

Contribution of the Darwin operator to non-leptonic decays of heavy quarks

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ABSTRACT: We compute the Darwin operator contribution ($1/m_b^3$ correction) to the width of the inclusive non-leptonic decay of a B meson (B^+ , B_d or B_s), stemming from the quark flavour-changing transition $b \rightarrow q_1 \bar{q}_2 q_3$, where $q_1, q_2 = u, c$ and $q_3 = d, s$. The key ideas of the computation are the local expansion of the quark propagator in the external gluon field including terms with a covariant derivative of the gluon field strength tensor and the standard technique of the Heavy Quark Expansion (HQE). We confirm the previously known expressions of the $1/m_b^3$ contributions to the semi-leptonic decay $b \rightarrow q_1 \ell \bar{\nu}_\ell$, with $\ell = e, \mu, \tau$ and of the $1/m_b^2$ contributions to the non-leptonic modes. We find that this new term can give a sizeable correction of about -4% to the non-leptonic decay width of a B meson. For B_d and B_s mesons this turns out to be the dominant correction to the free b -quark decay, while for the B^+ meson the Darwin term gives the second most important correction — roughly $1/2$ to $1/3$ of the phase space enhanced Pauli interference contribution. Due to the tiny experimental uncertainties in lifetime measurements the incorporation of the Darwin term contribution is crucial for precision tests of the Standard Model.

KEYWORDS: Heavy Quark Physics, Effective Field Theories, Nonperturbative Effects

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1 Introduction

The total decay rate of heavy hadrons can be described by the Heavy Quark Expansion (HQE), whose history goes back to the work of Shifman and Voloshin in the 1980ies [1, 2],¹ as the decay of the free heavy quark plus corrections that are suppressed by inverse powers of the heavy quark mass. Since the b -quark mass is large compared to the typical hadronic scale, the corrections are expected to be small and hence, back in 1986, the following b -hadron lifetime ratios were expected [2]

$$\left. \frac{\tau(B_s)}{\tau(B_d)} \right|_{\text{HQE 1986}} \approx 1, \quad \left. \frac{\tau(B^+)}{\tau(B_d)} \right|_{\text{HQE 1986}} \approx 1.1, \quad \left. \frac{\tau(\Lambda_b)}{\tau(B_d)} \right|_{\text{HQE 1986}} \approx 1, \quad (1.1)$$

which are in good agreement with the current experimental averages obtained by HFLAV (see the webupdate of ref. [4])

$$\left. \frac{\tau(B_s)}{\tau(B_d)} \right|_{\text{HFLAV 2020}} = 0.998 \pm 0.004, \quad \left. \frac{\tau(B^+)}{\tau(B_d)} \right|_{\text{HFLAV 2020}} = 1.076 \pm 0.004, \quad \left. \frac{\tau(\Lambda_b)}{\tau(B_d)} \right|_{\text{HFLAV 2020}} = 0.969 \pm 0.006. \quad (1.2)$$

¹For a more profound history see ref. [3].

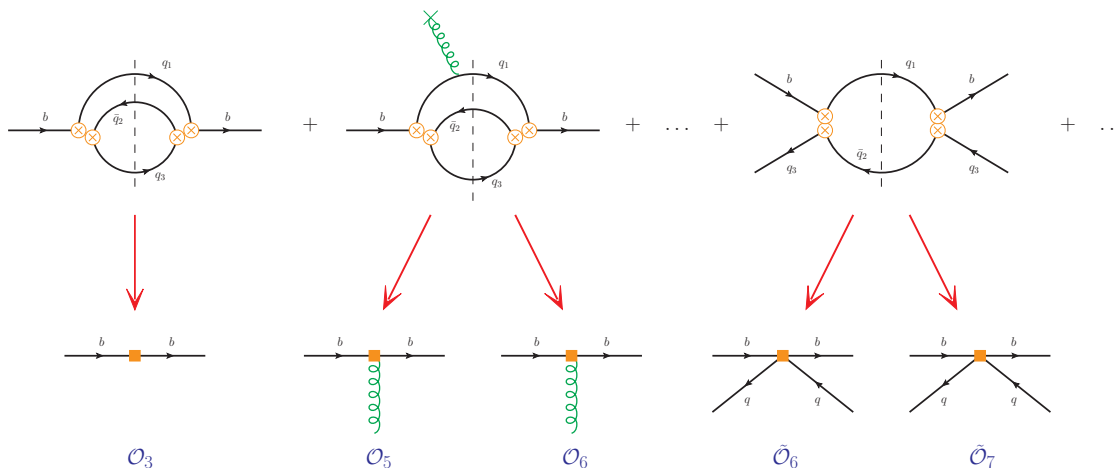


Figure 1. The diagrams describing contributions to the HQE in eq. (1.4). The crossed circles denote the $\Delta B = 1$ operators Q_i of the effective Hamiltonian while the squares denote the local $\Delta B = 0$ operators O_i and \tilde{O}_i . The two-loop and the phase space enhanced one-loop diagrams correspond respectively to the two-quark operators O_i and to the four-quark operators \tilde{O}_i in the HQE.

There has also been considerably progress on the theory side. The total width of a B meson with mass m_B and four-momentum p_B^μ is given by

$$\Gamma(B) = \frac{1}{2m_B} \sum_X \int_{PS} (2\pi)^4 \delta^{(4)}(p_B - p_X) |\langle X(p_X) | \mathcal{H}_{\text{eff}} | B(p_B) \rangle|^2, \quad (1.3)$$

where \mathcal{H}_{eff} represents the effective weak Hamiltonian [5] describing all possible b -quark decays. PS denotes the phase space integration and we have summed over all possible final states X into which the B meson can decay. Eq. (1.3) can be related, via the optical theorem, to the double insertion of the effective Hamiltonian and within the HQE framework one obtains

$$\Gamma(B) = \Gamma_3 + \Gamma_5 \frac{\langle \mathcal{O}_5 \rangle}{m_b^2} + \Gamma_6 \frac{\langle \mathcal{O}_6 \rangle}{m_b^3} + \dots + 16\pi^2 \left[\tilde{\Gamma}_6 \frac{\langle \tilde{\mathcal{O}}_6 \rangle}{m_b^3} + \tilde{\Gamma}_7 \frac{\langle \tilde{\mathcal{O}}_7 \rangle}{m_b^4} + \dots \right], \quad (1.4)$$

with the matrix elements of the $\Delta B = 0$ operators $\langle \mathcal{O}_Y \rangle = \langle B(p_B) | \mathcal{O}_Y | B(p_B) \rangle$. The structure of eq. (1.4) is diagrammatically represented in figure 1. The matrix elements of the operators O_i and \tilde{O}_i , denoting respectively two- and four-quark operators, are suppressed by $i - 3$ powers of the heavy quark mass m_b . The coefficients Γ_i and $\tilde{\Gamma}_i$ encode the corresponding short distance contributions. The leading term Γ_3 describes the free b -quark decay and does not contain any non-perturbative corrections — up to $\mathcal{O}(1/m_b)$ the matrix element of the operator $O_3 = \bar{b}b$ is simply one with the appropriate normalisation. O_5 refers to the dimension-five kinetic and chromo-magnetic operators, proportional to two covariant derivatives of the b -quark field. O_6 includes the dimension-six Darwin and spin-orbit operators with three covariant derivatives. So far the dependence on the spectator quark is only due to different values of the matrix elements and the coefficients $\Gamma_{3,5,6}$ are independent of the quark content of the B meson. The four-quark operators are phase-space enhanced (as indicated by the factor $16\pi^2$) and first arise at order $1/m_b^3$. The

possible topologies, specifically weak annihilation (WA), Pauli interference (PI) and weak exchange (WE), see figure 4, imply that the short distance coefficients are now dependent on the spectator quark. The dimension-seven operators $\tilde{\mathcal{O}}_7$ contain one covariant derivative compared to $\tilde{\mathcal{O}}_6$. Due to the larger phase space, it is typically expected that $\tilde{\mathcal{O}}_6$ gives the dominant contribution to the lifetime ratios [1–3, 6, 7].

Currently Γ_3 is known at NLO-QCD [8–15] for non-leptonic decays. NNLO-QCD corrections have been computed for semi-leptonic decays [16–25] and for non-leptonic decays the massless case was determined in full QCD (i.e. no effective Hamiltonian was used) in ref. [26]. Γ_5 was determined at LO-QCD for both semi-leptonic and non-leptonic decays [27–30] and we confirm these results. For the semi-leptonic modes even NLO-QCD corrections are available [31–33]. Γ_6 was first computed at LO-QCD in ref. [34] and recently the NLO-QCD corrections were determined in ref. [35], both for the semi-leptonic case only. $\tilde{\Gamma}_6$ is known at NLO-QCD for B -meson lifetimes [36, 37] and D meson lifetimes [38], while $\tilde{\Gamma}_7$ is only known at LO-QCD [39].

This work presents the first determination of Γ_6 for non-leptonic decays. An interesting subtlety of the computation is mixing between four- and two-quark operators. Namely, at dimension-six the renormalised one-loop matrix elements of the operators $\tilde{\mathcal{O}}_6$ contribute to the coefficient of the Darwin operator through the diagram in figure 5, ensuring the cancellation of the infrared (IR) divergences that otherwise would appear in Γ_6 . This feature has been intensively discussed for semi-leptonic decays [40–43] — also under the name of “intrinsic charm”, see e.g. refs. [44, 45].

Finally, for numerical analysis, the values of the non-perturbative matrix elements $\langle \mathcal{O}_Y \rangle$ are needed. $\langle \mathcal{O}_5 \rangle$ and $\langle \mathcal{O}_6 \rangle$ can be extracted from fits to the semi-leptonic spectrum for the case of B_d and B^+ mesons, see e.g. ref. [46]. In the literature one can also find lattice determinations [47–51] and sum rule estimates [52–54] for these parameters. For the $SU(3)_F$ violating ratios $\langle B_s | \mathcal{O}_{5,6} | B_s \rangle / \langle B_d | \mathcal{O}_{5,6} | B_d \rangle$ one can use the theory estimates from ref. [55]. The matrix elements of the four-quark operators $\langle \tilde{\mathcal{O}}_6 \rangle$ have been determined by Heavy Quark Effective Theory (HQET) sum rules [56]. Violations of $SU(3)_F$ are expected to yield visible effects and a calculation of these corrections with HQET sum rules — following ref. [57] — is currently been performed [58]. Corresponding lattice results for the matrix elements of the four-quark operators would be highly desirable.

Taking all the currently known contributions into account we arrive at significant improvements compared to the pioneering work in 1986, and the 2019 status of lifetime predictions reads [3, 56]

$$\left. \frac{\tau(B_s)}{\tau(B_d)} \right|_{\text{HQE 2019}} = 1.0006 \pm 0.0025, \quad \left. \frac{\tau(B^+)}{\tau(B_d)} \right|_{\text{HQE 2019}} = 1.082^{+0.022}_{-0.026}, \quad \left. \frac{\tau(\Lambda_b)}{\tau(B_d)} \right|_{\text{HQE 2019}} = 0.935 \pm 0.054, \quad (1.5)$$

which constitutes an impressive confirmation of the validity of the HQE.

The main motivations for this work are:

- In the case of the $\tau(B_s)/\tau(B_d)$ lifetime ratio several very pronounced cancellations are arising [3, 56, 59–61] among the contributions of $\tilde{\mathcal{O}}_6$ and \mathcal{O}_5 , which could in

principle make this ratio sensitive to the contribution of \mathcal{O}_6 — if $SU(3)_F$ violating corrections are large for this term, as indicated by ref. [55].

- It was found (see e.g. refs. [34, 41, 55, 62–64]) that the $1/m_b^3$ correction in semileptonic inclusive decays $B \rightarrow X_c \ell \bar{\nu}_\ell$ are of a similar size as the $1/m_b^2$ terms due to enhanced Wilson coefficients. This could lead to a visible effect in $\tau(B_s)/\tau(B_d)$, in particular if all other contributions are cancelling to a high degree.
- As indicated in eq. (1.2) the experimental precision achieved so far is very high, enabling thus precision tests and an even higher precision seems to be achievable, see e.g. the two most recent results from LHCb [65] and ATLAS [66], which interestingly differ significantly. The theory precision should of course cope with these experimental advancements.
- According to the above arguments the lifetime ratio $\tau(B_s)/\tau(B_d)$ might thus provide a unique opportunity to test directly higher orders in the HQE that are otherwise just noise compared to the leading, numerically dominant contributions. In that respect this might also increase our insights on the assumptions of quark hadron duality, severely questioned around 20 years ago — mostly due to a very low experimental value of the Λ_b lifetime, which seemed to be in severe conflict with the HQE expectation from eq. (1.1): the 2002 HFAG average gives e.g. $\tau(\Lambda_b)/\tau(B_d) = 0.798(34)$, more than 5σ away from the current experimental value given by eq. (1.2). This signal for a violation of quark hadron duality was a clear false alarm. Moreover, the first measurement of the decay rate difference of neutral B_s mesons, $\Delta\Gamma_s$, has excluded large duality violating effects, see e.g. the discussion in refs. [67, 68]. Smaller effects of duality violation can still appear and their potential size can be constrained by comparing experiment and theory for as many HQE observable as possible with a high precision, see e.g. ref. [69].
- Assuming the validity of quark hadron duality the $\tau(B_s)/\tau(B_d)$ lifetime ratio can be used to constrain invisible or hardly detectable B -meson decays, like $B_s \rightarrow \tau\tau$, at the per mille level, see e.g. refs. [70, 71] — see also ref. [72] for an alternative way to constrain the potential size of the $bs\tau\tau$ couplings.

We will not present updated lifetime ratio predictions in this work, but we will postpone a new numerical study until the $SU(3)_F$ violation ratio of the Bag parameters is available, see ref. [58].

The paper is structured as follows: in section 2 we outline the main ingredients of the calculation: in section 2.1 we explain how the double insertion of the effective Hamiltonian and the subsequent expansion in the external soft gluon field are performed. Section 2.2 is devoted to describe mixing between two- and four-quark operators at order $1/m_b^3$, which guarantees the cancellation of the IR divergences. Our results are presented in section 3 and we conclude in section 4. The expansion of the quark propagator in the external gluon field is discussed in more detail in appendix A, supplement material to section 2.2 and

section 3 can be found in appendix B and appendix C, respectively, and for completeness we also present the non-leptonic results for Γ_5 in appendix D.

2 Outline of the calculation

2.1 Contribution of two-quark operators up to order $1/m_b^3$

According to the optical theorem, the total width for the inclusive non-leptonic decay of a B meson induced at the quark level by the flavour-changing transition $b \rightarrow q_1 \bar{q}_2 q_3$ (with $q_1, q_2 = u, c$ and $q_3 = d, s$), can be computed from the discontinuity of the forward scattering matrix element:

$$\Gamma_{\text{NL}}(B) = \frac{1}{2m_B} \text{Im} \langle B(p_B) | i \int d^4x T \{ \mathcal{L}_{\text{eff}}(x), \mathcal{L}_{\text{eff}}(0) \} | B(p_B) \rangle. \quad (2.1)$$

The effective Lagrangian $\mathcal{L}_{\text{eff}}(x)$ reads

$$\mathcal{L}_{\text{eff}}(x) = -\frac{4G_F}{\sqrt{2}} V_{q_1 b}^* V_{q_2 q_3} [C_1 Q_1(x) + C_2 Q_2(x)] + \text{h.c.}, \quad (2.2)$$

where G_F is the Fermi constant, $V_{qq'}$ are the elements of the Cabibbo-Kobayashi-Maskawa (CKM) matrix [73, 74], $C_{1,2}(\mu_1)$ denote the Wilson coefficients at the renormalisation scale $\mu_1 \sim m_b$ and $Q_{1,2}$ are the effective $\Delta B = 1$ four-quark operators²

$$Q_1 = (\bar{q}_1^i \Gamma_\mu b^i) (\bar{q}_3^j \Gamma^\mu q_2^j), \quad (2.3)$$

$$Q_2 = (\bar{q}_1^i \Gamma_\mu b^j) (\bar{q}_3^j \Gamma^\mu q_2^i), \quad (2.4)$$

with i, j standing for colour indices and $\Gamma_\mu \equiv \gamma_\mu(1 - \gamma_5)/2$. In our notation, Q_1 is the colour singlet operator and Q_2 the colour rearranged one, opposite to e.g. the notation in ref. [5]. Three combinations of operators enter the decay rate in eq. (2.1): $Q_1 \otimes Q_1$, $Q_2 \otimes Q_2$ and $Q_1 \otimes Q_2$. We use the completeness property of the colour matrices t_{ij}^a ,³

$$t_{ij}^a t_{lm}^a = \frac{1}{2} \left(\delta_{im} \delta_{jl} - \frac{1}{N_c} \delta_{ij} \delta_{lm} \right), \quad (2.5)$$

to rewrite

$$Q_1 \otimes Q_2 = \frac{1}{N_c} (Q_1 \otimes Q_1) + 2 (Q_1 \otimes T), \quad (2.6)$$

where the colour octet operator T is given by

$$T = (\bar{q}_1^i \Gamma_\mu t_{ij}^a b^j) (\bar{q}_3^l \Gamma^\mu t_{lm}^a q_2^m). \quad (2.7)$$

Considering only the contribution of two-quark operators, the decay width takes the form

$$\Gamma_{\text{NL}}^{(2q)}(B) = \left[C_1^2 \Gamma_{11}^{(2q)} + 2 C_1 C_2 \left(\frac{1}{N_c} \Gamma_{11}^{(2q)} + 2 \Gamma_{1T}^{(2q)} \right) + C_2^2 \Gamma_{22}^{(2q)} \right], \quad (2.8)$$

²We consider only current-current operators with large Wilson coefficients.

³The colour matrices satisfy the following relations: $\text{Tr}[t^a] = 0$ and $\text{Tr}[t^a, t^b] = (1/2) \delta^{ab}$.

where

$$\Gamma_{11(1T)}^{(2q)} = -\frac{4G_F^2|V_{q_1b}|^2|V_{q_2q_3}|^2}{m_B} \text{Im} \langle B(p_B) | i \int d^4x \bar{b}(0) \Gamma_\mu(t^a) iS^{(q_1)}(0,x) \Gamma_\nu b(x) \times \text{Tr} \left[\Gamma^\mu(t^a) iS^{(q_3)}(0,x) \Gamma^\nu iS^{(q_2)}(x,0) \right] | B(p_B) \rangle + (x \leftrightarrow 0). \quad (2.9)$$

Note that in the case of $\Gamma_{1T}^{(2q)}$ the two colour matrices t^a appear on the r.h.s. of eq. (2.9). The corresponding expression for $\Gamma_{22}^{(2q)}$ is obtained from that of $\Gamma_{11}^{(2q)}$ by replacing $q_1 \leftrightarrow q_3$ as it follows by Fierz-transforming the operator Q_2 given in eq. (2.4). To compute dimension-six contributions we need to expand first each of the quark propagators up to one covariant derivative of the gluon field strength tensor, respectively defined as $iD_\mu = i\partial_\mu + A_\mu(x)$,⁴ and $G_{\mu\nu} = -i[iD_\mu, iD_\nu]$. In appendix A we derive the Fourier transform of the quark propagator in the soft external gluon field

$$S(x,0) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} S(k), \quad (2.10)$$

where

$$\begin{aligned} S(k) = & \frac{\not{k} + m}{k^2 - m^2} + \frac{1}{2(k^2 - m^2)^2} \left(-mG_{\rho\mu}\sigma^{\rho\mu} + 2\tilde{G}_{\rho\mu} k^\rho \gamma^\mu \gamma^5 \right) \\ & + \frac{2}{3} \frac{D_\rho G^{\rho\mu}}{(k^2 - m^2)^2} \left(\gamma_\mu - \frac{(\not{k} + 2m) k_\mu}{k^2 - m^2} \right) - \frac{2}{3} \frac{D_\nu G_{\rho\mu} k^\nu k^\rho \gamma^\mu}{(k^2 - m^2)^3} \\ & + \frac{2}{3} \frac{m D_\nu G_{\rho\mu}}{(k^2 - m^2)^3} (\gamma^\rho \gamma^\mu k^\nu - \gamma^\mu \gamma^\nu k^\rho) + 2i \frac{D_\nu \tilde{G}_{\rho\mu} k^\nu k^\rho \gamma^\mu \gamma^5}{(k^2 - m^2)^3} + \dots \quad (2.11) \end{aligned}$$

In the previous equation, $\tilde{G}_{\rho\mu} = (1/2)\epsilon_{\rho\mu\sigma\eta} G^{\sigma\eta}$ denotes the dual field strength tensor, with $\epsilon_{\rho\mu\sigma\eta}$ being the Levi-Civita tensor and $D_\rho G_{\mu\nu} = -[iD_\rho, [iD_\mu, iD_\nu]]$. From eq. (2.11) it follows that the quark propagator can be split up according to its colour structure as

$$S_{ij}(k) = S^{(0)}(k) \delta_{ij} + S^{(1)a}(k) t_{ij}^a, \quad (2.12)$$

where $S^{(0)}(k)$ denotes the free quark propagator and $S^{(1)a}(k)$ absorbs contributions with one gluon field (for more details see appendix A, also the original refs. [28, 29]). This representation allows a straightforward treatment of colour in eq. (2.9). In case of $Q_1 \otimes Q_1$ contribution, the colour flow factorises between the (bq_1) -quark line and the (q_2q_3) -loop, therefore one can only expand the propagator of q_1 up to terms linear in t^a , see l.h.s. of figure 2. For the $Q_2 \otimes Q_2$ combination the trace over colour indices involves the q_1 - and q_2 -quark propagators and only the gluon radiation from q_3 is non-vanishing, see r.h.s. of figure 2. Finally, in the case of the $Q_1 \otimes T$ contribution, the colour flow forces the gluon to be radiated off the (q_2q_3) -loop. Any interference term would be $\mathcal{O}(1/m_b^4)$, and it is thus sufficient to expand independently each of the two quark propagators in the loop (figure 3).

Using translation invariance of the quark field to write $b(x) = e^{-ipx} b(0)$, with p^μ being the b -quark four-momentum, and after performing x - and momentum integration,

⁴Note that the coupling constant g_s is absorbed in the definition of $A_\mu = A_\mu^a t^a$.

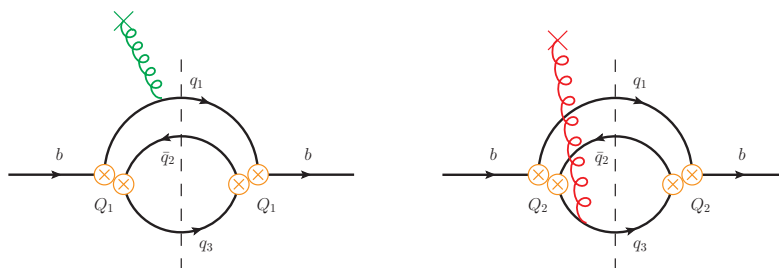


Figure 2. The two-loop diagrams describing power corrections up to dimension-six from $Q_1 \otimes Q_1$ (left) and $Q_2 \otimes Q_2$ (right) contributions.

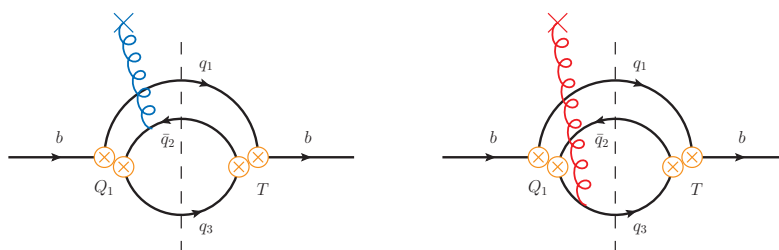


Figure 3. The two-loop diagrams describing power corrections up to dimension-six from $Q_1 \otimes T$ contribution.

we obtain two-loop tensor integrals of possible rank $r = 1, \dots, 4$. These are decomposed in terms of rank r tensors built with the metric tensor $g^{\mu\nu}$ and the external momentum p^μ . The corresponding coefficients represent scalar integrals of the type

$$\int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} \frac{f(p, k_1, k_2)}{[k_1^2 - m_1^2]^{n_1} [k_2^2 - m_2^2]^{n_2} [(p - k_1 - k_2)^2 - m_3^2]^{n_3}}, \quad (2.13)$$

where $f(p, k_1, k_2)$ is a function of all the possible scalar products. The integrals in eq. (2.13) are reduced by means of the integration-by-parts (IBP) technique to a linear combination of master integrals. The IBP reduction is implemented using the Mathematica package *LiteRed* [75, 76]. The discontinuity of the master integrals can be straightforwardly obtained from the imaginary part of the scalar sunset diagram $S(s; m_1, m_2, m_3)$, defined by eq. (2.13) with $f = 1$ and $n_1 = n_2 = n_3 = 1$ and given by,⁵ see ref. [77]

$$\text{Im } S(s; m_1, m_2, m_3) = \frac{1}{256\pi^3} \int_{(m_2+m_3)^2}^{(\sqrt{s}-m_1)^2} dt \frac{\sqrt{\lambda(t, m_2^2, m_3^2) \lambda(s, t, m_1^2)}}{t s}. \quad (2.14)$$

In the previous equation, $s = p^2 \geq (m_1 + m_2 + m_3)^2$ and $\lambda(a, b, c) \equiv a^2 + b^2 + c^2 - 2(ab + bc + ac)$ is the Källén function. One can easily compute analytically the integral in eq. (2.14) for two non-vanishing masses, while in the case of three non-vanishing masses the complexity highly increases and the solution involves elliptic functions, see e.g. refs. [78–81]. We emphasize that we set $d = 4$ from the beginning since the discontinuity of the

⁵For instance, the discontinuity of the master integral with $f = 1$, $n_1 = 2$ and $n_2 = n_3 = 1$ can be obtained by differentiating eq. (2.14) with respect to m_1^2 , etc.

diagrams at LO-QCD does not develop any ultraviolet divergences. On the other side, the gluon emission off a light quark propagator ($q = u, d$ or s) gives rise, at dimension-six, to infrared logarithmic divergences of the type $\log(m_q^2/m_b^2)$ which are cancelled by the renormalised one-loop matrix element of the corresponding four-quark operators, as shown in detail in section 2.2. Here we limit ourselves to state that for the computation of most of the diagrams of interest it is always possible to set one mass to zero and obtain an analytic expression for all the master integrals. This is not the case only for the gluon emission from the s quark in the $b \rightarrow c\bar{c}s$ transition, where we need to keep all masses finite in order to regularise the infrared divergence.

After computing the two-loop integrals we are left with the following matrix elements

$$\langle B(p_B) | \bar{b}_v(0) \mathcal{F}(p) b_v(0) | B(p_B) \rangle, \quad (2.15)$$

$$\langle B(p_B) | \bar{b}_v(0) \mathcal{F}_{\mu\nu}(p) (iD^\mu)(iD^\nu) b_v(0) | B(p_B) \rangle, \quad (2.16)$$

$$\langle B(p_B) | \bar{b}_v(0) \mathcal{F}_{\mu\nu\rho}(p) (iD^\mu)(iD^\nu)(iD^\rho) b_v(0) | B(p_B) \rangle, \quad (2.17)$$

where $\mathcal{F}(p)$, $\mathcal{F}_{\mu\nu}(p)$ and $\mathcal{F}_{\mu\nu\rho}(p)$ are functions of the quark masses and of the external momentum p^μ . We find that at order $1/m_b^2$ only the expansion of the anti-quark propagator contributes (in agreement with refs. [27–30]) and therefore the matrix element of dimension-five, eq. (2.16), vanishes in the case of both $Q_1 \otimes Q_1$ and $Q_2 \otimes Q_2$ contributions but not for $Q_1 \otimes Q_2$. On the other hand, the matrix elements of dimension-three, eq. (2.15), and of dimension-six, eq. (2.17), are non-vanishing for all three combinations of $\Delta B = 1$ operators. Following the standard technique of the HQE we decompose the b -quark momentum as

$$p^\mu = m_b v^\mu + iD^\mu, \quad (2.18)$$

where v^μ represents the four-velocity of the B meson and D^μ accounts for the soft interaction with the spectator quark. We have used the phase redefinition

$$b(x) = e^{-im_b v \cdot x} b_v(x), \quad (2.19)$$

to remove the large fraction of the b -field momentum, at $x = 0$ we trivially get $b(0) = b_v(0)$. We then expand eqs. (2.15)–(2.17) in the small quantity D^μ/m_b . Note that in the matrix elements with three covariant derivatives we can safely set $p^2 = m_b^2$ neglecting corrections $\mathcal{O}(1/m_b^4)$. The order of the covariant derivatives is fixed by

$$p^{\mu_1} p^{\mu_2} \dots p^{\mu_n} = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} p^{\sigma(\mu_1)} p^{\sigma(\mu_2)} \dots p^{\sigma(\mu_n)}, \quad (2.20)$$

where \mathcal{S}_n is the symmetric group of permutation of n elements. This leads to a systematic expansion of eq. (2.9), schematically

$$a \langle \bar{b}_v b_v \rangle + b_\mu \langle \bar{b}_v (iD^\mu) b_v \rangle + c_{\mu\nu} \langle \bar{b}_v (iD^\mu)(iD^\nu) b_v \rangle + d_{\mu\nu\rho} \langle \bar{b}_v (iD^\mu)(iD^\nu)(iD^\rho) b_v \rangle + \dots, \quad (2.21)$$

where $a, b_\mu, c_{\mu\nu}, d_{\mu\nu\rho}$ are now only functions of the quark masses and of the four-velocity v^μ .

Finally, the matrix elements in eq. (2.21) admit a series expansion in powers of $1/m_b$ given explicitly in ref. [62],⁶ from which we can readily obtain the coefficients of μ_π^2, μ_G^2 and of ρ_D^3, ρ_{LS}^3 , which are defined as

$$\begin{aligned} 2m_B \mu_\pi^2(B) &= -\langle B(p_B) | \bar{b}_v(iD_\mu)(iD^\mu)b_v | B(p_B) \rangle, \\ 2m_B \mu_G^2(B) &= \langle B(p_B) | \bar{b}_v(iD_\mu)(iD_\nu)(-i\sigma^{\mu\nu})b_v | B(p_B) \rangle, \end{aligned} \quad (2.22)$$

$$\begin{aligned} 2m_B \rho_D^3(B) &= \langle B(p_B) | \bar{b}_v(iD_\mu)(iv \cdot D)(iD^\mu)b_v | B(p_B) \rangle, \\ 2m_B \rho_{LS}^3(B) &= \langle B(p_B) | \bar{b}_v(iD_\mu)(iv \cdot D)(iD_\nu)(-i\sigma^{\mu\nu})b_v | B(p_B) \rangle, \end{aligned} \quad (2.23)$$

with $\sigma_{\mu\nu} = (i/2)[\gamma_\mu, \gamma_\nu]$. As already stressed, the numerical values of the non-perturbative parameters above depend on the spectator quark in the B meson.

We then arrive at the following form of eq. (2.8)

$$\begin{aligned} \Gamma_{\text{NL}}^{(2q)}(B) &= \Gamma_0 \left[\left(3C_1^2 + 2C_1C_2 + 3C_2^2 \right) C_0^{(q_1\bar{q}_2q_3)} \left(1 - \frac{\mu_\pi^2(B)}{2m_b^2} \right) \right. \\ &\quad + \left(3C_1^2 C_{G,11}^{(q_1\bar{q}_2q_3)} + 2C_1C_2 C_{G,12}^{(q_1\bar{q}_2q_3)} + 3C_2^2 C_{G,22}^{(q_1\bar{q}_2q_3)} \right) \frac{\mu_G^2(B)}{m_b^2} \\ &\quad \left. + \left(3C_1^2 C_{D,11}^{(q_1\bar{q}_2q_3)} + 2C_1C_2 C_{D,12}^{(q_1\bar{q}_2q_3)} + 3C_2^2 C_{D,22}^{(q_1\bar{q}_2q_3)} \right) \frac{\rho_D^3(B)}{m_b^3} \right], \end{aligned} \quad (2.24)$$

where

$$\Gamma_0 = \frac{G_F^2 m_b^5}{192 \pi^3} |V_{q_1 b}|^2 |V_{q_2 q_3}|^2. \quad (2.25)$$

In eq. (2.24), $C_0^{(q_1\bar{q}_2q_3)}$ refers to the partonic-level coefficient, while $C_{G,nm}^{(q_1\bar{q}_2q_3)}$ and $C_{D,nm}^{(q_1\bar{q}_2q_3)}$, $nm = \{11, 12, 22\}$, denote the coefficients of the chromo-magnetic and of the Darwin operators, respectively. The upper index $(q_1\bar{q}_2q_3)$ indicates the decay mode, e.g. $(c\bar{u}d)$ corresponds to the $b \rightarrow c\bar{u}d$ transition. Since we neglect the masses of the up, down and strange quark, all the coefficients in eq. (2.24) are functions of at most one dimensionless mass parameter $\rho = m_c^2/m_b^2$, apart from $C_{D,mn}^{(q_1\bar{q}_2q_3)}$, where the dependence on the light quark mass $m_q = m_{u,d,s}$ is still present in the form of divergent logarithms $\log(m_q^2/m_b^2)$. The cancellation of these terms is discussed in the next subsection. The complete LO-QCD expressions for $C_0^{(q_1\bar{q}_2q_3)}$ and $C_{G,nm}^{(q_1\bar{q}_2q_3)}$ can be found in appendix D. We point out that the adopted definition for the non-perturbative parameters in eqs. (2.22), (2.23) implies that the coefficient of ρ_{LS}^3 is found to vanish for all the $\Delta B = 1$ operator combinations.

2.2 Role of the four-quark operators

At order $1/m_b^3$ the gluon radiation from a light quark leads to IR divergences, namely

$$C_{D,nm}^{(q_1\bar{q}_2q_3)} = \mathcal{R}_{nm}^{(q_1\bar{q}_2q_3)} + \mathcal{D}_{nm}^{(q_1\bar{q}_2q_3)} \left(\log \left(\frac{m_q^2}{m_b^2} \right) \right), \quad (2.26)$$

⁶For reference we quote the expansion of the dimension-three matrix element up to order $1/m_b^3$, $\langle B(p_B) | \bar{b}_v b_v | B(p_B) \rangle = 2m_B \left(1 - \frac{\mu_\pi^2(B) - \mu_G^2(B)}{2m_b^2} \right)$.

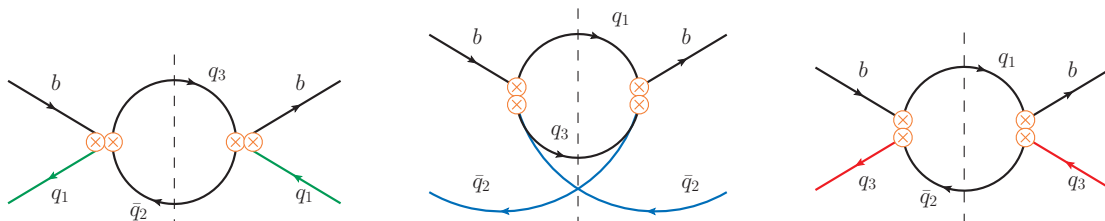


Figure 4. One-loop diagrams corresponding, from left to right, to the WA, PI and WE topology.

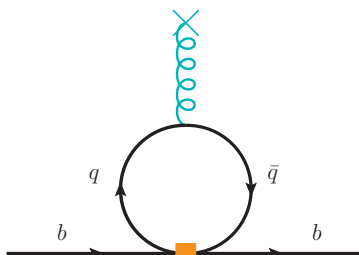


Figure 5. Diagram contributing to mixing between four- and two-quark operators at dimension-six.

where $\mathcal{R}_{nm}^{(q_1\bar{q}_2q_3)}$ are finite functions and $\mathcal{D}_{nm}^{(q_1\bar{q}_2q_3)}\left(\log(m_q^2/m_b^2)\right)$ absorb the contribution of the divergent logarithms, the latter are listed in appendix B. As discussed already in ref. [40], logarithmic infrared divergences signal mixing between operators of the same dimension. To see this in more detail, we start again from eq. (2.1),⁷

$$\Gamma_{\text{NL}}(B) = \frac{1}{2m_B} \langle B(p_B) | \text{Im} \hat{T} | B(p_B) \rangle, \quad (2.27)$$

with the transition operator \hat{T} being the time-ordered product of the double insertion of the effective Lagrangian. The expansion in inverse powers of m_b allows to express \hat{T} in terms of new local effective operators. At dimension-six one has⁸:

$$\text{Im} \hat{T}^{(d=6)} = \Gamma_0 C(\mu_1) \left[\mathcal{C}_{\rho_D}(\mu_1, \mu_0) \frac{\mathcal{O}_{\rho_D}(\mu_0)}{m_b^3} + \sum_{q=u,d,s} \sum_{i=1}^4 \mathcal{C}_{6,i}^{(q)}(\mu_1, \mu_0) \frac{\tilde{\mathcal{O}}_{6,i}^{(q)}(\mu_0)}{m_b^3} \right], \quad (2.28)$$

where $C(\mu_1) = \{3C_1^2, 2C_1C_2, 3C_2^2\}$ and μ_1 is the renormalisation scale at which the $\Delta B = 1$ Wilson coefficients of the weak Hamiltonian are determined. This scale arises also in loop corrections to the diagrams given in the upper line of figure 1. Up to the calculated order in the strong coupling the μ_1 dependence will cancel among these two sources. The second scale μ_0 ($\Lambda_{\text{QCD}} \ll \mu_0 \leq \mu_1 \sim m_b$) indicates the new factorisation scale of the $\Delta B = 0$ operators emerging in the HQE. The μ_0 dependence will cancel among loop corrections to the diagrams given in the lower line of figure 1 and the scale dependence of the $\Delta B = 0$ operators.

⁷For brevity we omit the indices $(q_1\bar{q}_2q_3)$ and (nm) , although they must be always understood.

⁸In principle the sum in eq. (2.28) includes also the c -quark but for $m_b \sim m_c \gg \Lambda_{\text{QCD}}$ the four-quark operators $\tilde{\mathcal{O}}_{6,i}^{(c)}$ are not relevant for the further discussion, see for more details ref. [44].

The l.h.s. of eq. (2.28) includes the contributions given by the two-loop diagrams in figure 2 and figure 3, which give rise directly to the Darwin operator \mathcal{O}_{ρ_D} ,⁹

$$\mathcal{O}_{\rho_D} = \bar{b}_v(iD_\mu)(iv \cdot D)(iD^\mu)b_v, \quad (2.29)$$

with the corresponding coefficient $\mathcal{C}_D(\mu_1)$ — see eq. (2.24). Since we are only working at LO-QCD, we do not find any explicit μ_1 -dependence at this stage. The l.h.s. of eq. (2.28) receives also contributions by the one-loop diagrams depicted in figure 4, corresponding to the weak-annihilation, Pauli interference and weak-exchange topologies. The coefficients $\mathcal{C}_{6,i}^{(q)}(\mu_1, \mu_0)$ do not develop any divergences at LO-QCD [6, 7] hence there is no explicit μ_1 - and μ_0 -dependence; it is however present in the NLO-QCD corrections determined in refs. [36, 37, 82]. We refer to appendix B for explicit expressions at LO-QCD. Integrating out these one-loop diagrams leads to the following $\Delta B = 0$ four-quark operators:¹⁰

$$\tilde{\mathcal{O}}_{6,1}^{(q)} = (\bar{b}_v^i \gamma_\mu (1 - \gamma_5) q^i)(\bar{q}^j \gamma^\mu (1 - \gamma_5) b_v^j), \quad \tilde{\mathcal{O}}_{6,2}^{(q)} = (\bar{b}_v^i \not{\psi} (1 - \gamma_5) q^i)(\bar{q}^j \not{\psi} (1 - \gamma_5) b_v^j), \quad (2.30)$$

$$\tilde{\mathcal{O}}_{6,3}^{(q)} = (\bar{b}_v^i \gamma_\mu (1 - \gamma_5) q^j)(\bar{q}^j \gamma^\mu (1 - \gamma_5) b_v^i), \quad \tilde{\mathcal{O}}_{6,4}^{(q)} = (\bar{b}_v^i \not{\psi} (1 - \gamma_5) q^j)(\bar{q}^j \not{\psi} (1 - \gamma_5) b_v^i). \quad (2.31)$$

The four-quark and the Darwin operators mix under operator renormalisation already at LO-QCD, through the diagram in figure 5. In dimensional regularisation with $d = 4 - 2\epsilon$ a factor of $\mu^{2\epsilon}$ must be included in order to compensate the mismatch in the dimension of $\tilde{\mathcal{O}}_{6,i}^{(q)}$ and \mathcal{O}_{ρ_D} , leading to explicitly μ -dependent renormalisation constants and thus a non-vanishing anomalous dimension matrix. The renormalised one-loop matrix element of $\tilde{\mathcal{O}}_{6,i}^{(q)}$ in the presence of a soft background gluon field, in NDR and in the $\overline{\text{MS}}$ scheme reads:

$$\langle \tilde{\mathcal{O}}_{6,i}^{(q)} \rangle^{\text{ren}} = \frac{a_i}{12\pi^2} \left[\log \left(\frac{\mu^2}{m_q^2} \right) + b_i \right] \langle \mathcal{O}_{\rho_D} \rangle + \mathcal{O} \left(\frac{1}{m_b} \right), \quad (2.32)$$

with $a_1 = 2$, $a_2 = -1$, $a_3 = a_4 = 0$ and $b_1 = -1$ and $b_2 = 0$. Note that the presence of the constant term b_i depends on the choice of the operator basis in eqs. (2.30), (2.31), see for instance refs. [42, 83]. At the matching scale $\mu = m_b$, we obtain

$$\mathcal{C}_{\rho_D}(\mu_1, m_b) \langle \mathcal{O}_{\rho_D}(m_b) \rangle = \mathcal{C}_D(\mu_1) \langle \mathcal{O}_{\rho_D}(m_b) \rangle - \sum_{q=u,d,s} \sum_{i=1}^2 \mathcal{C}_{6,i}^{(q)}(\mu_1, m_b) \langle \tilde{\mathcal{O}}_{6,i}^{(q)}(m_b) \rangle^{\text{ren}}, \quad (2.33)$$

where the coefficient \mathcal{C}_D has the divergent logarithmic dependence shown in eq. (2.26). From it follows that on the r.h.s. of eq. (2.33) all the logarithms $\log(m_q^2/m_b^2)$ cancel exactly, leaving $\mathcal{C}_{\rho_D}(\mu_1, m_b)$ free of any IR divergences. Finally one can solve the renormalisation

⁹We do not take into account the spin-orbit operator $\mathcal{O}_{\rho_{LS}} = \bar{b}_v(iD_\mu)(iv \cdot D)(iD_\nu)(-i\sigma^{\mu\nu})b_v$ since its contribution is vanishing within the adopted convention, see eq. (2.23).

¹⁰In the literature typically colour singlet and colour octet operators are used, see e.g. refs. [3, 6, 7, 56]; however, for our purposes it turns out to be advantageous to use instead the colour singlet and the colour rearranged operators. The trivial transformation between the two bases is given by eq. (2.5).

group equations to run $\mathcal{C}_{\rho_D}(\mu_1, m_b)$ down to the scale $m_b \geq \mu_0 \gg \Lambda_{\text{QCD}}$:

$$\mathcal{C}_{\rho_D}(\mu_1, \mu_0) = \mathcal{C}_{\rho_D}(\mu_1, m_b) - \frac{1}{12\pi^2} \log\left(\frac{\mu_0^2}{m_b^2}\right) \sum_{q=u,d,s} \left[2\mathcal{C}_{6,1}^{(q)}(\mu_1, m_b) - \mathcal{C}_{6,2}^{(q)}(\mu_1, m_b) \right]. \quad (2.34)$$

The coefficients $\mathcal{C}_{\rho_D}(\mu_1, \mu_0)$ are now IR-finite but scale dependent. This μ_0 -dependence cancels the corresponding one due to the running of the matrix element of the four-quark operators starting already at LO-QCD via the diagram in figure 5:

$$\langle \tilde{\mathcal{O}}_{6,i}^{(q)}(\mu_0) \rangle = \langle \tilde{\mathcal{O}}_{6,i}^{(q)}(m_b) \rangle + \frac{a_i}{12\pi^2} \log\left(\frac{\mu_0^2}{m_b^2}\right) \langle \mathcal{O}_{\rho_D}(m_b) \rangle. \quad (2.35)$$

Our results are presented in the next section. We conclude with the remaining dimension-six contribution to $\Gamma_{\text{NL}}(B)$ (see appendix C for the expression of the coefficients at LO-QCD):

$$\Gamma_{\text{NL}}^{(4q)}(B) = \frac{\Gamma_0}{2m_B} C(\mu_1) \sum_{q=u,d,s} \sum_{i=1}^4 \mathcal{C}_{6,i}^{(q)}(\mu_1, \mu_0) \frac{\langle \tilde{\mathcal{O}}_{6,i}^{(q)}(\mu_0) \rangle}{m_b^3}, \quad (2.36)$$

where the matrix element of the four-quark operators can be parametrised as

$$\langle B_{q'} | \tilde{\mathcal{O}}_{6,i}^{(q)} | B_{q'} \rangle = A_i m_B^2 f_B^2 \left(\mathcal{B}_i^{(q)}(B) \delta_{qq'} + \tau_i^{(q)}(B) \right), \quad q' = u, d, s, \quad (2.37)$$

with $A_1 = A_3 = 1$, $A_2 = A_4 = (m_B/(m_b + m_q))^2$ and f_B being the decay constant of the B meson. We have separated the contribution due to the valence and non-valence quarks, where $\mathcal{B}_i^{(q)}$ is non vanishing only for q equal to q' , the spectator quark in the B meson, and is expected to be of order one, see e.g. ref. [56], while $\tau_i^{(q)}$ accounts for the effects of an ‘‘intrinsic’’ q quark [43, 44] and is expected to be small. Its numerical value can be estimated via e.g. the calculation of the so-called eye contractions in the non-perturbative determination of the matrix elements, see ref. [58].

3 Results

The contribution of the Darwin operator to the inclusive non-leptonic decay $b \rightarrow q_1 \bar{q}_2 q_3$ is presented in the following form

$$\Gamma_{\text{NL}}^{(\rho_D)}(B) = \Gamma_0 \left(3C_1^2 \mathcal{C}_{\rho_D,11}^{(q_1 \bar{q}_2 q_3)} + 2C_1 C_2 \mathcal{C}_{\rho_D,12}^{(q_1 \bar{q}_2 q_3)} + 3C_2^2 \mathcal{C}_{\rho_D,22}^{(q_1 \bar{q}_2 q_3)} \right) \frac{\rho_D^3}{m_b^3}, \quad (3.1)$$

where ρ_D^3 is defined in eq. (2.23), $C_{1,2}(\mu_1)$ are the $\Delta B = 1$ Wilson coefficients, and

$$\mathcal{C}_{\rho_D,11}^{(u\bar{u}d)} = 6 + 8 \log\left(\frac{\mu_0^2}{m_b^2}\right), \quad (3.2)$$

$$\mathcal{C}_{\rho_D,12}^{(u\bar{u}d)} = -\frac{34}{3}, \quad (3.3)$$

$$\mathcal{C}_{\rho_D,22}^{(u\bar{u}d)} = 6 + 8 \log\left(\frac{\mu_0^2}{m_b^2}\right), \quad (3.4)$$

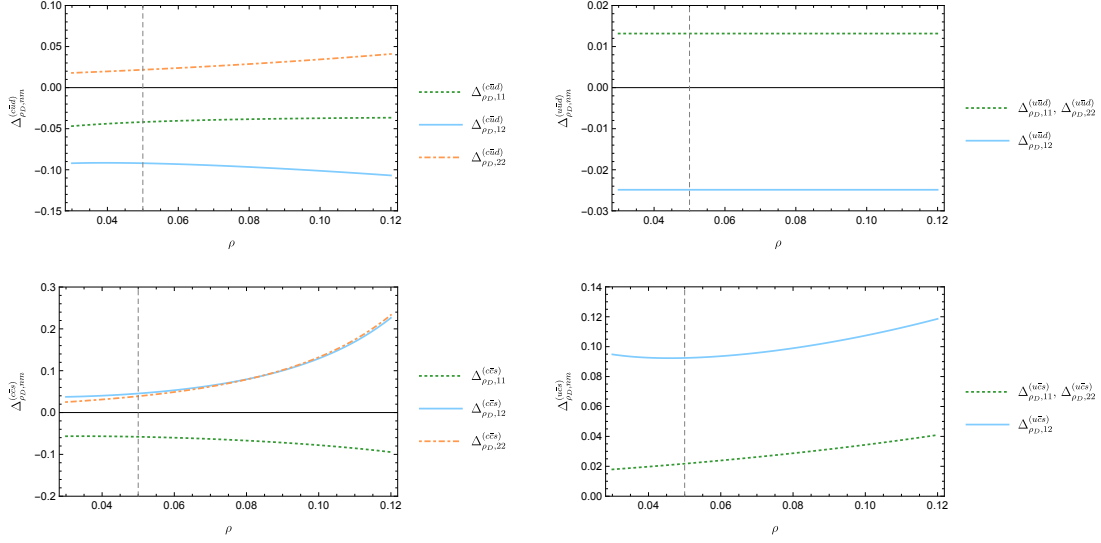


Figure 6. The Darwin term correction normalised to the partonic-level coefficient for different modes, namely $b \rightarrow c\bar{u}d$ (top left), $b \rightarrow u\bar{u}d$ (top right), $b \rightarrow c\bar{c}s$ (bottom left), and $b \rightarrow u\bar{c}s$ (bottom right). The green dotted line corresponds to $Q_1 \otimes Q_1$, the solid cyan line to $Q_1 \otimes Q_2$ and the dotted-dashed orange line to the $Q_2 \otimes Q_2$ contribution. For reference we have fixed the values $\mu_0 = m_b$, $m_b = 4.5 \text{ GeV}$ and $\rho_D^3 = 0.2 \text{ GeV}^3$. The dashed vertical line shows the approximate value $\rho = 0.05$ in the $\overline{\text{MS}}$ scheme.

$$\begin{aligned}
 C_{\rho_D,11}^{(c\bar{u}d)} &= \frac{2}{3}(1-\rho) \left[9 + 11\rho - 12\rho^2 \log(\rho) - 24(1-\rho^2) \log(1-\rho) - 25\rho^2 + 5\rho^3 \right] \\
 &\quad + 8(1-\rho)^2(1+\rho) \log\left(\frac{\mu_0^2}{m_b^2}\right), \tag{3.5}
 \end{aligned}$$

$$\begin{aligned}
 C_{\rho_D,12}^{(c\bar{u}d)} &= \frac{2}{3} \left[-41 - 12(2 + 5\rho + 2\rho^2 - 2\rho^3) \log(\rho) \right. \\
 &\quad \left. - 48(1-\rho)^2(1+\rho) \log(1-\rho) + 26\rho - 18\rho^2 + 38\rho^3 - 5\rho^4 \right] \\
 &\quad + 16(1-\rho)^2(1+\rho) \log\left(\frac{\mu_0^2}{m_b^2}\right), \tag{3.6}
 \end{aligned}$$

$$\begin{aligned}
 C_{\rho_D,22}^{(c\bar{u}d)} &= \frac{2}{3}(1-\rho) \left[9 + 11\rho - 12\rho^2 \log(\rho) - 24(1-\rho^2) \log(1-\rho) - 25\rho^2 + 5\rho^3 \right] \\
 &\quad + 8(1-\rho)^2(1+\rho) \log\left(\frac{\mu_0^2}{m_b^2}\right), \tag{3.7}
 \end{aligned}$$

$$C_{\rho_D,11}^{(u\bar{c}s)} = \frac{2}{3} \left[17 + 12 \log(\rho) - 16\rho - 12\rho^2 + 16\rho^3 - 5\rho^4 \right], \tag{3.8}$$

$$\begin{aligned}
 C_{\rho_D,12}^{(u\bar{c}s)} &= \frac{2}{3} \left[-9 + 12(1 - 3\rho^2 + \rho^3) \log(\rho) \right. \\
 &\quad \left. + 24(1-\rho)^3 \log(1-\rho) + 50\rho - 90\rho^2 + 54\rho^3 - 5\rho^4 \right] \\
 &\quad - 8(1-\rho)^3 \log\left(\frac{\mu_0^2}{m_b^2}\right), \tag{3.9}
 \end{aligned}$$

$$\begin{aligned} \mathcal{C}_{\rho_D,22}^{(c\bar{u}d)} &= \frac{2}{3}(1-\rho) \left[9 + 11\rho - 12\rho^2 \log(\rho) \right. \\ &\quad \left. - 24(1-\rho^2) \log(1-\rho) - 25\rho^2 + 5\rho^3 \right] \\ &\quad + 8(1-\rho)^2(1+\rho) \log\left(\frac{\mu_0^2}{m_b^2}\right), \end{aligned} \quad (3.10)$$

$$\begin{aligned} \mathcal{C}_{\rho_D,11}^{(c\bar{c}s)} &= \frac{2}{3} \left[\sqrt{1-4\rho} (17 + 8\rho - 22\rho^2 - 60\rho^3) \right. \\ &\quad \left. - 12(1-\rho - 2\rho^2 + 2\rho^3 + 10\rho^4) \log\left(\frac{1+\sqrt{1-4\rho}}{1-\sqrt{1-4\rho}}\right) \right], \end{aligned} \quad (3.11)$$

$$\begin{aligned} \mathcal{C}_{\rho_D,12}^{(c\bar{c}s)} &= \frac{2}{3} \left[\sqrt{1-4\rho} (-45 + 46\rho - 106\rho^2 - 60\rho^3) \right. \\ &\quad \left. + 12(1 + 4\rho^2 - 16\rho^3 - 10\rho^4) \log\left(\frac{1+\sqrt{1-4\rho}}{1-\sqrt{1-4\rho}}\right) \right] \\ &\quad + 8 \left[\mathcal{M}_{112}(\rho, \eta) - \sqrt{1-4\rho} \log(\eta) \right] \Big|_{\eta \rightarrow 0} \\ &\quad + 8\sqrt{1-4\rho} \log\left(\frac{\mu_0^2}{m_b^2}\right), \end{aligned} \quad (3.12)$$

$$\begin{aligned} \mathcal{C}_{\rho_D,22}^{(c\bar{c}s)} &= \frac{2}{3} \left[\sqrt{1-4\rho} (-3 + 22\rho - 34\rho^2 - 60\rho^3) \right. \\ &\quad \left. - 24\rho(1 + \rho + 2\rho^2 + 5\rho^3) \log\left(\frac{1+\sqrt{1-4\rho}}{1-\sqrt{1-4\rho}}\right) \right] \\ &\quad + 8 \left[\mathcal{M}_{112}(\rho, \eta) - \sqrt{1-4\rho} \log(\eta) \right] \Big|_{\eta \rightarrow 0} \\ &\quad + 8\sqrt{1-4\rho} \log\left(\frac{\mu_0^2}{m_b^2}\right). \end{aligned} \quad (3.13)$$

The dimensionless parameter $\eta = m_q^2/m_b^2$ and the master integral \mathcal{M}_{112} is defined as¹¹

$$\mathcal{M}_{112}(\rho, \eta) = - \int_{(\sqrt{\rho}+\sqrt{\eta})^2}^{(1-\sqrt{\rho})^2} dt \frac{(t^2 - 2(1+\rho)t + (1-\rho)^2)(t-\eta+\rho)}{t\sqrt{(t^2 - 2(1+\rho)t + (1-\rho)^2)(t^2 - 2t(\eta+\rho) + (\eta-\rho)^2)}}. \quad (3.14)$$

Note, that because of $m_d = m_s = 0$ the following relations hold:

$$\mathcal{C}_{\rho_D, nm}^{(c\bar{u}d)} = \mathcal{C}_{\rho_D, nm}^{(c\bar{u}s)}, \quad \mathcal{C}_{\rho_D, nm}^{(c\bar{c}s)} = \mathcal{C}_{\rho_D, nm}^{(c\bar{c}d)}, \quad \mathcal{C}_{\rho_D, nm}^{(u\bar{u}d)} = \mathcal{C}_{\rho_D, nm}^{(u\bar{u}s)}, \quad \mathcal{C}_{\rho_D, nm}^{(u\bar{c}s)} = \mathcal{C}_{\rho_D, nm}^{(u\bar{c}d)}.$$

The relative effect of the Darwin term with respect to the corresponding partonic-level contribution is given by

$$\Delta_{\rho_D, nm}^{(q_1 \bar{q}_2 q_3)} = \frac{\mathcal{C}_{\rho_D, nm}^{(q_1 \bar{q}_2 q_3)}}{\mathcal{C}_{\rho_D, nm}^{(q_1 \bar{q}_2 q_3)}} \frac{\rho_D^3}{m_b^3}. \quad (3.15)$$

¹¹An explicit analytic expression for $[\mathcal{M}_{112}(\rho, \eta) - \sqrt{1-4\rho} \log(\eta)] \Big|_{\eta \rightarrow 0}$ has been found in ref. [84].

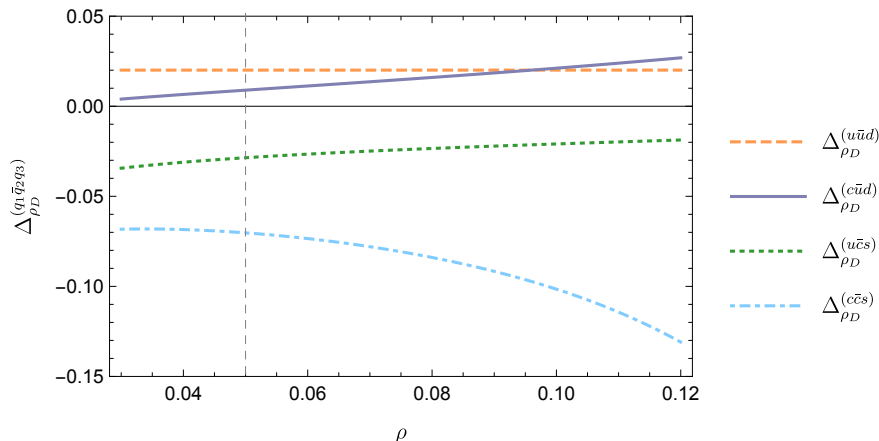


Figure 7. Total relative size of the Darwin term correction compared to the partonic-level contribution for different modes, including $b \rightarrow u\bar{u}d$ (dashed orange), $b \rightarrow c\bar{u}d$ (solid purple), $b \rightarrow u\bar{c}s$ (dotted green) and $b \rightarrow c\bar{c}s$ (dot-dashed cyan). For reference we have fixed the values $\mu_1 = \mu_0 = m_b$, $m_b = 4.5 \text{ GeV}$ and $\rho_D^3 = 0.2 \text{ GeV}^3$. The dashed vertical line shows the approximate value $\rho = 0.05$ in the $\overline{\text{MS}}$ scheme.

In figure 6, these ratios are plotted as functions of ρ for all the colour structures and the four modes, using for reference the values $\mu_0 = m_b$, $m_b = 4.5 \text{ GeV}$ and $\rho_D^3 = 0.2 \text{ GeV}^3$. Figure 7 shows the total relative contribution for each mode, namely

$$\Delta_{\rho_D}^{(q_1 \bar{q}_2 q_3)} = \frac{3 C_1^2 \mathcal{C}_{\rho_D, 11}^{(q_1 \bar{q}_2 q_3)} + 2 C_1 C_2 \mathcal{C}_{\rho_D, 12}^{(q_1 \bar{q}_2 q_3)} + 3 C_2^2 \mathcal{C}_{\rho_D, 22}^{(q_1 \bar{q}_2 q_3)}}{(3 C_1^2 + 2 C_1 C_2 + 3 C_2^2) \mathcal{C}_0^{(q_1 \bar{q}_2 q_3)}} \frac{\rho_D^3}{m_b^3}. \quad (3.16)$$

As one can see, the Darwin operator leads to sizeable corrections of the order 1 – 7% (for $\rho = 0.05$) to the $b \rightarrow q_1 \bar{q}_2 q_3$ decay width.

A final comment about the effect of the constant terms b_i in eq. (2.32) for the determination of the coefficients $\mathcal{C}_{\rho_D, nm}^{(q_1 \bar{q}_2 q_3)}$. We have already emphasised that their values depend on the choice of the four-quark operators basis. Consider as an example the coefficient $\mathcal{C}_{\rho_D, 12}^{(c\bar{u}d)}$. In our basis given by eqs. (2.30) and (2.31), $\mathcal{C}_{\rho_D, 12}^{(c\bar{u}d)} = -29.0$ for the reference value $\rho = 0.05$. If instead of eq. (2.30) we had considered the following operators (as it was chosen in ref. [84]):

$$\tilde{\mathcal{O}}_{6,1}^{(u)} = (\bar{b}_v^i \Gamma^\sigma \gamma^\mu \Gamma^\rho u^i) (\bar{u}^j \Gamma_\sigma \gamma_\mu \Gamma_\rho b_v^j), \quad \tilde{\mathcal{O}}_{6,2}^{(u)} = (\bar{b}_v^i \Gamma^\sigma \not{\rho} \Gamma^\rho u^i) (\bar{u}^j \Gamma_\sigma \not{\rho} \Gamma_\rho b_v^j), \quad (3.17)$$

then in eq. (2.32) we would have obtained $a_1 = 8$, $b_1 = -5/4$ and $a_2 = 2$, $b_2 = -3/2$, leading to $\mathcal{C}_{\rho_D, 12}^{(c\bar{u}d)} = -24.0$ (for the same value of ρ). This shift of $\sim 17\%$ should be compensated (up to higher orders) by the different value of the matrix element of the operators defined in these two bases.

4 Discussion and conclusion

This work presents the first determination of the Darwin term contribution to the non-leptonic decay $b \rightarrow q_1 \bar{q}_2 q_3$. Using the expansion of the quark propagator in the external

	$\mathcal{C}_{G,11}^{(q_1\bar{q}_2q_3)}$	$\mathcal{C}_{G,12}^{(q_1\bar{q}_2q_3)}$	$\mathcal{C}_{G,22}^{(q_1\bar{q}_2q_3)}$	$\mathcal{C}_{\rho_D,11}^{(q_1\bar{q}_2q_3)}$	$\mathcal{C}_{\rho_D,12}^{(q_1\bar{q}_2q_3)}$	$\mathcal{C}_{\rho_D,22}^{(q_1\bar{q}_2q_3)}$
$b \rightarrow u\bar{u}d$	-1.5	-9.5	-1.5	6	-11.3	6
$b \rightarrow u\bar{c}s$	-1.28	-6.23	-1.28	6.84	29.1	6.84
$b \rightarrow c\bar{u}d$	-1.28	-8.14	-1.28	-13.2	-29.0	6.84
$b \rightarrow c\bar{c}s$	-1.05	-4.99	-1.05	-11.5	-9.02	7.74

Table 1. Numerical values of the coefficients $\mathcal{C}_{G,nm}^{(q_1\bar{q}_2q_3)}$ and $\mathcal{C}_{\rho_D,nm}^{(q_1\bar{q}_2q_3)}$ for $\rho = 0.05$ and $\mu_0 = m_b$.

gluon field together with the standard technique of the HQE allows to obtain a systematic expansion in inverse powers of the heavy quark mass m_b . At order $1/m_b^3$ operator mixing ensures that the IR divergences arising from the expansion of the light quark propagators cancel, introducing though scale-dependence in the coefficient of ρ_D^3 , with the divergent $\log(m_q^2/m_b^2)$ being in fact replaced by $\log(\mu_0^2/m_b^2)$.¹² Preliminary numerical analysis reveals that this contribution is sizeable. For illustration purposes, we show the total non-leptonic decay width of the B^+ and B_d mesons up to dimension-six and just at LO-QCD. Inputs for the quark masses and the HQE parameters are taken from refs. [46, 56, 58] and we set $\mu_1 = \mu_0 = m_b$:

$$\begin{aligned}
 \Gamma_{\text{NL}}(B) &= \Gamma_3 \left[1 - 0.50 \frac{\mu_\pi^2}{m_b^2} - 0.44 \frac{\mu_G^2}{m_b^2} - 20.25 \frac{\rho_D^3}{m_b^3} + \underbrace{\dots}_{4\text{q-contr.}} \right] \\
 &= \Gamma_3 \left[1 - \underbrace{0.0112}_{\mu_\pi^2} - \underbrace{0.0071}_{\mu_G^2} - \underbrace{0.0373}_{\rho_D^3} - \underbrace{0.0029}_{\tau_i^{(q)}} - \underbrace{\begin{matrix} -0.1023 (B^+) \\ +0.0148 (B_d) \end{matrix}}_{\mathcal{B}_i^{(q)}} \right], \quad (4.1)
 \end{aligned}$$

where

$$\Gamma_3 = \left(3C_1^2 + 2C_1C_2 + 3C_2^2 \right) \sum_{\{q_1, q_2, q_3\}} \Gamma_0^{(q_1\bar{q}_2q_3)} \mathcal{C}_0^{(q_1\bar{q}_2q_3)}, \quad (4.2)$$

and $\Gamma_0^{(q_1\bar{q}_2q_3)}$ is given by eq. (2.25). In eq. (4.1) the effect of the different non-perturbative parameters is shown separately. We find that the new contribution due to the Darwin operator is significant, for the B_d meson it is even larger than the dimension-five and the weak-exchange contributions while for the B^+ meson the Pauli interference term still gives the dominant correction. To understand this better, in table 1 we compare the coefficients of the Darwin and of the chromo-magnetic operators¹³ for different modes. As already stated above, at order $1/m_b^2$, both diagrams in figure 2 vanish and the coefficient of the kinetic operator as well as $\mathcal{C}_{G,11}^{(q_1\bar{q}_2q_3)}$ and $\mathcal{C}_{G,22}^{(q_1\bar{q}_2q_3)}$ receive only the small contribution coming from the expansion of the matrix element of dimension-three, cf. eq. (2.15). Hence the dimension-five contribution stemming from the $Q_1 \otimes Q_1$ combination with large $3C_1^2$ is actually suppressed. A further pronounced source of cancellation lies in the combination

¹²Up to a finite polynomial in ρ , depending on the specific four-quark operator basis adopted.

¹³For simplicity, we do not show the coefficient of μ_π^2 being just $(-1/2)\mathcal{C}_0^{(q_1\bar{q}_2q_3)}$ for all three $\Delta B = 1$ operator combinations.

$3C_1^2\mathcal{C}_{G,11}^{(q_1\bar{q}_2q_3)} + 2C_1C_2\mathcal{C}_{G,12}^{(q_1\bar{q}_2q_3)}$ because of the negative value of C_2 and of the hierarchy $|C_1/C_2| \approx \mathcal{C}_{G,12}^{(q_1\bar{q}_2q_3)}/\mathcal{C}_{G,11}^{(q_1\bar{q}_2q_3)}$. No cancellations are present at order $1/m_b^3$ and this explains why, despite the power suppression, the Darwin contribution is larger than the one of dimension-five.

This result does not come as completely new, in fact it was already known in the well studied inclusive semi-leptonic decays, that the Darwin term constitutes a sizeable correction to the decay rate. For example, in the inclusive semil-leptonic $B \rightarrow X_c\ell\bar{\nu}_\ell$ transition (with $\ell = e, \mu$) the following hierarchy (up to $1/m_b^4$ and at LO-QCD) is found (see e.g. ref. [62])

$$\begin{aligned}
 \Gamma(B \rightarrow X_c\ell\bar{\nu}_\ell) &= \Gamma_3^{(c\ell\bar{\nu})} \left[1 - 0.5\frac{\mu_\pi^2}{m_b^2} - 1.9\frac{\mu_G^2}{m_b^2} - 19.1\frac{\rho_D^3}{m_b^3} - 20.5\frac{s_1}{m_b^4} \right. \\
 &\qquad\qquad\qquad \left. + 4.0\frac{s_2}{m_b^4} - 13.5\frac{s_3}{m_b^4} - 9.5\frac{s_5}{m_b^4} \right] \\
 &= \Gamma_3^{(c\ell\bar{\nu})} \left[1 - \underbrace{0.043}_{\text{dim-5}} - \underbrace{0.029}_{\text{dim-6}} + \underbrace{0.003}_{\text{dim-7}} \right], \tag{4.3}
 \end{aligned}$$

where $\Gamma_3^{(c\ell\bar{\nu})} = \Gamma_0^{(c\ell\bar{\nu})}\mathcal{C}_0^{(c\ell\bar{\nu})}$ and the numerical values of parameters are taken from ref. [85].

The size of the coefficients in eq. (4.3) confirms the suppression of the dimension-five contribution but does not seem to point towards an abnormal enhancement of the Darwin operator, whose coefficient is of similar size as the ones of dimension-seven. Note also that the second source of cancellation discussed above is now absent and the dimension-five contribution is larger than in the non-leptonic case. Finally, from eq. (4.3) it does not appear to be any problem with the convergence of the HQE.

To conclude, the inclusion of the Darwin operator contribution could lead to a sizeable modification of the theory prediction for the lifetime ratio $\tau(B_s)/\tau(B_d)$ compared to the high experimental precision, which reaches the accuracy of the four per mille level. An updated theoretical analysis of $\tau(B_s)/\tau(B_d)$ will be performed once the $SU(3)_F$ violation ratio of the Bag parameters and of the matrix elements of the Darwin operator will become available.

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A Expansion of the quark propagator in the external gluon field

Following refs. [28, 40], soft gluon interactions of the quark field can be accounted using the background field method. Assuming the b quark embedded in a weakly changing gluon

field allows to systematically expand the quark propagator as a series in the gluon field strength tensor $G_{\mu\nu} = -i[iD_\mu, iD_\nu]$. In the case of a massive quark the expansion of the propagator up to terms linear in $G_{\mu\nu}$ is given in ref. [28], while an expression including terms proportional to $D_\rho G_{\mu\nu} = -[iD_\rho, [iD_\mu, iD_\nu]]$ can be found in ref. [86].¹⁴ To compute the propagator, one starts from the Green-function equation,

$$(i\cancel{D} + \cancel{A}(x) - m)S(x, y) = \delta^{(4)}(x - y), \tag{A.1}$$

which admits a solution in form of the perturbative series

$$iS(x, y) = iS^{(0)}(x - y) + iS^{(1)}(x, y) + \dots \tag{A.2}$$

where $S^{(0)}(x - y)$ is the free-quark propagator and $S^{(1)}(x, y)$ the first order correction:

$$iS^{(1)}(x, y) = \int d^4z iS^{(0)}(x - z) i\cancel{A}(z) iS^{(0)}(z - y). \tag{A.3}$$

Using the Fock-Schwinger gauge i.e. $x^\mu A_\mu(x) = 0$, the gluon field is expressible directly in terms of the gluon field strength tensor, see ref. [40] for a detailed derivation:

$$A_\mu^a(z) = \int_0^1 d\alpha \alpha z^\rho G_{\rho\mu}^a(\alpha z). \tag{A.4}$$

Expanding $G_{\rho\mu}(\alpha z)$ around $z = 0$ and taking into account that in the Fock-Schwinger gauge $z^\mu \partial_\mu = z^\mu D_\mu$, yields:

$$A_\mu^a(z) = \frac{1}{2} z^\rho G_{\rho\mu}^a(0) + \frac{1}{3} z^\nu z^\rho D_\nu G_{\rho\mu}^a(0) + \mathcal{O}(DDG). \tag{A.5}$$

Substituting the previous expression in eq. (A.3) and setting $y = 0$ one obtains

$$S(x, 0) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} S(k), \tag{A.6}$$

where the quark propagator in momentum space reads

$$\begin{aligned} S(k) = & \frac{\cancel{k} + m}{k^2 - m^2} + \frac{1}{2(k^2 - m^2)^2} \left(-mG_{\rho\mu}\sigma^{\rho\mu} + 2\tilde{G}_{\rho\mu} k^\rho \gamma^\mu \gamma^5 \right) \\ & + \frac{2}{3} \frac{D_\rho G^{\rho\mu}}{(k^2 - m^2)^2} \left(\gamma_\mu - \frac{(\cancel{k} + 2m) k_\mu}{k^2 - m^2} \right) - \frac{2}{3} \frac{D_\nu G_{\rho\mu} k^\nu k^\rho \gamma^\mu}{(k^2 - m^2)^3} \\ & + \frac{2}{3} \frac{m D_\nu G_{\rho\mu}}{(k^2 - m^2)^3} (\gamma^\rho \gamma^\mu k^\nu - \gamma^\mu \gamma^\nu k^\rho) + 2i \frac{D_\nu \tilde{G}_{\rho\mu} k^\nu k^\rho \gamma^\mu \gamma^5}{(k^2 - m^2)^3} + \dots \end{aligned} \tag{A.7}$$

In the above, $\tilde{G}_{\rho\mu} = (1/2)\epsilon_{\rho\mu\sigma\eta} G^{\sigma\eta}$ and $\epsilon_{\rho\mu\sigma\eta}$ is the Levi-Civita tensor, while the ellipsis stands for terms with higher derivatives as well as higher powers of $G_{\mu\nu}$. In the limit $m \rightarrow 0$, eq. (A.7) correctly reproduces the massless expression given in ref. [40]. Finally,

¹⁴Note that their expression is not complete as it only contains terms relevant for the computation, namely with odd number of gamma-matrices.

we emphasise that the Fock-Schwinger gauge breaks explicitly the translation invariance of $S(x, y)$, namely:

$$S(0, x) = \int \frac{d^4 k}{(2\pi)^4} e^{ikx} \tilde{S}(k), \quad \tilde{S}(k) \neq S(k),$$

with

$$\begin{aligned} \tilde{S}(k) = & \frac{\not{k} + m}{k^2 - m^2} + \frac{1}{2(k^2 - m^2)^2} \left(-m G_{\rho\mu} \sigma^{\rho\mu} + 2 \tilde{G}_{\rho\mu} k^\rho \gamma^\mu \gamma^5 \right) \\ & + \frac{2}{3} \frac{D_\rho G^{\rho\mu}}{(k^2 - m^2)^2} \left(\gamma_\mu - \frac{\not{k} k_\mu}{k^2 - m^2} \right) - \frac{2}{3} \frac{D_\nu G_{\rho\mu} k^\nu k^\rho \gamma^\mu}{(k^2 - m^2)^3} \\ & - \frac{2}{3} \frac{m D_\nu G_{\rho\mu}}{(k^2 - m^2)^3} (\gamma^\rho \gamma^\mu k^\nu - \gamma^\mu \gamma^\nu k^\rho) - 2i \frac{D_\nu \tilde{G}_{\rho\mu} k^\nu k^\rho \gamma^\mu \gamma^5}{(k^2 - m^2)^3} + \dots \end{aligned} \quad (\text{A.8})$$

For completeness, we also present the equivalent representation of eq. (A.6) in coordinate space

$$\begin{aligned} S(x, 0) = & -\frac{i}{4\pi^2} \frac{m^2 K_1(m\sqrt{-x^2})}{\sqrt{-x^2}} - \frac{1}{4\pi^2} \frac{m^2 \not{x} K_2(m\sqrt{-x^2})}{x^2} \\ & - \frac{\tilde{G}_{\alpha\beta} x^\alpha \gamma^\beta \gamma^5}{8\pi^2} \frac{m K_1(m\sqrt{-x^2})}{\sqrt{-x^2}} - i \frac{G_{\alpha\beta} \sigma^{\alpha\beta}}{16\pi^2} m K_0(m\sqrt{-x^2}) \\ & + \frac{i}{24\pi^2} D_\alpha G^{\alpha\beta} \gamma_\beta K_0(m\sqrt{-x^2}) - \frac{i}{48\pi^2} D_\alpha G^{\alpha\beta} x_\beta \not{x} \frac{m K_1(m\sqrt{-x^2})}{\sqrt{-x^2}} \\ & - \frac{1}{24\pi^2} D_\alpha G^{\alpha\beta} x_\beta m K_0(m\sqrt{-x^2}) - \frac{i}{48\pi^2} D^\alpha G^{\beta\rho} \gamma_\rho x_\alpha x_\beta \frac{m K_1(m\sqrt{-x^2})}{\sqrt{-x^2}} \\ & - \frac{1}{16\pi^2} D^\alpha \tilde{G}^{\beta\rho} \gamma_\rho \gamma^5 x_\alpha x_\beta \frac{m K_1(m\sqrt{-x^2})}{\sqrt{-x^2}} - \frac{1}{48\pi^2} D^\alpha G^{\beta\rho} \gamma_\rho \gamma_\beta x_\alpha m K_0(m\sqrt{-x^2}) \\ & - \frac{1}{48\pi^2} D^\alpha G^{\beta\rho} \gamma_\rho \gamma_\alpha x_\beta m K_0(m\sqrt{-x^2}) + \mathcal{O}(DDG), \end{aligned} \quad (\text{A.9})$$

where $K_{0,1,2}(z)$ are the modified Bessel function of the second kind.

B Complementary material to section 2.2

The divergent coefficients $\mathcal{D}_{nm}^{(q_1 \bar{q}_2 q_3)}$ in eq. (2.26) read:

$$\begin{aligned} \mathcal{D}_{11}^{(u\bar{u}d)} &= 8 \log \left(\frac{m_u^2}{m_b^2} \right), \\ \mathcal{D}_{12}^{(u\bar{u}d)} &= 8 \left[\log \left(\frac{m_d^2}{m_b^2} \right) - \log \left(\frac{m_u^2}{m_b^2} \right) \right], \\ \mathcal{D}_{22}^{(u\bar{u}d)} &= 8 \log \left(\frac{m_d^2}{m_b^2} \right), \\ \mathcal{D}_{11}^{(u\bar{c}s)} &= 8 (1 - \rho)^2 (1 + \rho) \log \left(\frac{m_u^2}{m_b^2} \right), \end{aligned} \quad (\text{B.1})$$

$$\begin{aligned}
 \mathcal{D}_{12}^{(u\bar{c}s)} &= 8(1-\rho)^2(1+\rho) \left[\log\left(\frac{m_u^2}{m_b^2}\right) + \log\left(\frac{m_s^2}{m_b^2}\right) \right], \\
 \mathcal{D}_{22}^{(u\bar{c}s)} &= 8(1-\rho)^2(1+\rho) \log\left(\frac{m_s^2}{m_b^2}\right),
 \end{aligned} \tag{B.2}$$

$$\begin{aligned}
 \mathcal{D}_{12}^{(c\bar{u}d)} &= -16(1-\rho)^2 \log\left(\frac{m_u^2}{m_b^2}\right) + 8(1-\rho)^2(1+\rho) \log\left(\frac{m_d^2}{m_b^2}\right), \\
 \mathcal{D}_{22}^{(c\bar{u}d)} &= 8(1-\rho)^2(1+\rho) \log\left(\frac{m_d^2}{m_b^2}\right),
 \end{aligned} \tag{B.3}$$

$$\begin{aligned}
 \mathcal{D}_{12}^{(c\bar{c}s)} &= 8\sqrt{1-4\rho} \log\left(\frac{m_s^2}{m_b^2}\right), \\
 \mathcal{D}_{22}^{(c\bar{c}s)} &= 8\sqrt{1-4\rho} \log\left(\frac{m_s^2}{m_b^2}\right).
 \end{aligned} \tag{B.4}$$

The discontinuity of the WA, PI and WE diagrams in figure 4 at LO-QCD, respectively is

$$\begin{aligned}
 \Gamma^{\text{WA}} &= \frac{\Gamma_0}{2m_B} \frac{16\pi^2}{m_b^3} \sqrt{\lambda(1, z_2, z_3)} \left\{ \left(N_c C_1^2 + 2C_1 C_2 \right) \right. \\
 &\quad \times \left[\left((z_2 - z_3)^2 + z_2 + z_3 - 2 \right) \tilde{\mathcal{O}}_{6,1}^{(q_1)} \right. \\
 &\quad \quad \left. \left. - 2 \left(2(z_2 - z_3)^2 - 1 - z_2 - z_3 \right) \tilde{\mathcal{O}}_{6,2}^{(q_1)} \right] \right. \\
 &\quad \left. + C_2^2 \left[\left((z_2 - z_3)^2 + z_2 + z_3 - 2 \right) \tilde{\mathcal{O}}_{6,3}^{(q_1)} \right. \right. \\
 &\quad \quad \left. \left. - 2 \left(2(z_2 - z_3)^2 - 1 - z_2 - z_3 \right) \tilde{\mathcal{O}}_{6,4}^{(q_1)} \right] \right\},
 \end{aligned} \tag{B.5}$$

$$\begin{aligned}
 \Gamma^{\text{PI}} &= \frac{\Gamma_0}{2m_B} \frac{96\pi^2}{m_b^3} \sqrt{\lambda(1, z_1, z_3)} (1 - z_1 - z_3) \\
 &\quad \times \left[2C_1 C_2 \tilde{\mathcal{O}}_{6,1}^{(q_2)} + (C_1^2 + C_2^2) \tilde{\mathcal{O}}_{6,3}^{(q_2)} \right],
 \end{aligned} \tag{B.6}$$

$$\begin{aligned}
 \Gamma^{\text{WE}} &= \frac{\Gamma_0}{2m_B} \frac{16\pi^2}{m_b^3} \sqrt{\lambda(1, z_1, z_2)} \left\{ \left(2C_1 C_2 + N_c C_2^2 \right) \right. \\
 &\quad \times \left[\left((z_1 - z_2)^2 + z_1 + z_2 - 2 \right) \tilde{\mathcal{O}}_{6,1}^{(q_3)} \right. \\
 &\quad \quad \left. \left. - 2 \left(2(z_1 - z_2)^2 - 1 - z_1 - z_2 \right) \tilde{\mathcal{O}}_{6,2}^{(q_3)} \right] \right. \\
 &\quad \left. + C_1^2 \left[\left((z_1 - z_2)^2 + z_1 + z_2 - 2 \right) \tilde{\mathcal{O}}_{6,3}^{(q_3)} \right. \right. \\
 &\quad \quad \left. \left. - 2 \left(2(z_1 - z_2)^2 - 1 - z_1 - z_2 \right) \tilde{\mathcal{O}}_{6,4}^{(q_3)} \right] \right\},
 \end{aligned} \tag{B.7}$$

where $z_i = m_{q_i}^2/m_b^2$ and the four-quark operators $\tilde{\mathcal{O}}_{6,i}^{(q)}$ are defined in eqs. (2.30), (2.31). Note that, since we set $m_u = m_d = m_s = 0$, in our case z_i can be either equal to ρ or 0.

C Contribution of four-quark operators at order $1/m_b^3$

The four-quark operators contribution to the non-leptonic decay $b \rightarrow q_1 \bar{q}_2 q_3$ at order $1/m_b^3$ and at LO-QCD is written in the following form

$$\Gamma_{\text{NL}}^{(4q)}(B) = \frac{\Gamma_0}{2m_B} \left(3C_1^2 \mathcal{P}_{11}^{(q_1 \bar{q}_2 q_3)} + 2C_1 C_2 \mathcal{P}_{12}^{(q_1 \bar{q}_2 q_3)} + 3C_2^2 \mathcal{P}_{22}^{(q_1 \bar{q}_2 q_3)} \right), \quad (\text{C.1})$$

where

$$\begin{aligned} P_{11}^{(u\bar{u}d)} &= \frac{32\pi^2}{m_b^3} \left[-\langle \tilde{\mathcal{O}}_{6,1}^{(u)} \rangle + \langle \tilde{\mathcal{O}}_{6,2}^{(u)} \rangle + \langle \tilde{\mathcal{O}}_{6,3}^{(u)} \rangle - \frac{\langle \tilde{\mathcal{O}}_{6,3}^{(d)} \rangle - \langle \tilde{\mathcal{O}}_{6,4}^{(d)} \rangle}{3} \right], \\ P_{12}^{(u\bar{u}d)} &= \frac{32\pi^2}{m_b^3} \left[2\langle \tilde{\mathcal{O}}_{6,1}^{(u)} \rangle + \langle \tilde{\mathcal{O}}_{6,2}^{(u)} \rangle - \langle \tilde{\mathcal{O}}_{6,1}^{(d)} \rangle + \langle \tilde{\mathcal{O}}_{6,2}^{(d)} \rangle \right], \\ P_{22}^{(u\bar{u}d)} &= \frac{32\pi^2}{m_b^3} \left[-\langle \tilde{\mathcal{O}}_{6,1}^{(d)} \rangle + \langle \tilde{\mathcal{O}}_{6,2}^{(d)} \rangle + \frac{2\langle \tilde{\mathcal{O}}_{6,3}^{(u)} \rangle + \langle \tilde{\mathcal{O}}_{6,4}^{(u)} \rangle}{3} \right], \end{aligned} \quad (\text{C.2})$$

$$\begin{aligned} P_{11}^{(u\bar{c}s)} &= \frac{32\pi^2}{m_b^3} (1-\rho)^2 \left[(1+2\rho) \left(\langle \tilde{\mathcal{O}}_{6,2}^{(u)} \rangle + \frac{1}{3} \langle \tilde{\mathcal{O}}_{6,4}^{(s)} \rangle \right) - \left(1 + \frac{\rho}{2} \right) \left(\langle \tilde{\mathcal{O}}_{6,1}^{(u)} \rangle + \frac{1}{3} \langle \tilde{\mathcal{O}}_{6,3}^{(s)} \rangle \right) \right], \\ P_{12}^{(u\bar{c}s)} &= \frac{32\pi^2}{m_b^3} (1-\rho)^2 \left[(1+2\rho) \left(\langle \tilde{\mathcal{O}}_{6,2}^{(u)} \rangle + \langle \tilde{\mathcal{O}}_{6,2}^{(s)} \rangle \right) - \left(1 + \frac{\rho}{2} \right) \left(\langle \tilde{\mathcal{O}}_{6,1}^{(u)} \rangle + \langle \tilde{\mathcal{O}}_{6,1}^{(s)} \rangle \right) \right], \\ P_{22}^{(u\bar{c}s)} &= \frac{32\pi^2}{m_b^3} (1-\rho)^2 \left[(1+2\rho) \left(\langle \tilde{\mathcal{O}}_{6,2}^{(s)} \rangle + \frac{1}{3} \langle \tilde{\mathcal{O}}_{6,4}^{(u)} \rangle \right) - \left(1 + \frac{\rho}{2} \right) \left(\langle \tilde{\mathcal{O}}_{6,1}^{(s)} \rangle + \frac{1}{3} \langle \tilde{\mathcal{O}}_{6,3}^{(u)} \rangle \right) \right], \end{aligned} \quad (\text{C.3})$$

$$\begin{aligned} P_{11}^{(c\bar{u}d)} &= \frac{32\pi^2}{3m_b^3} (1-\rho)^2 \left[(1+2\rho) \langle \tilde{\mathcal{O}}_{6,4}^{(d)} \rangle - \left(1 + \frac{\rho}{2} \right) \langle \tilde{\mathcal{O}}_{6,3}^{(d)} \rangle + 3 \langle \tilde{\mathcal{O}}_{6,3}^{(u)} \rangle \right], \\ P_{12}^{(c\bar{u}d)} &= \frac{32\pi^2}{m_b^3} (1-\rho)^2 \left[(1+2\rho) \langle \tilde{\mathcal{O}}_{6,2}^{(d)} \rangle - \left(1 + \frac{\rho}{2} \right) \langle \tilde{\mathcal{O}}_{6,1}^{(d)} \rangle + 3 \langle \tilde{\mathcal{O}}_{6,1}^{(u)} \rangle \right], \\ P_{22}^{(c\bar{u}d)} &= \frac{32\pi^2}{m_b^3} (1-\rho)^2 \left[(1+2\rho) \langle \tilde{\mathcal{O}}_{6,2}^{(d)} \rangle - \left(1 + \frac{\rho}{2} \right) \langle \tilde{\mathcal{O}}_{6,1}^{(d)} \rangle + \langle \tilde{\mathcal{O}}_{6,3}^{(u)} \rangle \right], \end{aligned} \quad (\text{C.4})$$

$$\begin{aligned} P_{11}^{(c\bar{c}s)} &= \frac{32\pi^2}{3m_b^3} \sqrt{1-4\rho} \left[(1+2\rho) \langle \tilde{\mathcal{O}}_{6,4}^{(s)} \rangle - (1-\rho) \langle \tilde{\mathcal{O}}_{6,3}^{(s)} \rangle \right], \\ P_{12}^{(c\bar{c}s)} &= \frac{32\pi^2}{m_b^3} \sqrt{1-4\rho} \left[(1+2\rho) \langle \tilde{\mathcal{O}}_{6,2}^{(s)} \rangle - (1-\rho) \langle \tilde{\mathcal{O}}_{6,1}^{(s)} \rangle \right], \\ P_{22}^{(c\bar{c}s)} &= \frac{32\pi^2}{m_b^3} \sqrt{1-4\rho} \left[(1+2\rho) \langle \tilde{\mathcal{O}}_{6,2}^{(s)} \rangle - (1-\rho) \langle \tilde{\mathcal{O}}_{6,1}^{(s)} \rangle \right]. \end{aligned} \quad (\text{C.5})$$

The corresponding expressions for $\mathcal{P}_{nm}^{(u\bar{u}s)}$, $\mathcal{P}_{nm}^{(u\bar{c}d)}$, $\mathcal{P}_{nm}^{(c\bar{u}s)}$, $\mathcal{P}_{nm}^{(c\bar{c}d)}$ can be obtained from the above ones by replacing $\tilde{\mathcal{O}}_{6,i}^{(d)} \leftrightarrow \tilde{\mathcal{O}}_{6,i}^{(s)}$.

D Coefficients of the dimension-three and of the chromo-magnetic operators

Here we present the analytic expressions for the coefficients of the dimension-three and chromo-magnetic operators introduced in eq. (2.24). They read respectively

$$\mathcal{C}_0^{(u\bar{u}d)} = 1, \quad \mathcal{C}_{G,11}^{(u\bar{u}d)} = \mathcal{C}_{G,22}^{(u\bar{u}d)} = -\frac{3}{2}, \quad \mathcal{C}_{G,12}^{(u\bar{u}d)} = -\frac{19}{2}, \quad (\text{D.1})$$

$$\begin{aligned} \mathcal{C}_0^{(u\bar{c}s)} &= 1 - 8\rho - 12\rho^2 \log(\rho) + 8\rho^3 - \rho^4, \\ \mathcal{C}_{G,11}^{(u\bar{c}s)} = \mathcal{C}_{G,22}^{(u\bar{c}s)} &= -\frac{1}{2} \left(3 - 8\rho + 12\rho^2 \log(\rho) + 24\rho^2 - 24\rho^3 + 5\rho^4 \right), \\ \mathcal{C}_{G,12}^{(u\bar{c}s)} &= -\frac{1}{2} \left(19 + 16\rho + 12\rho(\rho + 4) \log(\rho) - 24\rho^2 - 16\rho^3 + 5\rho^4 \right), \end{aligned} \quad (\text{D.2})$$

$$\begin{aligned} \mathcal{C}_0^{(c\bar{u}d)} &= 1 - 8\rho - 12\rho^2 \log(\rho) + 8\rho^3 - \rho^4, \\ \mathcal{C}_{G,11}^{(c\bar{u}d)} = \mathcal{C}_{G,22}^{(c\bar{u}d)} &= -\frac{1}{2} \left(3 - 8\rho + 12\rho^2 \log(\rho) + 24\rho^2 - 24\rho^3 + 5\rho^4 \right), \\ \mathcal{C}_{G,12}^{(c\bar{u}d)} &= -\frac{1}{2} \left(19 - 56\rho + 12\rho^2 \log(\rho) + 72\rho^2 - 40\rho^3 + 5\rho^4 \right), \end{aligned} \quad (\text{D.3})$$

$$\begin{aligned} \mathcal{C}_0^{(c\bar{c}s)} &= \sqrt{1-4\rho} \left(1 - 14\rho - 2\rho^2 - 12\rho^3 \right) + 24\rho^2(1-\rho^2) \log \left(\frac{1+\sqrt{1-4\rho}}{1-\sqrt{1-4\rho}} \right), \\ \mathcal{C}_{G,11}^{(c\bar{c}s)} = \mathcal{C}_{G,22}^{(c\bar{c}s)} &= -\frac{1}{2} \left[\sqrt{1-4\rho} \left(3 - 10\rho + 10\rho^2 + 60\rho^3 \right) \right. \\ &\quad \left. - 24\rho^2(1-5\rho^2) \log \left(\frac{1+\sqrt{1-4\rho}}{1-\sqrt{1-4\rho}} \right) \right], \\ \mathcal{C}_{G,12}^{(c\bar{c}s)} &= -\frac{1}{2} \left[\sqrt{1-4\rho} \left(19 - 2\rho + 58\rho^2 + 60\rho^3 \right) \right. \\ &\quad \left. - 24\rho(2+\rho-4\rho^2-5\rho^3) \log \left(\frac{1+\sqrt{1-4\rho}}{1-\sqrt{1-4\rho}} \right) \right]. \end{aligned} \quad (\text{D.4})$$

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