TESTING ASYMMETRY IN DEPENDENCE WITH COPULA-COSKEWNESS

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Abstract. A new measure of asymmetry in dependence is proposed which is based on taking the difference between the margin-free coskewness parameters of the underlying copula. The new measure and a related test are applied to both a hydrological and a financial market data sample and we show that both samples exhibit systematic asymmetric dependence.

Keywords: Asymmetry, coskewness, exchangeability, copula, diversification.
JEL codes: C00, C12, C58.

1. Introduction

The past two decades have seen a steady increase in research on dependence modeling in general, and on copulas in particular. Not surprisingly, a vast amount of literature on inferential methodology for copulas has emerged, e.g., on parametric and nonparametric estimation (Genest et al., 1995; Omelka et al., 2009), on goodness-of-fit testing (Genest et al., 2009; Quessy and Bahraoui, 2014) or on estimation in and testing of nonparametric subclasses (Genest and Segers, 2009; Bücher et al., 2011), among many others. Possibly the main reason for the interest in copulas is the fact that, in contrast to correlation-based models, copulas allow for the modeling of different behaviours in the tail of a distribution.[1] For example, while the Gaussian copula is tail-independent and the Student's t copula is symmetrically tail dependent, other copulas like the Clayton or Gumbel–Hougaard copula are characterized by asymmetric tail dependence in the sense that the distribution's behaviour in the upper tail does not need to equal the behaviour in the lower tail. Despite the extensive work done on asymmetry in the tail dependence, however, asymmetry in the copula itself (also called exchangeability) and its importance in applied settings have so far received much less attention.

A copula is said to be asymmetric (or non-exchangeable) if $C(u, v) \neq C(v, u)$ holds for at least one pair $u, v, \in [0, 1]$, i.e., if the distribution exhibits different behaviours in the upper left and the lower right triangle of the unit square. In applications, such a case can easily occur if, for example, there exists a causal relation between the two random variables. Surprisingly, asymmetry in dependence is seldomly studied in economics and financial econometrics despite its obvious usefulness.[2] This is perhaps even more

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[1]Modeling the tail of a distribution is also relevant for various applications across disciplines such as risk management or actuarial science (see, e.g., Peng, 2008; Jong, 2012; Hua and Xia, 2014), which further explains the growing interest in those topics.

[2]This is in contrast to asymmetry in tail dependence, which has been extensively studied by, e.g., Demarta and McNeil (2004); Patton (2006); Christoffersen et al. (2012).

surprising given the fact that the asymmetry of a copula is closely connected with the coskewness of a bivariate random vector, a concept widely used in the context of asset pricing theory (see, e.g., Kraus and Litzenberger, 1976; Harvey and Siddique, 2000; Ang et al., 2006).

In this paper, we exploit the relation between the asymmetry of a copula and the coskewness of a random vector and propose a new measure of asymmetry in dependence that is based on the difference between two coskewness parameters. Consequently, our new measure is easy to interpret and allows for an analysis of the question into which direction the copula is skewed. Just like Spearman's correlation coefficient is related to Pearson's correlation coefficient, our proxy for the asymmetry of a copula is related to coskewness. In particular, no moment conditions are necessary for its definition and it is invariant with respect to strictly monotone transformations of the marginals. As a central contribution, we propose an estimator for the coskewness parameter, derive its asymptotic normality and, based on the normal approximation, propose a test for detecting asymmetry of the copula (see also Genest et al., 2011 or Quessy and Bahraoui, 2013 for related bootstrap-based tests).

In our empirical study, we illustrate the usefulness of our test for both the modeling of flooding events and losses in asset management by applying our test to two data samples from hydrology and finance. Our results, which should be of particular interest to non-life insurers, show that both samples exhibit strong asymmetry in their dependence that should be taken into account when modeling claims and losses.

The remaining part of the paper is structured as follows. In Section 2, we define our new measure of the asymmetry of a copula and discuss its properties. Section 3 presents a Monte Carlo simulation study in which we examine the finite sample performance of our test for asymmetry. In Section 4, we illustrate our new measure by applying it to a hydrological and a financial market data sample. Section 5 concludes. All proofs are deferred to an appendix.

2. A new measure of the asymmetry of a copula

Let (X, Y) be a random vector with joint cumulative distribution function (cdf) F and continuous marginal cdfs[3] F_X and F_Y and copula C. We recall Sklar's theorem which establishes the following relation between the joint distribution's cdf and its marginal cdfs:

$$F(x, y) = C\{F_X(x), F_Y(y)\} \qquad \text{for all } x, y \in \mathbb{R}, \tag{2.1}$$

where C is uniquely determined and given by the cdf of the random vector (U, V), with $U = F_X(X), V = F_Y(Y)$. A copula C is called symmetric or exchangeable if the following holds true:

$$C(u, v) = C(v, u) \qquad \text{for all } u, v \in [0, 1].^{4}$$

While many of the most common copula models allow for radial asymmetry (in particular, the lower and upper tail dependence may be different), they often do not allow for

[3] The case of possibly discontinuous margins is briefly treated in Remark 2.4 below.

[4] This is not to be confused with the related concept of radial symmetry, $C(u, v) = \overline{C}(u, v)$, where $\overline{C}$ denotes the survival copula associated with C.

asymmetry in the copula itself.[5] To detect such asymmetry, one could simply look at the difference of the two functions $(u, v) \mapsto C(u, v)$ and $(u, v) \mapsto C(v, u)$, which is zero for symmetric copulas, and define the following measure of asymmetry:

$$d_\infty(C) = 3 \sup_{(u,v)\in[0,1]^2} |C(u, v) - C(v, u)|.$$

In fact, any distance between the two functions $(u, v) \mapsto C(u, v)$ and $(u, v) \mapsto C(v, u)$ may be used to measure asymmetry. It can be shown that $d_\infty(C)$ takes values in $[0, 1]$ (whence the constant 3), and it is equal to zero if and only if C is symmetric (see Nelsen, 2007). Replacing C by the empirical copula allows to define a test that may detect departures from symmetry, which was proposed by Genest et al. (2011) with a recent application in Siburg et al. (2016).

Obviously, a positive coefficient $d_\infty(C)$ is hard to interpret when one is interested in the direction of the asymmetry. However, in many applications it is important to know for which points (u, v) or for which regions of the unit square the symmetry relation does not hold, and in which direction it points. Hence, we propose a simpler coefficient that allows for an easy interpretation and relates to the concept of coskewness: *copula-coskewness*. Our copula-coskewness measure is related to coskewness in a similar way the Spearman correlation is related to Pearson correlation. In particular, no moment conditions are necessary for its definition and it is invariant with respect to strictly monotone transformations of the marginals.[6]

We recall the definition of the two coskewness parameters (see, e.g., Miller, 2012):

$$\bar{s}_{X,Y} = \frac{\mathrm{E}[(X - \mu_X)^2(Y - \mu_Y)]}{\sigma_X^2 \sigma_Y}, \qquad \bar{s}_{Y,X} = \frac{\mathrm{E}[(X - \mu_X)(Y - \mu_Y)^2]}{\sigma_X \sigma_Y^2},$$

where μ_X, μ_Y and σ_X, σ_Y are the mean and standard deviation of X and Y, respectively. The coefficients are defined for any (X, Y) whose third marginal moments exist.

$\bar{s}_{X,Y}$ is positive, when large values of Y tend to occur jointly with either small or large values of X and vice versa for $\bar{s}_{Y,X}$. We define the copula-coskewness parameter as[7]

$$s_{X,Y} := \bar{s}_{U,V} = 12^{3/2} \times \mathrm{E}[(U - 1/2)^2(V - 1/2)]$$
$$s_{Y,X} := \bar{s}_{V,U} = 12^{3/2} \times \mathrm{E}[(U - 1/2)(V - 1/2)^2].$$

[5]An extensive overview of methods how to construct asymmetric copulas from given symmetric ones can be found in Liebscher (2008).

[6]One disadvantage of the copula-coskewness measure is that a value of zero does not necessarily imply symmetry in the copula. This, however, also holds for the common skewness parameter of a real-valued distribution. In the same way, independence of two random variables is not implied by a Spearman's rho that equals zero.

[7]Note that $\mu_U = \mu_V = 1/2$ and $\sigma_U^2 = \sigma_V^2 = 1/12$

Using simple algebra yields the following result:

$$\begin{aligned}\mathrm{E}[(U-1/2)^2(V-1/2)] &= \int (u-\tfrac{1}{2})^2(v-\tfrac{1}{2})\,\mathrm{d}C(u,v) \\ &= \int u^2v - uv + \tfrac{1}{4}v - \tfrac{1}{2}u^2 + \tfrac{1}{2}u - \tfrac{1}{8}\,\mathrm{d}C(u,v) \\ &= \int u^2v - uv\,\mathrm{d}C(u,v) + \tfrac{1}{12}.\end{aligned}$$

Here, and throughout, integration is over $[0,1]^2$ if not otherwise mentioned. The latter formula shows that the copula-coskewness parameters are simple functionals of the copula C. Further note that if D is an arbitrary continuous cdf on $[0,1]^2$ (not necessarily a copula), then $\int D(u,v)\,\mathrm{d}C(u,v) = \int C(u,v)\,\mathrm{d}D(u,v)$ (see, e.g., Lemma 1 in Remillard, 2010). Applying this twice to the last display, we also have

$$\mathrm{E}[(U-1/2)^2(V-1/2)] = \int (2u-1)C(u,v)\,\mathrm{d}(u,v) + \tfrac{1}{12}.$$

This second formula expresses the copula-coskewness parameters through the copula, but this time the copula appears in the integrand (which is convenient for proofs).

The copula-coskewness parameter $s_{U,V}$ attains values larger than zero when the distribution associated with the copula puts much of its mass to regions close to the points $(0,1)$ or $(1,1)$ of the unit square. Similarly, positive values of $s_{V,U}$ occur whenever much mass is concentrated near the points $(1,0)$ or $(1,1)$. The (possibly scaled) difference between the two parameters, $a_{X,Y}$, is hence an obvious choice for measuring the asymmetry of C:

$$a_{X,Y} \propto s_{U,V} - s_{V,U}\text{[8]}.$$

A positive value of $a_{X,Y}$ indicates that large values of Y occurring simultaneously with small values of X is more likely than large values of X occurring simultaneously with small values of Y, and vice versa for negative values. Clearly, it would be desirable to choose the constant in front of the difference in such a way that $a_{X,Y} \in [-1,1]$, with $a_{X,Y} = 0$ whenever the copula is symmetric. The choice of the constant is the topic of the subsequent Lemma 2.1.

After some simple algebra we obtain

$$\begin{aligned}s_{U,V} - s_{V,U} &\propto \mathrm{E}[(U-1/2)^2(V-1/2) - (U-1/2)(V-1/2)^2] \\ &= \mathrm{E}[(U-\tfrac{1}{2})(V-\tfrac{1}{2})(U-V)] \\ &= \mathrm{E}[U^2V - V^2U] \\ &= \tfrac{1}{3}\,\mathrm{E}[(V-U)^3],\end{aligned}$$

where the last equality follows from expanding $(V-U)^3$ and noting that $\mathrm{E}\,U^3 = \mathrm{E}\,V^3$. The copula-coskewness-based parameter of asymmetry $a_{X,Y}$ is hence a multiple of the

[8]One might alternatively think of defining a similar coefficient based on a copula-version of the cokurtosis, which employs fourth moments and is also widely used in the finance literature. However, the difference between two cokurtosis parameters cannot be related to the asymmetry in dependence, as large values of these parameters occur whenever two random variables deviate in the same direction. For this reason, we do not pursue the investigation of copula-cokurtosis any further within this paper.

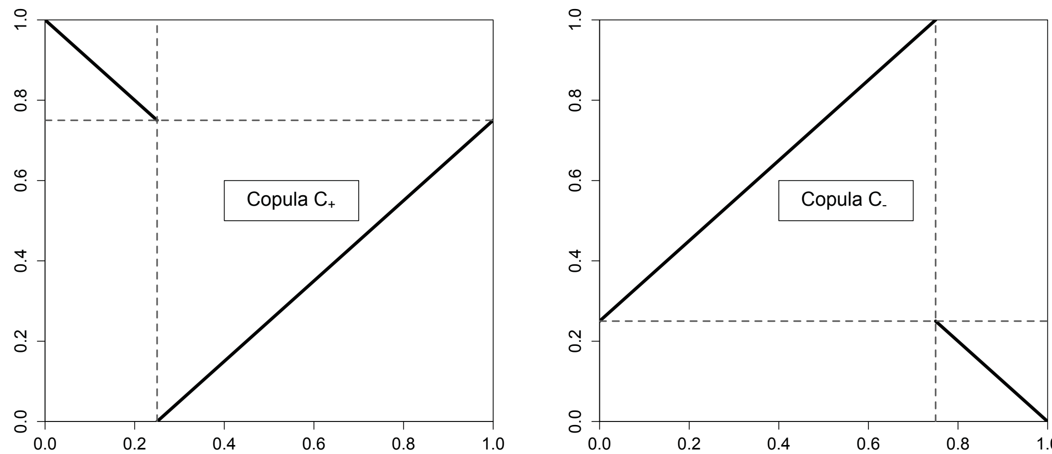

FIGURE 1. Support of the copula C_+ (left) and C_- (right). These are the maximal asymmetric elements with respect to the measure $a_{X,Y}$.

third (central) moment of $V-U$. The following Lemma establishes the range of possible values of $\mathrm{E}[(V-U)^3]$.

Lemma 2.1. *Let $\mathcal{C}$ denote the set of all copulas. Then*

$$\max\left\{\int (v-u)^3 \,\mathrm{d}C(u,v) : C \in \mathcal{C}\right\} = \frac{3^3}{4^4}.$$

The maximum is attained for the (shuffle of $\min(u,v)$*) copula C_+ whose support is the union of the lines $\{(u,v) \in (0,0.25)\times(0.75,1) : v = 1-u\}$ and $\{(u,v) \in (0.25,1)\times(0,0.75) : v = u-0.25\}$. Figure 1 provides a picture of the support. Similarly,*

$$\min\left\{\int (v-u)^3 \,\mathrm{d}C(u,v) : C \in \mathcal{C}\right\} = -\frac{3^3}{4^4}.$$

The minimum is attained for the (shuffle of $\min(u,v)$*) copula C_- whose support is the union of the lines $\{(u,v) \in (0,0.75)\times(0.25,1) : v = u+0.25\}$ and $\{(u,v) \in (0.75,1)\times(0,0.25) : v = 1-u\}$. Figure 1 shows a picture of the support.*

A proof of Lemma 2.1, relying on the theory of mass transportation problems, can be found in Appendix A. Lemma 2.1 suggests to scale the asymmetry parameter with $4^4/3^3$. Hence, from now on, let

$$a_{X,Y} := \frac{4^4}{3^3}\,\mathrm{E}[(V-U)^3] = \frac{256}{27}\,\mathrm{E}[(V-U)^3], \tag{2.2}$$

which attains values in $[-1,1]$ and which is equal to 0 whenever the copula is symmetric. By the preceding calculations, we have:

$$\begin{aligned} a_{X,Y} &= \frac{4^4}{3^3}\cdot 3 \cdot 12^{-3/2}(s_{X,Y}-s_{Y,X}) = \frac{32}{27\sqrt{3}}(s_{X,Y}-s_{Y,X}) \\ &\approx 0.6842(s_{X,Y}-s_{Y,X}). \end{aligned}$$

Estimation of $a_{X,Y}$ can be based on using an empirical analogue of the expression in (2.2). More precisely, we define

$$\hat{a}_{X,Y} = \frac{256}{27} \times \frac{1}{n(n+1)^3} \sum_{i=1}^{n} \{\mathrm{rank}(Y_i) - \mathrm{rank}(X_i)\}^3, \tag{2.3}$$

where $\mathrm{rank}(X_i)$ denotes the rank of X_i among $X_1, \dots, X_n$; and similarly for $\mathrm{rank}(Y_i)$. Note that in terms of the empirical copula $\hat{C}_n(u,v) = \frac{1}{n}\sum_{i=1}^n \mathbb{1}(\hat{U}_i \le u, \hat{V}_i \le v)$, where $\hat{U}_i = \mathrm{rank}(X_i)/(n+1)$ and $\hat{V}_i = \mathrm{rank}(Y_i)/(n+1)$, we can rewrite (2.3) in the following way:

$$\hat{a}_{X,Y} = \frac{256}{27} \times \frac{1}{n} \sum_{i=1}^{n} (\hat{V}_i - \hat{U}_i)^3 = \frac{256}{27} \times \int (v-u)^3 \,\mathrm{d}\hat{C}_n(u,v). \tag{2.4}$$

Due to the fact that the sample of pseudo-observations is not independent over i, asymptotic normality of $\hat{a}_{X,Y}$ cannot be deduced from a simple application of the central limit theorem. In fact, the asymptotic variance is quite complicated.

Proposition 2.2. *Let $(X_1, X_2), \dots, (X_n, Y_n)$ be independent and identically distributed with continuous margins and copula C. Then, with $(U_i, V_i) = (F(X_i), G(Y_i))$, we have the asymptotic expansion*

$$\sqrt{n}(\hat{a}_{X,Y} - a_{X,Y}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \{h(U_i, V_i) - \mathrm{E}[h(U_i, V_i)]\} + o_{\mathbb{P}}(1),$$

where $h(u,v) = (256/27) \cdot \{g_0(u,v) - 3g_1(u) + 3g_2(v)\}$ with $g_0(u,v) = (v-u)^3$ and

$$g_1(u) = \int (x-y)^2 \,\mathbb{1}(u \le x) \,\mathrm{d}C(x,y) = \mathrm{E}[(U-V)^2 \,\mathbb{1}(U \ge u)],$$
$$g_2(v) = \int (x-y)^2 \,\mathbb{1}(v \le y) \,\mathrm{d}C(x,y) = \mathrm{E}[(U-V)^2 \,\mathbb{1}(V \ge v)],$$

and where expectation on the right-hand side is with respect to $(U,V) \sim C$. As a consequence,

$$\sqrt{n}(\hat{a}_{X,Y} - a_{X,Y}) \rightsquigarrow \mathcal{N}(0, \sigma^2),$$

where $\sigma^2 = (256/27)^2 \cdot \mathrm{Var}\left\{(V-U)^3 - 3g_1(U) + 3g_2(V)\right\}$.[9]

A proof of the proposition can be found in Appendix A. Note that the result does not require any smoothness assumptions on the copula at all. This is in contrast to, for instance, the test for symmetry of Genest et al. (2011) which relys on the assumption of existing continuous first order partial derivatives of the copula C. The formal test that we are going to derive below can hence be used under far broader conditions than the test of Genest et al. (2011).

Making inference (confidence bands or testing) on basis of Proposition 2.2 requires estimation of the asymptotic variance. Motivated by the asymptotic expansion, we propose to estimate σ^2 by $\hat{\sigma}^2$, defined as the empirical variance of the (observable) sample $\hat{Z}_1, \dots \hat{Z}_n$, where

$$\hat{Z}_i = (256/27) \cdot \{g_0(\hat{U}_i, \hat{V}_i) - 3\hat{g}_1(\hat{U}_i) + 3\hat{g}_2(\hat{V}_i)\}, \qquad i = 1, \dots, n,$$

[9] Note that $\mathrm{E}[(V-U)^3 - 3g_1(U) + 3g_2(V)] = 4\,\mathrm{E}[(V-U)^3]$ by Fubini's theorem.

and

$$\hat{g}_1(\hat{U}_i) = \frac{1}{n-1} \sum_{j=1, j \neq i}^{n} (\hat{U}_j - \hat{V}_j)^2 \, \mathbb{1}(\hat{U}_j \geq \hat{U}_i),$$

$$\hat{g}_2(\hat{V}_i) = \frac{1}{n-1} \sum_{j=1, j \neq i}^{n} (\hat{U}_j - \hat{V}_j)^2 \, \mathbb{1}(\hat{V}_j \geq \hat{V}_i).$$

It can be shown that $\hat{\sigma}^2$ is consistent for σ^2. As a consequence, we can derive asymptotic one- or two-sided confidence bands for $a_{X,Y}$ and can formally test the hypothesis of symmetry by rejecting symmetry whenever $n\hat{a}^2_{X,Y}/\hat{\sigma}^2 > \chi^2_{1-\alpha}$, the $1-\alpha$-quantile of the χ^2-distribution. The test asymptotically holds its level and is consistent against any alternative with $a_{X,Y} \neq 0$.

Remark 2.3 (Modeling asymmetric dependence structures). Once significant asymmetry in the copula has been detected, a natural follow-up question is how such an asymmetric dependence structure should be modeled by the statistician. For this task, a suitable list of candidate asymmetric parametric copula families is needed. A lot of research has focussed on constructing asymmetric copulas (see Genest and Nešlehová, 2013; Liebscher, 2011 for recent overviews): for instance, one can simply rotate common symmetric copulas like the Gumbel or Clayton copulas by 90 or 270 degrees, or consider convex combinations thereof. Alternatively, one could use general construction methods like gluing (Siburg and Stoimenov, 2008), patchwork constructions (Durante et al., 2009), or a method known as Khoudraji's device (Khoudraji, 1995), implemented in the R-package `copula` (Hofert et al., 2016) and used throughout the simulation study in Section 3 below. The skewed t-copula also possesses an additional parameter that governs the skewness of the copula (Demarta and McNeil, 2004; Christoffersen et al., 2012).

For each candidate copula family, the parameters of the copula can then easily be estimated using standard procedures like, e.g., the Inference-for-Margins-method (Joe and Xu, 1996), pseudo-maximum-likelihood-estimation (Genest et al., 1995), or minimum-distance-estimation (Tsukahara, 2005). After the parameters of each candidate parametric copula family have been estimated, the statistician has to decide on the question which copula model fits the data best. For this task, one could test the goodness-of-fit of each model using copula-specific goodness-of-fit-tests (see, e.g., Genest et al., 2009) or employ model selection criteria like Akaike's Information Criterion.

Remark 2.4 (Extension to possibly discontinuous margins). Extensions of $a_{X,Y}$ and $\hat{a}_{X,Y}$ to discontinuous margins are possible, but should be interpreted with care: formula (2.1) from Sklar's theorem can be satisfied by several different copulas formally resulting in an identification problem for the copula; in particular, the hypothesis of symmetry is not well-defined. Still, one may be tempted to stick to a particular element from the class of all copulas satisfying (2.1), and the *multilinear extension copula* $C^{\maltese}$ (also known as *checkerboard copula*) provides a natural choice in this case. Formally, $C^{\maltese}$ is defined as the cdf of the random vector $(\psi_F(X, W_1), \psi_G(Y, W_2))$, where

$$\psi_F(x, w) = F(x-) + w\{F(x) - F(x-)\}, \quad \psi_G(y, w) = G(y-) + w\{G(y) - G(y-)\},$$

and where W_1, W_2 are iid standard uniform and independent of (X,Y),[10] see, e.g., Genest et al. (2013). The latter authors show that $C^{\maltese}$ plays a central role when analyzing dependence coefficients (see also Nešlehová, 2007). For purely discrete margins, it has further been shown in Genest et al. (2014) that $C^{\maltese}$ is equal to the independence copula if and only if X and Y are stochastically independent. One natural extension of $a_{X,Y}$ as defined in (2.2) to the case of discrete margins is therefore given by

$$a^{\maltese}_{X,Y} = \frac{256}{27} \int_{[0,1]^2} (v-u)^3 \, \mathrm{d}C^{\maltese}(u,v).$$

As before, symmetry of $C^{\maltese}$ implies that $a^{\maltese}_{X,Y} = 0$. For the estimation of $a^{\maltese}_{X,Y}$ in the case of purely discrete margins, one must replace $C^{\maltese}$ by the empirical checkerboard copula $\hat{C}^{\maltese}_n$, see Genest et al. (2014). Asymptotic normality may then be derived similarly as in the proof of Proposition 5.2 in Genest et al. (2014). However, calculation of the estimator is difficult and the interpretation of the hypothesis of symmetry of $C^{\maltese}$ is not obvious (even if $C^{\maltese}$ is symmetric, there may well be other copulas satisfying (2.1) which are asymmetric). Eventually, this is another illustration of the fact that the extension of copula methodology to discontinuous margins is difficult and yet mysterious; a lot of additional general research seems to be necessary (see also Section 6 in Genest et al., 2011).

3. Finite sample performance of the test for asymmetry

A Monte Carlo simulation study is performed to illustrate the finite-sample performance of the copula-coskewness test for asymmetry. In particular, the test is compared to the competing test from Genest et al. (2011), implemented in the R-package `copula` (Hofert et al., 2016), function `exchTest`; in the following shortly GNQ-test.

The following setup is used for the simulation study: we consider two two-parametric copula families constructed from Khoudraji's device (Khoudraji, 1995; Liebscher, 2011), namely

$$C^{\mathrm{CL}}_{\delta,\theta}(u,v) = u^{\delta} C^{\mathrm{CL}}_{\theta}(u^{1-\delta}, v), \qquad \delta \in [0,1], \theta \geq 1;$$
$$C^{\mathrm{GU}}_{\delta,\theta}(u,v) = u^{\delta} C^{\mathrm{GU}}_{\theta}(u^{1-\delta}, v), \qquad \delta \in [0,1], \theta > 0.$$

Here, C^{CL}_{θ} and C^{GU}_{θ} denote the Clayton and the Gumbel–Hougaard copula, respectively. As argued in Genest et al. (2011), Figure 1, both families are symmetric iff $\delta \in \{0,1\}$, with $\delta = 1$ corresponding to the independence copula. For the simulation study, we consider 42 different choices for the parameters: θ varies in such a way that Kendall's tau of the underlying Clayton (or Gumbel–Hougaard) copula is equal to 0.5 or 0.75 and δ varies in the set $\{0, 0.05, 0.1, \ldots, 0.95, 1\}$. Hence, various models from the null hypothesis and the alternative are covered by these choices. Figure 1 in Genest et al. (2011) shows that, for both models, the departure from symmetry is maximal for $\delta = 0.5$ and Kendall's tau equal to 0.75. Finally, three sample sizes are considered ($n = 50, 100, 200$), the number of bootstrap replications for the GNQ-test is set to $M = 500$ and the level of the tests is fixed to $\alpha = 0.05$.

[10]As a consequence, $C = C^{\maltese}$ is the unique copula in case of continuity of the margins.

In Figure 2, we present finite-sample simulation results concerning the empirical level/power of the test, based on $N = 3,000$ repetitions. Both tests decently hold their level under the null hypotheses corresponding to $\delta = 1$ (the independence copula) and are slightly conservative if $\delta = 0$ (the standard symmetric Clayton and Gumbel copula models). In the latter case, the smaller the sample size, the more conservative the tests. In terms of power, there is no clear winner between the two tests. One the one hand, the copula-coskewness tests is clearly preferably over the GNQ-test if the parameter τ is set to $\tau = 0.5$; uniformly over the sample sizes and the two models. On the other hand, for $\tau = 0.75$, the further we move away from independence (that is, for smaller values of δ), the GNQ-test begins to show a superior behavior. This advantage, however, becomes smaller with increasing sample size.

As a conclusion, the copula-coskewness tests provides a decent (and computationally attractive) competitor to the GNQ-test for symmetry of a copula.

4. Applications

In this section, we illustrate our new proxy for asymmetry in dependence by applying the concept of copula-coskewness in two major fields of research in which copulas have become one of the mainstays of dependence models: hydrology and finance. Both applications should yield interesting insights for non-life insurers. In our first study on hydrological data, we show how asymmetric dependence structures naturally appear in river systems making our new test of asymmetric a valuable tool for modeling and forecasting floods (and consequently claims). In our second empirical study, we show that portfolios of financial assets are frequently characterized by asymmetric dependence structures. Our new test can thus help asset managers to detect these asymmetries and chose a suiting asymmetric copula model for forecasting losses.

4.1. **Hydrology.** Consider a simple river system consisting of a main river and a tributary. We are interested in the bivariate dependence between flood events on the main river occurring upstream (X) and downstream (Y) of the confluence of the two rivers. A flood at station X necessarily yields a flood at station Y, whereas a flood at station Y may also well be caused by a flood stemming from the tributary, with a comparably low water level at station X. As a consequence, one would expect asymmetry in the copula between the two variables: mass may be attained to subsets of the unit cube close to the point $(0, 1)$, but not to subsets close to the point $(1, 0)$, resulting in a positive value for the copula-coskewness parameter $a_{X,Y}$. The strength of the asymmetry of course depends on the precise location of the gauges and the local climatic conditions.

The information that the dependence between X and Y is asymmetric can be of beneficial use for both property insurers and reinsurance companies. For example, non-life insurers are interested in accurate forecasts of floods, the probability of losses, and the size of ensuing claims. The accuracy of these forecasts, however, depends critically on the correct modeling of the claims and the factors causing them. While the univariate modeling of claims in non-life insurance with the use of skewed distributions has attracted considerable attention in the past (see, e.g., Eling, 2012), studies on the multivariate distribution of these data have not addressed the potentially asymmetric nature of the dependence inherent in claims and risk factors (see, e.g., Czado et al., 2012; Erhardt and Czado, 2012, for applications of symmetric copulas in actuarial science). In

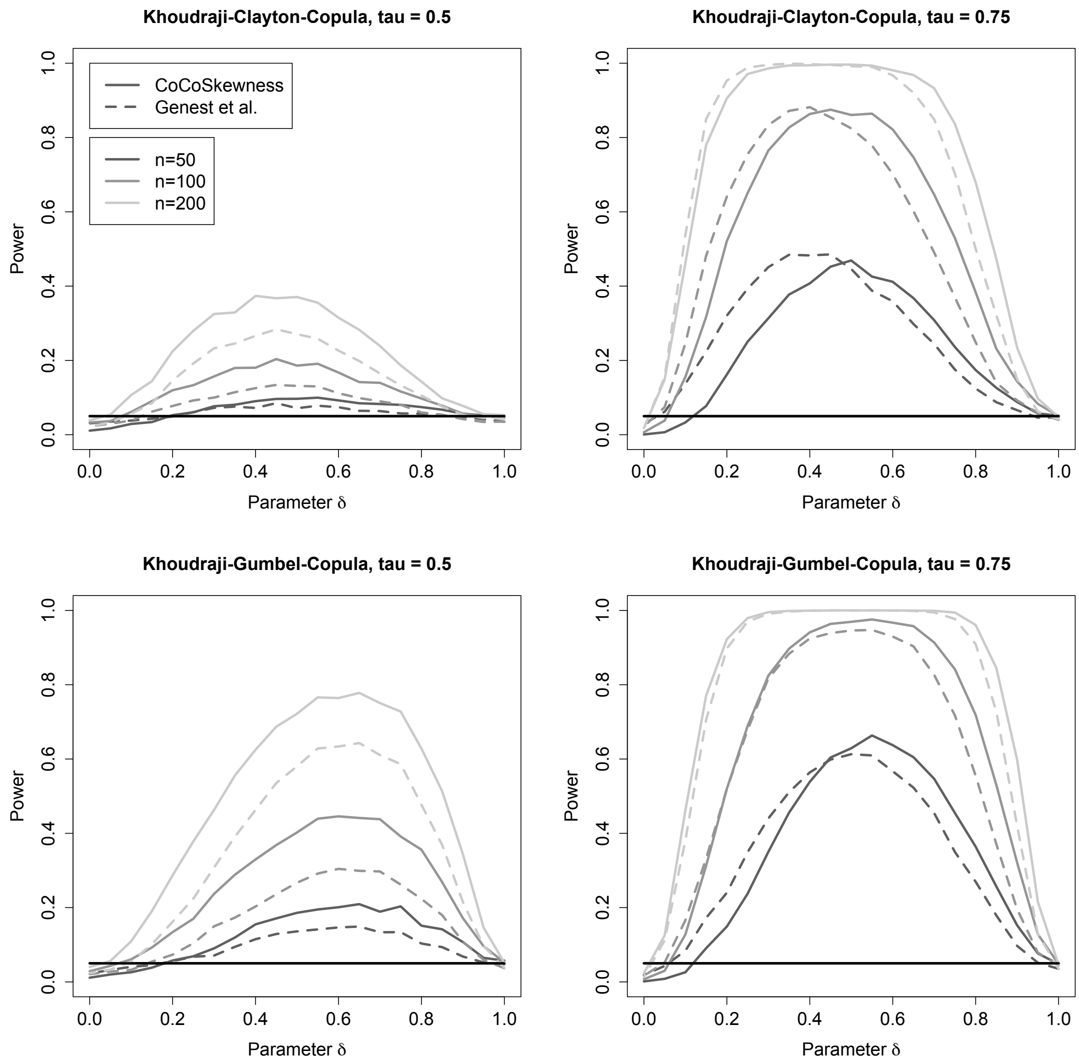

FIGURE 2. Empirical level of the copula-coskewness test for asymmetry (solid lines) and the GNQ-test proposed in Genest et al. (2011) (dashed lines). Both $\delta = 0$ and $\delta = 1$ correspond to the null hypothesis.

this section, we illustrate that bivariate hydrological data are indeed characterized by asymmetry in their dependence structure. A non-life insurer that aims at forecasting floods and insurance claims should thus consider a distributional model that includes a copula that can account for the found asymmetry.

We consider a data set which reveals that the above heuristic is in fact statistically significant. The data set consists of $n = 84$ bivariate maximal summer water flows (i.e., of flood events, measured in m^3/s) measured between 1929 and 2012 at the river Flöha in Saxony, Germany. The river station Pockau (X) lies approximately 20 km upstream of the river station Borstendorf (Y) and, in between, two smaller rivers join the river Flöha. Note that the river Flöha is an inflow of the river Elbe which caused severe

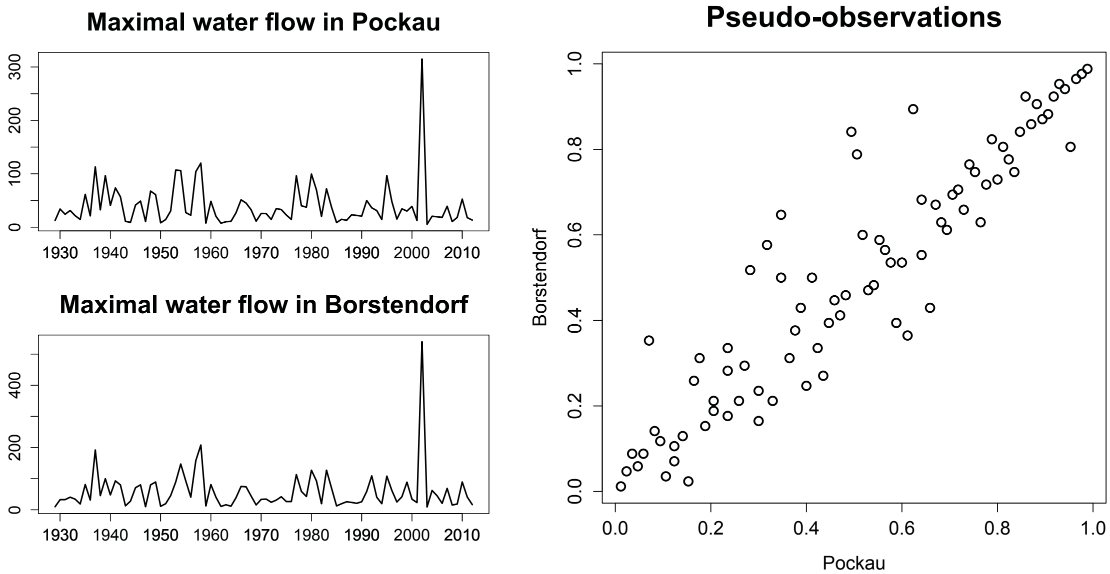

FIGURE 3. Left: Maximal summer flows in m^3/s measured at gauges in Pockau and Borstendorf, Germany. Right: Corresponding pseudo-observations.

floods in eastern Germany during 2006 and, most recently, 2013. The observations are illustrated in Figure 3. Note in particular that the marginal distributions are heavy tailed (ML-estimators of the tail index γ under a GEV-assumption are 0.52 and 0.54 with standard errors of 0.12 and 0.14, respectively) whence regular (second and) third moments and coskewness parameters do not exist; it is really mandatory to switch to (some) standard scale for the margins. From the plot of the pseudo-observations we can in fact observe that, from the subsample of observations far from the main diagonal (say, at distance larger than 0.1), a slightly higher proportion lies above than below the diagonal.

Applying the methodology developed in this paper, the null hypothesis of symmetry gets rejected with a p-value of 0.020. The estimated parameter value is $\hat{a}_{X,Y} = 0.012$, which is positive as expected by the above heuristics. The 95%-two-sided confidence interval based on the normal approximation in Proposition 2.2 is given by $[0.002, 0.024]$. The estimated value may appear rather low on first sight, which, however, is not too surprising, given that the two joining rivers are quite small (not longer than 30km), resulting in very similar climatic conditions at the entire river system. Floods will hence quite often occur simultaneously, as can also be seen from the plot of the pseudo-observations and an estimated value of Spearman's rho of 0.92. Finally, note that the Cramér-von-Mises test for symmetry of the copula by Genest et al. (2011) also rejects the null hypothesis, with a p-value of 0.005 based on $N = 50,000$ multiplier bootstrap repetitions. The latter test, however, does not give any insight into the direction of asymmetry.

4.2. **Finance.** Recall the interpretation of coskewness $\bar{s}_{X,Y}$ and $\bar{s}_{Y,X}$ of two assets X and Y. Whenever $\bar{s}_{X,Y}$ has positive and high values, large values of Y tend to occur jointly

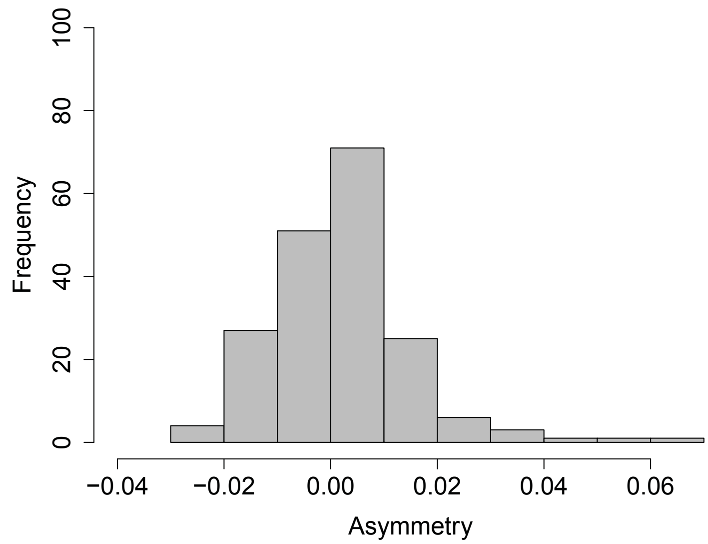

FIGURE 4. Histogram of asymmetry parameters.

with small and large values of X (which translates into more copula-mass around the points $(0,1)$ and $(1,1)$ of the unit cube). Assume for example that X and Y are daily returns of two financial assets (e.g., stocks or indices). The copula-coskewness-based asymmetry parameter $a_{X,Y}$ is defined as the difference of the two copula-coskewness parameters and thus, measures which joint movements of the returns are more likely. When $a_{X,Y}$ is zero, high returns of Y tend to occur jointly with higher returns of X in the same way high returns of X tend to occur jointly with lower returns of Y. When $a_{X,Y}$ is unequal to zero, we observe a diversification benefit for investors. When coskewness of the two assets X and Y is high, we expect higher returns of asset Y to be associated with small and large values of X. On the other hand, mediocre returns on asset X do not necessarily occur jointly with high returns of asset Y. In this way, Y may provide an opportunity to hedge extreme (negative) returns of asset X while normal returns of X are associated with mediocre returns of Y.

This kind of asymmetry could be found in returns of different markets, e.g., a thriving oil market may induce an increase in profits for industrial companies that provide respective equipment. In contrast, however, a successful industrial firm may not affect the oil market in the same way. Further, one might find asymmetry in the dependence of returns on stocks in a sub-sector and returns on a diversified market index (e.g., financial firms' stocks as part of the S&P 500 equity index) [11]

We apply our copula-coskewness-based asymmetry measure to a variety of financial market indices covering returns/yields on equity (e.g., S&P 500 Composite, MSCI World), commodities (e.g., TOPIX Oil & Coal, Gold Bullion LBM, Raw Sugar, Cotton), and U.S. treasury bonds. In total, we compile 21 financial time series from *Thomson Reuters Financial Datastream* that cover the time period from January 4, 1990

[11]See also Siburg et al. (2016) for a related interpretation of asymmetry in bivariate financial data.

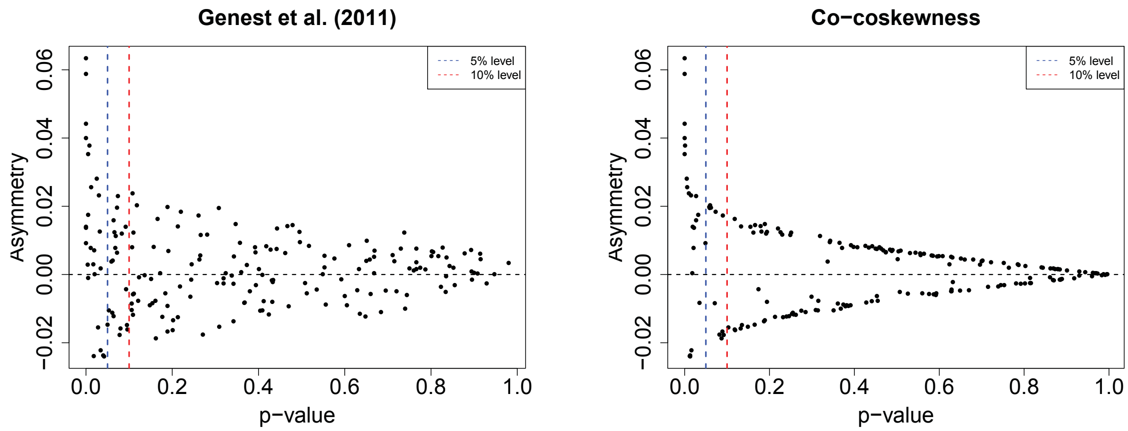

FIGURE 5. Left: Asymmetry parameters and p-values of the test of Genest et al. (2011) tests. Right: Asymmetry parameters and corresponding p-values of copula-coskewness-based test.

to December 31, 2015 (6,780 observations). We employ AR(3)- and GJR-GARCH(1,1)-processes and filter all univariate time series before computing rank-transformed pseudo-observations. These are then used to estimate our asymmetry measure for all bivariate pairs of the indexes in our sample. In Figure 4, we plot a histogram of the asymmetry parameters of all pairs of indexes. We can observe that the distribution of the asymmetry measure is slightly skewed towards negative values among the full sample. However, most of the values lie between zero and 0.02, with an average of about 0.002 across all pairs.

In total, we test our null hypothesis of symmetry for all 190 bivariate combinations of the 21 indices covering several markets using the copula-coskewness-based test and the asymmetry-test proposed in Genest et al. (2011). Using copula-coskewness, for 23 out of 190 pairs the null hypothesis of symmetric dependence is rejected at the 5% level. Thus, our test suggests asymmetry in dependence for about 12.1% of all combinations in our sample. In comparison, testing exchangeability using the test of Genest et al. (2011) yields 26 out of 190 pairs (13.7%) for which the null hypothesis is rejected at the 5% level. In Figure 5, we plot the resulting p-values of both tests against the asymmetry measure calculated for all pairs. Both pictures reveal that most of the p-values below the 5% level correspond with a positive value of the asymmetry measure $a_{X,Y}$. This is even more pronounced for our new test. As to be expected, higher p-values for the copula-coskewness-based test are associated with lower absolute values of $a_{X,Y}$, whereas the other test does not reveal an obvious pattern of test results being associated with asymmetry in dependence measured by $a_{X,Y}$. Finally, unreported results on the computational time needed by both tests shows that our test based on copula-coskewness is significantly faster than the alternative test of Genest et al. (2011) which requires about 34 times more computation time (with 1,000 multiplier iterations) than our test.

Table 1 provides a descriptive overview of the 23 pairs of assets that exhibit strong asymmetry. We observe that our measure of asymmetry is large and positive for five

(U.S.) equity indexes paired with the LMEX (London Metal Exchange) index. For example, the asymmetry in dependence is strongest for the S&P500 Composite or S&P500 Banks with the LMEX index with values of 0.063 and 0.059, respectively. Referring to our interpretations above, the equity indexes provide a diversification strategy for commodity markets. However, similar results hold true, e.g., for the U.S. 10 year T-Bill and Gold Bullion LBM ($a_{X,Y} = 0.024$) or LME-Aluminium ($a_{X,Y} = 0.023$)[12], which suggests another possibility: Pairing U.S. equity indexes or government bonds with commodity indexes from abroad, in this case the LMEX or LBM in the UK, yields strong and statistically significant positive asymmetries in dependence.[13] Although the directions of asymmetric dependence between these markets are not obvious on first sight, we empirically find that the U.S. market may provide diversification benefits for (commodity) indexes abroad. However, the same relation holds true for the MSCI World equity index excluding U.S. firms and thus, it is more likely that the equity market is a hedge against extreme negative returns in the commodity market.

Further, we observe that the combination of a broad equity index and smaller (bank) stock indexes exhibits positive and significant asymmetric dependence. For example, the asymmetry in dependence of the G12-DS Banks G7-DS Banks indexes is small, but significant and positive. Although the Russell 2000 index comprises more U.S. stocks than the S&P 500 Composite index, we still find that higher returns of the latter index are associated with both lower and higher returns of the Russell 2000 index, most likely due to the dominance of the S&P 500 index in the United States. However, the patterns in dependence of broader and smaller indexes documented above is consistent across most combinations of equity indexes in our sample. Further, we find this kind of relation between a broad index and smaller ones in commodity markets as well: Moody's Commodities Index exhibits significant positive asymmetric dependence when paired with LME-Aluminium and a cotton index.[14]

As mentioned above, the asymmetric dependence between two assets could be exploited for hedging purposes. For example, consider the case of oil and cotton for which we found a significantly negative asymmetric dependence between the two. Assuming an investor is long in oil, the negative asymmetric dependence shows that extreme drops in the price of oil will be offset by extreme price surges in cotton. Conversely, extreme increases in the price for oil will not coincide with extreme losses from an investment in cotton. Constructing a hedging strategy that exploits the information on the asymmetry of the dependence, however, is not straightforward as not only the (a)symmetry of the copula but also (e.g.) the correlation structure needs to be taken into account depending on the risk preferences of the investor when constructing the hedge. Moreover,

[12]Note that a negative asymmetry in dependence of X and Y is the same as a positive asymmetric dependence of Y and X.

[13]Similarly, we also find asymmetric dependence of the MSCI World Index without U.S. firms and an oil index.

[14]We also compare indexes of non-equity markets such as cotton or oil with LME-Aluminium and Gold Bullion LBM indexes. Oil and cotton exhibit a negative value of $a_{X,Y}$ and thus, cotton may be used to hedge against extreme drops in oil prices. The relation of oil prices and our gold or aluminium indexes is also clear. Higher values of our oil index are associated with both high and low prices of aluminium and gold. However, these findings are not as intuitive as other asymmetry coefficients of indexes and could also be the result of a false rejection.

the effectiveness of the hedge will of course depend critically on the absolute degree of asymmetry. We leave this issue for future research to explore.

Copula-coskewness	Index i	Index j
0.0634	S&P 500 COMPOSITE	LMEX Index
0.0588	S&P 500 BANKS	LMEX Index
0.0442	G7-DS Banks	LMEX Index
0.0400	RUSSELL 2000	LMEX Index
0.0378	G12-DS Banks	LMEX Index
0.0353	MSCI WORLD EX US	Gold Bullion LBM
0.0281	MSCI WORLD EX US	LME-Aluminium
0.0256	Crude Oil-Brent Cur. Month	LME-Aluminium
0.0238	Moody's Commodities Index	LME-Aluminium
0.0232	US T-Bill 10 Year	LME-Aluminium
0.0230	EU-DS Banks	Gold Bullion LBM
0.0175	MSCI WORLD EX US	TOPIX OIL & COAL PRDS.
0.0159	S&P 500 COMPOSITE	EU-DS Banks
0.0140	S&P 500 COMPOSITE	G12-DS Banks
0.0137	S&P 500 COMPOSITE	G7-DS Banks
0.0092	Moody's Commodities Index	Cotton
0.0078	S&P 500 COMPOSITE	RUSSELL 2000
0.0004	G12-DS Banks	G7-DS Banks
-0.0083	G7-DS Banks	MSCI WORLD EX US
-0.0222	Crude Oil-Brent Cur. Month	Cotton
-0.0237	Gold Bullion LBM	Crude Oil-Brent Cur. Month
-0.0239	US T-Bill 10 Year	Cotton
-0.0240	Gold Bullion LBM	US T-Bill 10 Year

Table 1. Asymmetry parameters for pairs with significant asymmetry in dependence (5% level)

5. Conclusion

In this paper, we have proposed to measure asymmetry in dependence by looking at differences in the coskewness of standardized bivariate random vectors. Our proxy for a data sample's degree of asymmetry of the copula is easy to interpret, signals the direction into which the probability mass of the copula is skewed, and the related test allows for a fast testing of the null hypothesis of symmetric dependence. In our two application studies, we have shown that both hydrological and financial market data may exhibit asymmetry in the underlying copulas. Both the interpretations of asymmetry in dependence being due to a causal relation between two random variables X and Y as well as asymmetry signaling diversification benefits during bearish market phases underline the importance of accounting for asymmetric dependence structures in financial applications. Thus, our test should constitute a helpful tool for non-life insurers for both the modeling of insurance claims and portfolio losses.

Future research could try to investigate the systematic nature of the found asymmetric dependence structures in more detail.

Appendix A. Proofs

Proof of Lemma 2.1. We only consider the claim for the maximum, the one for the minimum follows from symmetry. A simple calculation shows that $\int (v-u)^3 \, \mathrm{d}C_+ =$

$3^3/4^4$, whence it remains to be shown that C_+ is a maximizer of the map $C \mapsto \int (v - u)^3 \,\mathrm{d}C(u,v)$.

Let $B([0,1])$ denote the set of bounded Borel-measurable functions on $[0,1]$. Proposition 2 in Gaffke and Rüschendorf (1981) shows that the maximum in Lemma 2.1 is attained, and Corollary 3 in the same reference characterizes that maximum: C^* is a maximizer of $C \mapsto \int (v-u)^3 \,\mathrm{d}C(u,v)$ if and only if there exist functions $f, g \in B([0,1])$ such that $f(u) + g(v) \ge (v-u)^3$ and such that $f(u)+g(v) = (v-u)^3$ almost surely with respect to the measure induced by C^*.

An argumentation similar to the one in Example 1(a) of Gaffke and Rüschendorf (1981) suggests to define

$$f(u) = \begin{cases} -\frac{6}{64} - \frac{1}{2}(2u-1)^3, & u < \frac{1}{4} \\ \frac{1}{64} - \frac{3}{16}u, & u \ge \frac{1}{4} \end{cases}$$

and

$$g(v) = \begin{cases} \frac{1}{64} + \frac{3}{16}v, & v \le \frac{3}{4} \\ \frac{6}{64} + \frac{1}{2}(2v-1)^3, & v > \frac{3}{4}. \end{cases}$$

The proof of Lemma 2.1 is finished once we have shown that $f(u) + g(v) = (v-u)^3$ whenever $(u,v) \in \mathrm{supp}(C_+)$, and that $f(u) + g(v) \ge (v-u)^3$ for all $(u,v) \in [0,1]^2$.

First, let $(u,v) \in \mathrm{supp}(C_+) = A_1 \cup A_2$, where $A_1 = \{(u,v) \in (0, 0.25) \times (0.75, 1) : v = 1-u\}$ and where $A_2 = \{(u,v) \in (0.25, 1) \times (0, 0.75) : v = u - 0.25\}$. If $(u,v) \in A_1$, then

$$\begin{aligned} f(u) + g(v) = f(u) + g(1-u) = -\tfrac{6}{64} - \tfrac{1}{2}(2u-1)^3 + \tfrac{6}{64} + \tfrac{1}{2}(1-2u)^3 &= (1-2u)^3 \\ &= (v-u)^3. \end{aligned}$$

Similarly, if $(u,v) \in A_2$, then

$$\begin{aligned} f(u) + g(v) = f(u) + g(u - \tfrac{1}{4}) = \tfrac{1}{64} - \tfrac{3}{16}u + \tfrac{1}{64} + \tfrac{3}{16}(u - \tfrac{1}{4}) &= -\tfrac{1}{64} \\ &= (v-u)^3 \end{aligned}$$

It remains to be shown that $h(u,v) := f(u) + g(v) - (v-u)^3$ is nonnegative for all $(u,v) \in [0,1]^2$. Four cases need to be distinguished, we begin by $u \ge 1/4$ and $v \le 3/4$. Let $x = (v-u)$, a number in $[-1, 1/2]$. Then

$$h(u,v) = \tfrac{1}{64} - \tfrac{3}{16}u + \tfrac{1}{64} + \tfrac{3}{16}v - x^3 = \tfrac{1}{32} + \tfrac{3}{16}x - x^3.$$

The polynomial on the right-hand side can be easily seen to be nonnegative on $[-1, 1/2]$ (with two zeros at $x = -1/4$ and $x = 1/2$).

Now, consider $u < 1/4$ and $v > 3/4$. Let $x = 2v - 1 \in (1/2, 1]$ and $y = 1 - 2u \in (1/2, 1]$, such that $v - u = (2v - 1 + 1 - 2u)/2 = (x+y)/2$. By convexity of $t \mapsto t^3$ on the nonnegative numbers, we have

$$(v-u)^3 = \left\{\tfrac{1}{2}(x+y)\right\}^3 \le \tfrac{1}{2}x^3 + \tfrac{1}{2}y^3 = f(u) + g(v),$$

whence $h(u,v) \ge 0$.

Third, let $u < 1/4$ and $v \le 3/4$, then

$$h(u,v) = -\tfrac{5}{64} - \tfrac{1}{2}(2u-1)^3 + \tfrac{3}{16}v - (v-u)^3.$$

Since $\frac{\partial}{\partial u}h(u,v) = 3\{(v-u)^2 - (1-2u)^2\} = 3(v+1-3u)(v+u-1)$ is nonpositive on $[0,1/4] \times [0,3/4]$, the function $u \mapsto h(u,v)$ is nonincreasing for any v, whence

$$h(u,v) \ge h(\tfrac{1}{4}, v) = -\tfrac{1}{64} + \tfrac{3}{16} - (v - \tfrac{1}{4})^3.$$

The polynomial on the right-hand side can be easily seen to be nonnegative on $[0,3/4]$ (with two zeros at $v=0$ and $v=3/4$).

Finally, let $u \ge 1/4$ and $v > 3/4$, then

$$h(u,v) = \tfrac{7}{64} - \tfrac{3}{16}u + \tfrac{1}{2}(2v-1)^3 - (v-u)^3.$$

Since $\frac{\partial}{\partial v}h(u,v) = 3\{(2v-1)^2 - (v-u)^2\} = 3(2v-u+2)(3v+u-1)$ is nonnegative on $[1/4,1] \times [3/4,1]$, the function $v \mapsto h(u,v)$ is nondecreasing for any v, whence

$$h(u,v) \ge h(u, \tfrac{3}{4}) = \tfrac{11}{64} - \tfrac{3}{16} - (\tfrac{3}{4} - u)^3.$$

The polynomial on the right-hand side can be easily seen to be nonnegative on $[1/4,1]$ (with two zeros at $u = 1/4$ and $u = 1$). The proof is finished. □

Proof of Proposition 2.2. Let $\mathbb{C}_n(u,v) = \sqrt{n}(\hat{C}_n(u,v) - C(u,v)\}$ denote the empirical copula process. As a consequence of (2.4), we can write $\sqrt{n}(\hat{a}_{X,Y} - a_{X,Y}) = (256/27) \cdot \int (v-u)^3 \, \mathrm{d}\mathbb{C}_n(u,v)$. Random variables of the form $\int g(u,v) \, \mathrm{d}\mathbb{C}_n(u,v)$ are considered in Theorem 6 in Radulovic et al. (2014). For $g(u,v) = g_0(u,v) = (v-u)^3$, the functions $T_1(g)$ and $T_2(g)$ defined in formula (21) of that reference are given by

$$T_1(g) = -3g_1, \qquad T_2(g) = 3g_2.$$

It now follows from the proof of Theorem 6 in Radulovic et al. (2014) (in their notation: from the asymptotic equivalence of $\bar{\mathbb{Z}}_n$ and $\tilde{\mathbb{Z}}_n$) that

$$\int (v-u)^3 \, \mathrm{d}\mathbb{C}_n(u,v) = \int \{g_0(u,v) - 3g_1(u,v) + 3g_2(u,v)\} \, \mathrm{d}\alpha_n(u,v) + o_{\mathbb{P}}(1),$$

where $\alpha_n(u,v) = n^{-1/2} \sum_{i=1}^n \{\mathbb{1}(U_i \le u, V_i \le v) - C(u,v)\}$. The integral on the right-hand side of this display can be written as

$$n^{-1/2} \sum_{i=1}^n (27/256) \cdot \{h(U_i, V_i) - \mathrm{E}[h(U_i, V_i)]\},$$

which implies the assertion. □

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