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Stability in Kelvin–Voigt poroelasticity

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Abstract
Hölder continuous dependence of solutions upon the initial data is established for the linear theory of Kelvin–Voigt poroelasticity requiring only symmetry conditions upon the elastic coefficients. A novel functional is introduced to which a logarithmic convexity technique is employed.

Keywords Continuous dependence · Kelvin–Voigt · Improperly posed · Poroelasticity

Mathematics Subject Classification 74H25 · 74H55 · 35B30 · 35B35 · 35M13

1 Introduction

Improperly posed and related problems for partial differential equations have been investigated by many mathematicians, both in the past and relatively recently, e.g. Agmon [1], Agmon and Nirenberg [2], Ames and Epperson [3], Ames and Hughes [4], Ames and Straughan [5], Benrabah et al. [6], Caflisch et al. [7], Carasso [8–10], Chirita [11], Chirita and Zampoli [12], Fury [13], Fury and Hughes [14], Harfash [15], Hetrick and Hughes [16], Knops and Payne [17], Payne and Straughan [18–20], Straughan [21], Yang and Deng [22]. Such improperly posed problems are important in many real life applications. For example, Carasso [8,23,24], gives examples in the fields of indentification of groundwater pollution by reconstructing the contaminant plume history, or in deblurring an image in astrophysics or magnetic resonance imaging in brain scans.

In the field of classical linear elastodynamics Knops and Payne [17] established continuous dependence upon the initial data without requiring any definiteness conditions on the elastic coefficients, imposing instead only symmetry. In the modern literature it is recognised that for some bodies positive definiteness of the elastic coefficients is too restrictive. For example, in auxetic materials Poisson's ratio may be negative, see Xinchun and Lakes [25], and then the results of Knops and Payne [17], assume relevance due to their lack of definiteness requirement.

Many continuous bodies exhibit elastic behaviour but simultaneously demonstrate fluid like behaviour. In general, these are classed as viscoelastic materials. A particular subclass

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of such materials are the so called Kelvin–Voigt materials, cf. Chirita et al. [26], Chirita and Zampoli [12], Gal and Medjo [27], Su and Qin [28]. In the linear case these add a dissipation term to the equations of linear elasticity, see e.g. Chirita et al. [26], Chirita and Zampoli [12]. It is important to observe that Kelvin–Voigt theory is being used in various industrial applications to anayse real materials. For example, Gidde and Pawar [29] study viscoelastic properties of polydimethylsiloxane in a micropump by means of a Kelvin–Voigt model, Jayabal et al. [30] use the theory for computational skin modelling in the cosmetics industry, and Jozwiak et al. [31] use this theory to describe the dynamic behaviour of biopolymer materials.

The goal of this work is to establish a stability estimate and prove uniqueness for a solution to equations which describe a porous elastic body of Kelvin–Voigt type. We require only symmetry of the elastic coefficients and impose no definiteness whatsoever. In order to achieve this we introduce a novel functional and work with a logarithmic convexity method. The equations of poroelasticity couple the elastic displacement to the pressure field in the pores and are described by a second order in time system coupled to a first order one. Analysis of stability for this coupled system of partial differential equations necessarily requires a very different procedure to that which suffices in classical elastodynamics.

2 Governing equations

The governing equations for a multi-porosity elastic body are described by Svanadze [32] or by Straughan [33]. For a single porosity anisotropic elastic body of Kelvin–Voigt type the equations are, cf. Chirita and Zampoli [12], Straughan [33], p. 72,

$$\begin{aligned} \rho \ddot{u}_i &= (a_{ijkh}u_{k,h})_{,j} + (b_{ijkh}\dot{u}_{k,h})_{,j} - (\beta_{ij}p)_{,j} + \rho f_i, \\ \alpha \dot{p} &= (k_{ij}p_{,j})_{,i} - \beta_{ij}\dot{u}_{i,j} + \rho s, \end{aligned} \tag{1}$$

where $\rho, u_i, p, f_i, \alpha, s$ are the density, elastic displacement, pressure field in the pores, externally supplied body force, the inertia coefficient for the pressure, and the externally supplied heat source. The terms a_{ijkh} and b_{ijkh} are the elastic coefficients and the Kelvin–Voigt coefficients, respectively, β_{ij} is a coupling tensor, and k_{ij} is a pressure diffusion tensor. A superposed dot denotes partial time differentiation, partial differentiation with respect to x_i is written as $_{,i} \equiv \partial/\partial x_i$, and standard indicial notation is employed in conjunction with the Einstein summation convention.

Equation (1) hold on $\Omega \times (0, T]$, where $\Omega \subset \mathbb{R}^3$ is a bounded domain with boundary Γ sufficiently smooth to allow applications of the divergence theorem, and $T < \infty$ is a constant.

The solution (u_i, p) is subject to the following boundary conditions

$$u_i(\mathbf{x}, t) = h_i(\mathbf{x}, t), \qquad p(\mathbf{x}, t) = q(\mathbf{x}, t), \qquad \mathbf{x} \in \Gamma, \quad t \in (0, T], \tag{2}$$

where h_i and q are prescribed functions. The initial conditions are

$$u_i(\mathbf{x}, 0) = u_i^0(\mathbf{x}), \qquad \dot{u}_i(\mathbf{x}, 0) = v_i(\mathbf{x}), \qquad p(\mathbf{x}, 0) = r(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \tag{3}$$

where u_i^0, v_i and r are prescribed functions.

The boundary-initial value problem comprising (1)–(3) is denoted by $\mathcal{P}$.

The coefficient α satisfies the restriction

$$0 \le \alpha(\mathbf{x}) \le \alpha_U < \infty, \tag{4}$$

for a constant α_U and for all $\mathbf{x} \in \Omega$. In addition, the coefficients $a_{ijkh}, b_{ijkh}, \beta_{ij}, k_{ij}$ satisfy the symmetries

$$\begin{aligned} a_{ijkh} &= a_{khij} = a_{jikh}, \\ b_{ijkh} &= b_{khij} = b_{jikh}, \\ \beta_{ij} &= \beta_{ji}, \qquad k_{ij} = k_{ji}, \end{aligned} \tag{5}$$

whilst b_{ijkh} and k_{ij} satisfy the definiteness conditions

$$\begin{aligned} b_{ijkh}h_{ij}h_{kh} &\geq b\, h_{ij}h_{ij}, \qquad \forall\, h_{ij}, \\ k_{ij}q_iq_i &\geq k\, q_iq_i, \qquad \forall\, q_i, \end{aligned} \tag{6}$$

for constants $b, k > 0$. In addition we suppose the elastic coefficients are bounded above in the sense that

$$\max_{\mathbf{x}\in\bar{\Omega}} |a_{ijkh}| = A \tag{7}$$

for a constant A.

3 Uniqueness

To establish uniqueness for a soluition to $\mathcal{P}$ we let (u_i^1, p^1) and (u_i^2, p^2) be two solutions to $\mathcal{P}$ for the same boundary and initial data functions h_i, q, u_i^0, v_i and r. Define the difference (u_i, p) by

$$u_i = u_i^1 - u_i^2, \qquad p = p^1 - p^2.$$

Upon inspection one sees that (u_i, p) satisfies the boundary - initial value problem

$$\begin{aligned} \rho\ddot{u}_i &= (a_{ijkh}u_{k,h})_{,j} + (b_{ijkh}\dot{u}_{k,h})_{,j} - (\beta_{ij}p)_{,j} \\ \alpha\dot{p} &= (k_{ij}p_{,j})_{,i} - \beta_{ij}\dot{u}_{i,j} \end{aligned} \tag{8}$$

on $\Omega \times (0, T]$ together with the boundary conditions

$$u_i(\mathbf{x}, t) = 0, \quad p(\mathbf{x}, t) = 0, \qquad \mathbf{x} \in \Gamma,\ t \in (0, T], \tag{9}$$

and the initial conditions

$$u_i(\mathbf{x}, 0) = 0, \quad \dot{u}_i(\mathbf{x}, 0) = 0, \quad p(\mathbf{x}, 0) = 0, \qquad \mathbf{x} \in \Omega. \tag{10}$$

For vector functions u, v and scalar functions ϕ, ψ we define the notation

$$\begin{aligned} (u, Bv) &= \int_\Omega b_{ijkh}u_{i,j}v_{k,h}dx \\ (\phi, K\psi) &= \int_\Omega k_{ij}\phi_{,i}\psi_{,j}dx. \end{aligned} \tag{11}$$

Let $(\cdot, \cdot)$ and $\|\cdot\|$ denote the norm and inner product on $L^2(\Omega)$.

The first relation we require is obtained by multiplying $(8)_1$ by $\dot{u}_i$, by multiplying $(8)_2$ by p and integrating each over Ω. After use of the boundary conditions and an integration in time we find

$$\frac{1}{2}(\rho\dot{u}_i, \dot{u}_i) + \frac{1}{2}(a_{ijkh}u_{i,j}, u_{k,h}) + \frac{1}{2}(\alpha p, p) + \int_0^t (\dot{u}, B\dot{u})ds + \int_0^t (p, Kp)ds = 0. \tag{12}$$

To establish uniqueness we introduce a new logarithmic convexity functional. Let

$$\eta(\mathbf{x}, t) = \int_0^t p(\mathbf{x}, s)ds.$$

Then define $F(t)$ by

$$F(t) = (\rho u_i, u_i) + \int_0^t (u, Bu)ds + \int_0^t (\eta, K\eta)ds. \tag{13}$$

The first term in F is the weighted L^2 integral of the displacement which appears in classical linear elastodynamics. The second term involving (u, Bu) is necessary to handle the dissipation in $(8)_1$. Finally, the term in the gradient of the integrated pressure is introduced to deal with the first order in time equation $(8)_2$.

By differentiation

$$F'(t) = 2(\rho\dot{u}_i, u_i) + 2\int_0^t (\dot{u}, Bu)ds + 2\int_0^t (\eta, Kp)ds.$$

A further differentiation shows

$$F''(t) = 2(\rho\ddot{u}_i, u_i) + 2(\rho\dot{u}_i, \dot{u}_i) + 2(\dot{u}, Bu) + 2(\eta, Kp). \tag{14}$$

We integrate by parts on the last two terms in (14) to obtain

$$(\dot{u}, Bu) + (\eta, Kp) = (u_i, (b_{ijkh}\dot{u}_{k,h})_{,j}) - (p, (k_{ij}\eta_{,j})_{,i}). \tag{15}$$

Next integrate equation $(8)_2$ in time to see that

$$\alpha p = (k_{ij}\eta_{,j})_{,i} - \beta_{ij}u_{i,j}. \tag{16}$$

The next step uses $(8)_1$ to substitute for $\rho\ddot{u}_i$ in (14), we then employ (15) in (14), and further utilize (16) to obtain

$$F''(t) = 2(\rho\dot{u}_i, \dot{u}_i) - 2(a_{ijkh}u_{i,j}, u_{k,h}) - 2(\alpha p, p). \tag{17}$$

Now, substitute for the last two terms in (17) from (12) to derive

$$F''(t) = 4(\rho\dot{u}_i, \dot{u}_i) + 4\int_0^t (\dot{u}, B\dot{u})ds + 4\int_0^t (p, Kp)ds. \tag{18}$$

Now form the combination $FF'' - (F')^2$, to obtain

$$FF'' - (F')^2 = 4S^2,$$

where

$$\begin{aligned} S^2 =& \Big[(\rho u_i, u_i) + \int_0^t (u, Bu)ds + \int_0^t (\eta, K\eta)ds\Big] \\ &\times \Big[(\rho \dot{u}_i, \dot{u}_i) + \int_0^t (\dot{u}, B\dot{u})ds + \int_0^t (p, Kp)ds\Big] \\ &- \Big[(\rho u_i, \dot{u}_i) + \int_0^t (u, B\dot{u})ds + \int_0^t (p, K\eta)ds\Big]^2 \end{aligned}$$

and note $S^2 \geq 0$ by virtue of the Cauchy-Schwarz inequality. Thus,

$$FF'' - (F')^2 \geq 0. \tag{19}$$

To establish uniqueness from inequality (19) we employ a contradiction argument. The details are similar to those on p.21 of Straughan [33] to demonstrate $F \equiv 0$. Whence, $u_i \equiv 0$. Once $u_i \equiv 0$, p is given by equation $(8)_2$, and since this is a diffusion equation it follows in the usual way that $p \equiv 0$. Hence the solution to $\mathcal{P}$ is unique.

4 Hölder stability, when $E(0) \leq 0$

To analyse stability for a solution to equations (1) under conditions (2)–(7) we let (u_i^1, p^1) and (u_i^2, p^2) be two solutions to (1) which satisfy (2) for the same functions h_i and q but satisfy (3) for different initial data functions $u_i^1 = u_i^{10}$, $\dot{u}_i^1 = v_i^1$, $p^1 = r^1$, $u_i^2 = u_i^{20}$, $\dot{u}_i^2 = v_i^2$, $p^2 = r^2$. Define $u_i = u_i^1 - u_i^2$, $p = p^1 - p^2$, and $m_i = u_i^{10} - u_i^{20}$, $v_i = v_i^1 - v_i^2$, $r = r^1 - r^2$. Then the difference solution satisfies the equations (8) together with the boundary conditions (9), although the initial conditions are now

$$u_i(\mathbf{x}, 0) = m_i(\mathbf{x}), \quad \dot{u}_i(\mathbf{x}, 0) = v_i(\mathbf{x}), \quad p(\mathbf{x}, 0) = r(\mathbf{x}), \qquad \mathbf{x} \in \Omega. \tag{20}$$

We commence by deriving the energy equation using the procedure to arrive at (12), but now we derive

$$E(t) + \int_0^t (\dot{u}(s), B\dot{u}(s))ds + \int_0^t (p(s), Kp(s))ds = E(0), \tag{21}$$

where

$$E(t) = \frac{1}{2}(\rho \dot{u}_i, \dot{u}_i) + \frac{1}{2}(\alpha p, p) + \frac{1}{2}(a_{ijkh} u_{i,j}, u_{k,h}) \,. \tag{22}$$

We again let $\eta = \int_0^t p\, ds$ and integrate $(8)_2$ in time to obtain

$$\alpha p = (k_{ij}\eta_{,j})_{,i} - \beta_{ij} u_{i,j} + \alpha r(\mathbf{x}) + \beta_{ij} m_{i,j}(\mathbf{x}). \tag{23}$$

Now define P to be a solution to the equation

$$(k_{ij} P_{,j})_{,i} = \alpha r(\mathbf{x}) + \beta_{ij} m_{i,j}(\mathbf{x}), \tag{24}$$

on Ω, with $P = 0$ on Γ. Define now the function $\mu(\mathbf{x}, t)$ by

$$\mu(\mathbf{x}, t) = \eta(\mathbf{x}, t) + P(\mathbf{x}). \tag{25}$$

Observe from (23) and (24) that

$$\alpha p = (k_{ij} \mu_{,j})_{,i} - \beta_{ij} u_{i,j}. \tag{26}$$

In this section we define the function $F(t)$ by

$$\begin{aligned} F(t) =&(\rho u_i, u_i) + \int_0^t (u, Bu)ds \\ &+ \int_0^t (\mu, K\mu)ds + (T-t)\big[(v, Bv) + (P, KP)\big]. \end{aligned} \tag{27}$$

This is the function we manipulate with the logarithmic convexity arguments.

By differentiation,

$$F'(t) = 2(\rho \dot{u}_i, u_i) + 2\int_0^t (u, B\dot{u})ds + 2\int_0^t (p, K\mu)ds. \tag{28}$$

We differentiate again to find

$$F''(t) = 2(\rho \ddot{u}_i, u_i) + 2(\rho \dot{u}_i, \dot{u}_i) + 2(u, B\dot{u}) + 2(p, K\mu). \tag{29}$$

Next, substitute for $\rho \ddot{u}_i$ from equation $(8)_1$ to obtain after integration by parts and use of the boundary conditions

$$\begin{aligned} F''(t) =&4(\rho \dot{u}_i, \dot{u}_i) - 2(a_{ijkh}u_{k,h}, u_{i,j}) \\ &+ 2(p, \beta_{ij}u_{i,j}) + 2(k_{ij}\mu_{,j}, p_{,i}). \end{aligned} \tag{30}$$

Upon substitution for the a_{ijkh} term from (21) we may obtain from (30) after further integration by parts,

$$\begin{aligned} F''(t) =&4(\rho \dot{u}_i, \dot{u}_i) + 4\int_0^t (\dot{u}, B\dot{u})ds \\ &+ 4\int_0^t (p, Kp)ds - 4E(0). \end{aligned} \tag{31}$$

One now forms the combination $FF'' - (F')^2$ using (31), (28) and (27), and after some manipulation one may arrive at the equation

$$\begin{aligned} FF'' - (F')^2 =&4S^2 - 4E(0)F \\ &+ 4(T-t)\big[(v, Bv) + (P, KP)\big] \\ &\times \Big[(\rho \dot{u}_i, \dot{u}_i) + \int_0^t (\dot{u}, B\dot{u})ds + \int_0^t (p, Kp)ds\Big] \end{aligned} \tag{32}$$

where S^2, which is non-negative by virtue of the Cauchy - Schwarz inequality, is given by

$$\begin{aligned} S^2 =&\Big[(\rho \dot{u}_i, \dot{u}_i) + \int_0^t (\dot{u}, B\dot{u})ds + \int_0^t (p, Kp)ds\Big] \\ &\times \Big[(\rho u_i, u_i) + \int_0^t (u, Bu)ds + \int_0^t (\mu, K\mu)ds\Big] \\ &- \Big[(\rho u_i, \dot{u}_i) + \int_0^t (\dot{u}, Bu)ds + \int_0^t (p, K\mu)ds\Big]^2. \end{aligned}$$

Thus, from (32), when $E(0) \leq 0$,

$$FF'' - (F')^2 \geq 0$$

and so

$$(\log F)'' \geq 0.$$

Then one may deduce that

$$F(t) \leq [F(0)]^{1-\delta}[F(T)]^{\delta}, \qquad t \in [0, T), \tag{33}$$

where $\delta = t/T$, cf. Ames and Straughan [5], p. 17.

We now require the solution to $\mathcal{P}$ to belong to a constraint set such that

$$F(T) \leq M,$$

for a known constant M. This constraint is expected in an improperly posed problem, see e.g. Ames and Straughan [5], Carasso [24]. Estimate (33) then yields Hölder continuous dependence of the solution to $\mathcal{P}$ on compact subintervals of $[0, T)$ in the displacement measure of form

$$(\rho u_i, u_i) + \int_0^t (u, Bu)ds \leq M^{\delta}[F(0)]^{1-\delta}, \qquad t \in [0, T). \tag{34}$$

To establish Hölder continuous dependence in a measure of the pressure difference p we note from the energy equation (21) that

$$\begin{aligned} &\frac{1}{2}(\rho\dot{u}_i, \dot{u}_i) + \frac{1}{2}(\alpha p, p) + \int_0^t (\dot{u}, B\dot{u})ds \\ &\quad + \int_0^t (p, Kp)ds = E(0) - \frac{1}{2}(a_{ijkh}u_{i,j}, u_{k,h}). \end{aligned} \tag{35}$$

We now utilize conditions (6), (7) and (4), and Poincaré's inequality, to derive from (35)

$$(\alpha p, p) + \mu \int_0^t (\alpha p, p)ds \leq 2E(0) + A\|\nabla \mathbf{u}\|^2, \tag{36}$$

where $\gamma = 2k\lambda_1/\alpha_U$, with λ_1 being the first eigenvaluen in the membrane problem for Ω. Inequality (36) may be integrated with an integrating factor to obtain

$$\int_0^t (\alpha p, p)ds \leq \frac{2E(0)}{\gamma} + 2A\int_0^t e^{-\gamma(t-s)}\|\nabla \mathbf{u}\|^2 ds. \tag{37}$$

If we now use (6) on the b_{ijkh} term in (34) we obtain a Hölder continuous dependence estimate for p in the sense that

$$\int_0^t (\alpha p, p)ds \leq \frac{2E(0)}{\gamma} + \frac{2A}{b}M^{\delta}[F(0)]^{1-\delta}, \qquad t \in [0, T). \tag{38}$$

Together, the bounds (34) and (38) establish Hölder continuous dependence of a solution to $\mathcal{P}$ on compact subintervals of $[0, T)$.

Remark The results established here for a single porosity Kelvin–Voigt material may be extended to the equivalent theory for double or triple porosity. Such materials are attracting increasing attention, see e.g. Chirita [34], Svanadze [35], Svanadze [36], Svanadze [37], Svanadze [38]. Indeed double and triple porosity materials are being increasingly recognised as important in real life, see e.g. Durif et al. [39] where man made silicon carbide porous foams exhibiting double or triple porosity structure are analysed, or Navarro et al. [40] who employ a triple porosity model to an MX-80 betonite pellet mixture, and further examples may be found in Straughan [33].

5 Hölder stability, when $E(0) > 0$

When the initial energy satisfies $E(0) > 0$ the analysis leading to (32) still holds. However, we cannot immediately obtain an inequality like (33). Instead we employ an approach used in classical linear isothermal elastodynamics by Knops and Payne [17], cf. Straughan [33], pp. 28-29, where convexity is proved not of the basic functional F, but by modifying it to incorporate $E(0)$ in such a way as to still achieve continuous dependence. The functional F chosen here is of necessity very different from that used in classical elastodynamics by Knops and Payne [17].

Define the function $G(t)$ by

$$G(t) = \log\big[F(t) + 2E(0)\big] + t^2. \tag{39}$$

Then one may show

$$\begin{aligned}\big[F + 2E(0)\big]^2 G'' =& \big[F + 2E(0)\big]F'' - (F')^2 \\ &+ 2\big[F + 2E(0)\big]^2.\end{aligned} \tag{40}$$

We now substitute for $FF'' - (F')^2$ from (32) in (40) and may then show

$$\begin{aligned}\big[F + 2E(0)\big]^2 G'' = 4S^2 &+ 2F^2 + 4FE(0) \\ &+ 4\big\{(T-t)\big[(v, Bv) + (P, KP)\big] + 2E(0)\big\} \\ &\times \Big[(\rho\dot{u}_i, \dot{u}_i) + \int_0^t (\dot{u}, B\dot{u})ds + \int_0^t (p, Kp)ds\Big].\end{aligned} \tag{41}$$

Now since $E(0) > 0$ it follows that $G'' \geq 0$ and hence G is a convex function of t. Then one finds F satisfies the estimate

$$F(t) + 2E(0) \leq K\big[F(0) + 2E(0)\big]^{\xi}, \tag{42}$$

for t in a compact subinterval of $[0, T)$, with $0 < \xi < 1 - t/T < 1$, and with

$$K = \big[M + 2E(0)\big]^{t/T} \exp\big[t(T-t)\big],$$

see Knops and Payne [17], or Straughan [33], p. 29.

Estimate (42) establishes Hölder continuous dependence in the measure $(\rho u_i, u_i)$. One may now appeal to inequality (38) to establish Hölder continuous dependence in the p measure $\int_0^t (\alpha p, p)ds$.

Compliance with ethical standards

Conflict of interest There are no conflicts of interest.

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