Multidimensional Bargaining and Posted Prices*

Martin Hagen† Ángel Hernando-Veciana‡

CUNEF DUBS

July 6, 2021

Abstract

A seller and a buyer bargain over the quantities and prices of multiple goods. Both agents have private information about their preferences. Utility is quasilinear in money. We show that a deterministic mechanism satisfies (i) dominant-strategy incentive compatibility, (ii) ex-post individual rationality and (iii) ex-post budget balance if and only if it is a posted-price mechanism. A similar, more general result holds if (iii) is replaced by ex-post collusion-proofness and a no-free lunch condition. We provide a unified proof of both findings via the property of non-bossiness.

Keywords: mechanism design; bilateral trade; multidimensional private information; posted prices; dominant strategies; collusion-proofness.

JEL Classification: C72, D82.

---

*This paper was previously titled "Multi-Unit Bargaining and Posted Prices". We thank Pasha Andreyanov, Salvador Barberà, Toomas Hinnosaar, Byeong-hyeon Jeong, Antoine Loeper, Hervé Moulin, Antonio Romero-Medina, Shigehiro Serizawa and four anonymous referees for helpful comments. We gratefully acknowledge financial support from the Spanish Ministry of Economy and Competitiveness, grants MDM 2014-0431 and ECO2015-68406-P, as well as from Comunidad de Madrid, grant MadEco-CM.

†Colegio Universitario de Estudios Financieros (CUNEF). Calle de Leonardo Prieto Castro, 2, 28040 Madrid, Spain. Email: martin.hagen@cunef.edu

‡Durham University Business School, Mill Hill Lane, Durham DH1 3LB, UK. Email: angel.hernando-veciana@durham.ac.uk

1 Introduction

Implementation in dominant strategies can be seen as the holy grail of mechanism design. By awarding each agent with a strategy that is optimal no matter what the other agents do, dominant-strategy mechanisms induce straightforward decision problems. This is desirable for a number of reasons. For example, it reduces the costs of belief formation, mitigates the vulnerability to cognitive errors and prevents the exploitation of less sophisticated agents. The downside of dominant-strategy mechanisms is that they are often severely restricted, which can diminish social welfare. In this paper, we provide a sharp characterization of these restrictions for the case of bilateral bargaining over multidimensional issues.

In our model, a seller (female) and a buyer (male) can trade discrete units of multiple goods. Monetary transfers between the two agents are possible, and their utility functions are quasilinear in money. Both have private information about their preferences: the seller about the shape of her strictly increasing and convex cost function, and the buyer about the shape of his strictly increasing and concave value function. A (direct, deterministic) mechanism asks both agents to report their preferences and then specifies the bundle to be traded, the payment to be made by the buyer and the payment to be received by the seller. Our interest is to characterize the mechanisms that satisfy *(dominant-strategy) incentive compatibility* and *(ex-post) individual rationality*. That is, each agent should have a dominant strategy and, when both of them follow their dominant strategies, they should be at least as well off as in the status quo.

A natural additional requirement is *(ex-post) budget balance*, which requires the agents' payments to coincide. Our first main result (Corollary 1) shows that there is a unique type of mechanism that satisfies our ex-post notions of incentive compatibility, individual rationality and budget balance: A third party exogenously specifies a baseline bundle and its price. The two agents are only allowed to trade certain multiples of the baseline bundle. Both announce their optimal quantities and then trade the smaller of the two (at the exogenous price per bundle).

Budget balance is unnecessarily demanding if the third party can absorb a budget surplus or cover a budget deficit. It is well known that if the third party has deep enough pockets, the efficient amount of trade can be implemented in dominant strategies by means of the Vickrey-Clarke-Groves (VCG) mechanism. However, the VCG mechanism is vulnerable to collusion: by coordinating their reports, the two agents can abuse the third party's generosity and extract additional subsidies. It is therefore of interest to replace budget balance by *(ex-post) collusion-proofness*, which eliminates joint deviations that make at least one agent better off without making the other agent worse off. In addition, we impose a *no-free-lunch* condition on the buyer, which says that he should not receive any good for free. Its purpose is to rule out unappealing mechanisms in which the seller is a dictator and the buyer's individual-rationality constraint holds trivially.

Our second main result (Corollary 2) characterizes all mechanisms that satisfy incentive compatibility, individual rationality, collusion-proofness and no free lunch. These mechanisms are generalizations of the posted-price mechanisms described above. As before, the third party exogenously specifies a set of tradable bundles and their prices. Both agents then announce their optimum bundles and trade the smaller of the two. There are two main differences compared to the budget-balanced case: First, the tradable bundles need not be proportional to each other. But they must be totally ordered, that is, for any two bundles in the mechanism's range, one of them contains a larger quantity of all goods than the other. Second, the seller's and the buyer's prices need neither coincide nor be linear in quantity. But they must be strictly increasing and satisfy a particular form of concavity (convexity) for the seller (buyer). Their shape guarantees that both agents have quasiconcave utility over bundles.

Under both budget balance and collusion-proofness, prices must be exogenous. The reason is that both conditions restrict the extent to which an agent's report can affect the other agent's payment. These restrictions turn out to be so strong that they prevent any endogenous price formation. On a technical level, both budget balance and collusion-proofness imply a property called *non-bossiness*. In our setting, non-bossiness says that an agent's report can change the other agent's payment for a given bundle only if their own payment changes as well. We use this connection to provide a unified proof of both results (Theorem).

More precisely, incentive compatibility and non-bossiness together imply that the agents cannot affect the price at which a bundle is traded. Incentive compatibility then further restricts which bundles are tradable, how these bundles are priced and how the agents' preferences over bundles are aggregated. The most challenging step of our analysis is to prove that the mechanism's range is monotonic. That is, the tradable bundles are totally ordered, and larger bundles have larger prices. This feature does not follow from incentive compatibility alone. For example, in a mechanism where the seller is a dictator, the buyer's prices could be arbitrary. Our proof uncovers a technical condition sufficient for monotonicity: each agent must have a type that never trades. These *no-trade types* equip the agents with a form of veto power against "unambiguously bad" (i.e. non-monotonic) prices. The existence of no-trade types is implied by individual rationality together with either budget balance or collusion-proofness and no free lunch.

The difficulty of proving the monotonicity of the mechanism's range is in marked contrast to existing studies and due to the fact that our model features a multidimensional set of alternatives and quasilinear utility. Both aspects are natural in many economic environments but have so far only been analyzed separately. In the next section, we discuss the related literature in detail.

2 Related Literature

Two strands of the economic literature study dominant-strategy implementation in models with monetary transfers. The first one is associated with mechanism design: there is usually a single indivisible good, and the agents' preferences are quasilinear in money. The second strand is closer to social choice theory: there are often multiple units or goods, and the agents' preferences are classical (that is, continuous, strictly monotonic and strictly convex). We bring these two strands together by considering a model with multidimensional alternatives and quasilinear preferences.

Among the papers that adopt the mechanism design perspective, Hagerty and Rogerson's (1987) is closest to ours. In their model, a seller and a buyer with quasilinear preferences trade a single indivisible good. Hagerty and Rogerson show that every dominant-strategy incentive-compatible, ex-post individually rational and ex-post budget-balanced mechanism is a posted-price mechanism: a third party announces an exogenous price for the good, and trade occurs if and only if both agents approve. This is a special case of our Corollary 1. In the closely related problem of providing an indivisible public good, similar characterizations have been obtained by Bierbrauer and Hellwig (2016), Kuzmics and Steg (2017) and Drexl and Kleiner (2018). Even allowing for stochastic mechanisms does not upset the equivalence to (randomized) posted prices, as shown by Hagerty and Rogerson (1987) and Andreyanov et al. (2018).

All of these papers feature a single indivisible good and thus one-dimensional private information. In such models, it is standard to reformulate the incentive compatibility constraints via the envelope theorem (e.g. Milgrom and Segal, 2002). While this method can be extended to a multidimensional setting, the involved integrability conditions are technically complex (e.g. Jehiel et al., 1999; Manelli and Vincent, 2007). Instead, we obtain a more tractable characterization of incentive-compatible mechanisms by resorting to a version of the taxation principle (e.g. Rochet, 1985): the final allocation must be each agent's optimum among a menu offered by the other agent. Much of our analysis revolves around the structure of these menus.

This approach has precursors in the literature on social choice theory, which relates our analysis to the second strand of papers mentioned above. Particularly close is Barberà and Jackson (1995). In a two-person exchange economy with classical preferences, they provide a version of our Corollary 1. Their finding does not imply ours because they work on a larger preference domain. A critical feature of their proof is a specific transformation of utility functions which they call "simultaneous concavification" (p. 70). This transformation is not available on the quasilinear preference domain that we study. In consequence, it becomes much harder to show that the mechanism's range is monotonic (our Lemma 4). Furthermore, our analysis differs from Barberà and Jackson's in that we do not rely on budget balance. By employing the weaker condition of non-bossiness, we are able to identify a larger set of mechanisms.

Variants of posted-price mechanisms also arise in economies with one divisible public good and one divisible private good, where they are often referred to as "cost-sharing schemes". Notable examples are Moulin (1994) and Serizawa (1996, 1999). Like Barberà and Jackson (1995), their arguments exploit the scope of the classical preference domain. The sole exception is Serizawa's (1999) Proposition 1, which is also valid for quasilinear utility functions. It says that the agents equally share the cost of the public good. Since the cost is assumed to be increasing in the level of the public good, it follows that each agent's contribution is also increasing. Serizawa's proof hinges on the requirement that the mechanism be budget balanced and symmetric. We impose neither restriction, and the analogous monotonicity result turns out to be the technically most challenging step of our analysis (Lemma 4). An added complication is that we allow not only for multiple units but also for multiple goods.

Zhou (1991) studies a model with multiple divisible pure public goods. He shows that the range of any strategy-proof and non-dictatorial mechanism is one-dimensional. This feature also arises in our Corollary 1. Both findings are related because, under budget balance, bilateral trade can be transformed into a model with multiple public goods and no private good. Despite this similarity, Zhou's (1991) and our analyses are logically independent. Our model involves asymmetric agents, a different preference domain and—in the general case without budget balance—a private good (money).

Finally, multidimensional alternatives and quasilinear utility have been combined in a few other papers. Bierbrauer and Winkelmann (2020) study the provision of multiple binary public goods. Schummer (2000), Miyagawa (2001) and Svensson and Larsson (2002) consider the assignment of heterogeneous objects to agents who consume no more than one object. Miyagawa's (2001) contribution is particularly notable because he essentially extends Hagerty and Rogerson (1987)'s posted-price result to more than two agents. The main difficulty is to prove that the payment an agent makes or receives for a given object is exogenous—which is relatively easy with two agents. Conversely, the main difficulty in our model is to establish a monotonicity result on the price functions—which is trivial when each agent gets a single good. The combination of more than two agents and multidimensional demands is a challenging direction for future research.[1]

3 Model

There are two agents: a seller (s, female) and a buyer (b, male). They can trade discrete units of $\bar{g} \in \{1, 2, \dots\}$ different goods. The set of goods is denoted by $\bar{G} \equiv \{1, \dots, \bar{g}\}$. A bundle $q \equiv (q^g)_{g \in \bar{G}}$ specifies, for each good $g \in \bar{G}$, the quantity $q^g \in \mathbb{N}$ that the buyer acquires from the seller.[2] We

[1] Barberà and Jackson (1995) provide results on the classical preference domain.

[2] We adopt the following conventions: $\mathbb{N} \equiv \{0, 1, \dots\}$, $\mathbb{R}_+ \equiv [0, \infty)$ and $\mathbb{R}_{++} \equiv (0, \infty)$.

compare bundles (and any other vectors) according to the product order. That is, for all $q, \hat{q} \in \mathbb{N}^{\bar{g}}$, we write $q \leq \hat{q}$ if $q^g \leq \hat{q}^g$ for all $g \in \bar{G}$. Moreover, $q < \hat{q}$ if $q \leq \hat{q}$ and $q \neq \hat{q}$. Let $\bar{Q} \subseteq \mathbb{N}^{\bar{g}}$ denote the set of feasible bundles, meaning those that the seller could possibly give to the buyer. We make two assumptions on $\bar{Q}$. First, no trade is feasible, that is, $\mathbf{0} \in \bar{Q}$, where $\mathbf{0}$ denotes a $\bar{g}$-dimensional vector of zeros. Second, at least one agent is capacity constrained, that is, $\bar{Q}$ is bounded (and thus finite).[3]

An allocation (q, t_s, t_b) specifies the bundle $q \in \bar{Q}$ that is traded, the monetary transfer $t_s \in \mathbb{R}$ that the seller receives and the monetary transfer $t_b \in \mathbb{R}$ that the buyer makes. When considering one of the two agents only, we will occasionally use the term "allocation" also to refer to (q, t_s) or (q, t_b). Utility functions are quasilinear in money. Hence, the payoff that the seller derives from allocation (q, t_s, t_b) is given by $t_s - c(q)$, where $c(q)$ represents her monetary cost of delivering bundle q. Analogously, the buyer's payoff is given by $v(q) - t_b$, where $v(q)$ represents his monetary value of obtaining bundle q. The set of admissible cost functions for the seller is denoted by C and consists of all $c\colon \bar{Q} \to \mathbb{R}_+$ that satisfy $c(\mathbf{0}) = 0$ and are strictly increasing and convex. Analogously, the set of admissible value functions for the buyer is denoted by V and consists of all $v\colon \bar{Q} \to \mathbb{R}_+$ that satisfy $v(\mathbf{0}) = 0$ and are strictly increasing and concave.[4]

All of the above is common knowledge except for the realizations of $c \in C$ and $v \in V$, which are the private information of the seller and the buyer, respectively. We refer to c and v as "types", and to (c, v) as a "type profile".

4 Definitions

This section defines mechanisms and their properties. Our objective is to characterize the set of deterministic mechanisms whose induced game has an equilibrium in which each agent plays a dominant pure strategy.[5] By

[3]The assumption that $\bar{Q}$ is bounded is convenient but not essential. In Section 8.1, we explain how our analysis can be extended to any $\bar{Q} \subseteq \mathbb{N}^{\bar{g}}$ with $\mathbf{0} \in \bar{Q}$.

[4]A function $c\colon \bar{Q} \to \mathbb{R}_+$ is strictly increasing if $c(q) < c(\hat{q})$ for all $q, \hat{q} \in \bar{Q}$ with $q < \hat{q}$. It is convex if for all $\hat{q} \in \bar{Q}$, there exists $\hat{\gamma} \in \mathbb{R}^{\bar{g}}$ such that for all $q \in \bar{Q}$, $c(q) - c(\hat{q}) \geq \hat{\gamma} \cdot (q - \hat{q})$. It is concave if $-c$ is convex. Our definition of convexity is equivalent to convex-extensibility (Boyd & Vandenberghe, 2004, pp. 337–339). For other notions of convexity on discrete domains, see Murota (2016).

[5]We refer to a strategy as "dominant" if it is a best response to every action that the other agent can take. This is slightly different from the game-theoretic concept of "weak dominance"; see Börgers (2015, Chapter 4.1) for a brief discussion.

the revelation principle, there is no loss in restricting attention to direct mechanisms in which both agents find it optimal to truthfully report their types.

Definition 1. A (direct) **mechanism** is a function triple $(\varphi, \tau_s, \tau_b)\colon C \times V \to \bar{Q} \times \mathbb{R}^2$.

Definition 2. A mechanism $(\varphi, \tau_s, \tau_b)$ is (dominant-strategy) **incentive compatible** if for all $(c, v), (\hat{c}, \hat{v}) \in C \times V$,

$$\tau_s(c, v) - c\big(\varphi(c, v)\big) \geq \tau_s(\hat{c}, v) - c\big(\varphi(\hat{c}, v)\big), \qquad \text{IC}_s$$

$$v\big(\varphi(c, v)\big) - \tau_b(c, v) \geq v\big(\varphi(c, \hat{v})\big) - \tau_b(c, \hat{v}). \qquad \text{IC}_b$$

Incentive compatibility says that no agent benefits from misreporting their type individually. Likewise, collusion-proofness deters coordinated deviations. That is, if an individual or joint misreport increases the utility of one agent, then it must decrease the utility of the other agent.[6]

Definition 3. A mechanism $(\varphi, \tau_s, \tau_b)$ is (ex-post) **collusion-proof** if for all $(c, v), (\hat{c}, \hat{v}) \in C \times V$,

$$\begin{bmatrix} \tau_s(c, v) - c\big(\varphi(c, v)\big) \\ v\big(\varphi(c, v)\big) - \tau_b(c, v) \end{bmatrix} \not< \begin{bmatrix} \tau_s(\hat{c}, \hat{v}) - c\big(\varphi(\hat{c}, \hat{v})\big) \\ v\big(\varphi(\hat{c}, \hat{v})\big) - \tau_b(\hat{c}, \hat{v}) \end{bmatrix}.$$

Individual rationality requires the mechanism to make each agent at least as well off as the status quo (i.e. the allocation $(\mathbf{0}, 0, 0)$).

Definition 4. A mechanism $(\varphi, \tau_s, \tau_b)$ is (ex-post) **individually rational** if for all $(c, v) \in C \times V$,

$$\tau_s(c, v) - c\big(\varphi(c, v)\big) \geq 0, \qquad \text{IR}_s$$

$$v\big(\varphi(c, v)\big) - \tau_b(c, v) \geq 0. \qquad \text{IR}_b$$

Budget balance says that the seller receives exactly as much money as the buyer pays.

[6] None of our results changes if we weaken the definition of collusion-proofness by considering only coordinated deviations that are themselves immune to individual deviations ("double-crossing"). Formally, we can relax Definition 3 by imposing the non-inequality only on $(c, v), (\hat{c}, \hat{v}) \in C \times V$ such that $\tau_s(c, \hat{v}) - c(\varphi(c, \hat{v})) = \tau_s(\hat{c}, \hat{v}) - c(\varphi(\hat{c}, \hat{v}))$ and $v(\varphi(\hat{c}, v)) - \tau_b(\hat{c}, v) = v(\varphi(\hat{c}, \hat{v})) - \tau_b(\hat{c}, \hat{v})$. Similar notions of collusion-proofness have been used by Serizawa (2006) and Bierbrauer and Hellwig (2016).

Definition 5. A mechanism $(\varphi, \tau_s, \tau_b)$ is (ex-post) **budget balanced** if for all $(c, v) \in C \times V$, $\tau_s(c, v) = \tau_b(c, v)$.

In the absence of budget balance, the buyer's individual rationality constraint can trivially be satisfied by giving him the goods at no charge. Such "free lunch", however, is undesirable in most settings and ruled out by the following property.

Definition 6. A mechanism $(\varphi, \tau_s, \tau_b)$ satsfies **no free lunch** (for the buyer) if for all $(c, v) \in C \times V$, $\varphi(c, v) > 0$ implies that $\tau_b(c, v) > 0$.

No free lunch is a natural condition. It is satisfied by the VCG mechanism and many other bilateral-trading mechanisms commonly found in the literature. Furthermore, our analysis can be adjusted in obvious ways when no free lunch is replaced by the requirement that the mechanism not run an ex-post budget deficit. *No deficit* requires that for all $(c, v) \in C \times V$, $\tau_b(c, v) \geq \tau_s(c, v)$. Combined with the seller's individual-rationality constraint, no deficit implies no free lunch.[7] For this reason, our main result invoking no free lunch (Corollary 2) also holds under no deficit. We prefer no free lunch since it is less restrictive than no deficit.

We conclude this section with two technical conditions that are closely related to the properties above. We are going to use them to provide a unified proof of our two main results. First, non-bossiness requires that no agent can change the other agent's allocation without affecting their own.

Definition 7. A mechanism $(\varphi, \tau_s, \tau_b)$ is **non-bossy** if for all $(c, v), (\hat{c}, \hat{v}) \in C \times V$,

$$\Big[\varphi(c, v) = \varphi(\hat{c}, v) \text{ and } \tau_s(c, v) = \tau_s(\hat{c}, v)\Big] \implies \tau_b(c, v) = \tau_b(\hat{c}, v), \qquad \text{NB}_s$$

$$\Big[\varphi(c, v) = \varphi(c, \hat{v}) \text{ and } \tau_b(c, v) = \tau_b(c, \hat{v})\Big] \implies \tau_s(c, v) = \tau_s(c, \hat{v}). \qquad \text{NB}_b$$

Second, a mechanism has no-trade types if each agent can unilaterally prevent trade (by reporting a specific type).

Definition 8. A mechanism $(\varphi, \tau_s, \tau_b)$ has **no-trade types** if there exists $(c_0, v_0) \in C \times V$ such that for all $(c, v) \in C \times V$, $\varphi(c_0, v) = \varphi(c, v_0) = \mathbf{0}$.

[7]To see this, consider any $(c, v) \in C \times V$ such that $\varphi(c, v) > 0$. No deficit, IR_s and $\varphi(c, v) > 0$ imply that $\tau_b(c, v) \geq \tau_s(c, v) \geq c(\varphi(c, v)) > 0$. Thus, no free lunch holds.

5 Generalized Posted-Price Mechanisms

This section defines the focal mechanisms of our analysis. We call them "generalized posted-price mechanisms" because they exogenously fix the prices at which bundles are traded. The possible outcomes of a generalized posted-price mechanism are described by a "trade schedule", which specifies a set of tradable bundles $Q \subseteq \bar{Q}$ and two price functions $p_s, p_b : Q \to \mathbb{R}$. Q contains all bundles that the two agents can trade. It must include the zero vector and be totally ordered. For each bundle in Q, p_s and p_b determine a unique price for the seller and the buyer, respectively. These price functions must be strictly increasing and satisfy a particular form of concavity (convexity) for the seller (buyer).

Definition 9. A **trade schedule** (Q, p_s, p_b) specifies three exogenous objects:

(i) a **set of tradable bundles** $Q \equiv \{q_0, q_1, \ldots, q_n\} \subseteq \bar{Q}$, $n \in \mathbb{N}$, such that $q_0 = \mathbf{0}$ and $q_0 < q_1 < \cdots < q_n$,

(ii) a **price function for the seller** $p_s\colon Q \to \mathbb{R}$ which is strictly increasing and such that for all $k \in \{1, \ldots, n-1\}$ and $\gamma \in \mathbb{R}^{\bar{g}}_{++}$,

$$\frac{p_s(q_k) - p_s(q_{k-1})}{\gamma \cdot (q_k - q_{k-1})} \geq \frac{p_s(q_{k+1}) - p_s(q_k)}{\gamma \cdot (q_{k+1} - q_k)}, \tag{1}$$

(iii) a **price function for the buyer** $p_b\colon Q \to \mathbb{R}$ which is strictly increasing and such that for all $k \in \{1, \ldots, n-1\}$ and $\gamma \in \mathbb{R}^{\bar{g}}_{++}$,

$$\frac{p_b(q_k) - p_b(q_{k-1})}{\gamma \cdot (q_k - q_{k-1})} \leq \frac{p_b(q_{k+1}) - p_b(q_k)}{\gamma \cdot (q_{k+1} - q_k)}. \tag{2}$$

Every generalized posted-price mechanism is based on a trade schedule.[8] Since the prices are exogenously fixed, it only remains to determine which bundle the seller and the buyer trade. To this end, we now introduce notation that will be used extensively throughout the paper. Consider any trade schedule (Q, p_s, p_b) and type profile $(c, v) \in C \times V$. Note that when bundle

[8] It can be shown that (1) and (2) are equivalent to the following conditions: for all $k \in \{1, \ldots, n-1\}$ and $g \in \bar{G}$, $(g^g_{k+1} - g^g_k)[p_s(q_k) - p_s(q_{k-1})] \geq (g^g_k - g^g_{k-1})[p_s(q_{k+1}) - p_s(q_k)]$ and $(g^g_{k+1} - g^g_k)[p_b(q_k) - p_b(q_{k-1})] \leq (g^g_k - g^g_{k-1})[p_b(q_{k+1}) - p_b(q_k)]$. While these inequalities are more "explicit", they are less convenient to work with than (1) and (2).

$q \in Q$ is traded, the seller receives payoff $p_s(q) - c(q)$. For any two bundles $q, \hat{q} \in Q$, we write $q \succsim_c \hat{q}$ if $p_s(q) - c(q) \geq p_s(\hat{q}) - c(\hat{q})$, $q \succ_c \hat{q}$ if the inequality is strict, and $q \sim_c \hat{q}$ if it is an equality. Analogously, for the buyer, we write $q \succsim_v \hat{q}$ if $v(q) - p_b(q) \geq v(\hat{q}) - p_b(\hat{q})$, $q \succ_v \hat{q}$ if the inequality is strict, and $q \sim_v \hat{q}$ if it is an equality. The agents' optimal bundles in Q are defined by

$$\text{Opt}_s(c) \equiv \{q \in Q : \forall \hat{q} \in Q, q \succsim_c \hat{q}\},$$
$$\text{Opt}_b(v) \equiv \{q \in Q : \forall \hat{q} \in Q, q \succsim_v \hat{q}\}.$$

$\text{Opt}_s(c)$ and $\text{Opt}_b(v)$ may have multiple elements. Denote their respective minimum and maximum by

$$\underline{q}_s(c) \equiv \min\{\text{Opt}_s(c)\}, \qquad \bar{q}_s(c) \equiv \max\{\text{Opt}_s(c)\},$$
$$\underline{q}_b(v) \equiv \min\{\text{Opt}_b(v)\}, \qquad \bar{q}_b(v) \equiv \max\{\text{Opt}_b(v)\}.$$

We now make an important observation: For any trade schedule (Q, p_s, p_b), the particular shape of the price functions ensures that both agents have single-plateaued preferences over the bundles in Q.[9]

Lemma 1. *Consider any trade schedule* (Q, p_s, p_b).

(i) For all $c \in C$, $q_0 \prec_c \cdots \prec_c \underline{q}_s(c) \sim_c \cdots \sim_c \bar{q}_s(c) \succ_c \cdots \succ_c q_n$.

(ii) For all $v \in V$, $q_0 \prec_v \cdots \prec_v \underline{q}_b(v) \sim_v \cdots \sim_v \bar{q}_b(v) \succ_v \cdots \succ_v q_n$.

The formal proof of Lemma 1 is in Appendix A.1. The intuition is simple: The seller's payoff function on Q is given by $p_s - c$. By assumption, $-c$ is a concave function (on $\bar{Q}$, and thus also on Q). The particular form of concavity that (1) imposes on p_s guarantees that its sum with $-c$ is quasiconcave on Q. In other words, the seller's preferences over Q are single-plateaued. An analogous argument applies to the buyer.

We are now in a position to complete the definition of a generalized posted-price mechanism with trade schedule (Q, p_s, p_b). The bundle that the seller and the buyer trade can be generally determined as follows: Each agent announces one of their optimal bundles in Q, and the mechanism selects the smaller of the two. That is, $\varphi(c, v) = \min\{q_s(c), q_b(v)\}$ for some $q_s(c) \in$

[9] Our analysis does not exploit the full scope of single-plateauedness. It is irrelevant that preferences are decreasing to the right of the plateau. For a similar observation, see Berga and Serizawa (2000, Example 3, p. 49).

$\text{Opt}_s(c)$ and $q_b(v) \in \text{Opt}_b(v)$. There is one possible exception to this rule: if there exists a smaller bundle $q \in Q$ that both agents value exactly as much as $\min\{q_s(c), q_b(v)\}$, then q can also be chosen. Note that, since the agents' preferences are single-plateaued, q must lie between $\max\{\underline{q}_s(c), \underline{q}_b(v)\}$ and $\min\{q_s(c), q_b(v)\}$. This is formalized by the following definition.

Definition 10. A mechanism $(\varphi, \tau_s, \tau_b)$ is a **generalized posted-price mechanism** if there exists a trade schedule (Q, p_s, p_b) as well as two functions $q_s\colon C \to Q$ and $q_b\colon V \to Q$ such that the following conditions hold for all $(c, v) \in C \times V$: $q_s(c) \in \text{Opt}_s(c)$, $q_b(v) \in \text{Opt}_b(v)$,

$$\begin{aligned}\varphi(c, v) \in \Big\{q \in Q : \min\Big\{\min\big\{q_s(c), q_b(v)\big\}, \max\big\{\underline{q}_s(c), \underline{q}_b(v)\big\}\Big\} \\ \leq q \leq \min\big\{q_s(c), q_b(v)\big\}\Big\},\end{aligned} \tag{3}$$

$\tau_s(c, v) = p_s(\varphi(c, v))$ and $\tau_b(c, v) = p_b(\varphi(c, v))$.

6 Results

The following result constitutes the technical backbone of our analysis.

Theorem. *A mechanism satisfies incentive compatibility, non-bossiness and no-trade types if and only if it is a generalized posted-price mechanism.*

The "only if" is the challenging direction of the Theorem and will be proved in the next section. The "if" statement, on the other hand, is rather straightforward: Consider any generalized posted-price mechanism. First, it is non-bossy because every bundle in the mechanism's range (the set Q) is associated with an *exogenous* price for each agent. Second, no-trade types exist because the zero bundle is the unique optimum in Q for a seller with very large costs and a buyer with very low values. Neither type will ever trade, given that the agent with the smaller optimum determines the outcome. Third, incentive compatibility is clearly satisfied for the agent with the smaller optimum because they get their first choice. The other agent can only reduce the bundle through misreporting. But this moves the bundle further away from their optimum and, since preferences are single-plateaued (Lemma 1), reduces their utility. Hence, no agent has a profitable deviation. We formalize this proof in Appendix A.2.

Non-bossiness and the existence of no-trade types do not have much normative appeal.[10] However, both are implied by conjunctions of the desiderata from the previous section. This allows us to derive two meaningful corollaries of the Theorem. The first one invokes individual rationality and budget balance. Budget balance implies non-bossiness, while individual rationality guarantees no-trade types. Moreover, a generalized posted-price mechanism satisfies budget balance if and only if the seller's and the buyer's price functions coincide, that is, $p \equiv p_s = p_b$. Individual rationality excludes lump-sum payments to or from the agents, that is, $p(\mathbf{0}) = 0$. Combined with (1) and (2), it follows that the mechanism's range is linear: it consists of a "baseline allocation" $(q_1, p(q_1))$ and multiples thereof. The intuition is that only a linear price function can satisfy both the concavity property of (1) and the convexity property of (2). This discussion is summarized by the following result, whose formal proof is in Appendix A.3.

Corollary 1. *A mechanism satisfies incentive compatibility, individual rationality and budget balance if and only if it is a generalized posted-price mechanism such that*

(i) $p \equiv p_s = p_b$,

(ii) $p(\mathbf{0}) = 0$,

(iii) for all $k \in \{2, \ldots, n\}$, *there exists* $\mu_k \in (1, \infty)$ *such that* $(q_k, p(q_k)) = \mu_k(q_1, p(q_1))$.[11]

Our second corollary builds on the fact that non-bossiness is not only implied by budget balance but also by collusion-proofness. Without budget balance, however, individual rationality alone does not ensure the existence of the buyer's no-trade type. The reason is that the buyer trivially gets non-negative utility if he is never charged a positive payment. Hence, a dictatorial mechanism in which the agents trade the seller's optimum among an exogenous set of bundles can be incentive compatible, individually rational and collusion-proof but is not of the generalized posted-price form.[12] No free lunch rules out such mechanisms. It effectively gives bite to the buyer's

[10] The normative content of non-bossiness (or, rather, its lack thereof) is extensively discussed by Thomson (2016).

[11] Since $q_2 < \cdots < q_n$, it must be that $\mu_2 < \cdots < \mu_n$.

[12] If the seller has multiple optima, the best one for the buyer must be selected. Otherwise, collusion-proofness is violated.

individual rationality constraint, thus equipping him with a no-trade type. This allows us to apply Theorem and establish the following result, whose formal proof is in Appendix A.3.

Corollary 2. *A mechanism satisfies incentive compatibility, individual rationality, no free lunch*[13] *and collusion-proofness if and only if it is a generalized posted-price mechanism such that*

(i) $p_s(\mathbf{0}) \geq 0 \geq p_b(\mathbf{0})$,

(ii) $\forall q \in Q \setminus \{\mathbf{0}\}$, $p_b(q) > 0$,

(iii) $\forall (c, v) \in C \times V$, $q_s(c) = \bar{q}_s(c)$ *and* $q_b(v) = \bar{q}_b(v)$.

Condition (i) is due to individual rationality, (ii) to no free lunch, and (iii) to collusion-proofness. The first two should be clear. To understand the third, note that a generalized posted-price mechanism is not necessarily collusion-proof because it may select an "inefficiently low" bundle when an agent has multiple optima. For example, suppose that $Q = \{0, 1, 2\}$, $\text{Opt}_s(c) = \{0, 1\}$ and $\text{Opt}_b(v) = \{1\}$. If $q_s(c) = 0$, then $\varphi(c, v) = 0$. But any $\hat{c}$ with $\text{Opt}_s(\hat{c}) = \{1\}$ results in $\varphi(\hat{c}, v) = 1$. This misreport leaves the seller indifferent and makes the buyer better off. To prevent such joint deviations, collusion-proofness requires the traded bundle to be "as large as possible".

7 Proof of the Theorem: "only if"

In this section, we prove the "only if" part of the Theorem. We focus on the underlying intuition and relegate the more technical steps to Appendices A.4 to A.8. In a series of lemmas, we use incentive compatibility (IC), non-bossiness (NB) and no-trade types (NTT) to successively derive all elements of a generalized posted-price mechanism. For brevity, the relevant properties will be indicated in parentheses in the lemma heading. For example, "Lemma 2 (IC+NB)" means that Lemma 2 holds under incentive compatibility and non-bossiness.

[13] Corollary 2 holds almost unchanged if no free lunch is replaced by no deficit, which requires that for all $(c, v) \in C \times V$, $\tau_b(c, v) \geq \tau_s(c, v)$. The only modification is to replace (ii) by the following condition: $\forall q \in Q$, $p_b(q) \geq p_s(q)$.

7.1 From Allocations to Bundles

Fix a mechanism $(\varphi, \tau_s, \tau_b)$ and define its range of bundles by

$$Q \equiv \{q \in \bar{Q} : \exists (c, v) \in C \times V \text{ s.t. } \varphi(c, v) = q\}.$$

We are slightly abusing notation here because Q has already been defined as the set of tradable bundles in the trade schedule of a generalized posted-price mechanism (Definition 9). Both objects will eventually coincide.

Our first lemma captures the sole—yet crucial—role of non-bossiness in our analysis. Together with incentive compatibility, non-bossiness implies that transfers are completely pinned down by quantities.

Lemma 2 (IC+NB)**.** *For both $i \in \{s, b\}$, there exists a price function $p_i \colon Q \to \mathbb{R}$ such that $\tau_i(c, v) = p_i(\varphi(c, v))$ for all $(c, v) \in C \times V$.*

Proof. Consider any $(c_1, v_1), (c_2, v_2) \in C \times V$ such that $\hat{q} \equiv \varphi(c_1, v_1) = \varphi(c_2, v_2)$. We only show that $\tau_s(c_1, v_1) = \tau_s(c_2, v_2)$; the argument for the buyer is analogous. In Appendix A.4, we construct a type $\hat{c} \in C$ whose preference for bundle $\hat{q}$ (relative to the other bundles in Q) is stronger than both c_1's and c_2's preferences. Thus, by IC$_s$, $\varphi(\hat{c}, v_1) = \varphi(\hat{c}, v_2) = \hat{q}$. IC$_b$ then implies that $\tau_b(\hat{c}, v_1) = \tau_b(\hat{c}, v_2)$. Hence, by NB$_b$, $\tau_s(\hat{c}, v_1) = \tau_s(\hat{c}, v_2)$. Finally, since $\varphi(c_1, v_1) = \varphi(\hat{c}, v_1)$ and $\varphi(\hat{c}, v_2) = \varphi(c_2, v_2)$, IC$_s$ requires that $\tau_s(c_1, v_1) = \tau_s(\hat{c}, v_1)$ and $\tau_s(\hat{c}, v_2) = \tau_s(c_2, v_2)$, respectively. Therefore, $\tau_s(c_1, v_1) = \tau_s(c_2, v_2)$. □

Lemma 2 allows us to express the agents' preferences over allocations in terms of bundles only. For this purpose, we will use the preference notation introduced in Section 5: For any type profile $(c, v) \in C \times V$ and bundles $q, \hat{q} \in Q$, we write $q \succcurlyeq_c \hat{q}$ if $p_s(q) - c(q) \geq p_s(\hat{q}) - c(\hat{q})$, and $q \succcurlyeq_v \hat{q}$ if $v(q) - p_b(q) \geq v(\hat{q}) - p_b(\hat{q})$. The asymmetric part of $\succcurlyeq_c$ ($\succcurlyeq_v$) is represented by $\succ_c$ ($\succ_v$), and the symmetric part by $\sim_c$ ($\sim_v$). Moreover, we slightly abuse notation and denote the agents' optimal bundles in an arbitrary set $\hat{Q} \subseteq Q$ by

$$\text{Opt}_s(c, \hat{Q}) \equiv \{q \in \hat{Q} : \forall \hat{q} \in \hat{Q}, q \succcurlyeq_c \hat{q}\},$$
$$\text{Opt}_b(v, \hat{Q}) \equiv \{q \in \hat{Q} : \forall \hat{q} \in \hat{Q}, q \succcurlyeq_v \hat{q}\}.$$

As in Section 5, we abbreviate the agents' optima in the mechanism's range by $\text{Opt}_s(c) \equiv \text{Opt}_s(c, Q)$ and $\text{Opt}_b(v) \equiv \text{Opt}_b(v, Q)$. Finally, each agent's report determines a set of bundles available to the other agent. These conditional ranges, or **option sets**, are denoted by

$$Q_b(c) \equiv \{q \in Q : \exists v \in V \text{ s.t. } \varphi(c, v) = q\},$$
$$Q_s(v) \equiv \{q \in Q : \exists c \in C \text{ s.t. } \varphi(c, v) = q\}.$$

Based on the new notation, our next result provides a non-bossy version of the "taxation principle": the bundle chosen by the mechanism must be each agent's optimum among the options offered by the other agent.

Lemma 3 (IC+NB)**.** *For all* $(c, v) \in C \times V$*,*

$$\varphi(c, v) \in \text{Opt}_s\big(c, Q_s(v)\big) \cap \text{Opt}_b\big(v, Q_b(c)\big).$$

Proof. Consider any $(c, v) \in C \times V$. IC_s requires that $\varphi(c, v) \succcurlyeq_c \varphi(\hat{c}, v)$ for all $\hat{c} \in C$. By definition, this is equivalent to $\varphi(c, v) \in \text{Opt}_s(c, Q_s(v))$. Analogously, $\varphi(c, v) \in \text{Opt}_b(v, Q_b(c))$. □

7.2 Monotonicity

We now turn to the technically most challenging step of our analysis. For every incentive-compatible and non-bossy mechanism with no-trade types, the following result establishes monotonicity conditions on the tradable bundles and their associated prices.

Lemma 4 (IC+NB+NTT)**.** *There exists a labeling of bundles* $\{q_0, \dots, q_n\} = Q$*,* $n \in \mathbb{N}$*, such that* $q_0 = \mathbf{0}$ *and for all* $k \in \{1, \dots, n\}$*,*

(i) $q_{k-1} < q_k$*,*

(ii) $p_s(q_{k-1}) < p_s(q_k)$*,*

(iii) $p_b(q_{k-1}) < p_b(q_k)$*.*

Moreover, for all $k \in \{0, \dots, n\}$*,*

(iv) $\exists c_k \in C$ *s.t.* $Q_b(c_k) = \{q_0, \dots, q_k\}$ *and* $q_0 \prec_{c_k} \cdots \prec_{c_k} q_k$*,*

(v) $\exists v_k \in V$ *s.t.* $Q_s(v_k) = \{q_0, \dots, q_k\}$ *and* $q_0 \prec_{v_k} \cdots \prec_{v_k} q_k$*.*

To explain the intuition behind Lemma 4, suppose first that there is a single good (i.e. $\bar{g} = 1$). Specifically, let $Q = \{0, \ldots, n\}$. Bundles are totally ordered in this case, so (i) holds trivially. Parts (ii) and (iii) assert that the price functions are strictly increasing. If an agent's price function is not strictly increasing, then the mechanism's range contains two unanimously ranked allocations. For example, contrary to (ii), suppose that $p_s(0) \geq p_s(q)$ for some $q \in \{1, \ldots, n\}$. All seller types then prefer $(0, p_s(0))$ to $(q, p_s(q))$ because the former involves a lower cost and a weakly higher payment than the latter. To guarantee incentive compatibility, a seller type who provides $(q, p_s(q))$ must not have a misreport that yields $(0, p_s(0))$ instead. This can be achieved, for example, by letting the buyer be a dictator, so that the seller cannot influence which bundle is traded. The role of the no-trade type is to rule out such dictatorial mechanisms by giving veto power to the seller. More precisely, a no-trade seller type c_0 always provides zero units, that is, $Q_b(c_0) = \{0\}$. Hence, whenever $\varphi(c, v) = q$, seller type c can achieve $\varphi(c_0, v) = 0$ by reporting c_0. To discourage such misreports, we must have that $p_s(0) < p_s(q)$ for all $q \in \{1, \ldots, n\}$. The argument for the buyer is similar, although it is now the no-trade type who would misreport if he were charged a higher price for zero units than for some positive quantity.

The logic above can be applied inductively to prove that p_s and p_b are strictly increasing. Consider again the seller. It can be shown that, if $p_s(0) < p_s(q)$ for all $q \in \{1, \ldots, n\}$, there exists a seller type c_1 who always provides at most one unit, that is, $Q_b(c_1) = \{0, 1\}$. The role of c_1 is similar to that of the no-trade type in the previous paragraph. By reporting c_1, the seller can restrict the buyer's option set to prevent trading more than one unit if the price does not exceed $p_s(1)$. To discourage this misreport, it must be that $p_s(1) < p_s(q)$ for all $q \in \{2, \ldots, n\}$. Next, we can prove that $Q_b(c_2) = \{0, 1, 2\}$ for some c_2, and so on. Parts (ii) and (iv) of Lemma 4 are thus interrelated. In the buyer's case, there is a similar connection between (iii) and (v).

The arguments above still work when there is more than one good. In this case, Lemma 4(i) additionally requires the tradable bundles to be totally ordered. That is, for any two bundles in the mechanism's range, one of them must contain weakly more quantity of *all* goods (and strictly more quantity of at least one good) than the other. To illustrate why, suppose that $\bar{g} = 2$ and $Q = \{(0, 0), (0, 1), (1, 0)\}$. Consider a buyer type who prefers $(1, 0)$ to

$(0,1)$ to $(0,0)$. The set of bundles that he offers to the seller must include $(1,0)$ and $(0,0)$—the former because it is his optimum in Q, and the latter because the no-trade seller type accepts nothing else. The issue is that the buyer does not have a dominant strategy with respect to $(0,1)$. If he offers $(0,1)$, then a seller type who prefers $(0,1)$ to $(1,0)$ to $(0,0)$ will choose $(0,1)$ instead of $(1,0)$, which is bad for the buyer. But if he does not offer $(0,1)$, then a seller type who prefers $(0,1)$ to $(0,0)$ to $(1,0)$ will choose $(0,0)$ instead of $(0,1)$, which is also bad for the buyer. This example illustrates the essence of why there cannot be "unordered" bundles: they broaden the set of preferences an agent can express, ruining incentive compatibility for the other agent. The formal proof in Appendix A.5 follows this logic. The technical challenge is to prove that the types used in the example actually exist.

7.3 Option Sets

Parts (iv) and (v) of Lemma 4 describe the options offered by specific types of the seller and the buyer, respectively. Generalizing these statements, our next lemma shows that each type offers all bundles up to one of their optima, and none above.

Lemma 5 (IC+NB+NTT)**.** *There exist functions $q_s\colon C \to Q$ and $q_b\colon V \to Q$ such that, for all $(c,v) \in C \times V$,*

(i) $q_s(c) \in \mathrm{Opt}_s(c)$ *and* $Q_b(c) = \{q_0, \dots, q_s(c)\}$,

(ii) $q_b(v) \in \mathrm{Opt}_b(v)$ *and* $Q_s(v) = \{q_0, \dots, q_b(v)\}$.

To understand the intuition behind Lemma 5, consider the seller and suppose for simplicity that she has a unique optimal bundle in Q. If she offers a bundle above her optimum, a buyer type with sufficiently large marginal values will choose it. But he would pick the seller's optimum if she did not offer any larger bundle—and she can do this by deviating to one of the types from Lemma 4(iv). Hence, the largest bundle that the seller offers is her optimum. In addition, she would like to offer no bundles *below* her optimum because the buyer would then be forced to choose it. However, this would introduce profitable deviations for the types from Lemma 4(iv) with the same optimum. To prevent such misreports, the seller must offer *some* bundles below her optimum. Our proof in Appendix A.6 generalizes this idea to show that she must in fact offer *all* smaller bundles—just like the types from Lemma 4(iv).

7.4 Price Functions

Our next result establishes the last missing piece of a trade schedule (Definition 9): the seller's and the buyer's price functions must satisfy a particular form of concavity and convexity, respectively.

Lemma 6 (IC+NB+NTT)**.** *For all* $k \in \{1, \dots, n-1\}$ *and* $\gamma \in \mathbb{R}^{\bar{g}}_{++}$*,*

(i) $\dfrac{p_s(q_k) - p_s(q_{k-1})}{\gamma \cdot (q_k - q_{k-1})} \geq \dfrac{p_s(q_{k+1}) - p_s(q_k)}{\gamma \cdot (q_{k+1} - q_k)}$,

(ii) $\dfrac{p_b(q_k) - p_b(q_{k-1})}{\gamma \cdot (q_k - q_{k-1})} \leq \dfrac{p_b(q_{k+1}) - p_b(q_k)}{\gamma \cdot (q_{k+1} - q_k)}$.

Recall from Lemma 1 that any trade schedule induces single-plateaued preferences. This feature is critical for generalized posted-price mechanisms to be incentive compatible. The intuition behind Lemma 6 is that the converse is also true: if (i) or (ii) are violated, there exists a type whose preferences are not single-plateaued, which in turn breaks incentive compatibility. To see this more clearly, suppose that (i) does not hold. Then we can find a seller type $c \in C$ such that for some $k \in \{1, \dots, n-1\}$, $\text{Opt}_s(c) = \{q_{k+1}\}$ and $q_{k-1} \succ_c q_k$. By Lemma 5(i), $\text{Opt}_s(c) = \{q_{k+1}\}$ implies that $Q_b(c) = \{q_0, \dots, q_{k+1}\}$. Moreover, by Lemma 4(v), there exists a buyer type $v_k \in V$ such that $Q_s(v_k) = \{q_0, \dots, q_k\}$ and $q_0 \prec_{v_k} \cdots \prec_{v_k} q_k$. Note that $\text{Opt}_s(c, Q_s(v_k)) \subseteq \{q_0, \dots, q_{k-1}\}$ and $\text{Opt}_b(v_k, Q_b(c)) \subseteq \{q_k, q_{k+1}\}$. Lemma 3 thus yields that $\varphi(c, v) \in \emptyset$, which is impossible. The formal proof is in Appendix A.7.

7.5 Quantity Rule

Lemmas 2, 4 and 6 imply that every incentive-compatible and non-bossy mechanism with no-trade types is based on a trade schedule. Hence, Lemma 1 applies: the agents' preferences over bundles must be single-plateaued. We now use this observation to derive the quantity rule of a generalized posted-price mechanism. This is the final step in the proof of the "only if" part of the Theorem.

Lemma 7 (IC+NB+NTT)**.** *For all* $(c, v) \in C \times V$,

$$\varphi(c,v) \in \Big\{ q \in Q : \min\Big\{\min\{q_s(c), q_b(v)\}, \max\{\underline{q}_s(c), \underline{q}_b(v)\}\Big\} \leq q \leq \min\{q_s(c), q_b(v)\}\Big\}. \tag{3}$$

Lemma 7 is easy to prove when each agent has a unique optimum, that is, $\text{Opt}_s(c) = \{q_s(c)\}$ and $\text{Opt}_b(v) = \{q_b(v)\}$. In this case, (3) boils down to $\varphi(c, v) = \min\{q_s(c), q_b(v)\}$, so the agents trade the smaller of their two optima. To understand why, note that the seller offers all bundles up to her optimum (Lemma 5), and the buyer then chooses his preferred bundle from this set (Lemma 3), or vice versa. Thus, $\varphi(c, v) \in \text{Opt}_b\big(v, \{q_0, \ldots, q_s(c)\}\big)$. If $q_b(v) \leq q_s(c)$, the buyer's optimum is available, which implies that $\varphi(c, v) = q_b(v)$. On the other hand, if $q_s(c) < q_b(v)$, the buyer can only choose from bundles below his optimum. By Lemma 1, his utility is strictly increasing on this set. Hence, he picks the bundle closest to his optimum, which means that $\varphi(c, v) = q_s(c)$. In both cases, $\varphi(c, v) = \min\{q_s(c), q_b(v)\}$. The general proof, which allows for multiple optima, is in Appendix A.8.

8 Extensions

Before concluding the paper, we briefly discuss two extensions of our model. The first extension lifts the assumption that the feasible set is finite. The second extension considers the restriction to additively separable preferences.

8.1 Infinite set of feasible bundles

A technically convenient feature of our model is that the set of feasible bundles, $\bar{Q}$, is finite. This assumption can be decomposed into two parts: bounded and discrete. Let us discuss them in turn.

Requiring the feasible set to be bounded is common in the literature (e.g. Barberà and Jackson, 1995; Serizawa, 1999). Although we make heavy use of this assumption, our analysis extends to any $\bar{Q} \subseteq \mathbb{N}^{\bar{g}}$ with $\mathbf{0} \in \bar{Q}$. Some modifications are needed to accommodate the possibility that $\bar{Q}$ is unbounded. We now explain the two most noteworthy changes.[14]

[14] A detailed account of how the analysis changes is available upon request.

First, the definition of the buyer's no-trade type must be adjusted as follows: for all $\bar{q} \in \mathbb{N}^{\bar{g}}$, there exists $\bar{v}_0 \in V$ such that for all $c \in C$, $\varphi(c, \bar{v}_0) \in \{\mathbf{0}\} \cup \{q \in \mathbb{N}^{\bar{g}} : q \not\leq \bar{q}\}$. In words, $\bar{v}_0$ is a no-trade type relative to set $\{q \in \mathbb{N}^{\bar{g}} : q \leq \bar{q}\}$, so he may receive bundles outside that set. This modification weakens Definition 8 if, and only if, $\bar{Q}$ does not have an upper bound. Its purpose is to ensure that our two Corollaries can still be derived from the Theorem.[15] As it turns out, the proof of the Theorem itself mostly goes through. Only small adjustments need to be made, for example in the construction of certain types.

Second, the definition of a trade schedule must be amended by the following joint condition on (Q, p_s, p_b): if $\#Q = \infty$, then $\lim_{k\to\infty}[p_s(q_{k+1}) - p_s(q_k)]/[\gamma \cdot (q_{k+1} - q_k)] = 0$ for all $\gamma \in \mathbb{R}^{\bar{g}}_{++}$, or $\lim_{k\to\infty}[p_b(q_{k+1}) - p_b(q_k)]/[\gamma \cdot (q_{k+1} - q_k)] = \infty$ for all $\gamma \in \mathbb{R}^{\bar{g}}_{++}$. This condition is necessary and sufficient for at least one agent to have an optimum bundle in Q, that is, $\text{Opt}_s(c) \cup \text{Opt}_b(v) \neq \emptyset$ for all $(c, v) \in C \times V$. Otherwise, both agents want to trade infinitely much—which is impossible. A particular implication is that generalized posted-price mechanisms that satisfy budget balance must have a finite range.

Compared to the assumption of boundedness, the discreteness of $\bar{Q}$ plays a more substantive role in our analysis. We do not believe that our results will change fundamentally when $\bar{Q}$ is an arbitrary subset of the $\bar{g}$-dimensional non-negative reals. However, the formal proof of this conjecture seems daunting and thus a worthwhile endeavor for future research.

8.2 Additively separable preferences

Our model assumes that the agents' preferences over bundles can be represented by any strictly increasing and convex/concave function that takes the value 0 at the origin. We now show that our results do not fully extend to the sub-domain of additively separable preferences.[16]

Consider the following counterexample: There are two binary goods. The set of feasible bundles is $\bar{Q} = \{(0,0), (0,1), (1,0), (1,1)\}$. Let C^+ denote the set of additively separable cost functions $c\colon \bar{Q} \to \mathbb{R}_+$. That is, $c \in C^+$

[15] The issue is with Lemma A.1(iv). Its proof fails if $\bar{Q}$ is unbounded and we maintain the original definition of the buyer's no-trade type, but it goes through under our modified definition.

[16] We are grateful to an anonymous referee for this observation. The following discussion greatly benefited from their comments.

if and only if there exist $c^1, c^2 \in \mathbb{R}_{++}$ such that for all $q \equiv (q^1, q^2) \in \bar{Q}$, $c(q) = c^1 q^1 + c^2 q^2$. Analogously, $v \in V^+$ if and only if there exist $v^1, v^2 \in \mathbb{R}_{++}$ such that for all $q \in \bar{Q}$, $v(q) = v^1 q^1 + v^2 q^2$. Let $p \equiv (p^1, p^2) \in \mathbb{R}^2_{++}$ be an exogenous vector of prices, one for each good. Consider a mechanism $(\varphi^+, \tau_s^+, \tau_b^+)$ in which good $g \in \{1, 2\}$ is traded, at price p^g, if and only if both agents weakly benefit from trade. That is, for all $(c, v) \in C^+ \times V^+$, $\varphi^+(c, v) = (\mathbb{1}\{c^g \leq p^g \leq v^g\})_{g \in \{1,2\}}$ and $\tau_s^+(c, v) = \tau_b^+(c, v) = p \cdot \varphi^+(c, v)$. It is easy to verify that $(\varphi^+, \tau_s^+, \tau_b^+)$ is incentive compatible, individually rational and budget balanced. However, $(\varphi^+, \tau_s^+, \tau_b^+)$ is not a generalized posted-price mechanism because the range of bundles, $\bar{Q}$, is not totally ordered. Therefore, Corollary 1 fails when preferences are additively separable. Corollary 2 and the Theorem can be similarly disproved.[17]

Nonetheless, some of our results seem to carry over to the additively separable preference domain. Mechanism $(\varphi^+, \tau_s^+, \tau_b^+)$ can be interpreted as a further generalization of generalized posted-price mechanisms. It is still based on a trade schedule (Q, p_s, p_b). The main novelty is that the set of tradable bundles, Q, is not totally ordered—but it is a lattice. As a consequence, the two agents trade the meet, rather than the minimum, of their optimal bundles. Note that their preferences over bundles are multidimensional single-plateaued (in the sense of Barberà et al., 1993). We conjecture that these properties generalize. The main challenge in validating this conjecture is to precisely describe the permissible shapes of Q and then prove an analogue of Lemma 4. We consider this an intriguing open problem.

9 Conclusion

We have studied multidimensional bargaining between two agents with private information about their preferences. Our results show that ex-post notions of incentive compatibility, individual rationality and collusion-proofness (or budget balance) can only be satisfied by "generalized posted-price mechanisms". These mechanisms permit the agents to trade only bundles that are totally ordered. Every tradable bundle is associated with an exogenous pair of prices, one for each agent. The shape of these price functions is such

[17] Mechanism $(\varphi^+, \tau_s^+, \tau_b^+)$ is not incentive compatible on $C \times V$ because this domain allows for complementarities. Specifically, an agent may want to trade good 2 only if good 1 is also traded. This agent does not have a dominant strategy: their optimal response depends on whether the other agent wants to trade good 1 or not.

that each agent's utility over bundles is quasiconcave. Both agents then announce one of their optimal bundles and generally trade the smaller of the two. By combining multidimensional private information and quasilinear preferences, our characterizations extend findings in both mechanism design and social choice theory, which have rarely been connected.

Generalized posted-price mechanisms are easy to implement in practice because they request minimal information from the agents. All they have to report is their optimal bundles. This is in stark contrast to the VCG mechanism, which requires the seller and the buyer to reveal their entire cost and value functions, respectively. Generalized posted-price mechanisms are also easy to understand. In fact, their outcomes can be implemented in *obviously* dominant strategies (Li, 2017).[18] Simply consider a sequential procedure in which bargaining proceeds bundle by bundle until at least one of the agents is not willing to go further.

Generalized posted-posted price mechanisms have two main drawbacks. The first one is the requirement that bundles be totally ordered. As we explained in the previous section, this limitation disappears if the agents' preferences are additively separable. Exploring smaller preference domains in more detail is an important direction for future research. Another problem of generalized posted-price mechanisms is that they do not allow the prices to adjust to the agents' preferences. A natural question is whether this price exogeneity can be overcome by weaker notions of strategic robustness. In our model, interim implementation on all type spaces is not more permissive than dominant-strategy implementation (Bergemann & Morris, 2005). But other notions of robustness are. Examples include implementation in weakly undominated strategies (Börgers & Smith, 2012; Yamashita, 2015) and strategic simplicity (Börgers & Li, 2019). In this literature, bilateral trade is a popular application. Our results may thus prove valuable in providing the dominant-strategy benchmark against which alternative concepts of robustness can be evaluated.

[18]A strategy is obviously dominant if the worst possible outcome under this strategy is at least as good as the best possible outcome under any deviation (measured at the first information set where the two strategies differ).

A Appendix

A.1 Proof of Lemma 1

We only prove (i); (ii) is analogous. Consider any $c \in C$ and $k \in \{1, \dots, n-1\}$. Since c is strictly increasing and convex, it has a subgradient $\gamma_k \in \mathbb{R}^{\bar{g}}_{++}$ at q_k. Thus,

$$c(q_{k+1}) - c(q_k) \geq \gamma_k \cdot (q_{k+1} - q_k),$$
$$c(q_k) - c(q_{k-1}) \leq \gamma_k \cdot (q_k - q_{k-1}).$$

Combined with (1), it follows that

$$\frac{p_s(q_k) - p_s(q_{k-1})}{c(q_k) - c(q_{k-1})} \geq \frac{p_s(q_{k+1}) - p_s(q_k)}{c(q_{k+1}) - c(q_k)}. \tag{A.1}$$

Define $q_l \equiv \underline{q}_s(c)$, so $q_{l-1} \prec_c q_l$. This is equivalent to $\frac{p_s(q_l)-p_s(q_{l-1})}{c(q_l)-c(q_{l-1})} > 1$. From (A.1), it follows that $\frac{p_s(q_k)-p_s(q_{k-1})}{c(q_k)-c(q_{k-1})} > 1$ for all $k \in \{1, \dots, l\}$. Equivalently, $q_0 \prec_c \cdots \prec_c q_l$. Defining $q_m \equiv \bar{q}_s(c)$, an analogous argument shows that $q_m \succ_c \cdots \succ_c q_n$. Finally, $q_{m-1} \preccurlyeq_c q_m$ and (A.1) imply that $q_l \preccurlyeq_c \cdots \preccurlyeq_c q_m$. Since also $q_l \sim_c q_m$, it follows that $q_l \sim_c \cdots \sim_c q_m$.

A.2 Proof of the Theorem: "if"

Consider any generalized posted-price mechanism.

It is non-bossy because for all $q \in Q$ and $(c, v) \in C \times V$ with $\varphi(c, v) = q$, we have that $\tau_s(c, v) = p_s(q)$ and $\tau_b(c, v) = p_b(q)$.

The seller has a no-trade type because for any $c_0 \in C$ with $c_0(q) > p_s(q) - p_s(\mathbf{0})$ for all $q \in Q \setminus \{\mathbf{0}\}$, $\text{Opt}_s(c_0) = \{\mathbf{0}\}$ and thus $\varphi(c_0, v) = \mathbf{0}$ for all $v \in V$. Analogously, any $v_0 \in V$ such that $v_0(q) < p_b(q) - p_b(\mathbf{0})$ for all $q \in Q \setminus \{\mathbf{0}\}$ is a no-trade type for the buyer.

Finally, we verify incentive compatibility for the seller; the argument for the buyer is analogous. Consider any $(c, v) \in C \times V$. There are two cases.

First, suppose that $\underline{q}_s(c) \leq q_b(v)$. Since $\underline{q}_s(c) \leq q_s(c)$ by definition, it follows that $\underline{q}_s(c) \leq \min\{q_s(c), q_b(v)\}$. Hence,

$$\min\Big\{\min\{q_s(c), q_b(v)\}, \max\{\underline{q}_s(c), \underline{q}_b(v)\}\Big\}$$
$$\geq \min\Big\{\min\{q_s(c), q_b(v)\}, \underline{q}_s(c)\Big\} = \underline{q}_s(c).$$

Moreover, $\min\{q_s(c), q_b(v)\} \leq q_s(c) \leq \bar{q}_s(c)$. Thus, by (3), $\underline{q}_s(c) \leq \varphi(c,v) \leq \bar{q}_s(c)$. Recall from Lemma 1(i) that $\underline{q}_s(c) \sim_c \cdots \sim_c \bar{q}_s(c)$, so $\varphi(c,v) \in \mathrm{Opt}_s(c)$. Since type c gets her global optimum in Q, she does not have a profitable deviation.

Second, suppose that $q_b(v) < \underline{q}_s(c)$. Since $\underline{q}_s(c) \leq q_s(c)$ and $\underline{q}_b(v) \leq q_b(v)$ by definition, it follows that $q_b(v) < q_s(c)$ and $\underline{q}_b(v) < \underline{q}_s(c)$. Hence,

$$\begin{aligned}&\min\Big\{\min\{q_s(c), q_b(v)\}, \max\{\underline{q}_s(c), \underline{q}_b(v)\}\Big\}\\ =&\min\Big\{q_b(v), \underline{q}_s(c)\Big\} = q_b(v).\end{aligned}$$

Thus, by (3), $\varphi(c,v) = q_b(v)$. Moreover, for all $\hat{c} \in C$, $\varphi(\hat{c},v) \leq q_b(v)$. By Lemma 1(i), $q_b(v) < \underline{q}_s(c)$ implies that $q_0 \prec_c \cdots \prec_c q_b(v)$. Hence, $\varphi(\hat{c},v) \preccurlyeq_c \varphi(c,v)$ for all $\hat{c} \in C$.

A.3 Proof of Corollaries 1 and 2

Corollaries 1 and 2 are derived from the Theorem. The key observation is that non-bossiness and no-trade types are guaranteed under (i) incentive compatibility, individual rationality and budget balance, or (ii) incentive compatibility, individual rationality, no free lunch and collusion-proofness. Both statements are implied by the following result.

Lemma A.1.

(i) Budget balance implies non-bossiness.

(ii) Collusion-proofness implies non-bossiness.

(iii) Individual rationality and budget balance imply no free lunch.

(iv) Incentive compatibility, non-bossiness, individual rationality and no free lunch imply no-trade types.

Proof. Consider an arbitrary mechanism $(\varphi, \tau_s, \tau_b)$. (i) is obvious.

To prove (ii), suppose the seller is bossy at $(c,v) \in C \times V$. That is, there exists $\hat{c} \in C$ such that $\varphi(\hat{c},v) = \varphi(c,v)$, $\tau_s(\hat{c},v) = \tau_s(c,v)$ and $\tau_b(\hat{c},v) \neq \tau_b(c,v)$. If $\tau_b(\hat{c},v) < \tau_b(c,v)$, collusion-proofness is violated at (c,v). If $\tau_b(\hat{c},v) > \tau_b(c,v)$, collusion-proofness is violated at $(\hat{c},v)$. Hence, by contraposition, collusion-proofness implies non-bossiness.

To prove (iii), suppose $(\varphi, \tau_s, \tau_b)$ is budget balanced and individually rational (for the seller). Thus, for all $(c, v) \in C \times V$ such that $\varphi(c, v) > \mathbf{0}$, $\tau_b(c, v) = \tau_s(c, v) \geq c(\varphi(c, v)) > 0$, which establishes no free lunch.

To prove (iv), suppose $(\varphi, \tau_s, \tau_b)$ satisfies incentive compatibility, non-bossiness, individual rationality and no free lunch. Incentive compatibility and non-bossiness imply that for both $i \in \{s, b\}$, there exists $p_i \colon \bar{Q} \to \mathbb{R}$ such that $\tau_i(c, v) = p_i(\varphi(c, v))$ for all $(c, v) \in C \times V$. (This is shown in Lemma 2, Section 7.1, as part of the "only if" statement of the Theorem.) Let $c_0 \in C$ be such that $c_0(q) > p_s(q)$ for all $q > \mathbf{0}$. IR_s then requires that $\varphi(c_0, v) = \mathbf{0}$ for all $v \in V$, so c_0 is a no-trade seller type. Moreover, no free lunch is equivalent to $p_b(q) > 0$ for all $q > \mathbf{0}$. Hence, there exists $v_0 \in V$ such that $v_0(q) < p_b(q)$ for all $q > \mathbf{0}$. By IR_b, $\varphi(c, v_0) = \mathbf{0}$ for all $c \in C$, so v_0 is a no-trade buyer type. □

The next lemma captures the additional shape that individual rationality, budget balance, no free lunch and collusion-proofness impose on generalized posted-price mechanisms.

Lemma A.2. *A generalized posted-price mechanism satisfies*

(i) individual rationality if and only if $p_s(\mathbf{0}) \geq 0 \geq p_b(\mathbf{0})$,

(ii) budget balance if and only if $p \equiv p_s = p_b$ *and for all* $k \in \{2, \dots, n\}$, *there exists* $\mu_k \in (1, \infty)$ *such that* $q_k = \mu_k q_1$ *and* $p(q_k) - p(\mathbf{0}) = \mu_k[p(q_1) - p(\mathbf{0})]$.

(iii) no free lunch if and only if for all $q \in Q \setminus \{\mathbf{0}\}$, $p_b(q) > 0$,

(iv) collusion-proofness if and only if $q_s(c) = \bar{q}_s(c)$ *and* $q_b(v) = \bar{q}_b(v)$ *for all* $(c, v) \in C \times V$.

Proof. Consider any generalized posted-price mechanism.

First, we prove (i). In particular, we show that IR_s holds if and only if $p_s(\mathbf{0}) \geq 0$. Analogously, IR_b holds if and only if $p_b(\mathbf{0}) \leq 0$. By the Theorem, there exists a no-trade seller type $c_0 \in C$. Thus, $\varphi(c_0, \cdot) = \mathbf{0}$. If $p_s(\mathbf{0}) < 0$, IR_s is violated for c_0. Conversely, suppose that $p_s(\mathbf{0}) \geq 0$. By the Theorem, IC_s holds. Hence, for all $(c, v) \in C \times V$, $\tau_s(c, v) - c(\varphi(c, v)) \geq \tau_s(c_0, v) - c(\varphi(c_0, v)) = p_s(\mathbf{0}) \geq 0$. Thus, IR_s holds.

Second, we prove (ii). Clearly, budget balance holds if and only if $p \equiv p_s = p_b$. We now show that this equation, combined with (1) and (2), implies the additional conditions stated in (ii). These conditions are vacuously true if $n \in \{0,1\}$, so let $n \in \{2,3,\dots\}$. From (1) and (2), it follows that for all $k \in \{1,\dots,n-1\}$ and $\gamma \in \mathbb{R}^{\bar{g}}_{++}$,

$$\frac{p(q_{k+1}) - p(q_k)}{p(q_k) - p(q_{k-1})} = \frac{\gamma \cdot (q_{k+1} - q_k)}{\gamma \cdot (q_k - q_{k-1})}.$$

Since $q_0 = \mathbf{0}$, we get that for all $k \in \{0,\dots,n-1\}$,

$$p(q_{k+1}) - p(q_k) = \frac{p(q_1) - p(\mathbf{0})}{\gamma \cdot q_1}\big[\gamma \cdot (q_{k+1} - q_k)\big].$$

Hence, for all $l \in \{2,\dots,n\}$,

$$\begin{aligned} p(q_l) - p(\mathbf{0}) &= \sum_{k=0}^{l-1}\big[p(q_{k+1}) - p(q_k)\big] \\ &= \frac{p(q_1) - p(\mathbf{0})}{\gamma \cdot q_1}\sum_{k=0}^{l-1}\gamma \cdot (q_{k+1} - q_k) \\ &= \frac{\gamma \cdot q_l}{\gamma \cdot q_1}\big[p(q_1) - p(\mathbf{0})\big]. \end{aligned} \tag{A.2}$$

Define $\mu_l \in (1,\infty)$ via $p(q_l) - p(\mathbf{0}) = \mu_l\big[p(q_1) - p(\mathbf{0})\big]$. By (A.2), μ_l is such that for all $\gamma \in \mathbb{R}^{\bar{g}}_{++}$,

$$\gamma \cdot (q_l - \mu_l q_1) = 0. \tag{A.3}$$

We now show that $q_l = \mu_l q_1$. To the contrary, suppose there exists $g \in \bar{G}$ such that $q_l^g \neq \mu_l q_1^g$. For all $\epsilon > 0$, define $\gamma_\epsilon \in \mathbb{R}^{\bar{g}}_{++}$ by $\gamma_\epsilon^g = 1$ and $\gamma_\epsilon^h = \epsilon$ for all $h \in \bar{G} \setminus \{g\}$. For sufficiently small $\epsilon > 0$,

$$\gamma_\epsilon \cdot (q_l - \mu_l q_1) = q_l^g - \mu_l q_1^g + \epsilon \sum_{h \neq g}\big(q_l^h - \mu_l q_1^h\big) \approx q_l^g - \mu_l q_1^g \neq 0,$$

which contradicts (A.3). In conclusion, $q_l = \mu_l q_1$ and $p(q_l) - p(\mathbf{0}) = \mu_l\big[p(q_1) - p(\mathbf{0})\big]$ for all $l \in \{2,\dots,n\}$.

Third, (iii) is obvious.

Fourth, we prove (iv). Starting with the "if" part, consider any $(c,v) \in C \times V$ and suppose that $q_s(c) = \bar{q}_s(c)$ and $q_b(v) = \bar{q}_b(v)$. We distinguish two cases. First, suppose that $\max\big\{\underline{q}_s(c), \underline{q}_b(v)\big\} \leq \min\big\{\bar{q}_s(c), \bar{q}_b(v)\big\}$. By (3),

$$\varphi(c,v) \in \Big\{q \in Q : \max\big\{\underline{q}_s(c), \underline{q}_b(v)\big\} \leq q \leq \min\big\{\bar{q}_s(c), \bar{q}_b(v)\big\}\Big\}.$$

From Lemma 1, it follows that $\varphi(c,v) \in \text{Opt}_s(c) \cap \text{Opt}_b(v)$, so there is no jointly profitable deviation. Second, suppose that $\min\{\bar{q}_s(c), \bar{q}_b(v)\} < \max\{\underline{q}_s(c), \underline{q}_b(v)\}$. By (3), $\varphi(c,v) = \min\{\bar{q}_s(c), \bar{q}_b(v)\}$. Note that $\bar{q}_s(c) \neq \bar{q}_b(v)$. Without loss of generality, suppose that $\bar{q}_s(c) < \bar{q}_b(v)$. Then $\varphi(c,v) = \bar{q}_s(c) < \underline{q}_b(v)$, so the traded bundle is optimal for the seller but below the buyer's smallest optimum. Since the buyer's utility over bundles is single-plateaued (Lemma 1), he may benefit from a joint misreport $(\hat{c}, \hat{v}) \in C \times V$ only if $\varphi(\hat{c}, \hat{v}) > \varphi(c,v)$. But given that $\varphi(c,v) = \bar{q}_s(c)$, the seller is worse off at $(\hat{c}, \hat{v})$. Hence, collusion-proofness holds.

Conversely, suppose that $\underline{q}_s(c) < \bar{q}_s(c)$ for some $c \in C$; the argument for the buyer is analogous. Let $v \in V$ be such that $\text{Opt}_b(v) = \{\bar{q}_s(c)\}$. By (3), $\varphi(c,v) = \underline{q}_s(c) \notin \text{Opt}_b(v)$. But if the seller reports $\hat{c} \in C$ such that $\text{Opt}_s(\hat{c}) = \{\bar{q}_s(c)\}$, then $\varphi(\hat{c}, v) = \bar{q}_s(c) \in \text{Opt}_b(v)$.[19] Such a misreport makes the buyer better off, while leaving the seller indifferent. Hence, collusion-proofness is violated. □

Corollaries 1 and 2 follow from Lemma A.1, Lemma A.2 and the Theorem.

A.4 Proof of Lemma 2

Consider any $(c_1, v_1), (c_2, v_2) \in C \times V$ such that $\hat{q} \equiv \varphi(c_1, v_1) = \varphi(c_2, v_2)$. IC_s requires that for both $j \in \{1, 2\}$ and all $c \in C$,

$$c_j\big(\varphi(c_j, v_j)\big) - c_j\big(\varphi(c, v_j)\big) \leq \tau_s(c_j, v_j) - \tau_s(c, v_j). \tag{A.4}$$

Define $\hat{c} \in C$ by $\hat{c}(q) \equiv \sum_{g \in \bar{G}} [\epsilon q^g + \frac{1}{\epsilon} \max\{q^g - \hat{q}^g, 0\}]$ for all $q \in \bar{Q}$. If $\epsilon > 0$ is sufficiently small, then $\hat{c}(\hat{q}) - \hat{c}(q) < \min_{j \in \{1,2\}} \{c_j(\hat{q}) - c_j(q)\}$ for all $q \neq \hat{q}$. From (A.4), it follows that for both $j \in \{1, 2\}$ and all $c \in C$ such that $\varphi(c, v_j) \neq \varphi(c_j, v_j)$,

$$\hat{c}\big(\varphi(c_j, v_j)\big) - \hat{c}\big(\varphi(c, v_j)\big) < \tau_s(c_j, v_j) - \tau_s(c, v_j).$$

Hence, by IC_s for type $\hat{c}$, $\varphi(\hat{c}, v_j) = \varphi(c_j, v_j)$ for both $j \in \{1, 2\}$. The rest of the proof is in the main text.

[19] The types $v \in V$ and $\hat{c} \in C$ can be constructed as follows: for all $q \in \bar{Q}$, $v(q) \equiv \sum_{g \in \bar{G}} [\epsilon q^g + (1/\epsilon) \min\{q^g, \bar{q}_s^g(c)\}]$ and $\hat{c}(q) \equiv \sum_{g \in \bar{G}} [\epsilon q^g + (1/\epsilon) \max\{q^g - \bar{q}_s^g(c), 0\}]$, where $\epsilon > 0$ is sufficiently small.

A.5 Proof of Lemma 4

Define $n \equiv \#Q - 1$. Lemma 4 is equivalent to the following claim: there exists a labeling of bundles $\{q_0, \ldots, q_n\} = Q$ such that $q_0 = \mathbf{0}$ and for all $k \in \{0, \ldots, n\}$,

(a) $q_k < q$ for all $q \in Q \setminus \{q_0, \ldots, q_k\}$,

(b) if $k \neq 0$, then $p_s(q_{k-1}) < p_s(q)$ for all $q \in Q \setminus \{q_0, \ldots, q_{k-1}\}$,

(c) if $k \neq 0$, then $p_b(q_{k-1}) < p_b(q)$ for all $q \in Q \setminus \{q_0, \ldots, q_{k-1}\}$,

(d) $\exists c_k \in C$ s.t. $Q_b(c_k) = \{q_0, \ldots, q_k\}$ and $q_0 \prec_{c_k} \cdots \prec_{c_k} q_k$,

(e) $\exists v_k \in V$ s.t. $Q_s(v_k) = \{q_0, \ldots, q_k\}$ and $q_0 \prec_{v_k} \cdots \prec_{v_k} q_k$.

The proof is by induction.

Base case ($k = 0$). Define $q_0 \equiv \mathbf{0}$. (a), (b) and (c) hold trivially. (d) and (e) follow from NTT.

Induction step. Consider any $l \in \{0, \ldots, n-1\}$. Suppose that (a) to (e) hold for all $k \in \{0, \ldots, l\}$. In a series of claims, we show that there exists $q_{l+1} \in Q \setminus \{q_0, \ldots, q_l\}$ such that (a) to (e) also hold for $k = l+1$. Our first claim can be regarded as a preliminary version of condition (d) for $k = l+1$.

Claim A.1. *For all $\hat{q} \in Q \setminus \{q_0, \ldots, q_l\}$, there exists $\hat{c} \in C$ such that*

(i) $q_0 \prec_{\hat{c}} \cdots \prec_{\hat{c}} q_l$ *and if* $p_s(q_l) < p_s(\hat{q})$, *then* $q_l \prec_{\hat{c}} \hat{q}$,

(ii) *for all* $q \not\leq \hat{q}$, $q_0 \succ_{\hat{c}} q$,

(iii) $\text{Opt}_s(\hat{c}, Q_s(\hat{v})) = \{\hat{q}\}$ *for all* $\hat{v} \in V$ *such that* $\hat{q} \in Q_s(\hat{v})$,

(iv) $\{q_0, \ldots, q_l, \hat{q}\} \subseteq Q_b(\hat{c})$ *and for all* $q \not\leq \hat{q}$, $q \notin Q_b(\hat{c})$.

Proof. Consider any $\hat{q} \in Q \setminus \{q_0, \ldots, q_l\}$. Define $\hat{c} \in C$ as follows: for all $q \in \bar{Q}$,

$$\hat{c}(q) \equiv \sum_{g \in \bar{G}} \left[\epsilon q^g + \frac{1}{\epsilon} \max\{q^g - \hat{q}^g, 0\} \right].$$

The arguments below hold for sufficiently small $\epsilon > 0$. For all $q, q' \in Q$, we write $\hat{c}(q) \gg \hat{c}(q')$ if $\lim_{\epsilon \downarrow 0}[\hat{c}(q) - \hat{c}(q')] = \infty$, and $\hat{c}(q) \approx \hat{c}(q')$ if $\lim_{\epsilon \downarrow 0}[\hat{c}(q) - \hat{c}(q')] = 0$. Similar notation will be used throughout the proof of Lemma 4.

Proof of (i). Consider any $k \in \{1, \ldots, l\}$. By (a), $q_{k-1} < q_k < \hat{q}$. From the definition of $\hat{c}$, it follows that $\hat{c}(q_{k-1}) \approx \hat{c}(q_k)$. Moreover, by (b),

$p_s(q_{k-1}) < p_s(q_k)$. Hence, $q_{k-1} \prec_{\hat{c}} q_k$. Analogously, if $p_s(q_l) < p_s(\hat{q})$, then $q_l \prec_{\hat{c}} \hat{q}$.

Proof of (ii). For all $q \not\leq \hat{q}$, $\hat{c}(q) \gg \hat{c}(q_0)$ and thus $q_0 \succ_{\hat{c}} q$.

Proof of (iii). Consider any $\hat{v} \in V$ such that $\hat{q} \in Q_s(\hat{v})$. For all $q \in Q_s(\hat{v}) \setminus \{\hat{q}\}$, we show that $q \prec_{\hat{c}} \hat{q}$. This is obvious if $q \not\leq \hat{q}$ because then $\hat{c}(q) \gg \hat{c}(\hat{q})$. Suppose now that $q < \hat{q}$, so $\hat{c}(q) < \hat{c}(\hat{q})$. If $p_s(q) \geq p_s(\hat{q})$, then $q \succ_c \hat{q}$ and thus $\hat{q} \notin \text{Opt}_s(c, Q_s(\hat{v}))$ for all $c \in C$. From Lemma 3, it follows that $\hat{q} \notin Q_s(\hat{v})$, a contradiction. Hence, $p_s(q) < p_s(\hat{q})$. Since also $\hat{c}(q) \approx \hat{c}(\hat{q})$, $q \prec_{\hat{c}} \hat{q}$.

Proof of (iv). There are three steps. First, consider any $k \in \{0, \ldots, l\}$. By (e), there exists $v_k \in V$ such that $Q_s(v_k) = \{q_0, \ldots, q_k\}$. From (i), it follows that $\text{Opt}_s(\hat{c}, Q_s(v_k)) = \{q_k\}$. Hence, by Lemma 3, $q_k \in Q_b(\hat{c})$. Second, consider any $\hat{v} \in V$ such that $\hat{q} \in Q_s(\hat{v})$. By (iii), $\text{Opt}_s(\hat{c}, Q_s(\hat{v})) = \{\hat{q}\}$ and thus $\hat{q} \in Q_b(\hat{c})$. Third, by contradiction, suppose there exists $q \in Q_b(\hat{c})$ such that $q \not\leq \hat{q}$. Then $\varphi(\hat{c}, v) = q$ for some $v \in V$. By (d), there exists $c_0 \in C$ such that $Q_b(c_0) = \{q_0\}$ and thus $\varphi(c_0, v) = q_0$. Since (ii) states that $q_0 \succ_{\hat{c}} q$, IC_s is violated for $\hat{c}$. □

Drawing on Claim A.1, our next two results establish (b) and (c) for $k = l+1$.

Claim A.2. *For all $\hat{q} \in Q \setminus \{q_0, \ldots, q_l\}$, $p_s(q_l) < p_s(\hat{q})$.*

Proof. By contradiction, suppose that $p_s(q_l) \geq p_s(\hat{q})$ for some $\hat{q} \in Q \setminus \{q_0, \ldots, q_l\}$. By (a), $q_l < \hat{q}$. For all $c \in C$, it follows that $c(q_l) < c(\hat{q})$ and thus $q_l \succ_c \hat{q}$. Let $c_l \in C$ and $\hat{c} \in C$ be as in (d) and Claim A.1, respectively. Define $v \in V$ by $v(q) \equiv (1/\epsilon) \sum_{g \in \bar{G}} q^g$ for all $q \in \bar{Q}$. For sufficiently small $\epsilon > 0$, we have that $\varphi(c_l, v) = q_l$ and $\varphi(\hat{c}, v) = \hat{q}$. Since $q_l \succ_{\hat{c}} \hat{q}$, IC_s is violated for $\hat{c}$. □

Claim A.3. *For all $\hat{q} \in Q \setminus \{q_0, \ldots, q_l\}$, $p_b(q_l) < p_b(\hat{q})$.*

Proof. By contradiction, suppose that $p_b(q_l) \geq p_b(\hat{q})$ for some $\hat{q} \in Q \setminus \{q_0, \ldots, q_l\}$. Consider any $\hat{v} \in V$ such that $\hat{q} \in Q_s(\hat{v})$. Let $\hat{c} \in C$ be as in Claim A.1. Claim A.1(iii) implies that $\varphi(\hat{c}, \hat{v}) = \hat{q}$. Moreover, by (e), there exists $v_l \in V$ such that $Q_s(v_l) = \{q_0, \ldots, q_l\}$. From Claim A.1(i), it follows that $\text{Opt}(\hat{c}, Q_s(v_l)) = \{q_l\}$ and thus $\varphi(\hat{c}, v_l) = q_l$. Finally, by (a), $q_l < \hat{q}$ and thus $v_l(q_l) < v_l(\hat{q})$. Since also $p_b(q_l) \geq p_b(\hat{q})$, we have that $q_l \prec_{v_l} \hat{q}$. Hence, IC_b is violated for for v_l. □

In the remainder of the proof, the following definition is crucial: let bundle $q_{l+1} \in Q \setminus \{q_0, \ldots, q_l\}$ be such that for all $q \in Q \setminus \{q_0, \ldots, q_l\}$, either $p_b(q) > p_b(q_{l+1})$ or $[p_b(q) = p_b(q_{l+1})$ and $q \not> q_{l+1}]$. In words, among all bundles in $Q \setminus \{q_0, \ldots, q_l\}$, q_{l+1} entails the lowest payment for the buyer. If there are multiple such bundles, none of them is larger than q_{l+1}.

Our next claim establishes (e) for $k = l + 1$.

Claim A.4. *There exists $v_{l+1} \in V$ such that $Q_s(v_{l+1}) = \{q_0, \ldots, q_{l+1}\}$ and $q_0 \prec_{v_{l+1}} \cdots \prec_{v_{l+1}} q_{l+1}$.*

Proof. Type $v_{l+1} \in V$ is defined as follows: for all $q \in \bar{Q}$,

$$v_{l+1}(q) \equiv \sum_{g \in \bar{G}} \left[\epsilon q^g + \frac{1}{\epsilon} \min\{q^g, q_l^g\} + \frac{p_b(q_{l+1}) - p_b(q_l)}{\sum_{h \in \bar{G}} (q_{l+1}^h - q_l^h)} \min\{q^g, q_{l+1}^g\} \right],$$

where $\epsilon > 0$ is sufficiently small. Figure A.1 provides an illustration for $\bar{g} = 2$. Note that $v_{l+1} \in V$.

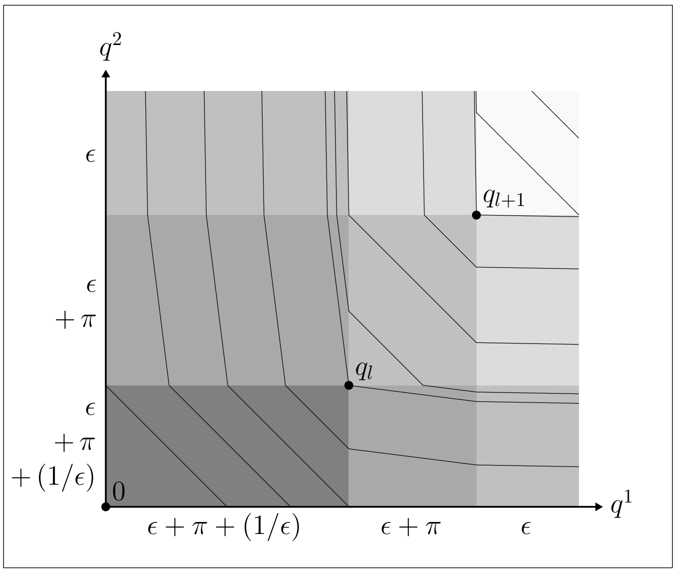

Figure A.1: [Proof of Claim A.4] Illustration of the isovalue curves of type v_{l+1} for $\bar{g} = 2$. The expressions along the axes refer to the two elements of the unique subgradient of v_{l+1} in the interior of the nine shaded rectangles. For example, the unique subgradient in the top-left rectangle is given by the vector $(\epsilon + \pi + (1/\epsilon), \epsilon)$, where $\pi \equiv [p_b(q_{l+1}) - p_b(q_l)] / \sum_{h \in \bar{G}} (q_{l+1}^h - q_l^h)$.

The proof has two main parts. First, we show that v_{l+1} satisfies the following three conditions:

$$q_0 \prec_{v_{l+1}} \cdots \prec_{v_{l+1}} q_l, \tag{A.5}$$

$$q_l \prec_{v_{l+1}} q_{l+1}, \tag{A.6}$$

$$q_l \succ_{v_{l+1}} \hat{q} \text{ for all } \hat{q} \in Q \setminus \{q_0, \ldots, q_{l+1}\}. \tag{A.7}$$

Based on (A.5) to (A.7), the second part of the proof then shows that $Q_s(v_{l+1}) = \{q_0, \ldots, q_{l+1}\}$.

Proof of (A.5). Consider any $k \in \{1, \ldots, l\}$. From (a), we know that $q_{k-1} < q_k \leq q_l$. Thus, by definition of v_{l+1}, $v_{l+1}(q_{k-1}) \ll v_{l+1}(q_k)$, which implies that $q_{k-1} \prec_{v_{l+1}} q_k$.

Proof of (A.6). By definition of v_{l+1} and (a),

$$\big[v_{l+1}(q_{l+1}) - p_b(q_{l+1})\big] - \big[v_{l+1}(q_l) - p_b(q_l)\big] = \epsilon \sum_{g \in \bar{G}} (q^g_{l+1} - q^g_l) > 0.$$

Proof of (A.7). Consider any $\hat{q} \in Q \setminus \{q_0, \ldots, q_{l+1}\}$. We are going to prove that $\lim_{\epsilon \downarrow 0}\{[v_{l+1}(q_{l+1}) - p_b(q_{l+1})] - [v_{l+1}(\hat{q}) - p_b(\hat{q})]\} > 0$. Since we know from the proof of (A.6) that $\lim_{\epsilon \downarrow 0}\{[v_{l+1}(q_{l+1}) - p_b(q_{l+1})] - [v_{l+1}(q_l) - p_b(q_l)]\} = 0$, it follows that $q_l \succ_{v_{l+1}} \hat{q}$.

By (a), $\hat{q} > q_l$. We distinguish two cases. First, if $\hat{q} > q_{l+1}$, then

$$\big[v_{l+1}(q_{l+1}) - p_b(q_{l+1})\big] - \big[v_{l+1}(\hat{q}) - p_b(\hat{q})\big] \xrightarrow{\epsilon \downarrow 0} p_b(\hat{q}) - p_b(q_{l+1}) > 0,$$

where the inequality follows from the definition of q_{l+1}. Second, if $\hat{q} \not> q_{l+1}$, then

$$\begin{aligned}&\big[v_{l+1}(q_{l+1}) - p_b(q_{l+1})\big] - \big[v_{l+1}(\hat{q}) - p_b(\hat{q})\big] \\ &\xrightarrow{\epsilon \downarrow 0} \underbrace{\big[p_b(\hat{q}) - p_b(q_{l+1})\big]}_{\geq 0} + \underbrace{\big[p_b(q_{l+1}) - p_b(q_l)\big]}_{>0} \underbrace{\frac{\sum_{g \in \bar{G}} \max\{q^g_{l+1} - \hat{q}^g, 0\}}{\sum_{h \in \bar{G}} (q^h_{l+1} - q^h_l)}}_{>0} > 0,\end{aligned}$$

where the weak inequality follows from the definition of q_{l+1}, the first strict inequality from Claim A.3, and the second strict inequality from $\hat{q} \not> q_{l+1}$ and (a). This concludes the proof of (A.5) to (A.7).

The proof of $Q_s(v_{l+1}) = \{q_0, \ldots, q_{l+1}\}$ has three parts.

First, consider any $k \in \{0, \ldots, l\}$. By (d), there exists $c_k \in C$ such that $Q_b(c_k) = \{q_0, \ldots, q_k\}$. (A.5) implies that $\text{Opt}_b(v_{l+1}, Q_b(c_k)) = \{q_k\}$ and thus $q_k \in Q_s(v_{l+1})$.

Second, consider any $c \in C$ such that $q_{l+1} \in Q_b(c)$. From (A.5) to (A.7), it follows that $\text{Opt}_b(v_{l+1}, Q_b(c)) = \{q_{l+1}\}$ and thus $q_{l+1} \in Q_s(v_{l+1})$.

Third, by contradiction, suppose that $\hat{q} \in Q_s(v_{l+1})$ for some $\hat{q} \in Q \setminus \{q_0, \dots, q_{l+1}\}$. Consider type $\hat{c} \in C$ from Claim A.1. By Claim A.1(iii), $\text{Opt}_s(\hat{c}, Q_s(v_{l+1})) = \{\hat{q}\}$ and thus $\varphi(\hat{c}, v_{l+1}) = \hat{q}$. Moreover, by (e), there exists $v_l \in V$ such that $Q_s(v_l) = \{q_0, \dots, q_l\}$. From Claim A.1(i), it follows that $\text{Opt}_s(\hat{c}, Q_s(v_l)) = \{q_l\}$ and thus $\varphi(\hat{c}, v_l) = q_l$. This violates IC_b because, by (A.7), $q_l \succ_{v_{l+1}} \hat{q}$. Therefore, $\hat{q} \notin Q_s(v_{l+1})$ for all $\hat{q} \in Q \setminus \{q_0, \dots, q_{l+1}\}$. □

Next, we show that Q does not contain a bundle "in between" q_l and q_{l+1}.

Claim A.5. *For all $q \in Q \setminus \{q_0, \dots, q_{l+1}\}$, $q \nleq q_{l+1}$.*

Proof. By contradiction, suppose there exists $\hat{q} \in Q \setminus \{q_0, \dots, q_{l+1}\}$ such that $\hat{q} < q_{l+1}$. Note that, by (a), $\hat{q} > q_l$. Without loss of generality, assume that for all $q \in Q \setminus \{q_0, \dots, q_{l+1}\}$ such that $q < q_{l+1}$, either $p_s(q) < p_s(\hat{q})$ or $[p_s(q) = p_s(\hat{q})$ and $q \nless \hat{q}]$. In words, among all bundles in between q_l and q_{l+1}, $\hat{q}$ entails the largest payment for the seller. If there are multiple such bundles, none of them is smaller than $\hat{q}$.

Define $c \in C$ as follows: for all $q \in \bar{Q}$,

$$c(q) \equiv \sum_{g \in \bar{G}} \left[\epsilon q^g + \left(\sum_{f \in \bar{G}} \epsilon \hat{q}^f + \frac{\max\{p_s(q_{l+1}) - p_s(\hat{q}), 0\}}{\sum_{h \in \bar{G}} (q^h_{l+1} - \hat{q}^h)} \right) \max\{q^g - \hat{q}^g, 0\} + \frac{1}{\epsilon} \max\{q^g - q^g_{l+1}, 0\} \right].$$

We are going to show below that for sufficiently small $\epsilon > 0$,

$$q_0 \prec_c \cdots \prec_c q_l \prec_c q_{l+1}, \tag{A.8}$$

$$\hat{q} \succ_c q \text{ for all } q \in Q \setminus \{\hat{q}\}. \tag{A.9}$$

(A.8) and (A.9) lead to a contradiction: By Claim A.4, there exists $v_{l+1} \in V$ such that $Q_s(v_{l+1}) = \{q_0, \dots, q_{l+1}\}$. (A.8) implies that $\text{Opt}_s(c, Q_s(v_{l+1})) = \{q_{l+1}\}$ and thus $\varphi(c, v_{l+1}) = q_{l+1}$. Moreover, consider any $\hat{v} \in V$ such that $\hat{q} \in Q_s(\hat{v})$. By (A.9), $\text{Opt}_s(c, Q_s(\hat{v})) = \{\hat{q}\}$ and thus $\varphi(c, \hat{v}) = \hat{q}$. However, $\hat{q} \prec_{\hat{v}} q_{l+1}$ because $v(\hat{q}) < v(q_{l+1})$ and $p_b(\hat{q}) \geq p_b(q_{l+1})$. Hence, IC_b is violated for type $\hat{v}$.

Proof of (A.8). For all $k \in \{1, \dots, l\}$, $p_s(q_{k-1}) < p_s(q_k)$ and $c(q_{k-1}) \approx c(q_k)$, so $q_{k-1} \prec_c q_k$. Moreover, $q_l \prec_c q_{l+1}$ because

$$\begin{aligned}
&\big[p_s(q_{l+1}) - c(q_{l+1})\big] - \big[p_s(q_l) - c(q_l)\big] \\
&= \underbrace{\min\big\{p_s(q_{l+1}), p_s(\hat{q})\big\} - p_s(q_l)}_{>0} \\
&\quad \underbrace{- \epsilon \sum_{g \in \bar{G}} \Big[q^g_{l+1} - q^g_l + (q^g_{l+1} - \hat{q}^g) \sum_{f \in \bar{G}} \hat{q}^f\Big]}_{\approx 0} > 0.
\end{aligned}$$

Proof of (A.9). First, if $q < \hat{q}$, then $p_s(q) < p_s(\hat{q})$ and $c(q) \approx c(\hat{q})$. Second, if $q \nleq q_{l+1}$, then $c(q) \gg c(\hat{q})$. Third, if $\hat{q} \ngeq q < q_{l+1}$, then $p_s(q) \leq p_s(\hat{q})$ and $c(q) > \sum_{f \in \bar{G}} \epsilon \hat{q}^f = c(\hat{q})$. In all three cases, $q \prec_c \hat{q}$. Finally, $q_{l+1} \prec_c \hat{q}$ because

$$\begin{aligned}
&\big[p_s(\hat{q}) - c(\hat{q})\big] - \big[p_s(q_{l+1}) - c(q_{l+1})\big] \\
&= \underbrace{\epsilon\Big(1 + \sum_{f \in \bar{G}} \hat{q}^f\Big) \sum_{g \in \bar{G}} (q^g_{l+1} - \hat{q}^g)}_{>0} + \underbrace{\max\big\{p_s(\hat{q}) - p_s(q_{l+1}), 0\big\}}_{\geq 0} > 0. \qquad \square
\end{aligned}$$

Claims A.1, A.2 and A.5 (with $\hat{q} = q_{l+1}$) ensure the existence of $c_{l+1} \in C$ such that $Q_b(c_{l+1}) = \{q_0, \dots, q_{l+1}\}$ and $q_0 \prec_{c_{l+1}} \cdots \prec_{c_{l+1}} q_{l+1}$. This establishes (d) for $k = l + 1$. Our final claim proves (a) for $k = l + 1$.

Claim A.6. *For all* $q \in Q \setminus \{q_0, \dots, q_{l+1}\}$, $q > q_{l+1}$.

Proof. By contradiction, suppose there exists $\hat{q} \in Q \setminus \{q_0, \dots, q_{l+1}\}$ such that $\hat{q} \not> q_{l+1}$. We know from Claim A.5 that $\hat{q} \nleq q_{l+1}$. For all $\epsilon, \rho \in \mathbb{R}_+$, define

$$X_\epsilon(\rho) \equiv \Big\{q \in \mathbb{R}^{\bar{g}}_+ : \exists \lambda \in [0, 1] \text{ s.t. } q \leq \rho\big[\lambda q_{l+1} + (1 - \lambda)\hat{q} + \epsilon \mathbf{1})\big]\Big\},$$

where $\mathbf{1}$ denotes a $\bar{g}$-dimensional vector of ones. Figure A.2 provides an illustration of X_0 for $\bar{g} = 2$. Importantly, since Q is discrete, we can let $\hat{q}$ be such that for sufficiently small $\epsilon > 0$, $X_\epsilon(1) \cap Q = \{q_0, \dots, q_{l+1}, \hat{q}\}$.[20]

[20] Setting $\epsilon > 0$ (instead of $\epsilon = 0$) ensures that for all $q \in \mathbb{R}^{\bar{g}}_+$, there exists $\rho \in \mathbb{R}_+$ such that $q \in X_\epsilon(\rho)$. This feature is important for the function r_ϵ below to be well defined.

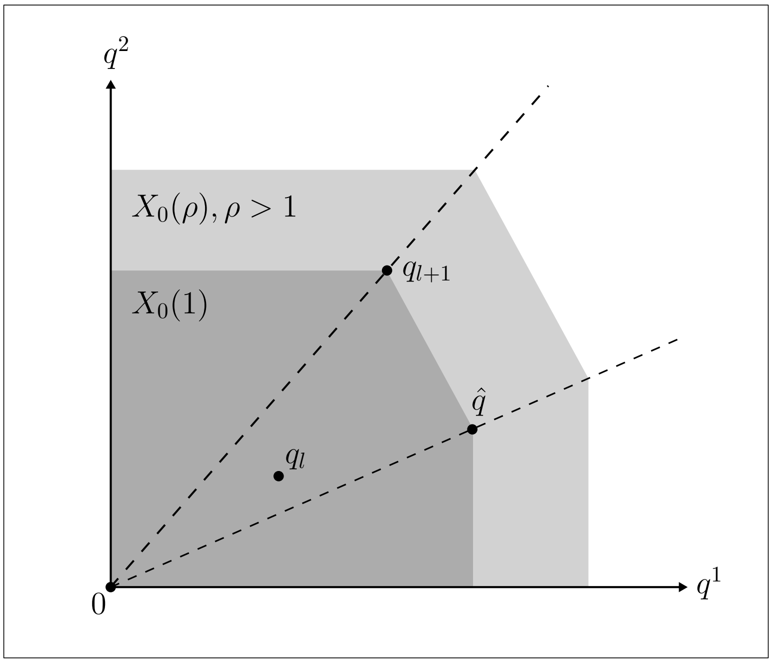

Figure A.2: [Proof of Claim A.6] Illustration of X_0 for $\bar{g} = 2$. The dark gray shaded area is $X_0(1)$. The union of the dark- and light gray shaded areas is $X_0(\rho)$, $\rho > 1$.

We are going to construct types $v \in V$ and $c \in C$ such that

$$q_0 \prec_v \cdots \prec_v q_l \prec_v \hat{q} \prec_v q_{l+1}, \tag{A.10}$$

$$q_0 \prec_c \cdots \prec_c q_l \prec_c q_{l+1} \text{ and } \hat{q} \succ_c q \text{ for all } q \in Q \setminus \{\hat{q}\}. \tag{A.11}$$

Before proving (A.10) and (A.11), we show that these two conditions are incompatible. By Claim A.1 and the definition of $\hat{q}$, there exists $\hat{c} \in C$ such that $Q_b(\hat{c}) = \{q_0, \ldots, q_l, \hat{q}\}$. (A.10) implies that $\text{Opt}_b(v, Q_b(\hat{c})) = \{\hat{q}\}$ and thus $\hat{q} \in Q_s(v)$. From (A.11), it follows that $\text{Opt}_s(c, Q_s(v)) = \{\hat{q}\}$ and thus $\varphi(c, v) = \hat{q}$. Moreover, by Claim A.4, there exists $v_{l+1} \in V$ such that $Q_s(v_{l+1}) = \{q_0, \ldots, q_{l+1}\}$. (A.11) implies that $\text{Opt}_s(c, Q_s(v_{l+1})) = \{q_{l+1}\}$ and thus $\varphi(c, v_{l+1}) = q_{l+1}$. Since $\hat{q} \prec_v q_{l+1}$, IC_b is violated for type v.

Proof of (A.10). Since $\hat{q} \ngeq q_{l+1}$, there exists $\gamma \in \mathbb{R}^{\bar{g}}_{++}$ such that $\gamma \cdot \hat{q} < \gamma \cdot q_{l+1}$. For all $q \in Q$, define $v(q) \equiv \frac{1}{\epsilon}(\gamma \cdot q)$, where $\epsilon > 0$ is sufficiently small. By (a), $q_0 < \cdots < q_l < \hat{q}$ and thus $v(q_0) \ll \cdots \ll v(q_l) \ll v(\hat{q})$. Moreover, by definition of γ, $v(\hat{q}) \ll v(q_{l+1})$.

Proof of (A.11). For all $q \in \bar{Q}$, define

$$c(q) \equiv \sum_{g \in \bar{G}} \left[\epsilon q^g + \left(\sum_{f \in \bar{G}} \epsilon \hat{q}^f + \frac{\max\{p_s(q_{l+1}) - p_s(\hat{q}), 0\}}{\sum_{h \in \bar{G}} \max\{q^h_{l+1} - \hat{q}^h, 0\}} \right) \max\{q^g - \hat{q}^g, 0\} \right] + \frac{1}{\epsilon} \max\{r_\epsilon(q) - 1, 0\},$$

where $r_\epsilon(q) \equiv \min\{\rho \in \mathbb{R}_+ : q \in X_\epsilon(\rho)\}$ and $\epsilon > 0$ is sufficiently small. The function $r_\epsilon \colon \mathbb{R}^{\bar{g}}_+ \to \mathbb{R}_+$ is convex,[21] which implies that $c \in C$.

Recall that for sufficiently small $\epsilon > 0$, $X_\epsilon(1) \cap Q = \{q_0, \dots, q_{l+1}, \hat{q}\}$. Hence, for all $q \in Q$, $r_\epsilon(q) > 1$ if and only if $q \in Q \setminus \{q_0, \dots, q_{l+1}, \hat{q}\}$. It follows that for all $q \in Q \setminus \{q_0, \dots, q_{l+1}, \hat{q}\}$, $c(q) \gg c(\hat{q})$ and thus $q \prec_c \hat{q}$.

It remains to show that $q_0 \prec_c \cdots \prec_c q_l \prec_c q_{l+1} \prec_c \hat{q}$. There are three steps. First, for all $k \in \{1, \dots, l\}$, $q_{k-1} \prec_c q_k$ because $p_s(q_{k-1}) < p_s(q_k)$ and $c(q_{k-1}) \approx c(q_k)$. Second, $q_l \prec_c q_{l+1}$ because

$$\begin{aligned} &\big[p_s(q_{l+1}) - c(q_{l+1})\big] - \big[p_s(q_l) - c(q_l)\big] \\ &= \underbrace{\min\big\{p_s(q_{l+1}), p_s(\hat{q})\big\} - p_s(q_l)}_{>0} \\ &\quad \underbrace{- \epsilon \sum_{g \in \bar{G}} \Big(q^g_{l+1} - q^g_l + \max\{q^g_{l+1} - \hat{q}^g, 0\} \sum_{f \in \bar{G}} \hat{q}^f \Big)}_{\approx 0} > 0. \end{aligned}$$

Third, $q_{l+1} \prec_c \hat{q}$ because

$$\begin{aligned} &\big[p_s(\hat{q}) - c(\hat{q})\big] - \big[p_s(q_{l+1}) - c(q_{l+1})\big] \\ &= \epsilon \Big[\underbrace{\sum_{g \in \bar{G}} q^g_{l+1}}_{>0} + \underbrace{\Big(\sum_{f \in \bar{G}} \hat{q}^f \Big)}_{>0} \underbrace{\Big(\sum_{g \in \bar{G}} \max\{q^g_{l+1} - \hat{q}^g, 0\} - 1 \Big)}_{\geq 0} \Big] \\ &+ \underbrace{\max\big\{p_s(\hat{q}) - p_s(q_{l+1}), 0\big\}}_{\geq 0} > 0. \end{aligned}$$

□

[21] It is straightforward to verify that (i) $r_\epsilon(\mathbf{0}) = 0$, (ii) for all $q \in \mathbb{R}^{\bar{g}}_+ \setminus \{\mathbf{0}\}$, $r_\epsilon(q) > 0$, (iii) r_ϵ is homogenenous of degree 1 and (iv) r_ϵ is quasiconvex. These four properties imply that r_ϵ is convex. The proof is standard and available from the authors upon request.

A.6 Proof of Lemma 5

We only prove (i); (ii) is analogous. Consider any $c \in C$. Clearly, there exists $v \in V$ such that $Q_s(v) \cap \mathrm{Opt}_s(c) \neq \emptyset$. IC_s requires that $\varphi(c, v) \in \mathrm{Opt}_s(c)$ and thus $Q_b(c) \cap \mathrm{Opt}_s(c) \neq \emptyset$. Define $l \in \{0, \dots, n\}$ by $q_l \equiv \max\{Q_b(c) \cap \mathrm{Opt}_s(c)\}$.

Next, we show that $Q_b(c) \subseteq \{q_0, \dots, q_l\}$. By contradiction, suppose that $q_m \in Q_b(c)$ for some $m \in \{l+1, \dots, n\}$. From the definition of q_l, it follows that $q_m \notin \mathrm{Opt}_s(c)$. Moreover, by Lemma 4(iv), there exists $c_l \in C$ such that $Q_b(c_l) = \{q_0, \dots, q_l\}$. Define $v \in V$ by $v(q) \equiv \sum_{g \in \bar{G}} \left[\epsilon q^g + \frac{1}{\epsilon} \min\{q^g, q_m^g\}\right]$ for all $q \in \bar{Q}$. In light of Lemma 4, parts (i) and (iii), we get that $q_0 \prec_v \cdots \prec_v q_l \prec_v \cdots \prec_v q_m \succ_v \cdots \succ_v q_n$ for sufficiently small $\epsilon > 0$. Hence, $\mathrm{Opt}_b(v, Q_b(c)) = \{q_m\}$ and $\mathrm{Opt}_b(v, Q_b(c_l)) = \{q_l\}$. From Lemma 3, it follows that $\varphi(c, v) = q_m$ and $\varphi(c_l, v) = q_l$, which violates IC_s for type c.

Finally, we show that $\{q_0, \dots, q_l\} \subseteq Q_b(c)$. The proof is by induction with basis $q_l \in Q_b(c)$. Consider any $k \in \{1, \dots, l\}$ and suppose that $q_k \in Q_b(c)$. We prove that $q_{k-1} \in Q_b(c)$. To the contrary, assume that $q_{k-1} \notin Q_b(c)$. By Lemma 4(iv), there exists $c_k \in C$ such that $Q_b(c_k) = \{q_0, \dots, q_k\}$ and $q_0 \prec_{c_k} \cdots \prec_{c_k} q_k$. Define $v \in V$ as follows: for all $q \in \bar{Q}$,

$$v(q) \equiv \sum_{g \in \bar{G}} \left[\epsilon q^g + \frac{1}{\epsilon} \min\{q^g, q_{k-1}^g\} + \left(\frac{p_b(q_k) - p_b(q_{k-1})}{\sum_{g \in \bar{G}} (q_k^g - q_{k-1}^g)} - 2\epsilon\right) \min\{q^g, q_k^g\}\right].$$

For sufficiently small $\epsilon > 0$, $q_{k-1} \succ_v q_k \succ_v q$ for all $q \in Q \setminus \{q_{k-1}, q_k\}$. Hence, $\mathrm{Opt}_b(v, Q_b(c)) = \{q_k\}$ and $\mathrm{Opt}_b(v, Q_b(c_k)) = \{q_{k-1}\}$. From Lemma 3, it follows that $\varphi(c, v) = q_k$ and $\varphi(c_k, v) = q_{k-1}$. Since $q_k \succ_{c_k} q_{k-1}$, IC_s is violated for c_k.

A.7 Proof of Lemma 6

We only prove (i); (ii) is analogous. By contradiction, suppose there exist $k \in \{1, \dots, n-1\}$ and $\gamma \in \mathbb{R}_{++}^{\bar{g}}$ such that

$$\Delta_{k-1}^k \equiv \frac{p_s(q_k) - p_s(q_{k-1})}{\gamma \cdot (q_k - q_{k-1})} < \frac{p_s(q_{k+1}) - p_s(q_k)}{\gamma \cdot (q_{k+1} - q_k)} \equiv \Delta_k^{k+1}.$$

Define $\alpha \equiv \frac{\gamma\cdot(q_k - q_{k-1})}{\gamma\cdot(q_{k+1}-q_{k-1})} \in (0,1)$. Then

$$\Delta_{k-1}^{k+1} \equiv \frac{p_s(q_{k+1}) - p_s(q_{k-1})}{\gamma\cdot(q_{k+1}-q_{k-1})} = \alpha\Delta_{k-1}^{k} + (1-\alpha)\Delta_{k}^{k+1} > \Delta_{k-1}^{k}.$$

Take $\delta \in \left(\Delta_{k-1}^{k}, \Delta_{k-1}^{k+1}\right)$. Define $c \in C$ as follows: for all $q \in \bar{Q}$,

$$c(q) \equiv \epsilon(\gamma\cdot q) + (\delta-\epsilon)\max\{\gamma\cdot(q-q_{k-1}),0\} + \frac{1}{\epsilon}\max\{\gamma\cdot(q-q_{k+1}),0\},$$

where $\epsilon \in (0,\delta)$ is sufficiently small. Figure A.3 provides a graphical illustration of type c for $\bar{g}=2$.

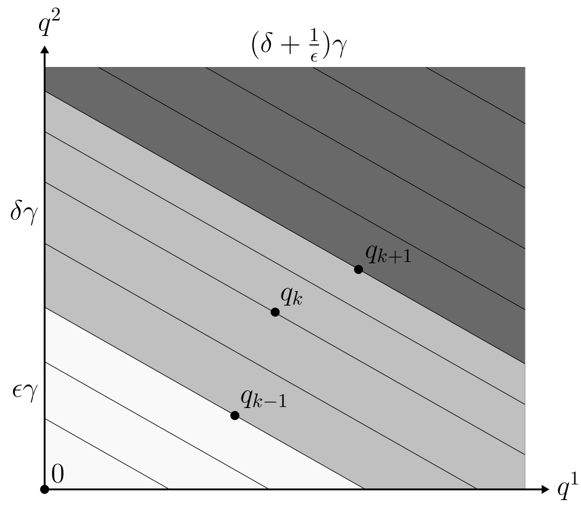

Figure A.3: [Proof of Lemma 6] Illustration of the isocost curves of type c for $\bar{g}=2$. The three expressions along the axes are the unique subgradients of c in the interior of the three shaded regions.

It is easy to check that $q_0 \prec_c \cdots \prec_c q_{k-1}$ and $q_{k+1} \succ_c \cdots \succ_c q_n$. In addition, $\delta > \Delta_{k-1}^{k}$ implies that $q_{k-1} \succ_c q_k$, and $\delta < \Delta_{k-1}^{k+1}$ implies that $q_{k-1} \prec_c q_{k+1}$. Since $\text{Opt}_s(c) = \{q_{k+1}\}$, Lemma 5(i) requires that $Q_b(c) = \{q_0,\ldots,q_{k+1}\}$. Moreover, by Lemma 4(v), there exists $v_k \in V$ such that $Q_s(v_k) = \{q_0,\ldots,q_k\}$ and $q_0 \prec_{v_k} \cdots \prec_{v_k} q_k$. Hence, $\text{Opt}_s(c, Q_s(v_k)) = \{q_{k-1}\}$ and $q_{k-1} \notin \text{Opt}_b(v_k, Q_b(c))$. Lemma 3 then implies that $\varphi(c, v_k) \in \emptyset$, which is impossible.

A.8 Proof of Lemma 7

Consider any $(c,v) \in C \times V$. By Lemma 3, $\varphi(c,v) \in Q_s(v) \cap Q_b(c)$. Moreover, defining $q_{sb} \equiv \min\{q_s(c), q_b(v)\}$, Lemma 5 implies that $Q_s(v) \cap Q_b(c) = \{q_0, \dots, q_{sb}\}$. Hence, $\varphi(c,v) \in \{q_0, \dots, q_{sb}\}$. Since also $\varphi(c,v) \in \mathrm{Opt}_s(c, Q_s(v))$, it follows that $\varphi(c,v) \in \mathrm{Opt}_s(c, \{q_0, \dots, q_{sb}\})$. Lemma 1(i) implies that $\mathrm{Opt}_s(c, \{q_0, \dots, q_{sb}\}) = \big\{\min\{q_{sb}, \underline{q}_s\}, \dots, q_{sb}\big\}$. Thus, $\varphi(c,v) \in \big\{\min\{q_{sb}, \underline{q}_s\}, \dots, q_{sb}\big\}$. Analogous reasoning for the buyer yields that $\varphi(c,v) \in \big\{\min\{q_{sb}, \underline{q}_b\}, \dots, q_{sb}\big\}$. Combine both expressions to obtain that

$$\varphi(c,v) \in \Big\{\max\big\{\min\{q_{sb}, \underline{q}_s\}, \min\{q_{sb}, \underline{q}_b\}\big\}, \dots, q_{sb}\Big\}. \tag{A.12}$$

Without loss of generality, suppose that $\underline{q}_s \leq \underline{q}_b$. Then $\min\{q_{sb}, \underline{q}_s\} \leq \min\{q_{sb}, \underline{q}_b\}$ and thus

$$\begin{aligned} \max\big\{\min\{q_{sb}, \underline{q}_s\}, \min\{q_{sb}, \underline{q}_b\}\big\} &= \min\{q_{sb}, \underline{q}_b\} \\ &= \min\big\{q_{sb}, \max\{\underline{q}_s, \underline{q}_b\}\big\}. \end{aligned}$$

Plug into (A.12) to conclude that

$$\varphi(c,v) \in \Big\{\min\big\{q_{sb}, \max\{\underline{q}_s, \underline{q}_b\}\big\}, \dots, q_{sb}\Big\}.$$

References

Andreyanov, P., Čopič, J., & Jeong, B.-h. (2018). *Optimal robust divisible mechanisms for public goods*. Working paper.

Barberà, S., Gul, F., & Stacchetti, E. (1993). Generalized median voter schemes and committees. *Journal of Economic Theory*, *61*(2), 262–289.

Barberà, S., & Jackson, M. O. (1995). Strategy-proof exchange. *Econometrica*, *63*(1), 51–87.

Berga, D., & Serizawa, S. (2000). Maximal domain for strategy-proof rules with one public good. *Journal of Economic Theory*, *90*(1), 39–61.

Bergemann, D., & Morris, S. (2005). Robust mechanism design. *Econometrica*, *73*(6), 1771–1813.

Bierbrauer, F., & Hellwig, M. (2016). Robustly coalition-proof incentive mechanisms for public good provision are voting mechanisms and vice versa. *Review of Economic Studies*, *83*(4), 1440–1464.

Bierbrauer, F., & Winkelmann, J. (2020). All or nothing: State capacity and optimal public goods provision. *Journal of Economic Theory*, *185*, 104955.

Börgers, T. (2015). *An introduction to the theory of mechanism design*. Oxford University Press.

Börgers, T., & Li, J. (2019). Strategically simple mechanisms. *Econometrica*, *87*(6), 2003–2035.

Börgers, T., & Smith, D. (2012). Robustly ranking mechanisms. *American Economic Review: Papers and Proceedings*, *102*(3), 325–329.

Boyd, S., & Vandenberghe, L. (2004). *Convex optimization*. Cambridge University Press.

Drexl, M., & Kleiner, A. (2018). Why voting? A welfare analysis. *American Economic Journal: Microeconomics*, *10*(3), 253–271.

Hagerty, K. M., & Rogerson, W. P. (1987). Robust trading mechanisms. *Journal of Economic Theory*, *42*(1), 94–107.

Jehiel, P., Moldovanu, B., & Stacchetti, E. (1999). Multidimensional mechanism design for auctions with externalities. *Journal of Economic Theory*, *85*(2), 258–293.

Kuzmics, C., & Steg, J.-H. (2017). On public good provision mechanisms with dominant strategies and balanced budget. *Journal of Economic Theory*, *170*, 56–69.

Li, S. (2017). Obviously strategy-proof mechanisms. *American Economic Review*, *107*(11), 3257–3287.

Manelli, A. M., & Vincent, D. R. (2007). Multidimensional mechanism design: Revenue maximization and the multiple-good monopoly. *Journal of Economic Theory*, *137*(1), 153–185.

Milgrom, P., & Segal, I. (2002). Envelope theorems for arbitrary choice sets. *Econometrica*, *70*(2), 583–601.

Miyagawa, E. (2001). House allocation with transfers. *Journal of Economic Theory*, *100*(2), 329–355.

Moulin, H. (1994). Serial cost-sharing of excludable public goods. *Review of Economic Studies*, *61*(2), 305–325.

Murota, K. (2016). Discrete convex analysis: A tool for economics and game theory. *Journal of Mechanism and Institution Design*, *1*(1), 149–271.

Rochet, J.-C. (1985). The taxation principle and multi-time Hamilton-Jacobi equations. *Journal of Mathematical Economics*, *14*(2), 113–128.

Schummer, J. (2000). Eliciting preferences to assign positions and compensation. *Games and Economic Behavior*, *30*(2), 293–318.

Serizawa, S. (1996). Strategy-proof and individually rational social choice functions for public good economies. *Economic Theory*, *7*(3), 501–512.

Serizawa, S. (1999). Strategy-proof and symmetric social choice functions for public good economies. *Econometrica*, *67*(1), 121–145.

Serizawa, S. (2006). Pairwise strategy-proofness and self-enforcing manipulation. *Social Choice and Welfare*, *26*(2), 305–331.

Svensson, L.-G., & Larsson, B. (2002). Strategy-proof and nonbossy allocation of indivisible goods and money. *Economic Theory*, *20*(3), 483–502.

Thomson, W. (2016). Non-bossiness. *Social Choice and Welfare*, *47*(3), 665–696.

Yamashita, T. (2015). Implementation in weakly undominated strategies: Optimality of second-price auction and posted-price mechanism. *Review of Economic Studies*, *82*(3), 1223–1246.

Zhou, L. (1991). Impossibility of strategy-proof mechanisms in economies with pure public goods. *Review of Economic Studies*, *58*(1), 107–119.