Degenerate nonlinear parabolic equations with discontinuous diffusion coefficients

Dohyun Kwon and Alpár Richárd Mészáros

Abstract

This paper is devoted to the study of some nonlinear parabolic equations with discontinuous diffusion intensities. Such problems appear naturally in physical and biological models. Our analysis is based on variational techniques and in particular on gradient flows in the space of probability measures equipped with the distance arising in the Monge–Kantorovich optimal transport problem. The associated internal energy functionals in general fail to be differentiable, therefore classical results do not apply directly in our setting. We study the combination of both linear and porous medium type diffusions and we show the existence and uniqueness of the solutions in the sense of distributions in suitable Sobolev spaces. Our notion of solution allows us to give a fine characterization of the emerging critical regions, observed previously in numerical experiments. A link to a three phase free boundary problem is also pointed out.

Contents

1. Introduction

In this paper, we investigate a class of degenerate nonlinear parabolic equations, with discontinuous diffusion intensities. These can be written formally as the Cauchy problem for the unknown $\rho : [0, T] \times \Omega \to [0, +\infty)$

$$\begin{cases} \partial_t \rho - \Delta \varphi(\rho) - \nabla \cdot (\nabla \Phi \rho) = 0, & \text{in } (0, T) \times \Omega, \\ (\nabla \varphi(\rho) + \nabla \Phi \rho) \cdot \mathbf{n} = 0, & \text{on } (0, T) \times \partial\Omega, \\ \rho(0, \cdot) = \rho_0, & \text{in } \Omega, \end{cases} \qquad (1.1)$$

where $T > 0$ is a given time horizon, $\Omega \subset \mathbb{R}^d$ is the closure of a bounded convex open set with smooth boundary, $\Phi : \Omega \to \mathbb{R}$ is a given Lipschitz continuous potential function, $\rho_0 \in \mathscr{P}(\Omega)$ is a nonnegative Borel probability measure and the diffusion intensity function $\varphi : [0, +\infty) \to \mathbb{R}$ is supposed to have a discontinuity at $\rho = 1$. Therefore, φ is extended to be a multi-valued function at the discontinuity and in addition, it is supposed to be monotone in the sense that if $\eta^i \in \varphi(\rho^i)$, then

$$(\eta^1 - \eta^2)(\rho^1 - \rho^2) \geqslant 0.$$

Our aim is to identify a large class of potentials Φ, nonlinearities φ and initial data ρ_0, for which we show the well-posedness of (1.1) in a suitable distributional sense. Furthermore, we aim to describe some fine properties of the solutions. Let us remark that our results are expected to be valid also in the case of $\Omega = \mathbb{R}^d$, without running into many technical difficulties, provided we work in the space of measures having enough uniform moment bounds, just as in the original works [**25**, **34**].

Such problems appear naturally in physical and biological models. Let us briefly describe two of these. In [**4**], the authors study so-called phenomena of *self-organized criticality*. These arise typically in *sandpile models*, in which the sand particles are subject to a constant diffusion only at regions where their density is greater than a given threshold, otherwise they remain still. At the macroscopic level, in the cited reference such models were described by equations similar to (1.1), with $\Phi = 0$ and $\varphi(\rho) = 0$, if $\rho < \rho_c$ and $\varphi = \text{const}$ if $\rho \geqslant \rho_c$ (where ρ_c is a given threshold value). Via an approximation procedure and numerical investigations, the authors observe the growth (in time) of the *critical region*, where $\rho = \rho_c$, therefore, they conclude that particles following this diffusion law 'self-organize into criticality'. Our main results in this paper will rigorously confirm such phenomena.

In [**17**], the authors study diffusion models for biological organisms that increase their motility when food or other resource is insufficient. They refer to such phenomena as *starvation driven diffusion*. At the mathematical level, their model consists in a system of reaction-diffusion equations for two species, where the diffusion rates are discontinuous functions depending on the (food supply)/(food demand) ratio in the global population. In this model, a Lotka–Volterra type competition is implemented and a particular example is provided when one species follows the starvation driven diffusion and the other follows the linear diffusion. The authors conclude, by means of numerical simulations, that in heterogeneous environments the starvation-driven diffusion turns out to be a better survival strategy than the linear one. Therefore, by this conclusion the authors would like to underline also the fact that in biological models, discontinuous diffusion rates might appear in a very natural way, resulting many times in a better description of competing biological systems.

Degenerate nonlinear parabolic problems like (1.1) received a lot of attention in the past couple of decades. For a nonexhaustive list of classical works on this subject, we refer to [**7–9, 12, 13**] and the references therein. In majority of the literature, however, the nonlinearity φ is taken to be a continuous function.

To the best of our knowledge, except in particular cases involving linear type diffusions and/or bounded initial data (see, for instance, in [**5, 6, 10**]), our model problem in its full generality has not been addressed previously in the literature. The solution obtained in the aforementioned references heuristically can be written as pairs (ρ, η_ρ) belonging to well-chosen function spaces, such that

$$\partial_t \rho - \Delta(\eta_\rho) - \nabla \cdot (\nabla \Phi \rho) = 0$$

is fulfilled either in the distributional or entropic sense and $\rho(t, x) \in \eta_\rho(t, x)$ almost everywhere (a.e.).

In this paper, we rely on the gradient flow structure of (1.1) in the space of probability measures, when equipped with the distance W_2 arising in the Monge–Kantorovich optimal

transport problem. To (1.1), we associate an entropy functional $\mathcal{E} : \mathscr{P}(\Omega) \to \mathbb{R} \cup \{+\infty\}$, defined as

$$\mathcal{E}(\rho) := \begin{cases} \displaystyle\int_\Omega S(\rho(x))\mathrm{d}x + \int_\Omega \Phi(x)\mathrm{d}\rho(x), & \text{if } S(\rho) \in L^1(\Omega), \\ +\infty, & \text{otherwise,} \end{cases} \tag{1.2}$$

where $S : [0, +\infty) \to \mathbb{R}$ is a given function. At the formal level, the relationship between φ and S can be written as

$$\varphi(\rho) = \rho S'(\rho) - S(\rho) + S(1) \text{ and } \varphi'(\rho) = \rho S''(\rho), \text{ if } \rho \neq 1.$$

We observe that the discontinuity of φ at $\rho = 1$ corresponds to the nondifferentiability of S at $\rho = 1$. Furthermore, as φ is monotone, we impose that S is convex and the multiple values of φ can be represented by the subdifferential of S. In this sense, throughout the paper, we consider S to be given which satisfies the following assumption.

ASSUMPTION 1.1. $S : [0, +\infty) \to \mathbb{R}$ *is continuous, strictly convex and superlinear, in the sense that* $\lim_{\rho \to +\infty} S(\rho)/\rho = +\infty$. *Furthermore, S is twice continuously differentiable in* $(0, +\infty) \setminus \{1\}$.

Let us note that in this article the internal energy part of the functional $\mathcal{E}$ in general will satisfy the well-known condition introduced by McCann [**32**], so it will be displacement convex. But this energy fails to be differentiable on $(\mathscr{P}(\Omega), W_2)$. Furthermore, in general we do not impose λ-convexity assumptions on the potential Φ (so the potential energy in general fails to be displacement λ-convex). Because of these two deficiencies, the classical results from [**2**] do not apply directly in our setting. The lack of geodesic λ-convexity in the context of Wasserstein gradient flows typically poses serious obstructions (as we can see, for instance, in [**18, 26, 30**]). Even though the existence of the gradient flow of $\mathcal{E}$ in $(\mathscr{P}(\Omega), W_2)$ is expected, the fine characterization of the density curves, their velocities and the *critical region* $\{\rho = 1\}$, in as general settings as possible, is a challenging task. Because of the same reasons, an approach by maximal monotone operators as in [**5**] would not be satisfactory in our setting either. In this context, ours seems to be the first contribution which gives fine characterization of the gradient flows of a general class of nondifferentiable internal energies in $(\mathscr{P}(\Omega), W_2)$.

In our analysis, we rely on the classical *minimizing movements* scheme of De Giorgi (see also [**25, 39**]). This, for a given $\rho_0 \in \mathscr{P}(\Omega)$ (and for a small parameter $\tau > 0$ and $N \in \mathbb{N}$ such that $N\tau = T$) iteratively constructs $(\rho_k)_{k=0}^N$ as

$$\rho_{k+1} = \operatorname{argmin}\left\{\mathcal{E}(\rho) + \frac{1}{2\tau}W_2^2(\rho_k, \rho) : \rho \in \mathscr{P}(\Omega)\right\}, \ k \in \{0, \dots, N-1\}. \tag{1.3}$$

In order to write down the first-order necessary optimality conditions associated to (1.3), in Section 2 as our first contribution in this paper, we give a precise characterization of the subdifferential of $\mathcal{E}$ in $(\mathscr{P}(\Omega), W_2)$ (cf. [**2**]) in various settings (depending on the growth condition of S and the summability of ρ_0). Our analysis in this section relies on classical results from convex analysis, carefully adapted to (1.3). As an intermediate result, we show (see Lemma 2.10) that optimizers of the problem (1.3) enjoy higher summability estimates than the *a priori* ones coming from the growth condition of S at $+\infty$.

In order to give a precise description of the optimality conditions associated to (1.3), we introduce a function p_k which encodes the 'transition' between the phases $\{\rho_k < 1\}$ and $\{\rho_k > 1\}$ through the *critical region* $\{\rho_k = 1\}$. This is very much inspired by the derivation of the *pressure variable* in recent models studying crowd movements under density constraints (see in [**19, 31, 33**]). Because of this similarity, throughout the paper, we sometimes use the abused terminology of *pressure* to refer to the variable p. Interestingly, numerical experiments

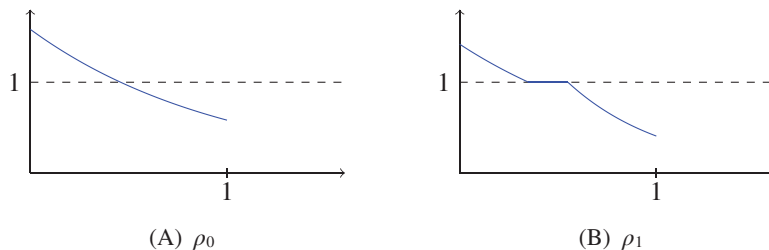(A) ρ_0 (B) ρ_1

FIGURE 1 (colour online). *One minimizing movement step in 1D, for $\Phi(x) = 2x$, $\Omega = [0,1]$ and S in* (1.7).

suggest (see Figure 1) that the critical region emerges in general already after one minimizing movement iteration.

After obtaining the necessary compactness results, we pass to the limit with the time discretization parameter $\tau \downarrow 0$ and we recover a PDE (which precisely describes the weak distributional solutions of (1.1)) satisfied by the limit quantities (ρ, p). This formally reads as

$$\begin{cases} \partial_t \rho - \Delta(L_S(\rho, p)) - \nabla \cdot (\nabla \Phi \rho) = 0, & \text{in } (0,T) \times \Omega, \\ \rho(0, \cdot) = \rho_0, & \text{in } \Omega, \\ (\nabla(L_S(\rho, p)) + \nabla \Phi \rho) \cdot \mathbf{n} = 0, & \text{in } [0,T] \times \partial\Omega. \end{cases} \tag{1.4}$$

Here, the operator L_S is defined pointwisely for functions $(\rho, p) : [0,T] \times \Omega \to \mathbb{R}$ by

$$L_S(\rho, p)(t, x) := [\rho(t,x)S'(\rho(t,x)) - S(\rho(t,x)) + S(1)]\mathbb{1}_{\{\rho \neq 1\}}(t,x) + p(t,x)\mathbb{1}_{\{\rho=1\}}(t,x) \tag{1.5}$$

and the pressure variable $p : [0,T] \times \Omega \to \mathbb{R}$ satisfies

$$\begin{cases} p = S'(1-) & \text{if } 0 \leqslant \rho < 1, \\ p \in [S'(1-), S'(1+)] & \text{if } \rho = 1, \\ p = S'(1+) & \text{if } \rho > 1. \end{cases} \tag{1.6}$$

Formally, (1.4) and (1.6) correspond to the *three phase free boundary problem*

$$\Delta p = -\Delta\Phi, \text{ in } \{\rho = 1\}, \quad p = S'(1-) \text{ in } \{\rho < 1\} \text{ and } p = S'(1+) \text{ in } \{\rho > 1\}.$$

Throughout the paper we distinguish cases depending on the diffusion rates in the two phases $\{\rho < 1\}$ and $\{\rho > 1\}$. We consider the combination of linear and porous medium type diffusions, which correspond to a behavior as $S(\rho) \sim \rho \log(\rho)$ and $S(\rho) \sim \rho^m$ (for $m > 1$), in $\{\rho < 1\}$ and $\{\rho > 1\}$. So, typical examples we have in mind include

$$S(\rho) := \begin{cases} \rho \log \rho, & \text{if } \rho \in [0,1], \\ \dfrac{\rho^m}{m-1} - \dfrac{1}{m-1}, & \text{if } \rho \in (1, +\infty), \end{cases} \quad \text{or}$$

$$S(\rho) := \begin{cases} \dfrac{\rho^m}{m-1}, & \text{if } \rho \in [0,1], \\ \dfrac{\rho^r}{r-1} - \dfrac{1}{r-1} + \dfrac{1}{m-1}, & \text{if } \rho \in (1, +\infty), \end{cases} \quad \text{for } m > r > 1.$$

Energies of only logarithmic type or power like ones with the same power on both phases will also be considered (as in (1.7) or (1.8)). The analysis in the case of general energies is quite involved. In the same time, optimizers in the minimizing movements scheme possess

different characteristics in the case of logarithmic type and porous medium type internal energies (such as fully supported vs. not fully supported; Lipschitz continuous vs. not Lipschitz continuous, etc.). As a result of this, we have to use different arguments to obtain the needed estimates. Therefore, to keep the paper as much readable as possible, we carefully break the cases (depending on the behavior of the internal energies) into specific sections.

In order to emphasize the main ideas of the paper, we present two toy problems in details. These turn out to be building blocks of our analysis for more general cases. Section 3 is devoted to the case when the entropy is of logarithmic type on both phases $\{\rho < 1\}$ and $\{\rho > 1\}$ and in particular S is given by

$$S(\rho) := \begin{cases} \rho \log \rho, & \text{if } \rho \in [0, 1], \\ 2\rho \log \rho, & \text{if } \rho \in (1, +\infty). \end{cases} \tag{1.7}$$

In this case, it turns out that the solution (ρ, p) satisfies $p = 1$ in $\{\rho < 1\}$, $p \in [1, 2]$ in $\{\rho = 1\}$, $p = 2$ in $\{\rho > 1\}$ and we have the simplified expression $L_S(\rho, p) = p\rho$.

Similarly, Subsection 5.1 presents the analysis in the case when S is given by

$$S(\rho) := \begin{cases} \dfrac{\rho^m}{m-1}, & \text{for } \rho \in [0, 1], \\ \dfrac{2\rho^m}{m-1} - \dfrac{1}{m-1}, & \text{for } \rho \in (1, +\infty), \end{cases} \tag{1.8}$$

for some $m > 1$. For this energy, the first equation of (1.4) can be written as

$$\partial_t \rho - \nabla \cdot \left(\rho \left[\nabla \left(\rho^{m-1} p \right) + \nabla \Phi \right] \right) = 0, \text{ in } (0, T) \times \Omega.$$

Furthermore, $p = \frac{m}{m-1}$ in $\{\rho < 1\}$, $p \in [\frac{m}{m-1}, \frac{2m}{m-1}]$ in $\{\rho = 1\}$ and $p = \frac{2m}{m-1}$ in $\{\rho > 1\}$.

Starting with Section 4, we consider general entropies. Assumptions are made on the growth of S in the two different phases $\{\rho < 1\}$ and $\{\rho > 1\}$. First, we impose

ASSUMPTION 1.2.

$$S : [0, +\infty) \to \mathbb{R} \text{ satisfies } \frac{\rho^{m-2}}{\sigma_2} < S''(\rho) \text{ if } \rho \in (0, 1) \text{ for some } m \geqslant 1 \text{ and } \sigma_2 > 0. \tag{1.9}$$

The imposed summability assumption on the initial data $\rho_0 \in \mathscr{P}(\Omega)$ plays also a crucial role in our analysis. If $\rho_0 \in L^\infty(\Omega)$, it turns out that the entire iterated sequence $(\rho_k)_{k=1}^N$ obtained in the scheme (1.3) remains essentially uniformly bounded, provided the potential Φ is regular enough. This fact does not depend on the differentiability of S and it is well known in the literature (see [38]). In this case, imposing only the assumption (1.9) on S is enough to obtain the well-posedness of (1.4)–(1.6).

The other 'extreme' case is when we only impose that ρ_0 has finite energy, that is, $\mathcal{E}(\rho_0) < +\infty$. We show that the iterated sequence will have improved summability estimates for $k \in \{1, \dots, N\}$ (see in Lemma 2.10), provided S satisfies the additional growth condition (1.10b)–(1.10a) below. These summability estimates on the iterated sequence will be enough to obtain the necessary a priori estimates and pass to the limit as $\tau \downarrow 0$ to obtain a weak solution to (1.4)–(1.6).

As a consequence of these arguments, we will always distinguish two cases with respect to the previous two summability assumptions when stating our main results. Our main result in the case of $\rho_0 \in L^\infty(\Omega)$ reads as:

THEOREM 1.1 (Theorems 3.1, 4.2, 5.1, 5.6 and Theorem 6.1). *Suppose that Assumptions 1.1-1.2 hold and Φ satisfies (2.4). For $\rho_0 \in L^\infty(\Omega)$, there exists $\rho \in L^\infty([0, T] \times \Omega)$,*

$\rho^m \in L^2([0,T]; H^1(\Omega))$ and $p \in L^2([0,T]; H^1(\Omega)) \cap L^\infty([0,T] \times \Omega)$ such that (ρ, p) is a unique solution of (1.4)–(1.6) in the sense of distributions.

For general initial data such that $\mathcal{E}(\rho_0) < +\infty$, we shall impose the following additional growth condition on S.

ASSUMPTION 1.3.

$$S : [0, +\infty) \to \mathbb{R} \text{ satisfies } \frac{\rho^{r-2}}{\sigma_1} \leqslant S''(\rho) \text{ if } \rho \in (1, +\infty) \text{ and} \tag{1.10a}$$

$$S''(\rho) \leqslant \sigma_1 \rho^{r-2} \text{ if } \rho \in (1, +\infty) \text{ for some } r, \sigma_1 \geqslant 1. \tag{1.10b}$$

Note that under (1.10) and $r > 1$, $\mathcal{E}(\rho_0) < +\infty$ is equivalent to $\rho_0 \in L^r(\Omega)$. Similarly to Theorem 1.1, we can formulate the corresponding well-posedness result.

THEOREM 1.2 (Theorems 3.1, 4.2, 5.1, 5.6 and Theorem 6.1). *Suppose that Assumptions 1.1–1.3 are fulfilled and*

$$m < r + \frac{\beta}{2} \tag{1.11}$$

hold true for $\beta > 1$ (its precise value is given in (2.23)). For $\rho_0 \in \mathscr{P}(\Omega)$ such that $\mathcal{E}(\rho_0) < +\infty$, there exists $\rho \in L^\beta([0,T] \times \Omega)$ and $p \in L^2([0,T]; H^1(\Omega)) \cap L^\infty([0,T] \times \Omega)$ such that (ρ, p) is a solution of (1.4)–(1.6) in the sense of distributions. Furthermore, we have

$$\rho^{m-\frac{1}{2}} \in L^2([0,T]; H^1(\Omega)), \text{ if } m \leqslant r \text{ and } \rho^{m-\frac{1}{2}} \in L^q([0,T]; W^{1,q}(\Omega)) \text{ if } r < m < r + \frac{\beta}{2}$$

for some $q \in (1, 2)$. If in addition $\beta \geqslant 2r$, then the pair (ρ, p) is unique.

Let us comment on the additional technical assumption (1.11) in the previous theorem. While this condition has to be required for purely technical reasons and we do not claim anything about its sharpness, we believe that it could be physically motivated. This would just mean that for unbounded initial data, the diffusion rate on the region $\{\rho < 1\}$ cannot be 'too much slower' than the one on the region $\{\rho > 1\}$. With other words, 'too fast' diffusion rate on $\{\rho > 1\}$ and 'too slow' diffusion on $\{\rho < 1\}$ might result in unphysical phenomena and in non existence of solutions.

It worth also noticing that the previous phenomenon is not expected for bounded solutions. Also, in particular from the definition of β in (2.23), we see that $\beta < +\infty$ can be arbitrary large if $d = 2$ and $\beta = +\infty$, if $d = 1$. Therefore, in such cases the previous theorem holds true without the additional assumption (1.11). The same is true in the case when $1 \leqslant m \leqslant r$.

Let us make a brief comment also on the proof of the previous theorems. In the case when the diffusion rates are equal on the two phases $\{\rho < 1\}$ and $\{\rho > 1\}$, that is, $m = r$, the derivation of the optimality conditions already gives us enough a priori estimates on gradients of suitable powers of the density variable. Then, these are enough to obtain the strong compactness of the interpolated curves of the discrete in time densities and pass to the limit as $\tau \downarrow 0$. The situation is way more challenging in the case when $m \neq r$. In these situations, we actually obtain the required estimates on the gradients of the discrete in time densities raised on a carefully chosen 'intermediate' power (depending on both m and r). This idea seems to be crucial in our analysis and this is one of the most technical parts of the paper.

It is worth to comment also on the fact that in Theorem 1.2 we obtain improved summability estimates of the density variable, even if one merely imposes L^r summability on ρ_0 and the diffusion rate in $\{\rho > 1\}$ is r, we obtain $\rho \in L^\beta([0,T] \times \Omega)$ (and β given in (2.23) satisfies $\beta > r$; in particular $\beta < +\infty$ is arbitrary large for $d = 2$ and $\beta = +\infty$ for $d = 1$). This improved

summability estimate (with respect to the summability of the initial data) seems to be well known in the case of standard porous medium equations (for instance, in the case of $\Phi = 0$, this is a consequence of [**40**, Theorem 8.7]). Our proof, which is based on purely optimal transport techniques, implies this estimate in particular also in the classical porous medium equation.

When studying the well-posedness of the system (1.4)–(1.6), one can ask the natural question whether these PDEs can be represented as continuity equations. Under suitable additional assumptions, this is always the case, as we can show in Theorem 4.9, Theorem 5.5 and Theorem 5.13 when (1.4) also reads as

$$
\begin{cases}
\partial_t \rho - \nabla \cdot \big(\rho \nabla \big(S'(\rho) \mathbb{1}_{\{\rho \neq 1\}} + p \mathbb{1}_{\{\rho = 1\}} \big) \big) - \nabla \cdot (\rho \nabla \Phi) = 0, & \text{in } (0, T) \times \Omega, \\
\rho(0, \cdot) = \rho_0, & \text{in } \Omega, \\
\rho \big[\nabla \big(S'(\rho) \mathbb{1}_{\{\rho \neq 1\}} + p \mathbb{1}_{\{\rho = 1\}} \big) + \nabla \Phi \big] \cdot \mathbf{n} = 0, & \text{in } [0, T] \times \partial \Omega.
\end{cases}
\tag{1.12}
$$

We underline that the required additional assumptions are needed to guarantee Sobolev estimates on $S'(\rho)$. We can summarize our results in this direction as follows.

THEOREM 1.3 (Theorems 4.9 and 5.13). *Let us suppose that we are in the setting of Theorem* 1.2 *and* (ρ, p) *is the solution of* (1.4)–(1.6). *If we additionally assume*

$$
m < r + \frac{1}{2}
\tag{1.13}
$$

and

$$
\beta > 2 \text{ and } m < \frac{\beta}{2} + \frac{1}{2},
\tag{1.14}
$$

then (ρ, p) *is a weak solution of* (1.12) *in the sense of distribution. The uniqueness of the solution holds under the same assumption as in Theorem* 1.2. *If in addition* $\rho_0 \in L^\infty(\Omega)$ *and* Φ *satisfies* (2.4), *we can drop* (1.14) *from the statement.*

In the same way as in Theorem 1.2 (by the definition of β in (2.23)), (1.14) holds for any $m, r \geqslant 1$ if $d = 1$ or $d = 2$. Moreover, when $r = m$, then the second inequality in (1.14) is satisfied for all $m \geqslant 1$ and $\beta > 2$ is equivalent to $m > \frac{3d-4}{2d}$.

The attentive reader could observe that in the statements of Theorems 1.1 and 1.2 we included the corresponding uniqueness results as well. Indeed, Section 6 is entirely devoted to this issue and in particular we obtain an L^1 contraction result for the density variable ρ (see in Theorem 6.1), implying its uniqueness. This will then imply the uniqueness of the corresponding p variable as well. Our approach is inspired by [**20**, Section 3] and [**40**, Theorem 6.5], and as expected, the *monotonicity* of the operator L_S (see Lemma 6.2) plays a crucial role in our argument. By the 'double degeneracy' of our problem, neither of the previously mentioned two approaches apply directly and a very careful combination of the two is required to obtain the desired L^1 contraction. Similarly as in [**40**, Theorem 6.5], in this analysis an additional summability assumption is needed on the density variable. Due to the extra L^β summability obtained in Theorem 1.2 or in the case of bounded solutions as in Theorem 1.1, this is automatically fulfilled in many cases. Let us mention that we expect a W_2-type contraction argument (in the spirit of [**14, 15**]) to hold in our setting as well, provided we impose some convexity assumptions on the potential Φ. The results from [**11**] imply that the Wasserstein contraction is equivalent to the geodesic convexity of the energy, provided the energy is smooth. Since the internal energies considered in this article, even though geodesically convex, in general fail to be differentiable, the results from [**11**] do not apply directly in our setting. These investigations represent the subject of future study. Finally, since in general we do not impose (λ-)convexity assumptions on Φ, our energy will in general lack the displacement (λ-)convexity property, which also motivates our search for an L^1-contraction instead.

Section 7 is devoted to further discussions on the models studied in this paper. In particular, we discuss examples where the emergence of the critical region $\{\rho = 1\}$ can be observed for positive times, even if that was not present in the case of the initial data, that is, $\mathscr{L}^d(\{\rho_0 = 1\}) = 0$. We illustrate this in dimension one and we describe stationary solutions (minimizers of the free energy) corresponding to suitable potential functions Φ, where the critical region is present. As we mentioned before, our problems can be linked to three phase-free boundary problems, and in this section we also derive these ones formally.

We end our paper with three small appendices, where we collected some well-known facts (or consequences of well-known results) from the theory of optimal transport (Appendix A) and convex analysis (Appendix B). In Appendix C we present also a suitable version of the classical Aubin–Lions lemma, which is repeatedly used throughout the paper to obtain compactness of families of time dependent functions in Lebesgue spaces.

2. *The minimizing movements scheme, optimality conditions and properties of the energy*

Throughout the paper $\Omega \subset \mathbb{R}^d$ is given, as the closure of a bounded, convex open set with smooth boundary. $\mathscr{P}(\Omega)$ denotes the space of Borel probability measures on Ω and $\mathscr{L}^d$ stands the Lebesgue measure on $\mathbb{R}^d$. We also use the notation $\mathscr{P}^{\mathrm{ac}}(\Omega) := \{\mu \in \mathscr{P}(\Omega) : \mu \ll \mathscr{L}^d \llcorner \Omega\}$. $T > 0$ is a fixed time horizon and we often use the notations $Q := [0, T] \times \Omega$ and $\mathbb{R}^+ := (0, +\infty)$.

As S' is strictly increasing in $\mathbb{R}^+ \setminus \{1\}$ from Assumption 1.1, $S'(0+)$ and $S'(1\pm)$ are well defined in $\mathbb{R} \cup \{-\infty\}$ and $\mathbb{R}$, respectively, as follows.

$$S'(0+) := \lim_{\varepsilon \to 0+} S'(\varepsilon), \quad S'(1-) := \lim_{\varepsilon \to 1-} S'(\varepsilon) \text{ and } S'(1+) := \lim_{\varepsilon \to 1+} S'(\varepsilon). \tag{2.1}$$

In particular, we have that $S'(1-) \leqslant S'(1+)$.

We define the corresponding internal energy $\mathcal{J} : \mathscr{P}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by

$$\mathcal{J}(\rho) := \begin{cases} \displaystyle\int_\Omega S(\rho(x))\,\mathrm{d}x & \text{if } \rho \in \mathscr{P}^{\mathrm{ac}}(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.2}$$

Furthermore, we suppose that there is given $\Phi : \Omega \to \mathbb{R}$ a potential function in $W^{1,\infty}(\Omega)$ and the associated potential energy $\mathcal{F} : \mathscr{P}(\Omega) \to \mathbb{R}$ given by

$$\mathcal{F}(\rho) := \int_\Omega \Phi(x)\,\mathrm{d}\rho(x).$$

Let $\rho_0 \in \mathscr{P}(\Omega)$ be given and consider a time discretization parameter $\tau > 0$ and $N \in \mathbb{N}$ such that $N\tau = T$. We define the *minimizing movements* $(\rho_k)_{k=1}^N$ of $\mathcal{J} + \mathcal{F}$ as follows: for $k \in \{1, \dots, N\}$ set,

$$\rho_k := \arg\min_{\rho \in \mathscr{P}(\Omega)} \left\{ \mathcal{J}(\rho) + \mathcal{F}(\rho) + \frac{1}{2\tau} W_2^2(\rho, \rho_{k-1}) \right\}. \tag{2.3}$$

Note that the existence and uniqueness of the solutions in the minimization problems (2.3) follow from standard compactness, lower semicontinuity and convexity arguments (similarly as in [**38**, Proposition 8.5], for instance).

In what follows, in our analysis we differentiate two cases with respect to the summability assumption on ρ_0. Since these need slightly different arguments, we separate them in two different subsections. In particular, if one assumes L^∞ summability on ρ_0, the presented results will hold true under no additional assumptions on S (other than in Assumption 1.1). However, in (2.3), we can allow general measure initial data, in which case an additional growth condition (see (1.10)) has to be imposed on S in order to obtain the same optimality conditions.

2.1. *Optimality conditions for $\rho_0 \in L^\infty(\Omega)$*

LEMMA 2.1. *Suppose that Assumption* 1.1 *takes place and $\rho_0 \in L^\infty(\Omega)$. If Φ is non-constant, let us assume that $\Phi \in C^1(\overline{\Omega})$ and*

$$\nabla\Phi(x_0) \cdot \boldsymbol{n}(x_0) > 0, \quad \forall x_0 \in \partial\Omega \ \ \text{and} \ \ \nabla\Phi \in BV(\Omega; \mathbb{R}^d) \text{ with } [\Delta\Phi]_+ \in L^\infty(\Omega), \qquad (2.4)$$

where $\boldsymbol{n}$ stands for the outward normal vector to $\partial\Omega$ and $[\Delta\Phi]_+$ denotes the positive part of the measure $\Delta\Phi$. Let $(\rho_k)_{k=1}^N$ be constructed via the scheme (2.3). *Then we have*

$$\|\rho_k\|_{L^\infty} \leqslant \|\rho_{k-1}\|_{L^\infty}(1 + \tau\|[\Delta\Phi]_+\|_{L^\infty})^d \leqslant \|\rho_0\|_{L^\infty}(1 + \tau\|[\Delta\Phi]_+\|_{L^\infty})^{kd}$$
$$\leqslant \|\rho_0\|_{L^\infty}e^{dT\|[\Delta\Phi]_+\|_{L^\infty}},$$

$\forall k \in \{1, \ldots, N\}$.

REMARK 2.1. Let us note that the second part of assumption (2.4) is sharp and it is very much related to the ones imposed in the work of Ambrosio (see [**1**]), as an improvement of the classical DiPerna–Lions theory [**21**], on transport equations with BV vector fields.

Proof of Lemma 2.1. The proof of this result in the case when $\Phi \equiv 0$ is essentially the same as the proof of [**38**, Proposition 7.32] (since that proof is not assuming any differentiability on S).

For general Φ, we use some ideas from the proof of [**16**, Theorem 1]. Let us approximate S with a sequence $(S_\varepsilon)_{\varepsilon>0}$ of smooth convex functions such that $S_\varepsilon'' \geqslant c_\varepsilon > 0$ for any $\varepsilon > 0$ with $S_\varepsilon'(0+) = -\infty$. Let Φ_ε be a smooth approximation of Φ which satisfies (2.4) and such that $\Phi_\varepsilon \to \Phi$, $\nabla\Phi_\varepsilon \to \nabla\Phi$, uniformly as $\varepsilon \downarrow 0$ and $\|[\Delta\Phi_\varepsilon]_+\|_{L^\infty} \leqslant \|[\Delta\Phi]_+\|_{L^\infty}$, for $\varepsilon > 0$. Let ρ_k^ε be the unique solution of (2.3), when we replace S with S_ε and Φ by Φ_ε. Writing down the optimality conditions, we obtain

$$S_\varepsilon'(\rho_k^\varepsilon) + \Phi_\varepsilon + \frac{\phi_k^\varepsilon}{\tau} = C \text{ a.e.},$$

where $\phi_k^\varepsilon \in \mathcal{K}(\rho_k^\varepsilon, \rho_{k-1})$. Let us suppose that $\phi_k^\varepsilon \in C^{2,\alpha}(\Omega)$, otherwise we approximate ρ_{k-1} by strictly positive $C^{0,\alpha}$ measures (and ρ_k^ε is Lipschitz continuous and strictly positive), and we use Caffarelli's regularity theory to deduce the desired regularity for the potential.

Now, let x_0 be a maximum point of ρ_k^ε. From the previous equality, since S_ε' is strictly increasing, we certainly have that x_0 is a minimum point of $\Phi_\varepsilon + \frac{\phi_k^\varepsilon}{\tau}$.

We claim that $x_0 \notin \partial\Omega$. Indeed, if x_0 would belong to $\partial\Omega$, we would have that

$$(\nabla\phi_k^\varepsilon(x_0) + \tau\nabla\Phi_\varepsilon(x_0)) \cdot \boldsymbol{n}(x_0) \leqslant 0.$$

However, by the convexity of Ω, we have that $(x_0 - \nabla\phi_k^\varepsilon(x_0)) \cdot \boldsymbol{n}(x_0) \leqslant 0$, from where $\nabla\phi_k^\varepsilon(x_0) \cdot \boldsymbol{n}(x_0) \geqslant 0$. This fact together with the assumption (2.4) yields a contradiction. Indeed, from the uniform convergence of $\nabla\Phi_\varepsilon \to \nabla\Phi$, we have that

$$\nabla\Phi_\varepsilon(x_0) \cdot \boldsymbol{n} \geqslant \nabla\Phi(x_0) \cdot \boldsymbol{n} - \varepsilon > 0,$$

for sufficiently small $\varepsilon > 0$.

Therefore, the maximum point x_0 of ρ_k^ε belongs to the interior of Ω. This implies that $\Delta\phi_k^\varepsilon(x_0) + \tau\Delta\Phi_\varepsilon(x_0) \geqslant 0$. Using the Monge–Ampère equation, we find

$$\|\rho_k^\varepsilon\|_{L^\infty} = \rho_k^\varepsilon(x_0) = \rho_{k-1}(x_0 - \nabla\phi_k^\varepsilon(x_0))\det\left(I_d - D^2\phi_k^\varepsilon(x_0)\right) \leqslant \|\rho_{k-1}\|_{L^\infty}(1 - \Delta\phi_k^\varepsilon(x_0))^d$$

$$\leqslant \|\rho_{k-1}\|_{L^\infty}(1 + \tau\Delta\Phi_\varepsilon(x_0))^d \leqslant \|\rho_{k-1}\|_{L^\infty}(1 + \tau\|[\Delta\Phi_\varepsilon]_+\|_{L^\infty})^d$$

$$\leqslant \|\rho_{k-1}\|_{L^\infty}(1 + \tau\|[\Delta\Phi]_+\|_{L^\infty})^d \leqslant \|\rho_0\|_{L^\infty}(1 + \tau\|[\Delta\Phi]_+\|_{L^\infty})^{kd}$$

$$\leqslant \|\rho_0\|_{L^\infty}e^{dT\|[\Delta\Phi]_+\|_{L^\infty}},$$

where in the first inequality we have used the inequality between the arithmetic and geometric means. Since the last three bounds depend only on the data, these will also remain valid also in the limit $\varepsilon \downarrow 0$ (since the minimizers of both the approximated and the original problems are unique). Therefore the thesis of the lemma follows. $\qquad\square$

Now, we state the main result of this subsection on the first-order necessary optimality conditions for the problems in (2.3).

THEOREM 2.2. *Suppose $\rho_0 \in L^\infty(\Omega)$. For all $k \in \{1, \dots, N\}$, there exists $C = C(k) \in \mathbb{R}$ and $\overline{\phi}_k \in \mathcal{K}(\rho_k, \rho_{k-1})$ such that*

$$\begin{cases} C - \dfrac{\overline{\phi}_k}{\tau} - \Phi \leqslant S'(0+) & \text{in } \{\rho_k = 0\}, \\[2mm] C - \dfrac{\overline{\phi}_k}{\tau} - \Phi \in [S'(1-), S'(1+)] & \text{in } \{\rho_k = 1\}, \\[2mm] C - \dfrac{\overline{\phi}_k}{\tau} - \Phi = S' \circ \rho_k & \text{otherwise.} \end{cases} \qquad (2.5)$$

Here, $\mathcal{K}(\rho_k, \rho_{k-1})$ is given in Definition A.1. Also, $S'(0+)$ and $S'(1\pm)$ are given in (2.1). Note that if $S'(0+) = -\infty$, then $\rho_k > 0$ a.e. (see Lemma A.4), and in this case the first inequality in (2.5) is not present.

The proof of the previous results relies on the precise derivation of the subdifferential of the corresponding objective functional in (2.3). Let us point out that the subdifferential of sum is not always the sum of subdifferentials (see, for instance [**38**, Example 7.22]). Therefore, we need to carefully choose the domain of definition of $\mathcal{J}$. In the spirit of Lemma 2.1, we consider it as a functional on $L^\infty(\Omega)$ instead of $\mathscr{P}(\Omega)$. The additive property of subdifferentials on $L^\infty(\Omega)$ holds under suitable conditions (cf. [**22**]).

REMARK 2.2. Let us underline that in our analysis we rely on the classical subdifferential calculus in L^p spaces rather than directly computing Wasserstein subdifferentials (cf. [**2**]). This is mainly because of the already available powerful classical results on precise representations of subdifferentials of integral functionals, such as $\mathcal{J}$, on L^p spaces (cf. [**35, 36**]). In the same time, this framework is well suited also for computing the subdifferential of $\rho \mapsto W_2^2(\cdot, \rho_{k-1})$ (see, for instance, in [**38**]).

It is worth mentioning, however, that at the heuristic level there is an intimate link between the two notions of subdifferentials, namely: if $\xi \in \partial\mathcal{J}(\rho)$ (that is, ξ is an element of the classical L^p subdifferential) is sufficiently regular, then $\nabla\xi \in \partial_{W_2}\mathcal{J}(\rho)$ (that is, $\nabla\xi$ is an element of the Wasserstein subdifferential). Nevertheless, this connection at this point remains only formal, because typically we do not have any a priori information on the regularity of ξ to justify this link.

PROPOSITION 2.3. *For all $k \in \{1, \dots, N\}$, we have*

$$\partial\left(\mathcal{J}(\rho) + \mathcal{F}(\rho) + \frac{1}{2\tau}W_2^2(\rho, \rho_{k-1})\right)\Big|_{\rho=\rho_k} = \partial\mathcal{J}(\rho_k) + \Phi + \frac{1}{2\tau}\partial(W_2^2(\rho, \rho_{k-1}))|_{\rho=\rho_k}. \qquad (2.6)$$

Proof. To simplify the writing, we consider only the case $k = 1$. Let us check that $\mathcal{J}$ and $W_2^2(\cdot, \rho_0)$ satisfy the assumptions in Lemma B.2. The convexity of S implies that of $\mathcal{J}$. Also, the continuity of $\mathcal{J}$ in $L^\infty(\Omega)$ follows from the continuity of S. From Lemma B.1, we conclude $\mathcal{J} \in \Gamma(L^\infty(\Omega))$. We have the same conclusion for the functional $\mathcal{F}$ (which is actually linear in ρ).

Let us show that $W_2^2(\cdot, \rho_0) \in \Gamma(L^\infty(\Omega))$. Define $H : L^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by

$$H(\phi) := -\int_\Omega \phi^c \, \mathrm{d}\rho_0. \tag{2.7}$$

Proposition A.3 implies that $H^* : L^\infty(\Omega) \to \mathbb{R} \cup \{+\infty\}$ is given (in the sense of (B.1)) by

$$H^* = \frac{1}{2} W_2^2(\cdot, \rho_0) \text{ on } L^\infty(\Omega).$$

We conclude $W_2^2(\cdot, \rho_0) \in \Gamma(L^\infty(\Omega))$.

Finally, choose $A \subseteq \Omega$ a Borel set such that $\mathscr{L}^d(A) \neq 1$ and define

$$\hat{\mu} := \frac{1}{\mathscr{L}^d(A)} \mathbb{1}_A. \tag{2.8}$$

$\mathcal{J}(\hat{\mu})$, $\mathcal{F}(\hat{\mu})$ and $W_2^2(\hat{\mu}, \rho_0)$ are finite. Furthermore, by the continuity of S in $\mathbb{R}^+$, $\mathcal{J}$ is continuous at $\hat{\mu}$. In the same way, $\mathcal{F}$ is also continuous at $\hat{\mu}$. Thus, we conclude (2.6) from Lemma B.2. □

Next, let us find the subdifferential of $W_2^2(\cdot, \rho_{k-1})$. While this subdifferential is expected to be the set of Kantarovich potentials $\mathcal{K}(\rho_k, \rho_{k-1})$, it is not straight forward to conclude about this as we consider the subdifferential for the functional on $L^\infty(\Omega)$. We rely on the ideas from [**38**, Proposition 7.17], tailored to our setting.

LEMMA 2.4 [**38**, Lemma 7.15]. *Let $\mathfrak{X}$ be a Banach space and $H : \mathfrak{X} \to \mathbb{R} \cup \{+\infty\}$ be convex and lower semicontinuous. Set $H^*(y) = \sup\limits_{x \in \mathfrak{X}} \{\langle x, y \rangle_{\mathfrak{X}, \mathfrak{X}^*} - H(x)\}$. Then, we have*

$$\partial H^*(y) = \arg\max_{x \in \mathfrak{X}} \{\langle x, y \rangle_{\mathfrak{X}, \mathfrak{X}^*} - H(x)\}. \tag{2.9}$$

LEMMA 2.5. *$H : L^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ given in (2.7) is convex and lower semicontinuous.*

Proof. The proof of convexity of H is the same as in [**38**, Proposition 7.17], where one needs to change only the definition of φ^c using essential infima.

Let us show now that H is lower semicontinuous For this, let $\varphi \in L^1(\Omega)$ and $(\varphi_n)_{n \in \mathbb{N}}$ a sequence in $L^1(\Omega)$ such that $\varphi_n \to \varphi$ strongly in $L^1(\Omega)$ as $n \to +\infty$.

Note first that by definition,

$$-\varphi^c(y) \geqslant \varphi(y), \text{ a.e. in } \Omega,$$

from where we have that $H(\varphi) > -\infty$. Because of the strong L^1 convergence, we know that there exists a subsequence of $(\varphi_n)_{n \in \mathbb{N}}$ (that we do not relabel), which is converging pointwise a.e. in Ω to φ. We shall work with this sequence from now on.

Writing the previous inequality for φ_n^c and φ_n, we have that

$$\liminf_{n \to +\infty} -\varphi_n^c(y) \geqslant \liminf_{n \to +\infty} \varphi_n(y) = \varphi(y), \text{ a.e. in } \Omega,$$

where we used the fact that $\varphi_n(y) \to \varphi(y)$ a.e. in Ω, as $n \to +\infty$.

Let us define $g : \Omega \to \mathbb{R} \cup \{+\infty\}$ as $g(y) := \liminf\limits_{n \to +\infty} -\varphi_n^c(y)$. Note that this is measurable function. Indeed, $(-\varphi_n^c)_{n \in \mathbb{N}}$ is a sequence of measurable functions (infima of measurable functions), and using Fatou's lemma for the non-negative sequence of measurable functions $(-\varphi_n^c - \varphi_n)_{n \in \mathbb{N}}$, one concludes that g is measurable and

$$\int_\Omega \varphi(y) \rho_0(y) \, \mathrm{d}y \leqslant \int_\Omega g(y) \rho_0(y) \, \mathrm{d}y \leqslant \liminf_{n \to +\infty} H(\varphi_n).$$

Claim. $\varphi(y) \leqslant -\varphi^c(y) \leqslant g(y)$ for a.e. $y \in \Omega$.

Proof of the claim. Actually the first inequality was shown before, thus we show only the second one. Thus, by Egorov's theorem, we have that for any $\delta > 0$ there exists a measurable set $B_\delta \subseteq \Omega$ such that $\mathscr{L}^d(B_\delta) < \delta$ and $(\varphi_n)_{n\in\mathbb{N}}$ converges uniformly to φ as $n \to +\infty$ on $\Omega \setminus B_\delta$. Let us fix a small $\delta > 0$. We have furthermore that for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\varphi(x) - \varepsilon \leqslant \varphi_n(x) \leqslant \varphi(x) + \varepsilon$$

for a.e. $x \in \Omega \setminus B_\delta$ and $n \geqslant N_\varepsilon$. Because of this, we have the following chain of inequalities for all $n \geqslant N_\varepsilon$

$$-\varphi_n^c(y) = \sup_{x\in\Omega}\left\{\varphi_n(x) - |x-y|^2\right\} \geqslant \sup_{x\in\Omega\setminus B_\delta}\left\{\varphi_n(x) - |x-y|^2\right\}$$

$$\geqslant \sup_{x\in\Omega\setminus B_\delta}\left\{\varphi(x) - \varepsilon - |x-y|^2\right\}.$$

Taking $\liminf_{n\to+\infty}$ of both sides, one obtains

$$g(y) \geqslant \sup_{x\in\Omega\setminus B_\delta}\left\{\varphi(x) - \varepsilon - |x-y|^2\right\}$$

for a.e. $y \in \Omega$. By the arbitrariness of ε and δ (in this order), one gets that

$$g(y) \geqslant \sup_{x\in\Omega}\left\{\varphi(x) - |x-y|^2\right\} = -\varphi^c(y),$$

as we claimed.

Note that we have proved the following: if $(\varphi_n)_{n\in\mathbb{N}}$ is converging to φ in $L^1(\Omega)$, then there exists a subsequence $(\varphi_{n_j})_{j\in\mathbb{N}}$ of the original sequence such that

$$H(\varphi) \leqslant \liminf_{j\to+\infty} H(\varphi_{n_j}).$$

This statement actually implies the lower semicontinuous of H on the full sequence. Indeed, observe that by the definition of lim inf, there exists a subsequence $(\varphi_{n_k})_{k\in\mathbb{N}}$ of the original sequence such that

$$\liminf_{n\to+\infty} H(\varphi_n) = \lim_{k\to+\infty} H(\varphi_{n_k}).$$

We have shown previously that there exists a subsequence $(\varphi_{n_{k_j}})_{j\in\mathbb{N}}$ of $(\varphi_{n_k})_{k\in\mathbb{N}}$ such that

$$H(\varphi) \leqslant \liminf_{j\to+\infty} H(\varphi_{n_{k_j}}).$$

On the other hand

$$\liminf_{j\to+\infty} H(\varphi_{n_{k_j}}) = \lim_{j\to+\infty} H(\varphi_{n_{k_j}}) = \lim_{k\to+\infty} H(\varphi_{n_k}) = \liminf_{n\to+\infty} H(\varphi_n),$$

thus the lower semicontinuous of H follows. $\qquad\square$

PROPOSITION 2.6. *For all $k \in \{1, \ldots, N\}$ we have*

$$\frac{1}{2}\partial(W_2^2(\rho, \rho_{k-1}))\big|_{\rho=\rho_k} = \mathcal{K}(\rho_k, \rho_{k-1}). \tag{2.10}$$

Proof. To simplify the notation, we set $k = 1$. Recall from Proposition A.3 that

$$\frac{1}{2}\partial(W_2^2(\rho, \rho_0))\big|_{\rho=\rho_1} = \partial H^*(\rho_1)$$

for H given in (2.7). From Lemmas 2.4 and 2.5, it holds that

$$\partial H^*(\rho_1) = \operatorname{argmax}_{\phi\in L^1(\Omega)}\left\{\int_\Omega \phi\,\mathrm{d}\rho_1 + \int_\Omega \phi^c\,\mathrm{d}\rho_0\right\}.$$

From Definition A.1, we conclude. $\qquad\square$

Finally, let us compute the subdifferential of $\mathcal{J}$ based on [**36**]. Before, we need the following preparatory result.

LEMMA 2.7 [**36**, Corollary 1B]. *Let ψ and Ψ be given as in (B.2). Assume that $\psi(\mu(x))$ is majorized by a summable function of x for at least one $\mu \in L^\infty(\Omega)$ and that $\psi^*(\zeta(x))$ is majorized by a summable function of x for at least one $\zeta \in L^1(\Omega)$. Then, an element $\xi \in L^\infty(\Omega)^*$ belongs to $\partial\Psi(\mu)$ given in (B.3) if and only if $\xi^{\mathrm{ac}}(x) \in \partial\psi(\mu(x))$ for a.e. $x \in \Omega$ where ξ^{ac} is the absolutely continuous component of ξ, and the singular component ξ^{s} of ξ attains its maximum at μ over*

$$\{\nu \in L^\infty(\Omega) : \Psi(\nu) < +\infty\}.$$

PROPOSITION 2.8. *For ρ_k is given in (2.3), if $\xi \in \partial\mathcal{J}(\rho_k) \cap L^1(\Omega)$, then it holds that*

$$\xi \in \begin{cases} [-\infty, S'(0+)] & \text{a.e. in } \{\rho_k = 0\}, \\ [S'(1-), S'(1+)] & \text{a.e. in } \{\rho_k = 1\}, \\ S' \circ \rho_k & \text{a.e. in } \{\rho_k \neq 1\}. \end{cases} \tag{2.11}$$

Proof. Let us show that S and S^* satisy assumptions on Lemma 2.7. Let $\mu = \zeta = \frac{1}{\mathscr{L}^d(\Omega)}\mathbb{1}_\Omega$, then $S(\mu)$ is finite, and thus in $L^1(\Omega)$. On the other hand, as S is superlinear, $S^* < +\infty$ in $[0, +\infty)$. Therefore, for any constant $c \in \mathbb{R}$, $S^*(c) \in L^1(\Omega)$.

By Lemma 2.7, $\xi^{\mathrm{ac}}(x) \in \partial S(\rho_k(x))$ a.e., where ξ^{ac} is the absolutely continuous part of ξ. From the direct computation of $\partial S(\rho_k(x))$, we conclude that ξ^{ac} satisfies the right-hand side of (2.11). As $\xi \in L^1(\Omega)$, the singular part of ξ is zero, $\xi^{\mathrm{ac}} = \xi$ and we conclude (2.11). $\qquad\square$

Proof of Theorem 2.2. We only consider the case that $k = 1$. By the optimality of ρ_1 in (2.3), it holds that

$$0 \in \partial\left(\mathcal{J}(\rho_1) + \mathcal{F}(\rho_1) + \frac{1}{2\tau}W_2^2(\rho_1, \rho_0)\right).$$

From Propositions 2.3 and 2.6, there exists $\xi \in \partial\mathcal{J}(\rho_1)$, $\overline{\phi}_1 \in \mathcal{K}(\rho_1, \rho_0)$ and $C \in \mathbb{R}$ such that

$$\xi + \frac{\overline{\phi}_1}{\tau} + \Phi - C = 0 \text{ a.e. on } \Omega.$$

As $\overline{\phi}_1, \Phi \in L^1(\Omega)$, $\xi \in \partial\mathcal{J}(\rho_1) \cap L^1(\Omega)$, Proposition 2.8 implies (2.5). $\qquad\square$

2.2. *Optimality conditions for $\rho_0 \in \mathscr{P}(\Omega)$ having finite energy*

In this subsection, we are imposing (1.10). Let us show first that $\mathcal{J}$ satisfying the additional assumption in (1.10) defines a continuous functional on $L^r(\Omega)$. In the previous subsection, the continuity of $\mathcal{J}$ in $L^\infty(\Omega)$ directly follows from the continuity of S.

LEMMA 2.9. *Let $\mathcal{J}$ be given in (2.2) satisfying (1.10b). Then $\mathcal{J}$ is continuous in $L^s(\Omega)$ for all*

$$s > r \text{ if } r = 1, \text{ and } s \geqslant r \text{ if } r > 1. \tag{2.12}$$

Proof. From (1.10b), there exists $c > 0$ such that for all $\rho \in [0, +\infty)$ (since S is also continuous, hence uniformly bounded on [0,1])

$$|S(\rho)| \leqslant c(\rho^s + 1), \tag{2.13}$$

for all s satisfying (2.12).

Consider a sequence $\{\mu_i\}_{i\in\mathbb{N}} \subset L^s(\Omega)$ such that

$$\mu_i \to \mu \text{ in } L^s(\Omega) \text{ as } i \to \infty \qquad (2.14)$$

These exists a subsequence $\{\mu_{i_j}\}_{j\in\mathbb{N}} \subset \{\mu_i\}_{i\in\mathbb{N}}$ such that

$$\mu_{i_j} \to \mu \text{ a.e. as } j \to \infty. \qquad (2.15)$$

From (2.13), it holds that for all $j \in \mathbb{N}$

$$0 \leqslant c(|\mu_{i_j}|^s + 1) - |S(\mu_{i_j})| \leqslant c(|\mu_{i_j}|^s + 1) \pm S(\mu_{i_j}). \qquad (2.16)$$

Let us apply Fatou's lemma into $c(|\mu_{i_j}|^s + 1) + S(\mu_{i_j})$. From (2.14), (2.15) and the continuity of S, it holds that

$$\int_\Omega c(|\mu(x)|^s + 1) + S(\mu(x))\,\mathrm{d}x \leqslant \liminf_{j\to\infty} \int_\Omega c(|\mu_{i_j}(x)|^s + 1) + S(\mu_{i_j}(x))\,\mathrm{d}x,$$

$$\leqslant \int_\Omega c(|\mu(x)|^s + 1)\,\mathrm{d}x + \liminf_{j\to\infty} \int_\Omega S(\mu_{i_j})\,\mathrm{d}x,$$

and we have

$$\mathcal{J}(\mu) \leqslant \liminf_{j\to\infty} \mathcal{J}(\mu_{i_j}).$$

Similarly to the argument at the end of the proof of Lemma 2.5, we conclude the lower semicontinuity along the full sequence, therefore

$$\mathcal{J}(\mu) \leqslant \liminf_{i\to\infty} \mathcal{J}(\mu_i). \qquad (2.17)$$

Applying Fatou's lemma again into $c(|\mu_{i_j}|^s + 1) - S(\mu_{i_j})$, we get

$$\mathcal{J}(\mu) \geqslant \limsup_{j\to\infty} \mathcal{J}(\mu_{i_j}), \qquad (2.18)$$

and as before, we deduce the upper semicontinuity along the full sequence. Therefore (2.17) and (2.18) imply that $\mathcal{J}(\mu) = \lim_{j\to\infty} \mathcal{J}(\mu_{i_j})$ and $\mathcal{J}$ is continuous in $L^s(\Omega)$. □

In what follows, we show that the minimizers of the of the minimizing movements scheme (2.3) enjoy higher order summability estimates (which are independent of ρ_0, but depend on τ). These will play a crucial role later when deriving the optimality conditions.

LEMMA 2.10. *Suppose that S satisfies Assumption 1.1 and (1.10a). Let $\rho_k \in \mathscr{P}(\Omega)$ be the minimizer in (2.3). Then $\rho_k \in L^\beta(\Omega)$, where $\beta := (2r-1)d/(d-2)$, if $d \geqslant 3$. If $d = 2$, then the statement is true for any $\beta < +\infty$ and $\beta = +\infty$ if $d = 1$. In particular, there exists $C > 0$ depending only on Ω, $\|\nabla\Phi\|_{L^\infty}$ and β such that if $\beta < +\infty$, then*

$$\int_\Omega (\rho_k)^\beta\,\mathrm{d}x \leqslant C + \frac{C}{\tau^2} W_2^2(\rho_k, \rho_{k-1}). \qquad (2.19)$$

Otherwise, for $d = 1$,

$$\|\rho_k\|_{L^\infty} \leqslant C.$$

REMARK 2.3. Let us note that the previous lemma gives an improvement on the summability of ρ_k. Indeed, in case when the internal energy is of logarithmic entropy type, we know a priori that $\rho_k \in L^1(\Omega)$, while in the case of power like entropies, we have a priori $\rho_k \in L^r(\Omega)$. In contrast to these, we clearly improve the summability exponents in both cases.

Proof of Lemma 2.10. For $\varepsilon > 0$, let $S_\varepsilon : [0, +\infty) \to \mathbb{R}$ smooth, strictly convex such that $S_\varepsilon'' \geqslant c_\varepsilon > 0$ (for some $c_\varepsilon > 0$), $S_\varepsilon'(0+) = -\infty$ and $S_\varepsilon \to S$ uniformly as $\varepsilon \to 0$. Let ρ_k^ε be the unique minimizer of the problem

$$\inf_{\rho \in \mathscr{P}(\Omega)} \left\{ \mathcal{E}_\varepsilon(\rho) := \int_\Omega S_\varepsilon(\rho) \, \mathrm{d}x + \mathcal{F}(\rho) + \frac{1}{2\tau} W_2^2(\rho, \rho_{k-1}) \right\}. \tag{2.20}$$

By the assumptions on S_ε, classical results imply that ρ_k^ε is Lipschitz continuous.

Without loss of generality, we can assume that S_ε satisfies the growth (1.10a) if $\rho > 2$. We can write the optimality condition

$$S_\varepsilon''(\rho_k^\varepsilon) \nabla \rho_k^\varepsilon + \nabla \Phi + \frac{\nabla \varphi_k^\varepsilon}{\tau} = 0 \quad \text{a.e.,} \tag{2.21}$$

where φ_k^ε is a Kantorovich potential in the transport of ρ_k^ε onto ρ_{k-1}. From here, there exists a constant $C > 0$ (depending only on r and σ_1) such that

$$\int_\Omega |S_\varepsilon''(\rho_k^\varepsilon) \nabla \rho_k^\varepsilon|^2 \rho_k^\varepsilon \, \mathrm{d}x \leqslant C \left(\|\nabla \Phi\|_{L^\infty}^2 + \frac{1}{\tau^2} W_2^2(\rho_k^\varepsilon, \rho_{k-1}) \right).$$

And in particular, for any $\ell > 2$, we have by setting $\Omega_\ell := \{\rho_k^\varepsilon > \ell\}$,

$$\int_{\Omega_\ell} |\nabla (\rho_k^\varepsilon)^{r-1/2}|^2 \, \mathrm{d}x \leqslant C \left(\|\nabla \Phi\|_{L^\infty}^2 + \frac{1}{\tau^2} W_2^2(\rho_k^\varepsilon, \rho_{k-1}) \right). \tag{2.22}$$

We know that the optimizers ρ_k^ε are Lipschitz continuous on their supports, therefore the super-level sets Ω_ℓ are open.

Moreover, once again using the fact that ρ_k^ε is Lipschitz continuous, we have that there exists $\delta > 0$ such that

$$\mathrm{dist}(\partial \Omega_\ell, \overline{\Omega}_{2\ell}) \geqslant 2\delta.$$

Indeed, otherwise if one supposes the contrary, then for any $n \in \mathbb{N}$, there exist $x_n \in \partial \Omega_\ell$ and $y_n \in \Omega_{2\ell}$ such that $\mathrm{dist}(x_n, y_n) < \frac{1}{n}$, then one would have that $|\rho_k^\varepsilon(x_n) - \rho_k^\varepsilon(y_n)| \leqslant \frac{1}{n} \|\nabla \rho_k^\varepsilon\|_{L^\infty(\Omega_\ell)} \to 0$, as $n \to +\infty$. However, this would be a contradiction since $\rho_k^\varepsilon(x_n) = \ell$ and $\rho_k^\varepsilon(y_n) \geqslant 2\ell$.

Now, by defining $\Omega_{\ell,\delta} := \{\chi_{\Omega_{2\ell}} \star \eta_\delta > s\}$ for some $s \in (0, 1/2)$ to be set later (where $\eta_\delta : \mathbb{R}^d \to \mathbb{R}$ is a mollifier obtained from a smooth even kernel $\eta : \mathbb{R}^d \to \mathbb{R}$ — such that $\int_{\mathbb{R}^d} \eta \, \mathrm{d}x = 1$, $\eta \geqslant 0$ and $\mathrm{spt}(\eta) \subset B_1(0)$ — by $\eta_\delta := (1/\delta^d) \eta(\cdot/\delta)$), we have that $\Omega_{2\ell} \subset \overline{\Omega}_{\ell,\delta} \subset \Omega_\ell$, $\Omega_{\ell,\delta}$ is an open set, and by Sard's theorem it has smooth boundary for $\mathscr{L}^1$-a.e. $s \in (0, 1/2)$. We choose such an s.

We have in particular from (2.22) that

$$\int_{\Omega_{\ell,\delta}} |\nabla (\rho_k^\varepsilon)^{r-1/2}|^2 \, \mathrm{d}x \leqslant C \left(\|\nabla \Phi\|_{L^\infty}^2 + \frac{1}{\tau^2} W_2^2(\rho_k^\varepsilon, \rho_{k-1}) \right),$$

and so the Sobolev embedding theorem implies (since ρ_k^ε is only uniformly bounded in $L^r(\Omega)$) that $(\rho_k^\varepsilon)^{r-1/2} \in L^{2^*}(\Omega_{\ell,\delta})$ from where $\rho_k^\varepsilon \in L^\beta(\Omega_{\ell,\delta})$, where $\beta := 2^*(r - 1/2)$, if $d \geqslant 3$ and $\beta < +\infty$ arbitrary if $d = 2$ and β can be taken $+\infty$ if $d = 1$. He we use the notation $2^* = 2d/(d-2)$.

From the above construction, we can claim that $\rho_k^\varepsilon \in L^\beta(\Omega)$. Indeed, we have

$$\int_\Omega (\rho_k^\varepsilon)^\beta \, \mathrm{d}x = \int_{\{\rho_k^\varepsilon \leqslant \ell\}} (\rho_k^\varepsilon)^\beta \, \mathrm{d}x + \int_{\Omega_{\ell,\delta}} (\rho_k^\varepsilon)^\beta \, \mathrm{d}x + \int_{\Omega_\ell \setminus \Omega_{\ell,\delta}} (\rho_k^\varepsilon)^\beta \, \mathrm{d}x$$

$$\leqslant (2^\beta + 1) \ell^\beta \mathscr{L}^d(\Omega) + C \left(\|\nabla \Phi\|_{L^\infty}^2 + \frac{1}{\tau^2} W_2^2(\rho_k^\varepsilon, \rho_{k-1}) \right).$$

Let us underline that this bound only depends on $W_2^2(\rho_k^\varepsilon, \rho_{k-1})$. Clearly, the previous inequality is valid only if $\beta < +\infty$. In the case of $d = 1$, that is, when $\beta = +\infty$, we first perform the

computation for $\beta < +\infty$ finite, and obtain the desired bound after taking β-root of the previous inequality and sending $\beta \to +\infty$.

Now, it is easy to see that because $S_\varepsilon \to S$ uniformly, we have that the objective functional in (2.20) Γ-convergences to the objective functional in the original problem as $\varepsilon \downarrow 0$, with respect to the weak-$*$ convergence of probability measures. Indeed, take a sequence $(\rho^\varepsilon)_{\varepsilon > 0}$ and ρ in $\mathscr{P}(\Omega)$ such that $\rho^\varepsilon \overset{*}{\rightharpoonup} \rho$ as $\varepsilon \downarrow 0$. Note that by the construction of the approximation S_ε, if $\mathcal{E}_\varepsilon(\rho^\varepsilon) \leqslant C$ (for a constant independent of ε), then we have that $(\rho^\varepsilon)_{\varepsilon > 0}$ is uniformly bounded in $L^r(\Omega)$. By the uniform convergence $S_\varepsilon \to S$, we have that for any $\delta > 0$ there exists ε_0 such that

$$S(\rho^\varepsilon) \leqslant S_\varepsilon(\rho^\varepsilon) + \delta, \ \forall \varepsilon < \varepsilon_0.$$

Therefore

$$\mathcal{E}(\rho) \leqslant \liminf_{\varepsilon \downarrow 0} \mathcal{E}(\rho^\varepsilon) \leqslant \liminf_{\varepsilon \downarrow 0} \mathcal{E}_\varepsilon(\rho^\varepsilon) + \delta \mathscr{L}^d(\Omega),$$

so the Γ-liminf inequality follows by the lower semicontinuity of the energy $\mathcal{E}$ and the arbitrariness of $\delta > 0$. For the Γ-limsup inequality, we use a constant sequence $\rho^\varepsilon = \rho$ as a recovery sequence such that $\mathcal{E}_\varepsilon(\rho)$ is finite for all $\varepsilon > 0$. Clearly $\lim_{\varepsilon \downarrow 0} \mathcal{E}_\varepsilon(\rho) = \mathcal{E}(\rho)$.

Finally, since both ρ_k and ρ_k^ε, the solutions of the original and the approximated problems, respectively, are unique, when $\varepsilon \downarrow 0$ we find that ρ_k also has the $L^\beta(\Omega)$ bound. The thesis of the lemma follows. □

Let us note that in Lemma 2.10 the L^β bounds on ρ_k depends only on $\frac{1}{\tau^2} W_2^2(\rho_k, \rho_{k-1})$ and the data. Therefore, when considering the piecewise constant interpolated curves $(\rho^\tau)_{\tau > 0}$ (see their precise definition in (3.8)), and integrating them in time and space, we find the following very important lemma.

LEMMA 2.11. *Suppose that $\rho_0 \in \mathscr{P}(\Omega)$ with $\mathcal{J}(\rho_0) < +\infty$ and (1.10) hold. The curves $(\rho^\tau)_{\tau > 0}$ are uniformly bounded in $L^\beta(Q)$ for β given in*

$$\beta := \begin{cases} (2r-1)\frac{d}{d-2} & \text{if } d \geqslant 3, \\ (0, \infty) & \text{if } d = 2 \\ +\infty & \text{if } d = 1. \end{cases} \tag{2.23}$$

Proof. Let β as in the statement of the lemma and let $(\rho^\tau)_{\tau > 0}$ stand for the piecewise constant interpolations as defined in (3.8). Then, Lemma 2.10 and (2.19) imply that

$$\int_0^T \int_\Omega (\rho^\tau)^\beta \, \mathrm{d}x \mathrm{d}t = \tau \sum_{k=1}^N \int_\Omega (\rho_k)^\beta \, \mathrm{d}x \leqslant \tau N C + C \sum_{k=1}^N \frac{1}{\tau} W_2^2(\rho_k, \rho_{k-1}),$$

where $C > 0$ depends only on the data and Ω. Since $\tau N = T$ and $\sum_{k=1}^N \frac{1}{\tau} W_2^2(\rho_k, \rho_{k-1})$ is uniformly bounded (see Lemma 3.4), we conclude. □

Under the above assumption, we show a result parallel to Theorem 2.2.

THEOREM 2.12. *Suppose that $\rho_0 \in \mathscr{P}(\Omega)$ such that $\mathcal{E}(\rho_0) < +\infty$ and (1.10) hold. Then, for all $k \in \{1, \ldots, N\}$ there exists $C = C(k) \in \mathbb{R}$ and $\overline{\phi}_k \in \mathcal{K}(\rho_k, \rho_{k-1})$ satisfying (2.5). Here, $\mathcal{K}(\rho_k, \rho_{k-1})$ and ρ_k are given in Definition A.1 and (2.3), respectively.*

We recall the following lemma from [35, 36] and compute the subdifferential of $\mathcal{J}$ explicitly. In comparison to the previous subsection, it holds that $(L^r(\Omega))^* = L^{r'}(\Omega)$ for $r \in (1, +\infty)$ where $r' := \frac{r}{r-1}$ and thus the argument below is simpler than Lemma 2.7.

LEMMA 2.13 [**35**, Theorem 2], [**36**, Equations (1.11) & (1.12)]. *Let ψ and Ψ be given as in* (B.2)*. Assume that $\psi(\mu(x))$ is majorized by a summable function of x for at least one $\mu \in L^\infty(\Omega)$ and that $\psi^*(\zeta(x))$ is majorized by a summable function of x for at least one $\zeta \in L^1(\Omega)$. Then, an element $\xi \in L^r(\Omega)^*$ belongs to $\partial\Psi(\mu)$ given in* (B.3) *if and only if $\xi(x) \in \partial\psi(\mu(x))$ for a.e. $x \in \Omega$.*

Proof of Theorem 2.12. Let us set $k = 1$. The first part of the proof is parallel to Propositions 2.3 and 2.6. Let us show

$$\partial\left(\mathcal{J}(\rho) + \mathcal{F}(\rho) + \frac{1}{2\tau}W_2^2(\rho, \rho_0)\right)\bigg|_{\rho=\rho_1} = \partial\mathcal{J}(\rho_1) + \Phi + \frac{1}{\tau}\mathcal{K}(\rho_1, \rho_0), \qquad (2.24)$$

where $\mathcal{K}$ is given in Definition A.1 and the subdifferential is defined in Definition B.1. Recall $\Gamma(\cdot)$ from Definition B.2 and its equivalent property in Lemma B.1. Note that $\mathcal{J} \in \Gamma(L^r(\Omega))$ follows from the convexity of S and Lemma 2.9. The same is true for $\mathcal{F}$.

Let us underline that it is crucial that we have *a priori* bounds on the optimizers of (2.3) in $L^\beta(\Omega)$ for some $\beta > 1$. Indeed, Lemma 2.10 yields that even if $r = 1$ (which corresponds to the logarithmic entropy type interaction energy), we have that the optimizers satisfy $\rho_k \in L^\beta(\Omega)$. In this case, without loss of generality, one considers the continuity of $\mathcal{J}$ and $\mathcal{F}$ in $L^\beta(\Omega)$. Otherwise, we gain $L^r(\Omega)$ bounds simply from the growth condition on S at $+\infty$, hence we can also refer to the continuity of $\mathcal{J}$ in this space.

Furthermore, from Proposition A.3, we have

$$H^* = \frac{1}{2}W_2^2(\cdot, \rho_0) \text{ on } L^\beta(\Omega)$$

for $H : L^{\beta'}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ given in (2.7) and $\beta' := \frac{\beta}{\beta-1}$. Thus we get $W_2^2(\cdot, \rho_0) \in \Gamma(L^\beta(\Omega))$. Finally, by the parallel argument in Lemma 2.5, H is also in $\Gamma(L^{\beta'}(\Omega))$. From Lemmas B.2 and 2.4, we conclude (2.24).

The rest of the proof is parallel to that of Theorem 2.2. From (2.24) and Lemma 2.13, there exists $\xi \in \partial\mathcal{J}(\rho_1)$ satisfying (2.11), $\overline{\phi}_1 \in \mathcal{K}(\rho_1, \rho_0)$ and $C \in \mathbb{R}$ such that

$$\xi + \frac{\overline{\phi}_1}{\tau} + \Phi - C = 0 \text{ a.e. on } \Omega.$$

and we conclude (2.5).   □

To give a fine characterization of the critical regions $\{\rho = 1\}$ arising in our models, we introduce a new *scalar pressure field p* defined via the subdifferential of $\mathcal{J}$ and its 'nontrivial' value on the set $\{\rho = 1\}$ is due to the nondifferentiability of S at $s_0 = 1$. This construction is inspired by recent works on crowd motion models with hard congestion effects (see, for instance, [**31, 33**]).

DEFINITION 2.4. Let $(\rho_k)_{k=1}^N$ be given by the minimizing movement scheme (2.3) and let $\overline{\phi}_k \in \mathcal{K}(\rho_k, \rho_{k-1})$. For $k \in \{1, \ldots, N\}$, let us define $p_k : \Omega \to \mathbb{R}$ by

$$p_k = p_k(\cdot; \tau) := \begin{cases} \max\left\{C - \dfrac{\overline{\phi}_k}{\tau} - \Phi, S'(1-)\right\} & \text{in } \rho_k^{-1}([0,1)), \\[3mm] C - \dfrac{\overline{\phi}_k}{\tau} - \Phi & \text{in } \rho_k^{-1}(\{1\}), \\[3mm] \min\left\{C - \dfrac{\overline{\phi}_k}{\tau} - \Phi, S'(1+)\right\} & \text{in } \rho_k^{-1}((1, +\infty)), \end{cases} \qquad (2.25)$$

where the constant $C \in \mathbb{R}$ might be different at each step. We observe that by the convexity of S and (2.5) shown in Theorem 2.12, p_k can be written in compact form as

$$p_k = \min \left\{ \max \left\{ C - \frac{\overline{\phi}_k}{\tau} - \Phi, S'(1-) \right\}, S'(1+) \right\}. \tag{2.26}$$

LEMMA 2.14. *For $k \in \{1, \dots, N\}$, $\overline{\phi}_k \in \mathcal{K}(\rho_k, \rho_{k-1})$ and p_k are Lipschitz continuous in Ω.*

Proof. From [**38**, Theorem 1.17], we have that $\overline{\phi}_k$ shares the modulus of continuity of the cost $(x, y) \mapsto |x - y|^2$. On the one hand, as $\overline{\Omega}$ is compact, we conclude that $\overline{\phi}_k$ is Lipschitz continuous. On the other hand, (2.26) implies that p_k is Lipschitz continuous. $\square$

REMARK 2.5. Let us remark that p_k is given in the same manner as in (2.25) or equivalently in (2.26) throughout the article. As the Lipschitz continuity of the Kantorovich potential is independent of the energy, the above lemma holds true for all the models we consider in this paper.

In the following lemma, we deduce more properties of the optimizers of the JKO scheme (2.3).

LEMMA 2.15. *Suppose that Assumption 1.1 takes place. Let $(\rho_k)_{k=1}^N$ be obtained via the minimizing movement scheme (2.3). For $k \in \{1, \dots, N\}$, let $\overline{\phi}_k \in \mathcal{K}(\rho_k, \rho_{k-1})$ given in Theorem 2.12. Then, we have:*

  (i)

$$\rho_k = \begin{cases} 0, & \text{in } f_k^{-1}((-\infty, S'(0+)]), \\ 1, & \text{in } f_k^{-1}([S'(1-), S'(1+)]), \\ (S')^{-1} \circ f_k, & \text{otherwise,} \end{cases} \tag{2.27}$$

  a.e. in Ω, where $f_k := C - \frac{\overline{\phi}_k}{\tau} - \Phi$, and $S'(0+)$ and $S'(1\pm)$ are given in (2.1);
  (ii) *ρ_k is continuous in Ω;*
  (iii) *the formula*

$$|\nabla \rho_k| = \frac{|\nabla f_k|}{S''(\rho_k)}, \tag{2.28}$$

  holds true a.e. in $\rho_k^{-1}(\mathbb{R}^+ \setminus \{1\})$;
  (iv) *if in addition we suppose that Assumption 1.2 takes place with some $\sigma_2 > 0$ and $m \in [1, 2]$, then ρ_k is Lipschitz continuous in Ω with a Lipschitz constant that might degenerate when $\tau \downarrow 0$.*

Proof. (i) By Assumption 1.1, S' is strictly increasing function in $\mathbb{R}^+ \setminus \{1\}$ and

$$S'(0+) < S'(a) < S'(1-) \leqslant S'(1+) < S'(b)$$

for all $a \in (0, 1)$ and $b > 1$. Thus, (2.5) shown in Theorem 2.12 implies that $\rho_k(x) = 0$ for $x \in f_k^{-1}((-\infty, S'(0+)])$ and $\rho_k(x) = 1$ for $x \in f_k^{-1}([S'(1-), S'(1+)])$. Also, S' is invertible in $\mathbb{R}^+ \setminus \{1\}$, therefore (2.5) implies

$$\rho_k(x) = (S')^{-1} \circ f_k(x) \text{ for } x \in f_k^{-1}(S'(\mathbb{R}^+ \setminus \{1\})),$$

and we conclude (2.27) a.e. in Ω.

(ii) Let us show that ρ_k is continuous in Ω. Define $\widehat{(S')^{-1}} : \mathbb{R} \to [0, +\infty)$ by

$$\widehat{(S')^{-1}} := \begin{cases} 0, & \text{in } (-\infty, S'(0+)], \\ 1, & \text{in } [S'(1-), S'(1+)], \\ (S')^{-1}, & \text{otherwise.} \end{cases} \tag{2.29}$$

Note that from (2.27), we have

$$\rho_k = \widehat{(S')^{-1}} \circ f_k \quad \text{a.e. in } \Omega. \tag{2.30}$$

From the continuity and invertibility of S' in $\mathbb{R}^+ \setminus \{1\}$, we conclude that $\widehat{(S')^{-1}}$ is continuous in $\mathbb{R}$. Furthermore, from Lemma 2.14, we know that $\bar{\phi}_k$ is Lipschitz continuous and Φ is Lipschitz continuous by assumption, therefore f_k is Lipschitz continuous. From (2.30), we conclude that ρ_k is continuous in Ω.

(iii) As S is strictly convex and twice differentiable in $\mathbb{R}^+ \setminus \{1\}$ (by Assumption 1.1), $(S')^{-1}$ is differentiable in $S'(\mathbb{R}^+ \setminus \{1\})$ and on this set we have

$$((\widehat{(S')^{-1}})' = ((S')^{-1})' = \frac{1}{S'' \circ (S')^{-1}}. \tag{2.31}$$

Therefore, by (2.30) we have (2.28).

(iv) Using Assumption 1.2, from (2.28), we conclude that a.e. in $\rho_k^{-1}(\mathbb{R}^+ \setminus \{1\})$ we can compute

$$|\nabla \rho_k| = \frac{|\nabla f_k|}{S''(\rho_k)} \leqslant \sigma_2 \rho_k^{2-m} |\nabla f_k|. \tag{2.32}$$

As f_k is Lipschitz continuous and ρ_k is bounded (since it is continuous in Ω), we conclude that ρ_k is Lipschitz continuous in Ω if $m \in [1, 2]$. $\qquad\square$

REMARK 2.6. Let us emphasize that the representation formula (2.27) is independent of the entropy function S, therefore it remains the same for all energies considered in the manuscript. As a consequence, the formula (2.28) holds also true for all the models throughout the paper.

3. Linear diffusion with discontinuities – a cornerstone of our analysis

In this section, we show the well-posedness of (1.4) in the most simple case considered in this paper, that is, when the associated internal energy is an entropy of logarithmic type. We give a fine characterization of the 'critical phase' $\{\rho = 1\}$ via a scalar *pressure field*. In the next sections, we shall see how the results and ideas from this sections will be important to build solutions for problems with more general nonlinearities.

In this section, we assume that $S : [0, +\infty) \to \mathbb{R}$ is defined by

$$S(\rho) := \begin{cases} \rho \log \rho, & \text{for } \rho \in [0, 1], \\ 2\rho \log \rho, & \text{for } \rho \in (1, +\infty). \end{cases} \tag{3.1}$$

Let us note that S defines a continuous superlinear function on $\mathbb{R}^+$ with $S'(1-) = 1$ and $S'(1+) = 2$.

Our main theorem from this section can be formulated as follows.

THEOREM 3.1. *For $\rho_0 \in \mathscr{P}(\Omega)$ such that $\mathcal{J}(\rho_0) < +\infty$ and S given in (3.1), there exists $\rho \in L^1(Q) \cap \mathrm{AC}^2([0,T]; \mathscr{P}(\Omega))$ and $p \in L^2([0,T]; H^1(\Omega)) \cap L^\infty(Q)$ with $\sqrt{\rho} \in L^2([0,T]; H^1(\Omega))$ such that (ρ, p) is a weak solution of*

$$\begin{cases} \partial_t \rho - \Delta(\rho p) - \nabla \cdot (\nabla \Phi \rho) = 0, & \text{in } (0,T) \times \Omega, \\ \rho(0, \cdot) = \rho_0, & \text{in } \Omega, \\ (\nabla(\rho p) + \nabla \Phi) \cdot \boldsymbol{n} = 0, & \text{in } [0,T] \times \partial\Omega, \end{cases} \tag{3.2}$$

in the sense of distribution. Furthermore, (ρ, p) satisfies

$$\begin{cases} p(t,x) = 1 & \text{a.e. in } \{0 < \rho(t,x) < 1\}, \\ p(t,x) \in [1,2] & \text{a.e. in } \{\rho(t,x) = 1\}, \\ p(t,x) = 2 & \text{a.e. in } \{\rho(t,x) > 1\}. \end{cases} \tag{3.3}$$

If in addition $\rho_0 \in L^\infty(\Omega)$ and Φ satisfies (2.4), then $\rho \in L^2([0,T]; H^1(\Omega)) \cap L^\infty(Q)$.

In the proof of the previous theorem, we rely on the minimizing movements scheme associated to the gradient flow of $\mathcal{J}$, defined in (2.3). As technical tools, we define different interpolations between the discrete in time densities $(\rho_k)_{k=0}^N$ and pressures $(p_k)_{k=0}^N$ and obtain a weak solution of (3.2) by sending $\tau \downarrow 0$.

By the definition of p_k in Definition 2.4, the optimality condition (2.5) in Theorem 2.2 applied to the energy from this section, can be simplified as follows.

LEMMA 3.2. *For all $k \in \{1, \ldots, N\}$, there exists $C \in \mathbb{R}$ such that*

$$p_k(1 + \log \rho_k) + \frac{\overline{\phi}_k}{\tau} + \Phi = C \quad \text{a.e.} \tag{3.4}$$

Proof. Note that a subdifferential $\partial S(\rho)$ of $S : [0, +\infty) \to \mathbb{R}$ is given by

$$\partial S(\rho) = \begin{cases} 1 + \log \rho & \text{for } 0 < \rho < 1, \\ [1,2] & \text{for } \rho = 1, \\ 2(1 + \log \rho) & \text{for } \rho > 1. \end{cases} \tag{3.5}$$

Thus, Theorem 2.2 and (2.25) imply

$$p_k = \begin{cases} 1 & \text{in } \rho_k^{-1}((0,1)), \\ C - \dfrac{\overline{\phi}_k}{\tau} - \Phi \in [1,2] & \text{in } \rho_k^{-1}(\{1\}), \\ 2 & \text{in } \rho_k^{-1}((1, +\infty)). \end{cases} \quad \text{a.e.} \tag{3.6}$$

Thus, we simplify (2.5) into (3.4). $\qquad \square$

An easy consequence of the above constructions is the following result, which can be seen as a simplified version of Lemma 2.15.

LEMMA 3.3. *For $k \in \{1, \ldots, N\}$, ρ_k is Lipschitz continuous in Ω. Here, ρ_k is given in (2.3).*

Proof. From (3.4) in Lemma 3.2, we have that

$$\rho_k(x) = \exp\left\{ \frac{1}{p_k(x)}\left(C - \frac{\overline{\phi}_k(x)}{\tau} - \Phi \right) - 1 \right\} \quad \text{a.e.} \tag{3.7}$$

As p_k and $\overline{\phi}_k$ are Lipschitz continuous from Lemma 2.14, Φ is Lipschitz continuous by the assumption and p_k has a lower bound $+1$ from (3.6), (3.7) implies that ρ_k is Lipschitz continuous. □

3.1. *Interpolations between the discrete in time densities, velocities, momenta and pressures*

As technical tools, similarly as it is done in the framework of models developed, for instance, in [**31, 33, 38**], we introduce two different kinds of interpolations between the objects in the title of the subsection. These interpolations actually are independent of the considered energies, and we refer back to them throughout the paper.

Piecewise constant interpolations. Let us define $\rho^\tau, p^\tau : Q \to \mathbb{R}$ and $\mathbf{v}^\tau, \mathbf{E}^\tau : Q \to \mathbb{R}^d$ as follows

$$
\begin{cases}
\rho^\tau(t,x) := \rho_k(x), \\
p^\tau(t,x) := p_k(x), \\
\mathbf{v}^\tau(t,x) := \dfrac{1}{\tau}\nabla\overline{\phi}_k(x), \\
\mathbf{E}^\tau(t,x) := \rho^\tau(t,x)\mathbf{v}^\tau(t,x)
\end{cases}
\qquad \text{for } (t,x) \in ((k-1)\tau, k\tau] \times \Omega \text{ and } k \in \{1,\dots,N\}, \qquad (3.8)
$$

for $(\rho_k)_{k=1}^N$ obtained in (2.3) and $\overline{\phi}_k \in \mathcal{K}(\rho_k, \rho_{k-1})$ given in Theorem 2.2.

By standard arguments on gradient flows (see for instance [**38**, Proposition 8.8], [**33**, Lemma 3.5]), we have the following.

LEMMA 3.4. *It holds that*

$$
\frac{1}{2\tau}\sum_{k=1}^N W_2^2(\rho_k, \rho_{k-1}) = \frac{1}{2\tau}\sum_{k=1}^N \int_\Omega |\nabla\overline{\phi}_k|^2 \,\mathrm{d}\rho_k(x) \leqslant \mathcal{J}(\rho_0) - \inf \mathcal{J}.
$$

Furthermore, there exists a constant $C > 0$ such that for any $0 \leqslant s < t \leqslant T$

$$
W_2(\rho^\tau(t), \rho^\tau(s)) \leqslant C(t - s + \tau)^{\frac{1}{2}}. \qquad (3.9)
$$

PROPOSITION 3.5. *Let $(\rho^\tau)_{\tau>0}$ and $(p^\tau)_{\tau>0}$ given (3.8) and (2.25), respectively. We have the followings.*

(1) *$(p^\tau)_{\tau>0}$ is uniformly bounded in $L^2([0,T]; H^1(\Omega)) \cap L^\infty(Q)$.*
(2) *$(\sqrt{\rho^\tau})_{\tau>0}$ is uniformly bounded in $L^2([0,T]; H^1(\Omega))$.*
(3) *If in addition $\rho_0 \in L^\infty(\Omega)$ and Φ satisfies (2.4), then $(\rho^\tau)_{\tau>0}$ is uniformly bounded in $L^2([0,T]; H^1(\Omega)) \cap L^\infty(Q)$.*

Proof. Step 1. Clearly, by construction, $(p^\tau)_{\tau>0}$ is uniformly bounded. Furthermore, if $\rho_0 \in L^\infty(\Omega)$, then Lemma 2.1 implies that $(\rho^\tau)_{\tau>0}$ is uniformly bounded by a constant depending only on the data for all $t \in [0,T]$.

Step 2. Now, let us show that $(\nabla\sqrt{\rho^\tau})_{\tau>0}$ and $(\nabla p^\tau)_{\tau>0}$ are uniformly bounded in $L^2(Q)$. Let $\overline{\phi}_k \in \mathcal{K}(\rho_k, \rho_{k-1})$. Lemmas 2.14 and 3.3 implies that $\overline{\phi}_k, \rho_k$ and p_k are Lipschitz continuous functions, and therefore by Rademacher's theorem one can differentiate these function a.e. in Ω. Note that $\{\rho_k \neq 1\}$ is an open by the continuity of ρ_k in Lemma 3.3 and thus (3.6) implies

$$
\nabla p_k = 0 \text{ a.e. in } \{\rho_k \neq 1\}. \qquad (3.10)
$$

Therefore, we get

$$
\log \rho_k \nabla p_k = 0 \text{ and } (\rho_k - 1)\nabla p_k = 0 \quad \text{a.e.} \qquad (3.11)
$$

Next, we claim that

$$\nabla p_k \cdot \nabla \rho_k = 0 \quad \text{a.e. in } \Omega. \tag{3.12}$$

From (3.6), the above holds in the open set $\{\rho_k \neq 1\}$ and in the interior of $\{\rho_k = 1\}$, but we point out that $\partial\{\rho_k = 1\}$ may have positive measure even though ρ_k is Lipschitz continuous. In order to show (3.12) in Ω, we apply the coarea formula and (3.10). As ρ_k is Lipschitz and ∇p_k is in $L^1(\Omega)$, we could use the coarea formula in [**27**, Corollary 5.2.6] and conclude that

$$\int_\Omega |\nabla p_k||\nabla \rho_k|\,\mathrm{d}x = \int_\mathbb{R} \int_{(\rho_k)^{-1}(s)} |\nabla p_k|\,\mathrm{d}\mathscr{H}^{d-1}\mathrm{d}s.$$

where $\mathscr{H}^{d-1}$ stands for the $(d-1)$-dimensional Hausdorff measure. From (3.10), we conclude (3.12).

Differentiating (3.4) and applying (3.11) and (3.12), we have

$$-\frac{\nabla\overline{\phi}_k}{\tau} - \nabla\Phi = \nabla(p_k(1 + \log \rho_k)) = \nabla p_k + \frac{p_k}{\rho_k}\nabla \rho_k \quad \text{a.e.} \tag{3.13}$$

From (3.13) and (3.12) again, we have

$$2\rho_k\left(\frac{|\nabla\overline{\phi}_k|^2}{\tau^2} + |\nabla\Phi|^2\right) \geqslant |\nabla p_k|^2 + \frac{p_k^2}{\rho_k}|\nabla \rho_k|^2 \quad \text{a.e.,} \tag{3.14}$$

from where we can write

$$2\rho_k\left(\frac{|\nabla\overline{\phi}_k|^2}{\tau^2} + |\nabla\Phi|^2\right) \geqslant |\nabla p_k|^2 + p_k^2|\nabla\sqrt{\rho_k}|^2 \quad \text{a.e.}$$

As $p_k \in [1, 2]$ (from (3.6)), we have

$$\int_\Omega \left(|\nabla p_k|^2 + |\nabla\sqrt{\rho_k}|^2\right) \leqslant 2\int_\Omega \frac{|\nabla\overline{\phi}_k|^2}{\tau^2}\rho_k\,\mathrm{d}x + 2\|\nabla\Phi\|_{L^\infty}^2.$$

From Lemma 3.4, we conclude that $(\sqrt{\rho^\tau})_{\tau>0}$ and $(p^\tau)_{\tau>0}$ are uniformly bounded in $L^2([0,T]; H^1(\Omega))$ for all $\tau > 0$.

Moreover, if $\rho_0 \in L^\infty(\Omega)$, we have $\|\rho_k\|_{L^\infty} \leqslant \|\rho_0\|_{L^\infty(\Omega)}e^{dT\|[\Delta\Phi]_+\|_{L^\infty}}$ (from Lemma 2.1), and therefore from (3.14) we get

$$\int_\Omega |\nabla p_k|^2\,\mathrm{d}x + \int_\Omega \frac{1}{\|\rho_0\|_{L^\infty(\Omega)}e^{dT\|[\Delta\Phi]_+\|_{L^\infty}}}|\nabla \rho_k|^2\,\mathrm{d}x \leqslant C, \tag{3.15}$$

from where we have $(\rho^\tau)_{\tau>0}$ is uniformly bounded in $L^2([0,T]; H^1(\Omega))$. □

COROLLARY 3.6. *Let* $(\rho^\tau)_{\tau>0}$ *and* $(p^\tau)_{\tau>0}$ *be as in the previous proposition. There exist* $p \in L^2([0,T]; H^1(\Omega))$ *and* $\rho \in L^1(Q)$ *such that*

$$\rho^\tau \to \rho \text{ in } L^1(Q), \text{ as } \tau \downarrow 0,$$

and

$$p^\tau \rightharpoonup p \text{ in } L^2([0,T]; H^1(\Omega)), \text{ as } \tau \downarrow 0.$$

along a subsequence. If in addition $\rho_0 \in L^\infty(\Omega)$ *and* Φ *satisfies* (2.4)*, then we also have* $\rho \in L^2([0,T]; H^1(\Omega))$ *and* $\rho^\tau \to \rho$ *in* $L^2(Q)$*, as* $\tau \downarrow 0$*.*

Proof. The weak sequential compactness of $(p^\tau)_{\tau>0}$ follows from the uniform boundedness in $L^2([0,T]; H^1(\Omega))$ in the previous proposition. Also, as $(\rho^\tau)_{\tau>0}$ has the 'quasi-Hölder' type estimates in Lemma 3.4 and $(\sqrt{\rho^\tau})_{\tau>0}$ is uniformly bounded in $L^2([0,T]; H^1(\Omega))$, we conclude the strong compactness of $(\rho^\tau)_{\tau>0}$ in $L^1(Q)$ by a consequence of a modified version of the

classical Aubin–Lions lemma in Lemma C.2, often used in similar context (see, for instance, [18, Proposition 4.8] and [29, Proposition 5.2]). If $\rho_0 \in L^\infty(\Omega)$, the last statement simply follows from the similar arguments. $\qquad\square$

As a consequence of the above results, we have the following.

LEMMA 3.7. (ρ, p) given in Corollary 3.6 satisfies (3.3).

Proof. Step 1. Let (ρ^τ, p^τ) be defined in (3.8) and (2.25). First, from (3.6), we have

$$(p^\tau - 2)(\rho^\tau - 1)_+ = (p^\tau - 1)(\rho^\tau - 1)_- = 0 \text{ in } Q. \tag{3.16}$$

As it holds that

$$|(\rho^\tau - 1)_+ - (\rho - 1)_+| \leqslant |\rho^\tau - \rho| \text{ and } |(\rho^\tau - 1)_- - (\rho - 1)_-| \leqslant |\rho^\tau - \rho|, \tag{3.17}$$

Proposition 3.5 implies that both $(\rho^\tau - 1)_+ \to (\rho - 1)_+$ and $(\rho^\tau - 1)_- \to (\rho - 1)_-$ in $L^1(Q)$ as $\tau \downarrow 0$ (up to passing to a subsequence).

Step 2. Let us show that for a.e. $t \in [0, T]$

$$\int_\Omega (p(t, x) - 2)(\rho(t, x) - 1)_+ \, dx = 0 \text{ and } \int_\Omega (p(t, x) - 1)(\rho(t, x) - 1)_- \, dx = 0. \tag{3.18}$$

We only show the first one as the parallel arguments work for the second one. From (3.16), we have

$$0 = \int_Q (p^\tau(t, x) - 2)(\rho^\tau(t, x) - 1)_+ \, dx \, dt. \tag{3.19}$$

Recall that up to passing to a subsequence, $(p^\tau)_{\tau > 0}$ convergences weakly$-\star$ in $L^\infty(Q)$ (see Proposition 3.5) and $((\rho^\tau(t, x) - 1)_+)_{\tau > 0}$ converges strongly (from Step 1) in $L^1([0, T \times \Omega])$ as $\tau \downarrow 0$. Combining these with (3.19), we conclude the first equation of (3.18).

As $p^\tau \in [1, 2]$ for p^τ given in (2.25), we have $p \in [1, 2]$ a.e. in Q. Thus, (3.18) implies

$$(p - 2)(\rho - 1)_+ = (p - 1)(\rho - 1)_- = 0 \quad \text{a.e.}$$

and we conclude (3.3). $\qquad\square$

PROPOSITION 3.8. *Let $\boldsymbol{E}^\tau$ be given in* (3.8). *Then up to passing to a subsequence, $(\boldsymbol{E}^\tau)_{\tau > 0}$ weakly-$\star$ converges to*

$$\boldsymbol{E} := -\nabla(p\rho) - \nabla\Phi\rho, \text{ in } \mathscr{D}'(Q; \mathbb{R}^d),$$

as $\tau \downarrow 0$ where and (ρ, p) is given in Corollary 3.6.

Proof. For any test function $\zeta \in C_c^\infty(Q; \mathbb{R}^d)$, we claim that up to passing to a subsequence,

$$\mathcal{I} := \int_Q \zeta \cdot d(\boldsymbol{E}^\tau - \boldsymbol{E}) \to 0, \text{ as } \tau \downarrow 0. \tag{3.20}$$

From (3.11), we have $\log \rho^\tau \nabla p^\tau = 0$ in a.e. in Q and thus it holds that

$$-\boldsymbol{E}^\tau = p^\tau \nabla \rho^\tau + \rho^\tau (1 + \log \rho^\tau) \nabla p^\tau + \nabla \Phi \rho^\tau = \nabla(\rho^\tau p^\tau) + \nabla \Phi \rho^\tau. \tag{3.21}$$

By the weak convergence of $(\rho^\tau)_{\tau > 0}$ to ρ, we already have that

$$\int_Q \zeta \cdot \nabla \Phi \, d\rho^\tau \, dt \to \int_Q \zeta \cdot \nabla \Phi \, d\rho \, dt, \ \tau \downarrow 0,$$

we only focus on the other term. By integration by parts and from the fact that $\zeta \in C_c^\infty(Q; \mathbb{R}^d)$, we study thus

$$\mathcal{I}_1 = \int_Q (\rho^\tau p^\tau - \rho p) \nabla \cdot \zeta \, \mathrm{d}x \, \mathrm{d}t.$$

By subtracting and adding the same term in the above equation, we get

$$\mathcal{I}_1 = \mathcal{I}_2 + \mathcal{I}_3 \text{ where } \mathcal{I}_2 = \int_Q (\rho^\tau - \rho) p^\tau \nabla \cdot \zeta \, \mathrm{d}x \, \mathrm{d}t \text{ and } \mathcal{I}_3 = \int_Q \rho(p^\tau - p) \nabla \cdot \zeta \, \mathrm{d}x \, \mathrm{d}t.$$

From the Hölder inequality, we have

$$|\mathcal{I}_2| \leqslant \|\rho^\tau - \rho\|_{L^1(Q)} \|p^\tau\|_{L^\infty(Q)} \|\nabla \cdot \zeta\|_{L^\infty(Q)}.$$

As $\rho^\tau \to \rho$ in $L^1(Q)$ as $\tau \downarrow 0$ and $\|p^\tau\|_{L^\infty(Q)}$ is uniformly bounded (Proposition 3.5), we conclude $\mathcal{I}_2 \to 0$ as $\tau \downarrow 0$. On the other hand, as $p^\tau \overset{\star}{\rightharpoonup} p$ in $L^\infty(Q)$ as $\tau \downarrow 0$ (Proposition 3.5), and $\rho \in L^1(Q)$ we have $\mathcal{I}_3 \to 0$ as $\tau \downarrow 0$ as well, and thus we conclude (3.20). $\qquad\square$

To arrive to the time continuous PDE satisfied by (ρ, p) from Corollary 3.6, as technical tools (inspired from [31, 33, 38]), we introduce a geodesic interpolation between $(\rho_k)_{k=1}^N$ and we consider the corresponding velocities and momenta as well.

More precisely, we define $\widetilde{\rho}^\tau : [0, T] \to \mathscr{P}(\Omega)$, $\tilde{\mathbf{v}}^\tau, \widetilde{\mathbf{E}}^\tau \in \mathscr{M}(Q; \mathbb{R}^d)$ as follows: for $t \in ((k-1)\tau, k\tau]$ and $k \in \{1, \dots, N\}$

$$(3.22) \quad \begin{cases} \widetilde{\rho}^\tau(t, x) := \left(\frac{k\tau - t}{\tau} \mathbf{v}^\tau(t, x) + \mathrm{id}\right)_{\#} \rho^\tau(t, x), \\ \tilde{\mathbf{v}}^\tau(t, x) := \mathbf{v}^\tau(t, x) \circ \left(\frac{k\tau - t}{\tau} \mathbf{v}^\tau(t, x) + \mathrm{id}\right)^{-1}, \\ \widetilde{\mathbf{E}}^\tau(t, x) := \widetilde{\rho}^\tau(t, x) \tilde{\mathbf{v}}^\tau(t, x), \end{cases}$$

where ρ^τ and $\mathbf{v}^\tau$ are given in (3.8).

Following the very same steps as in [38, Lemma 8.9] and [33, Step 2 in Theorem 3.1], we have the following.

LEMMA 3.9. *We have that:*

(i) *$(\widetilde{\rho}^\tau)_{\tau > 0}$ is uniformly bounded in $\mathrm{AC}^2([0, T]; \mathscr{P}(\Omega))$;*

(ii) *there exists $C > 0$ such that $\int_0^T \int_\Omega |\tilde{v}^\tau|^2 \, \mathrm{d}\widetilde{\rho}_t^\tau \, \mathrm{d}t \leqslant C$;*

(iii) *$(\widetilde{\mathbf{E}}^\tau)_{\tau > 0}$ is uniformly bounded in $\mathscr{M}(Q; \mathbb{R}^d)$.*

As a consequence, we have that along a subsequence

(iv) *$\sup_{t \in [0, T]} W_2(\widetilde{\rho}_t^\tau, \rho_t) \to 0$, as $\tau \downarrow 0$;*

(v) *$\widetilde{\mathbf{E}}^\tau \overset{\star}{\rightharpoonup} \mathbf{E}$, in $\mathscr{M}(Q; \mathbb{R}^d)$, as $\tau \downarrow 0$;*

where ρ is given in Proposition 3.5 and $\mathbf{E}$ is given in Proposition 3.8.

Now, we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let us underline that the main reason for introducing the interpolations $(\widetilde{\rho}^\tau, \widetilde{\mathbf{E}}^\tau)$ is that by construction, they satisfy the PDE

$$(3.23) \quad \begin{cases} \partial_t \tilde{\rho}^\tau + \nabla \cdot \widetilde{\mathbf{E}}^\tau = 0, & \text{in } (0, T) \times \Omega, \\ \tilde{\rho}^\tau(0, \cdot) = \rho_0, & \text{in } \Omega, \\ \widetilde{\mathbf{E}}^\tau \cdot \mathbf{n} = 0, & \text{on } [0, T] \times \partial\Omega, \end{cases}$$

in the distributional sense. Then, Lemma 3.9 and Proposition 3.8 allow us to conclude that (ρ, p) satisfies (3.2) in the distributional sense. Finally, from Lemma 3.7, we conclude that (ρ, p) satisfies (3.3). The thesis of the theorem follows. $\qquad\square$

4. Linear diffusion on $\{\rho < 1\}$ and porous medium type diffusion on $\{\rho > 1\}$

As we will see below, in this section the diffusion coefficients and the diffusion rates are not necessarily supposed to be the same in the regions $\{\rho < 1\}$ and $\{\rho > 1\}$. Therefore, a technical difficulty arises, because of the lack of a simple way (as in (3.4)) to derive the first-order necessary optimality conditions for the minimizing movement scheme. To overcome this issue, instead, we use a particular decomposition for S, which allows us to use the construction from Section 3.

In this section too, we impose Assumption 1.1. If $\rho_0 \notin L^\infty(\Omega)$, we impose additionally (1.10). Furthermore, throughout this section we suppose also the following: $S : [0, +\infty) \to \mathbb{R}$ satisfies

$$\frac{\rho^{-1}}{\sigma_2} \leqslant S''(\rho) \text{ in } (0, 1) \tag{4.1}$$

for some constant $\sigma_2 > \sigma_1$ for σ_1 given in (1.10a). This corresponds to (1.9) with $m = 1$.

A direct consequence of the above assumption is the following result.

LEMMA 4.1. $S : [0, +\infty) \to \mathbb{R}$ satisfies

$$S'(0+) = -\infty. \tag{4.2}$$

Proof. Integrating (4.1) from $\frac{1}{2}$ to ρ, it holds that

$$S'\left(\frac{1}{2}\right) - S'(\rho) \geqslant \frac{1}{\sigma_2}\left(\log \frac{1}{2} - \log \rho\right).$$

As $\sigma_2 > 0$, we conclude that

$$S'(\rho) \leqslant S'\left(\frac{1}{2}\right) - \frac{1}{\sigma_2} \log \frac{1}{2} + \frac{1}{\sigma_2} \log \rho \quad \to \quad -\infty \qquad \text{as } \rho \to 0^+. \qquad\square$$

EXAMPLE 4.1. For $m > 1$, $S : [0, +\infty) \to \mathbb{R}$ given by

$$S(\rho) := \begin{cases} \rho \log \rho, & \text{for } \rho \in [0, 1], \\ \dfrac{1}{m-1}(\rho^m - 1), & \text{for } \rho \in (1, +\infty). \end{cases}$$

Note that Assumption 1.1 follows from the smoothness and strict convexity of S in $\mathbb{R}^+ \setminus \{1\}$ and

$$S'(1-) = 1 < S'(1+) = \frac{m}{m-1}.$$

(4.1) is obtained by

$$\rho S''(\rho) = \begin{cases} 1, & \text{for } \rho \in (0, 1), \\ m\rho^{m-1} \geqslant m, & \text{for } \rho \in (1, +\infty). \end{cases}$$

(1.10) is also fulfilled with $r = m$.

In this case, $L_S(\rho, p)(x)$ is given by

$$L_S(\rho, p)(x) = \begin{cases} \rho(x), & \text{if } 0 < \rho(x) < 1, \\[2mm] p(x) \in \left[1, \dfrac{m}{m-1}\right], & \text{if } \rho(x) = 1, \\[2mm] \rho(x)^m + \dfrac{1}{m-1}, & \text{if } \rho(x) > 1. \end{cases}$$

Our main theorem from this section reads as:

THEOREM 4.2. *Suppose that* (1.10) *and* (4.1) *hold true. For* $\rho_0 \in \mathscr{P}(\Omega)$ *such that* $\mathcal{J}(\rho_0) < +\infty$, *there exists* $\rho \in L^\beta(Q) \cap \mathrm{AC}^2([0, T]; \mathscr{P}(\Omega))$ *for* β *given in* (2.23) *and* $p \in L^2([0, T]; H^1(\Omega)) \cap L^\infty(Q)$ *with* $\sqrt{\rho} \in L^2([0, T]; H^1(\Omega))$ *such that* (ρ, p) *is a weak solution of*

$$\begin{cases} \partial_t \rho - \Delta(L_S(\rho, p)) - \nabla \cdot (\nabla \Phi \rho) = 0, & \text{in } (0, T) \times \Omega, \\ \rho(0, \cdot) = \rho_0, & \text{in } \Omega, \\ (\nabla(L_S(\rho, p)) + \nabla \Phi \rho) \cdot \boldsymbol{n} = 0, & \text{in } [0, T] \times \partial\Omega, \end{cases} \quad (4.3)$$

in the sense of distribution. Furthermore, (ρ, p) *satisfies for a.e.* $(t, x) \in Q$

$$\begin{cases} p(t, x) = S'(1-) & \text{if } 0 < \rho(t, x) < 1, \\ p(t, x) \in [S'(1-), S'(1+)] & \text{if } \rho(t, x) = 1, \\ p(t, x) = S'(1+) & \text{if } \rho(t, x) > 1. \end{cases} \quad (4.4)$$

If in addition $\rho_0 \in L^\infty(\Omega)$ *and* Φ *satisfies* (2.4), *we can drop* (1.10) *from the statement and we obtain that* $\rho \in L^2([0, T]; H^1(\Omega)) \cap L^\infty(Q)$.

Let us briefly explain the outline of the proof. First, we define S_a and $S_b : [0, +\infty) \to \mathbb{R}$ by

$$S_a(\rho) := \begin{cases} S'(1-)\rho \log \rho, & \text{for } \rho \in [0, 1], \\ S'(1+)\rho \log \rho, & \text{for } \rho \in (1, +\infty), \end{cases} \quad (4.5)$$

and

$$S_b(\rho) := S(\rho) - S_a(\rho). \quad (4.6)$$

We show the convexity of S_a and twice differentiability of S_b in Lemma 4.4. This particular decomposition will be useful when deriving optimality conditions in our minimizing movement scheme. Under (4.1), we are able to apply similar arguments as the ones in Section 3.

We point out that Lemma 4.1 implies that $\rho_k > 0$ a.e. (see Lemma A.4). Theorem 2.2 and (4.1) yield that ρ_k satisfies the following lemma.

LEMMA 4.3. *Let* $(\rho_k)_{k=1}^N$ *be obtained via the minimizing movement scheme* (2.3). *For* $k \in \{1, \ldots, N\}$ *and* $\overline{\phi}_k \in \mathcal{K}(\rho_k, \rho_{k-1})$ *given in Theorem* 2.2, *we have that*

$$\rho_k = \begin{cases} 1, & \text{in } f_k^{-1}([S'(1-), S'(1+)]), \\ (S')^{-1} \circ f_k, & \text{otherwise,} \end{cases} \quad (4.7)$$

in Ω, *where* $f_k := C - \dfrac{\overline{\phi}_k}{\tau} - \Phi$, *and* $S'(1\pm)$ *is given in* (2.1). *In particular,* ρ_k *is Lipschitz continuous in* Ω *with a Lipschitz constant that might degenerate when* $\tau \downarrow 0$.

Proof. This result is a direct consequence of Lemma 2.15. Indeed, Lemma A.4 shows that $\rho_k > 0$ a.e. in Ω, therefore $\mathrm{spt}(\rho_k) = \Omega$. Also, in Assumption 1.2 we have $m = 1$. Moreover, since

Lemma 4.1 yields that $S'(0+) = -\infty$, and since we can eventually modify ρ_k on a negligible set, we can use the representation (4.7) for ρ_k everywhere in Ω. $\square$

The following properties hold for S_a and S_b.

LEMMA 4.4. S_a is convex and continuous in $\mathbb{R}^+$. Also, S_b is continuously differentiable and S_b' is locally Lipschitz continuous in $\mathbb{R}^+$. In particular, we have

$$S_b(1) = S(1) \text{ and } S_b'(1) = 0. \tag{4.8}$$

Proof. From convexity of S, it holds that $S'(1-) < S'(1+)$ and thus S_a is convex. It is obviously also continuous by construction.

On the other hand, by the construction in (4.5), $S_b(\rho)$ is differentiable on $\mathbb{R}^+ \setminus \{1\}$. Let us show that $S_b(\rho)$ is differentiable at $\rho = 1$. By differentiating (4.5) on $\mathbb{R}^+ \setminus \{1\}$, we have that

$$S_a'(\rho) = \begin{cases} S'(1-)(1 + \log \rho), & \text{for } \rho \in (0, 1), \\ S'(1+)(1 + \log \rho), & \text{for } \rho \in (1, +\infty). \end{cases}$$

Therefore, we conclude that

$$S_b'(1-) = S'(1-) - S_a'(1-) = 0 \text{ and } S_b'(1+) = S'(1+) - S_a'(1+) = 0$$

and S_b is continuously differentiable in $\mathbb{R}^+$. As both S' and S_a' are locally Lipschitz in $\mathbb{R}^+ \setminus \{1\}$, S_b' is also locally Lipschitz continuous in $\mathbb{R}^+ \setminus \{1\}$. As S_b' is continuous, we conclude that S_b' is locally Lipschitz continuous in $\mathbb{R}^+$. Finally, $S_b(1) = S(1)$ follows from $S_a(1) = 0$. $\square$

LEMMA 4.5. Let $(\rho_k)_{k=1}^N$ be obtained via the minimizing movement scheme (2.3) and let $(p_k)_{k=1}^N$ be constructed in (2.25). For $k \in \{1, \ldots, N\}$, we have that

$$p_k(1 + \log \rho_k) + S_b'(\rho_k) + \frac{\overline{\phi_k}}{\tau} + \Phi = C, \text{ a.e. in } \Omega. \tag{4.9}$$

Proof. We first note that Lemma 4.1 implies that $\rho_k > 0$ a.e. in Ω (see also Lemma A.4). From Theorem 2.2, we have

$$p_k = \begin{cases} S'(1-), & \text{in } \rho_k^{-1}((0, 1)), \\ C - \dfrac{\overline{\phi_k}}{\tau} - \Phi, & \text{in } \rho_k^{-1}(\{1\}), \\ S'(1+), & \text{in } \rho_k^{-1}((1, +\infty)). \end{cases} \tag{4.10}$$

As $S_b'(1) = 0$, (4.9) holds in $\rho_k^{-1}(\{1\})$ by (4.10).

Finally, from (4.10), in $\rho_k^{-1}(\mathbb{R}^+ \setminus \{1\})$ we have that

$$S_a'(\rho_k) = p_k(1 + \log \rho_k). \tag{4.11}$$

As $S' = S_a' + S_b'$ in $\rho_k^{-1}(\mathbb{R}^+ \setminus \{1\})$, we conclude (4.9) from Proposition 2.8. $\square$

REMARK 4.2. As S_b is differentiable, in the previous proof we also used the fact

$$\partial S = \partial S_a + S_b',$$

the proof of which can be found, for instance, in [**28**, Corollary 1.12.2].

Similarly as in Section 3, we construct piecewise constant and continuous in time interpolations $(\rho^\tau, \mathbf{v}^\tau, \mathbf{E}^\tau)$ and $(\tilde\rho^\tau, \tilde{\mathbf{v}}^\tau, \widetilde{\mathbf{E}}^\tau)$. Similarly to Proposition 3.5, we can formulate the following result.

PROPOSITION 4.6. $(\rho^\tau)_{\tau>0}$ and $(p^\tau)_{\tau>0}$ satisfy the exact same bounds as in Proposition 3.5.

Proof. Let us notice first that the uniform boundedness of $(p^\tau)_{\tau>0}$ in $L^\infty(Q)$ follows from the construction in (4.10).

Let us show the other estimates from Proposition 3.5. Note that both S_b' and ρ_k are locally Lipschitz continuous (as we have shown in Lemma 4.4 and Lemma 4.3). Thus, Lemma 4.5 implies

$$-\frac{\nabla\overline{\phi}_k}{\tau} - \nabla\Phi = \nabla p_k + \left(\frac{p_k}{\rho_k} + S_b''(\rho_k)\right)\nabla\rho_k, \quad \text{a.e. in } \Omega. \tag{4.12}$$

By the parallel computation as in (3.14), we conclude that

$$2\rho_k\frac{|\nabla\overline{\phi}_k|^2}{\tau^2} + 2\rho_k|\nabla\Phi|^2 \geqslant |\nabla p_k|^2 + \rho_k\left(\frac{p_k}{\rho_k} + S_b''(\rho_k)\right)^2|\nabla\rho_k|^2.$$

From Lemma 4.8, we have

$$\rho_k\left(\frac{p_k}{\rho_k} + S_b''(\rho_k)\right)^2|\nabla\rho_k|^2 \geqslant \frac{1}{\sigma_2^2\rho_k}|\nabla\rho_k|^2 \text{ a.e. in } \Omega.$$

The rest of arguments is parallel to *Step* 3 in Proposition 3.5, thus we conclude the thesis of the proposition. □

COROLLARY 4.7. *Up to passing to a suitable subsequence, the sequences* $(\rho^\tau)_{\tau>0}$ *and* $(p^\tau)_{\tau>0}$ *converge in the same sense as in Corollary* 3.6.

REMARK 4.3. From (4.12), we have

$$2\rho_k\frac{|\nabla\overline{\phi}_k|^2}{\tau^2} + 2\rho_k|\nabla\Phi|^2 \geqslant \frac{|\nabla(F(\rho_k,p_k))|^2}{\rho_k}, \text{ where } F(\rho,p) := p\rho + \rho S_b'(\rho) - S_b(\rho).$$

Then, if $\rho_0 \in L^\infty(\Omega)$, this observation together with the uniform L^∞ bounds on ρ^τ imply uniform $L^2([0,T]; H^1(\Omega))$ bounds on $F(\rho^\tau, p^\tau)$.

As the proof of Proposition 3.5, we rely on the coarea formula when proving the following result.

LEMMA 4.8. *For* $(\rho_k)_{k=1}^N$ *and* $(p_k)_{k=1}^N$ *given in* (2.3) *and* (2.25), *it holds that*

$$|p_k + \rho_k S_b''(\rho_k)||\nabla\rho_k| \geqslant \frac{1}{\sigma_2}|\nabla\rho_k| \text{ a.e. in } \Omega. \tag{4.13}$$

Proof. If $x \in \{\rho_k \neq 1\}$, then (4.11) implies

$$\frac{p_k(x)}{\rho_k(x)} + S_b''(\rho_k(x)) = S_a''(\rho_k(x)) + S_b''(\rho_k(x)) = S''(\rho_k(x)). \tag{4.14}$$

From (4.1), we conclude

$$|p_k + \rho_k S_b''(\rho_k)| \geqslant \frac{1}{\sigma_2} \text{ a.e. in } \{\rho_k \neq 1\}. \tag{4.15}$$

Recall that ρ_k is Lipschitz continuous (cf. Lemma 4.3) and thus

$$\nabla\rho_k = 0 \quad \text{a.e. in } \{\rho_k = 1\}$$

(see, for instance, [**23**, Theorem 4.(iv), Section 4.2.2]). Therefore, we conclude (4.13). □

Proof of Theorem 4.2. As an initial observation, let us remark that by similar arguments as in Lemma 3.9, one obtains the same estimates for the continuous in time interpolations $(\tilde{\rho}^\tau, \tilde{\mathbf{v}}^\tau, \widetilde{\mathbf{E}}^\tau)$, and by passing to the limit as $\tau \downarrow 0$, we obtain a continuity equation of the form

$$\partial_t \rho + \nabla \cdot \mathbf{E} = 0.$$

Since the limits of $(\tilde{\rho}^\tau, \widetilde{\mathbf{E}}^\tau)$ and $(\rho^\tau, \mathbf{E}^\tau)$ are the same, it remains to identify the limit of the latter one to get the precise form of our limit equation.

Step 1. From direct computation as in (3.21), we obtain that

$$-\mathbf{E}^\tau = \rho^\tau \nabla(S_b'(\rho^\tau) + p^\tau(1 + \log \rho^\tau)) + \rho^\tau \nabla \Phi$$

$$= \nabla(\rho^\tau S_b'(\rho^\tau) - S_b(\rho^\tau) + S_b(1) + p^\tau \rho^\tau) + \rho^\tau \nabla \Phi. \tag{4.16}$$

From Proposition 4.6 and Corollary 4.7, we can claim that

$$\nabla(\rho^\tau S_b'(\rho^\tau) - S_b(\rho^\tau) + S_b(1) + p^\tau \rho^\tau) \to \nabla(\rho S_b'(\rho) - S_b(\rho) + S_b(1) + p\rho), \tag{4.17}$$

as $\tau \downarrow 0$ in the sense of distribution. Indeed, using the strong $L^1(Q)$ compactness of $(\rho^\tau)_{\tau>0}$ and the weak-$\star$ compactness of $(p^\tau)_{\tau>0}$ in $L^\infty(Q)$, we can pass to the limit $\rho^\tau p^\tau$. Recall that $(\rho^\tau)_{\tau>0}$ in uniformly bounded in $L^\beta(Q)$ for β given in (2.23). As $r < \beta$, Corollary 4.7 yields the convergence of $(\rho^\tau)_{\tau>0}$ in $L^r(Q)$. As the growth rate of $\rho S_b'(\rho)$ and $S_b(\rho)$ is r, we conclude that $\rho^\tau S_b'(\rho^\tau) - S_b(\rho^\tau) \to \rho S_b'(\rho) - S_b(\rho)$ in $L^1(Q)$ as $\tau \downarrow 0$.

Step 2. Let us show that

$$\rho S_b'(\rho) - S_b(\rho) + S_b(1) + p\rho = L_S(\rho, p). \tag{4.18}$$

By parallel arguments as in Lemma 3.7, we conclude that (ρ, p) satisfies (4.4). Thus, it holds that

$$\rho S_a'(\rho) - S_a(\rho) = p\rho, \text{ a.e. in } \rho^{-1}(\mathbb{R}^+ \setminus \{1\}) \tag{4.19}$$

and we conclude (4.18) a.e. in $\rho^{-1}(\mathbb{R}^+ \setminus \{1\})$. From (4.17) and (4.18), we conclude (4.3).

Furthermore, from Lemma 4.4, we obtain that

$$\rho S_b'(\rho) - S_b(\rho) + S_b(1) + p\rho = p \text{ in } \rho^{-1}(\{1\}).$$

and we conclude (4.18) a.e. in $\rho^{-1}(\{1\})$. $\square$

In particular, (4.3) can be also represented in the form of a continuity equation, as we show below.

THEOREM 4.9. *Suppose that* (1.10) *and* (4.1) *hold true. Let* ρ_0 *and* (ρ, p) *be given in Theorem* 4.2. *If*

$$r \geqslant \frac{3d - 4}{2d}, \tag{4.20}$$

then (ρ, p) *also satisfies*

$$\begin{cases} \partial_t \rho - \nabla \cdot \left(\rho \nabla\left(S'(\rho)\mathbb{1}_{\{\rho \neq 1\}} + p\mathbb{1}_{\{\rho = 1\}}\right)\right) - \nabla \cdot (\rho \nabla \Phi) = 0, & \text{in } (0, T) \times \Omega, \\ \rho(0, \cdot) = \rho_0, & \text{in } \Omega, \\ \rho\left[\nabla\left(S'(\rho)\mathbb{1}_{\{\rho \neq 1\}} + p\mathbb{1}_{\{\rho = 1\}}\right) + \nabla \Phi\right] \cdot \boldsymbol{n} = 0, & \text{in } [0, T] \times \partial\Omega, \end{cases} \tag{4.21}$$

in the sense of distribution. If in addition $\rho_0 \in L^\infty(\Omega)$ *and* Φ *satisfies* (2.4), *we can drop* (1.10) *and* (4.20) *from the statement.*

REMARK 4.4. *Note that* (4.20) *is equivalent to* $\beta \geqslant 2$ *for* β *given in* (2.23) *and the inequality holds for any* $r \geqslant 1$ *if* $d = 1$ *or* $d = 2$.

Proof. Note that (4.4) and (4.5) imply that $S_a'(\rho) = p(1 + \log \rho)$ in $\rho^{-1}(\mathbb{R}^+ \setminus \{1\})$. Furthermore, from (4.6) and (4.8), it holds that

$$\mathcal{I}_1 := S'(\rho)\mathbb{1}_{\{\rho \neq 1\}} + p\mathbb{1}_{\{\rho = 1\}} = p(1 + \log \rho) + S_b'(\rho). \tag{4.22}$$

From $p \in L^2([0,T]; H^1(\Omega))$ and (4.4), we obtain

$$\rho \log \rho \nabla p = 0 \text{ a.e.} \tag{4.23}$$

From (4.22) and (4.23), we have

$$\rho \nabla \mathcal{I}_1 = \rho \nabla p + p \nabla \rho + \rho \nabla(S_b'(\rho)).$$

Next, we claim that

$$\rho \nabla p + p \nabla \rho + \rho \nabla(S_b'(\rho)) \in L^1(Q).$$

Consider the first term $\rho \nabla p$. Recall from Theorem 4.2 that $\nabla p \in L^2(Q)$. If $\rho_0 \in L^\infty(\Omega)$, then $\rho \in L^\infty(Q)$ from Lemma 2.1 and thus $\rho \nabla p \in L^1(Q)$. On the other hand, if (4.20) is fulfilled, then β given in (2.23) is greater than or equal to 2. As $\rho \in L^\beta(Q)$ from Lemma 2.11, we obtain $\rho \in L^2(Q)$ and thus $\rho \nabla p \in L^1(Q)$.

Furthermore, as $\nabla\sqrt{\rho} \in L^2(Q)$, $\nabla \rho = 2\rho^{\frac{1}{2}} \nabla\sqrt{\rho} \in L^1(Q)$ and the second term is in $L^1(Q)$. Finally,

$$\rho \nabla(S_b'(\rho)) = 2\rho^{\frac{3}{2}} S_b''(\rho) \nabla\sqrt{\rho}.$$

As the growth rate of $\rho^{\frac{3}{2}} S_b''(\rho)$ is $r - \frac{1}{2}$ and $r - \frac{1}{2} \leqslant \frac{\beta}{2}$, $\rho \in L^\beta(Q)$ implies $\rho^{\frac{3}{2}} S_b''(\rho) \in L^2(Q)$ and the last term is in $L^1(Q)$.

Finally, we have

$$\rho \nabla \mathcal{I}_1 = \nabla(\rho p + \rho S_b'(\rho) - S_b(\rho) + S_b(1)) = \nabla L_S(\rho, p)$$

for L_S given in (1.5). By Theorem 4.2, we conclude that (ρ, p) is a weak solution of (4.21). $\square$

5. Porous medium type diffusion on $\{\rho < 1\}$ and general diffusion on $\{\rho > 1\}$

Similarly to the classical porous medium equation, in this section we do not expect solutions to be fully supported. As in Section 3, let us first study an example with a particular nonlinearity.

5.1. Same diffusion exponent

In this subsection, we suppose that $S : [0, +\infty) \to \mathbb{R}$ is defined by

$$S(\rho) := \begin{cases} \dfrac{\rho^m}{m-1}, & \text{for } \rho \in [0,1], \\[2mm] \dfrac{2\rho^m}{m-1} - \dfrac{1}{m-1}, & \text{for } \rho \in (1, +\infty). \end{cases} \tag{5.1}$$

where $m > 1$.

Our main theorem in this section can be formulated as follows.

THEOREM 5.1. *For $\rho_0 \in \mathscr{P}(\Omega)$ such that $\mathcal{J}(\rho_0) < +\infty$ and S given in (5.1), there exists $\rho \in L^\beta(Q) \cap \mathrm{AC}^2([0,T]; (\mathscr{P}(\Omega), W_2))$ and $p \in L^2([0,T]; H^1(\Omega)) \cap L^\infty(Q)$ with $\rho^{m-\frac{1}{2}} \in L^2([0,T]; H^1(\Omega))$ such that (ρ, p) is a weak solution of*

$$\begin{cases} \partial_t \rho - \Delta([(m-1)\rho^m + 1]\frac{p}{m}) - \nabla \cdot (\nabla\Phi\rho) = 0, & \text{in } (0,T) \times \Omega, \\ \rho(0, \cdot) = \rho_0, & \text{in } \Omega, \\ (\nabla([(m-1)\rho^m + 1]\frac{p}{m}) + \nabla\Phi\rho) \cdot \boldsymbol{n} = 0, & \text{in } [0,T] \times \partial\Omega, \end{cases} \tag{5.2}$$

in the sense of distribution. Furthermore, (ρ, p) satisfies

$$
\begin{cases}
p(t, x) = \dfrac{m}{m-1} & \text{a.e. in } \{0 < \rho(t, x) < 1\}, \\[3mm]
p(t, x) \in \left[\dfrac{m}{m-1}, \dfrac{2m}{m-1} \right] & \text{a.e. in } \{\rho(t, x) = 1\}, \\[3mm]
p(t, x) = \dfrac{2m}{m-1} & \text{a.e. in } \{\rho(t, x) > 1\}.
\end{cases}
\tag{5.3}
$$

In addition, if $\rho_0 \in L^\infty(\Omega)$ and Φ satisfies (2.4), then $\rho \in L^\infty(Q)$ and $\rho^m \in L^2([0, T]; H^1(\Omega))$.

Let us recall the definition of $(\rho_k)_{k=1}^N$ and $(p_k)_{k=1}^N$ from (2.3) and (2.25), respectively. Let us underline that in the setting of this section due to the structure of the nonlinearity we typically expect $\mathrm{spt}(\rho_k)$ to be a proper subset of Ω, unlike in the case of Lemma A.4 which was used in Sections 3 and 4. For this reason, we expect the Lipschitz continuity of ρ_k^{m-1} instead of ρ_k. The following result is a simplified version of Lemma 2.15, tailored to the entropy function (5.1).

LEMMA 5.2. *For all $k \in \{1, \dots, N\}$, there exists $C \in \mathbb{R}$ such that*

$$
\rho_k^{m-1} p_k = \left(C - \frac{\overline{\phi}_k}{\tau} - \Phi \right)_+ \quad \text{a.e.}
\tag{5.4}
$$

In particular, ρ_k^{m-1} is Lipschitz continuous. Here, $\overline{\phi}_k$ is given in Theorem 2.2.

Proof. Note that

$$
\partial S(\rho) =
\begin{cases}
\dfrac{m}{m-1} \rho^{m-1} & \text{for } 0 < \rho < 1, \\[3mm]
\left[\dfrac{m}{m-1}, \dfrac{2m}{m-1} \right] & \text{for } \rho = 1, \qquad \text{and} \\[3mm]
\dfrac{2m}{m-1} \rho^{m-1} & \text{for } \rho > 1.
\end{cases}
$$

$$
p_k =
\begin{cases}
\dfrac{m}{m-1} & \text{in } \rho_k^{-1}([0, 1)), \\[3mm]
C - \dfrac{\overline{\phi}_k}{\tau} - \Phi & \text{in } \rho_k^{-1}(\{1\}), \qquad \text{a.e.} \\[3mm]
\dfrac{2m}{m-1} & \text{in } \rho_k^{-1}((1, +\infty)).
\end{cases}
\tag{5.5}
$$

for p_k given in (2.25). Then, Theorem 2.12 implies

$$
\rho_k^{m-1} p_k + \frac{\overline{\phi}_k}{\tau} + \Phi = C \text{ a.e. on } \mathrm{spt}(\rho_k)
\tag{5.6}
$$

for some constant $C \in \mathbb{R}$.

Moreover, if $\rho_k = 0$ a.e. on some set $A \subset \Omega$, then Theorem 2.12 and $S'(0+) = 0$ (from (5.5)) imply that

$$
C - \frac{\overline{\phi}_k}{\tau} - \Phi \leqslant 0 \text{ a.e. in } A,
$$

and we conclude (5.4).

Next, recall that Lemma 2.14 yields that $\overline{\phi}_k$ is Lipschitz continuous and thus $(C - \frac{\overline{\phi}_k}{\tau} - \Phi)_+$ is Lipschitz continuous as well. As Lemma 2.14 yields the Lipschitz continuity of p_k, and since

this has a positive lower bound $\frac{m}{m-1}$ (from (5.5) and (2.5)), we conclude that ρ_k^{m-1} is also Lipschitz continuous. $\square$

LEMMA 5.3. *Let* $(\rho^\tau)_{\tau>0}, (p^\tau)_{\tau>0}$ *stand for the piecewise constant interpolations given in* (3.8) *and* (2.25), *respectively. Then* $((\rho^\tau)^{m-\frac{1}{2}})_{\tau>0}$ *and* $(p^\tau)_{\tau>0}$ *are uniformly bounded in* $L^2([0,T]; H^1(\Omega))$.

Proof. From Lemma 5.2, it holds that

$$\mathcal{I}_1 := -\rho_k^{\frac{1}{2}}\nabla\Phi - \rho_k^{\frac{1}{2}}\frac{\nabla\overline{\phi}_k}{\tau} = \rho_k^{\frac{1}{2}}\nabla(\rho_k^{m-1}p_k) \quad \text{a.e.} \tag{5.7}$$

As p_k and ρ_k^{m-1} are Lipschitz continuous from Lemma 5.2, we have

$$\mathcal{I}_1 = \rho_k^{\frac{1}{2}}p_k\nabla(\rho_k^{m-1}) + \rho_k^{m-\frac{1}{2}}\nabla p_k \quad \text{a.e. on } \mathrm{spt}(\rho_k). \tag{5.8}$$

Furthermore, since we have the Lipschitz continuity of ρ_k^{m-1} and (5.5), we apply a parallel argument as in the proof of Proposition 3.5 and conclude that

$$(\rho_k^{m-\frac{1}{2}} - 1)\nabla p_k = 0 \text{ and } \nabla(\rho_k^{m-1})\cdot\nabla p_k = 0 \quad \text{a.e. on } \Omega. \tag{5.9}$$

From (5.8) and (5.9), we have that

$$\mathcal{I}_1^2 = p_k^2|\rho_k^{\frac{1}{2}}\nabla(\rho_k^{m-1})|^2 + |\nabla p_k|^2 \quad \text{a.e. on } \mathrm{spt}(\rho_k). \tag{5.10}$$

As $p_k \geqslant \frac{m}{m-1}$ a.e. in Ω as in (5.5), we conclude that

$$\mathcal{I}_1^2 \geqslant \left(\frac{m}{m-1}\right)^2|\rho_k^{\frac{1}{2}}\nabla(\rho_k^{m-1})|^2 + |\nabla p_k|^2 \quad \text{a.e. on } \mathrm{spt}(\rho_k). \tag{5.11}$$

(5.9) yields that $\nabla p_k = 0$ a.e. on $\mathrm{spt}(\rho_k)^c = \{\rho_k = 0\}$. Furthermore, as ρ_k^{m-1} is Lipschitz continuous (see Lemma 5.2), we have

$$\rho_k^{\frac{1}{2}}\nabla(\rho_k^{m-1}) = 0 \quad \text{a.e. on } \mathrm{spt}(\rho_k)^c.$$

Therefore, (5.11) holds a.e. on Ω.

On the other hand, applying Lemma 3.4, it holds that

$$\int_0^T\int_\Omega \mathcal{I}_1^2\,\mathrm{d}x\,\mathrm{d}t \leqslant 2(\mathcal{J}(\rho_0) - \inf\mathcal{J}) + T\mathscr{L}^d(\Omega)\|\nabla\Phi\|_{L^\infty}.$$

As $\rho_k^{\frac{1}{2}}\nabla(\rho_k^{m-1}) = \frac{m-1}{m-\frac{1}{2}}\nabla(\rho_k^{m-\frac{1}{2}})$ and since $(\rho^\tau)_{\tau>0}$ is uniformly bounded in $L^\beta(Q)$ (with $\beta > m - 1/2$, see Lemma 2.10) we conclude that $((\rho^\tau)^{m-\frac{1}{2}})_{\tau>0}$ and $(p^\tau)_{\tau>0}$ are uniformly bounded in $L^2([0,T]; H^1(\Omega))$ (since $(p^\tau)_{\tau>0}$ is also uniformly bounded) and therefore we conclude. $\square$

As a consequence of Lemma 5.3 and Lemma C.2, we have the following convergence.

COROLLARY 5.4. *Let* $(\rho^\tau)_{\tau>0}$ *and* $(p^\tau)_{\tau>0}$ *be as in the previous lemma. Then, there exists* $\rho \in L^m(Q)$ *and* $p \in L^2([0,T]; H^1(\Omega))$ *with* $\rho^{m-\frac{1}{2}} \in L^2([0,T]; H^1(\Omega))$, *such that*

$$\rho^\tau \to \rho \text{ in } L^m(Q), \text{ as } \tau \downarrow 0,$$

and

$$p^\tau \rightharpoonup p \text{ in } L^2([0,T]; H^1(\Omega)), \text{ as } \tau \downarrow 0.$$

along a subsequence.

Proof of Theorem 5.1. Note that (5.5) implies (5.3) for (ρ^τ, p^τ). Then, a similar argument as the one in Lemma 3.7 together with the convergence results from Corollary 5.4 reveals that (ρ, p) satisfies (5.3).

Furthermore, from Lemma 5.2, we can write that

$$\mathbf{E}^\tau = \rho^\tau \mathbf{v}^\tau = -\rho^\tau \nabla((\rho^\tau)^{m-1}p^\tau) - \nabla\Phi\rho^\tau = -\{(m-1)p^\tau(\rho^\tau)^{m-1}\nabla\rho^\tau + (\rho^\tau)^m\nabla p^\tau\} - \nabla\Phi\rho^\tau.$$

Note that (5.5) implies

$$((\rho^\tau)^m - 1)\nabla p^\tau = 0 \quad \text{a.e.} \tag{5.12}$$

From (5.12), we conclude that

$$(m-1)p^\tau(\rho^\tau)^{m-1}\nabla\rho^\tau + (\rho^\tau)^m\nabla p^\tau = (m-1)p^\tau(\rho^\tau)^{m-1}\nabla\rho^\tau + \frac{1}{m}\{(m-1)(\rho^\tau)^m + 1\}\nabla p^\tau,$$

$$= \frac{1}{m}\nabla([(m-1)(\rho^\tau)^m + 1]p^\tau). \tag{5.13}$$

As described in Proposition 3.8, up to passing to a subsequence and using the weak-$\star$ convergence of $(p^\tau)_{\tau>0}$ in $L^\infty(Q)$ and strong convergence of $((\rho^\tau)^m)_{\tau>0}$ in $L^1(Q)$ from Corollary 5.4, we conclude that $(\mathbf{E}^\tau)_{\tau>0}$ converges to

$$\mathbf{E} := -\frac{1}{m}\nabla([(m-1)\rho^m + 1]p) - \nabla\Phi\rho$$

in $\mathscr{D}'(Q; \mathbb{R}^d)$, as $\tau \downarrow 0$ where (ρ, p) is given in Corollary 5.4. The rest of argument is parallel to the proof of Theorem 3.1.

A last remark is that if $\rho_0 \in L^\infty(\Omega)$, then clearly $\rho \in L^\infty(Q)$ and thus $\rho^m \in L^2([0,T]; H^1(\Omega))$. $\qquad\square$

In particular, (5.2) can be also represented in the form of a continuity equation, as we show below. Note that the condition (5.14) below is equivalent to $\beta \geqslant 2m$.

THEOREM 5.5. *For S given in* (5.1), *let ρ_0 and (ρ, p) be given in Theorem* 5.1. *If*

$$d \leqslant 4m, \tag{5.14}$$

then (ρ, p) also satisfies

$$\begin{cases} \partial_t\rho - \nabla \cdot \left(\rho\left[\nabla\left(\rho^{m-1}p\right) + \nabla\Phi\right]\right) = 0, & \text{in } (0,T) \times \Omega, \\ \rho(0, \cdot) = \rho_0, & \text{in } \Omega, \\ \rho[\nabla\left(\rho^{m-1}p\right) + \nabla\Phi] \cdot \boldsymbol{n} = 0, & \text{in } [0,T] \times \partial\Omega, \end{cases} \tag{5.15}$$

in the sense of distribution. If in addition $\rho_0 \in L^\infty(\Omega)$ and Φ satisfies (2.4), *we can drop* (5.14) *from the statement.*

Proof. As $p \in L^2([0,T]; H^1(\Omega))$ and (ρ, p) satisfies (5.3) from Theorem 5.1, we have

$$\nabla p = 0 \quad \text{a.e. in } \{\rho \neq 1\} \tag{5.16}$$

and thus

$$(\rho^m - 1)\nabla p = 0 \quad \text{a.e.} \tag{5.17}$$

From the direct computation using (5.17), it holds that

$$\mathcal{I}_1 := \rho\nabla\left(\rho^{m-1}p\right) = (\rho^m)^{\frac{1}{m}}\nabla\left((\rho^m)^{\frac{m-1}{m}}p\right) = \rho^m\nabla p + \frac{m-1}{m}p\nabla(\rho^m).$$

We claim that $\mathcal{I}_1 \in L^1(Q)$, which is enough for the representation (5.15). Recall from Theorem 5.1 and Lemma 2.11, $\nabla p \in L^2(Q)$, $p \in L^\infty(Q)$ and $\rho \in L^\beta(Q)$ for β given in (2.23).

Consider the first term $\rho^m \nabla p$. If $\rho_0 \in L^\infty(\Omega)$, then $\rho \in L^\infty(Q)$ and thus $\rho^m \in L^2(Q)$. If (5.14) holds (which is automatically the case if $d = 1, 2$), then $\beta \geqslant 2m$ for β given in (2.23) ($r = m$ in this case) and thus $\rho^m \in L^2(Q)$. Furthermore, as $\nabla \rho^{m-\frac{1}{2}} \in L^2(Q)$ and

$$\nabla(\rho^m) = \frac{m}{m - \frac{1}{2}} \rho^{\frac{1}{2}} \nabla(\rho^{m-\frac{1}{2}}),$$

so the last term is also in $L^1(Q)$.

Finally, it is easy to see that

$$\mathcal{I}_1 = \frac{(m-1)\rho^m + 1}{m} \nabla p + \frac{m-1}{m} p \nabla(\rho^m)$$

$$= \nabla\left([(m-1)\rho^m + 1]\frac{p}{m}\right). \qquad \square$$

5.2. Porous medium type diffusion on $\{\rho < 1\}$ and general diffusion on $\{\rho > 1\}$

In this subsection, we suppose that Assumptions 1.1–1.3 hold true for some $r \geqslant 1$, for some $m > 1$ and a constants $\sigma_1, \sigma_2 > 0$. Note that S can be any function satisfying the assumptions, and in particular in the case of $r = 1$, S behaves as the logarithmic entropy when $\rho > 1$.

Our main theorem from this section reads as:

THEOREM 5.6. *Suppose that Assumptions 1.1–1.3 hold true for $m > 1$ and $r \geqslant 1$ such that*

$$m < r + \frac{\beta}{2} \tag{5.18}$$

is fulfilled for β given in (2.23). For $\rho_0 \in \mathscr{P}(\Omega)$ such that $\mathcal{J}(\rho_0) < +\infty$, there exists $\rho \in L^\beta(Q)$ and $p \in L^2([0, T]; H^1(\Omega)) \cap L^\infty(Q)$ such that (ρ, p) is a weak solution of

$$\begin{cases} \partial_t \rho - \Delta(L_S(\rho, p)) - \nabla \cdot (\nabla \Phi \rho) = 0, & \text{in } (0, T) \times \Omega, \\ \rho(0, \cdot) = \rho_0, & \text{in } \Omega, \\ (\nabla(L_S(\rho, p)) + \nabla \Phi \rho) \cdot \boldsymbol{n} = 0, & \text{in } [0, T] \times \partial\Omega, \end{cases} \tag{5.19}$$

in the sense of distribution. Furthermore, (ρ, p) satisfies

$$\begin{cases} p(t, x) = S'(1-) & \text{if } 0 \leqslant \rho(t, x) < 1, \\ p(t, x) \in [S'(1-), S'(1+)] & \text{if } \rho(t, x) = 1, \\ p(t, x) = S'(1+) & \text{if } \rho(t, x) > 1. \end{cases} \tag{5.20}$$

Here, L_S is given in (1.5). In particular,

$$\rho^{m-\frac{1}{2}} \in L^2([0, T]; H^1(\Omega)) \text{ if } m \leqslant r \text{ and } \rho^{m-\frac{1}{2}}$$

$$\in L^q([0, T]; W^{1,q}(\Omega)) \text{ if } r < m < r + \frac{\beta}{2} \tag{5.21}$$

for $q \in (1, 2)$ given in (5.36). If in addition $\rho_0 \in L^\infty(\Omega)$ and Φ satisfies (2.4), we can drop (1.10) and (5.18) from the statement and we obtain $\rho \in L^\infty(Q)$ and $\rho^m \in L^2([0, T]; H^1(\Omega))$.

EXAMPLE 5.1. As a nonlinearity, one can consider for instance the following one. For $m > r > 1$, let $S : [0, +\infty) \to \mathbb{R}$ be given by

$$S(\rho) := \begin{cases} \dfrac{\rho^m}{m-1}, & \text{for } \rho \in [0, 1], \\ \dfrac{\rho^r}{r-1} + \dfrac{1}{m-1} - \dfrac{1}{r-1}, & \text{for } \rho \in (1, +\infty). \end{cases}$$

This clearly satisfies Assumptions 1.1 and 1.2, since

$$S'(1-) = \frac{m}{m-1} < \frac{r}{r-1} = S'(1+).$$

In this case, the operator $L_S(\rho, p)$ becomes

$$L_S(\rho, p)(x) = \begin{cases} \rho(x)^m + \dfrac{1}{m-1} & \text{if } 0 < \rho(x) < 1, \\[2mm] p(x) \in \left[\dfrac{m}{m-1}, \dfrac{r}{r-1}\right] & \text{if } \rho(x) = 1, \\[2mm] \rho(x)^r + \dfrac{1}{r-1} & \text{if } \rho(x) > 1. \end{cases}$$

First, using similar ideas as in Section 4, we choose a constant l such that

$$1 < l < \beta \tag{5.22}$$

for β given in (2.23) and split the function S into S_a and $S_b : [0, +\infty) \to \mathbb{R}$ defined by

$$S_a(\rho) := \begin{cases} \dfrac{S'(1-)(\rho^l - 1)}{l}, & \text{for } \rho \leqslant 1, \\[3mm] \dfrac{S'(1+)(\rho^l - 1)}{l}, & \text{for } \rho > 1, \end{cases} \tag{5.23}$$

and

$$S_b(\rho) := S(\rho) - S_a(\rho). \tag{5.24}$$

Note that $S'(1+) > S'(1-)$. Then, as shown in Lemma 4.4, we conclude that S_a is convex and continuous in $[0, +\infty)$. Also, S_b is continuously differentiable and S_b' is locally Lipschitz continuous in $[0, +\infty)$.

Let us recall the definition of $(\rho_k)_{k=1}^N$ and $(p_k)_{k=1}^N$ from (2.3) and (2.25). Also, recall the definition of $\overline{\phi}_k$ given in Theorem 2.2.

LEMMA 5.7.   *For all $k \in \{1, \ldots, N\}$, there exists $C \in \mathbb{R}$ such that*

$$\rho_k^{l-1} p_k + S_b'(\rho_k) = \left(C - \frac{\overline{\phi}_k}{\tau} - \Phi\right)_+ \quad \text{a.e.} \tag{5.25}$$

In particular, p_k and ρ_k^{m-1} are Lipschitz continuous in Ω. If in addition $m \in (1, 2]$, then ρ_k is locally Lipschitz continuous in $\mathrm{spt}(\rho_k)$.

Proof.   First, (5.25) follows from the parallel argument in the proof of Lemma 5.2.

Note that $\overline{\phi}_k$ and p_k are Lipschitz continuous (cf. Lemma 2.14) and $f_k := C - \frac{\overline{\phi}_k}{\tau} - \Phi$ are Lipschitz continuous. From (2.28) in Lemma 2.15, we have that

$$|\nabla(\rho_k)^{m-1}| = (m-1)\rho_k^{m-2}|\nabla \rho_k| = (m-1)\rho_k^{m-2}\frac{|\nabla f_k|}{S''(\rho_k)} \quad \text{a.e. in } \rho_k^{-1}(\mathbb{R}^+ \setminus \{1\}).$$

On the one hand, this, together with (1.9) from Assumption 1.2, further implies

$$|\nabla(\rho_k)^{m-1}| \leqslant \sigma_2(m-1)|\nabla f_k| \quad \text{a.e. in } \{x \in \Omega : 0 < \rho_k < 1\}.$$

On the other hand, (1.10a) from Assumption 1.3 yields

$$|\nabla(\rho_k)^{m-1}| \leqslant \sigma_1(m-1)|\nabla f_k|\rho_k^{m-r}$$

$$\leqslant \sigma_1(m-1)|\nabla f_k| \max\{\|\rho_k\|_{L^\infty(\Omega)}^{m-r}, 1\}. \quad \text{a.e. in } \{x \in \Omega : \rho_k > 1\}.$$

Therefore, we conclude that ρ_k^{m-1} is Lipschitz continuous in Ω.

Finally, the following identity

$$|\nabla \rho_k| = \frac{\rho_k^{2-m}}{(m-1)}|\nabla(\rho_k)^{m-1}| \quad \text{a.e. in } \mathrm{spt}(\rho_k)$$

shows that ρ_k is locally Lipschitz continuous in $\mathrm{spt}(\rho_k)$, provided $m \in (1, 2]$. This in particular is also a consequence of Lemma 2.15(iv). $\qquad \square$

Proposition 5.8 below contains all the needed estimates and the compactness result on the sequence $(\rho^\tau, p^\tau)_{\tau>0}$ that are necessary to pass to the limit as $\tau \downarrow 0$ and prove the main theorem of this section, that is, Theorem 5.6. The proof of this proposition requires some intermediate results that are provided below in Lemmas 5.9–5.11.

PROPOSITION 5.8. *Let* $(\rho^\tau)_{\tau>0}, (p^\tau)_{\tau>0}$ *stand for the piecewise constant interpolations given in* (3.8) *and* (2.25), *respectively. Then,* $(p^\tau)_{\tau>0}$ *is uniformly bounded in* $L^2([0, T]; H^1(\Omega))$.

(1) *If* $r \geqslant m$, *then* $((\rho^\tau)^{m-\frac{1}{2}})_{\tau>0}$ *is uniformly bounded in* $L^2([0, T]; H^1(\Omega))$.
(2) *If* $r < m < r + \frac{\beta}{2}$, *then* $((\rho^\tau)^{m-\frac{1}{2}})_{\tau>0}$ *is uniformly bounded in* $L^q([0, T]; W^{1,q}(\Omega))$ *for some* $q \in (1, 2)$.
(3) *If in addition* $\rho_0 \in L^\infty(\Omega)$ *and* Φ *satisfies* (2.4), *then* $((\rho^\tau)^m)_{\tau>0}$ *is also uniformly bounded in* $L^2([0, T]; H^1(\Omega))$ *for any* $m > 1$ *and* $r \geqslant 1$.

Proof. From Lemma 5.7, it holds that

$$\mathcal{I}_1 := -\rho_k^{\frac{1}{2}}\frac{\nabla\overline{\phi}_k}{\tau} - \rho_k^{\frac{1}{2}}\nabla\Phi = \rho_k^{\frac{1}{2}}\nabla(\rho_k^{l-1}p_k + S_b'(\rho_k)) \quad \text{a.e.}$$

We follow the very same steps and in the proof of Lemma 5.2 (where we also use (5.8) and (5.9)). Therefore, we have

$$\mathcal{I}_1 = \frac{l-1}{m-1}\rho_k^{l-m+\frac{1}{2}}p_k\nabla(\rho_k^{m-1}) + \rho_k^{l-\frac{1}{2}}\nabla p_k + \rho_k^{\frac{1}{2}}\nabla(S_b'(\rho_k)) \quad \text{a.e. on } \mathrm{spt}(\rho_k). \qquad (5.26)$$

Note that

$$\rho_k^{\frac{1}{2}}\nabla(S_b'(\rho_k)) = \frac{1}{m-1}\rho_k^{\frac{5}{2}-m}S_b''(\rho_k)\nabla(\rho_k^{m-1}) \quad \text{a.e. on } \mathrm{spt}(\rho_k). \qquad (5.27)$$

From (5.26) and (5.27), it holds that

$$\mathcal{I}_1 = \frac{1}{(m-1)\rho_k^{m-2}}\big((l-1)\rho_k^{l-2}p_k + S_b''(\rho_k)\big)\rho_k^{\frac{1}{2}}\nabla(\rho_k^{m-1}) + \rho_k^{l-\frac{1}{2}}\nabla p_k \quad \text{a.e. on } \mathrm{spt}(\rho_k).$$

We can apply (5.9) and conclude (since $\nabla p_k = 0$ a.e. in $\{\rho_k \neq 1\}$) that

$$\mathcal{I}_1^2 = \frac{1}{(m-1)^2\rho_k^{2m-4}}\big((l-1)\rho_k^{l-2}p_k + S_b''(\rho_k)\big)^2\rho_k|\nabla(\rho_k^{m-1})|^2 + |\nabla p_k|^2 \quad \text{a.e. on } \mathrm{spt}(\rho_k).$$

(1) If $r \geqslant m$, then Lemma 5.9 below implies

$$\mathcal{I}_1^2 \geqslant \frac{\sigma_3^2}{(m-1)^2}|\nabla(\rho_k^{m-\frac{1}{2}})|^2 + |\nabla p_k|^2 \quad \text{a.e. on } \mathrm{spt}(\rho_k),$$

for σ_3 given in (5.29). By the parallel argument in Lemma 5.3, we conclude the uniform bound in $L^2([0, T]; H^1(\Omega))$.

(2) If $r < m < r + \frac{\beta}{2}$, then Lemma 5.10 below yields the uniform bound of $(\nabla(\rho^\tau)^{m-\frac{1}{2}})_{\tau>0}$ in $L^q(Q)$ for q given in (5.36). On the other hand, as $2r - 1 \leqslant \beta$, it holds that

$$\left(m - \frac{1}{2}\right)q = \frac{m - \frac{1}{2}}{\frac{m-r}{\beta} + \frac{1}{2}} = \beta \frac{2m - 1}{2m - 2r + \beta} \leqslant \beta.$$

As ρ^τ is uniformly bounded in $L^\beta(Q)$ from Lemma 2.11, $(\rho^\tau)^{m-\frac{1}{2}}$ is uniformly bounded in $L^q(Q)$.

(3) From Lemma 5.11, we conclude that

$$\mathcal{I}_1^2 \geqslant \frac{\sigma_4^2}{(m-1)^2}|\nabla(\rho_k^m)|^2 + |\nabla p_k|^2 \quad \text{a.e. on } \mathrm{spt}(\rho_k).$$

The same argument as before yields that $((\rho^\tau)^m)_{\tau>0}$ is uniformly bounded in $L^2([0,T]; H^1(\Omega))$. $\qquad\square$

LEMMA 5.9. *Let us suppose that we are in the setting of Proposition* 5.8. *If* $r \geqslant m$, *it holds that*

$$\left|\frac{1}{\rho_k^{m-2}}\left((l-1)\rho_k^{l-2}p_k + S_b''(\rho_k)\right)\right|\rho_k^{\frac{1}{2}}|\nabla(\rho_k^{m-1})| \geqslant \sigma_3|\nabla(\rho_k^{m-1/2})|, \tag{5.28}$$

where

$$\sigma_3 := \frac{m-1}{m-\frac{1}{2}}\min\left\{\frac{1}{\sigma_1}, \frac{1}{\sigma_2}\right\}. \tag{5.29}$$

Proof. We claim that

$$\left|\frac{1}{\rho_k^{m-2}}\left((l-1)\rho_k^{l-2}p_k + S_b''(\rho_k)\right)\right| \geqslant \min\left\{\frac{1}{\sigma_1}, \frac{1}{\sigma_2}\right\} \text{ in } \{\rho_k \neq 1\}. \tag{5.30}$$

Recall that

$$S_a''(\rho_k) = \begin{cases} (l-1)S'(1-)\rho_k^{l-2} & \text{if } \rho_k < 1, \\ (l-1)S'(1+)\rho_k^{l-2} & \text{if } \rho_k > 1, \end{cases}$$

and thus by the definition of p_k (see (2.25)) we have

$$(l-1)\rho_k^{l-2}p_k + S_b''(\rho_k) = S_a''(\rho_k) + S_b''(\rho_k) = S''(\rho_k) \quad \text{a.e. in } \{\rho_k \neq 1\}. \tag{5.31}$$

Thus, (1.9) implies

$$\frac{S''(\rho_k)}{\rho_k^{m-2}} \geqslant \frac{1}{\sigma_2} \quad \text{a.e. in } \{0 < \rho_k < 1\}. \tag{5.32}$$

Furthermore, as $r \geqslant m$, (1.10a) implies

$$\frac{S''(\rho_k)}{\rho_k^{m-2}} \geqslant \frac{\rho_k^{r-m}}{\sigma_1} \geqslant \frac{1}{\sigma_1} \quad \text{a.e. in } \{\rho_k > 1\}. \tag{5.33}$$

and we conclude (5.30).

Recall that ρ_k^{m-1} is Lipschitz continuous from Lemma 5.7. Thus, we have

$$\nabla(\rho_k^{m-1}) = 0 \quad \text{a.e. in } \{\rho_k = 1\} \tag{5.34}$$

(see, for instance, [**23**, Theorem 4(iv), Section 4.2.2]). As $\rho_k^{\frac{1}{2}}\nabla(\rho_k^{m-1}) = \frac{m-1}{m-\frac{1}{2}}\nabla(\rho_k^{m-\frac{1}{2}})$, (5.28) follows from (5.30) and (5.34). $\qquad\square$

LEMMA 5.10. *Let us suppose that we are in the setting of Proposition* 5.8. *If* $r < m < r + \frac{\beta}{2}$, *then*

$$\left\| \frac{1}{\rho_k^{m-2}} \big((l-1)\rho_k^{l-2} p_k + S_b''(\rho_k) \big) |\nabla(\rho_k^{m-\frac{1}{2}})| \right\|_{L^2(\Omega)} \geqslant C \|\nabla(\rho_k^{m-\frac{1}{2}})\|_{L^q(\Omega)} \qquad (5.35)$$

for some $q \in (1,2)$ *and a constant* $C > 0$.

Proof. From the relation between r and m, the constant q defined by

$$q := \frac{1}{\frac{m-r}{\beta} + \frac{1}{2}} \qquad (5.36)$$

is in the interval $(1,2)$. As shown in (5.31), it holds that

$$\mathcal{I}_2 := \frac{1}{\rho_k^{m-2}} \big((l-1)\rho_k^{l-2} p_k + S_b''(\rho_k) \big) |\nabla(\rho_k^{m-\frac{1}{2}})| = \frac{S''(\rho_k)}{\rho_k^{m-2}} |\nabla(\rho_k^{m-\frac{1}{2}})| \quad \text{a.e. in } \{\rho_k \neq 1\}. \qquad (5.37)$$

In $\{0 < \rho_k < 1\}$, (5.32) implies

$$\|\mathcal{I}_2\|_{L^2(\{0<\rho_k<1\})} \geqslant \frac{1}{\sigma_2} \left\| \nabla(\rho_k^{m-\frac{1}{2}}) \right\|_{L^2(\{0<\rho_k<1\})}$$

for σ_2 given in (1.9). As $q \in (1,2)$ and the domain is compact, the Hölder inequality yields that

$$\|\mathcal{I}_2\|_{L^2(\{0<\rho_k<1\})} \geqslant \frac{|\Omega|^{\frac{1}{2}-\frac{1}{q}}}{\sigma_2} \|\nabla(\rho_k^{m-\frac{1}{2}})\|_{L^q(\{0<\rho_k<1\})}. \qquad (5.38)$$

Next, we claim that

$$\|\mathcal{I}_2\|_{L^2(\{\rho_k>1\})} \geqslant C \|\nabla(\rho_k^{m-\frac{1}{2}})\|_{L^q(\{\rho_k>1\})} \qquad (5.39)$$

for some constant $C > 0$.

From (1.10a) and (5.37), it holds that

$$\|\mathcal{I}_2\|_{L^2(\{\rho_k>1\})} = \left\| \rho_k^{2-m} S''(\rho_k) \nabla(\rho_k^{m-\frac{1}{2}}) \right\|_{L^2(\{\rho_k>1\})} \geqslant \frac{1}{\sigma_1} \left\| \rho_k^{r-m} \nabla(\rho_k^{m-\frac{1}{2}}) \right\|_{L^2(\{\rho_k>1\})}. \qquad (5.40)$$

On the other hand, as

$$\frac{1}{2} + \frac{m-r}{\beta} = \frac{1}{q},$$

the Hölder inequality yields that

$$\left\| \rho_k^{r-m} \nabla(\rho_k^{m-\frac{1}{2}}) \right\|_{L^2(\{\rho_k>1\})} \|\rho_k^{m-r}\|_{L^{\frac{\beta}{m-r}}(\{\rho_k>1\})} \geqslant \|\nabla(\rho_k^{m-\frac{1}{2}})\|_{L^q(\{\rho_k>1\})}. \qquad (5.41)$$

As ρ_k is uniformly bounded in $L^\beta(\Omega)$ from Lemma 2.10, ρ_k^{m-r} is uniformly bounded in $L^{\frac{\beta}{m-r}}(\Omega)$. From (5.40) and (5.41), we conclude (5.39).

Finally, as (5.34) holds true, (5.35) follows from (5.38) and (5.39). $\qquad \square$

LEMMA 5.11. *Let us suppose that we are in the setting of Proposition* 5.8. *If* $\rho_0 \in L^\infty(\Omega)$ *and* Φ *satisfies* (2.4), *then it holds that*

$$\left| \frac{1}{\rho_k^{m-2}} \big((l-1)\rho_k^{l-2} p_k + S_b''(\rho_k) \big) \right| \rho_k^{\frac{1}{2}} |\nabla(\rho_k^{m-1})| \geqslant \sigma_4 |\nabla(\rho_k^m)|, \qquad (5.42)$$

where

$$\sigma_4 := \frac{m-1}{m} \min\left\{\frac{1}{\sigma_1}, \frac{1}{\sigma_2}\right\} \min\left\{\left(\|\rho_0\|_{L^\infty} e^{dT\|\Delta\Phi\|_{L^\infty}}\right)^{-\frac{1}{2}}, \left(\|\rho_0\|_{L^\infty} e^{dT\|\Delta\Phi\|_{L^\infty}}\right)^{r-m-\frac{1}{2}}\right\}.$$

Proof. Recall from Lemma 2.1 that if $\rho_0 \in L^\infty(\Omega)$, then we have

$$\|\rho_k\|_{L^\infty} \leqslant \|\rho_0\|_{L^\infty} e^{dT\|\Delta\Phi\|_{L^\infty}} =: C. \qquad (5.43)$$

On the other hand, from (5.31) and $\nabla(\rho_k^{m-1}) = \frac{m-1}{m}\nabla(\rho_k^m)$, it holds that

$$\mathcal{I}_3 := \frac{1}{\rho_k^{m-2}}\left((l-1)\rho_k^{l-2}p_k + S_b''(\rho_k)\right)\rho_k^{\frac{1}{2}}\nabla(\rho_k^{m-1}) = \frac{m-1}{m}\frac{S''(\rho_k)}{\rho_k^{m-\frac{3}{2}}}\nabla(\rho_k^m) \quad \text{a.e. in } \{\rho_k \neq 1\}. \qquad (5.44)$$

Then, (5.32) and (5.43) yield that

$$|\mathcal{I}_3| \geqslant \frac{m-1}{m\sigma_2}\rho_k^{-\frac{1}{2}}|\nabla(\rho_k^m)| \geqslant \frac{m-1}{m\sigma_2}C^{-\frac{1}{2}}|\nabla(\rho_k^m)| \quad \text{a.e. in } \{0 < \rho_k < 1\}. \qquad (5.45)$$

Furthermore, (1.10b) and (5.43) imply

$$|\mathcal{I}_3| \geqslant \frac{m-1}{m\sigma_1}\rho_k^{r-m-\frac{1}{2}}|\nabla(\rho_k^m)| \geqslant \frac{m-1}{m\sigma_1}C^{-\frac{1}{2}}\min\{C^{r-m}, 1\}|\nabla(\rho_k^m)| \quad \text{a.e. in } \{\rho_k > 1\}. \qquad (5.46)$$

Finally, as (5.34) holds, (5.42) follows from (5.45) and (5.46). $\qquad \square$

COROLLARY 5.12. *Let $(\rho^\tau)_{\tau>0}$ and $(p^\tau)_{\tau>0}$ be as in the previous proposition and (5.18) hold. There exists $\rho \in L^\beta(Q)$ and $p \in L^2([0,T]; H^1(\Omega))$ such that*

$$\rho^\tau \to \rho \text{ in } L^s(Q), \text{ as } \tau \downarrow 0,$$

and

$$p^\tau \rightharpoonup p \text{ in } L^2([0,T]; H^1(\Omega)), \text{ as } \tau \downarrow 0,$$

along a subsequence for any $s \in (0, \beta)$ and β given in (2.23).

Proof. Recall that Lemma 2.11 yields that $(\rho^\tau)_{\tau>0}$ is uniformly bounded in $L^\beta(Q)$. In both cases $r \geqslant m$ and $r < m < r + \frac{\beta}{2}$, Lemma C.2 and Proposition 5.8 yield $(\rho^\tau)_{\tau>0}$ is precompact in $L^s(Q)$ for any $s \in (0, \beta)$.

Indeed, first, we consider the case $r < m < r + \frac{\beta}{2}$. We apply Proposition 5.8(2) and Lemma C.2(1) to conclude that $(\rho^\tau)_{\tau>0}$ converges to ρ in $L^{(m-\frac{1}{2})q^*}(Q)$ along a subsequence, where $q^* := \frac{qd}{d-q}$ and $q \in (1, 2)$ is given in Proposition 5.8(2). Note that a direct computation shows that

$$q^* = \frac{2r-1}{2m-1}\frac{2d}{d-2} = \frac{\beta}{m-1/2}.$$

By a similar argument, we conclude the strong convergence of $(\rho^\tau)_{\tau>0}$ in $L^s(Q)$ along a subsequence, also in the case when $r \geqslant m$. $\qquad \square$

Proof of Theorem 5.6. Note that by the direct computation as in (5.13) and (4.16), we have

$$-\mathbf{E}^\tau = -\rho^\tau\mathbf{v}^\tau = \rho^\tau\nabla((\rho^\tau)^{l-1}p^\tau + S_b'(\rho^\tau)) + \rho^\tau\nabla\Phi$$

$$= \nabla\left(\frac{1}{l}((l-1)(\rho^\tau)^l + 1)p^\tau + \rho^\tau S_b'(\rho^\tau) - S_b(\rho^\tau) + S_b(1)\right) + \rho^\tau\nabla\Phi.$$

Then, we have $-\mathbf{E}^\tau = \nabla L_S(\rho^\tau, p^\tau) + \rho^\tau\nabla\Phi$ for L_S given in (1.5). Since $l, r < \beta$ from (5.18), Corollary 5.12 yields that $(\rho^\tau)^l$, $\rho^\tau S_b'(\rho^\tau)$ and $S_b(\rho^\tau)$ converge in $L^1(Q)$ as $\tau \downarrow 0$. As p^τ is

uniformly bounded, we conclude that

$$-\mathbf{E}^\tau \to \nabla\left(\frac{1}{l}((l-1)\rho^l + 1)p^\tau + \rho S_b'(\rho) - S_b(\rho) + S_b(1)\right) + \rho\nabla\Phi, \text{ as } \tau \to 0,$$

along a subsequence in $\mathscr{D}'(Q; \mathbb{R}^d)$. Note that we have $\rho \in L^\beta$ from the uniform boundedness in Lemma 2.11 and $p \in L^2([0, T]; H^1(\Omega)) \cap L^\infty(Q)$ from Proposition 5.8. As

$$L_S(\rho, p) = \frac{1}{l}((l-1)\rho^l + 1)p^\tau + \rho S_b'(\rho) - S_b(\rho) + S_b(1), \tag{5.47}$$

for L_S given in (1.5), we conclude that (ρ, p) satisfies (5.19). The rest of argument is parallel to Theorem 4.2. $\qquad\square$

In particular, (5.2) can be also represented in the form of a continuity equation, as we show below.

THEOREM 5.13. *Suppose that* (1.10) *and* (4.1) *hold true. Let ρ_0 and (ρ, p) be given in Theorem* 5.6. *If*

$$m < r + \frac{1}{2} \tag{5.48}$$

and

$$\beta > 2 \text{ and } m < \frac{\beta}{2} + \frac{1}{2} \tag{5.49}$$

hold, then (ρ, p) is a weak solution of

$$\begin{cases} \partial_t\rho - \nabla \cdot \left(\rho\nabla\left(S'(\rho)\mathbb{1}_{\{\rho\neq 1\}} + p\mathbb{1}_{\{\rho=1\}}\right)\right) - \nabla \cdot (\rho\nabla\Phi) = 0, & \text{in } (0, T) \times \Omega, \\ \rho(0, \cdot) = \rho_0, & \text{in } \Omega, \\ \rho\left[\nabla\left(S'(\rho)\mathbb{1}_{\{\rho\neq 1\}} + p\mathbb{1}_{\{\rho=1\}}\right) + \nabla\Phi\right] \cdot \mathbf{n} = 0, & \text{in } [0, T] \times \partial\Omega, \end{cases} \tag{5.50}$$

in the sense of distribution. If in addition $\rho_0 \in L^\infty(\Omega)$ and Φ satisfies (2.4), *we can drop* (5.49) *from the statement.*

Here, note that if $d = 1$ or 2, (5.49) holds true for any m.

Proof. (5.48) implies (5.18) and thus (5.21) follows from Proposition 5.8. From (5.49), we can choose l such that

$$\max\left\{1, m - \frac{1}{2}\right\} < l \leqslant \frac{\beta}{2}. \tag{5.51}$$

From (5.20) and the construction in (5.23), we have

$$S'(\rho)\mathbb{1}_{\{\rho\neq 1\}} + p\mathbb{1}_{\{\rho=1\}} = p\rho^{l-1} + S_b'(\rho).$$

We claim that

$$\rho\nabla(p\rho^{l-1} + S_b'(\rho)) \in L^1(Q). \tag{5.52}$$

By the direct computation, we obtain

$$\rho\nabla(p\rho^{l-1} + S_b'(\rho)) = \nabla p\rho^l + p\rho\nabla(\rho^{l-1}) + \rho\nabla(S_b'(\rho)), \tag{5.53}$$

$$= \nabla p\rho^l + \frac{l-1}{m - \frac{1}{2}}p\rho^{l-m+\frac{1}{2}}\nabla(\rho^{m-\frac{1}{2}}) + \frac{1}{m - \frac{1}{2}}S_b''(\rho)\rho^{\frac{5}{2}-m}\nabla(\rho^{m-\frac{1}{2}}). \tag{5.54}$$

As $\rho \in L^{\beta}(Q)$ and $\beta \geqslant 2l$, $\rho^l \in L^2(Q)$ and thus $\nabla p \rho^l \in L^1(Q)$. Now, let us consider the second term. From (5.21), $\nabla(\rho^{m-\frac{1}{2}}) \in L^q(Q)$ for q given in (5.36) and m, r satisfying (5.18). Recall that $p \in L^{\infty}(Q)$ and $\rho \in L^{\beta}(Q)$. As $l < \frac{\beta}{2}$ from (5.51), we have

$$\frac{1}{q} + \frac{l - m + \frac{1}{2}}{\beta} = \frac{m - r}{\beta} + \frac{1}{2} + \frac{l - m + \frac{1}{2}}{\beta} = \frac{l - r + \frac{1}{2}}{\beta} + \frac{1}{2} \leqslant 1.$$

Thus, from the Hölder inequality, we conclude that the second term in (5.53) is in $L^1(Q)$. Similarly, as $r - m + \frac{1}{2} > 0$ from (5.48), $S_b''(\rho)\rho^{\frac{5}{2}-m} \in L^{\frac{\beta}{r-m+\frac{1}{2}}}(Q)$ and

$$\frac{1}{q} + \frac{r - m + \frac{1}{2}}{\beta} = \frac{1}{2\beta} + \frac{1}{2} \leqslant 1,$$

we conclude that the third term is in $L^1(Q)$ and (5.52).

Next, we claim that

$$\rho\nabla(p\rho^{l-1} + S_b'(\rho)) = \nabla L_S(\rho, p) \tag{5.55}$$

for L_S given in (1.5). As $p \in L^2([0, T]; H^1(\Omega))$ and (ρ, p) satisfies (5.3) from Theorem 5.1, we have

$$\nabla p = 0 \quad \text{a.e. in } \{\rho \neq 1\}. \tag{5.56}$$

As $\rho^l \in L^2(Q)$ from Theorem 5.6 and (5.51), we have

$$(\rho^l - 1)\nabla p = 0 \quad \text{a.e.} \tag{5.57}$$

From the direct computation using (5.57), it holds that

$$\rho\nabla(\rho^{l-1}p) = (\rho^l)^{\frac{1}{l}}\nabla\left((\rho^l)^{\frac{l-1}{l}}p\right) = \rho^l\nabla p + \frac{l-1}{l}p\nabla(\rho^l)$$

$$= \frac{(l-1)\rho^l + 1}{l}\nabla p + \frac{l-1}{l}p\nabla(\rho^l)$$

$$= \nabla\left(((l-1)\rho^l + 1)\frac{p}{l}\right).$$

Therefore, we have

$$\nabla(p\rho^{l-1} + S_b'(\rho)) = \nabla\left(((l-1)\rho^l + 1)\frac{p}{l} + \rho S_b'(\rho) - S_b(\rho) + S_b(1)\right). \tag{5.58}$$

From (5.47) and (5.58), we conclude (5.55).

Finally, note that if $\rho_0 \in L^{\infty}(\Omega)$ and Φ satisfies (2.4), then $\rho \in L^{\infty}(Q)$ and thus $\rho^l \in L^2(Q)$ for any $l > 0$. Furthermore, we choose $l = m + 1$, then from Proposition 5.8.(3), we conclude that $p\rho\nabla(\rho^m) \in L^1(Q)$. Therefore, we show (5.52) without (5.49). $\qquad\square$

6. Uniqueness via an L^1-contraction

We construct an L^1 contraction result, inspired by [20, Section 3] and [40, Theorem 6.5]. In particular, this will imply the uniqueness of the solution of (4.3)–(4.4) and (5.19)–(5.20). Let us underline the fact that because of the generality of the previous two problems, on the one hand, the techniques from [20, Section 3] do not apply directly. On the other hand, because of the presence of the critical regimes $\{\rho^i = 1\}$, $i = 1, 2$, the construction from [40, Theorem 6.5] does not apply directly either. Therefore, we develop a careful combination of these two approaches to be able to provide an L^1-contraction for all the systems considered previously, with general initial data.

THEOREM 6.1. *Let* $(\rho^1, p^1), (\rho^2, p^2)$ *be solutions to* (1.4)–(1.6) *with initial conditions* $\rho_0^1, \rho_0^2 \in \mathscr{P}(\Omega)$ *such that* $\mathcal{J}(\rho_0^i) < +\infty$, $i = 1, 2$. *Suppose that* $L_S(\rho^i, p^i) \in L^2(Q)$, $i = 1, 2$ (*or equivalently* $\rho^i \in L^{2r}(Q)$, $i = 1, 2$). *Then we have*

$$\|\rho_t^1 - \rho_t^2\|_{L^1(\Omega)} \leqslant \|\rho_0^1 - \rho_0^2\|_{L^1(\Omega)}, \ \mathscr{L}^1 - \text{a.e. } t \in [0, T].$$

REMARK 6.1. It is worth noticing that the assumption $L_S(\rho^i, p^i) \in L^2(Q)$, $i = 1, 2$ in the statement of the previous theorem seems quite natural in the setting of L^1-type contractions for porous medium equations (see [40]). In our setting, because of the $L^\beta(Q)$ estimates on ρ^i (where β is defined in (2.23)) and because of the L^r-type growth condition on L_S at $+\infty$, this assumption is fulfilled already if $\beta \geqslant 2r$. In the same time, no such assumption is needed if the initial data are in $L^\infty(\Omega)$, since in that case L^∞ estimates hold true for ρ_t^i for a.e. $t \in [0, T]$ (see Lemma 2.1).

Proof of Theorem 6.1. Let (ρ^1, p^1) and (ρ^2, p^2) be two solutions to (1.4)–(1.6) with initial data ρ_0^1 and ρ_0^2, respectively. Let $\varphi \in C_c^2((0, T] \times \Omega)$ and using the notation

$$\mathcal{I}(\varphi, t) := \int_\Omega \varphi_t(\rho_t^1 - \rho_t^2) \, \mathrm{d}x$$

we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(\varphi, t) = \int_\Omega \partial_t\varphi(\rho^1 - \rho^2) + \varphi\partial_t(\rho^1 - \rho^2) \, \mathrm{d}x.$$

Now, using equation (4.3) and by integrating the above expression on $(0, t)$, we get

$$\mathcal{I}(\varphi, t) = \mathcal{I}(\varphi, 0) + \int_0^t \int_\Omega \partial_s\varphi(\rho^1 - \rho^2) + \Delta\varphi(L_S(\rho^1, p^1) - L_S(\rho^2, p^2))$$

$$- \nabla\varphi \cdot \nabla\Phi(\rho^1 - \rho^2) \, \mathrm{d}x \, \mathrm{d}s \tag{6.1}$$

$$= \mathcal{I}(\varphi, 0) + \int_0^t \int_\Omega (L_S(\rho^1, p^1) - L_S(\rho^2, p^2))[A\partial_s\varphi + \Delta\varphi - A\nabla\Phi \cdot \nabla\varphi] \, \mathrm{d}x \, \mathrm{d}s,$$

where we use the notation

$$A := \frac{\rho^1 - \rho^2}{L_S(\rho^1, p^1) - L_S(\rho^2, p^2)}, \tag{6.2}$$

with the convention $A = 0$, when $L_S(\rho^1, p^1) = L_S(\rho^2, p^2)$. Note that Lemma 6.2 below implies that if $L_S(\rho^1, p^1) = L_S(\rho^2, p^2)$ a.e., then $\rho^1 = \rho^2$ and $p^1 = p^2$ a.e. Furthermore, on this very particular set actually there is no contribution in the integral on the right-hand side of (6.1), so it is meaningful to set $A = 0$ there. Also, because of the monotonicity property of the operator L_S (see Lemma 6.2), we have that $A \geqslant 0$ a.e. in Q.

Similarly to the arguments from [20, Section 3], for $\zeta : \Omega \to \mathbb{R}$ smooth with $|\zeta| \leqslant 1$, we consider the dual backward equation as

$$\begin{cases} A\partial_t\varphi + \Delta\varphi - A\nabla\Phi \cdot \nabla\varphi = 0, & \text{in } (0, T) \times \Omega, \\ \nabla\varphi \cdot \mathbf{n} = 0, & \text{on } (0, T) \times \partial\Omega, \\ \varphi(T, \cdot) = \zeta, & \text{in } \Omega. \end{cases} \tag{6.3}$$

Let us note that if we are able to construct a suitable (weak) solution φ to (6.3), for which the computations in (6.1) remain valid, we can deduce the L^1-contraction result, after optimizing with respect to ζ. In general, one cannot hope for smoothness of A, and so (6.3) is degenerate. Therefore, we introduce suitable approximations which will allow to construct smooth test function.

Let us define two Borel sets
$$E_1 := \{\rho^1 \geqslant 1/2\} \cup \{\rho^2 \geqslant 1/2\}$$
and $E_2 := Q \setminus E_1$. We suppose that both sets E_1 and E_2 have positive measures with respect to $\mathscr{L}^{d+1}$, otherwise we simply do not consider the negligible one in the consideration below. First, by Lemma 6.3, we have that $A \llcorner E_1$ is bounded. Second we have the following

Claim. $A^{-1} \llcorner E_2 \in L^2(E_2)$.

Proof of the claim. Let us notice that we can write
$$E_2 = \left(\{\rho^1 < 1/2\} \cap \{\rho^2 \geqslant 1/2\}\right) \cup \left(\{\rho^1 \geqslant 1/2\} \cap \{\rho^2 < 1/2\}\right) \cup \left(\{\rho^1 < 1/2\} \cap \{\rho^2 < 1/2\}\right)$$
$$:= E_2^1 \cup E_2^2 \cup E_2^3.$$
We further decompose $E_2^1 := (\{\rho^1 < 1/2\} \cap \{1/2 \leqslant \rho^2 < 1\}) \cup (\{\rho^1 < 1/2\} \cap \{\rho^2 \geqslant 1\}) =: E_1^{11} \cup E_1^{12}$. For almost every (a.e.) $q \in E_1^{11}$, we have
$$A^{-1}(q) = \frac{L_S(\rho^1(q), p^1(q)) - L_S(\rho^2(q), p^2(q))}{\rho^1(q) - \rho^2(q)} = \tilde{\rho}(q) S''(\tilde{\rho}(q))$$
where $\tilde{\rho}(q)$ is between $\rho^1(q)$ and $\rho^2(q)$. Since restricted to E_1^{11} both ρ^1 and ρ^2 are bounded by 1, we have that $A^{-1} \llcorner E_1^{11} \in L^\infty(E_1^{11})$.

For a.e. $q \in E_1^{12}$, we have
$$A^{-1}(q) = \frac{L_S(\rho^1(q), p^1(q)) - L_S(\rho^2(q), p^2(q))}{\rho^1(q) - \rho^2(q)} \leqslant 2|L_S(\rho^1(q), p^1(q)) - L_S(\rho^2(q), p^2(q))|,$$
since restricted to this set $|\rho^1(q) - \rho^2(q)| \geqslant 1/2$ a.e. Therefore, by our assumption on $L_S(\rho^i, p^i)$, we have that $A^{-1} \llcorner E_2^{12} \in L^2(E_2^{12})$. Therefore, $A^{-1} \llcorner E_2^1 \in L^2(E_2^1)$.

Similarly, we can draw the same conclusion in the case of E_2^2, and so $A^{-1} \llcorner E_2^2 \in L^2(E_2^2)$.

For a.e. $q \in E_2^3$, we conclude similarly as in the case of E_2^{11}, that is, we have that
$$A^{-1}(q) = \frac{L_S(\rho^1(q), p^1(q)) - L_S(\rho^2(q), p^2(q))}{\rho^1(q) - \rho^2(q)} = \tilde{\rho}(q) S''(\tilde{\rho}(q)),$$
where $\tilde{\rho}(q)$ is between $\rho^1(q)$ and $\rho^2(q)$. Since restricted to E_2^3 both ρ^1 and ρ^2 are bounded by $1/2$, we have that $A^{-1} \llcorner E_2^3 \in L^\infty(E_2^3)$.

Therefore, combining all the previous arguments, one obtains that $A^{-1} \llcorner E_2 \in L^2(E_2)$, and the claim follows.

Let $\varepsilon > 0$ and let $K_1 := \|A\mathbb{1}_{E_1}\|_{L^\infty(Q)}$. Let $A_1^\varepsilon := \max\{\varepsilon, A\mathbb{1}_{E_1}\}$. Then, we have $\varepsilon \leqslant A_1^\varepsilon \leqslant K_1$ and $\|A_1^\varepsilon - A\mathbb{1}_{E_1}\|_{L^\infty(Q)} \leqslant \varepsilon$. In the same time, for $0 < \delta \leqslant K$ given, let $A_2^\varepsilon = A_2^\varepsilon(\delta, K)$ be smooth such that $\delta \leqslant (A_2^\varepsilon)^{-1} \leqslant K$ and
$$(A_2^\varepsilon)^{-1} \to [(A\mathbb{1}_{E_2})^{-1}]_{\delta, K} \text{ strongly in } L^q(E_2), \text{ as } \varepsilon \downarrow 0, \tag{6.4}$$
for any $q \in [1, +\infty)$ and in particular, $A_\varepsilon^{-1} \xrightarrow{\star} [(A\mathbb{1}_{E_2})^{-1}]_{\delta, K}$ weakly-$\star$ in $L^\infty(E_2)$ as $\varepsilon \downarrow 0$. Here, for a nonnegative function $f : Q \to [0, +\infty)$ we use the notation $f_{\delta, K} := \min\{\max\{f, \delta\}, K\}$.

Now, let us define $A_\varepsilon : Q \to [0, +\infty)$ as
$$A_\varepsilon := \begin{cases} A_1^\varepsilon, & \text{a.e in } E_1, \\ A_2^\varepsilon, & \text{a.e. in } E_2. \end{cases}$$
By construction, $\min\{\varepsilon; 1/K\} \leqslant A_\varepsilon \leqslant \max\{K_1, 1/\delta\}$. For $\theta > 0$, let A_θ (which depends also on ε, δ and K) be a smooth approximation of A_ε such that
$$\min\{\varepsilon; 1/K\} \leqslant A_\theta \leqslant \max\{K_1, 1/\delta\}, \text{ in } Q; \tag{6.5}$$
$$\varepsilon \leqslant A_\theta \leqslant K_1, \text{ a.e. in } E_1;$$
$$1/K \leqslant A_\theta \leqslant 1/\delta, \text{ a.e. in } E_2;$$

and $A_\theta \to A_\varepsilon$ strongly in $L^q(Q)$ for any $q \in [1, +\infty)$ and in particular

$$A_\theta \overset{\star}{\rightharpoonup} A_\varepsilon \text{ weakly} - \star \text{ in } L^\infty(Q), \text{ as } \theta \downarrow 0. \tag{6.6}$$

Moreover, we have

$$A_\theta^{-1} \to [(A\mathbb{1}_{E_2})^{-1}]_{\delta, K} \text{ in } L^q(E_2), \ \forall \ q \in [1, +\infty) \text{ and}$$

$$A_\theta^{-1} \overset{\star}{\rightharpoonup} [(A\mathbb{1}_{E_2})^{-1}]_{\delta, K} \text{ in } L^\infty(E_2), \text{ as } \max\{\theta, \varepsilon\} \downarrow 0. \tag{6.7}$$

To check this last claim, we argue as follows.

$$\|A_\theta^{-1} - [(A\mathbb{1}_{E_2})^{-1}]_{\delta, K}\|_{L^q(E_2)}$$

$$\leqslant \|A_\theta^{-1} - (A_2^\varepsilon)^{-1}\|_{L^q(E_2)} + \|(A_2^\varepsilon)^{-1} - [(A\mathbb{1}_{E_2})^{-1}]_{\delta, K}\|_{L^q(E_2)}$$

$$= \|(A_\theta - A_2^\varepsilon)/(A_\theta A_2^\varepsilon)\|_{L^q(E_2)} + \|(A_2^\varepsilon)^{-1} - [(A\mathbb{1}_{E_2})^{-1}]_{\delta, K}\|_{L^q(E_2)}$$

$$\leqslant K^2 \|A_\theta - A_2^\varepsilon\|_{L^q(E_2)} + \|(A_2^\varepsilon)^{-1} - [(A\mathbb{1}_{E_2})^{-1}]_{\delta, K}\|_{L^q(E_2)} \to 0,$$

as $\max\{\theta, \varepsilon\} \downarrow 0$, by the construction of A_θ and A_2^ε. We conclude similarly about the weak-$\star$ convergence as well.

Since $\Phi \in W^{1,\infty}(\Omega)$, we consider a smooth approximation of it, $(\Phi_\theta)_{\theta>0}$ such that $\nabla\Phi_\theta \to \nabla\Phi$, as $\theta \downarrow 0$, strongly in $L^{2r'}(\Omega)$.

Let us consider the regularized dual equation which reads as

$$\begin{cases} \partial_t \varphi_\theta + (1/A_\theta)\Delta\varphi_\theta - \nabla\Phi_\theta \cdot \nabla\varphi_\theta = 0, & \text{in } (0,T) \times \Omega, \\ \nabla\varphi_\theta \cdot \mathbf{n} = 0, & \text{on } (0,T) \times \partial\Omega, \\ \varphi_\theta(T, \cdot) = \zeta, & \text{in } \Omega. \end{cases} \tag{6.8}$$

Let φ_θ be the smooth solution of (6.8), when the coefficient function is A_θ and we use this in (6.1) as

$$\mathcal{I}(\varphi_\theta, T) - \mathcal{I}(\varphi_\theta, 0)$$

$$= \int_0^T \int_\Omega \partial_s \varphi_\theta (\rho^1 - \rho^2) + \Delta\varphi_\theta (L_S(\rho^1, p^1) - L_S(\rho^2, p^2)) - \nabla\varphi_\theta \cdot \nabla\Phi(\rho^1 - \rho^2) \mathrm{d}x\mathrm{d}s$$

$$= \int_{E_1} \partial_s \varphi_\theta (\rho^1 - \rho^2) + \Delta\varphi_\theta (L_S(\rho^1, p^1) - L_S(\rho^2, p^2)) - \nabla\varphi_\theta \cdot \nabla\Phi(\rho^1 - \rho^2) \mathrm{d}\mathscr{L}^{d+1}$$

$$+ \int_{E_2} \partial_s \varphi_\theta (\rho^1 - \rho^2) + \Delta\varphi_\theta (L_S(\rho^1, p^1) - L_S(\rho^2, p^2)) - \nabla\varphi_\theta \cdot \nabla\Phi(\rho^1 - \rho^2) \mathrm{d}\mathscr{L}^{d+1}$$

$$= \int_{E_1} (L_S(\rho^1, p^1) - L_S(\rho^2, p^2))[A\partial_s\varphi_\theta + \Delta\varphi_\theta - A\nabla\Phi \cdot \nabla\varphi_\theta]\mathrm{d}\mathscr{L}^{d+1}$$

$$+ \int_{E_2} (\rho^1 - \rho^2)\big[\partial_s\varphi_\theta + A^{-1}\Delta\varphi_\theta - \nabla\Phi \cdot \nabla\varphi_\theta\big]\mathrm{d}\mathscr{L}^{d+1} =: \mathcal{I}_1 + \mathcal{I}_2.$$

It remains to show that both $|\mathcal{I}_1|$ and $|\mathcal{I}_2|$ can be made arbitrary small. Because ϕ_θ solves (6.8) with the coefficient function A_θ, we have

$$\mathcal{I}_1 = \int_{E_1} (L_S(\rho^1, p^1) - L_S(\rho^2, p^2))[A\partial_s\varphi_\theta + \Delta\varphi_\theta - A\nabla\Phi \cdot \nabla\varphi_\theta]\mathrm{d}\mathscr{L}^{d+1}$$

$$- \int_{E_1} (L_S(\rho^1, p^1) - L_S(\rho^2, p^2))A\big[\partial_s\varphi_\theta + A_\theta^{-1}\Delta\varphi_\theta - \nabla\Phi_\theta \cdot \nabla\varphi_\theta\big]\mathrm{d}\mathscr{L}^{d+1}$$

$$= \int_{E_1} (L_S(\rho^1, p^1) - L_S(\rho^2, p^2))(A_\theta - A)A_\theta^{-\frac{1}{2}} A_\theta^{-\frac{1}{2}} \Delta\varphi_\theta \, \mathrm{d}\mathscr{L}^{d+1}$$

$$+ \int_{E_1} (L_S(\rho^1, p^1) - L_S(\rho^2, p^2))A\nabla\varphi_\theta \cdot (\nabla\Phi_\theta - \nabla\Phi)\mathrm{d}\mathscr{L}^{d+1}.$$

From here, by (6.5) we have

$$|\mathcal{I}_1| \leqslant \varepsilon^{-\frac{1}{2}} \|A_\theta^{-\frac{1}{2}} \Delta\varphi_\theta\|_{L^2(Q)} \left(\int_{E_1} |L_S(\rho^1, p^1) - L_S(\rho^2, p^2)|^2 |A_\theta - A|^2 \, \mathrm{d}\mathscr{L}^{d+1} \right)^{\frac{1}{2}}$$

$$+ \int_0^T \int_\Omega |\rho^1 - \rho^2||\nabla\varphi_\theta||\nabla\Phi_\theta - \nabla\Phi|\mathrm{d}x\,\mathrm{d}t.$$

By Lemma 6.4(1), the summability assumption on $\rho^i \in L^{2r}(Q)$ and the approximation $\nabla\Phi_\theta \to \nabla\Phi$, in $L^{2r'}(\Omega)$ as $\theta \downarrow 0$, we conclude that the second term in the previous inequality tends to 0 as $\theta \downarrow 0$. By Lemma 6.4(2), we have that $\|A_\theta^{-\frac{1}{2}} \Delta\varphi_\theta\|_{L^2(Q)} \leqslant C$ for some constant independent of θ and ε. Furthermore, by (6.6), by the summability assumption on $L_S(\rho^i, p^i)$ and by the construction of A_1^ε, for θ small enough we have

$$\int_{E_1} |L_S(\rho^1, p^1) - L_S(\rho^2, p^2)|^2 |A_\theta - A|^2 \mathrm{d}\mathscr{L}^{d+1}$$

$$\leqslant 2 \int_{E_1} |L_S(\rho^1, p^1) - L_S(\rho^2, p^2)|^2 |A_\theta - A_1^\varepsilon|^2 \mathrm{d}\mathscr{L}^{d+1}$$

$$+ 2 \int_{E_1} |L_S(\rho^1, p^1) - L_S(\rho^2, p^2)|^2 |A_1^\varepsilon - A|^2 \mathrm{d}\mathscr{L}^{d+1}$$

$$\leqslant \varepsilon^2 + C\varepsilon^2,$$

for some constant independent of ε, θ, K and therefore by the arbitrariness of ε, we conclude that $\mathcal{I}_1 = 0$.

In the case of $\mathcal{I}_2$, we argue as follows.

$$\mathcal{I}_2 = \int_{E_2} (\rho^1 - \rho^2)\big[\partial_s\varphi_\theta + A^{-1}\Delta\varphi_\theta - \nabla\Phi \cdot \nabla\varphi_\theta\big] \mathrm{d}\mathscr{L}^{d+1}$$

$$- \int_{E_2} (\rho^1 - \rho^2)\big[\partial_s\varphi_\theta + A_\theta^{-1}\Delta\varphi_\theta - \nabla\Phi_\theta \cdot \nabla\varphi_\theta\big] \mathrm{d}\mathscr{L}^{d+1}$$

$$= \int_{E_2} (\rho^1 - \rho^2)(A^{-1} - A_\theta^{-1})A_\theta^{\frac{1}{2}} A_\theta^{-\frac{1}{2}} \Delta\phi_\theta \, \mathrm{d}\mathscr{L}^{d+1}$$

$$+ \int_{E_2} (\rho^1 - \rho^2)\nabla\varphi_\theta \cdot (\nabla\Phi_\theta - \nabla\Phi)\mathrm{d}\mathscr{L}^{d+1}$$

$$= \int_{E_2} (\rho^1 - \rho^2)(A^{-1} - A_{\delta,K}^{-1})A_\theta^{\frac{1}{2}} A_\theta^{-\frac{1}{2}} \Delta\phi_\theta \, \mathrm{d}\mathscr{L}^{d+1}$$

$$+ \int_{E_2} (\rho^1 - \rho^2)(A_{\delta,K}^{-1} - A_\theta^{-1})A_\theta^{\frac{1}{2}} A_\theta^{-\frac{1}{2}} \Delta\phi_\theta \, \mathrm{d}\mathscr{L}^{d+1}$$

$$+ \int_{E_2} (\rho^1 - \rho^2)\nabla\varphi_\theta \cdot (\nabla\Phi_\theta - \nabla\Phi)\mathrm{d}\mathscr{L}^{d+1}$$

$$=: \mathcal{I}_{21} + \mathcal{I}_{22} + \mathcal{I}_{23}.$$

In the case of $\mathcal{I}_{23}$, we argue exactly as in the case of the second term of $\mathcal{I}_1$ to conclude that this term tends to 0 as $\theta \downarrow 0$. As for the other terms, let us note that by the definition of $A_{\delta,K}^{-1}$ (on E_2), we have that

$$\left| A^{-1} - A_{\delta,K}^{-1} \right| = \begin{cases} \delta & \text{a.e. in } \{0 \leqslant A^{-1} < \delta\} \cap E_2, \\ 0 & \text{a.e. in } \{\delta \leqslant A^{-1} \leqslant K\} \cap E_2, \\ A^{-1} - K & \text{a.e. in } \{K \leqslant A^{-1}\} \cap E_2, \end{cases}$$

and thus

$$\left| A^{-1} - A_{\delta,K}^{-1} \right| \leqslant \delta + (A^{-1} - K)_+, \text{ a.e. in } E_2. \tag{6.9}$$

Therefore, since $A_\theta^{\frac{1}{2}} \leqslant \delta^{-\frac{1}{2}}$, we obtain

$$|\mathcal{I}_{21}| \leqslant \|A_\theta^{-\frac{1}{2}} \Delta\phi_\theta\|_{L^2(Q)} \delta^{-\frac{1}{2}} \left(\delta \|\rho^1 - \rho^2\|_{L^2(E_2)} + \|(\rho^1 - \rho^2)(A^{-1} - K)\|_{L^2(\{K \leqslant A^{-1}\} \cap E_2)} \right) \to 0,$$

as $K \to +\infty$ and $\delta \downarrow 0$ (in this order). This is true indeed, by Lemma 6.4(2) and by the fact that

$$\int_{\{K \leqslant A^{-1}\} \cap E_2} (\rho^1 - \rho^2)^2 (A^{-1} - K)^2 \, d\mathscr{L}^{d+1} \leqslant \int_{\{K \leqslant A^{-1}\} \cap E_2} (\rho^1 - \rho^2)^2 (A^{-1})^2 \, d\mathscr{L}^{d+1}$$

$$\leqslant \int_{\{K \leqslant A^{-1}\} \cap E_2} (L_S(\rho^1, p^1) - L_S(\rho^2, p^2))^2 \, d\mathscr{L}^{d+1}.$$

Since $A^{-1} \in L^2(E_2)$, by Chebyshev's inequality $\mathscr{L}^{d+1}(\{K \leqslant A^{-1}\} \cap E_2) \to 0$, as $K \to +\infty$, so by the summability of $L_S^2(\rho^i, p^i)$ we deduce that for K large enough last term in the last inequality is smaller than δ^2. Therefore, by the arbitrariness of δ, we conclude that $\mathcal{I}_{21}$ has to be zero.

To show that $|\mathcal{I}_{22}|$ can be made arbitrary small, using again $A_\theta^{\frac{1}{2}} \leqslant \delta^{-\frac{1}{2}}$ a.e. on E_2 and Lemma 6.4(2), we have

$$|\mathcal{I}_{22}|^2 \leqslant \delta^{-1} C \int_{E_2} (\rho^1 - \rho^2)^2 (A_{\delta,K}^{-1} - A_\theta^{-1})^2 \, d\mathscr{L}^{d+1}.$$

By the fact that $A_{\delta,K}^{-1}, A_\theta^{-1} \in L^\infty(E_2)$, $\rho^1, \rho^2 \in L^2(E_2)$ and by the weak-$\star$ convergence of A_θ^{-1} to $A_{\delta,K}^{-1}$ in $L^\infty(E_2)$, we conclude that for θ small enough, the right-hand side of the previous inequality is smaller than δ, therefore by the arbitrariness of δ we conclude that $\mathcal{I}_{22} = 0$. $\square$

The next three lemmas (in an implicit or explicit form) are used in the proof of Theorem 6.1.

LEMMA 6.2. Let $(\rho^1, p^1), (\rho^2, p^2) : \Omega \to \mathbb{R}^2$ satisfy (1.6). Then L_S (defined in (1.5)) defines a monotone operator in the sense that

$$\text{if } \rho^1(x) < \rho^2(x), \text{ then } L_S(\rho^1, p^1)(x) < L_S(\rho^2, p^2)(x). \tag{6.10}$$

In particular, for $x \in \Omega$, if

$$L_S(\rho^1, p^1)(x) = L_S(\rho^2, p^2)(x), \tag{6.11}$$

then $\rho^1(x) = \rho^2(x)$ and $p^1(x) = p^2(x)$.

Proof. First of all, if we have (6.11) and $\rho^1(x) = \rho^2(x)$, then (1.5) and (1.6) imply $p^1(x) = p^2(x)$. Thus, it is enough to show that $\rho^1(x) = \rho^2(x)$.

Let us show now that L_S is a monotone operator in the sense of (6.10). First, note that $\rho \mapsto \rho S'(\rho) - S(\rho)$ is strictly increasing in $\mathbb{R}^+ \setminus \{1\}$. Indeed, from the strict convexity of S

(Assumption 1.1), one has that

$$\partial_\rho(\rho S'(\rho) - S(\rho)) = \rho S''(\rho) > 0 \text{ in } \mathbb{R}^+ \setminus \{1\}. \tag{6.12}$$

Therefore, (6.10) holds if $\rho^1(x), \rho^2(x) \in (0,1)$ or $\rho^1(x), \rho^2(x) \in (1,+\infty)$. Thus, it remains to treat the remaining cases.

Consider the case that $\rho^1(x) = 1 < \rho^2(x)$. Let us recall that S and S' are continuous in $\mathbb{R}^+$ and $\mathbb{R}^+ \setminus \{1\}$, respectively, (by Assumption 1.1). As $\rho \mapsto \rho S'(\rho) - S(\rho)$ is strictly increasing in $(1,+\infty)$, we have that

$$L_S(\rho^2, p^2) = \rho^2(x)S'(\rho^2(x)) - S(\rho^2(x)) + S(1) > \lim_{\rho \to 1+} \rho S'(\rho) - S(\rho) + S(1) = S'(1+)$$

$$\geqslant p^1(x) = L_S(\rho^1, p^1)(x). \tag{6.13}$$

So, from (6.13) and (1.6), we conclude (6.10).

Similar arguments show (6.10) in the case when $\rho^1(x) < \rho^2(x) = 1$.

Finally, by combining the inequalities in (6.10) for two previous cases, $\rho^1(x) = 1 < \rho^2(x)$ or $\rho^1(x) < 1 = \rho^2(x)$, we conclude (6.10) for $\rho^1(x) < 1 < \rho^2(x)$. $\qquad\square$

LEMMA 6.3. *We differentiate two cases.*

(1) *Assume $m = 1$ for m given in (1.9). Let (ρ^1, p^1) and (ρ^2, p^2) satisfy (1.6). Then we have*

$$0 \leqslant A \leqslant \max\{\sigma_1, \sigma_2\}, \text{ a.e. in } Q, \tag{6.14}$$

where $A = A(\rho^1, p^1, \rho^2, p^2)$ is given in (6.2) and σ_1, σ_2 are from Assumption (1.9)–(1.10).

(2) *Let $m > 1$. If there exist $c_0 > 0$ and a Borel set $E \subseteq Q$ such that $\rho^1, \rho^2 \geqslant c_0$ a.e. on E, then $A \llcorner E \in L^\infty(E)$ and $A \leqslant \max\{\sigma_1, \frac{\sigma_2}{c_0^{m-1}}\}$ a.e. in E, where $A = A(\rho^1, p^1, \rho^2, p^2)$ is given again in (6.2).*

Proof. Let us recall the definition of L_S from (1.5), that is,

$$L_S(\rho, p)(t, x) := [\rho(t,x)S'(\rho(t,x)) - S(\rho(t,x)) + S(1)]\mathbb{1}_{\{\rho \neq 1\}}(t,x) + p(t,x)\mathbb{1}_{\{\rho=1\}}(t,x).$$

The nonnegativity of A follows from the monotonicity of L_S shown in Lemma 6.2. We fix $q = (t, x) \in Q$ a Lebesgue for ρ^1, ρ^2, p^1, p^2 and assume $\rho^1(t,x) \geqslant \rho^2(t,x)$. If $q \in \{\rho^1 = 1\} \cap \{\rho^2 = 1\}$, there is nothing to check, since $A(q) = 0$ in both cases.

Let us show (1).

Case 1. If $q \in (\{\rho^1 > 1\} \cap \{\rho^2 > 1\}) \cup (\{\rho^1 < 1\} \cap \{\rho^2 < 1\})$, we have that

$$\rho^1(q)S'(\rho^1(q)) - S(\rho^1(q)) - \rho^2(q)S'(\rho^2(q)) + S(\rho^2(q))$$

$$= \tilde{\rho}S''(\tilde{\rho})(\rho^1(q) - \rho^2(q)) \geqslant \min\left\{\frac{1}{\sigma_1}, \frac{1}{\sigma_2}\right\}(\rho^1(q) - \rho^2(q)),$$

where $\tilde{\rho}$ is a constant between $\rho^1(q)$ and $\rho^2(q)$. Therefore, we get that $A(q) \leqslant \max\{\sigma_1, \sigma_2\}$.

Case 2. If $q \in \{\rho^1 > 1\} \cap \{\rho^2 = 1\}$, we have from (1.6) that

$$\rho^1(q)S'(\rho^1(q)) - S(\rho^1(q)) + S(1) - p^2(q) \geqslant \rho^1(q)S'(\rho^1(q)) - S(\rho^1(q)) - (S'(1+) - S(1)).$$

As $\rho \mapsto \rho S'(\rho) - S(\rho)$ is continuous in $[1, \rho^1(q)]$ and differentiable in $(1, \rho^1(q))$, the mean value theorem yields that

$$\rho^1(q)S'(\rho^1(q)) - S(\rho^1(q)) - p^2(q) \geqslant \tilde{\rho}S''(\tilde{\rho})(\rho^1(q) - 1) \geqslant \frac{1}{\sigma_1}(\rho^1(q) - 1),$$

where $\tilde{\rho}$ is between 1 and $\rho^1(q)$. Parallel arguments show (6.14) on the region $\{\rho^1 = 1\} \cap \{\rho^2 < 1\}$.

Case 3. If $q \in \{\rho^1 > 1\} \cap \{\rho^2 < 1\}$ from similar arguments as in Case 2, we have that

$$\rho^1(q)S'(\rho^1(q)) - S(\rho^1(q)) - (S'(1+) - S(1)) \geqslant \frac{1}{\sigma_1}(\rho^1(q) - 1)$$

and

$$(S'(1-) - S(1)) - [\rho^2(q)S'(\rho^2(q)) - S(\rho^1(q))] \geqslant \frac{1}{\sigma_2}(1 - \rho^2(q)).$$

As $S'(1+) \geqslant S'(1-)$, we conclude that

$$L_S(\rho^1, p^1)(q) - L_S(\rho^2, p^2)(q) \geqslant \sigma_1(\rho^1(q) - 1) + \sigma_2(1 - \rho^2(q)) = \min\left\{\frac{1}{\sigma_1}, \frac{1}{\sigma_2}\right\}(\rho^1(q) - \rho^2(q)).$$

The proof of (2) follows the very same steps as the one of (1). By the lower bound $c_0 > 0$ on the densities in E, we conclude that $A \leqslant \max\{\sigma_1, \frac{\sigma_2}{c_0^{m-1}}\}$. $\qquad\square$

LEMMA 6.4. *Let $\varepsilon > 0$ and let φ_ε be a smooth solution to* (6.8). *Then there exists a constant $C = C(T, \|\nabla\zeta\|_{L^2}) > 0$ such that:*

(1) $\sup_{t \in [0,T]} \|\nabla\varphi_\varepsilon\|_{L^2(\Omega)} \leqslant C$;

(2) $\|A_\varepsilon^{-\frac{1}{2}}\Delta\varphi_\varepsilon\|_{L^2(Q)} \leqslant C$.

Proof. The proof of this results follows the same lines as the one of [**20**, Lemma 3.1], therefore we omit it. $\qquad\square$

COROLLARY 6.5. *Let $\rho_0 \in \mathscr{P}(\Omega)$ satisfy $\mathcal{J}(\rho_0) < +\infty$. A solution pair to* (1.4)–(1.6) *such that $L_S(\rho, p) \in L^2(Q)$ is uniquely determined by ρ_0.*

Proof. From the contraction result in Theorem 6.1, we deduce the uniqueness of ρ. Now suppose that there exists to pressure fields p^1, p^2 solving (4.3) with the same ρ. Taking the difference of these two equations, we get

$$\Delta(L_S(\rho, p^1) - L_S(\rho, p^2)) = 0, \text{ in } \mathscr{D}'((0, T) \times \Omega).$$

For a.e. $t \in [0, T]$ and for any $\varphi \in C_c^2(\Omega)$, we have that

$$0 = \int_\Omega (L_S(\rho_t, p_t^1) - L_S(\rho_t, p_t^2))\Delta\varphi\,\mathrm{d}x = \int_{\{\rho_t = 1\}} (p_t^1 - p_t^2)\Delta\varphi\,\mathrm{d}x,$$

where in the last equality we used the fact that $p_t^1 = p_t^2$ a.e. in $\{\rho_t < 1\} \cup \{\rho_t > 1\}$. By the arbitrariness of φ, we conclude that $p_t^1 = p_t^2$ a.e. on $\{\rho_t = 1\}$ and therefore the uniqueness of p follows. $\qquad\square$

7. Discussions

7.1. *The emergence of the 'critical region' $\{\rho = 1\}$ – an example*

We consider $d = 1$ and we show that the critical region $\{\rho_t = 1\}$ is of positive measure, whenever the two regions $\{\rho_t > 1\}$ and $\{\rho_t < 1\}$ are also of positive measure. We will see that this also implies that the critical region is expected to emerge for positive times, even if $\mathscr{L}^1(\{\rho_0 = 1\}) = 0$ (and if $\mathscr{L}^1(\{\rho_0 < 1\}) > 0$ and $\mathscr{L}^1(\{\rho_0 > 1\}) > 0$). This phenomenon corresponds to the growth of the critical region for self-organized criticality in [**4**].

PROPOSITION 7.1. *Let $\Omega \subset \mathbb{R}$ and (ρ, p) be given in Theorem 3.1. If $t \in (0, T)$ is a Lebesgue point both for $t \mapsto \rho_t$ and $t \mapsto p_t$ with*

$$\mathscr{L}^1(\{\rho_t < 1\}) > 0 \quad \text{and} \quad \mathscr{L}^1(\{\rho_t > 1\}) > 0, \tag{7.1}$$

then $\mathscr{L}^1(\{\rho_t = 1\}) > 0$.

Proof. Let us show that $p(t, \cdot) \in C^{0, \frac{1}{2}}(\Omega)$ for a.e. $t \in [0, T]$. From Theorem 3.1, we know that $\partial_x p \in L^2(Q)$. As a consequence, we have that

$$\int_0^T \operatorname*{osc}_{x \in [a,b]} p(t, x) \, dt \leqslant \int_0^T \int_a^b |\partial_x p(t, x)| \, dt \leqslant \|\partial_x p\|_{L^2(Q)} T^{\frac{1}{2}} |b - a|^{\frac{1}{2}}.$$

Thus, $p \in L^1(0, T; C^{0, \frac{1}{2}}(\Omega))$ and we conclude.

Let $t \in (0, T)$ be a Lebesgue point for both $t \mapsto p_t$ and $t \mapsto \rho_t$ such that $\mathscr{L}^1(\{\rho_t < 1\}) > 0$ and $\mathscr{L}^1(\{\rho_t > 1\}) > 0$. Then (7.1) and (3.3) imply that there exists $\{U_i\}_{i \in \{1,2\}}$ subsets of Ω such that $\mathscr{L}^1(U_i) > 0$ and $p_t = i$ a.e. in U_i for $i \in \{1, 2\}$. As p_t is continuous in Ω for a.e. $t \in [0, T]$, there exists a point $x_0 \in \Omega$ such that $p_t(x_0) = 3/2$. Since $N := p_t^{-1}((5/4, 7/4))$ is a nonempty open set, N has a positive measure. From (3.3), we have that $N \subset \{\rho_t = 1\}$ and thus we conclude. $\square$

REMARK 7.1. A similar result can be stated in higher dimensions as well, based on the fact that Sobolev functions cannot take finitely many values, except if they are constants.

7.2. *Formal derivation of a free boundary problem — an example*

Next, we formally derive the free boundary motion corresponding to the particular problem in (3.2)–(3.3). For the analysis, we assume that ρ and p are continuous in Q and smooth in $\{p\rho < 1\}$, $\{1 < p\rho < 2\}$ and $\{p\rho > 2\}$, which also have smooth boundaries. Under this assumption, we deduce the following free boundary problem,

$$\begin{cases} \partial_t \rho - \Delta \rho - \nabla \cdot (\nabla \Phi \rho) = 0, & \text{in } \{p\rho < 1\}, \\ \rho = 1, & \text{in } \{1 < p\rho < 2\}, \\ \partial_t \rho - 2\Delta \rho - \nabla \cdot (\nabla \Phi \rho) = 0, & \text{in } \{p\rho > 2\}, \end{cases} \quad \text{and} \quad \begin{cases} p = 1, & \text{in } \{p\rho < 1\}, \\ -\Delta p = \Delta \Phi, & \text{in } \{1 < p\rho < 2\}, \\ p = 2, & \text{in } \{p\rho > 2\}, \end{cases} \tag{7.2}$$

with boundary conditions

$$\begin{cases} |D(p\rho)^{1+}| - |D(p\rho)^{1-}| &= 0, & \text{on } \{p\rho = 1\}, \\ |D(p\rho)^{2+}| - |D(p\rho)^{2-}| &= 0, & \text{on } \{p\rho = 2\}, \end{cases} \tag{7.3}$$

where for any $f : Q \to \mathbb{R}$ and $c \in \mathbb{R}$, denotes

$$Df^{c\pm}(t, x) := \lim_{\substack{(s,y) \to (t,x), \\ (s,y) \in \{\pm(f-c) > 0\}}} Df(s, y).$$

As the condition (3.3) implies

$$\begin{cases} \rho < 1, & p = 1 & \text{in } \{p\rho < 1\}, \\ \rho = 1, & 1 < p < 2, & \text{in } \{1 < p\rho < 2\}, \\ \rho > 1, & p = 2, & \text{in } \{p\rho > 2\}, \end{cases} \tag{7.4}$$

the first system of equations in (7.2) is a direct consequence of Theorem 3.1. Next, we consider the second system of equations in (7.2). For a test function $\xi \in C_c^\infty(Q)$ such that ξ is compactly

supported in $\{1 < p\rho < 2\}$, (7.4) implies

$$0 = \int_Q -\rho\partial_t\xi + D(p\rho) \cdot D\xi + D\Phi \cdot D\xi \,\mathrm{d}x\mathrm{d}t = \int_Q -\partial_t\xi + (\rho Dp + pD\rho + D\Phi) \cdot D\xi \,\mathrm{d}x\mathrm{d}t$$

$$= \int_Q (Dp + D\Phi) \cdot D\xi \,\mathrm{d}x\mathrm{d}t. \tag{7.5}$$

Thus, we conclude that $-\Delta p = \Delta\Phi$ in $\{1 < p\rho < 2\}$. The other cases follow from (7.4).

Finally, let us find the boundary condition (7.3) on $\{p\rho = 1\}$ and $\{p\rho = 2\}$. As [**24**, Theorem 3.1], we deduce the condition based on integration by parts. Note that the boundary condition (7.3) can be regarded as Rankine–Hugoniot conditions. For a test function $\xi \in C_c^\infty(Q)$, (3.2) implies

$$0 = \int_Q -\rho\partial_t\xi + [D(p\rho) + \rho D\Phi] \cdot D\xi \,\mathrm{d}x\mathrm{d}t = \int_{\{p\rho<1\}} -\rho\partial_t\xi + [D(p\rho) + \rho D\Phi] \cdot D\xi \,\mathrm{d}x\mathrm{d}t$$

$$+ \int_{\{1<p\rho<2\}} -\rho\partial_t\xi + [Dp + D\Phi] \cdot D\xi \,\mathrm{d}x\mathrm{d}t + \int_{\{p\rho>2\}} -\rho\partial_t\xi + [D(p\rho) + D\phi] \cdot D\xi \,\mathrm{d}x\mathrm{d}t. \tag{7.6}$$

For a set $N = \{p\rho < 1\}, \{1 < p\rho < 2\}$ or $\{p\rho > 2\}$, the smoothness of p and ρ in N and (7.2) imply

$$\int_N -\rho\partial_t\xi + [D(p\rho) + \rho D\Phi] \cdot D\xi \,\mathrm{d}x\mathrm{d}t = \int_{\partial N} (-\rho\mathbf{n}_t + [D(p\rho) + \rho D\Phi] \cdot \mathbf{n}_x)\xi \,\mathrm{d}\mathscr{H}^d, \tag{7.7}$$

where $\mathbf{n}_t$ and $\mathbf{n}_x$ are the outward normal vectors on ∂N in x and t directions, respectively. From (7.6) and (7.7), we conclude that

$$0 = \int_{\partial\{p\rho<1\}} [D(p\rho)^{1-} - D(p\rho)^{1+}] \cdot \mathbf{n}_x\xi \,\mathrm{d}\mathscr{H}^d + \int_{\partial\{p\rho>2\}} [D(p\rho)^{2+} - D(p\rho)^{2-}] \cdot \mathbf{n}_x\xi \,\mathrm{d}\mathscr{H}^d. \tag{7.8}$$

By the arbitrariness of ξ, (7.8) implies

$$\begin{cases} [D(p\rho)^{1+} - D(p\rho)^{1-}] \cdot \mathbf{n}_x & = 0 \text{ on } \{p\rho = 1\}, \\ [D(p\rho)^{2+} - D(p\rho)^{2-}] \cdot \mathbf{n}_x & = 0 \text{ on } \{p\rho = 2\}. \end{cases}$$

As $\mathbf{n}_x$ is parallel to $D(p\rho)$ on the level set of $p\rho$, we conclude (7.3).

7.3. A nontrivial stationary solution — an example

In this subsection, in one spacial dimension, we study stationary solutions to our problems. For simplicity, let us consider $\Omega := (0, l) \subset \mathbb{R}$ for $l > 0$. Let (ρ, p) be a solution to (3.2)–(3.3) with potential $\Psi(x) = 2x$, where we have associated energy functional,

$$\mathcal{J}(\rho(x)) = \int_0^l S(\rho(x))\,\mathrm{d}x + \int_0^l \Psi(x)\rho(x)\,\mathrm{d}x,$$

where S is given in (3.1). From Theorem 3.1, there exists a solution (ρ, p) of

$$\begin{cases} \partial_t\rho - \partial_x^2(\rho p) - 2\partial_x\rho = 0, & \text{in } (0, T) \times (0, l), \\ \rho(0, \cdot) = \rho_0, & \text{in } (0, l), \\ \partial_x(\rho p) + 2\rho = 0, & \text{in } [0, T] \times \partial(0, l), \end{cases} \tag{7.9}$$
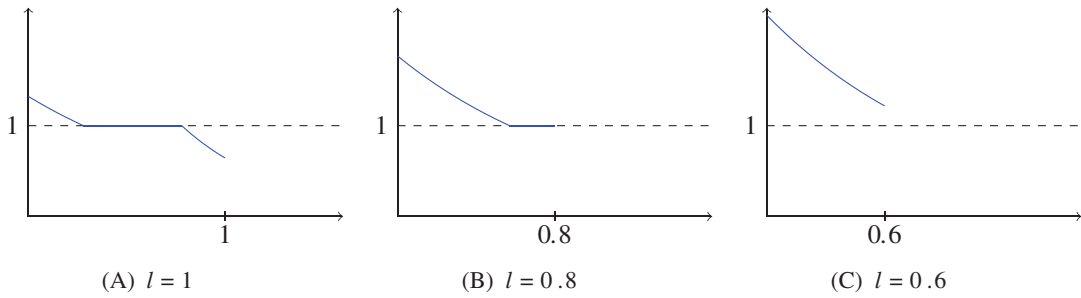
(A) $l = 1$ (B) $l = 0.8$ (C) $l = 0.6$

FIGURE 2 (colour online). *Stationary solutions.*

and (ρ, p) satisfies (3.3). A stationary problem associated to (7.9) is as follows: find $\rho, p : [0, l] \to \mathbb{R}$ satisfying (3.3) and

$$\begin{cases} \partial_x^2 (\rho p) + 2 \partial_x \rho = 0, & \text{in } (0, l), \\ \partial_x (\rho p) + 2 \rho = 0, & \text{at } x = 0 \text{ and } x = l. \end{cases} \qquad (7.10)$$

The solution (ρ, p) can be also characterized as minimizers of the free energy $\mathcal{J}$. Writing down the optimality conditions (using Lemma 3.2), we have

$$\rho = \begin{cases} \exp(A - x) & \text{in } [0, A) \cap [0, l], \\ 1 & \text{in } \left[A, A + \frac{1}{2}\right] \cap [0, l], \\ \exp(2A + 1 - 2x) & \text{in } \left(A + \frac{1}{2}, l\right] \cap [0, l], \end{cases} \qquad (7.11)$$

where A is chosen to satisfy

$$\int_0^l \rho \, \mathrm{d}x = 1. \qquad (7.12)$$

Depending on the value l, some cases in (7.11) may not be present (see Figure 2).

LEMMA 7.2. *If $l > \ln(\frac{3}{2}) + \frac{1}{2}$, then all three sets $\{\rho < 1\}$, $\{\rho = 1\}$ and $\{\rho > 1\}$ have positive measure.*

Proof. We consider the continuous function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \exp(x) - \frac{1}{2} \exp(2x + 1 - 2l) - 1.$$

We claim that there exists a solution of $f(x) = 0$ in $(0, l - \frac{1}{2})$. First, $f(0) = -\frac{1}{2} \exp(1 - 2l) < 0$. Also, as $l > \ln(\frac{3}{2}) + \frac{1}{2}$, it holds that

$$f\left(l - \frac{1}{2}\right) = \exp\left(l - \frac{1}{2}\right) - \frac{3}{2} > 0.$$

Therefore, by the continuity of f and by the intermediate value theorem, we can find $A \in (0, l - \frac{1}{2})$ such that $f(A) = 0$. For given A, we recall ρ from (7.11). Since we have

$$A, \quad A + \frac{1}{2} \in (0, l), \qquad (7.13)$$

all three sets $\{\rho < 1\}$, $\{\rho = 1\}$ and $\{\rho > 1\}$ have positive measure.

It remains to show that with this choice of A, ρ given in (7.11) satisfies (7.12). From (7.11) and (7.13), it holds that

$$\int_0^l \rho(x)\,\mathrm{d}x = \int_0^A \exp(A-x)\,\mathrm{d}x + \int_A^{A+\frac{1}{2}} 1\,\mathrm{d}x + \int_{A+\frac{1}{2}}^l \exp(2A+1-2x)\,\mathrm{d}x$$

$$= (\exp(A)-1) + \frac{1}{2} + \left(\frac{1}{2} - \frac{1}{2}\exp(2A+1-2l)\right) = f(A)+1 = 1.$$

and we conclude. $\qquad\square$

REMARK 7.2. By a parallel argument as above, one can check that a set $\{\rho > 1\}$ will have zero measure if $l \in (0, \ln(\frac{3}{2}) + \frac{1}{2}]$. To see this, let us differentiate two cases.

Case 1. If $l \in (0, \ln 2]$, then we have

$$\rho(x) = \frac{\exp(-x)}{1-\exp(-l)} \quad \text{in } [0,l].$$

Case 2. If $l \in (\ln 2, \ln(\frac{3}{2}) + \frac{1}{2}]$, then it holds that

$$\rho(x) = \begin{cases} \exp(A-x) & \text{in } [0,A), \\ 1 & \text{in } [A,l]. \end{cases}$$

where $A \in [l - \frac{1}{2}, l)$ is a solution of $\exp(A) - A = 2 - l$.

Appendix A. *Optimal transport toolbox*

Let us recall now some basic definitions and results from the theory of optimal transport. Let $\Pi(\mu,\nu)$ be the set of all Borel probability measure π on $\Omega \times \Omega$ such that

$$\pi(A \times \Omega) = \mu(A), \quad \pi(\Omega \times B) = \nu(B) \text{ for all measurable subsets } A, B \subset \Omega.$$

For $\mu, \nu \in \mathscr{P}_2(\Omega)$ we define the 2-Wasserstein or Monge–Kantorovich distance as

$$W_2(\mu,\nu) := \min\left\{ \int_{\Omega\times\Omega} |x-y|^2\,\mathrm{d}\gamma : \gamma \in \Pi(\mu,\nu) \right\}^{\frac{1}{2}}. \tag{A.1}$$

For $\phi : \Omega \to \mathbb{R}$ measurable, we use the notations

$$\phi^+(x) := \max\{\phi(x), 0\}, \phi^-(x) := \max\{-\phi(x), 0\} \text{ and } \phi^c(x) := \operatorname*{ess\,inf}_{y\in\Omega}\left\{\frac{1}{2}|x-y|^2 - \phi(y)\right\}$$

where $x \in \Omega$.

A.1. *Basic facts from optimal transport.*

Let us recall the definition and properties of Kantorovich potentials and optimal transport maps. The results are well known in the literature, we refer, for instance, to [**38**] for the proofs of the statements.

DEFINITION A.1. Let $\mu, \nu \in \mathscr{P}(\Omega)$ be given.

(1) We say that $\overline{\phi} : \Omega \to \mathbb{R}$ is a Kantorovich potential from μ to ν if $(\overline{\phi}, \overline{\phi}^c)$ is a maximizer of the *Kantorovich* problem:

$$\sup\left\{ \int_\Omega \phi\,\mathrm{d}\mu + \int_\Omega \psi\,\mathrm{d}\nu : (\phi,\psi) \in L^1_\mu(\Omega) \times L^1_\nu(\Omega), \phi(x) + \psi(y) \right.$$

$$\left. \leqslant \frac{1}{2}|x-y|^2,\ \mu \otimes \nu - \text{a.e. } (x,y) \in \Omega \times \Omega \right\}.$$

We denote the set of Kantorovich potential from μ to ν by $\mathcal{K}(\mu,\nu)$.

(2) We say that a Borel map $T : \Omega \to \Omega$ is a optimal transport map from μ to ν if T is a minimizer of the following problem:

$$\inf \left\{ \int_\Omega |x - T(x)|^2 \, \mathrm{d}\mu : T_\# \mu = \nu \right\}.$$

Here, $(T_\# \mu)(A) := \mu(T^{-1}(A))$ for any Borel set $A \subseteq \Omega$.

LEMMA A.1 [**38**]. *For $\mu \in \mathscr{P}^{\mathrm{ac}}(\Omega)$ and $\nu \in \mathscr{P}(\Omega)$, there exists a Lipschitz continuous Kantorovich potential $\overline{\phi}$ and an optimal transport map T from μ to ν. Also, it holds that*

$$x - T(x) = \nabla \overline{\phi}(x) \text{ for a.e. } x \in \mathrm{spt}(\mu) \text{ and } W_2(\mu, \nu) = \|\nabla \overline{\phi}\|_{L^2_\mu}. \tag{A.2}$$

LEMMA A.2 [**41**, Theorem 1.3], [**38**, Proposition 1.11]. *Let $\mu, \nu \in \mathscr{P}(\Omega)$. Define $\mathcal{L} : L^1_\mu(\Omega) \times L^1_\nu(\Omega) \to \mathbb{R}$ as*

$$\mathcal{L}(\phi, \psi) := \int_\Omega \phi \, \mathrm{d}\mu + \int_\Omega \psi \, \mathrm{d}\nu. \tag{A.3}$$

Then, it holds that

$$\frac{1}{2} W_2^2(\mu, \nu)$$

$$= \max \left\{ \mathcal{L}(\phi, \psi) : (\phi, \psi) \in C_b(\Omega) \times C_b(\Omega), \ \phi(x) + \psi(y) \leqslant \frac{1}{2}|x - y|^2 \text{ for all } x, y \in \Omega \right\},$$

$$= \sup \left\{ \mathcal{L}(\phi, \psi) : (\phi, \psi) \in L^1_\mu(\Omega) \times L^1_\nu(\Omega), \ \phi(x) + \psi(y) \right.$$

$$\left. \leqslant \frac{1}{2}|x - y|^2 \text{ for } \mu \otimes \nu - \text{a.e. } (x, y) \in \Omega \times \Omega \right\}.$$

PROPOSITION A.3. *For $r \in [1, +\infty]$, let $\mu \in L^r(\Omega) \cap \mathscr{P}(\Omega)$ and $\nu \in \mathscr{P}(\Omega)$. Then, it holds that*

$$\sup_{\phi \in L^{r'}(\Omega)} \mathcal{L}(\phi, \phi^c) = \frac{1}{2} W_2^2(\mu, \nu), \tag{A.4}$$

where $r' := \frac{r}{r-1}$ ($r' = 1$ if $r = +\infty$ and $r' = +\infty$ if $r = 1$) and $\mathcal{L}$ is given in (A.3).

Proof. Step 1. Let us show that

$$\frac{1}{2} W_2^2(\mu, \nu) = \mathcal{I}_1 \tag{A.5}$$

where

$$\mathcal{I}_1 := \sup \left\{ \mathcal{L}(\phi, \psi) : (\phi, \psi) \in L^{r'}(\Omega) \times L^1_\nu(\Omega), \ \phi(x) + \psi(y) \right.$$

$$\left. \leqslant \frac{1}{2}|x - y|^2 \text{ for } \mu \otimes \nu - \text{a.e. } (x, y) \in \Omega \times \Omega \right\}. \tag{A.6}$$

By Hölder's inequality, it holds that

$$\|\phi\|_{L^1_\mu(\Omega)} = \int_\Omega |\phi(x)| \mu(x) \, \mathrm{d}x \leqslant \|\phi\|_{L^{r'}(\Omega)} \|\mu\|_{L^r(\Omega)}. \tag{A.7}$$

As $\mu \in L^r(\Omega) \cap \mathscr{P}(\Omega)$, we conclude that

$$L^{r'}(\Omega) \subset L^1_\mu(\Omega) \text{ and thus } C_b(\Omega) \times C_b(\Omega) \subset L^{r'}(\Omega) \times L^1_\nu(\Omega) \subset L^1_\mu(\Omega) \times L^1_\nu(\Omega).$$

From Lemma A.2, we conclude (A.5).

Step 2. It remains to show that

$$\sup_{\phi \in L^{r'}(\Omega)} \mathcal{L}(\phi, \phi^c) = \mathcal{I}_1 \tag{A.8}$$

for $\mathcal{I}_1$ given in (A.6). Indeed, let us note that by density we have

$$\sup_{\phi \in L^{r'}(\Omega)} \mathcal{L}(\phi, \phi^c) = \sup_{\phi \in C_b(\Omega)} \mathcal{L}(\phi, \phi^c) = \max_{\phi \in C_b(\Omega)} \mathcal{L}(\phi, \phi^c),$$

and the latter two quantities are finite by [**38**, Proposition 1.11]. Therefore the thesis of the proposition follows. $\qquad\square$

A.2. *Some properties of minimizers in the minimizing movements scheme and optimality conditions.*

LEMMA A.4. *For ρ_k given in (2.3) and S satisfying (4.2), it holds that $\rho_k > 0$ a.e.*

Proof. The proof is inspired by [**38**, Lemma 8.6]. The difference is that we consider the sub-differential of S instead of its derivative.

Step 1. For simplicity, let us use the notation $\mu := \rho_k$ and consider a competitor

$$\mu_1 := \frac{1}{\mathscr{L}^d(\Omega)}. \tag{A.9}$$

Define $\mu_\varepsilon := (1-\varepsilon)\mu + \varepsilon\mu_1$ for $\varepsilon \in (0,1)$. From convexity of Wasserstein distance, we have

$$\mathcal{I}_1 := \mathcal{J}(\mu) - \mathcal{J}(\mu_\varepsilon) \leqslant \frac{1}{2\tau}W_2^2(\mu_\varepsilon, \rho_{k-1}) - \frac{1}{2\tau}W_2^2(\mu, \rho_{k-1})$$

$$\leqslant \varepsilon\left\{\frac{1}{2\tau}W_2^2(\mu_1, \rho_{k-1}) - \frac{1}{2\tau}W_2^2(\mu, \rho_{k-1})\right\}.$$

The compactness of Ω implies

$$\mathcal{I}_1 \leqslant C_1\varepsilon \text{ for some } C_1 > 0. \tag{A.10}$$

Step 2. Set $A := \{x \in \Omega : \mu > 0\}$ and $B := \{x \in \Omega : \mu = 0\}$. Let us show that $\mathscr{L}^d(B) = 0$. For sufficiently small $\varepsilon > 0$, it holds that $\varepsilon\mu_1 < 1$ and thus

$$\mathcal{I}_1 = \int_A S(\mu(x)) - S(\mu_\varepsilon(x)) + \Phi[\mu(x) - \mu_\varepsilon(x)]\mathrm{d}x + (S(0) - S(\varepsilon\mu_1))\mathscr{L}^d(\mathcal{B}) - \varepsilon\frac{1}{\mathscr{L}^d(\Omega)}\int_B \Phi\mathrm{d}x.$$

By convexity of S, it holds that

$$\mathcal{I}_1 \geqslant \varepsilon\int_A [\xi_\varepsilon(x) + \Phi](\mu(x) - \mu_1)\mathrm{d}x + (S(0) - S(\varepsilon\mu_1))\mathscr{L}^d(B) - \varepsilon\frac{1}{\mathscr{L}^d(\Omega)}\int_B \Phi\mathrm{d}x,$$

where $\xi_\varepsilon(x) \in \partial S(\mu_\varepsilon(x))$.

From (A.10), we conclude that for all $\xi_\varepsilon(x) \in \partial S(\mu_\varepsilon(x))$

$$\mathcal{I}_2 := \int_A [\xi_\varepsilon(x) + \Phi](\mu(x) - \mu_1)\mathrm{d}x + \frac{1}{\varepsilon}(S(0) - S(\varepsilon\mu_1))\mathscr{L}^d(\mathcal{B}) \leqslant C_1 + C. \qquad (\text{A.11})$$

Note that by the convexity of S, its subdifferential is monotone, therefore for all $\varepsilon \in [0,1]$,

$$(\xi_\varepsilon(x) - \xi_1)(\mu_\varepsilon(x) - \mu_1) \geqslant 0,$$

and thus

$$\xi_\varepsilon(x)(\mu(x) - \mu_1) \geqslant \xi_1(\mu(x) - \mu_1), \qquad (\text{A.12})$$

for a.e. $x \in \Omega$ where $\xi_1 \in \partial S(\mu_1)$. Therefore,

$$\mathcal{I}_2 \geqslant \int_A [\xi_1 + \Phi](\mu(x) - \mu_1)\mathrm{d}x + \frac{1}{\varepsilon}(S(0) - S(\varepsilon\mu_1))\mathscr{L}^d(B).$$

Since $S'(0+) = -\infty$ from (4.2), the right-hand side blows up as ε goes to zero unless $\mathscr{L}^d(B) = 0$. As $\mathcal{I}_2$ is bounded by $C_1 + C$ from (A.11), we conclude that $\mathscr{L}^d(B) = 0$, and thus $\mu > 0$ a.e. $\qquad \square$

Appendix B.   *Some results from convex analysis*

For a Banach space $\mathfrak{X}$ and $F : \mathfrak{X} \to \mathbb{R} \cup \{\pm\infty\}$, we say that $F^* : \mathfrak{X}^* \to \mathbb{R} \cup \{\pm\infty\}$ is a Legendre transform of F if

$$F^*(y) := \sup_{x \in \mathfrak{X}} \{\langle x, y \rangle_{\mathfrak{X}, \mathfrak{X}^*} - F(x)\} \text{ for } y \in \mathfrak{X}^*. \qquad (\text{B.1})$$

Here, $\mathfrak{X}^*$ stands for the topological dual space of $\mathfrak{X}$. We will denote by $C_b(\Omega)$ the space of bounded continuous functions in Ω. In the derivation of optimality conditions associated to the minimizing movement schemes, in Appendix B, we use subdifferential calculus in $L^r(\Omega)$ ($r \in [1, +\infty]$) spaces. Let us recall some basic results on this.

Let us recall the definition of subdifferentials on $L^r(\Omega)^*$ for $r \in [1, +\infty]$.

DEFINITION B.1 [**36**, (1.9), (1.10) & (1.13)]. For $\psi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, $r \in [1, +\infty]$ and $\Psi : L^r(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\Psi(\mu) := \int_\Omega \psi(\mu(x))\mathrm{d}x, \qquad (\text{B.2})$$

we say that $\xi \in L^r(\Omega)^*$ belongs to the subdifferential of Ψ at $\mu \in L^r(\Omega)$ if

$$\Psi(\nu) \geqslant \Psi(\mu) + \langle \xi, \nu - \mu \rangle_{L^r(\Omega)^*, L^r(\Omega)} \qquad (\text{B.3})$$

for every $\nu \in L^r(\Omega)$. We denote by $\partial\Psi(\mu)$ the set of subdifferentials of Ψ at the point $\mu \in L^r(\Omega)$.

DEFINITION B.2 [**22**, Definition 1.3.1]. Let $\mathfrak{X}$ be a Banach space. The set of functions $F : \mathfrak{X} \to \mathbb{R} \cup \{\pm\infty\}$ which are pointwise supremum of a family of continuous affine function is denoted by $\Gamma(\mathfrak{X})$.

LEMMA B.1 [**22**, Proposition 1.3.1]. *The following properties are equivalent to each other.*

(1) *$F \in \Gamma(\mathfrak{X})$.*
(2) *F is a convex lower semicontinuous function from $\mathfrak{X}$ into $\mathbb{R} \cup \{\pm\infty\}$ and if F takes the value $-\infty$, then F is identically equal to $-\infty$.*

LEMMA B.2 [**22**, Proposition 1.5.6]. *If* $F_1, F_2 \in \Gamma(\mathfrak{X})$ *and if there exists* $\hat{\mu} \in \mathfrak{X}$ *such that* $F_1(\hat{\mu}), F_2(\hat{\mu}) < +\infty$ *and either* F_1 *or* F_2 *is continuous at* $\hat{\mu}$, *then it holds that*

$$\partial F_1(\mu) + \partial F_2(\mu) = \partial(F_1 + F_2)(\mu) \text{ for all } \mu \in \mathfrak{X}.$$

Appendix C.   *An Aubin–Lions lemma and some of its consequences*

In [**37**], the authors presented the following version of the classical Aubin–Lions lemma (see [**3**]):

THEOREM C.1 [**37**, Theorem 2]. *Let* B *be a Banach space and* $\mathcal{U}$ *be a family of measurable* B-*valued function. Let us suppose that there exist a normal coercive integrand* $\mathfrak{F} : (0, T) \times B \to [0, +\infty]$, *meaning that:*

(1) $\mathfrak{F}$ *is* $\mathscr{B}(0, T) \otimes \mathscr{B}(B)$-*measurable, where* $\mathscr{B}(0, T)$ *and* $\mathscr{B}(B)$ *denote the* σ-*algebras of the Lebesgue measurable subsets of* $(0, T)$ *and of the Borel subsets of* B, *respectively;*
(2) *the maps* $v \mapsto \mathfrak{F}_t(v) := \mathfrak{F}(t, v)$ *are lower semicontinuous for a.e.* $t \in (0, T)$;
(3) $\{v \in B : \mathfrak{F}_t(v) \leqslant c\}$ *are compact for any* $c \geqslant 0$ *and for a.e.* $t \in (0, T)$,

and an lower semicontinuous map $g : B \times B \to [0, +\infty]$ *with the property*

$$[u, v \in D(\mathfrak{F}_t), \ g(u, v) = 0] \Rightarrow u = w, \text{ for a.e. } t \in (0, T).$$

If

$$\sup_{u \in \mathcal{U}} \int_0^T \mathfrak{F}(t, u(t)) \mathrm{d}t < +\infty \text{ and } \limsup_{h \downarrow 0} \sup_{u \in \mathcal{U}} \int_0^{T-h} g(u(t+h), u(t)) \mathrm{d}t = 0,$$

then $\mathcal{U}$ *is relatively compact in* $\mathcal{M}(0, T; B)$.

Many recent papers (including [**26, 29**]) on gradient flows in the Wasserstein space used the previous theorem to gain pre-compactness of interpolated curves. In our setting, we use the following result.

LEMMA C.2. *Let* $T > 0$ *and let* $q \in [1, +\infty)$ *and* $n > 0$ *be such that* $nq^* > 1$, *where* $q^* := \frac{qd}{d-q}$ *(with the convention* $q^* \in (0, +\infty)$ *is arbitrary if* $q \geqslant d$, *and therefore,* $n > 0$ *and* $nq^* > 1$ *can also be arbitrary). Suppose that* $(\rho^\tau)_{\tau > 0}$ *is a sequence of curves on* $[0, T]$ *with values in* $\mathscr{P}(\Omega)$ *and suppose that there exists* $C > 0$ *such that*

$$W_2^2(\rho_t^\tau, \rho_s^\tau) \leqslant C|t - s + \tau|, \ \forall \ 0 \leqslant s < t \leqslant T \tag{C.1}$$

and $((\rho^\tau)^n)_{\tau > 0}$ *is uniformly bounded in* $L^q([0, T]; W^{1,q}(\Omega))$ *by* C. *We suppose moreover that there exists* $\beta \geqslant 1$ *such that* $\|\rho_t^\tau\|_{L^\beta(\Omega)} \leqslant C$ *for a.e.* $t \in [0, T]$.

(1) *Then,* $(\rho^\tau)_{\tau > 0}$ *is pre-compact in* $L^\gamma(Q)$, *with* $1 \leqslant \gamma \leqslant \beta$ *if* $\beta < nq^*$ *and* $1 \leqslant \gamma < nq^*$, *if* $\beta \geqslant nq^*$.
(2) *If in addition,* $(\rho^\tau)_{\tau > 0}$ *is uniformly bounded in* $L^{\beta_2}(Q)$ *for some* $\beta_2 > \gamma$ *(where* γ *is given in (1)), then* $(\rho^\tau)_{\tau > 0}$ *is pre-compact in* $L^{\gamma_2}(Q)$, *for any* $1 \leqslant \gamma_2 < \beta_2$.

Proof. Let us use the previously stated Aubin–Lions lemma, that is, Theorem C.1. Let $1 \leqslant \alpha < q^*$ be fixed (that we set up later) and let us set $B := L^{n\alpha}(\Omega)$, $\mathfrak{F} : L^{n\alpha}(\Omega) \to [0, +\infty]$ defined as

$$\mathfrak{F}(\rho) := \begin{cases} \|\rho^n\|_{W^{1,q}(\Omega)}, & \text{if } \rho^n \in W^{1,q}(\Omega), \ \rho \in \mathscr{P}(\Omega), \\ +\infty, & \text{otherwise} \end{cases}$$

and $g : L^{n\alpha}(\Omega) \times L^{n\alpha}(\Omega) \to [0, +\infty]$ defined as

$$g(\mu, \nu) := \begin{cases} W_2(\mu, \nu), & \text{if } \mu, \nu \in \mathscr{P}(\Omega), \\ +\infty, & \text{otherwise.} \end{cases}$$

In this setting, $(\rho^\tau)_{\tau>0}$ and $\mathfrak{F}$ satisfy the assumptions of Theorem C.1. Indeed, from the assumption, one has in particular that $\int_0^T \|(\rho_t^\tau)^n\|_{W^{1,q}(\Omega)}^q \mathrm{d}t \leqslant C$. The injection $W^{1,q}(\Omega) \hookrightarrow L^\alpha(\Omega)$ is compact for any $1 \leqslant \alpha < q^*$, the injection $i : s \mapsto s^{\frac{1}{n}}$ is continuous from $L^\alpha(\Omega)$ to $L^{n\alpha}(\Omega)$ and the sub-level sets of $\rho \mapsto \|\rho^n\|_{W^{1,q}(\Omega)}$ are compact in $L^{n\alpha}(\Omega)$.

Moreover, by the fact that g defines a distance on $D(\mathfrak{F})$ and from (C.1), one has that g also satisfies the assumptions from Theorem C.1, hence the implication of the theorem holds and one has that $(\rho^\tau)_{\tau \geqslant 0}$ is pre-compact in $\mathscr{M}(0, T; L^{n\alpha}(\Omega))$. Let us note that (C.1) implies that there exists $\rho \in C([0, T]; \mathscr{P}(\Omega))$ such that up to passing to a subsequence $(\rho_\tau)_{\tau>0}$ converges uniformly (with respect to W_2) to ρ as $\tau > 0$. Up to passing to another subsequence, ρ is the limit also in $\mathscr{M}(0, T; L^{n\alpha}(\Omega))$.

From our assumption, we know that $\|\rho_t^\tau\|_{L^\beta(\Omega)} \leqslant C$ for a.e. $t \in [0, T]$. Now, if $\beta < nq^*$, then setting α such that $n\alpha = \beta$, Lebesgue's dominated convergence theorem implies the strong pre-compactness of $(\rho^\tau)_{\tau>0}$ in $L^\beta(Q)$. Otherwise, Lebesque's dominated convergence implies the strong pre-compactness in $L^\gamma(Q)$ for any $1 \leqslant \gamma < nq^*$. This concludes the proof of (1).

To show (2), we note that (1) already implies that $\rho^\tau \to \rho$, strongly in $L^\gamma(Q)$ as $\tau \downarrow 0$ and in particular a.e. in Q. Furthermore, by the uniform bounds in $L^{\beta_2}(\Omega)$, with $\beta_2 > \gamma$, for any $1 \leqslant \gamma_2 < \beta_2$ we have that

$$\int_Q (\rho^\tau)^{\gamma_2} \mathrm{d}x \mathrm{d}t \leqslant \left(T \mathscr{L}^d(\Omega)\right)^{1-\frac{\gamma_2}{\beta_2}} \|\rho^\tau\|_{L^{\beta_2}}^{\gamma_2},$$

which implies that $(\rho^\tau)^{\gamma_2}$ is uniformly integrable on Q. Therefore, Vitali's convergence theorem yields the claim. $\qquad\square$

References

1. L. AMBROSIO, 'Transport equation and Cauchy problem for *BV* vector fields', *Invent. Math.* 158 (2004) 227–260.
2. L. AMBROSIO, N. GIGLI and G. SAVARÉ, *Gradient flows in metric spaces and in the space of probability measures*, 2nd edn, Lecture notes in Mathematics ETH Zürich (Birkhäuser, Bassel, 2008).
3. J.-P. AUBIN, 'Un théorème de compacité', *C. R. Acad. Sci. Paris* 256 (1963) 5042–5044.
4. P. BANTAY and I. M. JÁNOSI, 'Avalanche dynamics from anomalous diffusion', *Phys. Rev. Lett.* 68 (1992) 2058–2061.
5. V. BARBU and M. RÖCKNER, 'Probabilistic representation for solutions to nonlinear Fokker-Planck equations', *SIAM J. Math. Anal.* 50 (2018) 4246–4260.
6. V. BARBU, M. RÖCKNER and F. RUSSO, 'Probabilistic representation for solutions of an irregular porous media type equation: the degenerate case', *Probab. Theory Related Fields* 151 (2011) 1–43.
7. PH. BÉNILAN, L. BOCCARDO and M. A. HERRERO, 'On the limit of solutions of $u_t = \Delta u^m$ as $m \to \infty$', *Rend. Semin. Mat. Univ. Politec. Torino*, (Special Issue) (1991) 1–13, (1989). (Some topics in nonlinear PDEs (Turin, 1989).)
8. PH. BÉNILAN, H. BRÉZIS and M. G. CRANDALL, 'A semilinear equation in $L^1(R^N)$', *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (4) 2 (1975) 523–555.
9. PH. BÉNILAN and M. G. CRANDALL, 'The continuous dependence on φ of solutions of $u_t - \Delta\varphi(u) = 0$', *Indiana Univ. Math. J.* 30 (1981) 161–177.
10. P. BLANCHARD, M. RÖCKNER and F. RUSSO, 'Probabilistic representation for solutions of an irregular porous media type equation', *Ann. Probab.* (2010) 1870–1900.

11. F. BOLLEY and J. A. CARRILLO, 'Nonlinear diffusion: geodesic convexity is equivalent to Wasserstein contraction', *Comm. Partial Differential Equations* 39 (2014) 1860–1869.

12. L. A. CAFFARELLI and L. C. EVANS, 'Continuity of the temperature in the two-phase Stefan problem', *Arch. Rational Mech. Anal.* 81 (1983) 199–220.

13. J. CARRILLO, 'Entropy solutions for nonlinear degenerate problems', *Arch. Ration. Mech. Anal.* 147 (1999) 269–361.

14. J. A. CARRILLO, R. J. MCCANN and C. VILLANI, 'Kinetic equilibration rates for granular media and related equations: entropy dissipation and mass transportation estimates', *Rev. Mat. Iberoam.* 19 (2003) 971–1018.

15. J. A. CARRILLO, R. J. MCCANN and C. VILLANI, 'Contractions in the 2-Wasserstein length space and thermalization of granular media', *Arch. Ration. Mech. Anal.* 179 (2006) 217–263.

16. J. A. CARRILLO and F. SANTAMBROGIO, 'L^∞ estimates for the JKO scheme in parabolic-elliptic Keller-Segel systems', *Quart. Appl. Math.* 76 (2018) 515–530.

17. E. CHO and Y.-J. KIM, 'Starvation driven diffusion as a survival strategy of biological organisms', *Bull. Math. Biol.* 75 (2013) 845–870.

18. M. DI FRANCESCO and D. MATTHES, 'Curves of steepest descent are entropy solutions for a class of degenerate convection–diffusion equations', *Calc. Var. Partial Differential Equations* 50 (2014) 199–230.

19. S. DI MARINO, B. MAURY and F. SANTAMBROGIO, 'Measure sweeping processes', *J. Convex Anal.* 23 (2016) 567–601.

20. S. DI MARINO and A. R. MÉSZÁROS, 'Uniqueness issues for evolution equations with density constraints', *Math. Models Methods Appl. Sci.* 26 (2016) 1761–1783.

21. R. J. DIPERNA and P.-L. LIONS, 'Ordinary differential equations, transport theory and Sobolev spaces', *Invent. Math.* 98 (1989) 511–547.

22. I. EKELAND and R. TÉMAM, *Convex analysis and variational problems*, Studies in Mathematics and its Applications 1 (North-Holland Publishing Co., Amsterdam-Oxford; American Elsevier Publishing Co., Inc., New York, 1976). (Translated from the French.)

23. L. C. EVANS and R. F. GARIEPY, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics (CRC Press, Boca Raton, FL, 1992).

24. L. C. EVANS and M. PORTILHEIRO, 'Irreversibility and hysteresis for a forward–backward diffusion equation', *Math. Models Methods Appl. Sci.* 14 (2004) 1599–1620.

25. R. JORDAN, D. KINDERLEHRER and F. OTTO, 'The variational formulation of the Fokker-Planck equation', *SIAM J. Math. Anal.* 29 (1998) 1–17.

26. I. KIM and A. R. MÉSZÁROS, 'On nonlinear cross-diffusion systems: an optimal transport approach', *Calc. Var. Partial Differential Equations* 57 (2018) 1–40.

27. S. G. KRANTZ and H. R. PARKS, *Geometric integration theory* (Springer Science & Business Media, Berlin, 2008).

28. A. Y. KRUGER, 'On Fréchet subdifferentials', *J. Math. Sci.* 116 (2003) 3325–3358.

29. M. LABORDE, 'On some non linear evolution systems which are perturbations of wasserstein gradient flows', *Topological optimization and optimal transport*, Radon Series on Computational and Applied Mathematics (eds M. Bergounioux, É. Oudet, M. Rumpf, G. Carlier, T. Champion and F. Santambrogio; De Gruyter, Berlin, 2017) 304–332.

30. D. MATTHES, R. MCCANN and G. SAVARÉ, 'A family of nonlinear fourth order equations of gradient flow type', *Comm. Partial Differential Equations* 34 (2009) 1352–1397.

31. B. MAURY, A. ROUDNEFF-CHUPIN and F. SANTAMBROGIO, 'A macroscopic crowd motion model of gradient flow type', *Math. Models Methods Appl. Sci.* 20 (2010) 1787–1821.

32. R. J. MCCANN, 'A convexity principle for interacting gases', *Adv. Math.* 128 (1997) 153–179.

33. A. R. MÉSZÁROS and F. SANTAMBROGIO, 'Advection-diffusion equations with density constraints', *Anal. PDE* 9 (2016) 615–644.

34. F. OTTO, 'The geometry of dissipative evolution equations: the porous medium equation', *Comm. Partial Differential Equations* 26 (2001) 101–174.

35. R. T. ROCKAFELLAR, 'Integrals which are convex functionals', *Pacific J. Math.* 24 (1968) 525–539.

36. R. T. ROCKAFELLAR, 'Integrals which are convex functionals. II', *Pacific J. Math.* 39 (1971) 439–469.

37. R. ROSSI and G. SAVARÉ, 'Tightness, integral equicontinuity and compactness for evolution problems in Banach spaces', *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) 2 (2003) 395–431.

38. F. SANTAMBROGIO, *Optimal transport for applied mathematicians: calculus of variations, PDEs, and modeling*, Progress in Nonlinear Differential Equations and their Applications 87 (Birkhäuser/Springer, Cham, 2015).

39. F. SANTAMBROGIO, 'Euclidean, metric, and Wasserstein gradient flows: an overview', *Bull. Math. Sci.* 7 (2017) 87–154.

40. J. L. VÁZQUEZ, *The porous medium equation: mathematical theory* (Oxford University Press, Oxford, 2007).

41. C. VILLANI, *Topics in optimal transportation*, Number 58 (American Mathematical Society, Providence, RI, 2003).

Dohyun Kwon
Department of Mathematics
University of Wisconsin-Madison
480 Lincoln Dr.
Madison, WI 53706
USA

dkwon7@wisc.edu

Alpár Richárd Mészáros
Department of Mathematical Sciences
Durham University
Upper Mountjoy Campus, Stockton Road
Durham DH1 3LE
United Kingdom

alpar.r.meszaros@durham.ac.uk