Expected Exit Time for Time-Periodic Stochastic Differential Equations and Applications to Stochastic Resonance

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Abstract

In this paper, we derive a parabolic partial differential equation for the expected exit time of non-autonomous time-periodic non-degenerate stochastic differential equations. This establishes a Feynman-Kac duality between expected exit time of time-periodic stochastic differential equations and time-periodic solutions of parabolic partial differential equations. Casting the time-periodic solution of the parabolic partial differential equation as a fixed point problem and a convex optimisation problem, we give sufficient conditions in which the partial differential equation is well-posed in a weak and classical sense. With no known closed formulae for the expected exit time, we show our method can be readily implemented by standard numerical schemes. With relatively weak conditions (e.g. locally Lipschitz coefficients), the method in this paper is applicable to wide range of physical systems including weakly dissipative systems. Particular applications towards stochastic resonance will be discussed.

Keywords: expected exit time; first passage time; time-inhomogeneous Markov processes; Feynman-Kac duality; stochastic resonance; locally Lipschitz; time-periodic parabolic partial differential equations.

1 Introduction

In many disciplines of sciences, (expected) exit time of stochastic processes from domains is an important quantity to model the (expected) time for certain events to occur. For example, time for chemical reactions to occur [Kra40, Gar09, Zwa01], biological neurons to fire [RS79, Sat78], companies to default [BC76, BR04], ions crossing cell membranes in molecular biology [Bre04] are all broad applications of exit times. For autonomous stochastic differential equations (SDEs), the expected exit time from a domain has been well-studied in existing literature. In particular, it is well-known that the expected exit time satisfies a second-order linear elliptic partial differential equation (PDE) [Has12, Gar09, Zwa01, Pav14, Ris96]. However, in existing literature, it appears that the expected exit time PDE is absent for non-autonomous SDEs and in particular time-periodic SDEs. Our novel contribution is the rigorous derivation of a second-order linear parabolic PDE obeyed by the expected exit time of time-periodic SDEs as its time-periodic solution. This establishes a Feynman-Kac duality for time-periodic SDEs for the expected duration. We expect that our approach and this duality go beyond this current paper to derive similar parabolic PDEs for other quantities associated to time-periodic SDEs. Conversely, we expect this duality provides stochastic

insight into existing time-periodic solutions of parabolic PDEs. In this paper, we also discuss briefly the ill-posedness of the PDE for the general non-autonomous SDE case and thereby explaining its absence in literature.

With typical and relatively weak SDE conditions e.g. non-degenerate diffusion and existence of continuous Markov transition density, the PDE can be rigorously derived. In the interest of many physical systems, we show that the results readily apply to weakly dissipative SDEs. The conditions required to solve the PDE are weaker than that to derive the PDE from the SDE. This is expected because from a PDE perspective, weak solutions of PDE on bounded domains can often be attained requiring coefficients to only be L^p or Hölder; and classical solutions may be obtained via Sobolev embedding. On the other hand, as a priori, it is not known if the process would exit the bounded domain in finite time or indeed have finite expectation. By considering the SDE and its Markov transition probability on the entire unbounded domain, we show that if the exit time has finite second moment then the PDE derivation can be rigorously justified. We show that irreducibility and the strong Feller property of the Markov transition probability are the key ingredients to conclude the exit time has finite second moment. While it is well-known that the strong Feller property holds provided the coefficients are globally Hölder and bounded with uniformly elliptic diffusion [Fri64, SV06], these conditions are too restrictive for the applications from a SDE perspective. The celebrated Hörmander's condition is a weak condition to deduce the strong Feller property for autonomous SDE [Hör85, Mal78, Hai11]. In the recent paper [HLT17], the authors extended Hörmander's condition to sufficiently imply the strong Feller property holds for non-autonomous SDEs. The smoothness SDE conditions of this paper is to invoke the result of [HLT17] while flexible enough for applications.

We provide two complementary approaches to prove that the parabolic PDE has a unique solution in a weak and classical sense. In the proofs, we keep as much generality as convenient to show the main ingredients for the well-posedness of the PDE and for straightforward application to similar problems. In one approach, we show that the time-periodic solution can be casted as a fixed point of the parabolic PDE evolution operator after a period. We prove that if the associated bilinear form is coercive, then the time-periodic solution exists and is unique by Banach Fixed Point Theorem. As coercivity can be difficult to verify in practice, we also take a calculus of variation approach. Specifically, we cast the problem as a convex optimisation problem by defining a natural cost functional and show that a unique minimiser exists and satisfies the PDE.

We emphasise that while our core results are theoretical in nature, the Banach fixed point and convex optimisation approach can be readily implemented by standard numerical schemes. Acquiring the tools to numerically compute the expected exit time is vital because explicit or even approximate closed form formulas for the expected time are rarely known, even in the autonomous case. The known cases include (autonomous) one-dimensional gradient SDEs with additive noise, where the expected exit time can be expressed as a double integral [Gar09] and has an approximate closed form solution given by Kramers' time, when the noise is small [Kra40]. Kramers' time has since been extended to higher dimensional gradient SDEs [Ber11]. However, to our knowledge, there are currently no-known exact formulae for the time-periodic case. Therefore, particularly for applications, there is an imperative to solving the PDE numerically.

This paper together with the periodic measure concept provides a novel mathematical approach to stochastic resonance, a phenomena that we now briefly describe. In a series of papers [BPSV81, BPSV82, BPSV83, Nic82], the paradigm of stochastic resonance was introduced to explain Earth's cyclical ice ages. In particular, the authors proposed a double-well potential SDE with periodic forcing to model the scientific observation that Earth's ice age transitions from "cold" and "warm" climate occurs abruptly and almost regularly every 10^5 years. The wells model the two metastable states, where the process typically stays at for large amount of time. The peri-

odic forcing corresponds to the annual mean variation in insolation due to changes in ellipticity of the earth's orbit, while noise stimulates the global effect of relatively short-term fluctuations in the atmospheric and oceanic circulations on the long-term temperature behaviour. In the absence of noise (with or without periodic forcing), these models do not produce transition between the two metastable states. Similarly, in the absence of periodic forcing, while the noise induces transitions between the stable states, the transitions are not periodic. It is the delicate interplay between periodicity and noise that explains the transitions between the metastable states to be periodic. Since the seminal papers, stochastic resonance has found applications in many physical systems including optics, electronics, neuronal systems, quantum systems amongst other applications [GHJM98, JH07, ZMJ90, Jun93, HIP05, Lon93].

The concept of periodic measures and ergodicity introduced in [FZ16] provides a rigorous framework and new insight for understanding such physical phenomena. Indeed, in [FZZ19], broad classes of SDEs were shown to possess a unique geometric periodic measure and specifically shown to apply double-well potential SDEs. The uniqueness of geometric periodic measure implies transition between the wells occurs [FZZ19]. While there is no standard definition [JH07, HI05], stochastic resonance is said to occur if the expected transition time between the metastable states is (roughly) half the period [CLRS17]. Indeed, the transition time between the wells is a special case of exit time. Applying the theory developed in this paper, we show that computationally solving the PDE and stochastic simulation for the expected transition time agrees. We then fine tune the noise intensity until the system exhibit stochastic resonance. Our PDE results also show the transition between cold and ward climates is indeed very abrupt when regime change happens.

Existing stochastic resonance literature often utilise Kramers' time, note however that Kramers' time applies only to autonomous gradient SDE case and in the small noise limit. For example in [MW89, CGM05] reduced the dynamics to "effective dynamics" two-state time-homogeneous Markov process and invoked a time-perturbed Kramers' time. More generally, utilising large deviation and specifically Wentzell–Freidlin theory [FW98], stochastic resonance and related estimates can be attained in the small noise limit. For example, [MS01] attained estimates for escape rates, a closely related quantity to expected transition time. Similarly, in [IP01] and [HI05, HIP05, HIPP14], the authors obtained estimates for the noise intensity for stochastic resonance by reducing to two-state Markov process and time-independent bounds respectively. In this paper, we retain the time-dependence of the coefficients and furthermore, small and large noise are permissible. In fact, the noise can even be state-dependent and exact exit time duration is obtained.

2 Expected Exit Time and Duration

2.1 Introduction

Consider a stochastic process $(X_t)_{t \geq s}$ on $\mathbb{R}^d$ with continuous sample-paths and an open non-empty (possibly unbounded) domain $D \subset \mathbb{R}^d$ with boundary ∂D. Without loss of generality, we assume throughout this paper that D is connected. Indeed if D is disconnected, one can solve separately on each connected subset. We define the first exit time from the domain D (or first passage time or first hitting time to the boundary) by

$$\eta_D(s, x) := \inf_{t \geq s}\{X_t \notin D | X_s = x\} = \inf_{t \geq s}\{X_t \in \partial D | X_s = x\}, \tag{2.1}$$

where $x \in D$ and the equality holds by sample-path continuity. We let $\eta_D(s, x) = \infty$ if X_t never exits D. While the absolute time in (2.1) is important, it is mathematically convenient and practically

useful to study instead the exit duration

$$\tau_D(s, x) := \eta_D(s, x) - s \tag{2.2}$$

directly. As D is generally fixed, where unambiguous, we omit the subscript D i.e. $\eta(s, x) = \eta_D(s, x)$ and $\tau(s, x) = \tau_D(s, x)$. By Début theorem, $\eta(s, x)$ is both a hitting time and a stopping time. In general, $\tau(s, x)$ is not. Thus some proofs and computations will be first done for $\eta(s, x)$, then related to $\tau(s, x)$ via (2.2). In this paper, we are interested in their expectations

$$\bar{\eta}(s, x) := \mathbb{E}[\eta(s, x)], \quad \bar{\tau}(s, x) := \mathbb{E}[\tau(s, x)]. \tag{2.3}$$

In conventional notation, one typically writes $\bar{\eta} = \mathbb{E}^{s,x}[\eta]$ and $\bar{\tau} = \mathbb{E}^{s,x}[\tau]$. For subsequent proofs, it is often more convenient that we keep the explicit dependence on the random variables.

In this paper, we are specifically interested in the expected exit and duration time for T-periodic non-degenerate SDEs on $\mathbb{R}^d$ of the form

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, & t \geq s, \\ X_s = x, & x \in D, \end{cases} \tag{2.4}$$

where W_t is a d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $b \in C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$ and $\sigma \in C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^{d \times d})$ are T-periodic i.e.

$$b(t, \cdot) = b(t + T, \cdot), \quad \sigma(t, \cdot) = \sigma(t + T, \cdot),$$

such that a unique solution $X_t = X_t^{s,x}$ exist. To avoid triviality, we always assume the coefficients collectively have a minimal period i.e. at least one of the coefficients have a minimal period.

When a unique solution of (2.4) exists, one can define the Markovian transition probability

$$P(s, t, x, \Gamma) := \mathbb{P}(X_t \in \Gamma | X_s = x), \quad s \leq t, \Gamma \in \mathcal{B}(\mathbb{R}^d). \tag{2.5}$$

If SDE (2.4) is T-periodic, then it is straightforward to show that

$$P(s, t, x, \cdot) = P(s + T, t + T, x, \cdot), \quad s \leq t. \tag{2.6}$$

We refer to SDEs as non-autonomous when there is an explicit time-dependence, periodic or otherwise. When the SDE coefficients are time-independent i.e. $b(t, \cdot) = b(\cdot)$ and $\sigma(t, \cdot) = \sigma(\cdot)$, then the SDE (2.4) is said to be autonomous. It is well-known that for autonomous SDEs, the expected exit time and expected duration coincide [Gar09, Pav14, Zwa01]. Denoting both the expected exit and duration time by $\bar{\tau}(x)$, it is moreover known that $\bar{\tau}(x)$ satisfies the following second-order elliptic PDE with vanishing boundaries [Has12, Gar09, Pav14, Zwa01, Ris96]

$$\begin{cases} L\bar{\tau} = -1, & \text{in } D, \\ \bar{\tau} = 0 & \text{on } \partial D, \end{cases} \tag{2.7}$$

where

$$Lf = \sum_{i=1}^{d} b^i(x)\partial_i f(x) + \frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(x)\partial_{ij}^2 f(x), \quad f \in C_0^2(\mathbb{R}^d), \tag{2.8}$$

is the usual infinitesimal stochastic generator with the conventional notation $\partial_i = \partial_{x_i}$ and $a(x) = (\sigma\sigma^T)(x)$.

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For non-autonomous SDEs however, due to the explicit dependence on time, expected exit time and expected duration no longer coincide. That is, $\bar{\tau}(s, x)$ generally depends on both initial time and initial state. In this non-autonomous case, we write explicitly the time-dependence and define the stochastic infinitesimal generator of (2.4) by

$$L(s)f(x) = \sum_{i=1}^{d} b^i(s, x)\partial_i f(x) + \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(s, x)\partial_{ij}^2 f(x), \quad f \in C_0^2(\mathbb{R}^d), \tag{2.9}$$

and its adjoint (on $C_0^2(\mathbb{R}^d)$), the Fokker-Planck operator by

$$L^*(s)f(x) = -\sum_{i=1}^{d} \partial_i(b^i(s, x)f(x)) + \frac{1}{2}\sum_{i,j=1}^{d} \partial_{ij}\left(a^{ij}(s, x)f(x)\right), \quad f \in C_0^2(\mathbb{R}^d). \tag{2.10}$$

It is easy but important to see that for non-autonomous SDEs, $\bar{\tau}(s, x)$ does not satisfy (2.7) even if L is replaced by $L(s)$. We note also one approach is to consider lifted coordinates $(t, X_t^{s,x})$. This autonomisation approach was noticed by two of the authors in their work [FZ16], where the terminology "lift" was used. In fact, the lift leads to Markovian RDS (Random Dynamical System) cocycle and homogeneous Markov semigroup. This allows us to obtain an invariant measure from the periodic measure. The operator $\partial_s + L(s)$ appeared naturally as the infinitesimal generator of the lifted semigroup and its spectral structure was also analysed to have an infinite number of equally placed pure imaginary eigenvalues. This very interesting phenomena is due to the degeneracy of $\partial_s + L(s)$ and periodic boundary condition. This agrees with their spectral structure of ergodic periodic measures of homogeneous Markov semigroup studied in the first part of the paper which was already published in [FZ20]. The lifting was heavily used in our earlier paper [FZZ19], where we discussed Fokker-Planck equation for the density of periodic measure. However for the problem we consider in this paper, we need to come back to the spatial state space due to technical challenges that arises when applying the techniques of autonomous system directly to the lifted coordinates (t, X_t). We briefly survey these issues below.

We note that while the Markov transition probability of lifted coordinates (t, X_t) is time-homogeneous, it immediately loses other important properties such as irreducibility and strong Feller property. This means classical results are not applicable. For instance, it is easy to show the lifted process can never be ergodic. Some papers such as [BDE01] observed that (t, X_t) is time-homogeneous and applied (2.7) with operator $\partial_s + L(s)$. However, upon close inspection of the derivation of (2.7) from texts including [Has12, Gar09, Pav14, Zwa01, Ris96], we note that (2.7) was derived only for autonomous systems, rather than non-autonomous systems.

Directly applying the autonomous results for non-autonomous SDEs with lifted coordinates (t, X_t) leads to technical boundary conditions issues. For instance, from a SDE perspective, while (2.1) is well-defined in spatial coordinates, it is not immediately clear how to define exit time in (t, X_t) coordinates. Possibly the most natural choice is $\pi_x(t, X_t)$, where π_x is the spatial projection. This adds unnecessary complexity and leads to further issues from a PDE perspective. Specifically, suppose we can extend the classic result from [Has12, Gar09, Pav14, Zwa01, Ris96] directly to (t, X_t), then it suggests we replace the boundary conditions of (2.7) with $u(s, x) = 0$ on $\partial(\mathbb{R}^+ \times D) = (\{0\} \times D) \cup (\mathbb{R}^+ \times \partial D)$. This is problematic. Treated as a parabolic PDE, this suggest that if the process starts at time $s = 0$, then for any $x \in D$, the exit time to leave the domain is zero. This cannot be the case. If persisted to interpreted as an elliptic equation, the elliptic PDE would be degenerate and the boundary conditions is insufficient (as conditions on $\{T\} \times D$ is missing, assuming we cap $\mathbb{R}^+$ to some finite T). If we diverge from [Has12, Gar09, Pav14, Zwa01, Ris96] slightly, the intuitive boundary condition is again $\pi_x u(s, x) = 0$. This is problematic in that there

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are minimal (if any) PDE theory that deals with such projection boundary conditions therefore making attempts to prove existence and uniqueness difficult. It is our assumption that particular to the boundary conditions is why [Has12, Gar09, Pav14, Zwa01, Ris96] and other similar texts have omitted the non-autonomous SDE case.

The novel contribution of this paper is the rigorous derivation of the second-order parabolic PDE (2.24) in which $\bar{\tau}$ satisfies for T-periodic SDEs. This is complete with boundary conditions in space and periodicity in the time domain to show the PDE is well-posed. Furthermore, using the framework we build in this paper, we provide numerical methods for solving the periodic solution of the parabolic PDE.

2.2 Expected Duration PDE Derivation

To rigorously derive the expected duration PDE, we first fix some standard nomenclature and notation. For the open domain $D \subset \mathbb{R}^d$ and open interval $I \subset \mathbb{R}^+$. We define their Cartesian product by $D_I := I \times D$. When $I = (0, T)$, we define $D_T := (0, T) \times D$. And we define $B_r(y) := \{x \in \mathbb{R}^d | \|x - y\| < r\}$ for the open ball of radius $r > 0$ centred at y, and denote for convenience $B_r := B_r(0)$. On $\mathbb{R}^d$, we let Λ be the Lebesgue measure. For matrices, we let $L_2(\mathbb{R}^d) := \{\sigma \in \mathbb{R}^{d \times d} | \|\sigma\|_2 < \infty\}$, where $\|\sigma\|_2 = \sqrt{\mathrm{Tr}(\sigma\sigma^T)} = \sqrt{\sum_{i,j=1}^d \sigma_{ij}^2}$ is the standard Frobenius norm. For $\theta_x \in (0, 1]$, denote by $C^{\theta_x}(D)$ the collection of all functions globally θ_x-Hölder continuous on D. For $\theta_t, \theta_x \in (0, 1]$, denote by $C^{\theta_t, \theta_x}(I \times D)$ the set of functions θ_t-Hölder and θ_x-Hölder functions in the t and x variable respectively.

Let $k_t, k_x \in \mathbb{N}$, we denote by $C^{k_t, k_x}(I \times D)$ to be the space of continuously k_t-differentiable functions in t and continuously k_x-differentiable function in x. For $\theta_t, \theta_x \in (0, 1]$, $C^{k_t + \theta_t, k_x + \theta_t}(I \times D)$ denotes the space of $C^{k_t, k_x}(I \times D)$ functions in which the k_t'th t-derivative and k_x'th x-derivatives are θ_t and θ_x are Hölder respectively. We also let $C_b^\infty(B_n)$ denote the space of bounded infinitely differentiable real-valued functions on B_n. Define for ease, $\|\sigma\|_\infty := \sup_{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d} \|\sigma(t,x)\|_2$. Following the conditions required of Theorem 1 of [HLT17], we say that drift is said to be locally smooth and bounded if for all $n \in \mathbb{N}$,

$$b(t, x) + \partial^\beta b(t, x) \quad \text{bounded on } \mathbb{R}^+ \times B_n, \tag{2.11}$$

where $\beta = (\beta_0, \beta_1, ..., \beta_d) \in \mathbb{N}^{d+1}$, $|\beta| := \sum_{i=0}^d \beta_i = d$ and $\partial^\beta := \frac{\partial^{|\beta|}}{\partial_t^{\beta_0} \partial_{x_1}^{\beta_1} ... \partial_{x_d}^{\beta_d}}$. Note that the partial derivative here refers to derivative in any spatial direction and in the time direction.

We say the SDE (2.4) satisfies the regularity condition if its coefficients b and σ are locally Lipschitz and there exists a function $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d, \mathbb{R}^+)$ and a constant $c > 0$ such that $\lim_{x \to \infty} V(t, x) = \infty$ for all fixed t and

$$L(t)V \leq cV, \quad \text{on } \mathbb{R}^+ \times \mathbb{R}^d. \tag{2.12}$$

It was shown in [Has12] that if SDE (2.4) satisfies the regularity condition (2.12), then the process is regular i.e. $\mathbb{P}^{s,x}\{\eta_\infty = \infty\} = 1$, where $\eta_\infty = \lim_{n \to \infty} \eta_{B_n}$. Moreover, there exists a unique almost surely finite solution. SDE (2.4) is said to be weakly dissipative if there exists a constant $c \geq 0$, $\lambda > 0$ such that

$$2\langle b(t, x), x \rangle \leq c - \lambda\|x\|^2 \quad \text{on } \mathbb{R}^+ \times \mathbb{R}^d. \tag{2.13}$$

If $c = 0$, then it is said to be dissipative. While weak dissipativity is a stronger condition than (2.12) and is also often easier to verify, particularly for many typical physical systems. It was shown that T-periodicity and weak dissipativity leads to the geometric ergodicity of periodic measures [FZZ19].

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We say σ is locally smooth and bounded if for all $n \in \mathbb{N}$

$$\sigma^{ij} \in C_b^\infty(B_n), \quad 1 \le i, j \le d. \tag{2.14}$$

Finally, we say σ is (globally) bounded with (globally) bounded inverse if

$$\max\{\|\sigma\|_\infty, \|\sigma^{-1}\|_\infty\} < \infty. \tag{2.15}$$

Observe (2.14) and (2.11) imply the respective functions are locally Lipschitz. Whenever we assume (2.14), we always demand that σ is a function of spatial variables only.

It appears that in numerous existing literature, almost surely finite exit time is implicitly assumed. Particularly for degenerate noise, it may well be that the exit time is infinite with positive probability or indeed almost surely. Utilising asymptotic stability of diffusion processes, it is easy to construct examples where the process never leaves a point or domain. We refer readers to [Mao07] for examples. In the following lemma, we give verifiable conditions to imply irreducibility and show further that η is almost surely finite with finite first and second moments.

Lemma 2.1. *Let $D \subset \mathbb{R}^d$ be a non-empty open bounded set. Assume that the T-periodic SDE (2.4) satisfies (2.12), (2.11), (2.14) and (2.15). Then $\eta(s,x)$ is finite almost surely for all $(s,x) \in \mathbb{R}^+ \times D$. Moreover, $\eta(s,x)$ has finite first and second moments.*

Proof. It was shown in [FZZ19] that (2.12) and (2.15) sufficiently implies P is irreducible i.e. $P(s,t,x,\Gamma) > 0$ for all $x \in \mathbb{R}^d$, $0 \le s < t < \infty$ and non-empty open set $\Gamma \in \mathcal{B}(\mathbb{R}^d)$. Then for any fixed $s \in \mathbb{R}^+$, for all $x \in \mathbb{R}^d$, it follows that there exists an $\epsilon(x) = \epsilon(s,x,D) \in (0,1)$ such that

$$\epsilon(x) = P(s, s+T, x, D).$$

By the results of [HLT17], [FZZ19] showed that when conditions (2.12), (2.11), (2.14) and (2.15) hold then P possesses a smooth density. This implies that P is strong Feller i.e. $P(s,t,\cdot,\Gamma)$ is continuous for all $s < t$ and $\Gamma \in \mathcal{B}(\mathbb{R}^d)$. Then it follows from the boundedness of D that the probability of staying within D in one period is at most

$$\epsilon := \sup_{x \in \bar{D}} \epsilon(x) > 0.$$

Since P is irreducible and $D \subset \mathbb{R}^d$ is non-empty open, we have that $P(s, s+T, x, D^c) > 0$ for any $x \in D$. Further, since D is bounded, we deduce that $\epsilon < 1$. By (2.6), $Z^{s,x} = (Z_n^{s,x}) := (X_{s+nT}^{s,x})_{n \in \mathbb{N}}$ is time-homogeneous Markov chain with one-step Markovian transition $P(s, s+T, x, \cdot)$. Define the exit time

$$\eta_Z = \eta_Z(s,x) := \min\{n \in \mathbb{N} : Z_n^{s,x} \notin D\}.$$

By sample-path continuity of X_t, it is clear that $X_t \notin D$ for at least one $t \in [s + (\eta_Z - 1)T, s + \eta_Z T]$. Hence $\tau(s,x) := \eta(s,x) - s \le \eta_Z(s,x) \cdot T$, in particular, we have

$$\{\eta_Z(s,x) \ge n\} \subset \{\eta(s,x) \ge s + nT\}.$$

Hence if $\mathbb{P}(\eta_Z < \infty) = 1$ then $\mathbb{P}(\eta < \infty) = 1$ i.e. if Z_n^s leaves D in almost surely finite time then X_t does also. For any $n \in \mathbb{N}$, it is easy to see that

$$\{\eta_Z = n\} = \{Z_n^s \in D^c\} \cap \bigcap_{m=1}^{n-1} \{Z_m^s \in D\}.$$

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Since $Z_0^s = x \in D$, by elementary time-homogeneous Markov chain properties,

$$\mathbb{P}(\eta_Z = n) = \mathbb{P}(Z_n^s \in D^c | \bigcap_{m=0}^{n-1} \{Z_m^s \in D\}) \mathbb{P}(\bigcap_{m=0}^{n-1} \{Z_m^s \in D\})$$

$$= \mathbb{P}(Z_n^s \in D^c | Z_{n-1}^s \in D) \prod_{m=1}^{n-1} \mathbb{P}(Z_m^s \in D | Z_{m-1}^s \in D)$$

$$\leq \epsilon^{n-1}. \tag{2.16}$$

This concludes that η is almost surely finite. Via (2.16), it is elementary to show that τ has finite first and second moments:

$$\mathbb{E}[\tau(s,x)] \leq T\mathbb{E}[\eta_Z(s,x)] = T\sum_{n=0}^{\infty} n\mathbb{P}(\eta_Z = n) \leq T\sum_{n=0}^{\infty} \frac{d}{d\epsilon}\epsilon^n = T\frac{d}{d\epsilon}\frac{1}{1-\epsilon} = \frac{T}{(1-\epsilon)^2} < \infty.$$

Similarly,

$$\mathbb{E}[\tau^2(s,x)] \leq T^2\mathbb{E}[\eta_Z^2(s,x)] = T^2\sum_{n=0}^{\infty} \left[\frac{d^2}{d\epsilon^2}\epsilon^{n+1} - \frac{d}{d\epsilon}\epsilon^n\right] = \frac{T^2(1+\epsilon)}{(1-\epsilon)^3} < \infty.$$

It follows that η has finite first and second moments. $\qquad\square$

Remark 2.2. Observe that Lemma 2.1 abstractly holds provided that P is irreducible and strong Feller. It is well-known that for autonomous SDEs, Hörmander's condition sufficiently implies the existence of a smooth density for P and therefore implies the strong Feller property. However, we note that Hörmander's condition is not sufficient for Lemma 2.1 to hold. Firstly, Hörmander's condition is insufficient to imply irreducibility, we refer readers to Remark 2.2 of [Hai11] for a counterexample. Secondly, Hörmander's condition was classically written for autonomous systems hence not directly applicable to the current non-autonomous case. Inclusive of the non-autonomous, it is well-known that density of P exists for uniformly elliptic diffusions with globally Hölder and bounded coefficients in $\mathbb{R}^d$ [FW98, SV06]. However, these conditions can be too restrictive for applications from the SDE perspective. On the other hand, Theorem 1 of [HLT17] extends Hörmander's condition to the case of non-autonomous SDEs with the conditions (2.11), (2.12) and (2.14). While these conditions require more smoothness than globally Hölder mentioned, the coefficients can be unbounded and is flexible enough for a wide range of SDEs.

Remark 2.3. It should be clear that Lemma 2.1 can be adapted to hold in the more general (not-necessarily T-periodic) non-autonomous case. Namely by picking any fixed $T > 0$, define $\epsilon_n := \sup_{x \in \overline{D}} P(s+(n-1)T, s+nT, x, D)$, then the same calculations via properties of the two-parameter Markov kernel yields $\mathbb{P}(\eta_Z = n) \leq \epsilon^n$, where $\epsilon := \max_{n \in \mathbb{N}} \epsilon_n$.

The Fokker-Planck equation is a well-known second-order linear parabolic PDE that describes the time evolution of the probability density function associated to SDEs [BKRS15, Ris96, Has12, Pav14, Gar09, Zwa01]. The existence and uniqueness of Fokker-Planck equation have been studied in many settings including irregular coefficients and time-dependent coefficients [LL08, BKRS15, RZ10, DR12]. In [FZZ19], it was shown that the periodic measure density is necessarily and sufficiently the time-periodic solution of the Fokker-Planck equation. Existence of time-periodic solution of the Fokker-Planck has been discussed in [Jun89, CHLY17, JQSY19].

To study the exit problem, we study the Fokker-Planck equation in the domain D and impose absorbing boundaries [Ris96, Gar09, Pav14]. Specifically, let $p_D(s,t,x,y)$ denote the probability

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density of the process starting at x at time s to y at time t that gets absorbed on ∂D. Then the density p_D satisfies the following Fokker-Planck equation

$$
\begin{cases}
\partial_t p_D(s,t,x,y) = L^*(t) p_D(s,t,x,y), & \\
p_D(s,s,x,y) = \delta_x(y), & x \in D, \\
p_D(s,t,x,y) = 0, & \text{if } y \in \partial D, t \geq s.
\end{cases}
\tag{2.17}
$$

Here $L^*(t)$ acts on forward variable y. To discuss the solvability of (2.17) and subsequent PDEs, we lay out typical PDE conditions that are weaker than conditions required by Lemma 2.1.

Condition A1: For some $\theta \in (0,1]$,

(i) Domain $D \in \mathcal{B}(\mathbb{R}^d)$ is non-empty and open with boundary $\partial D \in C^\theta(\mathbb{R}^{d-1})$.

(ii) The coefficients $a^{ij}, b^i \in C^{\frac{\theta}{2},\theta}(\bar{D}_T)$.

(iii) The matrix $a(s,x) = (a^{ij}(s,x))$ is uniformly elliptic i.e. there exists $\alpha > 0$ such that

$$
\langle a(s,x)\xi, \xi \rangle_{\mathbb{R}^d} \geq \alpha \|\xi\|_{\mathbb{R}^d}^2, \quad (s,x) \in D_T, \xi \in \mathbb{R}^d.
\tag{2.18}
$$

Particularly for adjoint operator $L^*(t)$ where more differentiability is required, we consider further

Condition A2: For some $\theta \in (0,1]$, Condition A1 holds and moreover $a^{ij}, b^i \in C^{1+\theta,2+\theta}(\bar{D}_T)$ and $\partial D \in C^{2+\theta}(\mathbb{R}^{d-1})$.

It is well-known that if Condition A2 holds, then there exists a unique solution $p_D(s,\cdot,x,\cdot) \in C^{1,2}(D_T)$ to (2.17). Moreover, $p_D(s,t,x,y)$ is jointly continuous in (x,y). For details, we refer readers to Section 7, Chapter 3 in [Fri64]. The following lemma and its proof are similar to the one presented in [Gar09, Pav14, Ris96] when the coefficients are time-independent. We prove for the time-dependent coefficients case. For clarity of the key ingredients of the following lemma, we assume η to have finite second moment rather than the conditions assumed in Lemma 2.1.

Lemma 2.4. *Assume that Condition A2 holds for SDE (2.4). Assume further that η has finite second moment. Then*

$$
\bar{\tau}(s,x) = \int_s^\infty \int_D p_D(s,t,x,y) dy dt,
\tag{2.19}
$$

where $p_D(s,\cdot,x,\cdot)$ is the unique solution to (2.17).

Proof. Let $G(s,t,x)$ be the probability that the process starting at x at time s is still within D at time $t \geq s$. In the derivation below, we treat (s,x) as fixed parameters so that G is only a function of t. By the absorbing boundary conditions of p_D, we have

$$
G(s,t,x) = \int_D p_D(s,t,x,y) dy.
\tag{2.20}
$$

On the other hand,

$$
G(s,t,x) = \mathbb{P}(\eta(s,x) > t) = 1 - \mathbb{P}(\eta(s,x) \leq t).
$$

Then, since p_D is t-differentiable, by (2.20), it is clear that a density $p_\eta(s,t,x)$ exists for $\eta(s,x)$ given by

$$
p_\eta(s,t,x) = -\partial_t G(s,t,x).
\tag{2.21}
$$

Note that if $x \in D$ then $G(s, s, x) = 1$. Note further that by Chebyshev's inequality,

$$G(s, t, x) = \mathbb{P}(\eta(s, x) > t) \leq \frac{1}{t^2} \mathbb{E}\left[\eta^2(s, x)\right], \quad t > s.$$

Since $G \geq 0$, it follows that $\lim_{t \to \infty} tG(s, t, x) = 0$, hence the following holds by an integration by parts

$$\begin{aligned}
\bar{\eta}(s, x) &= \int_s^\infty t p_\eta(s, t, x) dt \\
&= -\int_s^\infty t \partial_t G(s, t, x) dt \\
&= -tG(s, t, x)\big|_{t=s}^\infty + \int_s^\infty G(s, t, x) dt \\
&= s + \int_s^\infty G(s, t, x) dt.
\end{aligned}$$

Hence

$$\bar{\tau}(s, x) = \int_s^\infty G(s, t, x) dt. \tag{2.22}$$

The result follows by (2.20). $\qquad\square$

While finite first moment of η was not explicitly used in Lemma 2.4, we note that it is of course finite since it has finite second moment and applying Hölder's inequality. It is then obvious then that (2.22) is finite.

Let X^0 and X^1 be two random variables, we write $X^0 \sim X^1$ if they have the same distribution. Then we have the following intuitive lemma that was proved and presented in [FZZ19].

Lemma 2.5. *Let Condition A2 hold for T-periodic SDE (2.4). Let $\left(X_t^0\right)_{t \geq s}, \left(X_t^1\right)_{t \geq s+T}$ be two processes satisfying (2.4). If $X_s^0 \sim X_{s+T}^1$ then $X_{s+t}^0 \sim X_{s+T+t}^1$ for all $t \geq 0$.*

For T-periodic SDEs, we show in the next lemma, that the expected duration $\bar{\tau}$ is also T-periodic. While this holds in expectation, the same cannot be said of the sample-path realisations of τ. This is essentially because the noise realisation is not periodic! In the context of random dynamical systems, this can be proven rigorously. Indeed, if ω denotes the noise realisation and θ_t to be the Wiener shift, then one has $\tau(s, x, \omega) = \tau(s + T, x, \theta_T \omega)$, see [FZ16] for further details.

Lemma 2.6. *Assume that Condition A2 holds for T-periodic SDE (2.4). Assume further that η has finite second moment. Then $\bar{\tau}$ is also T-periodic.*

Proof. By Lemma 2.5 and (2.19), we have

$$\begin{aligned}
\bar{\tau}(s, x) &= \int_s^\infty \int_D p_D(s, r, x, y) dy dr \\
&= \int_s^\infty \int_D p_D(s + T, r + T, x, y) dy dr \\
&= \int_{s+T}^\infty \int_D p_D(s + T, r, x, y) dy dr \\
&= \bar{\tau}(s + T, x).
\end{aligned}$$

$\qquad\square$

For the following theorem, we recall Kolmogorov's backward equation

$$\partial_s p(s,t,x,y) + L(s)p(s,t,x,y) = 0, \tag{2.23}$$

where $L(s)$ acts on x variable.

We are now ready to derive the PDE in which $\bar{\tau}(s,x)$ satisfies. When the SDE is T-periodic, we show $\bar{\tau}(s,x)$ is the T-periodic solution of a second-order linear parabolic PDE. This contrasts with the autonomous case where the expected exit time satisfies the second-order linear elliptic PDE (2.7). To our knowledge the derived PDE and particularly its interpretation is new in literature. We note further that the following theorem establishes a Feynman-Kac duality for time-periodic SDEs for the expected duration.

Theorem 2.7. *Assume T-periodic SDE (2.4) satisfies the same conditions as Lemma 2.1. Then the expected duration $\bar{\tau}$ is the periodic solution of the following partial differential equation of backward type*

$$\begin{cases} \partial_s u(s,x) + L(s)u(s,x) = -1, & \text{in } D_T, \\ u = 0, & \text{on } [0,T] \times \partial D, \\ u(0,\cdot) = u(T,\cdot). & \text{on } D. \end{cases} \tag{2.24}$$

Proof. By Lemma 2.1, η has finite second moment and Condition A2 holds. Hence Lemma 2.4 holds. Thus, by (2.22), observe that for any $\delta > 0$,

$$\bar{\tau}(s+\delta,x) - \bar{\tau}(s,x) = \int_{s+\delta}^{\infty} \left(G(s+\delta,t,x) - G(s,t,x) \right) dt - \mathcal{G}(s+\delta),$$

where for clarity, $\mathcal{G}(r) := \int_s^r G(s,t,x)dt$. It follows by the fundamental theorem of calculus that

$$\begin{aligned} \partial_s \bar{\tau}(s,x) &= \int_s^{\infty} \partial_s G(s,t,x)dt - \mathcal{G}'(s) \\ &= \int_s^{\infty} \int_D \partial_s p_D(s,t,x,y)dydt - G(s,s,x) \\ &= \int_s^{\infty} \int_D \partial_s p_D(s,t,x,y)dydt - 1, \end{aligned}$$

where recall that G is expressed by (2.20) and $G(s,s,x) = 1$ since $x \in D$. Acting $L(s)$ on $\bar{\tau}$ by (2.19) and (2.23), we have

$$L(s)\bar{\tau}(s,x) = \int_s^{\infty} \int_D L(s)p_D(s,t,x,y)dydt = -\int_s^{\infty} \int_D \partial_s p_D(s,t,x,y)dydt.$$

Summing these quantities yields

$$(\partial_s + L(s))\bar{\tau}(s,x) = -1. \tag{2.25}$$

For T-periodic systems, Lemma 2.6 showed that $\bar{\tau}(s,\cdot) = \bar{\tau}(s+T,\cdot)$ for all $s \in \mathbb{R}^+$ hence deducing $\bar{\tau}$ satisfies (2.24) and u is T-periodic. By Lemma 2.6, this is sufficient by imposing $u(0,\cdot) = u(T,\cdot)$ and the result follows. $\square$

Remark 2.8. In the proof of Theorem 2.7, note that T-periodicity was not assumed until (2.25). This suggests that for general non-autonomous (not necessarily periodic) SDEs, $\bar{\tau}$ satisfies (2.25). However, as (2.25) is a parabolic PDE, in the absence of initial (or terminal) conditions, PDE

(2.25) alone is generally ill-posed. Indeed, in the general non-autonomous non-periodic case, both the time and the initial/terminal condition is part of the unknown. Indeed, if $\bar{\tau}(0, \cdot)$ is known, then this implies we already know the expected exit time when the system starts at time $s = 0$, however it is an unknown to-be solved. Resultantly, in some works such as [BDE01], it was assumed that for sufficiently large $\tilde{T}$ that $u(x, y, \tilde{T}) = 0$ for all $(x, y) \in D$. This is to impose some terminal condition of the parabolic PDE. This however suggests that for all $(x, y) \in D$, the expected exit time is a constant time $\tilde{T}$. While this assumption may hold in certain systems, it does not hold in general. For example, in the stochastic resonance problem that we shall discuss in Section 4.2 with Figure 2, (approximately) constant expected exit time only holds for a subdomain rather than across the entire domain. In fact, in these papers, one usually chooses $\tilde{T} > \sup \tau(x, y, t)$ by repeatedly running SDE simulations to find a sufficiently large $\tilde{T}$ greater than sample expected exit times. The results of this paper show that one can avoid simulations altogether and only solve the periodic solution of the PDE to attain the expected exit time. For time-periodic SDEs, the boundary conditions issue are partially resolved by Lemma 2.6, where initial and terminal conditions coincide, albeit unknown.

Remark 2.9. It should be clear that for coefficients with non-trivial time-dependence, the parabolic PDE (2.24) would generally imply that $\bar{\tau}(s, x) - \bar{\tau}(s', x) \neq (s - s')$ for $s \neq s'$. That is, the difference in initial starting time does not imply the same difference in expected time. This reinforce that initial time generally plays a non-trivial role in the expected duration.

As mentioned in the introduction, numerically solving PDE (2.24) can be an appealing alternative to stochastic simulations of the expected hitting time. We note further that solving (2.24) solves the expected hitting time for all initial starting points. On the other hand, direct simulation would (naively) require many simulations for each starting point.

Assuming a priori that the expected exit time is finite, then we can prove a converse of Theorem 2.7 via Dynkin's formula. In passing, this reassures that Theorem 2.7 is correct.

Proposition 2.10. *Assume η associated to T-periodic (2.4) has finite expectation. Then if (2.24) has a solution u. Then $u(s, x) = \bar{\tau}(s, x)$.*

Proof. Since η is a stopping time and has finite expectation, by Itô's and Dynkin's formula, then

$$\mathbb{E}^{s,x}\left[\varphi(\eta, X_\eta)\right] = \varphi(s, x) + \mathbb{E}^{s,x}\left[\int_s^\eta (\partial_t + L(t))\varphi(t, X_t)dt\right], \quad \varphi \in C_0^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d).$$

Remark 2.8 implies that there does not generally exist a $u \in C^{1,2}(D)$ such that $(\partial_s + L(s))u(s, x) = -1$ and vanishes on ∂D until we impose T-periodicity of u. Therefore if such a u exists, we have by (2.2)

$$0 = \mathbb{E}^{s,x}\left[u(\eta, X_\eta)\right] = u(s, x) + \mathbb{E}^{s,x}\left[\int_s^\eta -1dt\right] = u(s, x) - \bar{\tau}(s, x).$$

i.e. $u(s, x) = \bar{\tau}(s, x)$ and so the results follows. □

3 Well-Posedness of Expected Duration PDE

3.1 Fixed Point of an Initial Value Problem

In this section, utilising classical results for the well-posedness of initial value parabolic PDEs, we will show the existence of a unique solution to the expected duration PDE (2.24) for the associated T-periodic SDE. As mentioned in the introduction, we solve (2.24) with typical PDE conditions

rather than the stronger SDE conditions required for the rigorous derivation of the PDE. This has the advantage of a clearer exposition and key elements to solve the PDE.

In this subsection, we associate (2.24) with an initial value boundary PDE problem and show that (2.24) can be rewritten as a fixed point problem. We note however that (2.24), as an initial value problem, is a backward parabolic equation. Such equations are known to be generally ill-posed in typical PDE spaces. By reversing the time, we introduce a minus sign thus PDE is uniformly elliptic and hence more readily solvable in typical function spaces.

We give a general uniqueness and existence result via a spectral result of [Hes91] in $L^p(D)$. Specifically on $L^2(D)$, we show that if the associated bilinear form is coercive then one can apply a Banach fixed point argument to deduce the existence and uniqueness. This yields a practical way to numerically compute the desired solution.

To discuss the well-posedness of (2.24), we recall some standard Borel measurable function spaces. For any $1 \leq p < \infty$, we denote the Banach space $L^p(D)$ to be the space of functions $f : D \to \mathbb{R}$ such that its norm $\|f\|_{L^p(D)} := \left(\int_D |f(x)|^p dx\right)^{1/p} < \infty$. For $k \in \mathbb{N}$, we define as usual the Sobolev space $W^{k,p}(D)$ to contain all functions f in which its norm $\|f\|_{W^{k,p}(D)} := \left(\sum_{|\beta| \leq k} \|\partial^\beta f\|_{L^p(D)}^p\right)^{1/p} < \infty$. We let $W_0^{k,p}(D) = \{f \in W^{k,p}(D) | f = 0 \quad \text{on } \partial D\}$. For $p = 2$, $L^2(D)$ and $H_0^k(D) := W_0^{k,2}(D)$ are Hilbert spaces with inner-product $\langle f, g \rangle_{L^2(D)} := \left(\int_D f(x)g(x)dx\right)^{1/2}$ and $\langle f, g \rangle_{H_0^k(D)} := \sum_{|\beta| \leq k} \sum_{i=1}^d \langle \partial^\beta f, \partial^\beta g \rangle_{L^2(D)}$ respectively. Occasionally, we let $(H, \|\cdot\|_H)$ denote a generic Hilbert space. To avoid any possible confusion, we will be verbose with the norms and inner-products.

We begin by fixing $1 < p < \infty$ and define the time-reversed uniformly elliptic operator associated to (2.9) by

$$L_R(s) := L(T - s) = \sum_{i=1}^d b^i(T - s, x)\partial_i + \frac{1}{2} \sum_{i,j=1}^d a^{ij}(T - s, x)\partial_{ij}^2, \quad s \in [0, T]. \tag{3.1}$$

Note that $\mathcal{D}(L_R(s)) = W^{2,p}(D) \cap W_0^{1,p}(D) \subset L^p(D)$ for all $s \in [0, T]$. As mentioned, the initial boundary value problem (IBVP) associated to (2.24) is a backward hence ill-posed in $L^p(D)$. Suppose that u satisfies (2.24), consider the the time-reversed solution $v(s, x) = u(T - s, x)$. Then v satisfies

$$\begin{cases} \partial_s v - L_R(s)v = f, & \text{in } D_T, \\ v = 0 & \text{on } [0, T] \times \partial D, \\ v(0, \cdot) = v(T, \cdot), \end{cases} \tag{3.2}$$

where $f \equiv 1$. Clearly the solvability of (2.24) is equivalent to (3.2) up to time-reversal. Hence, for the rest of the paper we focus on showing existence and uniqueness of a solution to (3.2).

Due to the general applicability of the methods presented in this section, where possible, we retain a general inhomogeneous function $f : [0, T] \to L^p(D)$. We expect that this generality benefits some readers for solving similar problems.

The following IBVP associated to (3.2),

$$\begin{cases} (\partial_s - L_R(s))v = f, & \text{in } D_T, \\ v = 0, & \text{on } [0,T] \times \partial D, \\ v(0, \cdot) = v_0 & \text{on } D, \end{cases} \tag{3.3}$$

is a "forward" parabolic equation and is readily solvable. We say that v is a generalised solution of (3.3) if $v \in C([0, T], W^{2,p}(D) \cap W_0^{1,p}(D))$, its derivative $\frac{\partial v}{\partial s} \in C((0, T), L^p(D))$ exists and v satisfies

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(3.3) in $L^p(D)$ [Paz92, Ama95, DM92]. Consider also $\phi(s, x)$ satisfying the homogeneous PDE of (3.3) i.e.

$$
\begin{cases}
(\partial_s - L_R(s))\phi = 0, & \text{in } (r, T) \times D, \\
\phi = 0 & \text{on } [r, T] \times \partial D, \\
\phi(r, \cdot) = \phi_r, & \text{in } D.
\end{cases}
\tag{3.4}
$$

Given $\phi_r \in L^p(D)$, (3.4) is well-posed, we can define the evolution operator

$$
\Phi(r, s) : L^p(D) \to W^{2,p}(D) \cap W_0^{1,p}(D), \quad r \le s \le T,
\tag{3.5}
$$

by

$$
\Phi(r, s)\phi_r := \phi(s).
\tag{3.6}
$$

It is known that $\Phi(s, r)$ satisfies the semigroup property $\Phi(r, r) = Id$ and $\Phi(r, s) = \Phi(s, t)\Phi(r, s)$ for $r \le s \le t$. We refer readers to [Paz92] for regularity properties of Φ. When (3.3) is well-posed, it is well-known that by a variation of constants or Duhamel's formula [Ama95, DM92, Paz92], the solution to inhomogeneous problem (3.3) satisfies

$$
v(s) = \Phi(r, s)v_r + \int_r^s \Phi(r', s)f(r')dr'.
\tag{3.7}
$$

It is well-known that if Condition A1 holds and $f \in C^\gamma(0, T; L^p(D))$ for some $\gamma \in (0, 1)$, then (3.3) is well-posed [Paz92, Ama95]. Furthermore, we can define the solution operator after one period $\mathcal{A} : L^p(D) \to W^{2,p}(D) \cap W_0^{1,p}(D)$ by

$$
\mathcal{A}\varphi := \Phi(0, T)\varphi + \int_0^T \Phi(r, T)f(r)dr.
\tag{3.8}
$$

We discuss further conditions for regular solutions. Theorem 24.2 of [DM92] employed Schauder estimates and Sobolev embedding to show that if $p > d/2$, $\partial D \in C^2(\mathbb{R}^{d-1})$ then the solution to IBVP (3.3) with initial condition $v_0 \in W_0^{2,p}(D)$ satisfies the following regularity

$$
v \in C(\bar{D}_T) \cap C^{1+\frac{\theta}{2}, 2+\theta}((0, T] \times \bar{D}).
\tag{3.9}
$$

Furthermore, if $d < p < \infty$, by Sobolev embedding, then $v \in C^{\frac{1+\xi}{2}, 1+\xi}(\bar{D}_T) \cap C^{1+\frac{\theta}{2}, 2+\theta}((0, T] \times \bar{D})$ for some $\xi \in (0, 1)$, see [Hes91]. Thus, we write our first existence and uniqueness result.

Proposition 3.1. *Assume Condition A1 holds. Assume that $d < p < \infty$, $\partial D \in C^2(\mathbb{R}^{d-1})$ and $f \in C^\gamma(0, T; L^p(D))$ for some $\gamma \in (0, 1)$. Then there exists a unique regular solution satisfying (3.2). Moreover, if $f \ne 0$ then the solution is non-trivial.*

Proof. Since $f \in C^\gamma(0, T; L^p(D))$, by Condition A1, IBVP (3.3) is well-posed for any $v_0 \in L^p(D)$. Hence the evolution operator Φ defined by (3.6) is well-defined. In general, to solve T-periodic PDE (3.2), by Duhamel's formula (3.7), one wishes to find existence and uniqueness of a $v_0 \in L^p(D)$ such that

$$
v_0 = \mathcal{A}v_0.
\tag{3.10}
$$

For initial conditions in $W_0^{2,p}(D)$, by rearranging from (3.8), we have

$$
(I - \Phi(0, T))v_0 = \int_0^T \Phi(r, T)f(r)dr,
\tag{3.11}
$$

where $\Phi(0, s) : W_0^{2,p}(D) \to W_0^{2,p}(D)$ and $I : W_0^{2,p}(D) \to W_0^{2,p}(D)$ is the identity operator. With the current conditions, via Krein-Rutman theorem, it was shown in [Hes91] that $\lambda = \rho(\Phi(0, T)) \in (0, 1)$, where λ denotes the spectral radius of $\Phi(0, T)$. This implies that 1 is in the resolvent i.e. $(I - \Phi(0, T)) : W_0^{2,p}(D) \to W_0^{2,p}(D)$ is invertible. It follows that

$$v_0 = (I - \Phi(0, T))^{-1} \int_0^T \Phi(r, T) f(r) dr, \tag{3.12}$$

uniquely solves (3.10). By Sobolev embedding,

$$v(s, \cdot) = \Phi(0, s) v_0 + \int_0^s \Phi(r, s) f(r) dr, \quad s \in (0, T], \tag{3.13}$$

is a regular solution to (3.2). It is easy to see that (3.2) does not admit trivial solutions since (D is non-empty and) $v \equiv 0$ cannot satisfy (3.2) for $f \neq 0$. $\qquad\square$

As noted in [DM92], via the semigroup property, one can approximate $\Phi(0, T) \simeq \prod_{n=0}^{N-1} \Phi(t_n, t_{n+1})$ for $0 = t_0 < t_1 < ... < t_N = T$. Hence one can approximate the inverse in (3.12) by

$$(I - \Phi(0, T))^{-1} \simeq (I - \prod_{n=0}^{N-1} \Phi(t_n, t_{n+1}))^{-1}. \tag{3.14}$$

We note however computing (3.14) is generally computationally expensive.

We can gain more from (3.12). We recall the weak maximum principle: if the solution is regular and $f \geq 0$, then

$$\min_{(s,x)\in\bar{D}_T} v(s, x) = \min_{x\in D} v_r(x) \tag{3.15}$$

holds. We have seen that, by (3.11), the existence and uniqueness of $v_0 \in L^2(D)$ satisfying (3.10) requires the invertibility of $I - \Phi(0, T)$. By von Neumann series, we have

$$(I - \Phi(0, T))^{-1} = \sum_{k=0}^{\infty} \Phi^k(0, T),$$

where $\Phi^k(0, T)$ denotes the composition of the operator $\Phi(0, T)$.

It is well-known that parabolic PDEs experience parabolic smoothing (see e.g. [Paz92, Eva10]) i.e. the solution of parabolic equations are as smooth as the coefficients and initial data. For example if $p > d/2$ and $f \in C^\gamma(0, T; W^{2,p}(D))$, then $\Phi(s, t)f$ is a regular solution by (3.9). Moreover, if $f \geq 0$, by the maximum principle, $\Phi(s, t)f \geq 0$ for all $0 \leq s \leq t \leq T$. It follows that $\mathcal{I} := \int_0^T \Phi(r, T) f(r) dr \geq 0$ and $\Phi^k(0, T)\mathcal{I} \geq 0$ for all $k \in \mathbb{N}$. Moreover, it follows from (3.12) that

$$v_0 = \sum_{k=0}^{\infty} \Phi^k(0, T)\mathcal{I} \geq 0, \tag{3.16}$$

i.e. the solution to (3.10) is non-negative. Furthermore, if the coefficients and f are smooth then condition $p > d/2$ can be dropped and the same conclusion holds with a smooth solution [Paz92]. In particular, since $1 \in C^\infty((0, T) \times D)$ is non-negative, this aligns with physical reality that expected duration time $\bar{\tau}(0, \cdot) = v_0$ indeed is non-negative.

To gain further insight into solving (3.2) from both a theoretically and computational viewpoint, we progress our study with Hilbert spaces i.e. $p = 2$ and forego some of the regularity gained from Sobolev embedding e.g. (3.9). The following approach allows us to study (3.2).

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We start with a standard framework to deduce the existence and uniqueness of (3.2) on the Hilbert space $L^2(D)$. For convenience, we define the bilinear form $B_R : H_0^1(D) \times H_0^1(D) \to \mathbb{R}$ associated to $-L_R$ defined by

$$B_R[\varphi, \psi; s] = -\sum_{i=1}^{d} \int_D \tilde{b}^i(T-s, x)\partial_i\varphi(x)\psi(x)dx + \frac{1}{2}\sum_{i,j=1}^{d} \int_D a^{ij}(T-s, x)\partial_i\varphi(x)\partial_j\psi(x)dx, \quad (3.17)$$

where $\tilde{b}^i(s, x) = b^i(s, x) + \sum_{j=1}^{d} \partial_j a^{ij}(s, x)$ for each $1 \le i \le d$. We recall that a bilinear form $B_R : H_0^1(D) \times H_0^1(D) \to \mathbb{R}$ is coercive if there exists a constant $\alpha > 0$ such that

$$B_R[\varphi, \varphi; s] \ge \alpha \|\varphi\|_{H_0^1(D)}^2, \quad \varphi \in H_0^1(D), s \in [0, T]. \tag{3.18}$$

Assuming coercivity, we give the following existence and uniqueness theorem to (3.2).

Theorem 3.2. *Assume that $a^{ij}, b^i \in L^\infty(D_T)$ and $a(\cdot, \cdot)$ satisfies uniformly elliptic condition (2.18) and furthermore (3.17) is coercive for $s \in [0, T]$. Then for any $f \in L^2(0, T; L^2(D))$, there exists a unique solution $v \in C([0, T], H_0^1(D))$ to (3.2). If $f \neq 0$, then the solution is non-trivial.*

Proof. It is well-known (e.g. [Eva10]) that there exists a unique weak solution v to the IBVP (3.3) i.e. $v \in C([r, T]; L^2(D)) \cap L^2(r, T; H_0^1(D))$ such that $v(r) = v_r$, $\partial_s v \in L^2(r, T; H^{-1}(D))$ and for almost every $s \in [r, T]$,

$$\langle \partial_s v(s), \varphi \rangle_{H^{-1}(D) \times H_0^1(D)} + B_R[v, \varphi; s] = \langle f(s), \varphi \rangle_{H^{-1}(D) \times H_0^1(D)}, \quad \varphi \in H_0^1(D), \tag{3.19}$$

where $H^{-1}(D)$ is the space of linear functionals of the subspace $H_0^1(D)$ on $L^2(D)$ and $\langle \cdot, \cdot \rangle_{H^{-1}(D) \times H_0^1(D)} : H^{-1}(D) \times H_0^1(D) \to \mathbb{R}$ denotes the duality pairing between $H^{-1}(D)$ and $H_0^1(D)$. To prove our result, it is sufficient to assume $f \in L^2(D)$. To cast (3.10) in terms of a self-mapping, consider $\bar{\Phi}(0, T) : L^2(D) \to L^2(D)$ as the operator $\Phi(0, T)$ with its range enlarged to $L^2(D)$ and define $\bar{\mathcal{A}} : L^2(D) \to L^2(D)$ by

$$\bar{\mathcal{A}}\varphi := \bar{\Phi}(0, T)\varphi + \int_0^T \bar{\Phi}(r, T)f(r)dr. \tag{3.20}$$

We show there exists a unique fixed point of operator $\bar{\mathcal{A}}$. By Banach fixed point theorem, it suffices to show $\bar{\mathcal{A}}$ is a contraction on $L^2(D)$. Observe that this is sufficient provided $\bar{\Phi}(0, T)$ is a contraction mapping on $L^2(D)$ since

$$\left\| \bar{\mathcal{A}}\varphi - \bar{\mathcal{A}}\psi \right\|_{L^2(D)} = \left\| \bar{\Phi}(0, T)(\varphi - \psi) \right\|_{L^2(D)} \le \left\| \bar{\Phi}(0, T) \right\| \|\varphi - \psi\|_{L^2(D)}, \quad \varphi, \psi \in L^2(D).$$

In fact, we show that $\bar{\Phi}(0, s)$ is a contraction for any $s > 0$.

From (3.5), for initial condition $\phi_0 \in L^2(D)$, the homogeneous solution, $\phi(s) \in H^2(D) \cap H_0^1(D)$ satisfies (3.4). Then from (3.19), one has by coercivity

$$0 = \langle \partial_s \phi(s), \phi(s) \rangle + B_R[\phi(s), \phi(s); s]$$

$$\ge \frac{1}{2}\frac{d}{ds}\|\phi(s)\|_{L^2(D)}^2 + \alpha \|\phi(s)\|_{H_0^1(D)}^2$$

$$\ge \frac{1}{2}\frac{d}{ds}\|\phi(s)\|_{L^2(D)}^2 + \alpha \|\phi(s)\|_{L^2(D)}^2.$$

Gronwall's inequality then yields

$$\|\phi(s)\|_{L^2(D)}^2 \le e^{-2\alpha s} \|\phi_0\|_{L^2(D)}^2, \quad s \ge 0.$$

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Hence indeed

$$\left\|\bar{\Phi}(0,s)\right\| := \sup_{\phi_0 \in L^2(D)} \frac{\left\|\bar{\Phi}(0,s)\phi_0\right\|_{L^2(D)}}{\|\phi_0\|_{L^2(D)}} \leq e^{-\alpha s} < 1, \quad s > 0, \tag{3.21}$$

i.e. $\bar{\Phi}(0,s)$ is a contraction on $L^2(D)$. Therefore there exists a unique $v_0 \in L^2(D)$ satisfying (3.20). Since $\mathcal{A} : L^2(D) \to H_0^1(D) \subsetneq L^2(D)$, then by the right hand side of (3.10), it is easy to deduce that $v_0 \in H_0^1(D)$. Define v by (3.13), then $v \in C([0,T], H_0^1(D))$ is the unique solution to (3.2). Lastly, if $0 \neq f \in L^2(D)$, then v is non-trivial. $\qquad\square$

Theorem 3.2 offers not only a theoretical existence and uniqueness result on the solution to (3.2), by Banach fixed point, Theorem 3.2 immediately offers an iterative numerical approach to the solution. To numerically computing the next Banach fixed point iterate, one only requires to solve a IBVP for the parabolic PDE. Compared to (3.14), there are well-established numerical schemes for parabolic PDEs with known order of convergences.

We remark that coercivity is actually stronger than required. In the proof of Theorem 3.2, it is sufficient that $B[\varphi, \varphi; s] \geq \alpha \|\varphi\|_{L^2(D)}^2$. We give an example where coercivity is shown. We consider the example of a one-dimensional Brownian motion with periodic drift.

Example 3.3. Let $S \in C^1(\mathbb{R}^+)$ be a T-periodic function and $\sigma \neq 0$ and consider the one-dimensional T-periodic SDE

$$dX_t = S(t)dt + \sigma dW_t,$$

on some bounded interval D. Clearly Condition A2 is satisfied. By Theorem 3.2, it is sufficient to show the associated (time-reversed) bilinear form

$$B_R[\varphi, \psi; s] = -\int_D S(T-s)\partial_x\varphi(x)\psi(x)dx + \frac{\sigma^2}{2}\int_D \partial_x\varphi(x)\partial_x\psi(x)dx, \quad \varphi, \psi \in H_0^1(D),$$

is coercive. This is obvious by an integration by parts with vanishing boundaries and applying the Poincaré inequality

$$\begin{aligned}
B_R[\varphi, \varphi; s] &= -\frac{S(T-s)}{2}\int_D \partial_x(\varphi^2(x))dx + \frac{\sigma^2}{2}\|\partial_x\varphi\|_{L^2(D)}^2 \\
&\geq -\frac{S(T-s)}{2}\varphi^2(x)|_{\partial D} + \frac{\sigma^2}{4}\|\partial_x\varphi\|_{L^2(D)}^2 + \frac{\sigma^2}{4C_D}\|\varphi\|_{L^2(D)}^2 \\
&\geq \alpha\|\varphi\|_{H_0^1(D)}^2,
\end{aligned}$$

where C_D denotes the Poincaré constant for the domain D such that $\|\varphi\|_{L^2(D)}^2 \leq C_D \|\partial_x\varphi\|_{L^2(D)}^2$ and $\alpha = \min(\frac{\sigma^2}{4}, \frac{\sigma^2}{4C_D}) > 0$. Hence by Theorem 3.2, there exists a unique solution to (2.24).

3.2 Convex Optimisation

In Section 3.1, we showed that if the bilinear form associated to the PDE is coercive, then Theorem 3.2 yields a unique solution to (3.2). However, in general, coercivity of the associated bilinear form can be difficult to verify. Instead, we now seek to solve (3.2) by casting it as a convex optimisation problem with a natural cost functional. Convex optimisation has been a standard method to study solutions of elliptic PDEs.

In the presented convex optimisation framework, we show that the unique minimiser of the cost functional is a solution to (3.2). In this approach, coercivity of the functional holds almost

immediately, provided the maximum principle holds. When the maximum principle holds, it follows that the solution is a classical/strong solution as opposed to the weak solution given in Theorem 3.2.Furthermore, we show that the convex optimisation problem can be implemented readily by standard gradient methods.

We begin with a standard convex optimisation result on Hilbert spaces. Let $(H, \|\cdot\|_H)$ be a Hilbert space, $\mathscr{C} \subseteq H$ be a closed convex subset and $F : H \to \mathbb{R}$ be a functional. The functional F is said to be norm-like (or coercive) over $\mathscr{C}$ if

$$F(\varphi) \to \infty, \quad \text{as } \|\varphi\|_H \to \infty, \quad \varphi \in \mathscr{C}.$$

The functional F is Gateaux differentiable at $\varphi \in H$ if for any $\phi_0 \in H$, the directional derivative of F at φ in the direction ϕ_0, denoted by $DF(\varphi)(\phi_0)$, given by

$$DF(\varphi)(\phi_0) = \lim_{\epsilon \to 0} \frac{F(\varphi + \epsilon\phi_0) - F(\varphi)}{\epsilon} \tag{3.22}$$

exists. By definition, the gradient of F is $\frac{\delta F}{\delta \varphi} \in L^2(D)$ satisfying

$$\langle \frac{\delta F}{\delta \varphi}, \phi_0 \rangle = DF(\varphi)(\phi_0), \quad \phi_0 \in H. \tag{3.23}$$

We shall use the following standard result from convex optimisation theory (see e.g. [ET99, Tro10]).

Lemma 3.4. *Let H be a Hilbert space and $\mathscr{C} \subseteq H$ be a closed convex set. Let $F : H \to \mathbb{R}$ be a functional such that F is convex and norm-like over $\mathscr{C}$. Assume further that F is a (lower semi)continuous and bounded from below. Then there exists at least one $v_0 \in \mathscr{C}$ such that $F(v_0) = \inf_{\varphi \in \mathscr{C}} F(\varphi)$. If F is Gateaux differentiable, then for any such v_0, $DF(v_0)(\cdot) = 0$. If F is strictly convex then v_0 is unique.*

We now focus specifically on using Lemma 3.4 to solve (3.2). Recall that if Condition A1 holds then (3.3) is well-posed. We then associate to (3.3) the natural cost functional $F : L^2(D) \to \mathbb{R}$ defined by

$$F(\varphi) = \frac{1}{2}\|\mathcal{A}\varphi - \varphi\|_{L^2(D)}^2 = \frac{1}{2}\int_D [(\mathcal{A}\varphi)(x) - \varphi(x)]^2 dx, \tag{3.24}$$

where $\mathcal{A}$ is given by (3.8). This functional is natural to our periodic problem because if there exists $v_0 \in L^2(D)$ which minimises the functional to zero, it is a solution to (3.2) i.e.

$$F(v_0) = 0 \quad \Longleftrightarrow \quad \mathcal{A}v_0 = v_0,$$

i.e. v_0 solves (3.10) and therefore is a (possibly weak) solution to (3.2).

Optimisation briefly aside, we recommend using the cost functional F to quantify the convergence of the Banach iterates of Theorem 3.2.

In order to apply Lemma 3.4 on F, we recall some properties associated to linear parabolic PDEs. Suppose that (3.3) is well-posed. Since PDE (3.3) is linear, by the superposition principle, $\Phi(\cdot, \cdot)$ is a linear operator, i.e. $\Phi(s,t)[\lambda_1\phi_1 + \lambda_2\phi_2] = \lambda_1\Phi(s,t)\phi_1 + \lambda_2\Phi(s,t)\phi_2$. However, due to the inhomogeneous term, observe that $\mathcal{A}$ is not linear. Instead, if $\lambda_1, \lambda_2 \geq 0$ such that $\lambda_1 + \lambda_2 = 1$ then

$$\mathcal{A}(\lambda_1\varphi + \lambda_2\psi) = \Phi(0,T)[\lambda_1\varphi + \lambda_2\psi] + (\lambda_1 + \lambda_2)\int_0^T \Phi(r,T)f(r)dr = \lambda_1\mathcal{A}\varphi + \lambda_2\mathcal{A}\psi. \tag{3.25}$$

Since $\bar{\tau}$ is non-negative, we consider

$$\mathscr{C}(D) := \{\varphi \in H_0^2(D) | \varphi \geq 0\}.$$

It is easy to verify that $\mathscr{C}(D)$ is a closed convex Hilbert subspace of $L^2(D)$.

Theorem 3.5. *Let Condition A1 hold, $f \in C^\gamma(0, T; L^2(D))$ for some $\gamma \in (0, 1)$, $f \geq 0$ and $d \leq 3$. Let $F : \mathscr{C}(D) \subset L^2(D) \to \mathbb{R}$ be defined by (3.24). Then there exists a unique $v_0 \in \mathscr{C}(D)$ minimising F.*

Proof. Since Condition A1 holds, the IBVP (3.3) is well-posed. Hence the operators $\mathcal{A}$ and $\Phi(s, t)$ and thus F are all well-defined. We show that F satisfies the assumptions of Lemma 3.4. Obviously $F \geq 0$ and hence bounded from below. By the well-posedness of (3.3), it is clear that $\varphi \to \mathcal{A}\varphi$ and moreover $\varphi \to \mathcal{A}\varphi - \varphi$ are continuous from $L^2(D)$ to $L^2(D)$. It follows that F is continuous. Utilising the strong convexity of the quadratic function and (3.25), we show the strong convexity of F: for any $\lambda \in (0, 1)$ and $\varphi, \psi \in \mathscr{C}(D)$, we have that

$$
\begin{aligned}
F(\lambda\varphi + (1-\lambda)\psi) &= \frac{1}{2} \int_D [\mathcal{A}(\lambda\varphi + (1-\lambda)\psi) - (\lambda\varphi + (1-\lambda)\psi)]^2 dx \\
&= \frac{1}{2} \int_D [\lambda(\mathcal{A}\varphi - \varphi) + (1-\lambda)(\mathcal{A}\psi - \psi)]^2 dx \\
&< \frac{1}{2} \int_D \left[\lambda(\mathcal{A}\varphi - \varphi)^2 + (1-\lambda)(\mathcal{A}\psi - \psi)^2 \right] dx \\
&= F(\varphi) + (1-\lambda)F(\psi).
\end{aligned}
$$

Since $d \leq 3$, by Sobolev embedding and Schauder estimates, it follows from (3.9) (as $\frac{d}{2} < p = 2$) that for any $\varphi \in \mathscr{C}(D) \subset H^2(D)$ and $f \geq 0$, the solution to (3.3) with initial condition φ is regular. Therefore, the maximum principle (3.15) applies. Hence together with the homogeneous Dirichlet boundary conditions, it follows that $\mathcal{A}\varphi \geq 0$. Therefore, for any $x \in D$ and $\epsilon \in (0, 1)$, Young's inequality yields that

$$
\begin{aligned}
F(\varphi) &= \frac{1}{2} \int_D ((\mathcal{A}\varphi)^2 - 2(\mathcal{A}\varphi)\varphi + \varphi^2) dx \\
&\geq \frac{1}{2} \int_D ((1-\epsilon)\varphi^2 + (1-\epsilon^{-1})(\mathcal{A}\varphi)^2) dx \\
&= \frac{1-\epsilon}{2} \|\varphi\|_{L^2(D)}^2 + \frac{1-\epsilon^{-1}}{2} \|\mathcal{A}\varphi\|_{L^2(D)}^2.
\end{aligned}
$$

Hence it follows that $F(\varphi) \to \infty$ as $\|\varphi\|_{L^2(D)} \to \infty$. Then Lemma 3.4 yields a unique solution $v_0 \in \mathscr{C}(D)$ minimising F. $\qquad\square$

In the following proposition, we derive an expression for the directional derivative $DF(\varphi)(\phi_0)$. While it is then straightforward to apply the maximum principle to show that $DF(\varphi)(\cdot)$ is a linear continuous operator to deduce existence and uniqueness of the gradient (via Riesz representation theorem), we employ numerical analysts' adjoint state method (see e.g. [GP00, CLPS03, SFP14, Ple06]) to attain an expression for the gradient directly. From a numerical perspective, the gradient allows us to apply gradient methods to iteratively minimise F. Numerically, we note that it is not necessary to use adjoint state method to compute the gradient. However, it is well-known that adjoint state method is (generally) computationally efficient see e.g. [SFP14]. It is noted that comparing to Banach fixed point iterates of Theorem 3.2, the adjoint state method is computationally less efficient because a pair of IBVP is required to be solved rather than one.

To employ the adjoint state method, we recall that $L^*(s)$ is the Fokker-Planck operator given by (2.10). Akin to (2.17), if Condition A2 holds, then

$$
\begin{cases}
\partial_s w = L^*(s)w & \text{in } D_T, \\
w = 0 & \text{on } [0, T] \times \partial D, \\
w(0, \cdot) = w_0, & \text{on } D,
\end{cases}
\tag{3.26}
$$

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is well-posed for any $w_0 \in L^2(D)$[Paz92, Eva10, Ama95]. Hence, akin to (3.4) and (3.5), we define the evolution operator W for (3.26) i.e.

$$w(s, \cdot) = W(0, s)w_0, \quad s \geq 0, \tag{3.27}$$

where $W(0, s) : L^2(D) \to H^2(D) \cap H_0^1(D)$.

The following proposition was inspired by [AW10, BGP98] in employing the adjoint state method for periodic solutions of the Benjamin-Ono and autonomous evolution equations respectively. It plays a significant role to prove Theorem 3.7.

Proposition 3.6. *Let Condition A2 hold and F be defined by (3.24). Then for any $\varphi \in L^2(D)$, we have the expressions for the directional derivative*

$$DF(\varphi)(\phi_0) = \int_D (\mathcal{A}\varphi - \varphi)(\Phi(0, T)\phi_0 - \phi_0)dx, \quad \phi_0 \in L^2(D), \tag{3.28}$$

and the gradient

$$\frac{\delta F}{\delta \varphi} = W(0, T)w_0 - w_0, \tag{3.29}$$

with initial condition $w_0 = \mathcal{A}\varphi - \varphi$.

Proof. Utilising the linearity properties of $\mathcal{A}$ and $\Phi(0, T)$, from (3.8) and (3.22), we have

$$DF(\varphi)(\phi_0) = \lim_{\epsilon \to 0} \frac{1}{2} \int_D \frac{(\mathcal{A}(\varphi + \epsilon\phi_0) - (\varphi + \epsilon\phi_0))^2 - (\mathcal{A}\varphi - \varphi)^2}{\epsilon} dx$$

$$= \lim_{\epsilon \to 0} \frac{1}{2} \int_D \frac{((\mathcal{A}\varphi - \varphi) + \epsilon(\Phi(0, T)\phi_0 - \phi_0))^2 - (\mathcal{A}\varphi - \varphi)^2}{\epsilon} dx.$$

Hence (3.28) follows by collecting terms and taking the limit. We now wish to find $\frac{\delta F}{\delta \varphi} \in L^2(D)$ such that

$$DF(\varphi)(\phi_0) = \langle \frac{\delta F}{\delta \varphi}, \phi_0 \rangle = \int_D \frac{\delta F}{\delta \varphi}(x)\phi_0(x)dx, \quad \phi_0 \in L^2(D).$$

To compute the gradient, consider $w_R(s, x)$ satisfying the adjoint equation of PDE (3.4)

$$\begin{cases} -\partial_s w_R = L_R^*(s)w_R & \text{in } D_T, \\ w_R = 0, & \text{on } [0, T] \times \partial D, \\ w_R(T, \cdot) = w_T, & \text{on } D. \end{cases} \tag{3.30}$$

for some terminal condition $w_T \in L^2(D)$. Note that (3.30) is a backward equation. However, as terminal conditions are provided, (3.30) is well-posed provided Condition A2 are satisfied. This contrasts to the backward equation associated to (2.24) as a IBVP with initial conditions. For clarity, we introduce $w(s, \cdot) = w_R(T - s, \cdot)$. Then it is clear that w satisfies (3.26), since $L_R^*(T - s) = L^*(T - (T - s)) = L^*(s)$ by (3.1). In this form, it is clear that (3.26) and equivalently (3.30) are well-posed provided Condition A2 is satisfied. With the repeated time-reversal, $w(s, x)$ is understood to solve the Fokker-Planck equation forward in time.

Let ϕ be the homogeneous solution satisfying (3.4) with initial conditions $\phi(0, \cdot) = \phi_0$. By multiplying ϕ by w_R and integrating by parts over D_T, we have by (3.30)

$$
\begin{aligned}
0 &= \int_0^T \int_D (\partial_s \phi(s, x) - L_R(s)\phi(s, x)) \cdot w_R(s, x) dx ds \\
&= \int_D [\phi(s, x)w_R(s, x)]_{s=0}^T dx - \int_0^T \int_D \phi(s, x)\partial_s w_R(s, x) dx ds - \int_0^T \int_D \phi(s, x)L_R^*(s)w_R(s, x) dx ds \\
&= \int_D \phi(T, x)w_R(T, x) dx - \int_D \phi_0(x)w_R(0, x) dx.
\end{aligned}
$$

That is, in terms of w and Φ,

$$
\int_D \Phi(0, T)\phi_0(x) \cdot w(0, x) dx = \int_D \phi_0(x)w(T, x) dx. \tag{3.31}
$$

Impose the initial condition,

$$
w(0, \cdot) = w_T = \mathcal{A}\varphi - \varphi. \tag{3.32}
$$

Then it follows from (3.28), (3.31) and (3.32) that

$$
DF(\varphi)(\phi_0) = \int_D w_0(x)\Phi(0, T)\phi_0(x) dx - \int_D w_0(x)\phi_0(x) dx = \int_D [w(T, x) - w_0(x)] \phi_0(x) dx. \tag{3.33}
$$

Since ϕ_0 was arbitrary, by (3.27), we attain (3.29). $\qquad \square$

We note that while Lemma 3.4 yields a unique minimiser, it was not immediate whether F was minimised to zero. In the following theorem, we show indeed that the unique minimiser of F indeed minimises F to zero.

Theorem 3.7. *Let Condition A2 hold and $d \le 3$, $f \in C^\gamma(0, T; H^2(D))$ for some $\gamma \in (0, 1)$ and $f \ge 0$. Then $v_0 \in \mathscr{C}(D)$ obtained in Theorem 3.5 is the unique function in $L^2(D)$ satisfying (3.10). Moreover there exists a unique $v \in C(\bar{D}_T) \cap C^{1,2}(D_T)$ satisfying (3.2).*

Proof. By Theorem 3.5, the functional F has a unique minimiser $v_0 \in \mathscr{C}(D)$. By Lemma 3.4 and (3.33), it follows that

$$
DF(v_0)(\phi_0) = \int_D (w(T, x) - w_0(x))\phi_0(x) dx = 0, \quad \phi_0 \in L^2(D).
$$

By the fundamental lemma of calculus of variations, we have by (3.27)

$$
0 \equiv w(T, \cdot) - w_0(\cdot) = W(0, T)w_0 - w_0,
$$

i.e. w_0 is a fixed point of $W(0, T)$. Clearly $w_0 \equiv 0$ is a fixed point of $W(0, T)$. Let $w_0 \in H^2(D)$ be another fixed point solution to $W(0, T)$ and define $w(s, \cdot)$ by (3.27). With $d \le 3$, by (3.9), it follows that $w \in C(\bar{D}_T) \cap C^{1+\frac{\theta}{2}, 2+\theta}((0, T] \times \bar{D})$. In fact, since D is bounded, $w(s, \cdot) \in L^\infty(D)$ for $s \in [0, T]$. Note that p_D of (2.17) is a fundamental solution of (3.26), hence since w_0 is a fixed point of $W(0, T)$, it follows that

$$
w_0(x) = \int_D p_D(0, T, x, y)w_0(y) dy, \quad x \in D.
$$

Due to the absorbing boundaries of (3.26), note that for any $\Gamma \in \mathcal{B}(\mathbb{R}^d)$,

$$
\begin{aligned}
P_D(s,t,x,\Gamma) &:= \int_{D \cap \Gamma} p_D(s,t,x,y) dy \\
&= \mathbb{P}(\{X_t^{s,x} \in D \cap \Gamma\} \cap \bigcap_{r=s}^{t} \{X_r^{s,x} \in D\}) \\
&\leq \mathbb{P}(\{X_t^{s,x} \in D \cap \Gamma\}) \\
&= P(s,t,x,D).
\end{aligned}
$$

Hence with $\epsilon \in (0,1)$ from Lemma 2.1, it follows that

$$
\|w_0\|_\infty \leq \|w_0\|_\infty \int_D p(0,T,x,y) dy \leq \epsilon \|w_0\|_\infty.
$$

Thereby deducing $0 \in H^2(D)$ is the only fixed point of $W(0,T)$. Therefore, from (3.32), $v_0 \in \mathscr{C}(D)$ is the unique minimiser such that $\mathcal{A}v_0 = v_0$ and $F(v_0) = 0$. It follows then from (3.9) that

$$
v(s,x) := \Phi(0,s)v_0(x) + \int_0^s \Phi(r,s)f(r)dr \in C(\bar{D}_T) \cap C^{1+\frac{\theta}{2},2+\theta}((0,T] \times \bar{D})
$$

satisfies (3.2). We show that v_0 is the unique fixed point of $\mathcal{A}$ in the entire $L^2(D)$. Indeed suppose there exists another solution $\tilde{v}_0 \in L^2(D) \backslash \mathscr{C}(D)$ such that $\tilde{v}_0 = \mathcal{A}\tilde{v}_0$. By (3.9), $\mathcal{A}\tilde{v}_0 \in C^{2+\theta}(\bar{D}) \subset H^2(\bar{D})$ and satisfies the boundary conditions i.e. $\tilde{v}_0 \in H_0^2(D)$. Since $H^2(D) \ni f \geq 0$, it follows from (3.16) that $\tilde{v}_0 \geq 0$ i.e. $\tilde{v}_0 \in \mathscr{C}(D)$. Since uniqueness already holds in $\mathscr{C}(D)$, we conclude the uniqueness of v_0 satisfying (3.10) extends to $L^2(D)$. □

We summarise the PDE results in the context of expected duration $\bar{\tau}$ of SDEs. It will be clear that the SDE coefficients assumptions sufficiently implies Conditions A2 required for Theorem 3.7.

Theorem 3.8. *Let $d \leq 3$ and $D \subset \mathbb{R}^d$ be a non-empty open bounded set. Assume that the T-periodic SDE (2.4) satisfies (2.12), (2.11), (2.14) and (2.15). Then the expected duration $\bar{\tau}$ exists and is unique in $C(0,T;L^2(D))$. In fact, $\bar{\tau} \in C(\bar{D}_T) \cap C^{1,2}(D_T)$ is non-negative and non-trivial.*

Proof. Specific to expected duration, we let $f(s,x) = I_D(x)$, the indicator function on D for all $s \in [0,T]$. Since D is bounded, obviously $f \in C^\gamma(0,T;H^2(D))$ is non-negative. Then by Theorem 3.7, there exists a unique non-trivial non-negative solution $v \in C(\bar{D}_T) \cap C^{1,2}(D_T)$ satisfying (3.2). Then by time-reversal,

$$
\bar{\tau}(s,\cdot) := v(T-s,\cdot) = \Phi(0,T-s)v_0 + \int_0^{T-s} \Phi(r,T-s)I_D dr, \quad s \in [0,T],
$$

satisfies (2.24). □

4 Applications

4.1 Numerical Considerations

As discussed in the introduction, expected exit times have a range of applications including modelling the occurrence of certain events. Depending on the context, the problem is typically phrased as the stochastic process hitting a barrier or a threshold. While many physical problems have naturally

bounded domains, some applications have unbounded domains. For example, $D = (0, \infty)$ is a typical unbounded domain for species population or a wealth process, and exit from D implies extinction and bankruptcy respectively.

However, unbounded domain brings various technical difficulties for the expected duration PDE. Particularly from a computational viewpoint, any numerical PDE scheme requires a bounded domain. In the following remark, we show that the recurrency condition (4.1) below is sufficient to approximate the expectation duration from an unbounded domain to a bounded domain.

Remark 4.1. Suppose that there exists a radius $r_* > 0$ and $\epsilon > 0$ such that the coefficients of SDE (2.4) satisfies

$$2\langle b(t,x), x \rangle + \|\sigma\|_2^2 \leq -\epsilon \quad \text{on } \mathbb{R}^+ \times B_{r_*}^c. \tag{4.1}$$

Note that if b is continuous, then there exists a constant $M \geq -\epsilon$ such that $2\langle b(t,x), x \rangle + \|\sigma\|_2 \leq M$. Let D be an unbounded domain that is bounded in at least one direction hence $r_D := \inf_{y \in \partial D} \|y\|$ is finite. We suppose initial condition $x \in B_{r_I} \cap D$ for some fixed $r_I \geq 0$.

For any fixed $R_* > \max\{r_*, r_D, r_I\}$, define $\tilde{D} := D \cap B_{R_*}$. Note that $\tilde{D}$ is an open bounded domain with boundary $\partial\tilde{D} = \partial\tilde{D}_1 \cup \partial\tilde{D}_2$, where $\partial\tilde{D}_1 := \partial D \cap B_{R_*}$ and $\partial\tilde{D}_2 := D \cap \partial B_{R_*}$ are the subset of original boundary and "artificial" boundary to "close up" the original boundaries respectively. Observe that $\|x\| \in (r_D, R_*)$ for $x \in \partial\tilde{D}_1$ and $\|x\| = R_*$ for $x \in \partial\tilde{D}_2$.

For $\eta_{\tilde{D}}$ as defined by (2.1) for the domain $\tilde{D}$, by Itô's formula, we have

$$\|X_{t \wedge \eta_{\tilde{D}}}\|^2 = \|x\|^2 + \int_s^{t \wedge \eta_{\tilde{D}}} 2\langle b(r, X_r), X_r \rangle + \|\sigma\|_2^2 dr + \int_s^{t \wedge \eta_{\tilde{D}}} \langle X_r, \sigma dW_r \rangle.$$

Under the assumption (4.1), Corollary 3.2 of [Has12] implies that $\mathbb{E}^{s,x}(\eta_{\tilde{D}} - s) \leq \frac{\|x\|^2}{\epsilon}$. Since $X_{r \wedge \eta_{\tilde{D}}}$ is bounded for $r \in [s, t]$, it follows by taking expectations that

$$\mathbb{E}^{s,x}\|X_{t \wedge \eta_{\tilde{D}}}\|^2 \leq \|x\|^2 + M\mathbb{E}^{s,x}[\eta_{\tilde{D}} - s] \leq (1 + \frac{1}{\epsilon})\|x\|^2.$$

By Markov's inequality, it follows that $\mathbb{P}(\|X_{\eta_{\tilde{D}}}\|^2 \geq R_*^2) \leq R_*^{-2}(1 + \frac{1}{\epsilon})\|x\|^2 \leq R_*^{-2}(1 + \frac{1}{\epsilon})r_I^2 \to 0$ as $R_* \to \infty$. This implies that for sufficiently large R_*, the process exits $\tilde{D}$ via $\partial\tilde{D}_1$ rather than $\partial\tilde{D}_2$, thus

$$\bar{\tau}_D|_{\tilde{D}}(s, \cdot) \simeq \bar{\tau}_{\tilde{D}}(s, \cdot),$$

where $\bar{\tau}_D|_{\tilde{D}}$ denotes $\bar{\tau}_D$ restricted to $\tilde{D}$. In practice, $R_* = 2\max\{r_*, r_D, r_I\}$ is sufficient for weakly dissipative SDEs. We consider two examples and assume for simplicity that $r_I = r_*$.

It was shown in [FZZ19] that the periodic Ornstein-Uhlenbeck process possesses a geometric periodic measure [FZ16], furthermore it has a periodic mean reversion property akin to its classical counterpart with (constant) mean reversion. In applications, these properties are desirable for processes with underlying periodicity or seasonality. Indeed electricity prices in economics [BKM07, LS02] and daily temperature [BS07] were modelled by periodic Ornstein-Uhlenbeck processes. In [IDL14], the authors performed statistical inference of biological neurons modelled by Ornstein-Uhlenbeck processes with periodic forcing. In such models, one may be interested in the expected time in which a threshold is reached. For model parameter estimation, we refer readers to [DFK10].

Example 4.2. Consider the periodically forced multi-dimensional Ornstein-Uhlenbeck process

$$dX_t = (S(t) - AX_t)\, dt + \sigma dW_t,$$

where $S \in C(\mathbb{R}^+, \mathbb{R}^d)$ is T-periodic and $\sigma, A \in \mathbb{R}^{d \times d}$ with A positive definite i.e. there exists a constant $\alpha > 0$ such that $\langle Ax, x \rangle \geq \alpha \|x\|^2$ for all $x \in \mathbb{R}^d$. Denote $\|S\|_\infty = \sup_{t \in [0,T]} \|S(t)\|$. By Cauchy-Schwarz inequality and Young's inequality, it follows that

$$2\langle S(t) - Ax, x \rangle \leq 2\|S\|_\infty \|x\| - 2\alpha \|x\|^2 \leq \frac{\|S\|_\infty^2}{\alpha} - \alpha \|x\|^2,$$

i.e. weakly dissipative with coefficients $c = \frac{\|S\|_\infty^2}{\alpha}$ and $\lambda = \alpha$. Then

$$r_*^2 = \frac{1}{\alpha} \left(\frac{\|S\|_\infty^2}{\alpha} + \|\sigma\|_2 \right).$$

Remark 4.1 suggests one can reasonably approximate the unbounded domain $D = (0, \infty)^d$ by the bounded domain $\tilde{D} = (0, 2r_*)^d$. We remark also, to our current knowledge, there is no closed-form formula to solve (2.24) even in this simple example. As such, we provide the numerical solution to (2.24) in Section 4.2.

Example 4.3. Consider the stochastic overdamped Duffing oscillator

$$dX_t = \left(X_t - X_t^3 + A\cos(\omega t) \right) dt + \sigma dW_t, \tag{4.2}$$

where $A, \omega \in \mathbb{R}$ and $\sigma \in \mathbb{R} \backslash \{0\}$ are (typically small) parameters. By elementary calculus, it is straightforward to show (4.2) satisfies the weakly dissipative condition for any fixed $\lambda \in (0, 2)$ and $c = \frac{1}{2-\lambda} + 2|A| + \frac{\lambda}{4}$ for small A. Thus

$$r_*^2 = \frac{1}{\lambda} \left(\frac{1}{2-\lambda} + 2|A| + \|\sigma\|_2 \right) + \frac{1}{4}.$$

For concreteness, suppose that $A = 0.12$, $\omega = 10^{-3}$, $\sigma = 2.85$ and $\lambda = 1$, then $r_* = \sqrt{1.57} = 1.25$ (2 dp). Remark 4.1 suggests that the process exiting $D = (-1, \infty)$ can be approximated by the bounded domain $\tilde{D} = (-1, 2r_*)$.

Via Monte Carlo simulations, we numerically demonstrate Remark 4.1 for (4.2) to estimate $\bar{\tau}_{\tilde{D}}(0, x)$ for different bounded domains $\tilde{D}$. We partition $\tilde{D}$ into $N_x^{\text{sde}} = 100$ uniform initial conditions. For each fixed initial condition $x \in \tilde{D}$, we employ Euler–Maruyama method with time intervals of $\Delta t = 5 \cdot 10^{-3}$ to generate 1000 sample-paths of X_t until it exits $\tilde{D}$. We record and average the sample-path exit time to yield an estimate for $\bar{\tau}_{\tilde{D}}(0, x)$. Figure 1 shows that the estimation of $\bar{\tau}_{\tilde{D}}$ are "stable" for bounded domain $\tilde{D} = (-1, 2)$ and larger. Where the differences between these curves can be explained by the randomness of Monte Carlo simulations and sample size. On the other hand, the estimation of $\bar{\tau}_{\tilde{D}}$ differs significantly for $\tilde{D} = (-1, 1.5)$. Physically, this is interpreted as the artificial boundary $R_* = 1.5$ set too low and many sample-paths leave via this artificial boundary rather than via -1.

Finally, for subsequent analysis, while $\tilde{D} = (-1, 2)$ is sufficient, we will reduce D to $\tilde{D} = (-1, 3)$. We pick this larger domain to accommodate when we use $\sigma = 1$ where $r_* = 1.58$ (2 dp).

Remark 4.4. We remark that in Figure 1 for stochastic resonance problem such as (4.2), $\bar{\tau}(s, x)$ visually resembles a step-function is intuitively expected. We indeed expect $\bar{\tau}(s, x)$ small when the process starts near the left well because the process does not need to cross the potential well to exit. On the other hand, $\bar{\tau}(s, x)$ is large when the process starts in the right well. By the well-posedness of the PDE, $\bar{\tau}(s, x)$ is continuous and hence we expect a sharp increase between the two wells. Therefore, for the stochastic resonance problem, we expect $\bar{\tau}(s, x)$ to appear like a step-function.
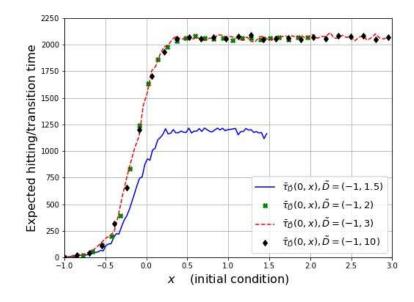
Figure 1: Monte Carlo estimation of $\bar{\tau}_{\tilde{D}}(0, x)$ with different $\tilde{D}$, plotted for $x \in (-1, 3) \cap \tilde{D}$.

4.2 Numerical Solution of Expected Duration PDE

In this section, we describe how one may numerically solve the PDE (3.2) using Theorem 3.2 and Theorem 3.7. For simplicity of this exposition, we do this in the one-dimensional case only. We remark that the presented numerical procedure does not claim originality nor optimal efficiency nor optimal accuracy as it is not the purpose of the current paper. We state and prove the following lemma that would be useful in applying Theorem 3.7 numerically. It provides an analytical step size in numerically computing the next iterate, given the initial point and iteration direction.

Lemma 4.5. *The function $\Psi(\gamma) := F(v + \gamma\varphi)$ for $v, \phi \in L^2(D)$ is minimised when*

$$\gamma = -\frac{\langle \mathcal{A}v - v, \Phi(0, T) - I)\phi \rangle_{L^2(D)}}{\|(\Phi(0, T) - I)\phi\|^2_{L^2(D)}}. \tag{4.3}$$

Proof. Using the identities (3.28) and $\mathcal{A}(v + \gamma\phi) = \mathcal{A}v + \gamma\Phi(0, T)\phi$ as well as the linearity property of $\Phi(0, T)$, we have that

$$\dot{\Psi}(\gamma) = DF(v + \gamma\phi)(\phi)$$
$$= \int_D (\mathcal{A}(v + \gamma\phi) - (v + \gamma\phi))(\Phi(0, T) - I)\phi dx$$
$$= \int_D (\mathcal{A}v - v)(\Phi(0, T) - I)\phi)dx + \gamma \int_D (\Phi(0, T) - I)\phi)^2 dx.$$

It follows that $\dot{\Psi}(\gamma) = 0$ when γ is given by (4.3). Note in fact, $F(v) = 0$ is already at a minimum when $\mathcal{A}v = v$ and hence $\gamma = 0$ using (4.3). $\qquad\square$

We now describe the numerical PDE scheme to solve (3.2). For given spatial domain $D = (d_{\min}, d_{\max})$, we partition it into $N_x^{\text{pde}} = 500$ uniform interior nodes and the time interval $[0, T]$ into $N_t^{\text{pde}} = 500$ uniform points. To evolve the parabolic IBVPs (3.4) and (3.26) in time, we implement

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a backward Euler scheme with central differencing. We note that in problems such as stochastic resonance where the period T is large, backward Euler method is preferred over Crank-Nicholson method to enable larger time steps and less prone to spurious oscillations.

To apply Banach Fixed Point numerically of Theorem 3.2, we seek an finite representation of $\mathcal{A}$ given by (3.8). We first discretise the continuous solution $v(t, x)$ to the IBVP (3.3) with

$$v_i^n := v(t^n, x_i), \quad 0 \le n \le N_t^{\text{pde}}, 0 \le i \le N_x^{\text{pde}},$$

where $x_i = d_{\min} + i\Delta x$, $i = 0, .., N_x^{\text{pde}} + 1$, $\Delta x = \frac{d_{\max} - d_{\min}}{N_x^{\text{pde}} + 1}$, $t^n = n\Delta t$, $n = 0, .., N_t^{\text{pde}}$ and $\Delta t = \frac{T}{N_t^{\text{pde}}}$. Using backward time-differencing and central spatial differencing, it is straightforward to attain the following parabolic PDE discretisation for (3.4)

$$\left(1 + a_i^n \frac{\Delta t}{(\Delta x)^2}\right) u_i^{n+1} = u_i^n + \left(\frac{a_i^n}{2} \frac{\Delta t}{(\Delta x)^2} + \frac{b_i^n}{2} \frac{\Delta t}{\Delta x}\right) u_{i+1}^{n+1} + \left(\frac{a_i^n}{2} \frac{\Delta t}{(\Delta x)^2} - \frac{b_i^n}{2} \frac{\Delta t}{\Delta x}\right) u_{i-1}^{n+1},$$

for $1 \le i \le N_x^{\text{pde}}, 0 \le n \le N_t^{\text{pde}}$ and $a_i^n = a(T - t^n, x_i)$ and $b_i^n = b(T - t^n, x_i)$ are the time-reversed coefficients evaluated at the grid points. We then define the tridiagonal matrix $\hat{\Phi}^{-1}(n) \in \mathbb{R}^{N_x^{\text{pde}} \times N_x^{\text{pde}}}$ for $1 \le n \le N_t^{\text{pde}}$ by

$$\hat{\Phi}^{-1}(n) := \begin{cases} i\text{'th diagonal:} & 1 + a_i^n \frac{\Delta t}{(\Delta x)^2}, \\ i\text{'th superdiagonal:} & -\frac{a_{i+1}^n}{2} \frac{\Delta t}{(\Delta x)^2} - \frac{b_{i+1}^n}{2} \frac{\Delta t}{\Delta x}, \\ i\text{'th subdiagonal:} & \frac{b_{i-1}^n}{2} \frac{\Delta t}{\Delta x} - \frac{a_{i-1}^n}{2} \frac{\Delta t}{(\Delta x)^2}. \end{cases}$$

Hence, for a given $f \in C^\gamma(0, T; L^p(D))$, define $\hat{f}^n = (f(T - t^n, x_1), ..., f(T - t^n, x_{N_x^{\text{pde}}}))^T$, together with homogeneous Dirichlet boundary conditions, we have the following linear system for (3.3)

$$\hat{\Phi}^{-1}(n)\hat{v}^{n+1} = \hat{v}^n + \Delta t \cdot \hat{f}^n, \quad n = 1, ..., N_t^{\text{pde}}. \tag{4.4}$$

Equation (4.4) implies that $\hat{v}^{n+1} = \hat{\Phi}(n)(v + \Delta t \cdot \hat{f}^n)$ and so $\hat{\Phi}(n)$ is the finite dimensional analogue of $\Phi(t^n, t^{n+1})$ given by (3.6). Since $\hat{\Phi}^{-1}(n)$ is tridiagonal, equation (4.4) can be efficiently solved by utilising lower decomposition or iterative methods which are generally more efficient than explicitly inverting the matrix. By iteratively solving, one evolves the IBVP (3.4) from initial time $t = 0$ to time $t = T$ for some given initial conditions. Define $\hat{\mathcal{A}}^{-1} := \hat{\Phi}^{-1}(1)\hat{\Phi}^{-1}(2) \cdots \hat{\Phi}^{-1}(N_t^{\text{pde}})$, then it follows that

$$\hat{v}^{N_t^{\text{pde}}} = \hat{\mathcal{A}}\hat{v}^0, \quad \hat{v}^0 \in \mathbb{R}^{N_x^{\text{pde}}}, \tag{4.5}$$

numerically solves (3.3) from initial time $t = 0$ to time $t = T$. Note further that $\hat{\mathcal{A}}$ is the finite counterpart of (3.8), the period solution map of (3.3). Given $\hat{\mathcal{A}}\hat{v}$, we define the finite dimensional cost functional

$$\hat{F}(\hat{v}) := \frac{\sqrt{\Delta x}}{2} \|\hat{\mathcal{A}}\hat{v} - \hat{v}\|_{\mathbb{R}^{N_x^{\text{pde}}}}^2,$$

where $\|\cdot\|_{\mathbb{R}^{N_x^{\text{pde}}}}$ is the standard $\mathbb{R}^{N_x^{\text{pde}}}$ Euclidean norm. Thus, to apply Banach Fixed Point iteration of Theorem 3.2, one can arbitrary choose initial vector $\hat{v}_0 \in \mathbb{R}^{N_x^{\text{pde}}}$ and iterate $\hat{v}_{n+1} = \hat{\mathcal{A}}\hat{v}_n$ until $\hat{F}(\hat{v}_n) \le 10^{-6}$. While $\hat{v}_0$ can be chosen arbitrarily, since $\bar{\tau}(s, x) \ge 0$, see (3.16), it is recommended to choose $\hat{v}_0 \ge 0$ also. Particular to stochastic resonance problems, by Remark 4.4, it is recommended

to pick $\hat{v}_0$ by a step-like function such as a scaled sigmoid function $\frac{1}{\sigma^2(1+e^{-x})}$. The scaling is due to our anticipation that the expected duration time is inversely proportional to the diffusion coefficient.

Given that the F is strictly convex and we have the gradient (3.29), we can numerically solve Theorem 3.7 by a gradient descent method scheme. For simplicity we assume $a = \text{const}$ and again apply backward time differencing and central spatial differencing, we have the discretisation for (3.26)

$$\left(1 + a(t^n, x_i)\frac{\Delta t}{(\Delta x)^2} + \partial_x b(t^n, x_i)\Delta t\right) u_i^{n+1}$$
$$= u_i^n + \left(\frac{a(t^n, x_i)}{2}\frac{\Delta t}{(\Delta x)^2} - \frac{b(t^n, x_i)}{2}\frac{\Delta t}{\Delta x}\right) u_{i+1}^{n+1} + \left(\frac{a(t^n, x_i)}{2}\frac{\Delta t}{(\Delta x)^2} + \frac{b(t^n, x_i)}{2}\frac{\Delta t}{\Delta x}\right) u_{i-1}^{n+1}.$$

Note here, the coefficients are not time-reversed. We similarly define the following tridiagonal matrices $W^{-1}(n) \in \mathbb{R}^{N_x^{\text{pde}} \times N_x^{\text{pde}}}$ for $n = 0, ..., N_t^{\text{pde}}$ by

$$\hat{W}^{-1}(n) : \begin{cases} i\text{'th diagonal:} & 1 + a(t^n, x_i)\frac{\Delta t}{(\Delta x)^2} + \partial_x b(t^n, x_i)\Delta t, \\ i\text{'th superdiagonal:} & -\frac{a(t^n, x_{i+1})}{2}\frac{\Delta t}{(\Delta x)^2}, \\ i\text{'th subdiagonal:} & -b(t^n, x_{i-1})\frac{\Delta t}{\Delta x} - \frac{a(t^n, x_{i-1})}{2}\frac{\Delta t}{(\Delta x)^2}. \end{cases}$$

We analogously define $\hat{W}^{-1}(n, m) := \hat{W}^{-1}(n)\hat{\Phi}_W^{-1}(n+1)\cdots \hat{W}^{-1}(m)$, where $0 \leq n < m < N_t^{\text{pde}}$ hence $\hat{W}(1, N_t^{\text{pde}})$ is the finite counterpart to $W(0, T)$ defined in (3.27). As with (4.5), we evolve (3.26) by solving the triangular system of equations. Unlike numerically solving Theorem 3.2, the initial conditions in (3.26) is not chosen arbitrarily. Instead given $\hat{v}_n$, the vector

$$\hat{w}_n := \hat{\mathcal{A}}\hat{v}_n - \hat{v}_n \in \mathbb{R}^{N_x^{\text{pde}}}$$

serves as the n'th initial condition to (3.26). Therefore, by (3.29), the finite dimensional gradient is given by

$$\frac{\delta \hat{F}}{\delta v_n} = \hat{W}(1, N_t^{\text{pde}})\hat{w}_n - \hat{w}_n \in \mathbb{R}^{N_x^{\text{pde}}}.$$

Then one can compute the gradient descent iterates by

$$\hat{v}_{n+1} = \hat{v}_n - \gamma_n \frac{\delta \hat{F}}{\delta v_n}, \tag{4.6}$$

that is we pick the next iterate in the direction $\phi = -\frac{\delta \hat{F}}{\delta v_n}$ with some step size γ_n. By Lemma 4.5, we can pick γ_n optimally with the formula

$$\gamma_n = \frac{\langle \hat{w}_n, (\hat{\Phi}(1, N_t^{\text{pde}}) - I)\frac{\delta \hat{F}}{\delta v_n}\rangle_{\mathbb{R}^{N_x^{\text{pde}}}}}{\|(\hat{\Phi}(1, N_t^{\text{pde}}) - I)\frac{\delta \hat{F}}{\delta v_n}\|_{\mathbb{R}^{N_x^{\text{pde}}}}^2},$$

as the finite dimensional counterpart of (4.3) replacing $L^2(D)$ inner-product with $\mathbb{R}^{N_x^{\text{pde}}}$ Euclidean inner-product. To compute γ_n, one just need to compute $(\hat{\Phi}(1, N_t^{\text{pde}})\frac{\delta \hat{F}}{\delta v_n}$ by again solving (4.4) where $\hat{f}^n \equiv 0$ and computing the inner-product. Thus to perform the gradient descent procedure, we again choose an arbitrary initial $\hat{v}_0 \in \mathbb{R}^{N_x^{\text{pde}}}$ and iterate (4.6) until $\hat{F}(v_n) \leq 10^{-6}$.

Using the procedure described above, we end this section numerically solve (3.2) for the expected duration for the Duffing Oscillator (4.2). Following Example 4.3, we choose the same parameters $A = 0.12$, $\omega = 10^{-3}$ and $\sigma = 0.285$. The same parameters was considered in [CLRS17]. As Example 4.3 and Figure 1 demonstrated, we reduce the unbounded domain to the bounded domain $\tilde{D} = (-1, 3)$. We then estimate the $s = 0$ cross-section $\bar{\tau}_{0.285}(0, \cdot)$ by three approaches. In this demonstration, we let $\bar{\tau}_{0.285}^{\text{sde}}$, $\bar{\tau}_{0.285}^{\text{bfp}}$ and $\bar{\tau}_{0.285}^{\text{grad}}$ to respectively represent the Monte Carlo simulation from Example 4.3, Banach fixed point iteration (Theorem 3.2) and gradient descent iteration via convex optimisation (Theorem 3.7). Figure 2 shows these approximations.
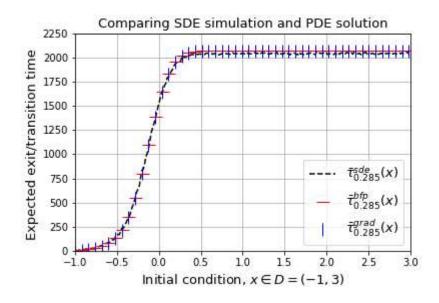

Figure 2: Numerical approximation of the expected transition time $\bar{\tau}_{0.285}(x)$ by $\bar{\tau}_{0.285}^{\text{sde}}(x)$, $\bar{\tau}_{0.285}^{\text{bfp}}(x)$ and $\bar{\tau}_{0.285}^{\text{grad}}(x)$ to SDE (4.2) with parameters $A = 0.12$, $\omega = 0.001$, $\sigma = 0.285$, $s = 0$, $D = (-1, 3)$.

For stochastic simulation of Figure 2, $\bar{\tau}_{0.285}^{\text{sde}}$ from Example 4.3 for the domain $D = (-1, 3)$ was reused. Figure 2 shows that $\bar{\tau}_{0.285}^{\text{bfp}}$ and $\bar{\tau}_{0.285}^{\text{grad}}$ closely approximate each other very well and in turn both visually approximate $\bar{\tau}_{0.285}^{\text{sde}}$ well, particularly for initial conditions starting in the right well. In the absence of an analytic formulae of $\bar{\tau}_{0.285}$, we take $\bar{\tau}_{0.285}^{\text{sde}}$ to be the "true" solution. Hence we estimated the relative errors $\frac{\|\bar{\tau}_{0.285}^{\text{sde}} - \bar{\tau}_{0.285}^{\text{bfp}}\|_{L^2(\tilde{D})}}{\|\bar{\tau}_{0.285}^{\text{sde}}\|_{L^2(\tilde{D})}} = 1.65\%$ (2 dp) and $\frac{\|\bar{\tau}_{0.285}^{\text{sde}} - \bar{\tau}_{0.285}^{\text{grad}}\|_{L^2(\tilde{D})}}{\|\bar{\tau}_{0.285}^{\text{sde}}\|_{L^2(\tilde{D})}} = 1.74\%$ (2 dp). The small relative error validates approximating expected transition time $\bar{\tau}_{0.285}$ by solving PDE (3.2) numerically by either $\bar{\tau}_{0.285}^{\text{bfp}}$ or $\bar{\tau}_{0.285}^{\text{grad}}$ for the Duffing Oscillator. It is anticipated that the (relative) errors can be explained by the particular numerical scheme and parameters used in the PDE and SDE methods. It may be particularly remarkable that the Banach fixed point iterates converges because it is not immediate whether the associated bilinear form is coercive.

Figure 2 is a cross-section of $\bar{\tau}_{0.285}(s, x)$ with $s = 0$. With the knowledge that the PDE solution approximates the SDE solution well, we produce Figure 3, a surface plot of $\bar{\tau}(s, x)$ to see how changing both initial time and initial state changes the expected duration time. As expected, the surface is indeed periodic in s.

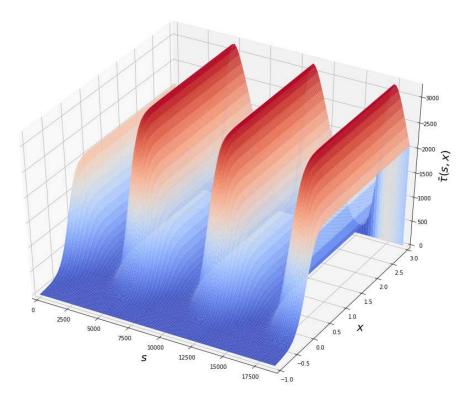Similarly, Figure 4 shows the surface plot for Example 4.2.

Figure 3: Numerical approximation of the expected duration $\bar{\tau}(s,x)$ of SDE (4.2) with parameters $A = 0.12$, $\omega = 0.001$, $\sigma = 0.285$, $s = 0$, $D = (-1,3)$ and $s \in [0,3T]$, where $T = \frac{2\pi}{\omega}$ is the period.

4.3 Stochastic Resonance

We now apply the results of this paper to study the physical phenomena of stochastic resonance. In the introduction, we discussed the modelling of stochastic resonance by a periodically-forced double-well potential SDEs and the interest in the expected duration time between the two wells. We refer to this time as expected transition time to align with the physical interpretation of the problem. In [FZZ19], it was shown that time-periodic weakly dissipative SDEs, which includes double-well potential SDEs, possesses a unique geometric periodic measure. The existence and uniqueness of geometric periodic measure of (4.2) implies that transitions between the metastable states do occur as well as asymptotic periodic behaviour [FZZ19]. Note however this does not imply that the transition time between the wells is periodic.

We consider specifically the stochastic overdamped Duffing Oscillator (4.2) as our model of stochastic resonance, this is a typical model in literature [BPSV82, BPSV81, BPSV83, CLRS17, GHJM98, HI05, HIP05]. It is easy to see that (4.2) is a gradient SDE

$$dX_t = -\partial_x V(t, X_t)dt + \sigma dW_t,$$

derived from the time-periodic double-well potential $V \in C^{1,2}(\mathbb{R} \times \mathbb{R})$ given by

$$V(t,x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 - Ax\cos(\omega t).$$

In the absence of the periodic forcing ($A = 0$), V has two local minimas at $x = \pm 1$ which are the metastable states and has a local maxima at $x = 0$, the unstable state. We consider the left and right well to be the intervals $(-\infty, 0)$ and $(0, \infty)$ respectively. Although the local minimas change over
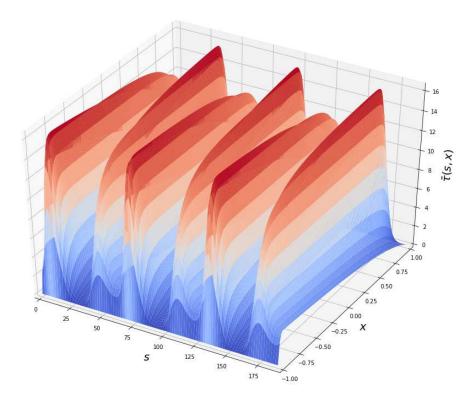
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Figure 4: Numerical approximation of the expected duration $\bar{\tau}(s, x)$ of SDE $dX_t = (\cos(0.1t) - X_t)\,dt + 0.285 \cdot dW_t$ with the domain $D = (-1, 1)$ and $s \in [0, 3T]$, where $T = \frac{2\pi}{\omega}$ is the period.

time, by the nature of the problem, we shall normalise the problem to have $x = -1, +1$ as the bottom of the left and right well respectively. It is well known (see e.g. [Gar09, Jun93, Ris96, Zwa01]) that in these double-well systems, the process quickly goes to the bottom of the well (if not already) and stays there for a long period of time before transitioning to the other well. This remains true even for $A \neq 0$, see [BPSV82, BPSV81, BPSV83, Nic82, Jun93, JH07, ZMJ90].

Currently, there appears to be neither standard nor rigorous definition of stochastic resonance [HI05, JH07]. In the context of this paper, a working definition is that the stochastic system is in stochastic resonance if the noise intensity is tuned optimally such that the expected transition time between the metastable states is (approximately) half the period [CLRS17]. We approach the stochastic resonance problem by solving the PDE for many fixed noise intensity values.

For the stochastic resonance problem, we take the starting time $s = 0$ (modulo the period), the time in which the right well is at its lowest point. In other words, we consider the process transition from the bottom of the right well to the left well. Formally, for $D = (-1, \infty)$, consider

$$\tau_\sigma(x) = \inf_{t \geq 0}\{X_t = -1 | X_0 = x\}, \quad x \in D,$$

associated to the SDE (4.2). We choose the same parameters $A = 0.12$, $\omega = 10^{-3}$ and keep the explicit σ subscript as we shall fine-tune σ to attain stochastic resonance. The same parameters was considered in [CLRS17].

We first study the sharpness of transition of the particle crossing the maxima point to reach the potential's minimum. We numerically solve the PDE (3.4) for different domains $D_b = (b, 3)$ for left boundary point $b = -1.0, -0.9, \cdots, 0.4, 0.5$. This is shown in Figure 5. The left graph shows the

expected transition times with respect to the different domains specified for all initial position within that domain. Observe that as b goes towards -1 from 0.5, the transition times accumulates quickly. The right graph fixes the initial position at the larger minima $x = 1$ of the double well potential and the curve is the expected transition time as a function of the left boundary point b of the domain D_b. It is the cross section of the left graph at $x = 1$. We see sharp increase of the expected transition time when b approaches 0, the maxima of the double well potential, from the right. This says that the particle would stay on the right hand side of well for long time. Note however that when b crosses 0 and towards -1, the expected transition time remains almost unchanged as the domain expands to $(-1, 3)$. This means that once the particle crosses the maxima, it quickly reaches and settles around the local minima $x = -1$. This is a reflection of the accumulating curves of the left graph. Note that the system does not need to exhibit stochastic resonance for these abrupt transitions and periods of long stability properties to occur. Furthermore, it is worth noting that this behaviour is expected by linearising the SDE around its the stationary points. In fact, we anticipate multiplicative ergodic theorem of non-autonomous random dynamical systems to rigorously explain this behaviour. Finally, we note that this this abrupt transition behaviour has been observed in the cyclical ice age climate change phenomena [BPSV82, Nic82].
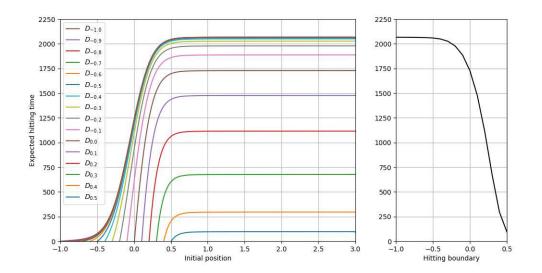

Figure 5: For SDE (4.2) with parameters $A = 0.12$, $\omega = 0.001$, $\sigma = 0.285$, we solve and plot the associated PDE (3.4) for expected transition time $\bar{\tau}^{\text{bfp}}_{0.285}(x)$ (left) and $\bar{\tau}^{\text{bfp}}_{0.285}(1)$ (right) for domains $D_b = (b, 3)$ for $b \in [-1, 0.5]$.

For the stochastic resonance problem, we consider also the transition from the left well to the right well. Specifically, consider the SDE

$$\begin{cases} dY_t = \left[Y_t - Y_t^3 + A\cos(\omega t) \right] dt + \sigma dW_t, & t \geq \frac{T}{2}, \\ Y_{\frac{T}{2}} = y, \end{cases}$$

and define

$$\tau_\sigma^{L \to R}(y) = \inf_{t \geq \frac{T}{2}} \left\{ Y_t \in \partial D | Y_{\frac{T}{2}} = y \right\} - \frac{T}{2}, \quad y \in D_L,$$

where $D_L = (-\infty, 1)$ and noting that $s = \frac{T}{2}$. Clearly, by a change of variables, $\tilde{Y}_t = -Y_{t + \frac{T}{2}}$ and

since $\cos(\omega(t + \frac{T}{2})) = -\cos(\omega t)$, we have that

$$\begin{cases} d\tilde{Y}_t = \left[\tilde{Y}_t - \tilde{Y}_t^3 + A\cos(\omega t)\right] dt - \sigma d\tilde{W}_t, \\ \tilde{Y}_0 = -y, \end{cases}$$

where $\tilde{W}_t = W_{t+\frac{T}{2}} - W_{\frac{T}{2}}$. It follows then that $\tau_\sigma^{L \to R}(y) = \inf_{t \geq 0}\{\tilde{Y}_t \in \partial D | \tilde{Y}_0 = -y\}$. Note that $\tilde{W}$ and W have the same distribution, hence it follows that

$$\bar{\tau}_\sigma(x) = \bar{\tau}_\sigma^{L \to R}(-x), \quad x \in D. \tag{4.7}$$

Indeed the same computation holds provided the drift is an odd function when $A = 0$.

Specifically for SDE (4.2) where $\omega = 0.001$, $T = 2000\pi$ is the period. Given (4.7), it is sufficient to cast the stochastic resonance problem as finding $\sigma_* \neq 0$ such that

$$\bar{\tau}_{\sigma_*}(1) \simeq \frac{T}{2} = 1000\pi. \tag{4.8}$$

i.e. the expected transition time between the wells is half the period.

To fine tune for stochastic resonance, we repeatedly solve the same PDE computations with the same numerical parameters and methods (as for Figure 2), changing only σ and considering the expected transition time $\bar{\tau}_\sigma^{\text{grad}}(1)$ as a function of σ. We vary σ in the σ-domain $[0.2, 1]$. We partition this σ-domain into two subintervals $[0.2, 0.3]$ and $[0.3, 1]$ and uniformly partition them into 50 and 100 points respectively. As a function of σ, we plot the expected transition time $\bar{\tau}_\sigma(1)$ in Figure 6.
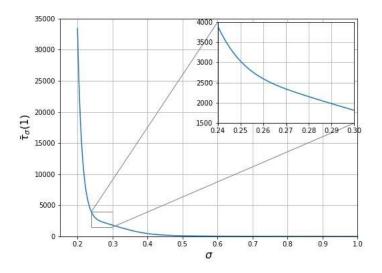

Figure 6: Plot of $\bar{\tau}_\sigma^{\text{grad}}(1)$ for $\sigma \in [0.2, 1]$ for SDE (4.2) with parameters $A = 0.12$ and $\omega = 0.001$, $s = 0$ and $\tilde{D} = (-1, 3)$.

It can be seen from Figure 6 that (4.8) is satisfied for some $\sigma_* \in [0.245, 0.25]$. We compute further $\bar{\tau}_\sigma^{\text{grad}}(1)$ on a finer partition of the interval $[0.245, 0.25]$ further and tabulate its numerical values in Table 1. Numerically, from Table 1, it can be seen that $\bar{\tau}_{0.2485}^{\text{grad}} \simeq \frac{T}{2} = 1000\pi$ to the nearest $5 \cdot 10^{-4}$.

σ	0.245	0.2455	0.246	0.2465	0.247	0.2475	0.248	0.2485	0.249	0.2495	0.25
$\bar{\tau}_\sigma^{\mathrm{grad}}(1)$	3388	3346	3306	3267	3230	3194	3159	3125	3093	3061	3030

Table 1: Computation of $\bar{\tau}_\sigma^{\mathrm{grad}}(1)$ (4sf) on a finer partition of $\sigma \in [0.245, 0.25]$.

5   Discussion

In this paper, we rigorously derived parabolic PDE (2.24) that the expected duration of time-periodic non-degenerate SDEs satisfies, complete with time-periodic boundary conditions and assumptions on the SDE coefficients. Furthermore, we proved that the PDE is well-posed by casting the problem as a fixed point problem in Theorem 3.2 as well as a convex optimisation problem in Theorem 3.7. In Section 4.2, we provided how these approaches can be applied numerically to solve PDE (2.24). In Section 4.3, we apply our theory and provide numerics to compute the expected duration time to switch from the regime and how it varies with the change of the noise intensity. This provides a fine-tuning method to find the noise intensity from historical events in reality. While it works very well for stochastic resonance problem, our theory is applicable to many real world and scientific problems where random periodicity is ubiquitous and exit duration is of important consideration.

Some conditions was assumed in this paper to simplify the exposition and to focus on the main ingredients and techniques used to attain the results given in this paper. For example, it was assumed the diffusion coefficient of SDE (2.4) is non-degenerate. We believe that this assumption can be relaxed along the lines of (time-dependent) Hörmander's condition and possibly even the UFG (Uniformly Finitely Generated) condition [KS82]. Similarly, we anticipate that the results in the paper can be extended to Lévy noise with a different infinitesimal generator. Another example is that in Theorem 3.7 we posed the solution of the (2.24) as convex optimisation problem in the $L^2(D)$ Hilbert space. We note that Lemma 3.4 applies more generally to reflective Banach spaces such as $L^p(D)$ for $1 < p < \infty$. If Theorem 3.7 can be established for $L^p(D)$, by Sobolev embedding, this would imply that (2.24) is well-posed for a wide range of systems and in higher dimensional systems.

On a broader view, there are a few different directions that this paper opens up. For instance, in Section 4.3, we implicitly assumed that the mapping $\sigma \to \bar{\tau}_\sigma(s, x)$ is continuous in the fine-tuning of σ for the stochastic resonance problem. While we expect this to be true, a rigorous proof would be of interest. From a theory perspective, we recall that Theorem 2.7 can be regarded as an instance of time-periodic Feynman-Kac duality in relating the expected duration of time-periodic SDEs to time-periodic solution of a parabolic PDE. We anticipate using a similar approach of this paper, other time-periodic Feynman-Kac dualities exist for other quantities of time-periodic SDEs.

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