Transport in Porous Media
https://doi.org/10.1007/s11242-020-01446-5

Nonlinear Stability for Thermal Convection in a Brinkman Porous Material with Viscous Dissipation

Brian Straughan[1]

Received: 25 April 2020 / Accepted: 2 July 2020
© The Author(s) 2020

Abstract
We investigate nonlinear stability in a model for thermal convection in a saturated porous material using Brinkman theory, taking into account viscous dissipation effects. There are (at least) two models for viscous dissipation available, and we include a derivation of one of these by assuming that the flow in the porous medium may be described by a theory for a mixture of an elastic solid and a linearly viscous fluid. A fully nonlinear stability result is provided when either of the viscous dissipation functions is taken into account, and it is shown that from the nonlinear energy stability viewpoint both models are, in a sense, equivalent.

Keywords Nonlinear stability · Viscous dissipation · Thermal convection · Brinkman porous media

1 Introduction

There has been much very interesting recent work on non-isothermal flow in a clear fluid or in a saturated porous medium when viscous dissipation effects are taken into account, see, e.g., Al-Hadrami et al. (2003), Barletta (2008), Barletta et al. (2009a, b, 2010, 2011a, b), Barletta and Mulone (2020), Barletta and Nield (2010), Barletta and Nield (2011), Breugem and Rees (2006), Hooman and Gurgenci (2007), Magyari and Rees (2006), Nield (2000a,b, 2004, 2007), Nield and Barletta (2010), Nield and Bejan (2017), Sect. 2.2.2, Nield et al. (2004), Nield and Simmons (2019), Sect. 3.1, and Rees and Magyari (2017). Since the viscous dissipation effects add some strongly nonlinear terms to the governing partial differential equations, an understanding of this is vital to fully explore thermal convection from a nonlinear point of view.

In the context of flow in a Brinkman porous medium, there is some controversy over the form the relevant nonlinear viscous dissipation terms should appear, see, e.g., Al-Hadrami et al. (2003), Barletta (2008), Barletta et al. (2011b), Breugem and Rees (2006), Hooman

Communicated by Edith van der Wal.

✉ Brian Straughan
brian.straughan@durham.ac.uk

[1] Department of Mathematics, University of Durham, Durham DH1 3LE, UK

Published online: 17 July 2020

and Gurgenci (2007), Magyari and Rees (2006) and Nield (2000a, b, 2004, 2007). It is not the purpose of this article to enter this controversy. We are interested in analyzing the fully nonlinear stability for the Brinkman theory, and so we allow both forms of nonlinear viscous dissipation. In fact, we show that, in a sense, from the viewpoint of nonlinear energy stability theory both forms of the viscous dissipation lead to exactly the same result and so in this sense are equivalent.

In the next section, we include a derivation of the Brinkman equations for thermal convection in a porous medium which is based on the theory for a mixture of an elastic solid and a viscous fluid. The viscous dissipation terms arise naturally and are essentially the same form as those of Al-Hadrami et al. (2003). We point out that this form of viscous dissipation is also derived in the analysis of Breugem and Rees (2006), who perform a rigorous averaging procedure. The comments of Breugem and Rees (2006) on whether the form of the viscous dissipation is correct are very enlightening and are lucidly described on pages 1 and 2 of their article.

Our main goal is to derive a nonlinear energy stability analysis for thermal convection in a Brinkman porous material, and we employ both forms of viscous dissipation function. A recent very interesting development of Barletta and Mulone (2020) establishes an analogous nonlinear energy stability analysis for thermal convection in a Darcy porous material when the relevant viscous dissipation term is present. A detailed analysis of thermal convection in a Brinkman porous material neglecting viscous dissipation effects is contained in Rees (2002), and the validity of the Brinkman equations is discussed in Nield (2000b). We stress that even in the absence of viscous dissipation effects, the Darcy and Brinkman theories are very different. For example, Straughan (2016) shows that one may lead to stationary convection, whereas the other yields oscillatory convection for the same problem of resonant convection. Thus, the Brinkman and Darcy equations represent different physical phenomena.

2 Derivation of the Brinkman Equations from Mixture Theory

There are various approaches to presenting a theory for a mixture of an elastic solid and a viscous fluid. We here describe that of Eringen (1994, 2004). Actually, Eringen (1994, 2004) develops his theory for a mixture of a fluid, a solid, and a gas. We restrict attention to only a fluid and a solid.

Throughout we employ standard indicial notation in conjunction with the Einstein summation convention. There are momentum equations for the fluid and for the solid, and letting s denote solid while f denotes fluid, these are (see also Straughan 2015a, pp. 20–22),

$$\begin{aligned} \rho^f\, {}''x_i^f &= \frac{\partial t_{ij}^f}{\partial x_j} + \rho^f b_i^f - p_i^f, \\ \rho^s\, {}''x_i^s &= \frac{\partial t_{ij}^s}{\partial x_j} + \rho^s b_i^s - p_i^s, \end{aligned} \tag{1}$$

where ρ, x_i, t_{ij}, b_i and p_i denote the respective, density, particle position, Cauchy stress tensor, body force, and interaction force for the fluid (f) and solid (s). The notation $''$ denotes the second derivative following the motion of the particular particle. In addition, the equation of energy balance is

$$\rho\dot{\epsilon} = \frac{\partial q_i}{\partial x_i} + t^f_{ij}v^f_{i,j} + t^s_{ij}v^s_{i,j} + p^f_i v^f_i + p^s_i v^s_i + \rho h, \tag{2}$$

where $\epsilon, q_i, v^f_i, v^s_i, h$ denote the internal energy, heat flux, fluid velocity, solid velocity, and externally supplied heat. The mixture density $\rho = \rho^f + \rho^s$. Eringen (1994, 2004) develops his theory in terms of the deformation gradient $\partial x_i/\partial X^s_K$, the temperature, the fluid density, the invariants of $d^f_{ij} = (v^f_{i,j} + v^f_{j,i})/2$, the temperature gradient, and the difference of fluid and solid velocity $v^f_i - v^s_i$. It is important to observe that $v^f_i - v^s_i$ is an objective quantity and so the velocity difference should enter the theory; from a mixture viewpoint, this term gives rise to the Darcy friction term.

The full equations for a deformed elastic body are given in Eringen (1994, 2004), see also Straughan (2015a, pp. 20–22). In this work, we are interested in flow through an undeformed skeleton which does not move; hence, $v^s_i \equiv 0$. Thus, we present a restricted constitutive theory to reflect this. The Cauchy stress $t^s_{ij} \equiv 0$ and so we require

$$\begin{aligned} t^f_{ij} &= -p^f\delta_{ij} + 2\hat{\mu}d^f_{ij},\\ q_i &= -kT_{,i},\\ p^f_i &= \hat{b}v^f_i, \end{aligned} \tag{3}$$

where p^f is the fluid pressure, $\hat{\mu}$, k and $\hat{b}$ are here taken as constants, k being the thermal conductivity. The relevant fluid momentum equation now becomes

$$\rho^f_0\dot{v}^f_i = t^f_{ij,j} + \rho^f b^f_i - \hat{b}v^f_i, \tag{4}$$

where $\dot{v}_i$ is the material derivative, ρ_0 is a reference density, and for an incompressible fluid the equation of continuity is

$$v^f_{i,i} = 0. \tag{5}$$

The energy equation now has form

$$\rho\dot{\epsilon} = q_{i,i} + t^f_{ij}v^f_{i,j} + \hat{b}v^f_i v^f_i + \rho h. \tag{6}$$

Equations (4) and (6) are reduced assuming a Boussinesq approximation, cf. Breugem and Rees (2006), Nield and Barletta (2010), Straughan (2015a, pp. 16–21). We assume the acceleration may be neglected in (4), (cf. Barletta et al. 2011b), and now additionally omit the sub- or superscript f. Adopting a Boussinesq approximation, Eqs. (4)–(6) may be reduced to, setting $h = 0$,

$$\begin{aligned} 0 &= -p_{,i} + \hat{\mu}\Delta v_i - \rho_0(1 - \alpha[T - T_0])gk_i - \hat{b}v_i,\\ v_{i,i} &= 0,\\ \rho_0 c_p\dot{T} &= k\Delta T + 2\hat{\mu}d_{ij}d_{ij} + \hat{b}v_i v_i, \end{aligned} \tag{7}$$

where g is gravity, $\mathbf{k} = (0, 0, 1)$, Δ is the Laplacian, $c_p = \partial\epsilon/\partial T$ is the specific heat at constant pressure, and the density is constant apart from the buoyancy term where we have written $\rho^f = \rho_0(1 - \alpha[T - T_0])$ with ρ_0, T_0 being reference density and temperature values, and α is the thermal expansion coefficient of the fluid. The term $\dot{T}$ is the material derivative $\dot{T} = T_{,t} + v_i T_{,i}$, and $d_{ij} = (v_{i,j} + v_{j,i})/2$.

Equations (7) are the equations for thermal convection in a Brinkman theory incorporating viscous dissipation. Note that the $\hat{b}v_iv_i$ term is already present in the Darcy theory, see, e.g., Barletta et al. (2009a), Barletta and Mulone (2020), Nield and Barletta (2010). Furthermore, observe that as $\hat{\mu} \to 0$ we recover the equations of Darcy theory when viscous dissipation is present, whereas when $\hat{b} \to 0$ the temperature equation assumes the correct form for non-isothermal flow in a clear fluid taking into account viscous dissipation effects.

3 Thermal Convection Equations

Suppose the saturated porous material is contained between the horizontal planes $z = 0$ and $z = d$ with the upper and lower temperatures retained at the constant values, $T = T_L$ °K, $z = 0$, and $T = T_U$ °K, $z = d$, where $T_L > T_U$. Define the temperature gradient, β, by

$$\beta = \frac{T_L - T_U}{d}\,.$$

Then, the steady (conduction) solution in whose stability we are interested is given by

$$\bar{v}_i \equiv 0, \qquad \bar{T} = -\beta z + T_L, \tag{8}$$

where the steady pressure, $\bar{p}(z)$, is a quadratic function derived from $(7)_1$.

To analyze thermal convection, we develop a stability analysis for the solution to (7). Thus, let (u_i, θ, π) be a perturbation to the steady solution $(\bar{v}_i, \bar{T}, \bar{p})$, i.e., $v_i = \bar{v}_i + u_i$, $T = \bar{T} + \theta$, $p = \bar{p} + \pi$. Then, the nonlinear perturbation equations are given by

$$\begin{aligned}
&0 = -\pi_{,i} + \hat{\mu}\Delta u_i + \rho_0 \alpha g k_i \theta - \hat{b}u_i,\\
&u_{i,i} = 0,\\
&\rho_0 c_p(\theta_{,t} + u_i\theta_{,i}) = k\Delta\theta + \rho_0 c_p \beta w + 2\hat{\mu}d_{ij}d_{ij} + \hat{b}u_iu_i\,.
\end{aligned} \tag{9}$$

Note that $\hat{\mu}$ is the Brinkman coefficient, whereas $\hat{b}$ represents the Darcy coefficient. Equations (9) may be non-dimensionalized in a standard way, see Straughan (2008), p. 163, to derive a non-dimensional version in terms of the Rayleigh number

$$Ra = R^2 = \frac{d^2 g \rho_0 \alpha c_p \beta}{\hat{b}k},$$

to obtain

$$\begin{aligned}
&0 = -\pi_{,i} + \lambda\Delta u_i - u_i + R\theta k_i,\\
&u_{i,i} = 0,\\
&\theta_{,t} + u_i\theta_{,i} = Rw + \Delta\theta + Bu_iu_i + Ad_{ij}d_{ij},
\end{aligned} \tag{10}$$

where Eqs. (10) hold on the domain $\{(x, y) \in \mathbb{R}^2\} \times \{z \in (0, 1)\} \times \{t > 0\}$, and λ, A, B are the non-dimensional forms of $\hat{\mu}$ and the viscous dissipation coefficients.

In this work, we assume the surfaces $z = 0, d$ are free from stress, and then, the boundary conditions are given by

$$u_{,z} = v_{,z} = w = \theta = 0, \qquad z = 0, 1, \tag{11}$$

where $\mathbf{u} = (u, v, w)$, and additionally the solution satisfies a plane tiling periodicity with a wave number a, Straughan (2004), p. 51, Straughan (2008), p. 152. The periodic cell so defined will be denoted by V. To exclude rigid motions, we require

$$\int_V u \,\mathrm{d}x = \int_V v \,\mathrm{d}x = 0.$$

4 Nonlinear Stability

Let $(\cdot,\cdot)$ and $\|\cdot\|$ be the inner product and norm on the Hilbert space $L^2(V)$.

To prove nonlinear stability, we multiply $(10)_1$ by u_i and integrate over V to find after using the boundary conditions (11)

$$\begin{aligned}\|\mathbf{u}\|^2 + \lambda\|\nabla\mathbf{u}\|^2 =& R(\theta, w)\\ \le& \frac{R}{2\epsilon}\|\theta\|^2 + \frac{R\epsilon}{2}\|w\|^2,\end{aligned}$$

where the arithmetic–geometric mean inequality has been employed. Select now $\epsilon = 1/R$ to obtain

$$\frac{1}{2}\|\mathbf{u}\|^2 + \lambda\|\nabla\mathbf{u}\|^2 \le \frac{R^2}{2}\|\theta\|^2\,. \tag{12}$$

Next, multiply $(10)_1$ by $-\Delta u_i$ and integrate over V to now obtain with the aid of (11),

$$\|\nabla\mathbf{u}\|^2 + \lambda\|\Delta\mathbf{u}\|^2 = R(\theta, \Delta w). \tag{13}$$

Note that in deriving (13), the boundary term $\oint_{\partial V} \pi\,\Delta w\, dA$ is encountered, where ∂V is the boundary of V. This is where the boundary conditions (11) are necessary to ensure this is zero. Upon using the arithmetic–geometric mean inequality on (13), one may then derive

$$\|\nabla\mathbf{u}\|^2 + \frac{\lambda}{2}\|\Delta\mathbf{u}\|^2 \le \frac{R^2}{2\lambda}\|\theta\|^2. \tag{14}$$

The next step involves multiplying $(10)_1$ by u_i, multiplying $(10)_3$ by θ, integrating each over V, and adding to obtain

$$\frac{d}{dt}\frac{1}{2}\|\theta\|^2 = RI - D + N, \tag{15}$$

where

$$I = 2(\theta, w), \qquad D = \|\mathbf{u}\|^2 + \lambda\|\nabla\mathbf{u}\|^2 + \|\nabla\theta\|^2, \tag{16}$$

and

$$N = A\int_V \theta d_{ij}d_{ij}\,\mathrm{d}x + B\int_V \theta u_i u_i\,\mathrm{d}x. \tag{17}$$

From Eq. (15) one proceeds to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\theta\|^2 \leq -D\Big(1-\frac{R}{R_E}\Big)+N, \tag{18}$$

where

$$\frac{1}{R_E}=\max_H \frac{I}{D}, \tag{19}$$

with H being the space of admissible solutions. The Euler–Lagrange equations are found from this, and one may show they are the same as the linearized version of (10). Then, the nonlinear critical Rayleigh number boundary is the same as the linear one, details may be found in, e.g., Straughan (2008, pp. 163–166).

To establish nonlinear stability, it remains to handle the nonlinear term N. To do this, we require the Poincaré inequality

$$\pi\|\theta\| \leq \|\nabla\theta\|, \tag{20}$$

the Wirtinger inequality

$$\pi\|\mathbf{u}\| \leq \|\nabla\mathbf{u}\|, \tag{21}$$

see Galdi and Straughan (1985), and the Sobolev inequalities

$$\int_V |\mathbf{u}|^4 \mathrm{d}x \leq c_1\|\mathbf{u}\|\,\|\nabla\mathbf{u}\|^3, \tag{22}$$

and

$$\int_V |\nabla\mathbf{u}|^4 \mathrm{d}x \leq c_2\|\nabla\mathbf{u}\|\,\|\Delta\mathbf{u}\|^3, \tag{23}$$

which may be established as in Galdi and Straughan (1985) and Payne and Straughan (2009), inequality (50), where c_1 and c_2 are constants. We also need the estimate, Galdi and Straughan (1985), inequality (5.27),

$$\sup_V |\mathbf{u}| \leq c_3\,\|\Delta\mathbf{u}\|. \tag{24}$$

We now bound the term N in (17). Firstly, by the Cauchy–Schwarz inequality

$$\begin{aligned}\int_V \theta u_i u_i\,\mathrm{d}x &\leq \|\theta\|\Big(\int_V |\mathbf{u}|^4\mathrm{d}x\Big)^{1/2}\\ &\leq c_1\|\theta\|\,\|\mathbf{u}\|^{1/2}\,\|\nabla\mathbf{u}\|^{3/2}.\end{aligned} \tag{25}$$

Then,

$$\int_V \theta d_{ij}d_{ij}\,\mathrm{d}x=\frac{1}{2}\int_V \theta u_{i,j}u_{i,j}\,\mathrm{d}x+\frac{1}{2}\int_V \theta u_{i,j}u_{j,i}\,\mathrm{d}x.$$

We integrate by parts the last term in this expression and use the conditions (11) to find

$$\int_V \theta d_{ij}d_{ij}\,\mathrm{d}x=\frac{1}{2}\int_V \theta u_{i,j}u_{i,j}\,\mathrm{d}x-\frac{1}{2}\int_V \theta_{,i}u_{i,j}u_j\,\mathrm{d}x. \tag{26}$$

To bound the last term, note

$$\int_V \theta_{,i}u_{i,j}u_j\,\mathrm{d}x \le c_3\|\Delta\mathbf{u}\|\,\|\nabla\theta\|\,\|\nabla\mathbf{u}\| \tag{27}$$

where the Cauchy–Schwarz inequality has also been used. Furthermore

$$\begin{aligned}\int_V \theta u_{i,j}u_{i,j}\,\mathrm{d}x \le&\|\theta\|\Big[\int_V (u_{i,j}u_{i,j})^2\mathrm{d}x\Big]^{1/2}\\ \le& c_2\|\theta\|\,\|\nabla\mathbf{u}\|^{1/2}\,\|\Delta\mathbf{u}\|^{3/2},\end{aligned} \tag{28}$$

where we have employed (23).

Utilizing (25)–(28), we may see that

$$\begin{aligned}N \le&\frac{A}{2}(c_3\|\Delta\mathbf{u}\|\,\|\nabla\theta\|\,\|\nabla\mathbf{u}\| + c_2\|\theta\|\,\|\nabla\mathbf{u}\|^{1/2}\,\|\Delta\mathbf{u}\|^{3/2})\\ &+ Bc_1\|\theta\|\,\|\mathbf{u}\|^{1/2}\,\|\nabla\mathbf{u}\|^{3/2}.\end{aligned} \tag{29}$$

Observe now from (12) and (14)

$$\|\mathbf{u}\| \le R\|\theta\|, \qquad \|\nabla\mathbf{u}\| \le \mu R\|\theta\|, \qquad \|\Delta\mathbf{u}\| \le \frac{R}{\lambda}\,\|\theta\|, \tag{30}$$

where $\mu\sqrt{2} = \min\{1, \lambda^{-1}\}$. Using the form for D in (16) together with (30) and inequality (20), we now derive

$$N \le hD\|\theta\|, \tag{31}$$

where

$$h = \frac{A}{2}\Big(\frac{c_3}{\lambda^2}\,R + \frac{c_2R^{3/2}}{\lambda^{3/2}\pi}\Big) + \frac{Bc_1}{\lambda^{3/4}}.$$

Suppose now $R < R_E$ so that $a = 1 - R/R_E > 0$. Then, from (18) using (31) one derives

$$\frac{d}{dt}\frac{1}{2}\|\theta\|^2 \le -D(a - h\|\theta\|). \tag{32}$$

Require now $\|\theta(0)\| < a/h$. Then, by a continuity argument, see, e.g., Straughan (2004, pp. 15, 16), from (32) we see that $\|\theta(t)\| \le \|\theta(0)\|$ and so (32) may be replaced by

$$\frac{d}{dt}\frac{1}{2}\|\theta\|^2 \le -bD \le -b\pi^2\|\theta\|^2, \tag{33}$$

where $b = a - h\|\theta(0)\| > 0$. From (32), we now show

$$\|\theta(t)\|^2 \le \|\theta(0)\|^2\exp\{-2b\pi^2 t\}, \tag{34}$$

and so $\|\theta(t)\|$ decays exponentially. Using (30), it follows $\|\mathbf{u}\|$, $\|\nabla\mathbf{u}(t)\|$, and $\|\Delta\mathbf{u}(t)\|$ also decay exponentially and we obtain nonlinear stability. In addition, using (24) we obtain decay of $\sup_V|\mathbf{u}|$ and so the perturbation velocity decays pointwise in time.

The conditions for nonlinear stability are

$$R < R_E \qquad \text{and} \qquad \|\theta(0)\| < \frac{a}{h}\,. \tag{35}$$

The first, $R < R_E$, corresponds to the linear instability threshold, cf. Straughan (2004, pp. 163–166). The second places a restriction on the magnitude of the temperature perturbation. Due to the severe nonlinearities of the viscous dissipation terms in $(10)_3$, we expect such a restriction.

Remark 1 A nonlinear energy stability analysis for the analogous problem for the Darcy equations has recently been provided by Barletta and Mulone (2020). The equations for the Darcy problem follow from (10) by setting λ and B equal to zero, and there are no boundary conditions on $u_{,z}, v_{,z}$ in (11). Their analysis hinges on use of the Sobolev inequality in $\mathbb{R}^3$ and a bound for $\nabla\mathbf{u}$ in terms of $\nabla\theta$.

The bound in Barletta and Mulone (2020) involves employing the differential equation on the boundary in its derivation. We here include an alternative derivation. For the Darcy system, $(10)_1$ is replaced by

$$u_i = -\pi_{,i} + R\theta k_i, \tag{36}$$

where now we only have $w = 0$ on $z = 0, 1$, but there is still periodicity in x and y. From (36), we take $\partial/\partial x_j$ and then multiply by $u_{i,j} - u_{j,i}$ to find, cf. Payne and Straughan (1996, pp. 230–232) and Straughan (2015b, pp. 117, 118),

$$\begin{aligned}\frac{1}{2}(u_{i,j} - u_{j,i}, u_{i,j} - u_{j,i}) =&(u_{i,j} - u_{j,i}, u_{i,j})\\ =&R(u_{i,j} - u_{j,i}, k_i\theta_{,j}),\end{aligned}$$

since $(u_{i,j} - u_{j,i})\pi_{,ij} = 0$. Then, we use the skewness of $u_{i,j} - u_{j,i}$ to write

$$\begin{aligned}\frac{1}{2}(u_{i,j} - u_{j,i}, u_{i,j} - u_{j,i}) &= \frac{R}{2}(u_{i,j} - u_{j,i}, k_i\theta_{,j} - k_j\theta_{,i}),\\ &\le \frac{R}{4\epsilon}(u_{i,j} - u_{j,i}, u_{i,j} - u_{j,i}) + \frac{R\epsilon}{4}(k_i\theta_{,j} - k_j\theta_{,i}, k_i\theta_{,j} - k_j\theta_{,i}),\end{aligned}$$

where in the last line the arithmetic–geometric mean inequality is used. Now take $\epsilon = R$ to find

$$(u_{i,j} - u_{j,i}, u_{i,j} - u_{j,i}) \le R^2(k_i\theta_{,j} - k_j\theta_{,i}, k_i\theta_{,j} - k_j\theta_{,i}).$$

Expanding these expressions,

$$\|\nabla\mathbf{u}\|^2 - \int_V u_{i,j}u_{j,i}\,\mathrm{d}x \le R^2\|\nabla\theta\|^2 - R^2\|\theta_{,z}\|^2.$$

Note

$$-\int_V u_{i,j}u_{j,i}\,\mathrm{d}x = -\oint_{\partial V} w_{,i}u_i\,dA = 0$$

since $w = 0$ on $z = 0, 1$. Then,

$$\|\nabla\mathbf{u}\|^2 \le R^2\int_V(\theta_x^2 + \theta_y^2)\mathrm{d}x \le R^2\|\nabla\theta\|^2.$$

This is the bound of Barletta and Mulone (2020). Note that we require more than this for the full system (10).

Remark 2 Breugem and Rees (2006) note that the viscous dissipation term in the Darcy formula can yield very high temperatures. The possibility of thermal runaway is discussed. Without the Brinkman terms, the Darcy version of Eq. (7) may be written in non-dimensional form as

$$\begin{aligned} p_{,i} &= Ra\,Tk_i - v_i, \\ v_{i,i} &= 0, \\ \dot{T} &= \Delta T + Bv_iv_i, \end{aligned} \tag{37}$$

cf. Barletta and Mulone (2020). It is to be expected high temperatures will be achieved due to the very strong forcing term Bv_iv_i. Indeed, one should ask whether finite time blow-up could occur, since, for example, it is well known that such blow-up does occur for equations like

$$\frac{\partial T}{\partial t} = \Delta T + CT^2,$$

with suitable boundary conditions and $C > 0$ constant.

We present a heuristic one-dimensional model arising from (37) which *suggests* that while T might become very large, blow-up in finite time will not occur. Consider the one-dimensional version of (37) without the pressure term

$$\begin{aligned} 0 &= Ra\,T - w, \\ \frac{\partial T}{\partial t} + w\frac{\partial T}{\partial x} &= \Delta T + Aw^2, \end{aligned}$$

where Δ is now the second derivative with respect to x. Eliminate w and derive

$$\frac{\partial T}{\partial t} + RaT\frac{\partial T}{\partial x} = \Delta T + BRa^2T^2\,. \tag{38}$$

Let this equation be defined on $(0, 1) \times \{t > 0\}$ with $T(x, 0)$ given and $T = 0$ at $x = 0, 1$. By using a weighted energy, Straughan (2004, pp. 16-19), shows that $\|T(t)\|^2$ remains bounded and T has a steep boundary layer near $x = 1$. In fact,

$$\|T(t)\|^2 \le O(\beta^3 e^{3\beta})$$

where $\beta = ARa$, is the asymptotic behavior of T as $t \to \infty$. There is a trade-off between the convective term TT_x and the viscous dissipation term BRa^2T^2. In fact, the convective term is preventing the viscous dissipation from inducing the solution to blow-up in a finite time.

5 Nonlinear Stability Analysis, Alternative Viscous Dissipation

There is an alternative viscous dissipation function which replaces the $d_{ij}d_{ij}$ term by one of the form $-u_i\Delta u_i$, see Nield (2007), Barletta et al. (2011b). In this case, the perturbation equations (10) are replaced by

$$\begin{aligned}&0=-\pi_{,i}+\lambda\Delta u_i-u_i+R\theta k_i,\\&u_{i,i}=0,\\&\theta_{,t}+u_i\theta_{,i}=Rw+\Delta\theta+Bu_iu_i-Au_i\Delta u_i,\end{aligned}\tag{39}$$

where the domain and boundary conditions are as in Sect. 4.

A nonlinear energy analysis proceeds exactly as in Sect. 4, and we arrive at Eq. (15) excepting now

$$N=B\int_V\theta u_iu_i\,\mathrm{d}x-A\int_V\theta u_i\Delta u_i\,\mathrm{d}x.\tag{40}$$

The B term is handled exactly as in Sect. 4. For the A term, we integrate by parts to find

$$-A\int_V\theta u_i\Delta u_i\,\mathrm{d}x=A\int_V\theta_{,j}u_iu_{i,j}\,\mathrm{d}x+A\int_V\theta u_{i,j}u_{i,j}\,\mathrm{d}x.\tag{41}$$

Observe that the form of the right-hand side of (41) is very similar to the right-hand side of (26). Indeed, the right-hand side of (41) is bounded in the same way as (27) and (28) to then arrive at (31). Thus, the analysis of energy stability for the alternative viscous dissipation function turns out to lead to exactly the same nonlinear stability result. In this sense, the two formulations are equivalent, although the actual solutions (u_i,θ,π) may not be.

Remark 3 Hooman and Gurgenci (2007) suggest employing a viscous dissipation function which is essentially a combination of the forms in Sects. 4 and 5. In our notation, this would give rise to a set of nonlinear perturbation equations which replace (10) of form,

$$\begin{aligned}&0=-\pi_{,i}+\lambda\Delta u_i-u_i+R\theta k_i,\\&u_{i,i}=0,\\&\theta_{,t}+u_i\theta_{,i}=Rw+\Delta\theta+Bu_iu_i+Ad_{ij}d_{ij}-Cu_i\Delta u_i\,.\end{aligned}\tag{42}$$

A nonlinear energy stability analysis may be worked out in a straightforward manner by following the techniques in Sects. 4 and 5. Again, the critical Rayleigh number found is the same as the one of linear instability theories.

6 Conclusions

We have analyzed the nonlinear stability for a solution to the Brinkman equations for thermal convection in a porous material incorporating viscous dissipation terms. The viscous dissipation consists of a term which is present in the analogous problem employing Darcy theory. Additionally, there is a term in the energy balance equation which is due entirely to the presence of the Brinkman contribution in the momentum equation. Barletta and Mulone (2020) recently established a rigorous nonlinear stability result for the Darcy problem. We here extend the method to derive an analogous rigorous nonlinear stability result for the Brinkman theory. In both cases, the nonlinear stability threshold is shown to be exactly the same as the linear instability one. For both situations, the size of the initial temperature field has to be suitably small to obtain rapid exponential decay in time. Given the nature of the strongly nonlinear viscous dissipation terms, this is to be expected as is the case in partial differential equation theory.

Acknowledgements This work was supported by an Emeritus Fellowship of the Leverhulme Trust, EM-2019-022/9.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

Al-Hadrami, A.K., Elliott, L., Ingham, D.B.: A new model for viscous dissipation in porous media across a range of permeability values. Trans. Por. Med. **53**, 117–122 (2003)

Barletta, A.: Comments on a paradox of viscous dissipation and its relation to the Oberbeck—Boussinesq approach. Int. J. Heat Mass Transf. **51**, 6312–6316 (2008)

Barletta, A., Mulone, G.: The energy method analysis of the Darcy - Bénard problem with viscous dissipation. Contin. Mech. Thermodyn. **40**, 1 (2020). https://doi.org/10.1007/s00161-020-00883-3

Barletta, A., Nield, D.A.: Convection—dissipation instability in the horizontal plane Couette flow of a highly viscous fluid. J. Fluid Mech. **662**, 475–492 (2010)

Barletta, A., Nield, D.A.: Thermosolutal convective instability and viscous dissipation effect in a fluid-saturated porous medium. Int. J. Heat Mass Transf. **54**, 1641–1648 (2011)

Barletta, A., Celli, M., Rees, D.A.S.: The onset of convection in a porous layer induced by viscous dissipation: a linear stability analysis. Int. J. Heat Mass Transf. **52**, 337–344 (2009a)

Barletta, A., Celli, M., Rees, D.A.S.: Darcy - Forchheimer flow with viscous dissipation in a horizontal porous layer: onset of convective instabilities. ASME J. Heat Transf. **131**, 072602-344 (2009b)

Barletta, A., Celli, M., Nield, D.A.: Unstably stratified Darcy flow with impressed horizontal temperature gradient, viscous dissipation and asymmetric thermal boundary conditions. Int. J. Heat Mass Transf. **53**, 1621–1627 (2010)

Barletta, A., Celli, M., Nield, D.A.: On the onset of dissipation thermal instability for the Poiseuille flow of a highly viscous fluid in a horizontal channel. J. Fluid Mech. **681**, 499–514 (2011a)

Barletta, A., Rossi di Schio, E., Celli, M.: Instability and viscous dissipation in the horizontal Brinkman flow through a porous medium. Trans. Por. Med. **85**, 105–119 (2011b)

Breugem, W.P., Rees, D.A.S.: A derivation of the volume-averaged Boussinesq equations for flow in porous media with viscous dissipation. Trans. Por. Med. **63**, 1–12 (2006)

Eringen, A.C.: A continuum theory of swelling porous elastic soils. Int. J. Eng. Sci. **32**, 1337–1349 (1994)

Eringen, A.C.: Corrigendum to "a continuum theory of swelling porous elastic soils". Int. J. Eng. Sci. **42**, 949–949 (2004)

Galdi, G.P., Straughan, B.: A nonlinear analysis of the stabilizing effect of rotation in the Bénard problem. Proc. R. Soc. Lond. A **402**, 257–283 (1985)

Hooman, K., Gurgenci, H.: Effects of viscous dissipation and boundary conditiond on forced convection in a channel occupied by a saturated porous medium. Trans. Por. Med. **68**, 301–319 (2007)

Magyari, E., Rees, D.A.S.: Effects of viscous dissipation on the Darcy free convection boundary - layer flow over a vertical plate with exponential temperature distribution in a porous medium. Fluid Dyn. Res. **38**, 405–429 (2006)

Nield, D.A.: Resolution of a paradox involving viscous dissipation and noninear drag in a porous medium. Trans. Por. Med. **41**, 349–357 (2000a)

Nield, D.A.: Modelling fluid flow and heat transfer in a saturated porous medium. J. Appl. Math. Decis. Sci. **4**, 165–173 (2000b)

Nield, D.A.: Comments on "a new model for viscous dissipation in porous media across a range of permeability values". Trans. Por. Med. **55**, 253–254 (2004)

Nield, D.A.: The modelling of viscous dissipation in a saturated porous medium. ASME J. Heat Transf. **129**, 1459–1463 (2007)
Nield, D.A., Barletta, A.: Extended Oberbeck—Boussinesq approximation study of convective instabilities in a porous layer with horizontal flow and bottom heating. Int. J. Heat Mass Transf. **53**, 577–585 (2010)
Nield, D.A., Bejan, A.: Convection in Porous Media, 5th edn. Springer, New York (2017)
Nield, D.A., Simmons, C.T.: A brief introduction to convection in porous media. Trans. Por. Med. **130**, 237–250 (2019)
Nield, D.A., Kuznetsov, A.V., Xiong, M.: Effects of viscous dissipation and flow work on forced convection in a channel filled by a porous medium. Trans. Por. Med. **56**, 351–367 (2004)
Payne, L.E., Straughan, B.: Stability in the initial—time geometry problem for the Brinkman and Darcy equations of flow in porous media. J. Math. Pures Appl. **75**, 225–271 (1996)
Payne, L.E., Straughan, B.: Decay for a Keller–Segal chemotaxis model. Stud. Appl. Math. **123**, 337–360 (2009)
Rees, D.A.S.: The onset of Darcy–Brinkman convection in a porous layer: an asymptotic analysis. Int. J. Heat Mass Transf. **45**, 2213–2220 (2002)
Rees, D.A.S., Magyari, E.: Hexagonal cell formation in Darcy–Bénard convection with viscous dissipation and form drag. Fluids **27**, 1 (2017). https://doi.org/10.3390/fluids2020027
Straughan, B.: The energy method, stability, and nonlinear convection, vol. 91, 21st edn. Springer, New York (2004)
Straughan, B.: Stability, and wave motion in porous media, vol. 165. Springer, New York (2008)
Straughan, B.: Convection with local thermal non-equilibrium and microfluidic effects. Advances in Mechanics and Mathematics Series, vol. 32. Springer, Cham, Switzerland (2015a)
Straughan, B.: Dependence on the reaction in porous convection. Boll. Unione Mat. Ital. **8**, 113–120 (2015b)
Straughan, B.: Importance of Darcy or Brinkman laws upon resonance in thermal convection. Ricerche Matem. **65**, 349–362 (2016)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.