Continuous dependence for the Brinkman - Darcy - Kelvin - Voigt equations backward in time

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Abstract

We show that the solution to the Brinkman - Darcy - Kelvin - Voigt equations backward in time depends Hölder continuously upon the final data. A logarithmic convexity technique is employed and uniqueness of the solution is simultaneously achieved.

1 Introduction

The problem of analysing the solution to an improperly posed problem for a system of partial differential equations has attracted many writers over the years and continues to do so recently,, see e.g. Agmon [1966], Agmon and Nirenberg [1967], Ames and Hughes [2005], Ames and Straughan [1997], Benrabah et al. [2020], Caflisch et al. [2017], Carasso [2013, 2019, 2020], Chirita [2014], Chirita and Zampoli [2015], Fury [2020], Fury and Hughes [2012],Hetrick and Hughes [2009], John [1960], Knops and Payne [1968], Yang and Deng [2017]cw., In particular, the pioneering paper of John [1960] showed how one could recover a restricted class of stable solutions by requiring an *a priori* bound at one particular place. This work has influenced many of the subsequent articles.

There has been a considerable amount of work dealing with solutions to the Navier - Stokes equations backward in time with regard to establishing uniqueness, stability (in a sense like that of John [1960]), and structural stability, cf. Ames and Payne [1994], Carasso [2020], Crispo et al. [2019], Galdi and Straughan [1988], Harfash [2013, 2014], Payne [1971, 1992, 1993, 1975], Payne and Straughan [1989, 1999], Straughan [1983]. Such problems are of practical value in extrapolating from the past and computational methods may be based on analytical results, as explained in detail by Carasso [2020].

In addition to the Navier - Stokes equations for flow of a linearly viscous incompressible fluid, there has been considerable interest in describing fluids which are viscous but remember some of their past history, so called viscoelastic fluids, see e.g. Amendola and Fabrizio [2010], Amendola et al. [2009], Fabrizio et al. [2015], Franchi et al. [2014, 2015a,b], Gatti et al. [2005]. A special class of these viscoelastic models is known as Kelvin - Voigt materials, Chirita and Zampoli [2015], and applied as extensions of the Navier - Stokes theory through the Navier - Stokes - Voigt equations, see e.g. Berselli and Bisconti [2012], Di Plinio et al. [2018], Sviridyuk and Sukacheva [1998].

The goal of this work is to establish an appropriate stability estimate for the solution to the Navier - Stokes - Voigt equations backward in time, which we do by implementing a suitable logarithmic convexity method. Such techniques are described in a variety of contexts by e.g. Agmon [1966], Ames and Straughan [1997], Carasso [1994, 1999, 2013, 2019, 2020], Crispo et al. [2019], Knops and Payne [1968], Payne [1975].

2 The Navier - Stokes -Voigt equations

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary Γ sufficiently smooth to allow use of the divergence theorem. The inner product and norm on $L^2(\Omega)$ will be denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively.

If the velocity field is denoted by v_i then the Navier - Stokes - Voigt equations may be written as

$$
\begin{aligned}
&v_{i,t} - \alpha \Delta v_{i,t} + v_j v_{i,j} = \Delta v_i - p_{,i}\,, \\
&v_{i,i} = 0,
\end{aligned}
\tag{1}
$$

where without loss of generality for the analysis herein the viscosity and density have been taken to have value 1, $\alpha > 0$ is a constant, p is the pressure field, Δ is the Laplacian, standard indicial notation is employed in conjunction with the Einstein summation convention, and $_{,t}$ denotes $\partial/\partial t$, whereas $_{,i}$ denotes $\partial/\partial x_i$.

In this work we reverse time to analyse a solution to the boundary - initial value problem backward in time. Di Plinio et al. [2018] allow (1) to also contain a Rayleigh friction term (linear in v_i) and we include this. The relevant equations backward in time may then be written

$$
\begin{aligned}
&v_{i,t} - \alpha \Delta v_{i,t} = v_j v_{i,j} - \Delta v_i + p_{,i} + \beta v_i\,, \\
&v_{i,i} = 0,
\end{aligned}
\tag{2}
$$

where β is a constant. Equations (2) hold on the domain $\Omega \times (0, T]$, for $T < \infty$, and the boundary - initial value problem is completed with boundary conditions

$$
v_i(\mathbf{x}, t) = h_i(\mathbf{x}, t), \qquad \mathbf{x} \in \Gamma,\ t > 0, \tag{3}
$$

and the initial conditions

$$
v_i(\mathbf{x}, 0) = g_i(\mathbf{x}), \qquad \mathbf{x} \in \Omega. \tag{4}
$$

The boundary - initial value problem consisting of (2) - (4) is denoted by $\mathcal{P}$.

3 Stability backward in time

To study stability of a solution to $\mathcal{P}$ we let u_i, p and v_i, q be two solutions for the same boundary data h_i, but for different initial data functions g_i^u and g_i^v. Define $w_i = u_i - v_i$ and $\pi = p - q$, and then from (2) - (4) one finds w_i satisfies the boundary - initial value problem

$$\begin{aligned} & w_{i,t} - \alpha\Delta w_{i,t} = u_j w_{i,j} + w_j v_{i,j} - \Delta w_i + \pi_{,i} + \beta w_i \, , \\ & w_{i,i} = 0, \end{aligned} \tag{5}$$

holding on $\Omega \times (0, T]$, with boundary conditions

$$w_i(\mathbf{x}, t) = 0, \qquad \mathbf{x} \in \Gamma, \; t > 0, \tag{6}$$

and initial conditions

$$w_i(\mathbf{x}, 0) = w_i^0(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \tag{7}$$

where $w_i^0 = g_i^u - g_i^v$.

Let $\|\cdot\|_p$ denote the norm on $L^p(\Omega)$. We now introduce the constraint classes for a function. We say $\phi_i \in \mathcal{M}_1$ if

$$\sup_{\Omega\times(0,T]} \phi_i\phi_i \;\le\; M_1^2, \tag{8}$$

and $\psi_i \in \mathcal{M}_2$ if

$$\sup_{\Omega\times(0,T]} \left\{ \|(\psi_{i,j} - \psi_{j,i})(\psi_{i,j} - \psi_{j,i})\|_{3/2} + \|\psi_{i,t}\psi_{i,t}\|_{3/2} \right\} \;\le\; M_2^2, \tag{9}$$

for constants M_1 and M_2.

Theorem 1

Suppose u_i, v_i satisfy $\mathcal{P}$ as outlined above with $u_i \in \mathcal{M}_1$, $v_i \in \mathcal{M}_2$. Let also M be an *a priori* bound for w_i at time T in the following sense,

$$\|\mathbf{w}(T)\|^2 + \alpha\|\nabla\mathbf{w}(T)\|^2 \;\le\; M < \infty. \tag{10}$$

Then the solution to $\mathcal{P}$ depends Hölder continuously upon the initial data on compact subsets of $[0, T)$.

Proof. Define the function $F(t)$ by

$$F(t) = \|\mathbf{w}(t)\|^2 + \alpha\|\nabla\mathbf{w}(t)\|^2 \, . \tag{11}$$

By differentiation,

$$F'(t) = 2(w_i, w_{i,t}) + 2\alpha(w_{i,j}, w_{i,jt}). \tag{12}$$

Next, integrate by parts the $w_{i,jt}$ term and then employ equations (5), to find after further integration by parts and use of the boundary conditions

$$F'(t) = 2\|\nabla\mathbf{w}\|^2 + 2(w_i, w_j v_{i,j}) + 2\beta\|\mathbf{w}\|^2.$$

Next differentiate again to obtain after some integration by parts

$$\begin{aligned}F''(t) =&2(w_{i,t}, w_j v_{i,j}) + 2(w_i, w_{j,t} v_{i,j}) + 2(w_i, w_j v_{i,jt})\\&- 4(w_{i,t}, \Delta w_i) + 4\beta(w_i, w_{i,t}).\end{aligned}$$

One now substitutes for Δw_i from $(2)_1$ and after further integration by parts and use of the boundary conditions one may arrive at the expression

$$\begin{aligned}F''(t) =&4\|\dot{\mathbf{w}}\|^2 + 4\alpha\|\nabla\dot{\mathbf{w}}\|^2 - 4(w_{i,t}, u_j w_{i,j})\\&- 2(w_{i,t}, w_j[v_{i,j} - v_{j,i}]) - 2(w_{i,j}, w_j v_{i,t}),\end{aligned}\tag{13}$$

where $\dot{\mathbf{w}} \equiv w_{i,t}$. Introduce the function χ_i by

$$\chi_i = w_{i,t} - \frac{1}{2}u_j w_{i,j} - \frac{1}{4}w_j(v_{i,j} - v_{j,i}).$$

Then F'' in (13) may be rewritten in the form

$$\begin{aligned}F''(t) =&4\|\chi\|^2 + 4\alpha\|\nabla\dot{\mathbf{w}}\|^2 - 2(w_j, w_{i,j} v_{i,t}) - \|\mathbf{u}\cdot\nabla\mathbf{w}\|^2\\&- (u_j w_{i,j}, w_k[v_{i,k} - v_{k,i}]) - \frac{1}{4}\|w_j(v_{i,j} - v_{j,i})\|^2,\end{aligned}\tag{14}$$

where

$$\|w_j(v_{i,j} - v_{j,i})\|^2 = \int_\Omega w_j(v_{i,j} - v_{j,i}) w_k(v_{i,k} - v_{k,i})dx.$$

Upon employing the arithmetic - geometric mean and Cauchy - Schwarz inequalities on the right of (14) we may arrive at

$$\begin{aligned}F''(t) \geq&4\|\chi\|^2 + 4\alpha\|\nabla\dot{\mathbf{w}}\|^2 - 2(w_j, w_{i,j} v_{i,t})\\&- 2\|\mathbf{u}\cdot\nabla\mathbf{w}\|^2 - \frac{1}{2}\|w_j(v_{i,j} - v_{j,i})\|^2.\end{aligned}\tag{15}$$

We next estimate the last three terms on the right of (15).

Firstly use the Cauchy - Schwarz and Hölder inequalities to see that

$$\begin{aligned}(w_j, w_{i,j} v_{i,t}) \leq&\frac{\epsilon}{2}\int_\Omega |\mathbf{w}|^2\,|\dot{\mathbf{v}}|^2 dx + \frac{1}{2\epsilon}\|\nabla\mathbf{w}\|^2\\ \leq&\frac{\epsilon}{2}\|\mathbf{w}\|_6^2\Big(\int_\Omega |\dot{\mathbf{v}}|^3 dx\Big)^{2/3} + \frac{1}{2\epsilon}\|\nabla\mathbf{w}\|^2.\end{aligned}$$

Now use the Sobolev inequality $\|\mathbf{w}\|_6 \leq c\|\mathbf{w}\|_{H^1}$ for c constant, and then select $\epsilon = 1/M_2 c$ where M_2 is defined in (9). In this way we derive

$$2(w_j, w_{i,j} v_{i,t}) \leq 2M_2 c\|\nabla\mathbf{w}\|^2.\tag{16}$$

Using (8) we deduce

$$2\|\mathbf{u}\cdot\nabla\mathbf{w}\|^2 \leq 2M_1^2\|\nabla\mathbf{w}\|^2.\tag{17}$$

Finally, using (9) and the Sobolev inequality,

$$
\begin{aligned}
\frac{1}{2}\|w_j(v_{i,j}-v_{j,i})\|^2 \le& \frac{1}{2}\|\mathbf{w}\|_6^2\Big(\int_\Omega \big[(v_{i,j}-v_{j,i})(v_{i,j}-v_{j,i})\big]^{3/2}dx\Big)^{2/3} \\
\le& \frac{1}{2}c^2M_2^2\|\nabla\mathbf{w}\|^2.
\end{aligned}
\tag{18}
$$

Employing (16) - (18) we find that the last three terms on the right of (15) may be bounded below by the term $-k_1\|\nabla\mathbf{w}\|^2$ where

$$
k_1 = 2M_2c + \frac{1}{2}c^2M_2^2 + 2M_1^2.
$$

Thus, from (15) we find

$$
\begin{aligned}
F''(t) \ge& 4\|\chi\|^2 + 4\alpha\|\nabla\dot{\mathbf{w}}\|^2 - k_1\|\nabla\mathbf{w}\|^2 \\
\ge& 4\|\chi\|^2 + 4\alpha\|\nabla\dot{\mathbf{w}}\|^2 - kF,
\end{aligned}
\tag{19}
$$

where $k = k_1/\alpha$.

We now form the combination $FF'' - (F')^2$ using (19), (12) and (11), and note that $(\chi_i, w_i) = (w_i, w_{i,t})$. Then from (19) we may establish the inequality

$$
FF'' - (F')^2 \ge 4S^2 - kF^2, \tag{20}
$$

where

$$
\begin{aligned}
S^2 =& (\|\mathbf{w}\|^2 + \alpha\|\nabla\mathbf{w}\|^2)(\|\chi\|^2 + \alpha\|\nabla\dot{\mathbf{w}}\|^2) \\
&- \big[(w_i,\chi_i) + \alpha((\nabla\mathbf{w},\nabla\dot{\mathbf{w}})\big]^2,
\end{aligned}
$$

which is non-negative by virtue of the Cauchy - Schwarz inequality. Whence from (20) we obtain

$$
FF'' - (F')^2 \ge -kF^2. \tag{21}
$$

Upon dividing by F^2 we may see that

$$
\frac{d^2}{dt^2}\Big[\log\big\{F\exp(kt^2/2)\big\}\Big] \ge 0. \tag{22}
$$

Using the properties of a convex function one may then show that

$$
F(t) \le \exp\Big[\frac{kt}{2}(T-t)\Big]\,\big[F(0)\big]^{(T-t)/T}\,\big[F(T)\big]^{t/(T-t)}. \tag{23}
$$

The *a priori* bound of the theorem, (10), is now utilized and we derive from (23)

$$
F(t) \le K\,\big[F(0)\big]^{(T-t)/T}, \qquad t\in[0,T), \tag{24}
$$

where $K(t) = M^{t/(T-t)}\exp\big[kt(T-t)/2\big]$. The inequality (24) establishes Hölder continuity for a solution to $\mathcal{P}$ on a compact sub-interval of $[0,T)$ for a solution in the measure $\|\mathbf{w}\|$ or $\|\nabla\mathbf{w}\|$ and the theorem is proved.

Corollary 1

Under the conditions of theorem 1 the solution to $\mathcal{P}$ is unique.

Proof. By contradiction. Suppose u_i is not identically zero and $u_i \neq 0$ for $t > 0$ whence $F(t) > 0$ for $t \in [\epsilon, T]$. In this case we divide (21) by F^2 which leads to (22) on $[\epsilon, T]$. Fix $t \in (\epsilon, T)$ and from (22) we know by properties of a convex function,

$$-\infty < \log\Big[F(t)\,\exp\Big(\frac{kt^2}{2}\Big)\Big] \leq \Big(\frac{T-t}{T-\epsilon}\Big)\,\log\Big[F(\epsilon)\,\exp\Big(\frac{k\epsilon^2}{2}\Big)\Big] + \Big(\frac{t-\epsilon}{T-\epsilon}\Big)\,\log\Big[F(T)\,\exp\Big(\frac{kT^2}{2}\Big)\Big].$$

For uniqueness $F(0) = 0$, so we let $\epsilon \to 0$ to obtain a contradiction since t is fixed, and the proof of the Corollary follows.

4 Conclusions

We have derived a precise estimate for a solution to the Navier - Stokes - Voigt equations backward in time to depend Hölder continuously on the initial data on compact subsets of the interval $[0, T)$. We also allowed for the generalized Navier - Stokes - Voigt system which includes the Rayleigh friction term suggested in the work of Di Plinio et al. [2018]. The estimate obtained in (24) could be combined with the ideas of Carasso [2020] to allow one to compute accurate solutions to the Navier - Stokes - Voigt equations backward in time.

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