Infinite-Parameter ADHM Transform

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Abstract

The Atiyah-Drinfeld-Hitchin-Manin (ADHM) transform and its various generalizations are examples of non-linear integral transforms between finite-dimensional moduli spaces. This note describes a natural infinite-dimansional generalization, where the transform becomes a map from boundary data to a family of solutions of the self-duality equations in a domain.

Dedicated to Michael Atiyah, in Memoriam

1   Introduction

One of many discoveries named after Michael Atiyah is the ADHM (Atiyah-Drinfeld-Hitchin-Manin) transform [1]. Starting with the work of Nahm [14, 15], it was subsequently generalized in various ways; see for example the review [12]. Independent of this, but related to it, was the Fourier-Mukai transform in algebraic geometry [13, 2].

Now ADHM is an integral transform, and as such is analogous to the Fourier, Radon and Penrose transforms, and also to the inverse scattering transform in soliton theory. These other examples are usually encountered in infinite-dimensional contexts: for example, the Fourier transform

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is an isomorphism of infinite-dimensional vector spaces. By contrast, the ADHM transform and its generalizations have mostly been studied in finite-dimensional contexts, giving correspondences between finite-dimensional moduli spaces. Indeed, it originally arose from algebraic geometry (the construction of algebraic vector bundles), and index theory was an important part of it. The ADHM construction and its various generalizations involve the kernels of Dirac operators, with part of the analysis being to prove that these operators are Fredholm, so that their kernels are finite-dimensional.

The purpose of this note is to point out that ADHM can in fact operate comfortably as an infinite-dimensional transform, just like its cousins. In fact, early work [18, 14, 16] already suggested an underlying local and infinite-dimensional structure. Of course, it has always been clear that one may take the basic ADHM transform, in which ADHM data of rank N correspond to instantons of charge N, and then implement some sort of $N \to \infty$ limit to obtain an infinite-dimensional system; but the aim here is to go beyond this bald observation, and to describe a specific scheme.

2 The Original ADHM Transform

This section summarizes the original ADHM construction [1], in order to describe the background and establish notation. Let $x_\mu = (x_1, x_2, x_3, x_4)$ denote Cartesian coordinates on $\mathbb{R}^4$, and let A_μ denote a gauge potential. For simplicity, we take the gauge group to be SU(2); so each of A_1, A_2, A_3 and A_4 is a 2×2 antihermitian tracefree matrix, and A_μ describes an SU(2) connection over $\mathbb{R}^4$. The corresponding gauge field (curvature) is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu],$$

where $\partial_\mu = \partial/\partial x_\mu$. The self-dual Yang-Mills equation is the condition that this 2-form be self-dual on Euclidean $\mathbb{R}^4$, namely that

$$\tfrac{1}{2}\varepsilon_{\alpha\beta\mu\nu}F_{\alpha\beta} = F_{\mu\nu}. \tag{1}$$

Here we are employing the Einstein convention of summing over repeated indices, and ε denotes the standard totally-skew tensor. The self-duality

equations (1) constitute a set of nonlinear partial differential equations for A_μ, invariant under gauge transformations $A_\mu \mapsto \Lambda^{-1}A_\mu\Lambda + \Lambda^{-1}\partial_\mu\Lambda$ with $\Lambda : \mathbb{R}^4 \to \mathrm{SU}(2)$, and also conformally invariant. If we impose boundary conditions which amount to saying that the field extends to the conformal compactification S^4, then the solutions are called instantons, and are classified topologically by an integer N.

There is an 'ansatz' (the 't Hooft-Corrigan-Fairlie-Wilczek ansatz [9]) which produces a subset of the solutions of (1). It has the form

$$A_\mu = \tfrac{1}{2}T_{\mu\nu}\,\partial_\nu \log \phi, \tag{2}$$

where $T_{\mu\nu}$ (defined below) is a certain constant tensor with values in the Lie algebra $\mathfrak{su}(2)$. Assuming this form for A_μ reduces (1) to the Laplace equation

$$\partial_\mu\partial_\mu\phi = 0 \tag{3}$$

on $\mathbb{R}^4$. This is a local result: no boundary conditions are required. To get an N-instanton solution, one may take ϕ to be a sum of N fundamental solutions

$$\phi(x) = 1 + \sum_{a=1}^{N} \frac{\lambda_a^2}{|x - x^{(a)}|^2}, \tag{4}$$

where the $x^{(a)}$ denote N distinct points in $\mathbb{R}^4$, $|x|^2 = x_\mu x_\mu$ denotes the Euclidean length-squared, and the λ_a are positive weights. The singularities at $x = x^{(a)}$ are removable by a gauge transformation on A_μ. This simple formula gives a $(5N)$-parameter family of instanton solutions, which we may think of as N instantons with locations $x^{(a)} \in \mathbb{R}^4$ and sizes λ_a, all in phase with one another. In the general N-instanton solution, which is not of the special form (2), each instanton acquires an individual $\mathrm{SU}(2)$ phase: this gives an extra $3N$ parameters, so the full moduli space has dimension $8N$ (or $8N - 3$, if we remove an overall phase).

The relevant aspects of the ADHM construction of $\mathrm{SU}(2)$ instantons may be summarized as follows [1, 4, 5, 6]; we mostly adhere here to the conventions of [6]. It is convenient to use quaternions. Let $e_a = (e_1, e_2, e_3)$ denote quaternions with $e_1e_2 = -e_3 = -e_2e_1$, $(e_1)^2 = -1$ etc, and define $e_\mu = (e_1, e_2, e_3, 1)$. A point $x \in \mathbb{R}^4$ corresponds to the quaternion $x = x_\mu e_\mu \in \mathbb{H}$.

The quaternion-valued 2-form $T_{\mu\nu}$ is defined by $e_\mu^* e_\nu = \delta_{\mu\nu} + T_{\mu\nu}$, where the 'star' denotes quaternionic conjugation (and will denote conjugate transpose when applied to a matrix). We identify the imaginary quaternions with the Lie algebra $\mathfrak{su}(2)$; in terms of the Pauli matrices σ_a, one may use the identification $e_a \equiv i\sigma_a$. So $T_{\mu\nu}$ is also a 2-form with values in $\mathfrak{su}(2)$, and this is the object appearing in (2).

Now let L be an N-row vector of quaternions, and M a symmetric $N \times N$ matrix of quaternions. They are required to satisfy the ADHM constraint, namely that

$$L^*L + M^*M \text{ is real.} \tag{5}$$

We also need an invertibility condition, to get a non-singular gauge field; but (5) is the crucial condition for generating solutions of (1). The ADHM data for an N-instanton solution consist of (L, M), subject to (5), and modulo the equivalence

$$M \equiv R^t M R, \quad L \equiv \alpha L R, \tag{6}$$

where $R \in \mathrm{O}(N)$ and α is a unit quaternion.

To obtain an instanton gauge potential A_μ from these data, we proceed as follows. For each $x \in \mathbb{H}$, let v be an N-row vector of quaternions, depending on x, and satisfying the linear equation

$$L + v(M + x) = 0. \tag{7}$$

Then set

$$A_\mu = \frac{1}{2\phi} \left[v \left(\partial_\mu v^* \right) - \left(\partial_\mu v \right) v^* \right], \tag{8}$$

where $\phi = 1 + vv^*$. This field A_μ is a 1-form with values in the imaginary quaternions, and hence in $\mathfrak{su}(2)$; and it is an instanton gauge potential. This is the ADHM transform: it in fact gives a one-to-one correspondence between ADHM data (L, M) of rank N, and SU(2) N-instantons up to gauge equivalence.

To get the special class of solutions (2), we take the components λ_a of L to be real and positive, and $M = \mathrm{diag}(-x^{(1)}, -x^{(2)}, \ldots, -x^{(N)})$; these data satisfy the ADHM constraints (5). The solution of (7) is then $v = [v_1, \ldots, v_N]$, where

$$v_a = \lambda_a (x^{(a)} - x)^{-1};$$

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and evaluating (8) gives (2) and (4).

3 Generalized ADHM — Ansatz Case

Our aim is to describe an infinite-parameter generalization of the standard ADHM transform. The starting-point is the ansatz (2), which produces a subclass of self-dual gauge fields from harmonic functions ϕ. The ADHM construction involves harmonic functions of the form (4), namely finite sums of fundamental solutions. By contrast, the general smooth solution of (3) in a domain D in $\mathbb{R}^4$ is determined by arbitrary functions, namely data on the boundary $S = \partial D$. This suggests a generalization of ADHM in which the data, or at least most of it, correspond to arbitrary functions on the 3-surface S. It is this idea that we shall pursue in what follows, beginning with the ansatz (2), and then moving on to a more general class of fields. For simplicity we will take D to be the unit ball $|x| \leq 1$, so that S is the unit 3-sphere $|x| = 1$; however, the idea works just as well for other domains.

The self-duality equation (1) can be re-written as a non-linear second-order equation for a Hermitian matrix-valued function on D; this form is sometimes referred to as the Hermitian Yang-Mills equation. It is known that the corresponding Dirichlet problem has a unique solution [7]. So for example in the SU(2) case, the general (local) solution of (1) depends on three arbitrary functions of three variables, namely the boundary data. The aim in this note is to describe a generalized ADHM transform which may be viewed as an implementation of the map from boundary data to the solution in the interior.

Our generalized setup is as follows. As coordinates on S we use unit quaternions: $y \in \mathbb{H}$ with $|y| = 1$. The N-vector v of the previous section becomes a square-integrable quaternion-valued function $v(y)$ on S, depending also on x. The real N-vector L becomes a smooth real-valued function $L(y)$ on S. The linear equation (7) defining v is replaced by

$$L(y) + v(y)(-y + x) = 0. \tag{9}$$

So, in effect, the matrix M has become $-y$ times a three-dimensional delta-

function on S. Clearly the solution of (9) is

$$v(y) = L(y)(y - x)^{-1}. \tag{10}$$

To get a gauge potential $A_\mu(x)$ from v, we define a quaternionic product by

$$\langle v, w \rangle = \int_S v(y)\, w(y)^* \, |dy|^3,$$

where $|dy|^3$ denotes the standard Euclidean measure on $S = S^3$; and then set

$$A_\mu = \frac{1}{2\phi} \left[\langle v, \partial_\mu v \rangle - \langle \partial_\mu v, v \rangle \right], \tag{11}$$

where ϕ is the real-valued function $\phi(x) = 1 + \langle v, v \rangle$. As before, the partial derivatives ∂_μ in (11) are with respect to x_μ. With v given by (10), the function ϕ is

$$\phi(x) = 1 + \int_S \frac{L(y)^2}{|x - y|^2} \, |dy|^3, \tag{12}$$

and the expression (11) then reduces to the ansatz form (2), with ϕ given by (12).

Now (12) is simply the Green's function formula for solutions of the Laplace equation in the domain D, with Robin boundary data on $S = \partial D$. More precisely, $L(y)$ is determined by ϕ as

$$2\pi^2 L(y)^2 = \phi(y) + \partial_n \phi(y) - 1, \tag{13}$$

where $\partial_n \phi = y_\mu \partial_\mu \phi$ denotes the outward normal derivative of ϕ on S. So for this class of self-dual gauge fields, the ADHM data, consisting essentially of the real-valued function L up to sign, may be interpreted as boundary data for the gauge field.

One may make contact with the original, finite, version (4) as follows. Suppose that we are given a real-valued function L on S, giving rise to the harmonic function ϕ as in (12), and the gauge potential A_μ. For each positive integer N, choose a uniformly-distributed sample of N points $x^{(1)}, \ldots, x^{(N)}$ on S, and define 'finite' ADHM data $L^{(N)} = [\lambda_1, \ldots, \lambda_N]$ and $M^{(N)}$ by

$$\lambda_i = \pi \sqrt{\frac{2}{N}} L(x^{(i)}), \quad M^{(N)} = \mathrm{diag}(-x^{(1)}, -x^{(2)}, \ldots, -x^{(N)}). \tag{14}$$

Then the standard ADHM construction gives an N-instanton field $A_\mu^{(N)}$ as in (2) and (4), involving the harmonic function

$$\phi^{(N)}(x) = 1 + \sum_{i=1}^{N} \frac{\lambda_i^2}{|x - x^{(i)}|^2}. \tag{15}$$

Now $\phi^{(N)} \to \phi$ as $N \to \infty$, this being simply a Monte Carlo evaluation of the integral (12); and therefore we also have $A_\mu^{(N)} \to A_\mu$ as $N \to \infty$. In effect, the expression (15) approximates the general ansatz solution in D in terms of N instantons on the boundary S, all in phase with one another.

This picture could be extended further: in the Green's formula, we could add a finite number of delta-function sources in the interior of D. Then ϕ would have singularities of the form (4) inside D, but would still produce a smooth gauge field, incorporating instantons inside D. We then get a combination of the 'finite-parameter' original version of the ADHM construction and the 'infinite-parameter' version introduced above. This extended story is reminiscent of the inverse scattering transform for solitons, where the scattering data consists of a finite-parameter soliton part, plus an infinite-parameter radiation part. In our case, we would have a finite number of instantons located inside D, plus infinitely many other degrees of freedom.

4 Generalized ADHM — Full Version

The aim here is to extend the structure of the previous section, so that it is no longer restricted to fields of the ansatz type (2). By analogy with the standard case, this can be done by allowing $L(y)$ to be quaternion-valued rather than real-valued. Then M will no longer be 'diagonal', but rather becomes a symmetric quaternion-valued generalized function $M(z, y)$, thought of as the kernel of an integral operator acting on the function v by

$$(vM)(y) = \int_S v(z)M(z, y)\, |dz|^3. \tag{16}$$

The data (L, M) are required to satisfy the analogue of (5), namely that

$$L(y)^*L(z) + \int_S M(y, s)^*M(s, z)\, |ds|^3 \text{ is real.} \tag{17}$$

This will guarantee that the corresponding gauge field is self-dual, wherever it is defined. The linear equation satisfied by v then becomes

$$L + vM + vx = 0, \qquad (18)$$

and the self-dual gauge potential is given, as before, by (11).

It is straightforward to generalize the 'discrete approximation' (14, 15) by taking a sample of N points on S, and this illustrates how the structure described above may be viewed as an $N \to \infty$ limit of the N-instanton construction.

Now in the general case, we have the problem of solving the non-linear ADHM constraint equation (17), and this is difficult: indeed, already very difficult for the original finite system (5). In the discussion below, we first examine the simplest example; and then describe the linearized version, which gives an indication of 'how many' self-dual gauge fields the construction produces.

The simplest non-trivial example with $L(y)$ not real-valued is

$$L(y) = \kappa y, \quad (vM)(y) = -v(y)y + \lambda \int_S v(z)(y+z) \, |dz|^3, \qquad (19)$$

where κ and λ are real constants satisfying

$$2\pi^2\lambda^2 - 2\lambda + \kappa^2 = 0, \quad 0 < \kappa\pi\sqrt{2} < 1, \quad 0 < 2\pi^2\lambda < 1.$$

It is easily checked that this (L, M) satisfies the constraint equation (17). Clearly there is a high degree of symmetry in this solution, in particular under the rotation group $SO(4)$ acting on S; and this implies that the corresponding gauge field must be the 1-instanton located at the origin $x = 0$. The only parameter in this solution is the instanton size, and this is determined by κ, or equivalently by λ. Explicitly implementing the ADHM construction with $x = 0$, to obtain the gauge field there, reveals that the instanton size is in fact given by

$$\rho = \frac{1 - 2\pi^2\lambda}{\sqrt{2\lambda(1 - \pi^2\lambda)}}$$

in terms of λ. Note that $\rho \to 0$ as $\lambda \to 1/(2\pi^2)$, and $\rho \to \infty$ as $\lambda \to 0$. As was pointed out previously, one can obtain the 1-instanton at $x = 0$ by

putting it in 'by hand' as a delta-function source. But we see here that solutions including instantons inside D can also, and perhaps more neatly, be obtained from smooth data such as (19).

Now let us consider the linearized version. The details of this are somewhat analogous to those of the finite (instanton) case described in [4]. Let ϵ be a parameter with $0 < \epsilon \ll 1$, and take L to be a quaternion-valued function with $|L(y)| = O(\epsilon)$: we shall work to lowest order in ϵ. (The scale is set by the volume of S, which here is of order unity.) If we write

$$M(y, z) = -y\, \delta(y - z) + P(y, z),$$

then the constraint equation (17) is equivalent to

$$
\begin{aligned}
2(y^* - z^*)P(y, z) \;=\; & L(y)^*L(z) - L(z)^*L(y) \\
& + \int_S (P(y, s)^*P(z, s) - P(z, s)^*P(y, s))\, |ds|^3,
\end{aligned}
$$

and this can be solved iteratively for P, order-by-order in ϵ. For our purposes here, it is sufficient to observe that $P(y, z) = O(\epsilon^2)$. The solution of (18) then has the form

$$v(y) = L(y)(y - x)^{-1} + O(\epsilon^3),$$

and this can then be used in (11) to compute the leading term $\mathcal{A}_\mu$ in the gauge potential A_μ, which will be of order ϵ^2. The calculation is straightforward, and the details will be omitted here; the result can be written as follows.

Define $C_\mu^a[\phi]$ by

$$C_\mu^a[\phi]e_a = \tfrac{1}{2}T_{\mu\nu}\partial_\nu\phi,$$

which is just the linearized version of (2). So C_μ^1 produces a self-dual U(1) gauge field $a_\mu = C_\mu^1[\phi]$ from a harmonic function ϕ; and furthermore, given any self-dual U(1) gauge field a_μ, there exists a harmonic function ϕ such that $a_\mu = C_\mu^1[\phi]$. The same is true of C_μ^2 and C_μ^3. Now given $L(y)$, define a quaternion-valued solution Φ of the Laplace equation (3) by

$$\Phi(x) = 1 + \int_{\mathbb{R}^3} \frac{L(y)^2}{|x - y|^2}\, |dy|^3, \tag{20}$$

and let Φ_μ be its quaternionic components, in other words $\Phi = \Phi_\mu e_\mu$. Then our SU(2) gauge potential is $A_\mu = \mathcal{A}_\mu^a e_a + O(\epsilon^4)$, where

$$\mathcal{A}_\mu^a = C_\mu^a[\Phi_4] + \varepsilon_{abc}C_\mu^b[\Phi_c]. \tag{21}$$

In other words, the leading $O(\epsilon^2)$ part of A_μ just consists of the three self-dual U(1) gauge fields $\mathcal{A}_\mu^a$ given by the formula (21). Note that the generalized ADHM data $L(y)$ corresponds to boundary data, via the obvious quaternionic generalization of the formula (13).

Now if L is real-valued, then (20) becomes (12), and (21) just reduces to the linearized version of (2), as expected. In this case, there is effectively only one independent gauge field: for example $\mathcal{A}_\mu^1$ determines $\mathcal{A}_\mu^2$ and $\mathcal{A}_\mu^3$. To put this another way, each of the three $\mathcal{A}_\mu^a$ is determined by the single harmonic function Φ_4.

If L is quaternion-valued rather than real, then there are four harmonic functions Φ_μ appearing in the formula (21). So one might expect to obtain a more general class of linearized self-dual SU(2) fields than those corresponding to the ansatz (2). This is indeed the case, but only partially: two of the resulting $\mathcal{A}_\mu^a$ are independent, but not all three of them. To see that at least two of the $\mathcal{A}_\mu^a$ are independent is easy: for example, given $\mathcal{A}_\mu^1$ and $\mathcal{A}_\mu^2$, set $\Phi_2 = \Phi_3 = 0$, choose Φ_4 such that $\mathcal{A}_\mu^1 = C_\mu^1[\Phi_4]$, and then choose Φ_1 such that $\mathcal{A}_\mu^2 = C_\mu^2[\Phi_4] + C_\mu^3[\Phi_1]$. In other words, the formula (21) can produce arbitrary $\mathcal{A}_\mu^a$ for $a = 1, 2$. But it cannot do so for $a = 1, 2, 3$; and consequently (21) does not yield the most general linearized self-dual SU(2) fields in D. This is somewhat less obvious; a sketch of the reasoning is as follows. If one could generate three independent self-dual U(1) gauge fields $\mathcal{A}_\mu^a$ via (21), then in particular one could get $\mathcal{A}_\mu^1 = \mathcal{A}_\mu^2 = 0$, while $\mathcal{A}_\mu^3 \neq 0$. But imposing $\mathcal{A}_\mu^1 = \mathcal{A}_\mu^2 = 0$ in (21) leads, after some algebra, to $\mathcal{A}_\mu^3 = 0$.

It is reasonable, therefore, to conjecture that the analogous result is true for the full nonlinear system, namely that the generalized ADHM construction described in this section produces an infinite-dimensional class of solutions of (1) in D from their boundary data: a class larger than that of the ansatz (2), but not the whole solution space. In fact, the conjecture is that we get a family of solutions depending on two arbitrary functions of three variables, whereas the ansatz solutions depend on one such function, and the general self-dual gauge field on three such functions.

5   Comments

The aim above has been to describe a specific infinite-dimensional version of the ADHM construction, and to promote the claim that Atiyah's original finite-dimensional algebraic-geometrical picture extends rather naturally to a local infinite-dimensional one. This generalization could, in some sense, be viewed as a non-linear version of the Green's function formula for solutions of the Laplace equation on a bounded domain D in $\mathbb{R}^4$, in terms of boundary data on ∂D. The scheme described above is not a complete solution of the boundary-value problem: it does not yield the general solution in the interior. So an intriguing question is whether there is a further generalization of ADHM which does amount to a complete solution.

Another local and infinite-dimensional generalization of the ADHM construction, which on the face of it is quite distinct from the one presented here, involves the Nahm equations with values in the Lie algebra sdiff(S^2) of Hamiltonian vector fields on S^2, and their relation to 'abelian monopole bags' [17, 8, 10, 11, 3]. In this case, we have a hodograph transformation which transforms the Nahm equation to the Laplace equation on a domain in $\mathbb{R}^3$ (or in 3-dimensional hyperbolic space). Although this looks rather different, it may possibly be related to the four-dimensional scheme described above.

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