_1(x) \Leftrightarrow \mathcal{K}_\infty(x) \geq 0.$$

Our theorem implies that the implication "$\Rightarrow$" holds already under conditions (i) and (ii) and we have an example that the implication "$\Longleftarrow$" is no longer true if one drops condition (iii). Note also that our theorem is a local result, meaning that the assumption is made for an arbitrary vertex x (in contrast to Ralli's result where the assumption is made for the entire graph). However, if a graph satisfies the assumption of our theorem for all vertices x, then it is equivalent to the graph satisfying Ralli's assumption. In other words, there are no entire graphs where Theorem 4.5 holds and Ralli's does not.

Proof of Theorem 4.5. The implications $\Longleftarrow$ in (a) and (b) follow immediately from Theorem 3.4 and Proposition 4.1.

Let us now prove the forward implication in (a). Let $x \in V$ be given with $d = d_x$ and $S_1(x) = \{y_1, ..., y_d\}$. The property $\kappa_0(x, y) = 0$ for all $y \in S_1(x)$ implies that we have perfect matchings $\sigma_i : S_1(x) \to S_1(y_i)$ for all $1 \le i \le d$. In particular, we can assume that these perfect matchings σ_i satisfy the following property:

Property (P): If there exists a perfect matching between $S_1(x) \backslash \{y_i\}$ and $S_1(y_i) \backslash \{x\}$, then $\sigma_i(y_i) = x$.

Our goal is to show that we can modify these perfect matchings in such a way that $\sigma_i(y_j) = \sigma_j(y_i)$ for all $i \ne j$. Defining then $\eta_i : B_1(x) \to B_1(y_i)$ as $\eta_i(x) = y_i$ and $\eta_i(y) = \sigma_i(y)$ for $y \in S_1(x)$ provide (S)-Ricci flatness.

We first prove the following crucial fact:

Fact: Let $i \ne j$. We have $\sigma_i(y_j) = x$ if and only if y_i and y_j are not linked.

This fact can be shown as follows: We first prove the easier "$\Longleftarrow$" implication. Assume y_i and y_j are not linked. Then $\sigma_i(y_j) \sim y_i$, y_j cannot be in $S_2(x)$ and we must have therefore $\sigma_i(y_j) = x$. For the "$\Rightarrow$" implication, we provide an indirect proof: If y_i and y_j were linked by $z \in S_2(x)$, then the σ_i-preimage of $z \in S_1(y_i)$ must be in $\{y_i, y_j\}$ but we know that $\sigma_i(y_j) = x$. Therefore $\sigma_i(y_i) = y_j$. Defining then the map $\tilde{\sigma}_i : S_1(x) \to S_1(y_i)$ via

$$\tilde{\sigma}_i(y_k) = \begin{cases} \sigma_i(y_k) & \text{if } k \ne i, j, \\ z & \text{if } k = j, \\ x & \text{if } k = i, \end{cases}$$

induces a perfect matching between $S_1(x) \backslash \{y_i\}$ and $S_1(y_i) \backslash \{x\}$. This would imply $\sigma_i(y_i) = x$ contradicting to $\sigma_i(y_j) = x$.

Now we prove our goal.

We first show that $\sigma_i(y_j) = x$ implies $\sigma_j(y_i) = x$: Since $\sigma_i(y_j) = x$, y_i and y_j are not linked by our Fact which, in turn, implies $\sigma_j(y_i) = x$ by our Fact, again.

We deal with all other pairs (i, j), $i \ne j$ as follows: If $\sigma_i(y_j) = \sigma_j(y_i)$, we do not change the assignments $\sigma_i(y_i)$, $\sigma_i(y_j)$, $\sigma_j(y_i)$, $\sigma_j(y_j)$. Now we assume that $\sigma_i(y_j) =: z \ne \sigma_j(y_i) := z'$. Note that $z, z' \in S_2(x)$ and they both are links of y_i and y_j. Since $z \in S_1(y_j)$ and $d_x^-(z) \le 2$, we must have $\sigma_j^{-1}(z) \in \{y_i, y_j\}$. Since σ_j is injective and $\sigma_j(y_i) = z'$, we must have $\sigma_j^{-1}(z) = y_j$. So we must have

$$\sigma_j(y_j) = z. \tag{13}$$

Similarly, we conclude that $\sigma_i(y_i) = z'$. Now we modify σ_i as follows: $\sigma_i(y_i) = z$ and $\sigma_i(y_j) = z'$. This preserves property (P) of the perfect matching σ_i and establishes

$\sigma_i(y_j) = \sigma_j(y_i)$ for this pair of indices (i, j). Note that if (i, j) and (k, l) are two different pairs with $\sigma_i(y_j) \neq \sigma_j(y_i)$ and $\sigma_k(y_l) \neq \sigma_l(y_k)$, then $\{i, j\} \cap \{k, l\} = \varnothing$ for, otherwise, if $k = i$, there is no perfect matching between $S_1(x)$ and $S_1(y_i)$ since the four links between y_i, y_j and y_i, y_l can only have three possible preimages under σ_i. This guarantees that we can repeat this process for all such pairs (i, j) simultaneously and we will end up with the required symmetric arrangement.

Finally, it remains to prove the forward implication of (b). The assumption $\kappa_{\mathrm{LLY}}(x, y) = \frac{2}{d}$ for all $y \in S_1(x)$ implies $\kappa_0(x, y) = 0$ by Theorem 1.1. The existence of perfect matchings between $S_1(x) \backslash \{y_i\}$ and $S_1(y_i) \backslash \{x\}$ for all $1 \leq i \leq d$ from Remark 4.3 further imply that our chosen maps σ_i satisfy $\sigma_i(y_i) = x$ for all i. In this situation, we can disregard the above possibility of $z = \sigma_i(y_j) \neq \sigma_j(y_i) = z'$ with $z, z' \in S_2(x)$, since this would imply (13), which contradicts $\sigma_j(y_j) = x$. Therefore, the maps σ_i do not need to be modified and the induced maps $\eta_i : B_1(x) \rightarrow V$ satisfy both symmetry and reflexivity. $\square$

Remark 4.7.

(a) The reverse of the implication in Theorem 1.2(a) is not true since we have a triangle-free 2-ball in Figure 4 with $\mathcal{K}_\infty(x) = 0, d_x^-(z) = 2$ for all $z \in S_2(x)$ and $\kappa_0(x, y) < 0$ for all $y \in S_1(x)$ as a counterexample. Note that $S_1(x) = \{v_1, ..., v_6\}$.
(b) All conditions in Theorem 4.5(a) are necessary:
 (i) If x is contained in a triangle, we have the icosidodecahedral graph (see Figure 1b) as a counterexample with $\kappa_0(x, y) = 0$ for all edges $\{x, y\}$ but $\mathcal{K}_\infty(x) < 0$ for all vertices x, which means that x cannot be Ricci flat by Theorem 3.5.
 (ii) If we drop $d_x^-(z) \leq 2$ for all $z \in S_2(x)$, Figure 5 provides a counterexample with $\kappa_0(x, y) = 0$ for all $y \in S_1(x)$ and $\mathcal{K}_\infty(x) < 0$.
(c) All conditions in Theorem 4.5(b) are necessary. Since in the case of triangles we have the following upper bound:

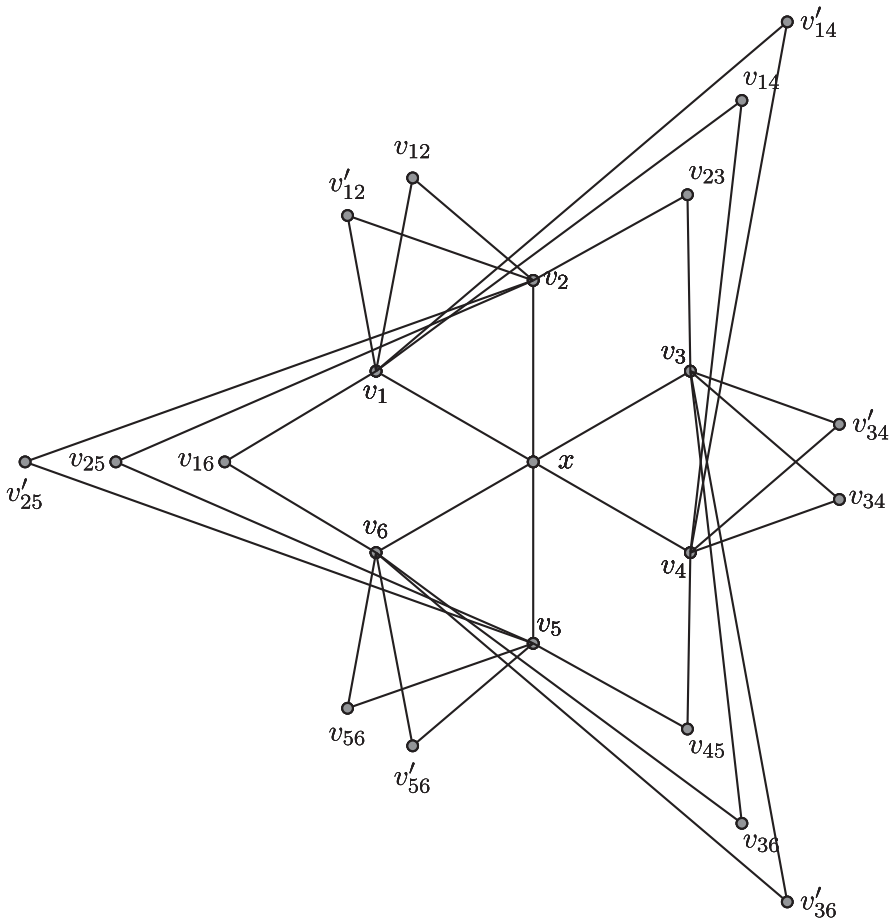$$\kappa_{\mathrm{LLY}}(x, y) \leq \frac{2 + \#_\Delta(x, y)}{d},$$

a natural generalization of the equivalence in the case of triangles would be the following statement:

$$\kappa_{\mathrm{LLY}}(x, y) = \frac{2 + \#_\Delta(x, y)}{d} \quad \text{for all } y$$

$$\in S_1(x) \text{ is equivalent to } x \text{ being (RS)-Ricci flat}.$$

 (i) If x is contained in a triangle, we have $K_3 \times K_3$ with $d = 4$ as a counterexample:

$$\kappa_{\mathrm{LLY}}(x, y) = \frac{3}{4} = \frac{2 + \#_\Delta(x, y)}{d}$$

for all edges $\{x, y\}$, but no vertex of $K_3 \times K_3$ is (RS)-Ricci flat.

FIGURE 4 Example with $\mathcal{K}_\infty(x) = 0$, $d_x^-(z) = 2$ for all $z \in S_2(x)$ and $\kappa_0(x, v_i) = -\frac{1}{3}$

(ii) If we drop $d_x^-(z) \leq 2$ for all $z \in S_2(x)$, the 6-regular incidence graph of the (11, 6, 3)-design provides a counterexample with $\kappa_{\mathrm{LLY}}(x, y) = \frac{1}{3}$ for all $y \in S_1(x)$, but x is not (RS)-Ricci flat (see Example 6.3).

5 | GRAPH PRODUCTS

This section is concerned with three natural products of two graphs G and H: the tensor product $G \otimes H$, the Cartesian product $G \times H$, and the strong product $G \boxtimes H$. We will see that Ricci flatness is preserved under all three products. However, while Cartesian products preserve the nonnegativity of both Bakry–Émery and Ollivier–Ricci curvatures, we will see that this property fails to be true in the case of strong products.

Let us start with the definitions of these graph products:

Definition 5.1. Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs. The vertex set of each of the three products $G \otimes H$ (*tensor product*), $G \times H$ (*Cartesian product*) and $G \boxtimes H$ (*strong product*) is given by $V_G \times V_H$. To define the edge sets for each of these products, let
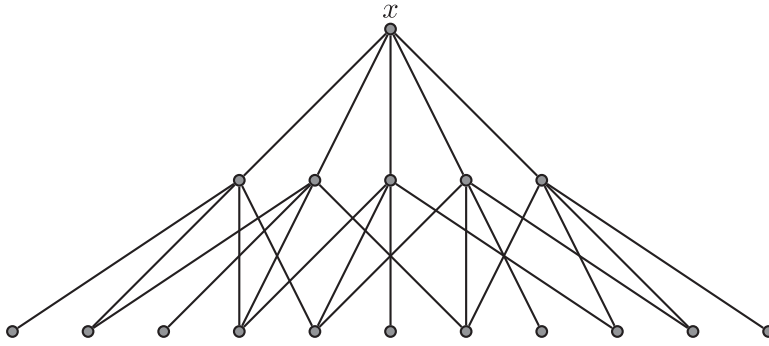
FIGURE 5 Example with $\mathcal{K}_\infty(x) = -0.194 < 0$ and $\kappa_p(x, y) = 0 \ \forall \ p \in [0, 1], y \sim x$

$$E_{\text{hor}} := \{\{(x_1, y), (x_2, y)\} | x_1 \overset{G}{\sim} x_2\},$$
$$E_{\text{vert}} := \{\{(x, y_1), (x, y_2)\} | y_1 \overset{H}{\sim} y_2\},$$
$$E_{\text{diag}} := \{\{(x_1, y_1), (x_2, y_2)\} | x_1 \overset{G}{\sim} x_2 \ \text{and} \ y_1 \overset{H}{\sim} y_2\}$$

denote the set of *horizontal*, *vertical* and *diagonal* edges, respectively. Then

$$G \otimes H := (V_G \times V_H, E_{\text{diag}}),$$
$$G \times H := (V_G \times V_H, E_{\text{hor}} \cup E_{\text{vert}}),$$
$$G \boxtimes H := (V_G \times V_H, E_{\text{hor}} \cup E_{\text{vert}} \cup E_{\text{diag}}).$$

Note that, in the case of a d_G-regular graph G and a d_H-regular graph H, the products $G \otimes H, G \times H$ and $G \boxtimes H$ are $(d_G d_H)$-regular, $(d_G + d_H)$-regular and $(d_G + d_H + d_g d_H)$-regular, respectively.

Our first result is concerned with preservance of Ricci flatness:

Theorem 5.2. *Let G, H be two Ricci flat graphs. Then the graph products $G \otimes H, G \times H$ and $G \boxtimes H$ are again Ricci flat. Similarly, all three graph products preserve also (R)-, (S)- and (RS)-Ricci flatness.*

Proof. Assume that G and H are Ricci flat at $x \in V_G$ and at $y \in V_H$, respectively, that is, there exist maps $\eta_i^G : B_1(x) \to V_G \ (1 \le i \le d_G)$ and $\eta_k^H : B_1(y) \to V_H \ (1 \le k \le d_H)$ satisfying the conditions (i)–(iii) in Definition 3.1.

Note that we have the inclusions

$$B_1^{G \times H}(x, y), \quad B_1^{G \otimes H}(x, y) \subset B_1^{G \boxtimes H}(x, y).$$

We define the following maps $\eta'_i, \eta''_k, \eta_{jl}^{\otimes} : B_1^{G \boxtimes H}(x, y) \to V_G \times V_H$ (for $1 \le i, j \le d_G, 1 \le k, l \le d_H$):

$$\eta'_i(u, v) := \left(\eta_i^G(u), v\right),$$
$$\eta''_k(u, v) := \left(u, \eta_k^H(v)\right),$$
$$\eta_{jl}^{\otimes}(u, v) := \left(\eta_j^G(u), \eta_l^H(v)\right).$$

Note that

$$\eta_{jl}^{\otimes} = \eta'_j \circ \eta''_l = \eta''_l \circ \eta'_j.$$

We only consider the strong product case here, since all other products can be dealt with similarly by restrictions of the relevant η-maps to the corresponding 1-balls. We now check properties (i)–(iii) of Definition 3.1 for these maps on $B_1^{G \boxtimes H}(x, y)$.

To verify (i), we observe that $(u, v) \sim \eta'_i(u, v)$ represents a horizontal edge in $G \boxtimes H$, $(u, v) \sim \eta''_k(u, v)$ represents a vertical edge and $(u, v) \sim \eta_{jl}^{\otimes}(u, v)$ represents a diagonal edge.

Next, we verify (ii): The above observation implies that $\eta'_i(u, v)$, $\eta''_k(u, v)$ and $\eta_{jl}^{\otimes}(u, v)$ are mutually distinct for any choices of i, j, k, l. Moreover, it is easy to check that

$$\eta'_i(u, v) \neq \eta'_j(u, v), \quad \eta''_k(u, v) \neq \eta''_l(u, v), \quad \eta_{ik}^{\otimes}(u, v) \neq \eta_{jl}^{\otimes}(u, v)$$

for any choice of $i \neq j$ and $k \neq l$.

Now we verify (iii): We have

$$\bigcup_{j,l} \eta_{jl}^{\otimes}\left(\eta_{ik}^{\otimes}(x, y)\right) = \bigcup_j \eta_j^G \eta_i^G x \times \bigcup_l \eta_l^H \eta_k^H y$$
$$= \bigcup_j \eta_i^G \eta_j^G x \times \bigcup_l \eta_k^H \eta_l^H y$$
$$= \bigcup_{j,l} \eta_{ik}^{\otimes}\left(\eta_{jl}^{\otimes}(x, y)\right).$$

Similar commutation properties hold for the other families of η-maps, that is, we have

$$\bigcup_* \eta^* \eta^{**}(x, y) = \bigcup_* \eta^{**} \eta^*(x, y),$$

where η^* and η^{**} are the maps within the families η'_i, η''_k and $\eta_{jl}^{\otimes}$. Combining these results, we obtain

$$\bigcup_j \eta'_j\left(\eta_{ik}^{\otimes}(x, y)\right) \cup \bigcup_j \eta''_l\left(\eta_{ik}^{\otimes}(x, y)\right) \cup \bigcup_{j,l} \eta_{jl}^{\otimes}\left(\eta_{ik}^{\otimes}(x, y)\right)$$
$$= \bigcup_j \eta_{ik}^{\otimes}(\eta'_j(x, y)) \cup \bigcup_j \eta_{ik}^{\otimes}(\eta'_j(x, y)) \cup \bigcup_{j,l} \eta_{ik}^{\otimes}\left(\eta_{jl}^{\otimes}(x, y)\right)$$

and

$$\bigcup_j \eta'_j(\eta'_i(x, y)) \cup \bigcup_j \eta''_l(\eta'_i(x, y)) \cup \bigcup_{j,l} \eta_{jl}^{\otimes}(\eta'_i(x, y))$$
$$= \bigcup_j \eta'_i(\eta'_j(x, y)) \cup \bigcup_j \eta'_i(\eta''_l(x, y)) \cup \bigcup_{j,l} \eta'_i\left(\eta_{jl}^{\otimes}(x, y)\right)$$

and

$$\bigcup_j \eta'_j(\eta''_k(x, y)) \cup \bigcup_j \eta''_l(\eta''_k(x, y)) \cup \bigcup_{j,l} \eta_{jl}^{\otimes}(\eta''_k(x, y))$$
$$= \bigcup_j \eta''_k(\eta'_j(x, y)) \cup \bigcup_j \eta''_k(\eta''_l(x, y)) \cup \bigcup_{j,l} \eta''_k\left(\eta_{jl}^{\otimes}(x, y)\right).$$

In conclusion, Ricci flatness is preserved for all three graph products.

Finally, we verify the preservance of (R)-, (S)- and (RS)-Ricci flatness. Assume (R)-Ricci flatness at $x \in V_G$ and $y \in V_h$. (R)-Ricci flatness at (x, y) follows now from

$$\left(\eta_{jl}^{\otimes}\right)^2(x, y) = \left(\left(\eta_j^G\right)^2(x), \left(\eta_l^H\right)^2(y)\right) = (x, y),$$

and $(\eta'_i)^2(x, y) = (\eta''_k)^2(x, y) = (x, y)$ can be checked similarly. Preservance of (S)-Ricci flatness follows from

$$\eta^*\eta^{**}(x, y) = \eta^{**}\eta^*(x, y),$$

where η^* and η^{**} are the maps within the families η'_i, η''_k and $\eta_{jl}^{\otimes}$. $\qquad\square$

In the case of Cartesian products of two regular graphs G, H, there are explicit curvature formulas in terms of curvatures of the factors: Bakry–Émery curvature $\mathcal{K}_\infty^{G \times H}(x, y) = \min\left\{\mathcal{K}_\infty^G(x), \mathcal{K}_\infty^H(y)\right\}$ can be found in [10, Corollary 7.13] and Ollivier–Ricci curvature $\kappa_0^{G \times H}(x, y)$ and $\kappa_{\mathrm{LLY}}^{G \times H}(x, y)$ can be found in [14, Claims 1 and 2 in Proof of Theorem 3.1]. In particular, nonnegativity of each of these curvature notions is preserved under Cartesian products. In our next result, we provide lower curvature bounds for horizontal and vertical edges of the strong product $G \boxtimes H$:

Theorem 5.3. *Let G and H be two regular graphs with vertex degrees d_G and d_H, respectively. Lower Ollivier–Ricci curvature bounds on horizontal edges and vertical edges are given by*

$$\kappa_*((x_1, y_1), (x_2, y_1)) \geq \frac{d_G(d_H + 1)}{d_{G \boxtimes H}} \kappa_*^G(x_1, x_2),$$

$$\kappa_*((x_1, y_1), (x_1, y_2)) \geq \frac{d_H(d_G + 1)}{d_{G \boxtimes H}} \kappa_*^H(y_1, y_2),$$

where κ_* may refer to κ_0 or κ_{LLY} and $d_{G \boxtimes H} = d_G + d_H + d_G d_H$ is the vertex degree of $G \boxtimes H$.

Proof of Theorem 5.3. Let us consider a horizontal edge $(x_1, y_1) \sim (x_2, y_1)$ where $x_1 \overset{G}{\sim} x_2$. We will prove this argument for Lin–Lu–Yau curvature first. Let $\pi_G \in \Pi\left(\mu_{x_1}^{1/(1+d_G)}, \mu_{x_2}^{1/(1+d_G)}\right)$ be an optimal transport plan, that is, its cost is equal to $W_1^G\left(\mu_{x_1}^{1/(1+d_G)}, \mu_{x_2}^{1/(1+d_G)}\right)$. Now we define a function $\pi : (V_G \times V_H)^2 \to [0, \infty)$ as follows:

$$\pi((w_1, z_1), (w_2, z_2)) := \begin{cases} \dfrac{1 + d_G}{1 + d_{G \boxtimes H}} \pi_G(w_1, w_2), & \text{if } z_1 = z_2 \in B_1(y_1), \\ 0, & \text{otherwise}. \end{cases}$$

Now we verify the following marginal constraints showing that π is indeed a transport plan $\pi \in \Pi\left(\mu_{(x_1,y_1)}^{1/(1+d_{G\boxtimes H})}, \mu_{(x_2,y_2)}^{1/(1+d_{G\boxtimes H})}\right)$: for fixed $(w_1, z_1) \in V_G \times V_H$,

$$\sum_{w_2,z_2} \pi((w_1, z_1), (w_2, z_2)) = \frac{1 + d_G}{1 + d_{G\boxtimes H}} \cdot \mathbb{1}_{B_1(y_1)}(z_1) \sum_{w_2} \pi_G(w_1, w_2)$$

$$= \frac{1 + d_G}{1 + d_{G\boxtimes H}} \cdot \mathbb{1}_{B_1(y_1)}(z_1) \cdot \mu_{x_1}^{1/(1+d_G)}(w_1)$$

$$= \frac{1}{1 + d_{G\boxtimes H}} \cdot \mathbb{1}_{B_1(y_1)}(z_1) \cdot \mathbb{1}_{B_1(x_1)}(w_1)$$

$$= \frac{1}{1 + d_{G\boxtimes H}} \cdot \mathbb{1}_{B_1(x_1,y_1)}(w_1, z_1)$$

$$= \mu_{(x_1,y_1)}^{1/(1+d_{G\boxtimes H})}(w_1, z_1)$$

and, similarly,

$$\sum_{w_1,z_1} \pi((w_1, z_1), (w_2, z_2)) = \mu_{(x_2,y_1)}^{1/(1+d_{G\boxtimes H})}(w_2, z_2).$$

The cost of this transport plan can then be calculated as

$$\text{cost}(\pi) = \sum_{(w_2,z_2)} \sum_{(w_1,z_1)} \text{dist}_{G\boxtimes H}((w_1, z_1), (w_2, z_2))\pi((w_1, z_1), (w_2, z_2))$$

$$= \sum_{z_1 \in B_1(y_1)} \sum_{w_1,w_2} \text{dist}_G(w_1, w_2)\frac{1 + d_G}{1 + d_{G\boxtimes H}}\pi_G(w_1, w_2)$$

$$= \frac{(1 + d_H)(1 + d_G)}{1 + d_{G\boxtimes H}} \sum_{w_1,w_2} \text{dist}_G(w_1, w_2)\pi_G(w_1, w_2)$$

$$= \text{cost}(\pi_G).$$

Recall that π_G is assumed to be an optimal transport plan and, therefore,

$$W_1^{G\boxtimes H}\left(\mu_{(x_1,y_1)}^{1/(1+d_{G\boxtimes H})}, \mu_{(x_2,y_1)}^{1/(1+d_{G\boxtimes H})}\right) \le \text{cost}(\pi) = \text{cost}(\pi_G) = W_1^G\left(\mu_{x_1}^{1/(1+d_G)}, \mu_{x_2}^{1/(1+d_G)}\right).$$

This inequality translates via Definition 2.3 and relation (5) into

$$\kappa_{\text{LLY}}((x_1, y_1), (x_2, y_1)) \ge \frac{d_G(d_H + 1)}{d_{G\boxtimes H}}\kappa_{\text{LLY}}^G(x_1, x_2),$$

which gives the desired lower bound for κ_{LLY} on the horizontal edge $(x_1, y_1){\sim}(x_2, y_1)$.

Now we prove a similar lower bound for κ_0. Let $\pi_G^0 \in \Pi\left(\mu_{x_1}^0, \mu_{x_2}^0\right)$ be an optimal transport plan, whose cost is

$$\text{cost}\left(\pi_G^0\right) = \sum_{\substack{w_1,w_2 \in V_G \\ (w_1,w_2)\ne(x_1,x_2)}} \text{dist}_G(w_1, w_2)\pi_G^0(w_1, w_2),$$

where the condition $(w_1, w_2) \neq (x_1, x_2)$ on the summation can be imposed because $\pi_G^0(x_1, x_2) = 0$ due to marginal constraints of π_G^0.

Define a function $\pi^0 \colon (V_G \times V_H)^2 \to [0, \infty)$ as follows:

$$
\pi^0((w_1, z_1), (w_2, z_2))
$$
$$
:= \begin{cases}
\dfrac{d_G}{d_{G \boxtimes H}} \pi_G^0(w_1, w_2) & \text{if } z_1 = z_2 \in B_1(y_1) \text{ and } (w_1, w_2) \neq (x_1, x_2), \\[2mm]
\dfrac{1}{d_{G \boxtimes H}} & \text{if } z_1 = z_2 \in S_1(y_1) \text{ and } (w_1, w_2) = (x_1, x_2), \\[2mm]
0 & \text{otherwise}.
\end{cases}
$$

Now we verify that $\pi^0 \in \Pi\left(\mu_{(x_1, y_1)}^0, \mu_{(x_2, y_1)}^0\right)$: Let $(w_1, z_1) \in V_G \times V_H$. We distinguish two cases:

1. If $w_1 \neq x_1$ we have

$$
\begin{aligned}
\sum_{w_2, z_2} \pi^0((w_1, z_1), (w_2, z_2)) &= \frac{d_G}{d_{G \boxtimes H}} \cdot \mathbb{1}_{B_1(y_1)}(z_1) \sum_{w_2} \pi_G^0(w_1, w_2) \\
&= \frac{d_G}{d_{G \boxtimes H}} \cdot \mathbb{1}_{B_1(y_1)}(z_1) \cdot \mu_{x_1}^0(w_1) \\
&= \frac{1}{d_{G \boxtimes H}} \cdot \mathbb{1}_{B_1(y_1)}(z_1) \cdot \mathbb{1}_{S_1(x_1)}(w_1) \\
&= \mu_{(x_1, y_1)}^0(w_1, z_1).
\end{aligned}
$$

The last equality follows from the fact that $w_1 \neq x_1$ implies

$$
\begin{aligned}
\mathbb{1}_{B_1(y_1)}(z_1) \cdot \mathbb{1}_{S_1(x_1)}(w_1) &= \mathbb{1}_{B_1(y_1)}(z_1) \cdot \mathbb{1}_{B_1(x_1)}(w_1) \\
&= \mathbb{1}_{B_1(x_1, y_1)}(w_1, z_1) = \mathbb{1}_{S_1(x_1, y_1)}(w_1, z_1).
\end{aligned}
$$

2. If $w_1 = x_1$ we have

$$
\begin{aligned}
&\sum_{w_2, z_2} \pi^0((x_1, z_1), (w_2, z_2)) \\
&= \mathbb{1}_{S_1(y_1)}(z_1) \frac{1}{d_{G \boxtimes H}} + \mathbb{1}_{B_1(y_1)}(z_1) \frac{d_G}{d_{G \boxtimes H}} \underbrace{\sum_{w_2 \neq x_2} \pi_G^0(\underbrace{x_1}, w_2)}_{=0} \\
&= \frac{1}{d_{G \boxtimes H}} \mathbb{1}_{S_1(y_1)}(z_1) = \mu_{(x_1, y_1)}^0(x_1, z_1).
\end{aligned}
$$

The verification of

$$
\sum_{w_1, z_1} \pi^0((w_1, z_1), (w_2, z_2)) = \mu_{(x_2, y_1)}^0(w_2, z_2)
$$

is done similarly. The cost of π^0 can then be calculated as

$$\text{cost}(\pi^0) = \sum_{(w_2, z_2)} \sum_{(w_1, z_1)} \text{dist}_{G \boxtimes H}((w_1, z_1), (w_2, z_2)) \pi^0((w_1, z_1), (w_2, z_2))$$

$$= \sum_{z_1 \in B_1(y_1)} \sum_{(w_1, w_2) \neq (x_1, x_2)} \text{dist}_G(w_1, w_2) \frac{d_G}{d_{G \boxtimes H}} \pi_G^0(w_1, w_2)$$

$$+ \sum_{z_1 \in S_1(y_1)} \text{dist}_G(x_1, x_2) \frac{1}{d_{G \boxtimes H}}$$

$$= \frac{(1 + d_H) d_G}{d_{G \boxtimes H}} \text{cost}(\pi_G^0) + \frac{d_H}{d_{G \boxtimes H}}.$$

Therefore, we have

$$W_1^{G \boxtimes H}\left(\mu_{(x_1, y_1)}^0, \mu_{(x_2, y_1)}^0\right) \leq \text{cost}(\pi^0) = \frac{(1 + d_H) d_G}{d_{G \boxtimes H}} W_1^G\left(\mu_{x_1}^0, \mu_{x_2}^0\right) + \frac{d_H}{d_{G \boxtimes H}},$$

or equivalently

$$\kappa_0((x_1, y_1), (x_2, y_1)) \geq \frac{d_G(d_H + 1)}{d_{G \boxtimes H}} \kappa_0^G(x_1, x_2),$$

which gives the desired lower bound for κ_0.

In the same way we obtain analogous results for vertical edges:

$$\kappa_*((x_1, y_1), (x_1, y_2)) \geq \frac{d_H(d_G + 1)}{d_{G \boxtimes H}} \kappa_*^H(y_1, y_2).$$

$\square$

Corollary 5.4. *Let G and H be two regular graphs with nonnegative κ_0 (or κ_{LLY}). Then all horizontal and vertical edges of $G \boxtimes H$ have also nonnegative κ_0 (or κ_{LLY}).*

It turns out, however, that the statement of Corollary 5.4 is no longer true for diagonal edges, as the following example shows.

Example 5.5. Let G be a 4-regular graph with an induced 2-ball $B_2(v_0) = \{v_0, \ldots, v_9\}$ as shown in Figure 6. Then $\kappa_0(v_0, v_i) \geq 0$ for $1 \leq i \leq 4$ and $\mathcal{K}_\infty(v_0) > 0$. Let $H = P_\infty$ be the bi-infinite paths with vertices w_j, $j \in \mathbb{Z}$. Then $\kappa_0(w_0, w_{\pm 1}) = \kappa_{\text{LLY}}(w_0, w_{\pm 1}) = 0$ and $\mathcal{K}_\infty(w_0) = 0$.
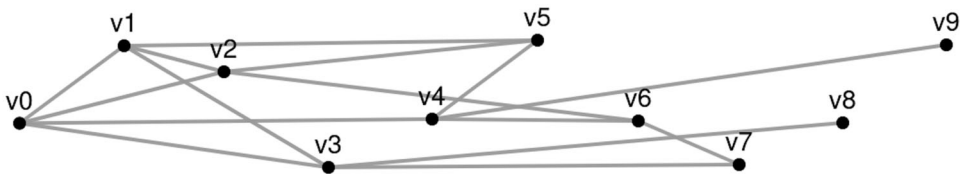
FIGURE 6 Induced 2-ball of a quartic graph with $\mathcal{K}_\infty(v_0) = 0.013$ and $\kappa_{\text{LLY}}(v_0, v_1) = 2\kappa_0(v_0, v_1) = 1$, $\kappa_{\text{LLY}}(v_0, v_2) = 2\kappa_0(v_0, v_2) = 0.5$ and $\kappa_{\text{LLY}}(v_0, v_3) = \kappa_0(v_0, v_3) = \kappa_{\text{LLY}}(v_0, v_4) = \kappa_0(v_0, v_4) = 0$

However, the strong product $G \boxtimes H$ has negative Ollivier–Ricci curvatures on the following diagonal edges (see Figure 7):

$$\kappa_0((v_0, w_0), (v_3, w_{\pm 1})) = \kappa_{\mathrm{LLY}}((v_0, w_0), (v_3, w_{\pm 1})) = -0.071,$$

and negative Bakry–Émery curvature at (v_0, w_0) (see Figure 8):

$$\mathcal{K}_\infty(v_0, w_0) = -0.062.$$

Remark 5.6. The previous example shows for strong products that the nonnegativity of curvatures is generally not preserved for diagonal edges. The same example can be used to show that this phenomenon appears also in the case of tensor products, where only diagonal edges are present.

Another interesting question about graphs products is the following: In the case of Cartesian products, the full curvature function (as a function of the dimension $\mathcal{N}$) at a vertex (x, y) is completely determined by the curvature functions of the factors at the vertices x and y (see [10, Theorem 7.9]):

$$\mathcal{K}^{G \times H}_{(x,y)} = \mathcal{K}^{G}_x * \mathcal{K}^{H}_y,$$

where * is a special operation defined in [10, Definition 7.1]. We would like to know whether a similar formula (with a suitably defined operation) can be proved for tensor products and strong products.
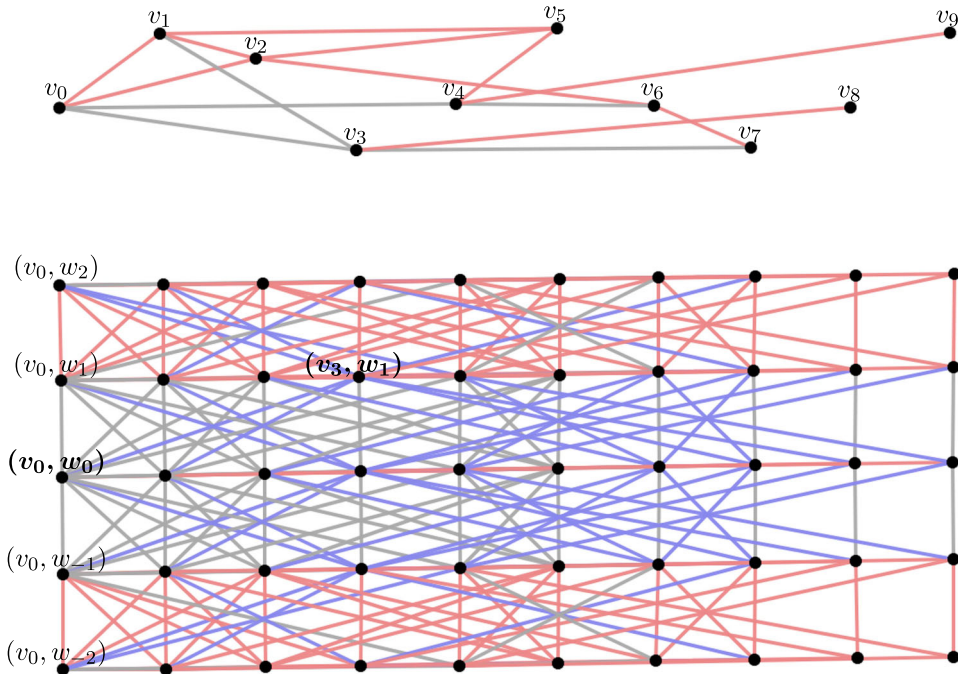
FIGURE 7 Local Ollivier–Ricci curvatures κ_{LLY} of G and $G \boxtimes P_\infty$ at edges incident to v_0 and (v_0, w_0), respectively. Positive/negative/zero curvatures of edges are represented by the colours red/blue/grey. Every horizontal line of the lower graph represents a projection of G [Color figure can be viewed at wileyonlinelibrary.com]
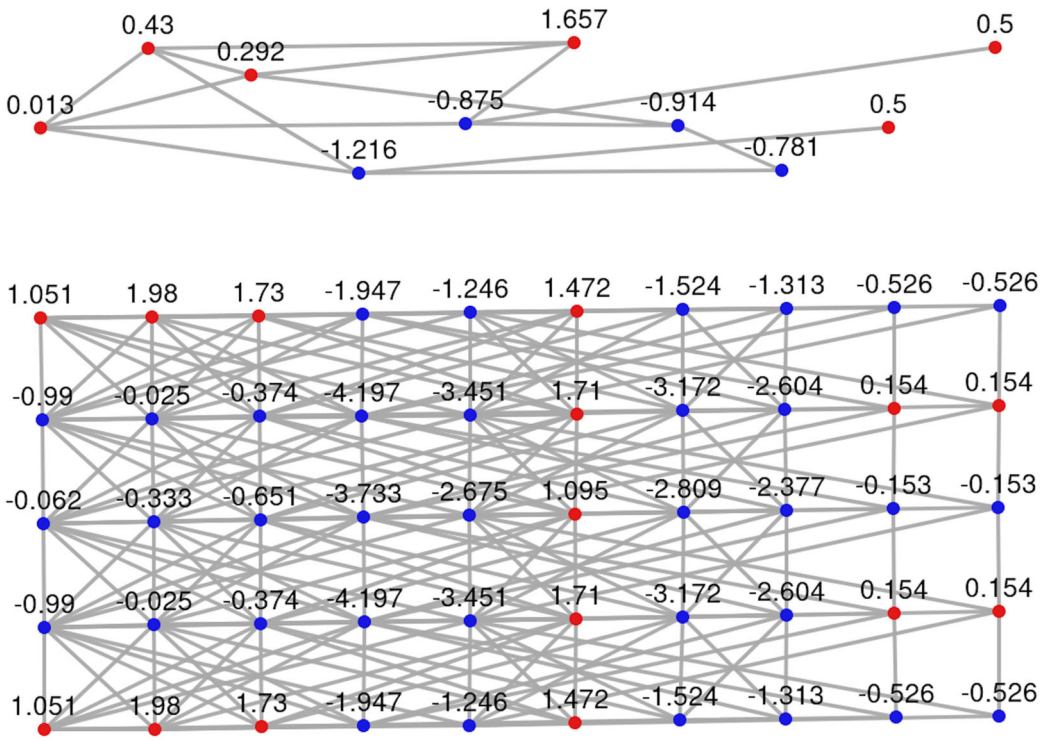
FIGURE 8 Local Bakry–Émery curvatures of G and $G \boxtimes P_\infty$ at v_0 and (v_0, w_0). Positive/negative curvatures of vertices are represented by the colours red/blue. Every horizontal line of the lower graph represents a projection of G [Color figure can be viewed at wileyonlinelibrary.com]

6 | DISTANCE-REGULAR GRAPHS

In this section we turn our focus on distance-regular graphs of girth 4, which is an interesting family of triangle-free graphs with maximal curvature values for κ_0, κ_{LLY} and $\mathcal{K}_\infty$. Distance-regular graphs are defined as follows:

Definition 6.1. A regular graph $G = (V, E)$ is called *distance-regular* if, for any pair $x, y \in V$ of vertices and any $r, t \geq 0$ the cardinality of $S_r(x) \cap S_t(y)$ depends only on $r, t, d(x, y)$.

The *intersection array* of a distance-regular graph $G = (V, E)$ of vertex degree d is defined as an array of integers:

$$\{b_0, b_1, ..., b_{d-1}; c_1, ..., c_d\}$$

defined as follows: Fix $x \in V$. Then, for $0 \leq i \leq d - 1$ and $1 \leq j \leq d$, we set $b_i = d_x^+(z)$ for every $z \in S_i(x)$ and $c_j = d_x^-(z)$ for every $z \in S_j(x)$.

Theorem 6.2. *Let $G = (V, E)$ be a distance-regular graph of vertex degree d and girth 4. Then we have*

$$\kappa_0(x, y) = 0 \quad \text{and} \quad \kappa_{LLY}(x, y) = \frac{2}{d} \quad \text{for all } \{x, y\} \in E \tag{14}$$

and

$$\mathcal{K}_\infty(x) = 2 \quad \text{for all } x \in V. \tag{15}$$

Note that the curvature values in (14) and (15) are upper curvature bounds for any triangle-free d-regular graph by Proposition 4.1. Theorem 6.2 is a generalization of [2, Theorem 4.10] and [10, Corollary 11.7(i) in the arXiv version], which are both concerned with the special case of strongly regular graphs. Even though the proofs for this special case carry over to the much larger class of distance-regular graphs, we present them here for the reader's convenience.

Proof. Let $G = (V, E)$ be a distance-regular graph of vertex degree d and girth 4 and $\{x, y\} \in E$. By Remark 4.3(b), it suffices to show the existence of a perfect matching between $S_1(x) \backslash \{y\}$ and $S_1(y) \backslash \{x\}$ to conclude

$$\kappa_{\mathrm{LLY}}(x, y) = \frac{2}{d}. \tag{16}$$

Let H be the induced subgraph of the union of $S_1(x) \backslash \{y\}$ and $S_1(y) \backslash \{x\}$. Note that H is bipartite since G is triangle-free. Let $X \subset S_1(x) \backslash \{y\}$ and Y be the set of neighbours of X in $S_1(y) \backslash \{x\}$. The set Y is nonempty due to the girth 4 assumption. Then we have the following double-counting of the edges between X and Y:

$$\sum_{w \in X} d_w^H = |E(X, Y)| \le \sum_{z \in Y} d_z^H, \tag{17}$$

where d_w^H is the vertex degree of w in H. Using distance-regularity, we obtain $d_w^H = d_z^H = c_2 - 1$ and (17) implies $|X| \le |Y|$. We can now apply Hall's Marriage Theorem to conclude that there is a perfect matching between $S_1(x) \backslash \{y\}$ and $S_1(y) \backslash \{x\}$.

By Theorem 1.1, (16) implies $\kappa_0(x, y) \ge 0$. Combining this with Proposition 4.1(i), we conclude $\kappa_0(x, y) = 0$.

For the calculation of the Bakry–Émery curvature we employ the method presented at the beginning of Section 8 of [10] (which is Section 9 in the corresponding arXiv version) and the notation introduced there. In view of Theorem 8.1(i) in [10], we only need to verify that the second smallest eigenvalue $\lambda_1 = \lambda_1(\Delta_{S''_1(x)}) \ge \frac{d}{2}$ (note that $\lambda_0 = 0$), since then

$$\mathcal{K}_\infty(x) = \frac{3 + d - \mathrm{av}_1^+(x)}{2} = \frac{3 + d - (d - 1)}{2} = 2.$$

Triangle-freeness of G implies

$$\Delta_{S''_1(x)} = \Delta_{S_1(x)} + \Delta_{S'_1(x)} = \Delta_{S'_1(x)},$$

where $\Delta_{S'_1(x)}$ is the weighted Laplacian on the 1-sphere $S_1(x)$ with the following weights:

$$w'_{y_1 y_2} = \sum_{\substack{z \in S_2(x) \\ y_1 \sim z \sim y_2}} \frac{1}{d_x^-(z)} \quad \text{for all } y_1, y_2 \in S_1(x), y_1 \ne y_2.$$

Since G is distance-regular, we obtain $d_x^-(z) = c_2$ and

$$|\{z \in S_2(x) : y_1 \sim z \sim y_2\}| = c_2 - 1.$$

This implies $w'_{y_1 y_2} = \frac{c_2 - 1}{c_2}$ and, therefore, the Laplacian $\Delta_{S'_1(x)}$ is $\frac{c_2 - 1}{c_2} \Delta_{K_d}$, where Δ_{K_d} is the nonnormalized Laplacian of the complete graph K_d. Consequently, we have

$$\lambda_1 \left(\Delta_{S'_1(x)} \right) = \frac{c_2 - 1}{c_2} \lambda_1(\Delta_{K_d}) = \frac{c_2 - 1}{c_2} d \geq \frac{d}{2},$$

since $c_2 \geq 2$ because G has girth 4. $\qquad\square$

It is tempting to assume that distance-regular graphs of girth 4 are always (R)-Ricci flat and then using Theorems 3.4 and 3.5(b) to conclude the statement of Theorem 6.2. However, the following example shows that this assumption is *not* always true. It remains an open question, however, whether every distance-regular graph of girth 4 is Ricci flat.

Example 6.3 (Incidence graph of $(11, 6, 3)$-design). This is a distance-regular graph with intersection array $\{6, 5, 3; 1, 3, 6\}$ (see [11]).

The structure of the incomplete 2-ball around a vertex x is given by

$$S_1(x) = \{v_1, \ldots, v_6\} \quad \text{and} \quad S_2(x) = \{v_7, \ldots, v_{16}\},$$
$$v_1 \sim v_8, v_{11}, v_{13}, v_{14}, v_{15},$$
$$v_2 \sim v_7, v_{10}, v_{11}, v_{12}, v_{13},$$
$$v_3 \sim v_9, v_{10}, v_{11}, v_{15}, v_{16},$$
$$v_4 \sim v_7, v_8, v_{10}, v_{14}, v_{16},$$
$$v_5 \sim v_8, v_9, v_{12}, v_{13}, v_{16},$$
$$v_6 \sim v_7, v_9, v_{12}, v_{14}, v_{15}.$$

We give an indirect prove that this graph is not (R)-Ricci flat. Assume otherwise, that is, there exists an associated matrix A with only 0 entries on diagonal. The other possible entries of A listed as below:

$$\left(\begin{array}{c|c|c|c|c|c|c}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 0 & \mathbf{\mathit{11, 13}} & 11, 15 & 8, 14 & 8, 13 & 14, 15 \\
2 & 11, 13 & 0 & 10, 11 & 7, 10 & 12, 13 & 7, 12 \\
3 & 11, 15 & 10, 11 & 0 & 10, 16 & 9, 16 & 9, 15 \\
4 & 8, 14 & 7, 10 & 10, 16 & 0 & 8, 16 & 7, 14 \\
5 & 8, 13 & 12, 13 & 9, 16 & 8, 16 & 0 & 9, 12 \\
6 & 14, 15 & 7, 12 & 9, 15 & 7, 14 & 9, 12 & 0
\end{array}\right),$$

$$\left(\begin{array}{c|c|c|c|c|c}
0 & \mathbf{\mathit{11, 13}} & 11, 15 & 8, 14 & 8, 13 & 14, 15 \\
11, 13 & 0 & 10, 11 & 7, 10 & 12, 13 & 7, 12 \\
11, 15 & 10, 11 & 0 & 10, 16 & 9, 16 & 9, 15 \\
8, 14 & 7, 10 & 10, 16 & 0 & 8, 16 & 7, 14 \\
8, 13 & 12, 13 & 9, 16 & 8, 16 & 0 & 9, 12 \\
14, 15 & 7, 12 & 9, 15 & 7, 14 & 9, 12 & 0
\end{array}\right),$$

Recall that the matrix A cannot have repeated entries in any row and column. If the entry of A_{12} is chosen to be **11**, then all entries for the first three rows are uniquely determined as the numbers in red. Then the entry of A_{46} cannot be either 7 or 14, due to appearance of them in the sixth column. Contradiction!

Similarly, if the entry of A_{12} is chosen to be **13**, then all entries for the first three rows must be the numbers in blue. Then the entry of A_{45} cannot be either 8 or 16 due to the fifth column. Contradiction!

In conclusion, the Incidence graph of $(10, 6, 3)$-design is not (R)-Ricci flat, even though it is triangle-free and has both maximum possible Bakry–Émery curvature $\mathcal{K}_{\infty}(x) = 2$ and maximum possible Ollivier–Ricci curvature $\kappa_{\mathrm{LLY}}(x, y) = \frac{2}{d}$.

However, the vertices of this graph are Ricci flat via the following matrix choice for A:

$$\begin{pmatrix} 11 & 0 & 15 & 8 & 13 & 14 \\ 13 & 12 & 11 & 10 & 0 & 7 \\ 0 & 11 & 10 & 16 & 9 & 15 \\ 14 & 10 & 16 & 7 & 8 & 0 \\ 8 & 13 & 9 & 0 & 16 & 12 \\ 15 & 7 & 0 & 14 & 12 & 9 \end{pmatrix}.$$

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