Published for SISSA by Springer

Received: October 13, 2020
Accepted: December 6, 2020
Published: January 29, 2021

G_4 flux, algebraic cycles and complex structure moduli stabilization

A.P. Braun[a] **and R. Valandro**[b,c]

[a] *Department of Mathematical Sciences, Durham University, Lower Mountjoy, Stockton Rd, Durham DH1 3LE, U.K.*

[b] *Dipartimento di Fisica, Università di Trieste, Strada Costiera 11, I-34151 Trieste, Italy*

[c] *INFN, Sezione di Trieste, Via Valerio 2, I-34127 Trieste, Italy*

E-mail: andreas.braun.physics@gmail.com, roberto.valandro@ts.infn.it

Abstract: We construct G_4 fluxes that stabilize all of the 426 complex structure moduli of the sextic Calabi-Yau fourfold at the Fermat point. Studying flux stabilization usually requires solving Picard-Fuchs equations, which becomes unfeasible for models with many moduli. Here, we instead start by considering a specific point in the complex structure moduli space, and look for a flux that fixes us there. We show how to construct such fluxes by using algebraic cycles and analyze flat directions. This is discussed in detail for the sextic Calabi-Yau fourfold at the Fermat point, and we observe that there appears to be tension between M2-tadpole cancellation and the requirement of stabilizing all moduli. Finally, we apply our results to show that even though symmetric fluxes allow to automatically solve several F-term equations, they typically lead to flat directions.

Keywords: Flux compactifications, M-Theory, F-Theory

ArXiv ePrint: 2009.11873

https://doi.org/10.1007/JHEP01(2021)207

JHEP01(2021)207

Contents

1 Introduction **1**

2 Fluxes and moduli stabilization **4**

3 Fermat sextic fourfold and algebraic cycles **7**
3.1 Algebraic cycles at the Fermat point 8
3.2 Some properties of algebraic cycles 10
3.3 Algebraic cycles and their Hodge loci 11

4 Residues of rational forms and complex structure deformations **12**
4.1 Middle cohomology from residues of rational forms 12
4.2 Group actions and residues 13
4.3 Eigencycles and algebraic cycles 15
4.4 Complex structure moduli 16
4.5 Period integrals 18

5 Algebraic fluxes and stabilization of complex structure moduli **20**
5.1 One linear algebraic cycle 20
5.2 A sum of two linear algebraic cycles 21
5.3 Fluxes respecting group actions 21
5.4 Stabilizing all moduli using linear algebraic cycles 22
5.5 Tadpole issues 24

6 Moduli stabilization and symmetry actions **25**

7 Conclusions and future directions **27**

A Rational forms, residues, and cohomology of hypersurfaces **28**

1 Introduction

M-theory compactified on a Calabi-Yau (CY) fourfold X has $h^{1,3}(X)$ complex structure moduli, which can be thought of as variations of the holomorphic top form Ω. In such models, one can include four-form fluxes G_4 as part of the background, which preserve the Calabi-Yau metric up to warping [1]. Such fluxes give a potential to the complex structure moduli at tree level, which can be expressed in the resulting three-dimensional $\mathcal{N} = 2$ theory in terms of the Gukov-Vafa-Witten (GVW) superpotential [2]

$$W_{\mathrm{GVW}} = \int_X G_4 \wedge \Omega\,. \tag{1.1}$$

The minima of the induced scalar potential are solutions of the F-term equations $D_I W = 0$, $I = 1, \ldots, h^{1,3}(X)$. They are supersymmetric Minkowski vacua if furthermore $W_{\rm GVW} = 0$. This implies that the complex structure must be such that $G_4 \in H^{2,2}(X)$. It is commonly believed that a typical G_4 flux fixes all of the complex structure moduli. The argument for this is simple: there are as many constraints as there are complex structure moduli. The implicit assumption which enters this argument is that each of the F-term equations is linearly independent, which is expected to hold for a 'generic' choice of G_4.

As a consequence of flux quantization [3], which says that $G_4 + \frac{c_2(X)}{2} \in H^4(X, \mathbb{Z})$, sensible choices of G_4 form a lattice, which begs the questions what precisely is meant by a 'generic' choice of flux in this context. Complicating matters even more, there is the consistency condition commonly refereed to as $M2$-tadpole cancellation [1], which bounds the length squared of possible flux choices from above. Although it is always possible to find lattice vectors such that all F-term equations become linearly independent, this might require to pick lattice sites which are far away from the origin and hence too long to satisfy the bound imposed by the $M2$-tadpole.[1] The relevant question is hence: 'is there a choice of flux such that all F-term equations are independent and the bound imposed by $M2$-tadpole cancellation is satisfied ?'.

This is a difficult question to study in general, and it may well be that the tadpole constraint has a strong selective power. This observation becomes particularly interesting when the fourfold X is elliptically fibered and used as an F-theory background. In such compactifications, the four-dimensional gauge sector is engineered by appropriate singularities of X, and (part of) the complex structure moduli space of X corresponds e.g. to adjoint Higgs fields. Complex structure moduli that do not receive a potential from (1.1) hence give rise to flat directions in the gauge sector, and the inability to stabilize all complex structure moduli corresponds to such flat directions inevitably being present.[2] On the other hand, loci of enhanced gauge symmetry are typically at very high codimension in the moduli space [5–7] and it would be a fascinating scenario if the consistency conditions only allowed fluxes that would select such loci for us [5, 8].

The difficulty in working through explicit examples to shed light on the issues sketched above is mainly a technical one. Among the (known) Calabi-Yau fourfolds, a typical number of complex structure moduli is of the order of 1000s. Evaluating (1.1) then requires to solve Picard-Fuchs equations of a ridiculously high degree. Furthermore, it is in general highly non-trivial to identify which elements of $H^4(X)$ are integral, so that they can be used to define an appropriately quantized flux. One method to find such a basis is given by mirror symmetry [9].

The main motivation of the present work is to further explore an alternative approach. The crucial idea underlying this approach is as follows: at supersymmetric Minkowski vacua, the properly quantized flux must be an element of $H^{2,2}(X) \cap H^4(X, \mathbb{Z})$ up to a shift $\frac{1}{2}c_2(X)$. The group $H^{2,2}(X) \cap H^4(X, \mathbb{Z}) \equiv H_{Hodge}(X)$ of Hodge cycles is not constant

[1]There is also the possibility that the F-term equations have no solutions, as explained in [4] for M-theory compactifications on $K3 \times K3$.

[2]Some of the open string moduli may sit in matter multiplets that must be massless at the classical level to be consistent with some phenomenological requirements.

JHEP01(2021)207

throughout complex structure moduli space, but may be enhanced at specific loci, called Hodge loci. This is analogous to the enhancement of the Picard lattice of K3 surfaces at Noether-Lefschetz loci. If we identify such a locus and switch on a flux which is proportional to one of the Hodge cycles appearing there, the model cannot be deformed away from this locus, as the flux is only of type $(2,2)$ on the Hodge locus, so that the associated F-term equations are necessarily violated away from it. Instead of picking a flux in $H^4(X)$ and asking where it drives the model, the strategy we want to use is to *identify loci in the moduli space where supersymmetric fluxes are possible, and then ask if we can find a flux that traps it there.* See [10, 11] for a beautiful exposition of this idea.

For K3 surfaces, the Torelli theorem implies that demanding for a single lattice vector in $H^2(K3,\mathbb{Z})$ to be in $H^{1,1}(K3)$ fixes one complex structure modulus. This is not true for fourfolds, where the number of complex structure moduli we need to tune for a single element η in $H^4(X,\mathbb{Z})$ to be in $H^{2,2}(X)$ depends on both X and η. Fixing all complex structure moduli then corresponds to finding a so-called 'general' Hodge cycles for which the associated Hodge locus is just a point in the complex structure moduli space of X. If such a cycle furthermore satisfies the $M2$-tadpole constraint (after adding the piece $\frac{c_2(X)}{2}$), there is a G_4-flux that stabilizes all complex structure moduli.

In order to identify Hodge cycles and their Hodge loci, we will make use of algebraic cycles of complex dimension two. These are Poincaré dual to forms of Hodge type $(2,2)$ and it is not hard to find instances which only appear at special loci in the moduli space. Such an approach was followed in [12], and we will extend this work in several aspects. In [12], the number of stabilized moduli was simply counted by working out how many polynomial deformations are frozen by the existence of a given algebraic cycle. As this tacitly assumes the validity of a version of the Hodge conjecture, such a method is insufficient for a reliably counting. This point which was adressed in [12] by using the relationship of complex structure moduli of F-Theory compactifications to open string moduli in IIB orientifolds, a way of reasoning that is not available for general M-Theory backgrounds on Calabi-Yau fourfolds. Furthermore, one may consider fluxes which are Poincaré dual to some linear combination of algebraic cycles. In this instance, studying polynomial deformations is simply not powerful enough to detect all flat directions.

Working with the sextic fourfold X_6 at the Fermat point as a simple example, we show how to address both of these issues by directly evaluating the rank of the matrix

$$G_{IJ} \equiv \int_X G_4 \wedge D_I D_J \Omega\,, \tag{1.2}$$

which counts the number of fixed complex structure moduli. The crucial ingredient needed to evaluate these integrals are the periods of variations of Ω over algebraic cycles, which have been computed for the sextic fourfold at the Fermat point in [13, 14]. For the simplest class of algebraic cycle we show how to recover the periods (up to overall normalization) by exploiting the automorphism group of X_6, and construct fluxes that stabilize all complex structure moduli. These fluxes, however, significantly overshoot the tadpole constraint originating from the cancellation of $M2$-brane charge. Although a computation that confirms this in some form of generality is computationally too demanding to be within the scope of the present work, we take this as evidence for the tension between the $M2$ tadpole cancellation constraint and the desire to stabilize all complex structure moduli.

As a further application we consider the interplay between fluxes and symmetries. In [15, 16] it was suggested to use fluxes respecting some symmetries of the complex structure moduli space, in order to stabilize all moduli. The trick is that one needs to solve only the F-term equations of the invariant moduli, as the (many) F-terms of non-invariant complex structure deformations automatically vanish at a symmetric point. However, the argument does not take into account possible flat directions. In fact, we show that such flat directions are typically present in such setups. We give an example for the Fermat sextic fourfold. This shows that caution has to be taken in using the trick of turning on symmetric fluxes to claim full complex structure moduli stabilization.

After reviewing some aspects of flux compactification in M-Theory on Calabi-Yau fourfolds in section 2, we discuss algebraic cycles at the Fermat point of the sextic fourfold in section 3. In section 4, we describe the middle cohomology of X, the span of algebraic cycles, and variations of the holomorphic top-form Ω using residues of holomorphic forms with poles on $\mathbb{P}^5$. Some technical background on residues and rational forms are contained in an appendix. After introducing expressions for periods of residue forms on algebraic cycles, we apply these to several examples in section 5, and give some estimates that quantify the tension between complete moduli stabilization and tadpole cancellation. Moduli stabilization in the presence of fluxes respecting a symmetry is disussed in section 6. We close with a discussion of open issues and future directions.

2 Fluxes and moduli stabilization

In this paper we consider M-theory compactified on a CY fourfold X. The resulting low energy theory is a three dimensional (3d) $\mathcal{N} = 2$ supergravity, i.e. a theory with four supercharges. The metric deformations preserving the Calabi-Yau condition are called metric moduli and become massless scalars in the 3d theory. For CY fourfolds X, the metric moduli are encoded in the $h^{1,1}(X)$ periods of the Kähler form J and the $h^{1,3}(X)$ independent deformations of the holomorphic $(4,0)$-form Ω. These moduli are called *Kähler moduli* and *complex structure moduli*, respectively. There are also $h^{1,1}(X)$ axionic moduli that come from the dimensional reduction of the eleven-dimensional (11d) sugra six-form C_6 (the dual of C_3), which complexify the *Kähler moduli*.

The dynamics of the moduli is determined by the Kähler potential

$$K = K_{c.s.} + K_K \,, \tag{2.1}$$

with

$$K_{c.s.} = -\ln\left(\int_X \Omega\wedge\bar{\Omega}\right) \qquad \text{and} \qquad K_K = -3\ln\left(\frac{1}{4!}\int_X J\wedge J\wedge J\wedge J\right) . \tag{2.2}$$

One can switch on a non-zero vev for the four-form flux $G_4 = dC_3$, that is quantized according to

$$G_4 + \frac{c_2(X)}{2} \in H^4(X,\mathbb{Z})\,. \tag{2.3}$$

where $c_2(X)$ is the second Chern class of the tangent bundle of X.

A non zero flux along internal directions generates a potential for the metric moduli after compactification [17]. This can be understood from the 11d C_3 kinetic term $\int G_4 \wedge * G_4$, which depends on the metric through the Hodge star operator $*$). The minima of the supergravity scalar potential are given by the solutions of the following equations

$$\begin{cases} D_I W = 0 \quad I = 1, \ldots, h^{3,1} \\ \\ D_k \tilde{W} = 0 \quad k = 1, \ldots, h^{1,1} \end{cases} \tag{2.4}$$

where

$$W = \int_X G_4 \wedge \Omega \qquad \text{and} \qquad \tilde{W} = \int_X G_4 \wedge J \wedge J\,. \tag{2.5}$$

Here W is the GVW superpotential [2] and $(D_I, D_k) = (\partial_I + \partial_I K, \partial_k + \partial_k K)$, with K the Kähler potential (2.1). The index I runs over the complex structure moduli, and the index k runs over the Kähler moduli.

These minima are at zero cosmological constant (i.e. they are Minkowski vacua). They are furthermore supersymmetric if the vevs of W and $\tilde{W}$ vanish, i.e. $W|_{\text{min}} = 0$ and $\tilde{W}|_{\text{min}} = 0$. This condition together with (2.4) can be rephrased by saying that the four-form flux must lie in $H^{2,2}_{\text{prim}}(X)$, i.e. G_4 *must be a primitive four-form of Hodge type* $(2,2)$. We now explain this. The same can be done in the dual type IIB compactification on CY orientifolds [18], see also [19] for an overview over the classic literature on the subject.

We first explain why G_4 is of Hodge type $(2,2)$:

- The condition $W = 0$ means
$$\int_X G_4 \wedge \Omega = 0;$$
this implies that the $(0,4)$ component of G_4 vanishes. Since G_4 is real, also its $(4,0)$ component is zero.

- The condition $D_I W = 0$ means
$$\int_X G_4 \wedge D_I \Omega = 0 \qquad \forall I;$$
since the forms $D_I \Omega$ give a basis of $H^{3,1}(X)$ [20, 21], the $(1,3)$ and $(3,1)$ components of G_4 vanish.

We then see that only the $(2,2)$ part of G_4 survives.

As regarding the primitivity condition, expand first the Kähler form J in a basis of harmonic $(1,1)$-forms ω_k: $J = t^k \omega_k$. t^k are the $h^{1,1}(X)$ Kähler moduli. After imposing $\tilde{W} = 0$, the second condition in (2.4) becomes $\partial_k \tilde{W} = 0$, that means

$$\int_X G_4 \wedge J \wedge \omega_k = 0 \qquad \forall k \tag{2.6}$$

that implies $G_4 \wedge J = 0$, i.e. G_4 is a primitive form.

JHEP01(2021)207

When the flux, as required, belongs to $H^{2,2}_{\rm prim}(X)$, then it is also *self-dual*, i.e. $*G_4 = G_4$. This, in particular, implies that the contribution of G_4 to the M2-charge, i.e.

$$Q_{M2}^{\rm flux} = \frac{1}{2}\int_X G_4 \wedge G_4 \,, \tag{2.7}$$

is positive definite. In order to be possible to satisfy the M2-tadpole cancellation condition,

$$Q_{M2}^{\rm flux} + N_{M2} = \frac{\chi(X)}{24} \tag{2.8}$$

without introducing anti-branes, $Q_{M2}^{\rm flux}$ must be smaller than the contribution coming from the geometry, i.e. $Q_{M2}^{\rm flux} \leq \frac{\chi(X)}{24}$.[3]

Let us now concentrate on the complex structure moduli. We choose a point in the complex structure moduli space that satisfies $D_I W = 0$ and $W = 0$. We take coordinates s_I such that this point is at $\mathbf{s} = (s_0, s_1, s_2, \ldots) = 0$. The holomorphic $(4,0)$-form at a generic point is $\Omega(s)$ and $W(s) = \int_X G_4 \wedge \Omega(s)$. We then have

$$D_I W(s)|_{s=0} = 0 \,. \tag{2.9}$$

A flat direction of the potential is a curve $s(t)$ in the moduli space passing through $s = 0$ at $t = 0$ that satisfies the minimum condition for all t in a neighborhood of $t = 0$, i.e.

$$D_I W(s(t)) = 0 \qquad\qquad \forall t \,. \tag{2.10}$$

Expanding around $t = 0$ and keeping the leading term at small t, one finds the infinitesimal expression for (2.10), i.e.

$$\dot{s}_J(0)\partial_J D_I W(0) = 0 \,. \tag{2.11}$$

Notice that $\partial_J D_I W(0) = D_J D_I W(0)$, since the two expressions differ by $(\partial_J K) D_I W(0)$ which vanishes because of (2.9).[4] The vectors $\dot{s}_J(0)$ solving (2.11) give the flat directions.

We hence conclude that in order to have no flat directions at $s = 0$, a sufficient condition is that the matrix

$$G_{IJ} := D_J D_I W(s)|_{s=0} \tag{2.12}$$

has maximal rank. In general, one may ask how many flat directions remain in the presence of G_4. This is called the dimension of the Hodge locus of G_4 in the math literature, see [22, 23] for a review. In the example that we treat in detail in this work, the Fermat locus of the sextic fourfold, the rank of G_{IJ} equals the (complex) codimension of the Hodge locus.[5]

The Poincaré dual of an algebraic four-cycle is a four-form of type $(2,2)$. When the fourfold is at a specific point in the complex structure moduli space, one may be able to

[3] Introducing anti-M2 branes may introduce a bad instability related to the explicit breaking of supersymmetry in the four-dimensional effective theory.

[4] In general $D_I W$ is not holomorphic, as there is the term $(\partial_I K)W$ with $\partial_I K$ non-holomorphic in the complex structure moduli chiral superfields. Hence one may expect in (2.11) a term involving $\bar{\partial}_J D_I W$. However the extra term is $(\partial_I \partial_{\bar{J}} K)W$, which vanishes at $s = 0$ because of the $W = 0$ condition.

[5] Note that strictly speaking, we are only analysing this problem for linear deformations of the F-terms, and there might be terms of higher order in s. This doesn't happen for the sextic fourfold at the Fermat point [23].

JHEP01(2021)207

construct explicit algebraic cycles, as we will do for the sextic fourfold. One can then use them to define a choice of properly quantized flux that is a primitive $(2,2)$-form at that specific point. The question we want to address here, is how many moduli are stabilized once such a flux is introduced: any deformation that originates in a G_4 flux not purely of type $(2,2)$ is lifted by the flux potential.

Let us come back to the GVW superpotential that generates the minima condition for the complex structure moduli. The part of the flux G_4 that contributes to the superpotential, the F-term conditions and the stability condition is the one that has non-zero intersection with $\Omega(s)$ and its derivatives. Here by 'intersection' we mean the product given by the inner form $a_1 \cdot a_2 \equiv \int_X a_1 \wedge a_2$. The holomorphic four-form and its derivatives do not span the full middle cohomology $H^4(X)$, but only the primary horizontal subspace [20, 21]. In contrast, forms of Hodge type $(2,2)$ defined by intersections of divisors lie in the primary vertical subspace, which is perpendicular to the primary horizontal subspace.[6] To study stabilization of complex structure moduli we hence need to consider G_4 fluxes that lie in the horizontal subspace of $H^4(X)$ (apart from the piece $\frac{1}{2}c_2(X)$ that is forced on us by quantization). The algebraic cycles we consider here are exactly of this type [12].

3 Fermat sextic fourfold and algebraic cycles

The manifold of interest to us in this paper is the sextic fourfold. A sextic fourfold at a generic point in its moduli space is defined by the vanishing of a homogeneous polynomial of degree 6 in $\mathbb{P}^5$:

$$\mathcal{X}_6 : \quad x_0^6 + x_1^6 + x_2^6 + x_3^6 + x_4^6 + x_5^6 + \sum_{\mathbf{a}} c_{\mathbf{a}} \prod x_i^{a_i} = 0 \tag{3.1}$$

where $\mathbf{a} = (a_1, \cdots, a_5)$ are integers such that $\sum a_i = 6$ and the $c_{\mathbf{a}}$ are complex coefficients that can be thought of as deformations of the complex structure. The topological numbers of $\mathcal{X}_6$ are

$$h^{1,1}(\mathcal{X}_6) = 1 \qquad h^{2,1}(\mathcal{X}_6) = 0 \qquad h^{3,1}(\mathcal{X}_6) = 426 \qquad h^{2,2}(\mathcal{X}_6) = 1752\,. \tag{3.2}$$

It follows that $\chi(\mathcal{X}_6) = 2610$ and $b^4_+(\mathcal{X}_6) = 1754$, $b^4_-(\mathcal{X}_6) = 852$.

The single class in $h^{1,1}(\mathcal{X}_6)$ is generated by the restriction of the hyperplane class H of $\mathbb{P}^5$, and any Kähler form on X_6 is necessarily proportional to H. There is a unique generator of the primary vertical subspace $H^{2,2}_V(\mathcal{X}_6)$ which is given by $H \cdot H \equiv H^2$ and which is always proportional to the square of the Kähler form.

The orthogonal directions to H^2 in $H^4(\mathcal{X}_6)$ are hence all primitive, i.e. $h^{2,2}_{\text{prim}}(\mathcal{X}_6) = 1751$, and can be shown[7] to all belong to the primary horizontal subspace $H^4_H(\mathcal{X}_6)$, which has dimension 1751. The second Chern character of $\mathcal{X}_6$ is

$$c_2(\mathcal{X}_6) = 15H^2\,. \tag{3.3}$$

[6] In general, there can be directions in $H^{2,2}(X)$ which do not lie in either subspace [5].

[7] We will see this explicitly in section 4.4. This can also be shown by computing that the dimension of the primary vertical subspace of the mirror, $h^{2,2}_V(X^\vee) = 1751$ and using that primary vertical and horizontal subspaces are swapped by the mirror map.

The term $\frac{1}{2}c_2(\mathcal{X}_6)$ is hence not integral, so that flux quantization forces us to include a half-integral flux proportional to H^2.

For a typical choice of the $c_{\mathbf{a}}$, the only algebraic cycles contained in $\mathcal{X}_6$ are complete intersections of $\mathcal{X}_6$ with multiples of the hyperplane divisor in $\mathbb{P}^5$. On $\mathcal{X}_6$ the classes of these are proportional to H^2, so that the rank of $H^{2,2}(\mathcal{X}_6)\cap H^4(\mathcal{X}_6,\mathbb{Z})_{prim}$ is zero. As H^2 is never primitive, there are furthermore no supersymmetric fluxes along this direction.[8] If we tune the $c_{\mathbf{a}}$ to special values, the situation changes and the rank of $H^{2,2}(\mathcal{X}_6)\cap H^4(\mathcal{X}_6,\mathbb{Z})_{prim}$ becomes non-zero.

Let us hence make a specific choice and set all $c_{\mathbf{a}}=0$, which puts us on the Fermat point[9] of the moduli space of the sextic. We will denote the sextic fourfold at the Fermat point by:

$$X_6:\ \ x_0^6+x_1^6+x_2^6+x_3^6+x_4^6+x_5^6=0\,. \tag{3.4}$$

As the above equation describes a smooth submanifold of $\mathbb{P}^5$, the topological numbers of the Fermat sextic are the same as those of $\mathcal{X}_6$. Only the group $H^{2,2}(X_6)\cap H^4(X_6)_{prim}$ is different from the case of a generic sextic fourfold: it has the maximal possible rank of 1751.

3.1 Algebraic cycles at the Fermat point

It is not hard to find the simplest type of algebraic cycle sitting inside X_6. Take e.g. $x_0=\alpha x_1$, $x_2=\beta x_3$ and $x_4=\gamma x_5$ for $\alpha^6=\beta^6=\gamma^6=-1$. In this case, these three equations imply (3.4), so that they define a subvariety of complex codimension 3 inside $\mathbb{P}^5$, which is complex codimension 2 in X_6.

Using the large group of automorphisms of X_6, we can immediately write down the general form of such cycles as

$$C^{\boldsymbol{\ell}}_\sigma:\ x_{\sigma(0)}=e^{i\pi/6}e^{i\pi\ell_0/3}x_{\sigma(1)}\qquad x_{\sigma(2)}=e^{i\pi/6}e^{i\pi\ell_1/3}x_{\sigma(3)}\qquad x_{\sigma(4)}=e^{i\pi/6}e^{i\pi\ell_2/3}x_{\sigma(5)} \tag{3.5}$$

Here the $\ell_i\in\{0,1,2,3,4,5\}$ specify which sixth root of unity we are using and σ is a permutation of $\{0,1,2,3,4,5\}$ which specifies which coordinates are paired to form $C^{\boldsymbol{\ell}}_\sigma$.

The existence of such algebraic cycles can also be inferred by writing the defining equation of X_6 (3.4) in the following 'factorized' form

$$\begin{aligned}\prod_{\ell_0=0}^{5}\left(x_{\sigma(0)}-e^{i\pi/6}e^{i\pi\ell_0/3}x_{\sigma(1)}\right)+\prod_{\ell_1=0}^{5}\left(x_{\sigma(2)}-e^{i\pi/6}e^{i\pi\ell_1/3}x_{\sigma(3)}\right)&\\+\prod_{\ell_2=0}^{5}\left(x_{\sigma(4)}-e^{i\pi/6}e^{i\pi\ell_2/3}x_{\sigma(5)}\right)&=0\,.\end{aligned} \tag{3.6}$$

[8] A similar observation regarding hypersurfaces in $\mathbb{P}^2\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ has been made in [24].

[9] This is also called the Gepner point (if one thinks in terms of the worldsheet CFT of strings propagating on $\mathcal{X}_6$) or the Brieskorn-Pham point (if one things in terms of singularity theory).

This gives a hint of how other instances of algebraic cycles can be found. Another factorization of the defining equation for X_6 is:

$$0 = \left(x_0^3 + e^{i\pi k} x_1^3 + i x_2^3\right)\left(x_0^3 + e^{i\pi k} x_1^3 - i x_2^3\right) + \prod_{m=0}^{2}\left(x_3^2 - 2^{1/3} e^{i\pi k} e^{\frac{2\pi i m}{3}} x_0 x_1\right) + \prod_{\ell=0}^{6}\left(x_4 - e^{i\pi/4} e^{i\pi\ell/2} x_5\right) \tag{3.7}$$

up to permutation of the four coordinates and for $k = 0, 1$. One then realizes the existence of the algebraic cycles

$$C_\sigma^{kjm\ell}: \quad \begin{aligned} x_{\sigma(0)}^3 + e^{i\pi k} x_{\sigma(1)}^3 + i e^{i\pi j} x_{\sigma(2)}^3 &= 0 \\ x_{\sigma(3)}^2 - 2^{1/3} e^{i\pi k} e^{\frac{2\pi i m}{3}} x_{\sigma(0)} x_{\sigma(1)} &= 0 \\ x_{\sigma(4)} - e^{i\pi/6} e^{i\pi\ell/3} x_{\sigma(5)} &= 0 \end{aligned} \tag{3.8}$$

for $k, j \in \mathbb{Z}/2\mathbb{Z}$, $m \in \mathbb{Z}/3\mathbb{Z}$ and $\ell \in \mathbb{Z}/6\mathbb{Z}$. Note that $x_{\sigma(4)}$ and $x_{\sigma(5)}$ are paired in a similar way as before, whereas a more complicated factorization is used for the remaining four coordinates. These cycles are a lift of the cycles that were used to construct the Néron-Severi group of Fermat sextic surfaces in [25], where famously using only lines is no longer sufficient [26].

An example of a completely non-linear factorization of (3.4) is given by

$$0 = \prod_{s=0}^{2}\left(x_0^2 + e^{\frac{2\pi i}{3}(k_1+s)} x_1^2 + e^{\frac{2\pi i}{3}(k_2+2s)} x_2^2\right) + \prod_{s=0}^{2}\left(x_3^2 + e^{\frac{2\pi i}{3}(k_4+s)} x_4^2 + e^{\frac{2\pi i}{3}(k_5+2s)} x_5^2\right) + 3\prod_{n=0}^{1}\left(i\, e^{i\pi n} e^{\frac{i\pi}{3}(k_1+k_2)} x_0 x_1 x_2 + e^{\frac{i\pi}{3}(k_4+k_5)} x_3 x_4 x_5\right) \tag{3.9}$$

for some $k_1, k_2, k_4, k_5 \in \mathbb{Z}/3\mathbb{Z}$. We then find that the Fermat sextic fourfold contains the algebraic cycles

$$C_\sigma^{k_1k_2k_4k_5n}: \quad \begin{aligned} x_{\sigma(0)}^2 + e^{\frac{2\pi i}{3}k_1} x_{\sigma(1)}^2 + e^{\frac{2\pi i}{3}k_2} x_{\sigma(2)}^2 &= 0 \\ x_{\sigma(3)}^2 + e^{\frac{2\pi i}{3}k_4} x_{\sigma(4)}^2 + e^{\frac{2\pi i}{3}k_5} x_{\sigma(5)}^2 &= 0\ , \\ i\, e^{i\pi n} e^{\frac{i\pi}{3}(k_1+k_2)} x_{\sigma(0)} x_{\sigma(1)} x_{\sigma(2)} + e^{\frac{i\pi}{3}(k_4+k_5)} x_{\sigma(3)} x_{\sigma(4)} x_{\sigma(5)} &= 0 \end{aligned} \tag{3.10}$$

with $k_i \in \mathbb{Z}/3\mathbb{Z}$ and $n \in \mathbb{Z}/2\mathbb{Z}$.

There are further algebraic cycles of the form $f_0 = f_1 = f_2 = 0$ contained in X_6 which can be seen by constructing other factorizations of the form

$$X_6 : f_0 P_0 + f_1 P_1 + f_2 P_2 = 0\,, \tag{3.11}$$

see [27] and [23] for a more systematic treatment. Note also that all of the examples of algebraic cycles we have given are complete intersections inside the ambient $\mathbb{P}^5$, which points to another direction of generalization: algebraic cycles which are not complete intersections. Over $\mathbb{Q}$, it is known, however, that all of $H^{2,2}_{prim}(X_6)$ is generated by the above algebraic cycles [25, 27, 28].

JHEP01(2021)207

3.2 Some properties of algebraic cycles

Having introduced some algebraic cycles on the Fermat sextic, let us study some of their properties. We will limit our discussion mostly to the 'linear' algebraic cycles $C_\sigma^{\boldsymbol{\ell}}$.

As each of the $C_\sigma^{\boldsymbol{\ell}}$ is given by three linear equations inside $\mathbb{P}^5$, each such cycle has the topology of $\mathbb{P}^2$. To compute intersection numbers, we can use the following trick. Consider a complete intersection of X_6 with $x_0 - \alpha x_1 = 0$ for $\alpha^6 = -1$ and $x_2 - \beta x_3 = 0$ for $\alpha^6 = \beta^6 = -1$. The resulting cycle on X_6 is in the class H^2 restricted to X_6. Using (3.4), however, we see that this cycle is reducible into a sum of six of the $C_\sigma^{\boldsymbol{\ell}}$. We may hence write

$$H^2 = \sum_{\ell_0=0}^{5} C_\sigma^{\boldsymbol{\ell}} \tag{3.12}$$

for any choice of σ and every ℓ_1 and ℓ_2. As $H^2 \cdot H^2 = 6$ on X_6 and $H^2 \cdot C_\sigma^{\boldsymbol{\ell}}$ is the same for every $C_\sigma^{\boldsymbol{\ell}}$ by symmetry, it follows that

$$H^2 \cdot C_\sigma^{\boldsymbol{\ell}} = 1\,. \tag{3.13}$$

Using the observation that $C_\sigma^{\boldsymbol{\ell}} \cdot C_\sigma^{\boldsymbol{\ell}'} = 0$ if $\boldsymbol{\ell}$ and $\boldsymbol{\ell}'$ differ in all three components (together with a similar rule when intersection algebraic cycles employing different permutations σ), the above can be iterated to find that the intersection numbers follow the pattern

$\dim(C_\sigma^{\boldsymbol{\ell}} \cap C_{\sigma'}^{\boldsymbol{\ell}'})$	$C_\sigma^{\boldsymbol{\ell}} \cdot C_{\sigma'}^{\boldsymbol{\ell}'}$
2	21
1	-4
0	1
$\emptyset$	0

. (3.14)

i.e. the dimension of the intersection of two algebraic cycles in $\mathbb{P}^5$ determines the intersection number between the associated homology classes.[10] For any pair of permutations σ and σ', the intersection numbers can also be expressed in terms of relations on the ℓ_i and ℓ'_j.

Although one can work out the details using the same approach, such a simple pattern is not obeyed by the other algebraic cycles introduced in the last section. Self-intersections of any algebraic cycle C_{f_0,f_1,f_2} of complete intersection type given by $f_0 = f_1 = f_2 = 0$ can however be worked out using adjunction, and the result is [23]

$$C_{f_0,f_1,f_2} \cdot C_{f_0,f_1,f_2} = d_0 d_1 d_2 (36 - 6(d_0 + d_1 + d_2) + d_0 d_1 + d_0 d_2 + d_1 d_2) \tag{3.15}$$

where d_i are the degrees of the polynomials f_i.

[10] If two algebraic cycles do not intersect transversely, there is always a pair of homologous (typically non-holomorphic) cycles that do intersect transversely.

JHEP01(2021)207

3.3 Algebraic cycles and their Hodge loci

Let us now try to see how many moduli we expect to be stabilized by demanding that any of the cycles C_σ^ℓ remains of type $(2,2)$.[11] We can work out the number of polynomial deformations which are obstructed by demanding that C_σ^ℓ is an algebraic cycle as follows. First one observes that it does not matter which C_σ^ℓ we are talking about as they are all equivalent modulo automorphisms of the Fermat sextic. For $\alpha^6=-1$, let us hence consider the cycle

$$C: \qquad x_0-\alpha x_1=0\,, \qquad x_2-\alpha x_3=0\,, \qquad x_4-\alpha x_5=0\,. \tag{3.16}$$

We can introduce a new set of coordinates:

$$(y_0,y_1,y_2,y_3,y_4,y_5)=(x_0-\alpha x_1,x_0+\alpha x_1,x_2-\alpha x_3,x_2+\alpha x_3,x_4-\alpha x_5,x_4+\alpha x_5)\,, \tag{3.17}$$

in terms of which the Fermat sextic equation (3.4) becomes

$$y_0y_1(3y_0^4+10y_0^2y_1^2+3y_1^4)+y_2y_3(3y_2^4+10y_2^2y_3^2+3y_3^4)+y_4y_5(3y_4^4+10y_4^2y_5^2+3y_5^4)=0 \tag{3.18}$$

Polynomial deformations are counted by counting monomials of degree 6 modulo the Jacobi ideal.[12] There are $\binom{11}{6}=462$ possible monomials of degree 6 in 6 variables. The Jacobi ideal is generated by

$$(3y_{2\kappa+1}^5+30y_{2\kappa+1}^3y_{2\kappa}^2+15y_{2\kappa+1}y_{2\kappa}^4,\,3y_{2\kappa}^5+30y_{2\kappa}^3y_{2\kappa+1}^2+15y_{2\kappa}y_{2\kappa+1}^4) \text{ with } \kappa=0,1,2. \tag{3.19}$$

We can use the Jacobi ideal to eliminate all monomials proportional to y_i^5, and there are 36 such monomials. Hence the number of complex structure moduli is 426, which equals $h^{3,1}(X)$ as expected.

We now demand the cycle C_σ^ℓ to persist as an algebraic cycle. This is the case only if the deformed fourfold is of the form

$$y_0P_5(y_0,\dots,y_5)+y_2Q_5(y_0,\dots,y_5)+y_4R_5(y_0,\dots,y_5)=0 \tag{3.20}$$

where P,Q,R are homogeneous polynomial of degree 5. This means that we can use only the monomials that have a factor of y_0, y_2 or y_4 to deform the Fermat sextic. The obstructed deformations are then monomials of degree 6 in the three coordinates y_1,y_3,y_5. This gives $\binom{8}{6}=28$ deformations. We have to subtract the $3\times 3=9$ monomials that are in the Jacobi ideal. We then obtain that 19 moduli are fixed by demanding that any of the C_σ^ℓ persists as an algebraic cycle.

Again, there is a general version of this method that can be applied to any cycle C_{f_0,f_1,f_2}. The result only depends on the degrees of the polynomials f_i and can be found in [23].

[11] Straightforwardly using C_σ^ℓ as a flux is at odds with primitivity and flux quantization. To achieve a primitive flux, we would need to choose $G_4=C_\sigma^\ell-\frac{1}{6}H^2$. This is at odds with flux quantization, which requires G_4 to be integral up to a piece $\frac{1}{2}H^2$. One would hence need to consider $G_4=3C_\sigma^\ell-\frac{1}{2}H^2$. Any piece proportional to H^2 does however not influence complex structure deformations, and the number of stabilized moduli is the same for C_σ^ℓ and $3C_\sigma^\ell$, so that we prefer to simply ask about the 'Hodge Locus' of C_σ^ℓ here.

[12] This way of counting deformations is explained in some more detail in section 4. It gives the same result as counting monomials modulo automorphism of $\mathbb{P}^5$, but is more convenient here.

The main issue with this approach which requires us to work harder is that we are interested in stabilizing all complex structure moduli, which forces us to consider linear combinations of algebraic cycles. Demanding that a single cycle C^{ℓ}_{σ} be algebraic only fixes some, but not all of the moduli. Similar results are obtained for other cycles C_{f_0,f_1,f_2}, so that we are led to consider linear combinations of (the Poincaré duals of) algebraic cycles. Merely counting polynomial deformations then becomes useless and we need a method to evaluate (2.12) in order to treat such situations.

A further issue that deserves some discussion concerns the Hodge conjecture. While the Hodge conjecture over $\mathbb{Q}$ has been proven for the Fermat sextic [27, 28] (see section 4 for more details), we do not know if it is true in general. This means for other points in the moduli space of the sextic or for other fourfolds, the number of polynomial deformations that are fixed by demanding a cycle stays algebraic may not equal the number of complex structure deformations that are fixed by demanding that the dual integral $(2,2)$ form stays of type $(2,2)$. Of course every algebraic cycle must be dual to an integral form of type $(2,2)$, but it is not clear that every integral form of type $(2,2)$ can be represented by a linear combination of algebraic cycles. In our context this implies that there might be extra flat directions that cannot be detected from polynomial deformations. Again, being able to evaluate (2.12) settles this issue.

4 Residues of rational forms and complex structure deformations

In this section we will use the techniques of rational forms to explicitly describe the middle cohomology of X, some background on these techniques is given in appendix A. This is then used to describe complex structure deformations and moduli stabilization for fluxes defined by (sums of) algebraic cycles. Throughout this section, X is the Fermat sextic fourfold (3.4).

4.1 Middle cohomology from residues of rational forms

As reviewed in appendix A, primitive forms of Hodge type $(p, 4-p)$ on the Fermat sextic are described as residues of rational forms

$$\varphi = \frac{P(x)}{Q(x)^{5-p}}\Omega_0 \,, \tag{4.1}$$

on $\mathbb{P}^5$. Here $Q = 0$ is the hypersurface equation defining the Fermat sextic fourfold X, Ω_0 is a fixed differential form on $\mathbb{P}^5$ that is completely antisymmetric in the homogeneous coordinates x_i, and P is a homogeneous polynomial of degree $6(4-p)$. The residue map is linear, maps surjectively to the primitive forms in the middle cohomology of X, and becomes injective when restricting to polynomials P which are not contained in the Jacobi ideal of Q.

Let us apply these statements to reproduce the Hodge numbers of the sextic. To work out the dimensions of the rings we are going to consider, it is beneficial to know that there are

$$\#(\text{degree l in m variables}) = \binom{l+m-1}{l} \tag{4.2}$$

terms in a homogeneous polynomial of degree l in m variables.

The existence of a unique $(4,0)$-form up to scaling follows from the fact that for $p=4$, P is just a number. To find classes of $(3,1)$-forms, we hence need to consider the case $p=3$, i.e. homogeneous polynomials of degree 6 modulo the Jacobi ideal of Q:

$$H^{3,1}_{\rm prim}(X) = \frac{\mathbb{C}[x_0,\cdots,x_5]_6}{\langle \partial_i Q\rangle} = \frac{\mathbb{C}[x_0,\cdots,x_5]_6}{\langle x_0^5,\cdots,x_5^5\rangle}\,. \tag{4.3}$$

The ring of homogeneous polynomials of degree 6 in 6 variables has dimension 462. At the Fermat point, the Jacobi ideal is generated by the polynomials x_i^5 for all i, so that $6\cdot 6 = 36$ generators of $\mathbb{C}[x_0,\cdots,x_5]_6$ are contained in the Jacobi ideal of Q. We hence recover the familiar number $h^{3,1}_{\rm prim}(X) = h^{3,1}(X) = 426$. Finally, for $H^{2,2}_{\rm prim}(X)$ we have $p=2$, so that we need to count polynomials of degree 12 modulo the Jacobi ideal:

$$H^{2,2}_{\rm prim}(X) = \frac{\mathbb{C}[x_0,\cdots,x_5]_{12}}{\langle x_0^5,\cdots,x_5^5\rangle}\,. \tag{4.4}$$

We can work this out by noting that for each variable, the number of terms in $\mathbb{C}[x_0,\cdots,x_5]_{12}$ which are in the ideal x_i^5 is given by a homogeneous polynomial of degree 7. Using this we need to take into account that for each pair of variables x_i and x_j there are terms $x_i^5 x_j^5 P_2(x)$ for a polynomial $P_2(x)$ of degree 2, which are in both the ideal generated by x_i^5 and x_j^5. Hence

$$\left|\frac{\mathbb{C}[x_0,\cdots,x_5]_{12}}{\langle x_0^5,\cdots,x_5^5\rangle}\right| = \binom{17}{12} - 6\cdot\binom{12}{7} + 15\cdot\binom{7}{2} = 1751 = h^{2,2}_{\rm prim}(X)\,. \tag{4.5}$$

4.2 Group actions and residues

The content of the last subsection can be rephrased by considering the natural group action of $G = \mu^6/\mu$ for $\mu = \mathbb{Z}/6\mathbb{Z}$ by coordinatewise multiplication:

$$(x_0,\cdots,x_5) \to (\zeta_0 x_0,\cdots,\zeta_5 x_5) \tag{4.6}$$

for $\zeta_i^6 = 1$ an 6-th root of unity. The quotient arises because elements of μ^6 for which all ζ_i are equal are inside the $\mathbb{C}^*$ acting on the homogeneous coordinates of $\mathbb{P}^5$.[13]

Let us consider the character group A of G, which is the group of representations by complex valued functions of G:

$$A = \{\mathbf{a} = (a_0,\cdots,a_5)\,|\, a_i \in \mathbb{Z}/6\mathbb{Z} \;\text{ and }\; \sum_i a_i = 0 \bmod 6\}\,. \tag{4.7}$$

The elements of A are functionals on G that associate to an element $g\in G$ the phase

$$\mathbf{a}(g) = \prod_i \zeta_i^{a_i}\,. \tag{4.8}$$

Note that the condition $\sum_i a_i = 0 \bmod 6$ guarantees that the unit element of G (i.e. $\zeta_0 = \ldots = \zeta_5$) is mapped to 1, i.e. that this is indeed a group homomorphism.

[13] Note that this group is larger than the group μ^4 which is used in the Green-Plesser mirror construction: as we do not have a term proportional to $\prod_i x_i$ there is no need to impose $\prod_i \zeta_i = 1$.

JHEP01(2021)207

The action of G on X induces an action on the middle cohomology $H^4(X,\mathbb{C})$. One can use the character group A to describe such an action. G is an abelian group, so its elements can be diagonalized simultaneously. The elements $\mathbf{a}$ play the role of 'eigenvalues'. We may define 'eigencycles' relative to $\mathbf{a} \in A$ to be those classes η for which

$$g^*\eta = \mathbf{a}(g)\eta \qquad \forall g \in G\,. \tag{4.9}$$

For a given $\mathbf{a}$, we denote the span of the cycles which satisfy the above relation by $V(\mathbf{a})$.

The spaces $V(\mathbf{a})$ have the nice property that any pair $V(\mathbf{a})$ and $V(\mathbf{a}')$ is orthogonal except when $\mathbf{a} = -\mathbf{a}'$. To see this, take $\eta_{\mathbf{a}}$ in $V(\mathbf{a})$ and $\eta_{\mathbf{a}'}$ in $V(\mathbf{a}')$. The inner form (given by the integral of their wedge product) then transforms as

$$\int_X \eta_{\mathbf{a}} \wedge \eta_{\mathbf{a}'} \to \int_X \eta_{\mathbf{a}} \wedge \eta_{\mathbf{a}'} \prod_i \zeta_i^{a_i+a'_i} \qquad \forall g \in G\,. \tag{4.10}$$

However, as the inner form is merely a number which hence must be invariant under the action of G, it follows that $\mathbf{a} = -\mathbf{a}'$ is a necessary condition for the integral to be non-zero.

To see the relation between the forms realized as residues and the eigenspaces under the character group, let us use a monomial basis for the polynomials P in (4.1). For $\mathbf{b} = (b_1, \cdots, b_5)$, there is an associated monomial $\mu_{\mathbf{b}} = x_0^{b_0} \cdots x_5^{b_5}$ with $\sum_i b_i = \deg P = 6(4-p)$. To such a monomial, we can associate a differential form

$$\varphi_{\mathbf{a}} = \frac{\mu_{\mathbf{b}}}{Q(x)^{5-p}}\, \Omega_0\,. \tag{4.11}$$

where $\mathbf{a} = (1,1,1,1,1,1) + \mathbf{b}$. Under the group action (4.6), $\varphi_{\mathbf{a}}$ has the simple transformation behavior

$$\varphi_{\mathbf{a}} \to \mathbf{a}(g)\varphi_{\mathbf{a}} \tag{4.12}$$

which follows from the fact that the Fermat polynomial Q of degree n is invariant and Ω_0 transforms as

$$\Omega_0 \to \left(\prod_i \zeta_i\right) \Omega_0\,. \tag{4.13}$$

As $\deg P = k \cdot \deg\ Q - 6 = 6k - 6$, we hence have that

$$|a| \equiv \frac{1}{6}\sum_i a_i = \frac{1}{6}(6 + \deg\ P) = k = 5 - p\,. \tag{4.14}$$

As long as $b_i < 5$, we can furthermore associate $\mu_{\mathbf{b}}$ with a generator of $\mathbb{C}[x_0, \cdots, x_5]_{|\mathbf{b}|}/\langle \partial_i Q\rangle$. When this is satisfied we have $a_i < 6$, so that we conclude that

$$\varphi_{\mathbf{a}} \in H^{p,4-p}(X)\,. \tag{4.15}$$

This recovers the following result of [28–31], which can be phrased in the present context as follows: let A^* be the subset of the character group for which all of the $a_i \neq 0$. Then (Theorem 1 of [28]):

a) $\dim_{\mathbb{C}} V(\mathbf{a}) = 1$ if and only if $\mathbf{a} \in A^*$; $\dim_{\mathbb{C}} V(\mathbf{a}) = 0$ otherwise.

JHEP01(2021)207

b) The Hodge type of $V(\mathbf{a})$ is given by

$$(p,q) = (5 - |a|, |a| - 1) \tag{4.16}$$

and the canonical representative with $1 \leq a_i \leq 5$ should be chosen for each a_i in the above formula. Note that $|a|$ is always an integer as $\sum a_i = 0$ modulo 6. Together with a), the above implies that for $\eta_{\mathbf{a}} \in V(\mathbf{a})$, $\bar{\eta}_{\mathbf{a}}$ is proportional to $\eta_{-\mathbf{a}}$.

It is not hard to use this description to simply enumerate primitive forms by counting appropriate tuples $\mathbf{a}$, one finds

$\lvert\mathbf{a}\rvert$	# elements in A^*
1	1
2	426
3	1751
4	426
5	1

(4.17)

4.3 Eigencycles and algebraic cycles

Elements of $V(\mathbf{a})$ can not only be realized in terms of $\varphi_{\mathbf{a}}$ but also by forming appropriate linear combinations of algebraic cycles, which links the two descriptions of elements of $H^{2,2}(X)$. Furthermore, this allows us to work out the span of the algebraic cycles. Finally, finding representatives for all $V(\mathbf{a})$ with $|\mathbf{a}| = 3$ in terms of algebraic cycles proves the Hodge conjecture (over $\mathbb{Q}$) for the Fermat sextic, see [25, 27, 28, 31] for more details and generalizations to other Fermat varieties.

The description of forms $\eta_{\mathbf{a}} \in V(\mathbf{a})$ in terms of algebraic cycles works by putting restrictions on the tuples $\mathbf{a} \in A^*$. We call an element $\mathbf{a} \in A^*$ n-decomposable if the elements of $\mathbf{a}$ can be decomposed into pairs such that maximally n of them satisfy

$$a_i + a_j = 0 \tag{4.18}$$

modulo 6. For the Fermat sextic, $\mathbf{a} \in A^*$ with $|\mathbf{a}| = 3$, so that it corresponds to a $(2,2)$-form, can be 3-decomposable, 1-decomposable, or indecomposable.[14] Using a computer makes it easy to enumerate them, the resulting numbers and their general forms (up to permutations and taking the inverse) are given below

type	number	standard form
3 − decomposable	1001	$(r, 6-r, s, 6-s, t, 6-t)$
1 − decomposable	720	$(t, 6-t, 1, 3, 4, 4)$
indecomposable	30	$(1, 1, 4, 4, 4, 4)$

(4.19)

As they should, these sum up to the total 1751 primitive classes in $H^{2,2}(X)$.

[14] As $\sum a_i = 0$ mod 6, 2-decomposable implies 3-decomposable.

Let us consider 3-decomposable elements of A^*. We can write a general 3-decomposable $\mathbf{a}$ as

$$a_{\sigma(0)} + a_{\sigma(1)} = 0 \qquad a_{\sigma(2)} + a_{\sigma(3)} = 0 \qquad a_{\sigma(4)} + a_{\sigma(5)} = 0\,, \tag{4.20}$$

for some permutation σ. The corresponding element of $V(\mathbf{a})$ is

$$\eta_{\mathbf{a}} = \sum_{\ell_0\ell_1\ell_2} e^{-\frac{i\pi}{3}(a_{\sigma(1)}\ell_0 + a_{\sigma(3)}\ell_1 + a_{\sigma(5)}\ell_2)} C_\sigma^{\boldsymbol{\ell}}\,, \tag{4.21}$$

where $C_\sigma^{\boldsymbol{\ell}}$ are the linear algebraic cycles defined in (3.5). Using the transformation behavior of the $C_\sigma^{\boldsymbol{\ell}}$ it is not hard to see that it is crucial for the defining relation (4.9) of eigencycles to hold that we are only talking about 3-decomposable $\mathbf{a}$ here.

This result can be immediately used to constrain the possible intersections between the $C_\sigma^{\boldsymbol{\ell}}$ and the residues of the forms $\varphi_{\mathbf{a}}$, and we shall see how these are in fact fixed up to normalization later. A second application concerns the linear relations between the $C_\sigma^{\boldsymbol{\ell}}$. We have already seen that they obey the 'sum rule' (3.12) using elementary methods. This is insufficient to work out the dimensionality of span of all of the $C_\sigma^{\boldsymbol{\ell}}$, however. The above proves that its dimension is 1001 and shows how further linear relations arise: whenever $\mathbf{a}$ is 3-decomposable in more than one way, we can write down $\eta_{\mathbf{a}}$ in two independent ways in terms of the $C_\sigma^{\boldsymbol{\ell}}$ using different permutations. As $V(\mathbf{a})$ is complex one-dimensional, this implies that the two expressions must be proportional.

Following the formulae in [27], it is possible to write down similar expressions for eigencycles for 1- or in-decomposable $\mathbf{a}$ using the non-linear algebraic cycles (3.8) and (3.10).

4.4 Complex structure moduli

Having explained how to capture the middle cohomology in terms of residues and sketched the relationship to algebraic cycles, let us now discuss complex structure deformations in this language. We focus again on the Fermat sextic hypersurface in $\mathbb{P}^5$ and consider deforming away from the Fermat locus. We may parametrize a general deformation as

$$Q(x;s) = \sum_i x_i^6 + \sum_{\mathbf{b}_I} s_I\ \mu_{\mathbf{b}_I} \tag{4.22}$$

for complex parameters s_I and monomials

$$\mu_{\mathbf{b}_I} = x_0^{(b_I)_0} \cdots x_5^{(b_I)_5} \tag{4.23}$$

which are such that $|\mathbf{b}_I| = 1$ and $(b_I)_i < 5$.

This corresponds to complex structure deformations, which may be represented by deformations of the holomorpic top-form Ω, which in turn can be written as a residue

$$\Omega(s) = \mathrm{Res}\left[\frac{\Omega_0}{Q(x;s)}\right] = \mathrm{Res}\left[\varphi_{\mathbf{1}}\right] \tag{4.24}$$

throughout the moduli space. Setting $s = 0$ in the above, we recover the holomorphic top-form at the Fermat locus.

JHEP01(2021)207

The variation of Hodge structure is described by choosing a topological basis γ_k of $H^4(X)$ and studying the variation of the integrals

$$\int_{\gamma_k} \varphi = \int_{\gamma_k} \mathrm{Res}\left[\frac{P}{Q(x;s)^{5-p}}\Omega_0\right] . \tag{4.25}$$

as we vary Q. This defines the Hodge bundle and we may locally choose a trivialization by identifying the topological cycles γ_k in nearby sextics. There is a flat connection ∇_I on this bundle, called the Gauss-Manin connection, which acts on residues as

$$\nabla_I \varphi = \mathrm{Res}\left[\partial_I \frac{P}{Q(x;s)^{5-p}}\Omega_0\right] . \tag{4.26}$$

The flatness of this connection simply follows from the commutativity of the differential operators.

An infinitesimal deformation of

$$\Omega = \mathrm{Res}\left[\varphi_{\mathbf{1}}\right] = \mathrm{Res}\left[\frac{1}{Q(x;s)}\Omega_0\right] \tag{4.27}$$

at the Fermat point can hence be written as

$$\varphi = \varphi_{\mathbf{1}}|_{s=0} + \sum_I s_I\, \partial_{s_I} \varphi_{\mathbf{1}}|_{s=0} . \tag{4.28}$$

Note that

$$\partial_I \varphi_{\mathbf{1}}|_{s=0} = -\frac{\mu_{\mathbf{b}_I}}{Q(x;0)^2}\Omega_0 = -\varphi_{\mathbf{a}_I} , \tag{4.29}$$

which gives a $(3,1)$-form upon taking the residue at the Fermat point as we have restricted to $b_i < 5$ in (4.22). We hence recover that deformations of the complex structure are given by $(3,1)$-forms.

One could also include terms in the sum in (4.22) for which $\mu_{\mathbf{b}_I}$ is in the Jacobi ideal of Q. Deforming by such terms again adds a term to Ω which is given by the residue of a rational form, but now the pole order of this rational form can be reduced to 1 (see appendix A). This implies that the residue does not produce a $(3,1)$ form, but a $(4,0)$ form. Such deformations would hence only rescale Ω.

In physics, one is usually interested in the covariant derivative $D_I = \nabla_I + \partial_I K$, which by definition maps

$$D : H^{p,4-p} \to H^{p-1,4-p+1} . \tag{4.30}$$

When working at the Fermat point and acting on $H^{4,0}$, we have just seen that we must have $\partial_I K = 0|_{s=0}$ as ∇_I alone already has the property of mapping purely to $H^{3,1}$ when using the basis of monomials $\mathbf{b}_I$ with $(\mathbf{b}_I)_i < 5$ to define local coordinates on the complex structure moduli space.[15] In the monomial basis we have chosen, the action of covariant derivatives is hence particularly simple.

[15]Note that this does not imply that $\partial_I K = 0$ holds for any choice of coordinates on complex structure moduli space, as such coordinate changes can give Kähler transformation that map $K(s,\bar{s}) \to K(s,\bar{s}) + f(s) + \bar{f}(\bar{s})$ for a holomorphic function $f(s)$.

This structure becomes slightly more complicated when considering second derivatives of Ω

$$\partial_I \partial_J \varphi_{\mathbf{1}}(s)\big|_{s=0} = 2\, \frac{\mu_{\mathbf{b}_I+\mathbf{b}_J}}{Q^3}\Omega_0\bigg|_{s=0} = 2\, \varphi_{\mathbf{1}+\mathbf{b}_I+\mathbf{b}_J}\big|_{s=0}\,. \tag{4.31}$$

As long as all components of $\mathbf{b}_I + \mathbf{b}_J$ are smaller than 5, this form is of pure type $(2,2)$. Whenever this is not the case, however, $\mu_{\mathbf{b}_I+\mathbf{b}_J}$ is in the Jacobi ideal of Q and we may reduce the pole order leading to a form of degree $(3,1)$. To define a covariant derivative acting on $(3,1)$-forms, we need to subtract the $(3,1)$-pieces of the derivatives. This means we need to set the derivative to zero whenever it produces a form for which $\mu_{\mathbf{b}_I+\mathbf{b}_J}$ is in the Jacobi ideal of Q.

In summary, the covariant derivative acts on forms as

$$D_I : \mathrm{Res}\,[\varphi_{\mathbf{a}}] \to \mathrm{Res}\,[\varphi_{\mathbf{a}+\mathbf{b}_I}] \tag{4.32}$$

as long as $(\mathbf{a}+\mathbf{b}_I)_i < 6$ for all i and it sends them to zero otherwise.

We need to evaluate the rank of the matrix (2.12) in order to find the (co)-dimension of the Hodge locus and hence the number of stabilized moduli. From the above it follows that it can be simply written as

$$G_{IJ} = D_I D_J \int_X G_4 \wedge \Omega = 2 \int_X G_4 \wedge \mathrm{Res}\left[\frac{\mu_{\mathbf{b}_I+\mathbf{b}_J}}{Q^3}\Omega_0\right] \tag{4.33}$$

evaluated at the Fermat point. Note that we might as well have written partial derivatives as the integral automatically picks out the $(2,2)$ piece of the derivatives acting on Ω. In a similar vein, any term for which one of the $(\mathbf{b}_I+\mathbf{b}_J)_i \geq 5$ vanishes. See [22, 23, 32–34] for an in-depth discussion of the above result.

4.5 Period integrals

In order to evaluate the integral in (4.33), we need to know the period integrals of algebraic cycles, i.e. for an algebraic cycle C, we need to know

$$\int_C \mathrm{Res}\left[\frac{\mu_{\mathbf{b}_I+\mathbf{b}_J}}{Q^3}\Omega_0\right]. \tag{4.34}$$

The periods of forms such as (4.31) over the linear algebraic cycles $C^{\boldsymbol{\ell}}_\sigma$ have been computed in [13, 14] using results of [32]. The upshot is that for $|\mathbf{b}| = 2$, we have that

$$\frac{1}{(2\pi i)^2}\int_{C^{\boldsymbol{\ell}}_\sigma} \frac{\mu_{\mathbf{b}}}{Q^3}\Omega_0 = \begin{cases} \frac{\mathrm{sgn}(\sigma)}{6^3 2!}\, e^{\frac{i\pi}{6}\left(\sum_{e=0}^{2}(b_{\sigma(2e)}+1)(2\ell_e+1)\right)} & \text{if } b_{\sigma(2e-2)} + b_{\sigma(2e-1)} = 4 \\ 0 & \text{otherwise.}\end{cases} \tag{4.35}$$

Up to the overall normalization, this can also be derived by using the automorphism group $(\mathbb{Z}/6\mathbb{Z})^6/(\mathbb{Z}/6\mathbb{Z}) \rtimes \mathcal{S}_6$ of the sextic. One must have that

$$\sigma \circ g\left(\int_{C^{\boldsymbol{\ell}}_\sigma} \frac{\mu_{\mathbf{b}}}{Q^3}\Omega_0\right) = \int_{C^{\boldsymbol{\ell}}_\sigma} \frac{\mu_{\mathbf{b}}}{Q^3}\Omega_0 \qquad g \in (\mathbb{Z}/6\mathbb{Z})^6/(\mathbb{Z}/6\mathbb{Z})\,,\ \ \sigma \in \mathcal{S}_6\,, \tag{4.36}$$

with $\sigma \circ g \left(\int_{C^{\boldsymbol{\ell}}_\sigma} \frac{\mu_{\mathbf{b}}}{Q^3} \Omega_0 \right) \equiv \int_{(C^{\boldsymbol{\ell}}_\sigma)'} \frac{\mu'_{\mathbf{b}}}{Q^3} \Omega'_0$, where the prime quantities are the ones transformed by g and σ.

Let us first consider permutations. After acting with any permutation σ, we may simply relabel the coordinates x_i in the r.h.s. of (4.36) to undo the permutation again. This produces the same expression we started from, except for Ω_0, which produces a sign $\mathrm{sgn}(\sigma)$ as it is completely antisymmetric in the x_i. This explains the corresponding factor in (4.35).

Now consider the action by $g \in G = (\mathbb{Z}/6\mathbb{Z})^6/(\mathbb{Z}/6\mathbb{Z})$. This will both act on the differential form under the integral, as well as the cycle $C^{\boldsymbol{\ell}}_\sigma$. We can write

$$g \left(\int_{C^{\boldsymbol{\ell}}_\sigma} \frac{\mu_{\mathbf{b}}}{Q^3} \Omega_0 \right) = \int_{(C^{\boldsymbol{\ell}}_\sigma)'} \frac{\mu'_{\mathbf{b}}}{Q^3} \Omega_0 = \mathbf{a}(g) \int_{C^{\boldsymbol{\ell}'}_\sigma} \frac{\mu_{\mathbf{b}}}{Q^3} \Omega_0 \tag{4.37}$$

for some $\boldsymbol{\ell}'$, where $\mathbf{a}$ is the element of the character group associated with $\mathbf{a} = \mathbf{b} + (1,1,1,1,1,1)$. To check if this makes (4.36) consistent with (4.35), it is enough to consider one of the generators of G, all other cases can be found by analogous computations or repeated application of this action. As we have already understood the action of permutations, let us furthermore choose σ as the trivial permutation $\sigma = \mathrm{id}$ and investigate integrals over the cycles $C^{\boldsymbol{\delta}}_{\mathrm{id}}$. Consider the map $\zeta_0 : x_0 \to e^{i\pi/3} x_0$, which generates one of the $(\mathbb{Z}/6\mathbb{Z}) \subset G$. ζ_0 maps $C^{\boldsymbol{\ell}}_\sigma$ to $C^{\boldsymbol{\ell}'}_\sigma$ where $\delta'_0 = \delta_0 - 1$. We can hence write

$$\begin{aligned}
a(\zeta_0) \int_{C^{\boldsymbol{\delta}'}_{\mathrm{id}}} \frac{\mu_{\mathbf{b}}}{Q^3} \Omega_0 &= e^{\frac{i\pi}{3} a_0} \int_{C^{\boldsymbol{\delta}'}_{\mathrm{id}}} \frac{\mu_{\mathbf{b}}}{Q^3} \Omega_0 \\
&= e^{\frac{i\pi}{3}(b_0+1)} \frac{1}{6^3 2!} e^{\frac{i\pi}{6}\left[(b_0+1)(2\ell'_0+1)+(b_2+1)(2\ell'_1+1)+(b_4+1)(2\ell'_2+1)\right]} \\
&= e^{\frac{i\pi}{3}(b_0+1)} \frac{1}{6^3 2!} e^{\frac{i\pi}{6}[(b_0+1)(2\ell_0-1)+(b_2+1)(2\ell_1+1)+(b_4+1)(2\ell_2+1)]} \\
&= \frac{1}{6^3 2!} e^{\frac{i\pi}{6}[(b_0+1)(2\ell_0+1)+(b_2+1)(2\ell_1+1)+(b_4+1)(2\ell_2+1)]} \\
&= \int_{C^{\boldsymbol{\delta}}_{\mathrm{id}}} \frac{\mu_{\mathbf{b}}}{Q^3} \Omega_0
\end{aligned} \tag{4.38}$$

that is exactly what (4.36) says.

We then need to know the integral of $\varphi_{\mathbf{a}}$ on a single cycle $C^{\boldsymbol{\ell}}_\sigma$ to compute the integrals of $\varphi_{\mathbf{a}}$ over all the cycles $C^{\boldsymbol{\ell}'}_\sigma$ in the same orbit. One simply uses

$$\int_{C^{\boldsymbol{\ell}'}_\sigma} \varphi_{\mathbf{a}} = \mathbf{a}(g)^{-1} \int_{C^{\boldsymbol{\ell}}_\sigma} \varphi_{\mathbf{a}} \,. \tag{4.39}$$

that is derived by (4.36). This shows that in fact all relative coefficients of (4.35) are fixed by $G \rtimes \mathcal{S}_6$, as it acts transitively on the $C^{\boldsymbol{\ell}}_\sigma$. It is not true, however, that $G \rtimes \mathcal{S}_6$ acts transitively on a basis of algebraic cycles for $H^{2,2}(X) \cap H^4(X,\mathbb{Q})$. If we want to study periods of such a basis up to a global normalization, we hence need more than the relative factors between periods of the $C^{\boldsymbol{\ell}}_\sigma$.

Note that the condition $b_{\sigma(2e-2)} + b_{\sigma(2e-1)} = 4$ for all $e \in \{0,1,2\}$ implies that the intersections of $C^{\boldsymbol{\ell}}_\sigma$ with the eigenspace $V(\mathbf{a})$ is non-zero only if $\mathbf{a}$ is 3-decomposable. This is not unexpected, as we have seen, $V(\mathbf{a})$ for $\mathbf{a}$ 3-decomposable can be constructed from the C^ℓ_σ, whereas eigenspaces for $\mathbf{a}$ 1-decomposable or indecomposable are constructed from other algebraic cycles. The vanishing statement hence strengthens the observation that $V(\mathbf{a})$ and $V(\mathbf{a}')$ are orthogonal except $\mathbf{a}' = \bar{\mathbf{a}}$. This can also be seen directly as follows. Let us consider the case where σ is the trivial permutation. The action of $\zeta_0^k\zeta_1^k$ on C^ℓ_σ is trivial in this case. It follows that

$$\int_{C^\ell_\sigma} \frac{\mu_\mathbf{b}}{Q^3}\Omega_0 = \zeta_0^k\zeta_1^k \int_{C^\ell_\sigma} \frac{\mu_\mathbf{b}}{Q^3}\Omega_0 = e^{\frac{i\pi}{3}\cdot k(b_0+b_1+2)} \int_{C^\ell_\sigma} \frac{\mu_\mathbf{b}}{Q^3}\Omega_0 \tag{4.40}$$

so that the integral can only be non-zero when $a_0 + a_1 = 0 \mod 6$. We can make the same argument for the other two pairs x_2, x_3 and x_4, x_5. The same argument applies (with different pairings) for other permutations, and implies that $\mathbf{a} = \mathbf{b} + 1$ must be 3-decomposable for the integral to be non-zero.

The above can be generalized to arbitrary algebraic cycles of complete intersection type [14], i.e. cycles of the type $f_0 = f_1 = f_2 = 0$ inside a hypersurface (3.11). The result is

$$\frac{1}{(2\pi i)^2}\int_Z \frac{\mu_\mathbf{b}}{Q^3}\Omega_0 = c\cdot\frac{5^6}{2}\,, \tag{4.41}$$

where c is the unique number which satisfies

$$\mu_\mathbf{b}\det(\partial_i H_j) = c\ \det\left(\mathrm{Hess}(Q)\right) \qquad \mod \langle\partial_i Q\rangle\,, \tag{4.42}$$

the vector H is given by $H = (f_0, P_0, f_1, P_1, f_2, P_2)$ and Hess denotes the Hessian matrix. For the linear cycles C^ℓ_σ, this reproduces the normalization of (4.35) from the general formula (4.41).

5 Algebraic fluxes and stabilization of complex structure moduli

With the tools we have collected in the previous section, we are now ready to directly address how algebraic cycles can be used as fluxes and how many moduli they stabilize. All one needs to do after defining a flux which is appropriately quantized and primitive, is to evaluate the period integrals (4.34) needed to compute the rank of the 426×426 matrix G_{IJ} (4.33).

5.1 One linear algebraic cycle

As a first example, let us revisit the case of using a single linear algebraic cycle C^ℓ_σ (3.5) as a G_4 flux.[16] Using the period integrals (4.35) we find

$$\mathrm{rk} G_{IJ}\left(C^\ell_\sigma\right) = 19\,, \tag{5.1}$$

which is precisely the same number we obtained by analyzing obstructed polynomial deformations.

In fact, this is the lowest possible value G_{IJ} can have for any algebraic cycle. This is not surprising as linear algebraic cycles are the simplest type that can exist for the Fermat sextic (see Proposition 7 'Olympiad problem' of [34]).

[16]The same issues as discussed in footnote 11 apply.

5.2 A sum of two linear algebraic cycles

With the material we have collected, it is straightforward to work out what happens when we add two different linear algebraic cycles $C^{\ell}_{\sigma}+C^{\ell'}_{\sigma'}$. A direct computation (in combination with the automorphism group) shows that the rank of G_{IJ} only depends on the mutual intersection between the two, and we can make the following table

$C^{\ell}_{\sigma} \cdot C^{\ell'}_{\sigma'}$	rk $G_{IJ}\left(C^{\ell}_{\sigma}+C^{\ell'}_{\sigma'}\right)$
21	19
-4	32
1	38
0	38

. (5.2)

The first row corresponds to the case $C^{\ell}_{\sigma}=C^{\ell'}_{\sigma'}$. As the number of flat directions for a linear combination of two cycles is at least equal to the number of flat directions common to both of them, the rank of the matrix G_{IJ} must be subadditive:

$$\text{rk } G_{IJ}\left(C^{\ell}_{\sigma}\right)+\text{rk } G_{IJ}\left(C^{\ell'}_{\sigma'}\right)\geq \text{rk } G_{IJ}\left(C^{\ell}_{\sigma}+C^{\ell'}_{\sigma'}\right)\,, \tag{5.3}$$

which is indeed the case for the numbers we find.

5.3 Fluxes respecting group actions

The sextic moduli space has the symmetry group $G=\mu^6/\mu$ for $\mu=\mathbb{Z}/6\mathbb{Z}$, that we discussed in section 4.2. Consider the Greene-Plesser subgroup $G_{PG}=(\mathbb{Z}/6\mathbb{Z})^4$ [35]. It is generated by $\alpha_i^6=1$ for $i=1,2,3,4$ with action

$$(x_0,x_1,x_2,x_3,x_4,x_5)\to\left((\alpha_1\alpha_2\alpha_3\alpha_4)^{-1}x_0,\alpha_1x_1,\alpha_2x_2,\alpha_3x_3,\alpha_4x_4,x_5\right) \tag{5.4}$$

on the homogeneous coordinates of X_6. Famously, only a single complex structure deformation, corresponding to the monomial $\prod_i x_i$, i.e. $\mathbf{b}=(1,1,1,1,1,1)$, is symmetric under the action of this group, while all others are projected out. The obvious way to construct a flux that is even under the action of G_{PG} is to start with the orbit of any of the linear cycles C^{ℓ}_{σ} under G_{PG}. It turns out that this is not the minimal choice and one can repeatedly use the sum rule (3.12) to show that

$$\sum_{g\in G_{GP}} g(C^{\ell}_{\sigma})=4\,C_{eee} \tag{5.5}$$

where

$$C_{eee}=\sum_{\ell_0,\ell_1,\ell_2\in(0,2,4)} C^{\ell}_{\sigma}\,. \tag{5.6}$$

Using (3.12) and the intersection numbers (3.14) one can also show directly that C_{eee} is even under G_{PG} and that $C_{eee}\cdot C_{eee}=3^5$. As there are 3^3 terms in the sum, we have $C_{eee}\cdot H^2=27$. A symmetric flux that is primitive and properly quantized is

$$G_4^{sym}=C_{eee}-9/2H^2\,, \tag{5.7}$$

and the induced tadpole is hence $243/4$, which is well within the allowed range.

Directly evaluating $G_{IJ}(C_{eee})$ using (4.35) we find that indeed

$$\text{rk } G_{IJ}\,(G_4^{sym}) = 141\,. \tag{5.8}$$

In particular, there is a single entry that obstructs the unique deformation that is symmetric under G_{PG}. Hence the symmetric flux fixes the invariant modulus; this corresponds to solving $D_{s^{\rm i}}W = 0$ explicitly (see at the end of section 2 for notation) and finding that the solution sits at the Fermat point. Our computation goes further: it is true that the Fermat point (that belongs to the fixed point set of G_{PG}) is a solution of $D_{s_a^{\rm ni}}W = 0$ (with $a = 1, \ldots, 425$), but only 140 out of the 425 non-symmetric deformations under G_{PG} are fixed by G_4^{sym}, the other 285 ones are flat directions.

Finally, one may wonder about using a flux that is symmetric under the entire automorphism group $(\mathbb{Z}/6\mathbb{Z})^6/(\mathbb{Z}/6\mathbb{Z}) \rtimes \mathcal{S}_6$ of the sextic at the Fermat point. As all of the forms $\varphi_{\mathbf{a}}$ with $|a| = 3$ have non-trivial transformations already under the scaling part, it follows that the matrix G_{IJ} can only contain zeros in such a case. The same can be seen by noting that $(\mathbb{Z}/6\mathbb{Z})^6/(\mathbb{Z}/6\mathbb{Z}) \rtimes \mathcal{S}_6$ acts transitively on the $C_\sigma^{\boldsymbol{\ell}}$. Using the sum rule (3.12) one can argue that summing over an orbit results in a cycle proportional to H^2 (the only invariant cycle), so that G_{IJ} vanishes for all I, J. This implies that there are no invariant fluxes that are primitive for this group.

5.4 Stabilizing all moduli using linear algebraic cycles

Let us now see if we can find a flux stabilizing all moduli employing only linear algebraic cycles. From the subadditivity (5.3), it follows that we need to consider a linear combination of at least 23 of the $C_\sigma^{\boldsymbol{\ell}}$. As we have seen in sections 4.2 and 4.3, only a subspace of dimension 1001 within $H^{2,2}_{\rm prim}(X)$ is spanned by the linear algebraic cycles, and this subspace precisely corresponds to 3-decomposable tuples $\mathbf{a}$. By a simple scan over all possibilities, one can find that for every b_I with $|\mathbf{b}_I| = 1$, there is a b_J with $|\mathbf{b}_J| = 1$ such that

$$\mathbf{a}_{IJ} = (1,1,1,1,1,1) + b_I + b_J \tag{5.9}$$

is 3-decomposable. In other words, linear cycles in principle allow us to constrain all complex structure deformations.

Due to the large number of linear algebraic cycles, there are $6^3 \cdot 15 = 3240$ of them, a simple scan is computationally much too expensive. Besides randomly sampling choices, the following (semi-)systematic method can be used. It is an experimental fact that the inequality (5.3) becomes an equality if we consider sums of linear algebraic cycles such that all of them are mutually orthogonal, i.e.

$$\text{rk } G_{IJ}\left(\sum_{i\in I} C_{\sigma_i}^{\boldsymbol{\ell}_i}\right) = |I|\cdot 19 \qquad \text{if} \qquad C_{\sigma_i}^{\boldsymbol{\ell}_i}\cdot C_{\sigma_j}^{\boldsymbol{\ell}_j} = 0 \tag{5.10}$$

for all $i \neq j \in I$. Correspondingly, the maximal size of a set with this property is 22, one

JHEP01(2021)207

possible choice being

$$
\begin{aligned}
I_{\max} = \{ &[[0,0,0]\,, [0,1,2,3,4,5]]\,, [[0,0,0]\,, [0,2,1,5,3,4]]\,, [[0,0,1]\,, [0,3,1,2,4,5]]\,, \\
&[[0,0,1]\,, [0,5,1,3,2,4]]\,, [[0,1,0]\,, [0,4,1,2,3,5]]\,, [[1,0,1]\,, [0,1,2,5,3,4]]\,, \\
&[[1,0,2]\,, [0,3,1,4,2,5]]\,, [[1,1,1]\,, [0,2,1,4,3,5]]\,, [[1,1,4]\,, [0,4,1,3,2,5]]\,, \\
&[[1,2,2]\,, [0,5,1,2,3,4]]\,, [[2,0,5]\,, [0,1,2,4,3,5]]\,, [[2,1,4]\,, [0,3,1,5,2,4]]\,, \\
&[[2,2,2]\,, [0,2,1,4,3,5]]\,, [[2,3,1]\,, [0,4,1,3,2,5]]\,, [[3,2,3]\,, [0,1,2,4,3,5]]\,, \\
&[[3,2,5]\,, [0,4,1,3,2,5]]\,, [[3,3,2]\,, [0,5,1,4,2,3]]\,, [[3,4,3]\,, [0,2,1,5,3,4]]\,, \\
&[[3,5,5]\,, [0,3,1,2,4,5]]\,, [[4,3,4]\,, [0,4,1,2,3,5]]\,, [[5,3,3]\,, [0,4,1,5,2,3]]\,, \\
&[[5,5,4]\,, [0,2,1,5,3,4]] \}
\end{aligned}\,, \tag{5.11}
$$

where each entry is of the form $[\ell_i, \sigma_i]$. Consistent with (5.10), a sum of the associated linear algebraic cycles gives a matrix G_{IJ} of rank 418.

We can use this as a starting point to construct a flux stabilizing all moduli by adding a further linear algebraic cycle. We are looking for a C_e such that for

$$C = \sum_{i \in I_{\max}} C^{\boldsymbol{\ell}_i}_{\sigma_i} + C_e \tag{5.12}$$

the flux

$$G_4 = C + nH^2 \tag{5.13}$$

is primitive and appropriately quantized. Quantization requires n to be half-integer, $n = m/2$ for m odd, and primitivity requires

$$3m = 22 + C_e \cdot H^2\,, \tag{5.14}$$

so that we can choose $C_e = -C^{\boldsymbol{\ell}_e}_{\sigma_e}$, which gives $m = 7$. The tadpole of such a configuration is given by

$$N_{D3} = \frac{1}{2}\left(\sum_{i \in I_{\max}} C^{\boldsymbol{\ell}_i}_{\sigma_i} - C^{\boldsymbol{\ell}_e}_{\sigma_e} - \frac{7}{2}H^2\right)^2 = \frac{1}{2}\left(\left(\sum_{i \in I_{\max}} C^{\boldsymbol{\ell}_i}_{\sigma_i} - C^{\boldsymbol{\ell}_e}_{\sigma_e}\right)^2 - 6\frac{7^2}{4}\right) \tag{5.15}$$

which is minimal if we choose $C^{\boldsymbol{\ell}_e}_{\sigma_e} \cdot \sum_{i \in I_{\max}} C^{\boldsymbol{\ell}_i}_{\sigma_i}$ to be as large as possible. There is a unique such $C^{\boldsymbol{\ell}_e}_{\sigma_e}$ which has $C^{\boldsymbol{\ell}_e}_{\sigma_e} \cdot \sum_{i \in I_{\max}} C^{\boldsymbol{\ell}_i}_{\sigma_i} = 11$, it is given by

$$[\sigma_e, \ell_e] = [[2,5,3], [0,2,1,4,3,5]]\,. \tag{5.16}$$

Working out $G_{IJ}(C) = G_{IJ}(G_4)$ one finds that it has maximal rank, 426. The resulting tadpole is computed to be

$$N_{D3} = \frac{1}{2} G_4 \wedge G_4 = \frac{775}{4}\,. \tag{5.17}$$

Although such a flux would stabilize all complex structure moduli at the Fermat point, it significantly overshoots the available tadpole

$$\chi(X)/24 = 435/4\,. \tag{5.18}$$

5.5 Tadpole issues

The result of the last section is at the same time encouraging and disappointing: while it is not hard to find a primitive flux with proper quantization that stabilizes all moduli, it generates a tadpole that is almost twice the maximal allowed value. Scanning over random linear combinations of linear algebraic cycles gives many more examples with the same properties. This result can already be anticipated from rough estimates.

Consider stabilizing all moduli by a combination of linear algebraic fluxes

$$C = \sum_{i \in I} f_i C_{\sigma_i}^{\boldsymbol{\ell}_i} \tag{5.19}$$

and let $\sum f_i = 3m$ for an odd integer m. We can then find a primitive properly quantized flux by setting

$$G_4 = C - \frac{m}{2} H^2 \,. \tag{5.20}$$

The induced tadpole is then

$$\frac{1}{2} G_4^2 = \frac{1}{2}\left(C^2 - \frac{3}{2}m^2\right) \,. \tag{5.21}$$

Ignoring the contribution[17] from $C_{\sigma_i}^{\boldsymbol{\ell}_i} \cdot C_{\sigma_j}^{\boldsymbol{\ell}_j}$ to C^2, and assuming that all $f_i = 1$, we can write this as

$$\frac{1}{2} G_4^2 = \frac{1}{2}\left(63m - \frac{3}{2}m^2\right) \,. \tag{5.22}$$

As every $C_\sigma^{\boldsymbol{\ell}}$ stabilizes at most 19 moduli, we need to have $3m > 426/19$. This roughly reproduces (5.17) for the minimal choice of m.

The negative contribution in (5.22) points at a potential way out by letting m become sufficiently large. As we have seen, there are at most 22 mutually orthogonal linear algebraic cycles and we need to let $3m$ be significantly larger to bring the tadpole down sufficiently. It turns out that ignoring mutual intersections between the terms in C become increasingly unjustified, so that the tadpole contribution of such fluxes is again far too large to give a viable model.

Until now, we have completely ignored non-linear algebraic cycles. Performing a similar rough estimate gives a comparable result to what we have found for the linear algebraic cycles. There, the crucial ratio was that of the number of moduli that could be fixed with a single linear algebraic cycle 19, to the square of such a cycle, 21. These ratios can also be computed for non-linear algebraic cycles, the result is that for a complete intersection algebraic cycle $C_{f_0 f_1 f_2}$ given by $f_0 = f_1 = f_2 = 0$ for homogeneous polynomials with

[17]Mutual intersections between the $C_\sigma^{\boldsymbol{\ell}}$ can be negative, so one may hope that this helps in lowering the tadpole contribution. However, the savings in the tadpole are counter-weighted by the decrease in stabilized moduli for linear combinations of cycles with negative intersections, see table 5.2. Nonetheless, we have made some scans trying to exploit mutual negative intersections, with the result that the decrease in stabilized moduli becomes even more significant the more cycles are added.

JHEP01(2021)207

degrees d_i [36]

(d_0, d_1, d_2)	$C^2_{f_0 f_1 f_2}$	rk G_{IJ} $(C_{f_0 f_1 f_2})$
$(1,1,1)$	21	19
$(1,1,2)$	34	32
$(1,1,3)$	39	37
$(1,2,2)$	56	54
$(1,2,3)$	66	62
$(1,3,3)$	81	71
$(2,2,2)$	96	92
$(2,2,3)$	120	106
$(2,3,3)$	162	122
$(3,3,3)$	243	141

. (5.23)

As the ratio of these two numbes stays roughly the same, we can anticipate to find similar results using non-linear algebraic cycles. As the maximal allowed tadpole of the flux is $\chi(X)/24 = 435/4 = \frac{1}{2}C^2$, but we need to stabilize 426 moduli, the ratio between C^2 and rk $G_{IJ}(C)$ should be roughly $\frac{1}{2}$ rather than the ratio of ~ 1 (and larger) observed above.

The above results do not imply that there cannot be a properly quantized and primitive flux stabilizing all moduli that also satisfies the tadpole constraint. Much more work is needed to make such a claim. What we can say (at least for the sextic fourfold we studied), however, is that it is not completely straightforward to construct such a flux.

6 Moduli stabilization and symmetry actions

In this section we address the problem of moduli stabilization in cases with symmetry in some more generality. In particular, we explain why fluxes that are invariant under some group actions typically leave some flat directions in the effective potential. This will give a conceptual way of understanding the result found in section 5.3.

Let X_p be a Calabi-Yau fourfold at a point p in its complex structure moduli space, and G any subgroup of the automorphism group of X_p. Although what we are going to say can be put in slightly more general terms, let us assume for simplicity that X_p is a hypersurface in a toric variety T for which we can represent all forms in the middle cohomology of X_p as residues. Let us furthermore assume that G acts by rescaling the homogeneous coordinates x_i of the ambient space by roots of unity and preserves the holomorphic top form $\Omega_X|_p$ at p (as well as the Kähler form of X_p).

We can write a family in the vicinity of p as

$$X: \; Q = Q_0 + \sum_N s_N \mu_N + \sum_\Phi t_\Phi \nu_\Phi = 0 \tag{6.1}$$

where X_p is given by $\mathbf{s} = \mathbf{t} = 0$, the monomials μ_N are invariant under the action of G, and the monomials ν_Φ are not, but transform as

$$\nu_\Phi \to \alpha_\Phi(g)\nu_\Phi \qquad \text{(no summation)}. \tag{6.2}$$

By assumption we can write

$$\Omega_X|_p = \mathrm{Res}\left[\frac{1}{Q_0}\Omega_T\right], \tag{6.3}$$

for some fixed holomorphic form Ω_T on the ambient space T. As both Ω_X at p and Q_0 are invariant under the action of G, it follows that Ω_T must be preserved by G as well.

Let us now consider switching on a flux G_4 which is invariant under the action of G. This implies that the GVW superpotential (1.1) is invariant under G as well. The F-terms in the non-invariant directions t_Φ at p transform as

$$F_\Phi = \int_{X_p} G_4 \wedge D_\Phi \Omega_X|_p = -\int_{X_p} G_4 \wedge \mathrm{Res}\left[\frac{\nu_\Phi}{Q_0^2}\Omega_T\right] \to \alpha_\Phi(g) F_\Phi. \tag{6.4}$$

However, the above integral at p simply yields a number that cannot change under any automorphism, so that it follows that $F_\Phi = 0$ for all Φ. This argument was used in [15, 16, 37, 38] to argue that[18] the F-term equations in the non-invariant directions are automatically satisfied and one only needs to take care of the invariant directions.

As we have seen in the example in section 5.3, this does not imply that there are no flat directions along the non-invariant directions t_Φ in complex structure moduli space. We can repeat a similar argument as above for the matrix G_{IJ} to see why. Let us first consider the mixed terms $G_{N\Phi}$ between invariant and non-invariant directions. They transform as

$$G_{N\Phi} = \int_{X_p} G_4 \wedge \mathrm{Res}\left[\frac{\mu_n \nu_j}{Q_0^3}\Omega_T\right] \to \alpha_\Phi(g)\, G_{N\Phi}\,. \tag{6.5}$$

As these have a non-trivial scaling they must vanish, so that G_{IJ} is block-diagonal among invariant and non-invariant directions. For the matrix elements between two non-invariant directions we find

$$G_{\Phi\Xi} = \int_{X_p} G_4 \wedge \mathrm{Res}\left[\frac{\nu_\Phi \nu_\Xi}{Q_0^3}\Omega_T\right] \to \alpha_\Phi(g)\alpha_\Xi(g) G_{\Phi\Xi}\,. \tag{6.6}$$

These can hence only be non-vanishing if $\alpha_\Phi(g)\alpha_\Xi(g) = 1$ for all $g \in G$. This is a strong condition, and for many ν_Φ there is no ν_Ξ such that it holds, which implies that both t_Φ and t_Ξ are flat directions.

Let us now come back to the example discussed in section (5.3), where X_p is the sextic fourfold at the Fermat point and G is the Greene-Plesser group G_{GP}. In this case there is only a single invariant monomial $\mu_N = \prod_i x_i$. All other monomials ν_Φ correspond to non-invariant directions. In order to have a non-zero matrix elements $G_{\Phi\Xi}$ we need that

$$\nu_\Phi \nu_\Xi = \prod_i x_i^2\,, \tag{6.7}$$

as this is the only monomial of appropriate degree that is invariant under G_{GP}. The only non-invariant complex structure deformations that have non-zero elements in G_{IJ} hence

[18] These papers deal with Calabi-Yau threefolds. However their arguments and their conclusions directly apply to fourfolds as well.

correspond to pairs of tuples $\mathbf{b}_\Phi$ and $\mathbf{b}_\Xi$ different from $(1,1,1,1,1,1)$ with $\sum_i (\mathbf{b}_\Phi)_i = 6$ and $\sum_i (\mathbf{b}_\Xi)_i = 6$ such that

$$\mathbf{b}_\Phi + \mathbf{b}_\Xi = (2,2,2,2,2,2)\,. \tag{6.8}$$

It turns out that there are precisely 70 such pairs, so that the rank of G_{IJ} can be at most 141 for any flux that is symmetric under G_{GP}, i.e. there are at least 285 flat directions in this case. For the example we have chosen in section 5.3, this is precisely what was found by explicitly evaluating G_{IJ}.

7 Conclusions and future directions

In this work we have begun to explore how to use algebraic cycles as fluxes on Calabi-Yau fourfolds. We have reviewed methods which allow us to compute the number of stabilized moduli and the induced tadpole for fluxes proportional to linear combinations of algebraic cycles. We have analyzed in detail the sextic fourfold. We have found fluxes that stabilize all the complex structure moduli at a specific point in the complex structure moduli space, without the need of dealing with Picard-Fuchs equations of (very) high rank. What is striking about this analysis is that it appears very hard to find a flux that satisfies all consistency constraints and stabilizes all of the complex structure moduli. In particular, in the example we have considered, we have noticed tension between tadpole cancellation and the desire to stabilize all complex structure moduli.

The above is far from a complete analysis, and there are several crucial points that need to be addressed for a complete picture. First of all, it is in principle straightforward (but tedious) to work out rk $G_{IJ}(C_\Sigma)$, C_Σ^2, and $H \cdot C_\Sigma$ for any linear combination of algebraic cycles C_Σ. Having access to all fluxes defined via algebraic cycles is not sufficient, as the integral Hodge conjecture for the Fermat sextic is presently unanswered. It has been shown to be correct, however, for the quartic and quintic Fermat fourfolds in [39], and it is possible to extend their methods to the sextic. With a proof of the integral Hodge conjecture for the Fermat sextic, it is then possible to compute rk $G_{IJ}(G_4)$ for all fluxes satisfying the tadpole constraint. This naively seems like a task that is computationally too demanding to be undertaken, but a clever exploitation of the large automorphism group of the sextic might make it feasible. We intend to attack this problem in future work.

Thinking even further ahead, it is highly desirable to extend the methods we have reviewed to other points in the moduli space of the sextic with maximal $H^{2,2}(\mathbb{Z}) \cap H^4(X,\mathbb{Z})$, and even to other Calabi-Yau fourfolds. In particular, it would be exciting to find criteria which can distinguish which points in the moduli space can and which cannot be stabilized using fluxes that satisfy the tadpole constraint. Such criteria would have far reaching implications for the existence and structure of the string landscape.

Finally, let us note that the Fermat sextic we considered in this paper is known to be modular [40]. In recent work, it was conjectured that in fact all flux vacua correspond to modular varieties [41], although the converse to this statement is not true [42]. A similar statement holds for attractive $K3$ surfaces [43], which appear in the study of flux vacua on the 'toy' fourfold $K3 \times K3$ [11] (see also [8]), as well as the closely related attractor points on Calabi-Yau manifolds [44]. At its core, modularity is concerned with Galois representations, which are again related to algebraic cycles according to the Tate conjecture. It should be fascinating to explore this relationship further.

Acknowledgments

We wish to thank Hossein Movasati for discussion and explanations, as well as Michele Cicoli and Savdeep Sethi for comments. The work of R.V. is partially supported by "Fondo per la Ricerca di Ateneo - FRA 2018"(UniTS) and by INFN Iniziativa Specifica ST&FI.

A Rational forms, residues, and cohomology of hypersurfaces

In this section we review some classic material concerning rational differential forms on $\mathbb{P}^n$, i.e. forms with poles, and their residues. This is based on [45, 46], a beautiful exposition of which can be found in [23, 47–49].

The basic idea of residues of forms is to extend the residue formula

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z} = 1\,, \tag{A.1}$$

for γ a closed curve encircling the origin, to integrals of differential forms, i.e.

$$\frac{1}{2\pi i}\int_\gamma \frac{dz\wedge\alpha}{z} = \alpha\,, \tag{A.2}$$

for a smooth differential form α. In this way, rational differential forms on $\mathbb{C}^n$ with poles along $z=0$ are naturally identified with smooth forms on the locus $z=0$. The following essentially deals with properly formulating this idea for hypersurfaces X of $\mathbb{P}^{d+1}$. The upshot is that we can write differential forms on X as differential forms with poles on $\mathbb{P}^{d+1}$.

The setting we will be interested concerns differential forms with poles ('rational forms') on complex projective space. By a result of [45], rational $d+1$-forms on $\mathbb{P}^{d+1}$ can always be written in terms of the unique holomorphic $d+1$-form Ω_0

$$\varphi = \frac{P(x)}{R(x)}\Omega_0 \tag{A.3}$$

for homogeneous polynomials $P(x)$ and $R(x)$ with $\deg R = \deg P + (d+2)$. The form Ω_0 is given by

$$\Omega_0 = \sum_{j=0}^{d+1}(-1)^j\; x_j\; dx_0\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_{d+1}\,, \tag{A.4}$$

where our notation is supposed to indicate that $\widehat{dx_j}$ is omitted from the $\wedge$ product of dx_j. There are similar expression for n-forms with $n<d+1$. The constraint on the degrees of P and R hence guarantees that φ is invariant under the $\mathbb{C}^*$ acting on the homogeneous coordinates x_i.

Depending on the choice of the denominator, φ can have poles of various orders along a hypersurface $X\subset\mathbb{P}^{d+1}$. Working modulo exact forms, the pole order can sometimes be reduced, as summarized in the following statements [45, 46].

a) For any rational $d+1$-form φ there exists a η such that $\varphi + d\eta$ has pole order $d+1$.

b) If a rational $d+1$-form φ has pole order k along X and there exists and η such that $\varphi + d\eta$ has pole order $k-1$, we can choose η to have pole order $k-1$.

c) If X is given by $Q(x) = 0$ and φ has pole order k we can write

$$\varphi = \frac{P(x)}{Q(x)^k}\,\Omega_0\,. \tag{A.5}$$

There exists an η such that $\varphi+d\eta$ has pole order $k-1$ if and only if $P(x)$ is contained in the Jacobi ideal of $Q(x)$, i.e. the ideal generated by the polynomials $\partial Q/\partial x_j$.

Elements of $H^{d+1}(\mathbb{P}^{d+1} - X)$ can hence be represented by forms such as (A.5). If X is described by a polynomial $Q(x) = 0$ and $Q(x)$ has degree l, then $\mathrm{deg}P = kl - (d+2)$. As we always reduce the pole degree of φ modulo exact forms, k is at most $d+1$, due to the property a).

The residue map is defined as

$$\mathrm{Res} : H^{d+1}(\mathbb{P}^{d+1} - X) \to H^d_{\mathrm{prim}}(X) \tag{A.6}$$

as follows. For any d-cycle Γ on the hypersurface X we set

$$\int_\Gamma \mathrm{Res}(\varphi) = \int_{T(\Gamma)} \varphi\,, \tag{A.7}$$

where $T(\Gamma)$ is a tube, i.e. a circle bundle over Γ. It can be shown that such a tube always exists and the definition of the residue is independent of this choice. The image of the residue map is not all of $H^d(X)$, but only maps to the primitive cohomology $H^d_{\mathrm{prim}}(X)$, i.e. to those forms perpendicular to the restriction of the hyperplane class. As shown in [45] the residue map is surjective on the primitive cohomology

$$\mathrm{im}\,(\mathrm{Res}) = H^d_{\mathrm{prim}}(X)\,. \tag{A.8}$$

Let us see in some more detail how this definition of the residue realizes (A.2). Consider a rational n-form of pole order k in a small neighborhood containing $Q = 0$. There we can choose coordinates such that φ becomes

$$\varphi = \frac{dQ \wedge \alpha}{Q^k} + \frac{\beta}{Q^{k-1}} = \frac{1}{k-1} d\left(\frac{\alpha}{Q^{k-1}}\right) + \frac{\beta + \frac{1}{k-1} d\alpha}{Q^{k-1}} \tag{A.9}$$

for some smooth forms α and β. Hence we may always reduce the pole order of holomorphic forms locally. Using a partition of unity, one can show that this can in fact be done globally, but at the expense of holomorphicity. Iterating this procedure, it follows that we may write (up to exact forms)

$$\varphi = \frac{\gamma \wedge df}{Q} + \delta \tag{A.10}$$

for some smooth forms γ and δ. The residue is then simply

$$\mathrm{Res}(\varphi) = \gamma|_X\,. \tag{A.11}$$

JHEP01(2021)207

This explains why the residue of a holomorphic rational form φ such as (A.5) on $\mathbb{P}^n$ is not necessarily holomorphic, except when $k = 1$.

Let us now define A_k^{d+1} to be the additive group of rational $d+1$-forms of pole order at most k along X. We can then form the 'cohomology groups'

$$\mathcal{H}_k(X) = \frac{A_k^{d+1}(X)}{dA_{k-1}^d(X)}\,. \tag{A.12}$$

The $\mathcal{H}_k(X)$ for different k have a filtration

$$\mathcal{H}_0 \subset \mathcal{H}_1 \subset \cdots \subset \mathcal{H}_{d+1}\,, \tag{A.13}$$

which precisely maps to the Hodge filtration of the primitive cohomology under the residue map:

$$\mathrm{Res}\,(\mathcal{H}_k) = \mathcal{F}^{d+1-k} H^d_{\mathrm{prim}}(X)\,, \tag{A.14}$$

where

$$\mathcal{F}^{d+1-k} H^d = \bigoplus_{i \geq d+1-k} H^{i,d-i}(X)\,. \tag{A.15}$$

Hence the residue map takes $\mathcal{H}_1(X)$ to $H^{d,0}(X)$, while $\mathcal{H}_2(X)$ maps to $H^{d,0}(X) \oplus H^{d-1,1}(X)$, etc. The forms of maximal pole order, $k = d+1$, are mapped to $\mathcal{F}^0 H^d_{\mathrm{prim}}(X) = H^d_{\mathrm{prim}}(X)$.

We can isolate the Hodge cohomology groups of primitive forms by forming the quotients

$$H^{p,d-p}_{\mathrm{prim}}(X) = \frac{\mathcal{F}^p H^d_{\mathrm{prim}}(X)}{\mathcal{F}^{p+1} H^d_{\mathrm{prim}}(X)} = \frac{\mathcal{H}_{d+1-p}(X)}{\mathcal{H}_{d-p}(X)}\,. \tag{A.16}$$

where the last equality is realized by applying the residue map.

The result of the above is that we can associate Hodge cohomology groups in the middle cohomology with polynomials P of appropriate degree modulo the Jacobi ideal of the polynomial Q defining the hypersurface equation. Consider a rational form of pole degree k written as (A.5). Such a form defines an element in $A_k^{d+1}(X)$ and hence an element in $\mathcal{H}_k(X)$. If P is contained in the Jacobi ideal of Q, there exists an η such that $\varphi + d\eta$ has a pole of degree $k-1$. This implies that φ is equivalent to an element of A_{k-1}^{d+1} in $\mathcal{H}_k(X)$, which in turn implies that φ is also contained in $\mathcal{H}_{k-1}(X)$. But this means that φ is zero in (A.16), so that the statement we started the paragraph with follows.

References

[1] K. Becker and M. Becker, *M theory on eight manifolds*, *Nucl. Phys. B* **477** (1996) 155 [hep-th/9605053] [INSPIRE].

[2] S. Gukov, C. Vafa and E. Witten, *CFT's from Calabi-Yau four folds*, *Nucl. Phys. B* **584** (2000) 69 [*Erratum ibid.* **608** (2001) 477] [hep-th/9906070] [INSPIRE].

[3] E. Witten, *On flux quantization in M-theory and the effective action*, *J. Geom. Phys.* **22** (1997) 1 [hep-th/9609122] [INSPIRE].

[4] A.P. Braun, A. Hebecker, C. Lüdeling and R. Valandro, *Fixing D7 brane positions by F-theory fluxes*, *Nucl. Phys. B* **815** (2009) 256 [arXiv:0811.2416] [INSPIRE].

[5] A.P. Braun and T. Watari, *The vertical, the horizontal and the rest: anatomy of the middle cohomology of Calabi-Yau fourfolds and F-theory applications*, *JHEP* **01** (2015) 047 [arXiv:1408.6167] [INSPIRE].

[6] A.P. Braun and T. Watari, *Distribution of the number of generations in flux compactifications*, *Phys. Rev. D* **90** (2014) 121901 [arXiv:1408.6156] [INSPIRE].

[7] T. Watari, *Statistics of F-theory flux vacua for particle physics*, *JHEP* **11** (2015) 065 [arXiv:1506.08433] [INSPIRE].

[8] A.P. Braun, Y. Kimura and T. Watari, *The Noether-Lefschetz problem and gauge-group-resolved landscapes: F-theory on $K3 \times K3$ as a test case*, *JHEP* **04** (2014) 050 [arXiv:1401.5908] [INSPIRE].

[9] N. Cabo Bizet, A. Klemm and D. Vieira Lopes, *Landscaping with fluxes and the $E8$ Yukawa point in F-theory*, arXiv:1404.7645 [INSPIRE].

[10] K. Dasgupta, G. Rajesh and S. Sethi, *M theory, orientifolds and G-flux*, *JHEP* **08** (1999) 023 [hep-th/9908088] [INSPIRE].

[11] P.S. Aspinwall and R. Kallosh, *Fixing all moduli for M-theory on $K3 \times K3$*, *JHEP* **10** (2005) 001 [hep-th/0506014] [INSPIRE].

[12] A.P. Braun, A. Collinucci and R. Valandro, *G-flux in F-theory and algebraic cycles*, *Nucl. Phys. B* **856** (2012) 129 [arXiv:1107.5337] [INSPIRE].

[13] H. Movasati and R. Villaflor Loyola, *Periods of linear algebraic cycles*, arXiv:1705.00084.

[14] R. Villaflor Loyola, *Periods of complete intersection algebraic cycles*, arXiv:1812.03964.

[15] A. Giryavets, S. Kachru, P.K. Tripathy and S.P. Trivedi, *Flux compactifications on Calabi-Yau threefolds*, *JHEP* **04** (2004) 003 [hep-th/0312104] [INSPIRE].

[16] F. Denef, M.R. Douglas and B. Florea, *Building a better racetrack*, *JHEP* **06** (2004) 034 [hep-th/0404257] [INSPIRE].

[17] M. Haack and J. Louis, *M theory compactified on Calabi-Yau fourfolds with background flux*, *Phys. Lett. B* **507** (2001) 296 [hep-th/0103068] [INSPIRE].

[18] S.B. Giddings, S. Kachru and J. Polchinski, *Hierarchies from fluxes in string compactifications*, *Phys. Rev. D* **66** (2002) 106006 [hep-th/0105097] [INSPIRE].

[19] F. Denef, *Les Houches lectures on constructing string vacua*, *Les Houches* **87** (2008) 483 [arXiv:0803.1194] [INSPIRE].

[20] A. Strominger, *Special geometry*, *Commun. Math. Phys.* **133** (1990) 163 [INSPIRE].

[21] B.R. Greene, D.R. Morrison and M. Plesser, *Mirror manifolds in higher dimension*, *Commun. Math. Phys.* **173** (1995) 559 [hep-th/9402119] [INSPIRE].

[22] C. Voisin, *Hodge loci*, *Handbook Moduli* **3** (2010) 507.

[23] H. Movasati, *Hodge theory*, International Press, Boston, MA, U.S.A. (2020).

JHEP01(2021)207

[24] Y. Kimura, *Gauge groups and matter spectra in F-theory compactifications on genus-one fibered Calabi-Yau 4-folds without section: hypersurface and double cover constructions*, Adv. Theor. Math. Phys. **22** (2018) 1489 [arXiv:1607.02978] [INSPIRE].

[25] N. Aoki and T. Shioda, *Generators of the Néron-Severi group of a Fermat surface*, in *Arithmetic and geometry*, Birkhäuser, Boston, MA, U.S.A. (1983), pg. 1.

[26] T. Shioda, *On the Picard number of a Fermat surface*, *J. Fac. Sci. Univ. Tokyo* **28** (1982) 725.

[27] N. Aoki, *Some new algebraic cycles on Fermat varieties*, J. Math. Soc. Jpn. **39** (1987) 385.

[28] T. Shioda, *The Hodge conjecture for Fermat varieties*, Math. Annalen **245** (1979) 175.

[29] N.M. Katz, *On the intersection matrix of a hypersurface*, Ann. Sci. École Norm. Sup. **2** (1969) 583.

[30] A. Ogus, *Griffiths transversality in crystalline cohomology*, Ann. Math. **108** (1978) 395.

[31] Z. Ran, *Cycles on Fermat hypersurfaces*, Compos. Math. **42** (1980) 121.

[32] J. Carlson, M. Green, P.A. Griffiths and J. Harris, *Infinitesimal variations of Hodge structure (I)*, Compos. Math. **50** (1983) 109.

[33] C. Voisin, *Hodge theory and complex algebraic geometry II*, Cambridge University Press, Cambridge, U.K. (2003).

[34] H. Movasati, *Gauss-Manin connection in disguise: Noether-Lefschetz and Hodge loci*, arXiv:1411.1766.

[35] B.R. Greene and M.R. Plesser, *Duality in Calabi-Yau moduli space*, Nucl. Phys. B **338** (1990) 15 [INSPIRE].

[36] H. Movasati, *Why should one compute periods of algebraic cycles?*, arXiv:1602.06607.

[37] J. Louis, M. Rummel, R. Valandro and A. Westphal, *Building an explicit de Sitter*, JHEP **10** (2012) 163 [arXiv:1208.3208] [INSPIRE].

[38] M. Cicoli, D. Klevers, S. Krippendorf, C. Mayrhofer, F. Quevedo and R. Valandro, *Explicit de Sitter flux vacua for global string models with chiral matter*, JHEP **05** (2014) 001 [arXiv:1312.0014] [INSPIRE].

[39] E. Aljovin, H. Movasati and R.V. Loyola, *Integral Hodge conjecture for Fermat varieties*, J. Symbol. Comput. **95** (2019) 177 [arXiv:1711.02628].

[40] R. Schimmrigk, *Emergent spacetime from modular motives*, Commun. Math. Phys. **303** (2011) 1 [arXiv:0812.4450] [INSPIRE].

[41] S. Kachru, R. Nally and W. Yang, *Supersymmetric flux compactifications and Calabi-Yau modularity*, arXiv:2001.06022 [INSPIRE].

[42] R. Schimmrigk, *On flux vacua and modularity*, JHEP **09** (2020) 061 [arXiv:2003.01056] [INSPIRE].

[43] R. Livné, *Motivic orthogonal two-dimensional representations of Gal*$(\bar{\mathbb{Q}}/\mathbb{Q})$, Israel J. Math. **92** (1995) 149.

[44] P. Candelas, X. de la Ossa, M. Elmi and D. Van Straten, *A one parameter family of Calabi-Yau manifolds with attractor points of rank two*, JHEP **10** (2020) 202 [arXiv:1912.06146] [INSPIRE].

[45] P.A. Griffiths, *On the periods of certain rational integrals: I*, Ann. Math. **90** (1969) 460.

[46] P.A. Griffiths, *On the periods of certain rational integrals: II*, Ann. Math. **90** (1969) 496.

[47] D. Cox and S. Katz, *Mirror symmetry and algebraic geometry*, Mathematical surveys and monographs, American Mathematical Society, U.S.A. (1999).

[48] P. Candelas, X. de la Ossa and F. Rodriguez-Villegas, *Calabi-Yau manifolds over finite fields.* 1, hep-th/0012233 [INSPIRE].

[49] C. Doran, B. Greene and S. Judes, *Families of quintic Calabi-Yau* 3*-folds with discrete symmetries*, Commun. Math. Phys. **280** (2008) 675 [hep-th/0701206] [INSPIRE].

JHEP01(2021)207