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What Problem Did Ladd-Franklin (Think She) Solve(d)?

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1 Introduction

This paper stems from a casual remark that Frederique Janssen-Lauret (Manchester) made to me once when we were discussing women working in logic and foundations of math in the late 19th and early 20th century. The conversation naturally turned to Christine Ladd-Franklin, a student of C.S. Peirce's at Johns Hopkins. Janssen-Lauret called Ladd's PhD dissertation a tour de force in which she 'solved a problem first raised by Aristotle which had baffled logicians for two thousand years: how to reduce all forms of the syllogism to one'.

As a historian of logic (albeit focusing many centuries earlier than Ladd), this immediately caught my attention because an achievement like that is something that absolutely every logician should know about, and I was embarrassed—especially in my position as someone who is outspoken about the need for logicians to be familiar with the history of their field!—that I wasn't more familiar with this result.

So I did what any self-respecting logician would do, found Ladd's dissertation (which was published in 1883, a year after she completed it, even though since she was not allowed to be formally registered as a student at the time, she could not be awarded the degree of Dr.) on googlebooks, downloaded it, and searched the PDF until I found her solution:

Theorem 1.1 *The argument of inconsistency,*

$$(a \mathbin{\overline{\vee}} b)(\bar{b} \mathbin{\overline{\vee}} c)(c \vee a)\overline{\vee}$$

is the single form to which all the ninety-six valid syllogisms (both universal and particular) may be reduced [Ladd(1883), p. 40].

Proof Any given syllogism is immediately reduced to this form by taking the contradictory of the conclusion, and by seeing that the universal propositions are expressed with a negative copula and particular propositions with an affirmative copula. [Ladd(1883), p. 40] □

At this point, I was immediately faced with two problems, which formed the investigative basis of the present paper:

1. Not only did I have no idea what this solution was,
2. I also had no idea what problem it solved.

And thus was born the present paper, out of a desire to understand the 'solution' *and* the problem it was purported to solve.

2 Syllogisms: A primer

To take the second problem first: As presented, it was a problem about reducing all forms of syllogism to one form, but the idea of 'form' here is confusing: Syllogisms are typically spoken of as having *figure* and *mood*, not 'form'. So this brings us to an initial question: **What is the 'form' of a syllogism?**

Before we attempt to answer this question (first by discarding two initially plausible but eventually unacceptable possibilities), we first give a brief primer on syllogisms, for those who need a refresher. (Those who don't can skip to §3).

Syllogisms are a specific type of argument form built up out of pairs of categorical propositions.

Definition 2.1 A *categorical proposition* is a subject-predicate proposition consisting of a subject term, a predicate term, a quality, and a quantity.

Any categorematic term can serve as a subject or predicate term in a categorical proposition—(alternatively, one can *define* a categorematic term as one which can be used as the subject or predicate of a categorical proposition without any addition)—and we usually use S and P to stand for arbitrary subject and predicate terms, respectively.

In Aristotle, the subject and predicate terms S and P are always what medieval logicians called 'finite' (*finitus*), that is they are limited or bounded in their application. Words such as 'cat', 'number', 'Hobbit', etc., are all finite terms, even given that 'number' can be predicated of infinitely many objects. Medieval authors also allowed 'infinite' (*infinitus*) terms, e.g., 'non-S' and 'non-P'—remember this for what comes later!

There are four types of categorical proposition recognised by Aristotle:

a: 'All S are P' (universal affirmative)
e: 'No S is P' (universal negative)
i: 'Some S is P' (partial[1] affirmative)
o: 'Some S is not P' (partial negative)

The a, e, i, and o are called 'copulas'. Note that these English expressions—with the subject term first and the predicate term second—reverse the order of the terms with respect to how they occur in Aristotle's Greek predications. Instead of using the more usual English verbiage, Aristotle speaks of predicates belonging to subjects, e.g., 'P belongs to all S'.

Syllogisms are built out of these categorical propositions.[2]

Definition 2.2 A *syllogism* is a set of three categorical propositions which share amongst them three terms that each occur exactly twice. Two of the propositions are designated the premises, and the other is the conclusion.

The predicate term of the conclusion is the 'major' term; the subject term of the conclusion is the 'minor' term; the term that occurs only in the premises is the 'middle term'. It is a convention that the premise with the major term in it, the major premise, is written first.

In the traditional Aristotelian approach, syllogisms have both *figure* (the disposition of the terms within the three categorical propositions that make up the syllogism) and *mood* (the disposition of copulas into a figure, along with the choice of specific major, minor, and middle terms). Aristotle identified three relative arrangements of the major, minor, and middle terms that a syllogism can have, that is, three figures. From Antiquity and into the Middle Ages, Aristotelian commentators also discussed a fourth figure, though they were divided as to whether the moods in this figure consituted a genuinely distinct figure, or whether syllogisms in the fourth figure were merely 'indirect' syllogisms of the first figure [Henle(1949)]. The four figures are depicted in Figure 1; we follow the Aristotelian ordering of the terms whereby the predicate term is listed first.

	1st Figure	2nd Figure
Major premise:	P—M	M—P
Minor premise:	M—S	M—S
Conclusion:	$\therefore P$—S	$\therefore P$—S

	3rd Figure	4th Figure
Major premise:	P—M	M—P
Minor premise:	S—M	S—M
Conclusion:	$\therefore P$—S	$\therefore P$—S

Figure 1 The Four Figures

Each figure can be turned into a mood by inserting a copula between the terms, thereby creating categorical propositions and actual syllogisms. For instance, the mood $P\,a\,M, M\,a\,S \therefore P\,a\,S$ is formed from the first figure by inserting the universal affirmative copula into each gap, and it is a valid syllogism (it is the syllogism traditionally known as 'Barbara'). Of the possible syllogisms that can be so constructed, 24 now are standardly taken to be valid.

Let us now return to the question with which we began: What, then, is the *form* of a syllogism?

3 What is the 'form' of a syllogism?

Given the preceding, there are immediately two possible options:

1. Is it 'figure'?
2. Is it 'mood'?

We'll treat each in turn.

Question 3.1 Is it 'figure'?

No. 'Form' here clearly can't be 'figure', because if we identify the 'form' of a syllogism with its figure, then the problem has not baffled logicians for thousands of years: since it was already well-known that every non-first figure syllogism can be reduced to a first-figure syllogism. Aristotle himself provided a method for reducing every non-first-figure syllogism into a first-figure syllogism, thus reducing all figures of the syllogism to one. (This is what much of the early chapters of the *Prior Analytics* is devoted to.)

For instance, Aristotle knew how to reduce both

Baroco	**Bocardo**
All P are M	Some M is not P
Some S is not M	All M are S
$\therefore$ Some S is not P	$\therefore$ Some S is not P

to

Barbara
All M are P
All S are M
$\therefore$ All S are P

via *reductio ad absurdum*—that is, taking the contradictory of the conclusion and replacing one of the premises with it, and then making the contradictory of the replaced premise the conclusion. The other syllogisms can be reduced to one of the so-called 'perfect' first-figure syllogisms (Barbara, Celarent, Dario, and Ferio), whose validity is self-evident, through the use of simple conversion (which swaps the subject and predicate terms) and accidental conversion (which swaps the subject and predicate terms and changes the quality of the copula).

So the long-standing *problem* cannot about this sort of reduction, interpreting 'form' as 'figure', because the solution to this problem has long been known.

Question 3.2 Is it 'mood'?

If not figure, how about mood? If we identify the 'form' of the syllogism with its mood, then we *do* have an open question, namely, whether it is possible to reduce all syllogistic moods to a single mood. When Aristotle reduces all the non-first-figure syllogisms, he reduces them to one of *two* of the perfect syllogisms, but no further. Must he have stopped there? Could he have taken his reduction process a few steps further? This is a legitimate question which Aristotle does not attempt to answer (in part because he does not pose it: once reduction to two of the perfect moods is achieved, he is satisfied).

But while the question whether it is possible to reduce all the valid moods to a single valid mood is certainly an interesting one, and one left open by Aristotle's works, this does not mean that 'mood' is a candidate for 'form' here. This is because not only was Aristotle not interested in the question 'can all syllogisms be reduced to a single mood', it's also not one that has exercised logicians for two millennia. Certainly I do not know of any medieval logician who was particularly worried about such a fine-grained reduction, with most being content to reduce just to the four perfect syllogisms, Barbara, Celarent, Darii, and Ferio. So even interpreted in this way, forming a genuine question, it isn't the *right* question, because it does not have the right historical background. We must consider one further option:

Question 3.3 Is it something else?

The problem with this option is that there is no clear candidate what this 'something else' might be that would satisfy the requirement of identifying a continuous interest. One tempting candidate is the notion of 'form' that occurs in the hylomorphic distinction between form and matter. This distinction was introduced by Aristotle, and philosophers post-Aristotle were anxious to adapt Aristotelian hylomorphism to many contexts Aristotle did not originally intend the doctrine to cover, including to linguistic, rather than metaphysical, entities such as arguments (specifically syllogisms). Dutilh Novaes traces the development of 'mereological logical hylomorphism' from Aristotle's 'non-mereological metaphysical hylomorphism' [Dutilh Noves(2012), p. 389] and notes that the first person to apply the form-matter distinction to arguments was not Aristotle but Alexander of Aphrodisias, who likened the figure of a syllogism to 'a sort of common matrix' into which appropriate matter could be inserted to form a syllogism [Dutilh Noves(2012), p. 400].

While hylomorphism introduces 'form' into the vocabulary of words used to discuss syllogisms, it doesn't introduce any new *concept* into our armory: What Alexander picks out as the form of a syllogism just *is* its figure. Other ancient commentators followed Alexander, though by the thirteenth century some medieval logicians (including Robert Kilwardby and Albert the Great) experimented with the other option, namely, that the form of a syllogism was its mood, that is figure + copulas [Geudens(2020), p. 122], and this view came to dominate by the fourteenth century [Dutilh Noves(2012), p. 403]. In neither approach did the late antique or medieval philosophers ever concern themselves with the question of whether all syllogisms could be reduced to a single 'form'. So as tempting as it is, this isn't a viable option: Not only was no medieval logician concerned with the reduction of syllogisms to a single form, with 'form' being distinct from either figure or mood, Aristotle himself did not apply the distinction to syllogisms in this way [Dutilh Noves(2012), p. 399]. This therefore cannot have been the source of a problem that had plagued logicians *since Aristotle*.

As a result, we still do not have a legitimate alternative to figure and mood to answer the question 'What is the form of a syllogism?'. We have no evidence that 'form' was ever used to pick out something other than 'mood' or 'figure' by late antique and medieval logicians, which means interpreting 'form' as something other than figure or mood makes it *even more unlikely* that the question is one that had bothered logicians for millennia: Certainly, if it *had*, this puzzlement left no trace in more than fifteen hundred years of documentary record.

To sum up, throughout centuries of commentary on Aristotle throughout late anqituity and the Middle Ages, (a) everyone recognized Aristotle's success at reducing non-first-figure syllogisms to the first figure and (b) no one thought that they could be reduced to the *same* first figure mood. Thus, if we take 'form' to be either 'mood' or 'figure', the problem everyone says Ladd solved just *isn't* a problem. If we take 'form' in the way that medieval logicians used the term in connection with syllogisms, we *still* don't have a reduction problem that puzzled logicians since Aristotle.

But Janssen-Lauret isn't the only one to speak of Ladd's achievement in such approbative terms. Probably the most well-known modern commentator on Ladd, Russinoff, has this to say about Ladd's contributions:

> In 1883, while a student of C.S. Peirce at Johns Hopkins University, Christine Ladd-Franklin published a paper titled *On the Algebra of Logic*, in which she develops an elegant and powerful test for the validity of syllogisms that constitutes the most significant advance in syllogistic logic in two thousand years... In this paper, I bring to light the important work of Ladd-Franklin so that she is justly credited *with having solved a problem over two millennia old.* [Russinoff(1999), p. 451, emphasis added]

Russinoff characterizes this two millennia old problem thus:[3]

> The problem that Aristotle posed and attempted to solve is to give a general characterization of the valid syllogisms... though he did not succeed in providing a unified and complete treatment of the syllogistic argument... [Russinoff(1999), pp. 452–454]

Later , Pietarinen takes up Russinoff's account, and says that Ladd's result in her thesis was:

> the *ground-breaking* discovery involving the reduction of Aristotelian syllogistics into a single formula [Pietarinen(2013), p. 3, emphasis added].[4]

What can possibly explain the discrepancy between the laudatory terms used by Russinoff and Pietarinen, and the fundamental uncertainty of what, exactly, this two-millennia-old problem is?

4 Ladd's algebras

Clearly Ladd achieved something important, so let's return to the other question I had, which is, 'What was her solution?' Maybe if we can understand the solution, we can figure out what problem it was solving.

Working through Ladd's thesis is a laborious task, even for a logician, because the language and vocabulary that she used is no longer standard. Having worked through the thesis in great detail (at a rate of about 2–3 page a day), in this section I give an exposition in the contents in a way which is—hopefully—accessible to contemporary logicians trained in modern vocabulary and techniques, and also to philosophers who are not necessarily algebraists.

The topic of Ladd's dissertation is algebras of logic. She begins by identifying five algebras of logic, due to:

1. Boole [Boole(1854)]
2. Jevons [Jevons(1864)]
3. Schröder [Schröder(1877)]
4. McColl [McColl(1877)]
5. Peirce [Peirce(1867)]

The latter four are 'all modifications, more or less slight, of that of Boole' [Ladd(1883), p. 17]. The purpose of Ladd's dissertation is to introduce a sixth algebra, one that addresses what she sees as drawbacks in the previous attempts. Ladd's algebra will most resemble Schröder's, she says, but differs in the use of the copula as well as how conclusions are expressed.

The basic components of Ladd's algebras are subject and predicate terms, in the same way that these terms are the foundation of Aristotle's categorical syllogisms. Atomic subject and predicate terms (hereafter simply called 'terms') are indicated by, e.g., a, b, c. Ladd follows Wundt and Peirce in using ∞ as a term to represent the domain of discourse [Ladd(1883), p. 19]. ∞ is itself a term, and can occur in propositions.

Every simple term is a sum of objects and a product of qualities:

Definition 4.1 If the term a denotes the objects $a_1, a_2, a_3, \ldots$ and connotes the qualities $\alpha_1, \alpha_2, \alpha_3, \ldots$, then

$$a = (a_1 + a_2 + \ldots)\alpha_1\alpha_2\alpha_3 \ldots$$

[Ladd(1883), p. 28]

Complex terms can be formed from atomic terms as follows:

- $\bar{a}$:= 'what is not a'.[5]
- $a \times b$:= 'what is both a and b'.
- $a + b$:= 'What is either a or b'.

Infinite series of $\times$ or $+$, or combinations of the two, are allowed. Taking terms as classes or sets, in modern set-theoretic notation these are $\mathcal{D} \setminus a$ (where $\mathcal{D}$ indicates the domain), $a \cap b$, and $a \cup b$, respectively. Nevertheless, throughout what follows we will continue to use Ladd's notation. At times, as in Definition 4.1, $a \times b$ will be represented as ab. $\overline{\infty}$ is given its own symbol: 0 [Ladd(1883), p. 19].

Of the many things that can be done with these terms, three are primary:

1. The identity of the subject and predicate terms can be affirmed.
2. The identity of the subject and predicate terms can be denied.
3. Complex terms can be negated.

Regarding (3), Ladd identifies three ways in which complex terms can be negated, one used by both Boole and Jevons, the others due to DeMorgan and Schröder, separately. We discuss the negation of complex terms below, but focus first on the first two actions: The affirmation and denial of the identity of subject and predicate terms. These two actions correspond to two types of propositions in Ladd's algebra: 'those which affirm the identity of the subject and predicate, and those which do not' [Ladd(1883), p. 17].

4.1 Identity and non-identity of terms In all six of the algebras under consideration (the original five plus Ladd's), identity propositions are expressed the same way, via equality:

$$a = b \tag{1}$$

Note that while Ladd uses a and b here, these identity propositions are not restricted to atomic terms; complex terms may also be used in place of a and b.[6] Note also that propositions of the form (1) should not be understood (as they would in modern notation) as the identification of two objects, picked out via constants in some logical language. Instead, propositions of this form should be interpreted as indicating the intersubstitutability of the two logical expressions a and b, *salve veritate*. That is, (1) is equivalent to the following two English propositions [Ladd(1883), p. 18]:

$$\text{There is no } a \text{ which is not } b. \tag{2}$$

and

$$\text{There is no } b \text{ which is not } a. \tag{3}$$

In modern first-order logic, therefore, $a = b$ would be written as $\forall x(Ax \leftrightarrow Bx)$. Thus, whenever $\forall x(Ax \leftrightarrow Bx)$ is true, $a = b$ is true.

The following are some true affirmations of identity involving only positive terms:

$$\begin{array}{l|l}
aaa = a & a + a + a = a \\
abc = bca = cba & a + b + c = b + c + a = c + b + a \\
a(b + c) = (ab + ac) & a + bc = (a + b)(a + c)
\end{array}$$

That is, $\times$ and $+$ are both idempotent, associative, and commutative, and $\times$ and $+$ both distribute over each other.

We further have the following basic laws and identities:

$$\begin{array}{l|l}
a\bar{a} = 0 & a + \bar{a} = \infty \\
a = a \times \infty = a(b + \bar{b})(c + \bar{c})\ldots & a = a + 0 = a + b\bar{b} + c\bar{c} + \ldots \\
0 = a0 = ab\bar{b}c\bar{c}\ldots & \infty = a + \infty = a + (b + \bar{b}) + (c + \bar{c})\ldots
\end{array}$$

$$ab + a\bar{b} + \bar{a}b + \bar{a}\bar{b} = (a + \bar{a}) + (b + \bar{b}) = \infty \tag{4}$$

$$(\bar{a} + \bar{b})(\bar{a} + b)(a + \bar{b})(a + b) = a\bar{a} + b\bar{b} = 0 \tag{5}$$

$$a + ab + abc + \cdots = a \qquad \text{(Law of absorption)}$$

$$a(a + b)(a + b + c)\cdots = a \qquad \text{(Law of absorption)}$$

Each of these principles is straightforward.

All of these identities can also be negated to form non-idenity statements.

4.2 Negative complex terms As noted above, Ladd identifies three ways in which complex terms can be negated. Negated terms involve the notion of a *complete development*, which Ladd introduces:

Definition 4.2 The *complete development* of n terms $(a+\bar{a})(b+\bar{b})(c+\bar{c})\ldots$ is the sum of 2^n combinations of n terms each. [Ladd(1883), p. 19]

Example 4.3 The complete development of $(a + \bar{a})(b + \bar{b})$ is:

$$ab + a\bar{b} + \bar{a}b + \bar{a}\bar{b}$$

The first way a complex term can be negated is followed by Boole and Jevons [Ladd(1883), p. 20]. Note that the complete development of $(a+\bar{a})(b+\bar{b})$, that is, $ab + a\bar{b} + \bar{a}b + \bar{a}\bar{b}$ exhausts the entire domain (everything is either both a and b, a but not b, b but not a, or neither a nor b). From this it follows [Ladd(1883), (12)]:

$$\begin{aligned} \overline{ab} &= a\bar{b} + \bar{a}b + \bar{a}\bar{b} && (6) \\ \overline{ab + a\bar{b}} &= \bar{a}b + \bar{a} + \bar{b} && (7) \\ \overline{ab + a\bar{b} + \bar{a}b} &= \bar{a}\bar{b} && (8) \end{aligned}$$

So that the negation of a multiplicative term is simply equivalent to the negation of each of the multiplicands.

In the second way, due to DeMorgan, 'the negative of a product is the sum of the negatives of the terms, and the negative of a sum is the product of the negatives of the terms' [Ladd(1883), p. 20], as follows:

$$\begin{aligned} \overline{ab} &= \bar{a} + \bar{b} && (9) \\ \overline{a\bar{b}} &= \bar{a} + b && (10) \\ \overline{a + b} &= \bar{a}\bar{b} && (11) \\ \overline{a + \bar{b}} &= \bar{a}b && (12) \end{aligned}$$

This is, of course, a variation of DeMorgan's laws, and can be derived from the equations listed under the first way. The dualism between $\times$ and $+$ that these laws give, in the presence of negation, 'has been pointed out by Schröder' [Ladd(1883), p. 21], and demonstrates that one of the two binary connectives can always be rewritten away.

The third way, due to Schröder, is via the following equation:

$$\overline{pab + qa\bar{b} + r\bar{a}b + s\bar{a}\bar{b}} = \bar{p}ab + \bar{q}a\bar{b} + \bar{r}\bar{a}b + \bar{s}\bar{a}\bar{b} \qquad (13)$$

Ladd explains this as follows: 'That is, consider any number of the letters as the elements of a complete development, and take the negative of their coefficients' [Ladd(1883), p. 21]. This also reduces to the first way because:

$$pab + \bar{p}ab = ab$$

and hence formula added to its negation will result in the complete universe, ∞. If any part of the complete development is missing, its coefficient is taken to be 0, and the negative of 0 is ∞ [Ladd(1883), p. 22].

Ladd notes that this form is the easiest to apply when the expression you want to negate 'closely resembles a complete development. When the expression does not, then it may be simpler to use one of the other two methods.

4.3 Simple forms Finally, we introduce the notion of a 'simple form' of an expression (whether a proposition or a term).

Definition 4.4 An expression may be said to be *in its simplest form* when it is represented by the smallest possible number of letters. [Ladd(1883), p. 22]

The simplest form of an expression need not be unique. For instance, the following three terms are all equivalent to $a(b+\bar{b})+(a+\bar{a})b$:

$$a+b \tag{14}$$

$$a+\bar{a}b \tag{15}$$

$$a\bar{a}b+b \tag{16}$$

The first may be simplest in terms of number of symbols, but it is redundant in the sense that it can be factored out to:

$$a(b+\bar{b})+(a+\bar{a})b$$

which is equivalent to

$$ab+a\bar{b}+ab+\bar{a}b$$

which contains two copies of ab. In contrast, the second of the three equations above is equivalent to

$$a(b+\bar{b})+\bar{a}b$$

which is equivalent to

$$ab+a\bar{b}+\bar{a}b$$

which contains no redundant combinations.

In many cases, reduction of an expression to its simplest form can be performed by inspection; but when the reduction cannot be so formed, the simplest form can be obtained by 'taking the negative of the expression, reducing it, and then restoring it to the positive form' [Ladd(1883), p. 23]. This result Ladd quotes from [McColl(1878), p. 21].

These provide us with the basic building blocks of Ladd's algebras. Next, we look at how categorical propositions can be expressed in these terms.

	Traditional	Boole / Schröder	Jevons	McColl	Peirce
Uni-	All a is b	$a = vb$	$a = ab$	$a \colon b$	$a \prec b$
versal	No a is b	$a = v\bar{b}$	$a = a\bar{b}$	$a \colon \bar{b}$	$a \prec \bar{b}$
Part-	Some a is b	$va = vb$	$ca = cab$	$a \div \bar{b}$	$a \overline{\prec} \bar{b}$
ial	Some a is not b	$va = v\bar{b}$	$ca = ca\bar{b}$	$a \div b$	$a \overline{\prec} b$

Table 1 Methods of formalizing categorical propositions.

5 Categorical propositions and syllogisms

In the previous section we have given definitions of simple terms, complex terms, and their negations, as well as what identity statements between terms can be made. But the subject matter of Boolean algebras—and Ladd's result—are the categorical propositions that go into the Aristotelian syllogistic, so this doesn't get at the meat of what we want to be able to express. A statement such as $a = b$ only gives us co-extensiveness; it does not give us a quality/quantity relationship between a subject and a predicate. Therefore, in order to capture categorical statements, and hence categorical syllogisms, we must extend the basic algebraic language with a means of indicating quantity.

There are two ways in which quantity can be added to terms in order to generate universal or partial propositions, according to Ladd. We can either 'assign the expression of the "quantity" of propositions to the copula or [assign it] to the subject' [Ladd(1883), p. 23]. If quantity is assigned to the copula, then two copulas are necessary, one for expressing universal quantities, and one for expressing partial quantities. If, however, quantity is assigned to the subject term, the only copula that is needed is identity.

Both McColl and Peirce adopt the first way, assigning the expression of the quantity of the proposition to the copula, while Boole, Schröder, and Jevons adopt the second way; a summary of their notation is given in Table 1 (reproduced from [Ladd(1883), p. 24]; in Boole and Schröder's notation, the symbol v should not be taken as a categorical term like a or b, but rather a special term that picks out some arbitrary indefinite class. Jevons's c works similarly, but he does not distinguish it in the way that v is distinguished; it can be any other class term [Ladd(1883), p. 24].)

While there is an advantage to assigning quantity to the subject term, in that only one copula is necessary, that being identity, Ladd identifies two advantages to the method assigning quantity to the copula. First, copulas that include their quantity can be used to link either terms *or* propositions[7], so that, e.g., $a \overline{\prec} b$ can be read either 'a is not wholly contained under b' or 'a does not imply b' [Ladd(1883), p. 24]. Second, there is a correspondence between the quantity of the copula and its quality. The universal copulas are positive (affirmative), and the partial copulas are negative [Ladd(1883), p. 25].

According to Ladd, these two advantages outweigh the advantage gained from have a single copula, but neither McColl nor Peirce exploited all the possible advantages that could be gained from having an asymmetric copula. Ladd's contribution in her dissertation is to introduce a new—and wholly symmetric—way of representing categorical propositions, by keeping the two-copula situation (one for each quantity) but reversing the qualities of the copulas.[8] That is, instead of taking as basic:

$$\begin{array}{lll} (a) & a \prec b & \text{`}a\text{ is wholly }b\text{'} \\ (o) & a \overline{\prec} b & \text{`}a\text{ is not wholly }b\text{'} \end{array}$$

we could instead take instead:

$$\begin{array}{lll} (e) & a \mathbin{\overline{\vee}} b & \text{`}a\text{ is-wholly-not }b\text{'} \\ (i) & a \vee b & \text{`}a\text{ is-partly }b\text{'} \end{array}$$

Both $\vee$ and $\overline{\vee}$ are *symmetric* combinators, in that 'the propositions $a \mathbin{\overline{\vee}} b$, $a \vee b$, may be read either forward or backward' [Ladd(1883), p. 26]. Later, she would describe her contribution thus: 'The secret... is wholly contained in the fruitful idea that subject and predicate are not necessarily indivisible wholes, but that they can be broken up and their separate elements shifted at pleasure from one side of the copula to the other' [Ladd Franklin(1889), p. 545]. This is because propositions using $\overline{\vee}$ are always statements of exclusion [Ladd(1883), p. 26]. Let us thus call statements using $\prec$ *inclusions*, which is why $\prec$, unlike $\vee$, is asymmetric. Importantly, it doesn't matter which notation we take as basic: Inclusions can always be converted into exclusions by changing the copula and the sign of the predicate [Ladd(1883), p. 27]:

$$a \prec b = a \mathbin{\overline{\vee}} \bar{b}$$

That is, $a \prec b$ is equivalent to $a \mathbin{\overline{\vee}} \bar{b}$. Going the other direction, every exclusion is equivalent to a pair of inclusions, differing on which of the two terms you take as the predicate:

$$\begin{array}{ccccc} a \mathbin{\overline{\vee}} b & = & a \prec \bar{b} & = & b \prec \bar{a} \\ a \vee b & = & a \overline{\prec} \bar{b} & = & b \overline{\prec} \bar{a} \end{array}$$

A corollary of this is that in exclusions, the subject and predicate always have the same quantity; while in inclusions they always have different quantity (the subject have universal quantity and the predicate having 'indeterminate' quantity) [Ladd(1883), p. 28]. A further point in favor of exclusions over inclusions is that 'the copula $\prec$ must be taken in an inverted sense according as subject and predicate are taken in extension or in intension' [Ladd(1883), p. 28], but there is no such requirement for $\overline{\vee}$, because of its symmetric nature.

This notation was first introduced in order to allow us to express not only equalities between terms but also categorical propositions. The notation is powerful enough, though, that we can do more than that. We can express existence and nonexistence claims via ∞. For instance:

$$x \mathbin{\overline{\vee}} \infty \tag{17}$$

means 'x does not, under any circumstances, exist', and

$$x \vee \infty \tag{18}$$

means that 'x is at least sometimes existent' [Ladd(1883), p. 29].

Because, unlike in the algebra of terms, the algebra of propositions does not have 0 [Ladd(1883), p. 29], we can drop reference to ∞ in contexts where it can be restored without ambiguity. Therefore, we can rewrite (17) and (18) as:

$$x\overline{\vee} \tag{17$'$}$$

and

$$x\vee \tag{18$'$}$$

This notation allows us to translate from categorical propositions (e.g., $a \mathbin{\overline{\vee}} b$ 'No a is b') into statements about the existence and nonexistence of terms (e.g., $ab\overline{\vee}$ 'The combination ab does not exist') [Ladd(1883), p. 30]. This extends to complex terms as well:

$$abc\overline{\vee} = a \mathbin{\overline{\vee}} bc = ca \mathbin{\overline{\vee}} b = \dots \tag{19}$$

(That is, saying 'the combination abc does not exist' is the same as saying 'a is excluded from the combination bc' is the same as saying 'the combination ca is excluded from b', etc.) This is because:

> the factors of a combination which is excluded or not excluded may be written in any order, and the copula may be inserted at any point, or it may be written at either end. [Ladd(1883), p. 30]

It is also possible to rewrite identity statements as statements involving exclusions [Ladd(1883), p. 31]. For example, both $a = b$

and $\bar{a} = \bar{b}$ can be rewritten as follows:

$$(a = b) = (a \bigtriangledown \bar{b})(\bar{a} \bigtriangledown b) = (\bar{a} = \bar{b}) \tag{20}$$

It thus follows that

$$(a = b) = (\bar{a} = \bar{b}) \tag{21}$$

This allows us to formulate two of the fundamental underpinning principles of classical logic: The Principle of Non-Contradiction (that no proposition is both true and false) and the Law of Excluded Middle (that every proposition is either true or false):

$$a\bar{a}\bigtriangledown \tag{PNC}$$

$$\overline{a + \bar{a}}\bigtriangledown \tag{LEM}$$

Finally, there is one further advantage of Ladd's formal system, rooted in the fact that it treats both categoricals and hypotheticals identically [Ladd(1883), p. 23].[9] That is, we can use a, b, c, etc., to pick out not only terms, as we have above, but also propositions. For example, if we let p denote a premise and c a conclusion following from p, then we can express this consequence as:

$$p \bigtriangledown \bar{c}$$

namely, that the negation of the consequent is inconsistent with the antecedent (is excluded from the antecedent).

Note, however, that the symmetry of the copula allows us to write this exclusion equivalently as

$$\bar{c} \bigtriangledown p$$

which, as Ladd points out, lacks the fact that 'the word inference... implies proceeding in a definite direction in an argument,—either from the premise to the conclusion, or from the negative of the conclusion to the negative of the premise' [Ladd(1883), p. 29]. It then follows that the mutual inconsistency of two propositions can be indicated by:

$$p \bigtriangledown c$$

Mutual inconsistency does *not* have the same sort of directionality that inference or implication does; so perhaps it is better to think of inference in this non-directional sense, as merely stating the mutual inconsistency of the premise(s) with the negation of the conclusion. This observation—that we can express inference or consequence (generally understood as a *directional* relationship, from the premises to the conclusion) in a *non-directional way*—is the key to understanding Ladd's solution, which we discuss in the next section.

With her chosen notation identified and motivated, Ladd identifies three subjects of interest for any symbolic logic [Ladd(1883), p. 30]:

1. the uniting and separating of propositions.
2. the insertion or omission of terms, or immediate inference.
3. elimination with the least possible loss of content, or syllogism.

We discuss each of these in turn, briefly in the case of the first two, and in more detail with the last, as it is elimination that interests us the most, as that is where her crucial result occurs.

First, note the following two identities, drawing on the definitions of sum and product that hold for both terms and propositions:

$$(a\overline{\vee})(b\overline{\vee}) = (a + b\overline{\vee}) \tag{22}$$

$$(a\vee) + (b\vee) = (a + b\vee) \tag{23}$$

In (22), the left-hand side states that a doesn't exist and b doesn't exist; while the right-hand side says that neither a nor b exist (that is, this is an instance of DeMorgan's laws). In (23), the left-hand says that either a exists or b exists, while the right-hand side says that either a or b exists; both of these identities should be understood as expressing schemata, with a and b replaceable with any logical expression of any degree of complexity (whether a proposition or a term). An immediate corollary of these two identities is the following:

Corollary 6.1 *'A combination of any number of universal propositions, or an alternation of any number of particular propositions, is then expressed as a single proposition by taking the sum of the elements of the separate propositions' [Ladd(1883), p. 32].*

This gives us a form of inference ('if it should be called inference at all' [Ladd(1883), p. 32]) where the conclusion is identical to the premise(s); and in fact (22) and (23) are, by (21), the same proposition:

Proof

$$(a\overline{\vee})(b\overline{\vee}) \quad = \quad (a + b\overline{\vee}) \quad (22)$$

is the same as:

$$\overline{(a\overline{\vee})(b\overline{\vee})} \quad = \quad \overline{(a + b\overline{\vee})} \quad (21)$$

which is the same as:

$$(a\vee) + b(\vee) \quad = \quad \overline{(a + b\overline{\vee})} \quad (\S 4.2)$$

which is the same as

$$(a\vee) + b(\vee) \quad = \quad (a + b\vee) \quad (\S 4.2)$$

$\square$

Therefore, of (22) and (23), only one is necessary as we can always derive the other. We can also factor both (22) and (23) into the following inconsistencies:

$$(a\overline{\vee})(b\overline{\vee})\,\overline{\vee}\,(a + b\vee) \tag{24}$$

and

$$(a\vee)(b\vee)\,\overline{\vee}\,(a + b\overline{\vee}) \tag{25}$$

However, while Corollary 6.1 shows that products of universal propositions and sums of partial propositions can be represented in a single proposition, 'there is no single expression in this algebra for a sum of universal propositions or a product of particular propositions' [Ladd(1883), p. 33].

Next, Ladd considers when terms can be added or dropped from logical expressions while maintaining equivalence in truth value.

If several terms are disjointly inconsistent with each other, then any subset of them will also be inconsistent:

$$(a + b + c\overline{\vee})\,\overline{\vee}\,(a + b\vee) \tag{26}$$

Similarly, if two terms are jointly inconsistent with each other, then conjoining another term to them will also be jointly inconsistent:

$$(abc\vee)\,\overline{\vee}\,(ab\overline{\vee}) \tag{27}$$

These inconsistencies can be expressed as inferences:

$$\text{If } a + b + c\,\overline{\vee}\,\text{, then } a + b\overline{\vee} \tag{28}$$

$$\text{If } ab\,\overline{\vee}\,\text{, then } abc\overline{\vee} \tag{29}$$

$$\text{If } a + b\vee\text{, then } a + b + c\vee \tag{30}$$

$$\text{If } abc\vee\text{, then } ab\vee \tag{31}$$

That is: (28) If three terms are disjointly inconsistent, then any two of them will be disjointly inconsistent; (29) if two terms are jointly inconsistent, then adding another term will not change the inconsistency; (30) if two terms are disjointly consistent, then they will be disjointly consistent with a third; and (31) if three terms are jointly consistent, then any two of them will be jointly consistent. These are all expressions of monotonicity conditions.

From these conditions we can derive the following theorem (equivalent, Ladd notes, to Theorem I of [Peirce(1867), XXI], 'if a is b and c is d, then ac is bd' [Ladd(1883), p. 34].):

$$(a \mathbin{\overline{\vee}} b)(c \mathbin{\overline{\vee}} d) \mathbin{\overline{\vee}} (ac \vee b + d) \tag{I}$$

At this point Ladd points out a benefit of stating propositions in terms of $\overline{\vee}$, namely, that it can treat the terms both intensionally and extensionally. More interestingly, the inconsistency (I) 'may be put into an inference in four different ways, according as both universals, one universal, one universal and the particular, or the particular alone, is taken as premise and the negative of what remains as a conclusion' [Ladd(1883), p. 36]:

If the argument is formed with two universal premises, it is of the following form:

$$\begin{aligned} &a \mathbin{\overline{\vee}} b \\ &c \mathbin{\overline{\vee}} d \\ \therefore &ac \mathbin{\overline{\vee}} b + d \end{aligned} \tag{32}$$

If it has one universal and one partial premise, then it is of the following form:

$$\begin{aligned} &a \mathbin{\overline{\vee}} b \\ &ac \vee b + d \\ \therefore &c \vee d \end{aligned} \tag{33}$$

and we'll show below how these can be converted into syllogisms.

Alternatively, these inferences could be construed in a one-premise format, folding the other premise into the conclusion. These 'one premise' versions do not give rise to syllogisms, but to a different type of argument, with a disjunctive conclusion. This gives us [Ladd(1883), p. 43]:

$$\begin{aligned} &a \mathbin{\overline{\vee}} b \\ \therefore &(ac \mathbin{\overline{\vee}} b + d) + (c \vee d) \end{aligned} \tag{34}$$

$$\begin{aligned} &ac \vee b + d \\ \therefore &(a \vee b) + (c \vee d) \end{aligned} \tag{35}$$

What all the forgoing demonstrates is the expressive power of Ladd's notation, which can account for both inconsistencies and inferences, and can correlate them together. What is interesting is that the correlation of inconsistencies and inferences is not one-one:

> Argument by way of inconsistencies, therefore, whatever may be thought of its naturalness, is at least $2(n-1)$ times more condensed than argument in the usual form. [Ladd(1883), p. 37]

Now, consider the premises of a basic syllogism like Barbara; between the two propositions there are three terms, one of which occurs in both of the premises (the 'middle' term). This term does not occur in the conclusion of the syllogism—it has been *eliminated*. This brings us to the final subject of symbolic logic, namely, the elimination of terms or, the syllogism. By 'elimination' we mean the absence of terms in the premise(s) of an inference from the conclusion. For example, in (32) there is no elimination, as all of the terms found in the premises are also found in the conclusion. But in (33), a and b occur in the premises but not in the conclusion; as Ladd notes, the first premise has been entirely eliminated and the second premise partially [Ladd(1883), p. 37].

Building upon this, Ladd makes a very interesting claim about the purpose of reasoning:

> The most common object in reasoning is to eliminate a single term at a time—namely, one which occurs in both premises. [Ladd(1883), p. 37]

The only way that it makes sense to identify this as the most common object of logic or reasoning is if it is a descriptive claim concerning the nature of reasoning at the time that Ladd was working, which was in many respects predominantly syllogistic. This goal of logic can be accomplished via the inference form in (I) by setting d equal to $\bar{b}$, so that $b+d=\infty$. We can then rewrite (I) as:

$$(a \,\overline{\vee}\, b)(\bar{b} \,\overline{\vee}\, c)(c \vee a)\overline{\vee} \tag{II}$$

(going via the intermediate equation $(a \,\overline{\vee}\, b)(c \,\overline{\vee}\, \bar{b})(ac \vee \infty)\overline{\vee}$).

Elimination is intimately linked to the argument from inconsistency. For any two of the three propositions whose inconsistency is asserted in Theorem 1.1, the third one lacks a term which is present in both of the other two (the middle term, in Aristotelian terminology). As a result, 'when any two of the inconsistent propositions in (1.1) are taken as premises, the negative of the remaining one is the conclusion' [Ladd(1883), p. 38].

Further, 'there are, therefore, two distinct forms of inference with elimination of a middle term, special cases of (32) and (33)' [Ladd(1883), p. 38]. If we let x stand for an arbitrary middle term, these two distinct forms of inference are the following. The first has two universal premises and a universal conclusion:

$$\begin{aligned} a \mathbin{\overline{\vee}} x \\ b \mathbin{\overline{\vee}} \bar{x} \\ \therefore ab\overline{\vee} \end{aligned} \tag{36}$$

The second has one universal and one partial premise, and a partial conclusion:

$$\begin{aligned} a \mathbin{\overline{\vee}} x \\ b \vee x \\ \therefore b\bar{a}\vee \end{aligned} \tag{37}$$

For those that are familiar with the traditional Aristotelian syllogistic mnemonics, when a and b are atomic terms, these are Barbara/Celarent and Darii/Ferio, respectively.

Just as (32) and (33) can be rewritten with an arbitrary middle term x and setting d to $\bar{b}$, so too can (34) and (35) be rewritten [Ladd(1883), p. 43]:

$$\begin{aligned} a \mathbin{\overline{\vee}} b \\ \therefore (a \mathbin{\overline{\vee}} x) + (\bar{b} \vee x) \end{aligned} \tag{38}$$

and

$$\begin{aligned} ac\vee \\ \therefore (a \vee x) + (c \vee b\bar{x}) \end{aligned} \tag{39}$$

One thing to note here is that whether the middle term occurs positively or negatively does not change its fundamental role as the middle term. The following rule is given for the validity of syllogisms of the form (36):

Definition 6.2 (Validity) [Ladd(1883), p. 39] A syllogism of the form (36) is valid if:

1. The middle term has unlike signs in the two premises.
2. The other terms have the same sign in the conclusion as in the premises.

Correspondingly, the rule for the validity of syllogisms of the form (37) is as follows:

Definition 6.3 (Validity) [Ladd(1883), p. 39] A syllogism of the form (37) is valid if:

1. The middle term must have the same sign in both premises.
2. The other term of the universal premise only has its sign changed in the conclusion.

At this point, we are now getting close to the main result in Ladd's thesis, the argument from inconsistency (Theorem 1.1). Theorem 1.1 is an *antilogism*, a word that Ladd introduces; it is in a sense the *opposite* of a syllogism, instead asserting the inconsistency of three propositions, that is, the combination of any two with the negation of the third will be true. Whichever combination is taken, the term that is present in the two will be 'eliminated' from the negation of the third. Ladd's theorem is that every valid syllogism can be reduced to this statement of trifold inconsistency, and this antilogism is the argument of inconsistency with which we opened the paper.

From Theorem 1.1 a corollary follows, in the form of an easy to apply rule, stated in ordinary English, for identifying whether any syllogism is valid:

Rule 6.4 Take the contradictory of the conclusion, and see that the universal propositions are expressed with a negative copula and particular propositions with an affirmative copula. If two of the propositions are universal and the other particular, and if that term only which is common to the two universal propositions has unlike signs, then, and only then, the syllogism is valid. [Ladd(1883), p. 41]

A syllogism that has been rewritten into an antilogism according to this rule Ladd calls an 'antilogism *in the canonical form*' [Ladd-Franklin(1928), p. 533]. Note that some rewriting will be involved to ensure that universal propositions are always expressed with a negative copula and partial propositions with an affirmative: If we admit 'infinite' terms, then this rewriting into symmetric propositions is always possible, by taking the complement of the predicate and changing the quality of the copula, as 'All S are non-P' is equivalent to 'No S is P', and 'Some S is not P' is equivalent to 'Some S is non-P' (cf. [Reichenbach(1952), p. 1]).[10] For instance, the partial negative claim 'Some men are not Greek' is equivalent to the partial affirmative claim 'Some men are non-Greek', and the universal affirmative claim 'All men are mortal' is equivalent to the universal negative claim 'All men are-not non-mortal'.

Ladd gives two examples of how this method works in practice:

Example 6.5 The syllogism Baroco[11]:

$$\text{All } P \text{ is } M$$
$$\text{Some } S \text{ is not } M$$

$$\therefore \text{Some } S \text{ is not } P$$

is equivalent to the inconsistency

$$(P \,\overline{\vee}\, \bar{M})(S \vee \bar{M})(S \,\overline{\vee}\, P)\overline{\vee}$$

Example 6.6 The syllogism Bocardo[12]:

$$\text{Some } M \text{ is not } P$$
$$\text{All } M \text{ is } S$$
$$\therefore \text{Some } S \text{ is not } P$$

is equivalent to the inconsistency

$$(M \vee \bar{P})(M \,\overline{\vee}\, \bar{S})(S \,\overline{\vee}\, \bar{P})\overline{\vee}$$

Ladd also gives two examples of invalid syllogisms that can be demonstrated to be invalid according to this method [Ladd(1883), pp. 41–42]. We consider only the first here, since the second is invalid in the same way:

> Only Greeks are brave,
> All Spartans are Greeks,
> Therefore all Spartans are brave.

is equivalent to the following inconsistency:

> Non-Greeks are-not brave,
> Spartans are-not non-Greeks,
> Some Spartans are not-brave.

(Note that the first premise in the original syllogism is not properly categorical, as it contains the exclusive 'only'; however, it is properly converted into a categorical in the inconsistency.) This fails the test in two ways; first, 'Greeks' appears in both premises, and it has the same sign in both; second, 'brave' appears with different signs, and yet it is not the middle term.

All other valid syllogisms can be reduced to an inconsistent triad in a similar way, and *this* is the solution that Ladd has been widely lauded for.

7 What problem is this a solution for?

In the preceding sections I have outlined and explained Ladd's contribution to the development of logical algebras in the 19th century. We have identified what the antilogism is, and situated it within the context of her definition of a new type of algebra with a symmetric copula. At this point, while the reader may understand what the 'solution' is, they may be forgiven for still being uncertain as to what the problem it is a solution for. In fact, it was about this point in my research that I found myself mired in a pit of metaphysical despair: Not only had I forsaken the logical high road of symbols and proof, I found myself asking such questions as: *What counts as a problem?* In particular, this coalesced into two pairs of questions: (1) Does a problem have to be recognised as a problem for it to be a problem? Or is it possible that Ladd solved a problem without realising it? (2) Assuming that Ladd addressed a lacuna in Aristotle, did that 'problem' exist through the two millennia in which no one was bothered by it? Or did it only become a problem once someone found it problematic?

In this section, we depart from Ladd exegesis and attempt to address some of these questions, as well as the motivating question of the paper, namely: Why do people think that Ladd's solution is a solution to a problem of Aristotle? This is not a question of logic but rather of history—or rather, historiography—of logic: It's an issue of not merely of what the actual historical events were, but how they came to be interpreted in the way that they are. In order to motivate our reconstruction of the historical events that led to this (mis)interpretation, there are three points we wish to make in this section:

1. Aristotle already had the rudiments of this solution.
2. Ladd thought she was solving a problem due to *Jevons*, not Aristotle.
3. The attribution of the solution as one of an Aristotelian problem post-dates the solution by some fifty years.

By taking a detour through a bit of historiography, we will show how Russinoff and recent scholars others came to repeat this incorrect attribution, by failing to take sufficient account of the history of syllogistic logic and of what actually occurs in Ladd's thesis.

We will take each of these points in turn, each in their own subsection.

7.1 The solution was already in Aristotle To the first point: One reason why we should be hesitant to describe what Ladd did as a solution to a problem that plagued Aristotle is because the rudiments of this method are already incorporated in the Aristotelian reductions, as we pointed out in §3, and was also noticed by Kattsoff in 1936:

> This method is actually the method of indirect reduction which is denoted by the letter 'k' in the mnemonic names *Baroko* and *Bokardo*[13] of the Aristotelian logic. The name antilogism was given to this by Mrs. Ladd-Franklin. [Kattsoff(1936), p. 385]

The difference is merely that Aristotle took Barbara as basic and Baroco and Bocardo as derived, while Ladd showed that one can take as 'basic' the inconsistency of the following three claims:

All M are P
All S are M
Not all S are P

Any two of these propositions entails the denial of the third; which is to say that the contradictory of any of the propositions follows from the other two, which gives us all three syllogisms.

What's the benefit of doing it this way instead of Aristotle's way? First, Ladd says:

> If for the usual three statements consisting of *two premises and a conclusion* one substitutes the equivalent *three statements that are together incompatible*. . . one has a formula which has this great advantage: the order of the statements is immaterial—the *relation* is a perfectly symmetrical one. [Ladd-Franklin(1928), p. 532]

In addition to the symmetry of the relation, the result is

> a source of great simplicity—there is only *one* valid form of the antilogism instead of the fifteen valid forms of the syllogism which common logic requires us to bear in mind. [Ladd(1883), p. 532]

Thirdly, both the simplicity and the symmetry can be improved upon if all of the three claims can be written as either (e) 'No S are P' or (i) 'Some S is P' claims, which can be simply converted. (Alternatively, the symmetric forms 'All but S is P' and 'Not all but S is P' can be taken).

The fact that the rudiments of the solution were already in Aristotle, combined with the fact that this so-called 'problem' of reduction was not picked up by any of the main ancient or medieval commentators on Aristotle gives us reason to think that whatever Ladd's solution was for, it was *not* for a problem of Aristotle's.

7.2 What did Ladd think she was doing? So let us now turn to what problem *Ladd herself* thinks she solved. It is quite clear from her thesis that Ladd did not think she was solving a problem of Aristotle (in fact, 'Aristotle' is not mentioned once in her dissertation, and 'Aristotelian' only once [Ladd(1883), p. 66]); she thought she was solving a problem of *Jevons*, and did not see herself as solving a problem that had plagued the syllogism for two millennia. Instead, she saw her work as firmly situated in the developments initiated by Boole. In another paper, written thirty-five years after the publication of Boole's *Laws of Thought*[14], Ladd sums up Boole's contribution to logic thusly:

> The task which Boole accomplished was the complete solution of the problem:—given any number of statements, involving any number of terms mixed up indiscriminately in the subjects and the predicates, to eliminate certain of those terms, that is, to see exactly what the statements amount to irrespective of them, and then to manipulate the remaining statements so that they shall read as a description of a certain

> other chosen term (or terms) standing by itself in a subject or predicate. [Ladd Franklin(1889), p. 543]

The simplest example of this type of manipulation is found in the syllogism—three propositions, three distinct terms, each occurring twice, and one term which is eliminated from the premises when generating the conclusion. The difficulty in Boole's solution is determining what is the term (or terms) to be eliminated; once the term is identified, 'an ordinary syllogism would suffice to put it to flight' [Ladd Franklin(1889), p. 544]. This difficulty was taken up by others working in the Boolean algebra tradition, including by Jevons. According to Ladd, her algebra, including the antilogism, 'contains a solution of what Mr. Jevons calls the 'inverse logical problem' ' [Ladd(1883), p. 50]. The *Inverse Problem* is:

> given certain combinations inconsistent with conditions to determine those conditions. [Jevons(1880), p. 252]

The solution that Jevons provides 'consists in inventing laws and trying whether their results agree with those before us' [Jevons(1880), p. 252]. A more precise characterisation of the *Inverse Logical Problem Involving Three Terms* is given in [Jevons(1874), p. 157]:

> Three terms and their negatives may be combined... in eight different combinations, and the effect of laws or logical conditions is to destroy any one or more of these combinations. Now we may make selections from eight things in 2^8 or 256 ways; so that we have no less than 256 different cases to treat, and the complete solution is at least fifty times as troublesome as with two terms... The test of inconsistency is that each of the letters A, B, C, a, b, c[15] shall appear somewhere in the series of combinations; but I have not been able to discover any mode of calculating the number of cases in which inconsistency would happen... an exhaustive examination of the combinations in detail is the only method applicable. [Jevons(1874), pp. 157–158]

Ladd's solution is a solution to this problem because the antilogism gives rise to Rule 6.4, which gives a systematic method for determining conditions inconsistent with certain combinations. Because any given syllogism can be reduced to a syllogism of the form given in Theorem 1.1, one need not blindly invent laws and see if they agree with results.

7.3 Why do people think this was Aristotle's problem? Having established that Ladd thought she was solving a problem in Boolean algebra articulated by Jevons, we are then left with the following question: How do we get from what *actually* happened to the statements of, e.g., Russinoff and Pietarinen that Ladd solved a 'millennia old' problem, and that the problem was one of Aristotle's?

In the earliest review of Ladd's dissertation, [Anonymous(1883)], no specific mention is made of the result in Theorem 1.1. The (unidentified) reviewer introduces Ladd's new notation, $\vee$ and $\overline{\vee}$, gives its semantics and formation rules, and notes that 'with these she is able to write algebraically all the old forms of statement, and to perform the customary operations of symbolic logic with great brevity and facility' [Anonymous(1883), p. 514]. Schröder devotes a section of his *Vorlesungen über die Algebra der Logik* to Ladd's 'Formel' and the application of it to the reduction of syllogisms, in 1891 [Schröder(1891), §43]. Drawing on Schröder's work, Müller in the early 20th century uses Ladd's method in his discussion of *Das Eliminationsproblem und die Syllogistik*, calling it 'die Ladd-Franklin'sche Formel' [Müller(1901)]. The singling out of the antilogism as a fundamental contribution is first (as far as I can tell) made by Brown some twenty-five years after the publication of her thesis: 'when Mrs. Ladd-Franklin has demonstrated that one simple form underlies all syllogism. . . ' [Brown(1909), p. 304].

But something had happened by the 1920s. One of the first detailed discussions of Ladd's antilogism in English is found in [Shen(1927)].[16] In this short (6-page) article, Shen gives an account of Ladd-Franklin's eight-fold propositional structure as superior to the four-fold (that is, *a*, *e*, *i*, and *o* categorical propositions of Aristotle) structure of what he calls, derogatorily, 'common logic'. He introduces a slightly different notation from what is used in [Ladd(1883)], and proceeds to demonstrate a number of equivalent notations for the basic proposition types:

a: No p is q.
b: All but p is q.
c: All p is q.
d: None but p is q.
$\boldsymbol{\alpha}$**:** Some p is q.
$\boldsymbol{\beta}$**:** Not all but p is q.
$\boldsymbol{\gamma}$**:** Not all p is q.
$\boldsymbol{\delta}$**:** Some besides p is q.

> A term represents any concept, which may happen to be an object, an aggregate of objects (a class), a quality (attribute), a congeries of qualities, or a proposition. The use of the copula 'is' in the verbal translation of the symbols must not be taken as an indication that they can only apply to the logic of extension. [Shen(1927), p. 55]

The transformations used to generate the equivalent forms can also be used to show that 'all valid moods of the syllogism are reducible to, or deducible from, the one single formula of the Antilogism' [Shen(1927), p. 58]. The difference between a syllogism and the antilogism is that a syllogism 'states that two premises are sufficient for the conclusion' while the antilogism states that 'two premises are incompatible with the negative of the conclusion' [Shen(1927), p. 58]. Shen gives a nice way of how we can read or understand the antilogism in English: 'Unless at least one of the two premises is rejected, the conclusion is inevitable' [Shen(1927), p. 58, fn. 1]. An antilogism is valid when the following three rules are satisfied [Shen(1927), p. 59]:

1. Any two propositions have one term the same and one term different.
2. One and only one proposition is *particular*.
3. A term that appears in the *particular* proposition has the same sign in a *universal* proposition; a term that appears in both *universal* propositions has opposite signs.

Shen then quotes the 'late Professor Josiah Royce of Harvard', (1855–1916, who, in addition to being the pre-eminent metaphysician at the turn of the 19th century in the USA, was regarded as "a logician of the first rank" ' [Pratt(2007), p. 133]), in a reference that, I argue, provides us the key to where the contemporary confusion regarding Ladd arises:

> There is no reason why this should not be accepted as the definitive solution of the problem of the reduction of syllogisms. It is rather remarkable that the crowning activity in a field worked over since the days of Aristotle should be the achievement of an American woman. [Shen(1927), p. 60]

This gives us an indication of what the problem was and how long it had been thought to be a problem.

Shen doesn't give a source for this quote of Royce, but it appears to have been taken from a newspaper article (perhaps two), written around the time Ladd finally received her degree, in 1926. Pietarinen identifies one source for this quote in the *Hartford Courant*:

> In a newspaper clip 'To Get Her Degree Earned Years Ago', Josiah Royce is quoted as describing her thesis work as 'the crowning activity in a field worked over since the days of Aristotle'. 'The [Aristotelian] system was never fully demonstrated until Mrs. Ladd-Franklin worked out the whole method at Johns Hopkins' (*The Hartford Courant*, February 21, 1926, p. 20). [Pietarinen(2013), fn. 6]

Pietarinen goes on to say:

> See Russinoff (1999) on how, in her dissertation, Ladd-Franklin in fact managed to solve—or at least to see the solution to—the problem that was over two millennia old, though she did not give, nor could she have given the proof in such a rigorous form that is possible nowadays in the semantic terms of possible interpretations in varying domains. [Pietarinen(2013), fn. 6]

This is, again, a glowing recommendation of Ladd's work, but it assumes that *we know what the problem is.*

A slightly different source for the quote from Royce is given by Spillman in a footnote:

> Royce told his students, 'It is rather remarkable that the crowning activity in a field worked over since the days of Aristotle should be the achievement of an American woman. 'Professor Royce on an American Woman's Work,' *New York Evening Post*, n.d., Box 14, CLF-FF Papers. [Spillman(2012), fn. 29]

Royce himself died nearly a decade earlier, and it is not clear that this newspaper quote can be any further substantiated. Nevertheless, it is clear that there was something of a sea-change in how Ladd's fundamental contribution in her thesis was viewed that happened sometime in the 1910s and 1920s. Whether Royce is simply reflecting a common characterization of the antilogism from his time, or whether he is the *origin* of it, we cannot say; but either way, this profuse praise of his in her honour—brought to light again around the time that Ladd was finally being recognized formally for her achievements fifty years earlier, unsurprisingly caught the attention of later scholars, who—perhaps not having looked beyond Ladd's thesis to the wider context that she was working in—were happy to take his analysis of her contribution at his word.

Even so, it took awhile for Royce's characterization of Ladd's contribution to gain hold; even in the middle of the 20th century, many logicians still recognized Ladd's work in its rightful guise, as a contribution to the development of Boolean algebra. For instance, Beth describes Ladd's contribution thus:

> It should be mentioned here, that the first adherents of symbolic logic had no luck in their treatment of classical syllogism. The reason for this drawback was, that they exclusively dealt with equations, whereas in the symbolism of logical algebra an existential statement can only be expressed by an inequality. In 1883, Mrs Christine Ladd-Franklin gave the first adequate treatment of classical syllogism, by means of a symbolism, created *ad hoc*. [Beth(1947), p. 23]

The antilogism isn't even mentioned, because Beth understood it as it was—a *part* of a wider project, and not the crowning achievement of the project, nor the solution of a long-lasting problem.

8 Conclusion

We can then sum up the historical facts as follows:

- In her 1883 dissertation, Ladd-Franklin introduced to Boolean algebra a pair of symmetric copulas.

- This allowed her to define the 'antilogism', an 'inconsistent triad' that could be used to represent every valid syllogism.
- People recognised the utility of this representation soon after her work.
- Within 30 years, people made the leap to her formula being a *solution* to a *problem*.
- Within 40 years, people attributed the problem to Aristotle.
- At some point after that, the problem attributed to Aristotle was attributed as a problem to all intervening logicians, too.
- But while she might have solved a problem, it certainly wasn't Aristotle's, nor had it vexed people for millennia.

To conclude, I would like to just say that none of the foregoing is meant to take away from Ladd *or* her contribution. Ladd is justly valued as a brilliant algebraist whose new copulas represent a significant advancement on the other algebras of her time. The ability to represent both terms and propositions in an entirely analogous fashion is genuinely novel and also extremely powerful. There is much in her thesis that we have not explored here that is worth further study. So I want to be clear that I have no quarrel with either her contributions or her importance to the history of logic. I only want to set the historical record straight as to what it was that contribution was, and why this contribution is important. It has nothing to do with Aristotle, and everything to do with algebra. If we're going to sing the praises of *an American woman*'s contributions to logic, let us do it for what she actually did.[17]

Notes

1. This quantity is called by many, Ladd inclued, 'particular'. We follow contemporary usage in prefering 'partial', trusting that no confusion will arise.

2. There is some dispute amongst scholars of Aristotle whether a syllogism consists in two premise *plus a conclusion* which follows validly from those premises, or whether the syllogism is *just the pair of premises*, which can give rise to potentially more than one conclusion. It is not necessary to adjudicate the issue here, as nothing of import turns on the definition.

3. One could object to this characterisation of Aristotle's results. For, in one sense he did provide a unified and complete treatment of the syllogistic argument: He introduced conversion rules, and showed how one could reduce every valid syllogism to one of the four perfect ones, and he gave meta proofs showing that no reduction could be given for the ones that are invalid.

4. It is interesting to note that this quote comes from the preprint version of the paper; in the published version, the result is couched in much less fabulous terms: ‘The result was the reduction of the Aristotelian syllogistics into a single formula’ [Pietarinen(2013), p. 142]. It would be interesting to know at what point of the publication process this statement was changed, and why.

5. The appropriate way to generate complements of complex terms is discussed in §4.2.

6. This lack of a distinction in notation between atomic and complex terms is but one of the ways in which reading Ladd’s work is difficult for a modern logician.

7. Though it might be argued that this is a drawback, rather than an advantage.

8. Note that at this point in her paper, Ladd switches from using a and b for terms/propositions, to using A and B. She does not comment on this switch, and we continue to use a and b to hopefully reduce confusion.

9. Some people may see this as identical treatment as an advantage. A disadvantage is that this takes Ladd’s system another step away from Aristotle’s.

10. A brief correction to Reichenbach’s history of the syllogism. Reichenbach notes that ‘Aristotle does not refer to these forms [i.e., infinite terms] in his discussion of the syllogism in the *Analytics*. This omission seems never to have been corrected until modern times’ [Reichenbach(1952), p. 2]. This is categorically false: The use of infinite terms was routine in medieval discussions of the syllogism.

11. Note: Ladd calls it ‘Baroko’. We have departed from her nomenclature to use the now-usual mnemonic name, which hearkens back to the medieval mnemonics with ‘c’ mnemonically picking out Latin *per **c**ontradictione*.

12. Note: Ladd calls it ‘Bokardo’.

13. Cf. fn. 8.

14. The paper, [Ladd Franklin(1889)], is itself a really nice introduction to the development of symbolic logic from Boole to the present date, i.e., a 35 year period; Ladd’s own developments are only briefly referenced. For anyone who is interested in the history of logical developments in the second half of the 19th century, I highly recommend this paper as a basically first-hand account.

15. In Jevons’ notation, A stands for the presence of the general term that A denotes, and a stands for its absence, cf. [Jevons(1874), p. 154].

16. The antilogism is also discussed in Johnson’s textbook a few years earlier, [Johnson(1922), pp. 78, 87]; however, he does not credit Ladd with the discovery, as she rightly complains about in [Ladd-Franklin(1928), p. 532]: ‘I take it very ill of Mr. W.E. Johnson that he has robbed me, without acknowledgment, of my beautiful word ‘antilogism’.’

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