SIAM J. MATH. ANAL.
Vol. 53, No. 5, pp. 5646–5681

ERGODICITY OF SUBLINEAR MARKOVIAN SEMIGROUPS*

CHUNRONG FENG† AND HUAIZHONG ZHAO‡

Abstract. In this paper, we study the ergodicity of an invariant sublinear expectation of sublinear Markovian semigroups. For this, we first develop an ergodic theory of an expectation preserving map on a sublinear expectation space. Ergodicity is defined as any invariant set in which either itself or its complement has 0 capacity. We prove, under a general sublinear expectation space setting, the equivalent relation between ergodicity and the corresponding transformation operator having simple eigenvalue 1, and also with Birkhoff-type strong law of large numbers if the sublinear expectation is regular. For a sublinear Markov process, we prove that its ergodicity is equivalent to the Markovian semigroup having eigenvalue 1, and the eigenvalue is simple in the space of bounded measurable functions. As an example we show that G-Brownian motion $\{B_t\}_{t\geq 0}$ on the unit circle has an invariant expectation and is ergodic if and only if $\mathbb{E}(-(B_1)^2) < 0$. Moreover, it is also proved in this case that the invariant expectation is regular and the canonical stationary process has no mean-uncertainty under the invariant expectation.

Key words. invariant sublinear expectation, spectrum, sublinear Markovian semigroup, G-Brownian motion, mean-uncertainty, fully nonlinear PDEs

AMS subject classifications. 60H10, 60J65, 37H05, 37A30

DOI. 10.1137/20M1356518

1. Introduction.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. The measure theoretical ergodic theory deals with a measure preserving map $\theta : (\Omega, \mathcal{F}) \to (\Omega, \mathcal{F})$ such that $\theta P = P$. Recall that the measurable dynamical system $\{\theta^n\}_{n\in\mathbb{N}}$ on $(\Omega, \mathcal{F}, P)$ is called ergodic if any invariant set $A \in \mathcal{F}$, i.e., $\theta^{-1}A = A$, has either full measure or zero measure. Ergodicity describes the indecomposable property of the system (cf. Walters [41]). The well-known result of Birkhoff's theorem says that a dynamical system is ergodic if and only if, in the long run, the time average of a function along its trajectory is the same as the spatial average on the entire space with respect to the stationary measure (Birkhoff [2], von Neumann [39, 40]).

Due to the spreading nature of random forcing, ergodicity is an important common feature of stochastic systems. It has aroused enormous interests of mathematicians (cf. Da Prato and Zabczyk [8], Durrett [12], Feng and Zhao [21]). For a Markovian random dynamical system, it is well known that 1 is a simple eigenvalue of the Markovian semigroup if and only if the stochastic system is ergodic, and is a unique eigenvalue on the unit circle and is simple if and only if the stochastic system is weakly mixing. The latter statement is equivalent to the Koopman–von Neumann theorem. Recently, the ergodic theory for periodic measures was obtained, in which it was proved that the Markovian semigroup has eigenvalues, $\{e^{i\frac{2m\pi}{\tau}t}\}_{m\in\mathbb{Z}}$ for a $\tau > 0$, on the unit circle apart from the eigenvalue 1 (Feng and Zhao [21]). Moreover, invariant measures of quasi-periodic stochastic systems were observed in Feng, Qu, and Zhao [19].

*Received by the editors July 30, 2020; accepted for publication (in revised form) July 6, 2021; published electronically October 5, 2021.
https://doi.org/10.1137/20M1356518

Funding: This work was supported by the Royal Society Newton Fund through grant NA15034 and by the EPSRC through grant EP/S005293/2.

†Department of Mathematical Sciences, Durham University, Durham, DH1 3LE, UK (chunrong.feng@durham.ac.uk).

‡Department of Mathematical Sciences, Durham University, Durham, DH1 3LE, UK, and Research Centre for Mathematics and Interdisciplinary Sciences, Shandong University, Qingdao, 266237, China (huaizhong.zhao@durham.ac.uk).

On a completely different topic, the concept of sublinear expectation is central in probability and statistics under uncertainty, measures of risk, and superhedging in finance (Artzner et al. [1], Chen and Epstein [6], El Karoui, Peng, and Quenez [15], Follmer and Schied [23]). A coherent risk measure, which is defined as a real-valued (monetary value) functional with properties of constant preserving (cash invariance), monotonicity, convexity, and positive homogeneity, is equivalent to a sublinear expectation. A systematic stochastic analysis of nonlinear/sublinear expectation and G-Brownian motion has been given in the substantial work of Peng [34, 35, 36].

It is clear now that the corresponding partial differential equations (PDEs) of G-diffusions are fully nonlinear parabolic PDEs. They provide us with the Markovian semigroup of G-diffusion processes (Peng [34, 35, 36]). It is noted that fully nonlinear PDEs have been intensively studied in literature, e.g., in Cafarelli and Cabre [3], Krylov [30, 31], and Lions [32, 33]. More recently, the viscosity solution of path-dependent fully nonlinear PDEs has been of great interest (Ekren, Touzi, and Zhang [13, 14], Peng [37]). However, study of the dynamical properties of long time behavior of G-diffusion processes is still missing. In this context, an ergodic theory under the sublinear expectation setting will be key to this study. Our results will give the invariant properties, equilibrium, and statistical property of the stochastic systems under uncertainty.

It is worth noting that economists have already observed "nonlinearities" in the behavior of real-world trading in financial markets due to heterogeneity of expectation-formation processes (Cutler, Poterba, and Summers [7], De Long et al. [10], Frankel and Froot [24], Greenwood and Shleifer [26], Williams [42]). Potentially biased beliefs of future price movements drive the decision of stock-market participants and create ambiguous volatility. Using sublinear expectations and G-Brownian motions to model ambiguity has been attempted in the mathematical finance literature; see, e.g., Chen and Epstein [6] and Epstein and Ji [17].

In this paper, we will go beyond the measure space framework to study an ergodic theory in a nonlinear functional setting. The lack of the dominated convergence and the Riesz representation adds a lot of difficulty to the analysis. But the topology of a sublinear expectation space is still rich enough for us to define the ergodicity. We will establish its equivalence with the indecomposable property and characterization in terms of spectrum of transformation operators. We will prove the law of large numbers (LLN) also implies ergodicity, but the converse holds under a regularity assumption. This setup is a natural framework for the ergodicity of invariant expectation of continuous time sublinear Markovian semigroups such as that of G-diffusions.

It is noted here that the convergence of the LLN we study in this paper is in the pathwise sense quasi-surely. Convergence of the LLN in the sense of distribution was obtained by Peng [36] for independent and identically distributed (i.i.d.) random variable sequences. But an i.i.d. random variable sequence may not be stationary in nonadditive probability settings (Feng, Wu, and Zhao [20]). On the other hand, in our theory the independence assumption is not needed. Thus the LLN in the ergodic sense we study here and Peng's LLN for i.i.d. random variable sequences have different conclusions under different assumptions.

We study Markovian stochastic dynamical systems with noise over a sublinear expectation space. A canonical sublinear expectation space with an expectation preserving map is constructed from an invariant expectation by the nonlinear Kolmogorov extension theorem onto the lifted path space. There is a natural expectation preserving dynamical system on the canonical sublinear expectation space. The ergodicity of the stochastic system is then given by that of the canonical system. Its equivalence with a spectral property of the Markovian semigroup is also established.

As an example we show that the G-Brownian motion $B_t = \sqrt{t}\xi$ on the unit circle, where ξ has normal distribution $N(0, [\underline{\sigma}^2, \bar{\sigma}^2])$, has an ergodic invariant expectation if and only if $\underline{\sigma}^2 > 0$. Moreover, the invariant expectation and its extension on the canonical path space are regular, so a Birkhoff-type LLN holds. It is also noted that the canonical stationary process, which is the process corresponding to the large time behavior, has no mean-uncertainty under the invariant expectation.

This paper is the first paper to study the ergodic theory on a sublinear expectation space. This study is very general in order to include both discrete time and continuous time cases. Extending ideas of this paper on the discrete time case, ergodicity for capacity—especially upper probability—has been obtained in Feng, Wu, and Zhao [20]. Inspired by this work, the ergodicity of upper expectations generated from periodic measures has also been obtained (Feng, Qu, and Zhao [18]).

2. Dynamical systems on sublinear expectation spaces and ergodicity.

We first briefly recall the concept of sublinear expectation for convenience. Let $(\Omega, \mathcal{F})$ be a measurable space, and let $\mathcal{D}$ be the linear space of all $\mathcal{F}$-measurable real-valued functions. In particular, the indicator functions of any $\mathcal{F}$-measurable sets which will be used in this paper are included in $\mathcal{D}$.

Definition 2.1 (cf. Peng [36]). *A sublinear expectation* $\mathbb{E}$ *is a functional* $\mathbb{E} : \mathcal{D} \to \mathbb{R}$ *satisfying*

(i) *monotonicity:*

$$\mathbb{E}[X] \geq \mathbb{E}[Y] \quad \textit{if } X \geq Y;$$

(ii) *constant preserving:*

$$\mathbb{E}[c] = c \quad \textit{for } c \in \mathbb{R};$$

(iii) *subadditivity: for each* $X, Y \in \mathcal{D}$,

$$\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y];$$

(iv) *positive homogeneity:*

$$\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X] \quad \textit{for } \lambda \geq 0.$$

The triple $(\Omega, \mathcal{D}, \mathbb{E})$ *is called a sublinear expectation space.*

The ergodicity concept in the sublinear situation is very subtle due to the absence of the linearity for functionals. The essence of the ergodicity is indecomposibility of dynamical systems. However, unlike in the classical ergodic theory, a set A satisfying $\mathbb{E}\mathrm{I}_A = 1$ does not imply $\mathbb{E}\mathrm{I}_{A^c} = 0$ as the sublinear expectation $\mathbb{E}$ only satisfies

$$\mathbb{E}\mathrm{I}_A + \mathbb{E}\mathrm{I}_{A^c} \geq 1. \tag{2.1}$$

In fact it is quite possible that $\mathbb{E}\mathrm{I}_A = 1$ and $\mathbb{E}\mathrm{I}_{A^c} = 1$. As a consequence, it is not viable to extend the classical definition of ergodicity, which says that any invariant set A has either probability 0 or 1 to $\mathbb{E}I_A = 0$ or 1 in the sublinear case.

The nonadditivity also creates a lot of technical difficulty in the analysis of its dynamics due to the lack of many important analysis tools such as dominated convergence and the Riesz representation. But the topology of a sublinear expectation space is still rich enough for us to define the ergodicity in which the indecomposibility is the most important property to survive. This is in line with the classical definition in measure theoretical ergodic theory. We observe that three different forms of ergodicity in terms of invariant sets, spectrum of transformation operators, and strong LLN are still equivalent under the sublinear expectation setting with slightly stronger functionals satisfying the regularity given below. Without assuming conditions (iii) and (iv) in Definition 2.1, it is still not clear how one can define ergodicity in line with indecomposibility.

The representation result (Artzner et al. [1], Delbaen [9], Follmer and Schied [22]) says that there exists a family of linear expectations $\{E_\theta : \theta \in \Theta\}$ defined on $\mathcal{D}$ such that

$$\mathbb{E}[X] = \sup_{\theta\in\Theta} E_\theta[X].$$

If it is assumed further that

(2.2)

$$\mathbb{E}[X_i] \to 0 \text{ for each sequence of measurable functions such that } X_i(\omega) \downarrow 0 \text{ for each } \omega,$$

by the Daniell–Stone theorem, the following *representation as upper integrals* holds: there exists a family of σ-additive probability measures $\mathcal{P}$ on $(\Omega, \mathcal{F})$ such that

$$\mathbb{E}[X] = \sup_{P\in\mathcal{P}} E_P[X]. \tag{2.3}$$

The representation as upper integrals is not essential to proceed with our theory. We only need this in the proof of the LLN from the ergodicity. We introduce the regularity of the following form.

DEFINITION 2.2. *The functional* $\mathbb{E}[\cdot]$ *is said to be regular if, for any* $A_n \in \mathcal{F}$, $A_n \downarrow \emptyset$, *we have* $\mathbb{E}[I_{A_n}] \downarrow 0$.

Remark 2.3. (i) Definition 2.2 is equivalent to saying that if, for any $A_n \in \mathcal{F}$, $A_n \downarrow A$ and $\mathbb{E}I_A = 0$, we have $\mathbb{E}[I_{A_n}] \downarrow 0$. This can be seen from

$$|\mathbb{E}[I_{A_n}] - \mathbb{E}[I_A]| \le \mathbb{E}[I_{A_n\setminus A}].$$

(ii) A similar condition to that of Definition 2.2 in the capacity setting was also used in the literature; see, e.g., Cerreia-Vioglio, Maccheroni, and Marinacci [4], where this was called continuous.

(iii) In Lemma 4.1 and Proposition 4.5, we will prove that the semigroup and the invariant expectation for G-Brownian motion on S^1 are regular.

(iv) The main results of this section are the relationships of ergodicity (E), the simpleness of eigenvalue 1 of the transformation operator U_t on the bounded and measurable function space (SE), and the law of large numbers (LLN). We prove (E) $\iff$ (SE) $\Leftarrow$ (LLN) without the regular condition, which is needed only in the proof of (E) $\Rightarrow$ (LLN).

Now we introduce a measurable transformation $\theta : \Omega \to \Omega$ that preserves the sublinear expectation $\mathbb{E}$, i.e.,

$$\theta\mathbb{E} = \mathbb{E}. \tag{2.4}$$

Here $\theta\mathbb{E}$ is defined as

$$\theta\mathbb{E}[X(\cdot)] = \mathbb{E}[X(\theta\cdot)] \quad \text{for any } X \in \mathcal{D}.$$

Set the transformation operator $U_1 : \mathcal{D} \to \mathcal{D}$ by

$$U_1\xi(\omega) = \xi(\theta\omega),\ \xi \in \mathcal{D}.$$

Then expectation preserving of θ is equivalent to

$$\mathbb{E}[U_1\xi] = \mathbb{E}[\xi] \text{ for any } \xi \in \mathcal{D}.$$

Define $\theta^n = \theta \circ \theta \circ \cdots \circ \theta$, $n \in \mathbb{N}$. Then $\{\theta^n\}_{n\in\mathbb{N}}$ forms a family of measurable transformations from $(\Omega, \mathcal{F})$ to itself and satisfies the expectation preserving property and the semigroup property:

$$\theta^{m+n} = \theta^m \circ \theta^n \text{ for } n, m \in \mathbb{N}. \tag{2.5}$$

Thus $\{\theta^n\}_{n\in\mathbb{N}}$ is a dynamical system on $(\Omega, \mathcal{D}, \mathbb{E})$ and preserves the sublinear expectation. In the following $S = (\Omega, \mathcal{D}, \mathbb{E}, \{\theta^n\}_{n\in\mathbb{N}})$ denotes the dynamical system.

We say that a statement holds quasi-surely if it is true for all $\omega \in \Omega \setminus A$ for a set A with $\mathbb{E}[I_A] = 0$ and v-almost surely (v-a.s.) if it is true for all $\omega \in \Omega \setminus A$ for a set A with $\mathbb{E}[-I_A] = 0$.

If a set $B \in \mathcal{F}$ satisfies

$$\theta^{-1}B = B, \tag{2.6}$$

then we say the set B is invariant with respect to the transformation θ. First we prove the following result.

THEOREM 2.4. *If $\theta : \Omega \to \Omega$ is a measurable expectation preserving transformation of the sublinear expectation space $(\Omega, \mathcal{D}, \mathbb{E})$, then the statements*

(i) *any invariant measurable set $B \in \mathcal{F}$ with respect to θ satisfies either $\mathbb{E}\mathrm{I}_B = 0$ or $\mathbb{E}\mathrm{I}_{B^c} = 0$;*
(ii) *if $B \in \mathcal{F}$ and $\mathbb{E}\mathrm{I}_{\theta^{-1}B\triangle B} = 0$, then either $\mathbb{E}\mathrm{I}_B = 0$ or $\mathbb{E}\mathrm{I}_{B^c} = 0$;*
(iii) *for every $A \in \mathcal{F}$ with $\mathbb{E}\mathrm{I}_A > 0$, we have $\mathbb{E}\mathrm{I}_{(\bigcup_{n=1}^{\infty} \theta^{-n}A)^c} = 0$;*
(iv) *for every $A, B \in \mathcal{F}$ with $\mathbb{E}\mathrm{I}_A > 0$ and $\mathbb{E}\mathrm{I}_B > 0$, there exists $n \in \mathbb{N}^+$ such that $\mathbb{E}\mathrm{I}_{(\theta^{-n}A\cap B)} > 0$,*

have the following relations: (i) *and* (ii) *are equivalent;* (iii) *implies* (iv); (iv) *implies* (i). *Moreover, if $\mathbb{E}$ is regular, then* (ii) *implies* (iii) *and all four statements are equivalent.*

In (ii) above, $\cdot \triangle \cdot$ means the symmetric difference. The proof of this theorem is postponed to the appendix. The result is similar to that in classical ergodic theory, but it needs to deal with issues due to the lack of additivity of probability. It is crucial to establish relations of these four statements, especially their equivalence when $\mathbb{E}$ is regular under the sublinear expectation setting. We will see that the combination of sublinearity and statement (i) enables us to establish the ergodic theory. We now discuss statement (i) more closely. Note that if the set B is invariant, then it is easy to see that $\theta^{-1}(B^c) = B^c$. Thus in the case that $0 < \mathbb{E}\mathrm{I}_B \le 1$ and $0 < \mathbb{E}\mathrm{I}_{B^c} \le 1$, we could study θ by studying two simpler transformations $\theta|_B$ and $\theta|_{B^c}$ separately. On the contrary, if $\mathbb{E}\mathrm{I}_B = 0$ and $\mathbb{E}\mathrm{I}_{B^c} = 1$, we only need to study $\theta|_{B^c}$. Similarly, if $\mathbb{E}\mathrm{I}_B = 1$ and $\mathbb{E}\mathrm{I}_{B^c} = 0$, we only need to study $\theta|_B$. In the latter two cases, the transformation is indecomposable. However, it is noted that $\mathbb{E}\mathrm{I}_B = 0$ implies $\mathbb{E}\mathrm{I}_{B^c} = 1$ and $\mathbb{E}\mathrm{I}_{B^c} = 0$ implies $\mathbb{E}\mathrm{I}_B = 1$. With the above observations, we give the following definition.

DEFINITION 2.5. *Let $(\Omega, \mathcal{D}, \mathbb{E})$ be a sublinear expectation space. An expectation preserving transformation θ of $(\Omega, \mathcal{D}, \mathbb{E})$ is called ergodic if any invariant measurable set $B \in \mathcal{F}$ satisfies either $\mathbb{E}\mathrm{I}_B = 0$ or $\mathbb{E}\mathrm{I}_{B^c} = 0$.*

THEOREM 2.6. *If* $(\Omega, \mathcal{D}, \mathbb{E})$ *is a sublinear expectation space and the measurable map* $\theta : \Omega \to \Omega$ *is expectation preserving, then the following statements are equivalent:*

(i) *The map* θ *is ergodic.*

(ii) *Whenever* $\xi : \Omega \to \mathbb{R}$ (*or* $\mathbb{C}$) *is measurable, bounded quasi-surely, and* $U_1\xi = \xi$*, then* ξ *is constant quasi-surely.*

If $\mathbb{E}$ *is regular, then* (i) *and* (ii) *are equivalent to statement* (iii).

(iii) *Whenever* $\xi : \Omega \to \mathbb{R}$ (*or* $\mathbb{C}$) *is measurable and* $U_1\xi = \xi$ *quasi-surely, then* ξ *is constant quasi-surely.*

Proof. It is trivial to see that (iii)⇒(ii).

(ii)⇒(i) Consider $A \in \mathcal{F}$ as an invariant set. Note that I_A is bounded measurable and satisfies $U_1\mathrm{I}_A = \mathrm{I}_A$ quasi-surely. Thus I_A is constant quasi-surely. So $\mathrm{I}_A = 0$ or 1. If $\mathrm{I}_A = 0$ quasi-surely, then $\mathbb{E}\mathrm{I}_A = 0$. If $\mathrm{I}_A = 1$ quasi-surely, then $\mathrm{I}_{A^c} = 1 - \mathrm{I}_A = 0$ quasi-surely, so $\mathbb{E}\mathrm{I}_{A^c} = 0$. That is to say either $\mathbb{E}\mathrm{I}_A = 0$ or $\mathbb{E}\mathrm{I}_{A^c} = 0$. Thus θ is ergodic.

(i)⇒(iii) Now we assume $\mathbb{E}$ is regular. Let θ be ergodic, let ξ be measurable, and let $U_1\xi = \xi$ quasi-surely. We assume ξ to be real-valued since if ξ is complex-valued, then we can consider the real and imaginary parts separately. We will prove ξ is a constant. For a number $\alpha \in \mathbb{R}$, define $A_\alpha = \{\omega : \xi(\omega) > \alpha\}$ and $A_\alpha^c = \{\omega : \xi(\omega) \leq \alpha\}$. Note that $\xi(\theta\omega) = \xi(\omega)$ quasi-surely and $(\theta^{-1}A_\alpha) \triangle A_\alpha \subset \{\omega : \xi(\theta\omega) \neq \xi(\omega)\}$, and we have $\mathbb{E}\mathrm{I}_{(\theta^{-1}A_\alpha)\triangle A_\alpha} = 0$. By assumption that θ is ergodic and by Theorem 2.4, we know that $\mathbb{E}[\mathrm{I}_{A_\alpha}] = 0$ or $\mathbb{E}[\mathrm{I}_{A_\alpha^c}] = 0$. Thus $\mathbb{E}[\mathrm{I}_{A_\alpha}] = 0$ or 1. Let $J := \{\alpha : \mathbb{E}[\mathrm{I}_{A_\alpha}] = 0\}$. By the regular property of $\mathbb{E}$, we have

$$0 = \mathbb{E}[\mathrm{I}_{\{\omega:\xi(\omega)=\infty\}}] = \mathbb{E}[\mathrm{I}_{\cap_{n=1}^\infty A_n}] = \lim_{n\to\infty} \mathbb{E}[\mathrm{I}_{A_n}].$$

Thus there exists $n \in \mathbb{N}$ such that $\mathbb{E}[\mathrm{I}_{A_n}] = 0$, that is, $n \in J$, which implies $J \neq \emptyset$. So set $\alpha_* = \inf J$ and immediately $\alpha_* \in J$ by monotone (increasing) convergence of sublinear expectation. Hence for any $\alpha > \alpha_*$, we have $\mathbb{E}[\mathrm{I}_{A_\alpha}] = 0$, and for any $\alpha < \alpha_*$, we have $\mathbb{E}[\mathrm{I}_{A_\alpha}] = 1$ and $\mathbb{E}[\mathrm{I}_{A_\alpha^c}] = 0$ by ergodicity. By monotone (increasing) convergence of sublinear expectation again, we have $\mathbb{E}[\mathrm{I}_{\{\omega:\ \xi(\omega)<\alpha_*\}}] = 0$. Combining $\mathbb{E}[\mathrm{I}_{\{\omega:\ \xi(\omega)>\alpha_*\}}] = 0$ and the subadditivity of $\mathbb{E}$, we have $\mathbb{E}[\mathrm{I}_{\{\omega:\ \xi(\omega)\neq\alpha_*\}}] = 0$. Thus ξ is constant quasi-surely.

From the proof (i)⇒(iii), we can see that the regular assumption is used to prove $J \neq \emptyset$. But if ξ is bounded quasi-surely, this is true automatically. So we do not need the regular assumption to obtain the equivalence of (i) and (ii). ◻

We now give the definition of the strong law of large numbers (SLLN).

DEFINITION 2.7. *A dynamical system* $S = \{\Omega, \mathcal{F}, \mathbb{E}, (\theta^n)_{n\in\mathbb{N}}\}$ *is said to satisfy the strong law of large numbers (SLLN) if, for any bounded measurable function* ξ*, there exists a constant* c *such that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \xi(\theta^n\omega) = c \quad \textit{quasi-surely.} \tag{2.7}$$

Remark 2.8. (i) The SLLN in general may have random limit. But here we are interested in the relationship between the SLLN and ergodicity. So in this paper, we define the SLLN in a strong sense in which the limit is constant.

(ii) In fact, it will be shown that the ergodicity and the SLLN are equivalent if $\mathbb{E}$ is regular. Without the regularity assumption, the SLLN still implies ergodicity, but it is not clear whether the converse is true. Thus, unlike the classical case, the SLLN may not be used as the definition of the ergodicity of the dynamical system $\{\theta^n\}_{n\in\mathbb{N}}$ unless it is regular.

Let $L_b(\mathcal{F})$ be the space of all $\mathcal{F}$-measurable real-valued functions such that $\sup_{\omega\in\Omega}|X(\omega)| < \infty$. Since $U_1 1 = 1$ by definition of U_1, it is obvious that 1 is an eigenvalue of $U_1 : L_b \to L_b$. The following result is almost obvious, but fundamental.

Theorem 2.9. *If S satisfies the SLLN, then the eigenvalue 1 of U_1 on L_b is simple and $\hat{\theta}$ is ergodic.*

Proof. Consider ξ satisfying

$$U_1\xi = \xi,$$

and ξ is a bounded measurable random variable. Thus by the SLLN assumption, we have that ξ is constant quasi-surely. Therefore the eigenvalue 1 of U_1 is simple. Finally, by Theorem 2.6, θ is ergodic. ∎

We now investigate the converse part of Theorem 2.9. For this we study the Birkhoff ergodic theorem under sublinear expectation. Before doing this, we need the following lemma. The expectation preserving property of θ is not required in Lemmas 2.10 and 2.12.

Lemma 2.10 (maximal ergodic lemma). *Let $\xi \in L^1(\Omega)$, $\xi_j(\omega) = \xi(\theta_j\omega)$, and $S_0 = 0$,*

$$S_k(\omega) = \xi_0(\omega) + \cdots + \xi_{k-1}(\omega) \quad \text{for } k \geq 1, \tag{2.8}$$

$$M_k(\omega) = \max_{0\leq j\leq k} S_j(\omega). \tag{2.9}$$

Then for $k \geq 1$,

$$\mathbb{E}[\xi \mathrm{I}_{\{M_k(\omega)>0\}}] \geq 0.$$

Proof. The proof is similar to that in the case of linear expectation given by Garsia [25], and so is omitted here. ∎

We say that a random variable ξ has no mean-uncertainty under $\mathbb{E}$ if $\mathbb{E}[\xi] = -\mathbb{E}[-\xi]$. Define the space for some $p \geq 1$, $L^p(\Omega) := \{\xi \in \mathcal{D} : \mathbb{E}|X|^p < \infty\}$,

$$\mathcal{H}^p := \{\xi \in L^p(\Omega) : \xi \text{ has no mean-uncertainty}\}$$

and

$$\mathcal{H}^p_{\mathbb{C}} := \{\xi \in L^p_{\mathbb{C}}(\Omega) : \xi \text{ has no mean-uncertainty}\}.$$

We have the following lemma which will be used later. Note here that we do not need the regularity assumption.

Lemma 2.11. *The space $\mathcal{H}^p$ (and $\mathcal{H}^p_{\mathbb{C}}$) is a Banach space.*

Proof. First note $\mathcal{H}^p$ ($\mathcal{H}^2_{\mathbb{C}}$) is a linear subspace of $L^p(\Omega)$ ($L^p_{\mathbb{C}}(\Omega)$). We only need to prove the real-valued random variable case. To see this, assume $\xi_1, \xi_2 \in L^p(\Omega)$ satisfy

$$\mathbb{E}[\xi_1] = -\mathbb{E}[-\xi_1], \qquad \mathbb{E}[\xi_2] = -\mathbb{E}[-\xi_2];$$

then by the sublinearity of $\hat{\mathbb{E}}$

$$\mathbb{E}[\xi_1 + \xi_2] \leq \mathbb{E}[\xi_1] + \mathbb{E}[\xi_2] = -\mathbb{E}[-\xi_1] - \mathbb{E}[-\xi_2] \leq -\mathbb{E}[-(\xi_1 + \xi_2)].$$

So

$$\mathbb{E}[\xi_1 + \xi_2] + \mathbb{E}[-(\xi_1 + \xi_2)] \leq 0.$$

But

$$\mathbb{E}[\xi_1 + \xi_2] + \mathbb{E}[-(\xi_1 + \xi_2)] \geq 0.$$

Therefore

$$\mathbb{E}[\xi_1 + \xi_2] + \mathbb{E}[-(\xi_1 + \xi_2)] = 0,$$

i.e., $\xi_1 + \xi_2$ has no mean-uncertainty. Since ξ_2 has no mean-uncertainty, neither does $-\xi_2$. Thus from what we have proved, we conclude that $\xi_1 - \xi_2$ has no mean-uncertainty.

Consider $\lambda_1, \lambda_2 > 0$. Note that $\mathbb{E}[\lambda_1\xi_1] = \lambda_1\mathbb{E}[\xi_1]$ and $\mathbb{E}[-\lambda_1\xi_1] = \lambda_1\mathbb{E}[-\xi_1]$. Thus if ξ_1 has no mean-uncertainty, then neither does $\lambda_1\xi_1$. Similarly if ξ_2 has no mean-uncertainty, then neither does $\lambda_2\xi_2$. Then by what we have proved, $\lambda_1\xi_1 + \lambda_2\xi_2$ has no mean-uncertainty. Now when $\lambda_1 > 0, \lambda_2 < 0$, if ξ_1 and ξ_2 have no mean-uncertainty, then $\lambda_1\xi_1$ and $-\lambda_2\xi_2$ have no mean-uncertainty. Hence $\lambda_2\xi_2$ has no mean-uncertainty. Thus $\lambda_1\xi_1 + \lambda_2\xi_2$ have no mean-uncertainty. This claim is also true for $\lambda_1 < 0$, $\lambda_2 > 0$, and $\lambda_1, \lambda_2 < 0$. Therefore $\lambda_1\xi_1 + \lambda_2\xi_2 \in \mathcal{H}^p$.

Assume $\xi_n \in \mathcal{H}^p$ is a Cauchy sequence and with the limit $\xi \in L^p(\Omega)$, i.e.,

$$\lim_{n\to 0} \mathbb{E}|\xi - \xi_n|^p = 0. \tag{2.10}$$

Then we show that ξ also has no mean-uncertainty. In fact,

$$\begin{aligned} \mathbb{E}[\xi] &\leq \mathbb{E}[\xi - \xi_n] + \mathbb{E}[\xi_n] \\ &= \mathbb{E}[\xi - \xi_n] - \mathbb{E}[-\xi_n] \\ &\leq \mathbb{E}[\xi - \xi_n] + \mathbb{E}[-\xi + \xi_n] - \mathbb{E}[-\xi]. \end{aligned}$$

Then, letting $n \to \infty$, we know the first two terms in the above will go to 0 because of (2.10). Thus $\mathbb{E}[\xi] \leq -\mathbb{E}[-\xi]$. But $\mathbb{E}[\xi] \geq -\mathbb{E}[-\xi]$, so $\mathbb{E}[\xi] = -\mathbb{E}[-\xi]$, i.e., ξ has no mean-uncertainty so that $\xi \in \mathcal{H}^p$. ◻

The following result is an extension of the Birkhoff ergodic theorem to the case of sublinear expectation with the regularity assumption and the representation as upper integrals. Let $\mathcal{I} \subset \mathcal{F}$ be the collection of such sets A such that $\mathbb{E}\mathrm{I}_{(\theta^{-1}A)\triangle A} = 0$. It is easy to check that $\mathcal{I}$ is a σ-field and $X \in \mathcal{I}$ if and only if $X(\theta\omega) = X(\omega)$ quasi-surely. Therefore, for any $\xi \in L^1(\Omega)$ and each $P \in \mathcal{P}$, as $E_P[\xi|\mathcal{I}]$ is $\mathcal{I}$-measurable, so $E_P[\xi|\mathcal{I}](\omega) = E_P[\xi|\mathcal{I}](\theta\omega)$ quasi-surely. Define $\bar{\xi}^*, \underline{\xi}^*$ to be $\mathcal{I}$-measurable random variables such that

$$\underline{\xi}^* \leq E_P[\xi|\mathcal{I}] \leq \bar{\xi}^*,$$

quasi-surely for each $P \in \mathcal{P}$. The proof of the following lemma is given in the appendix.

Lemma 2.12. *Assume $\mathbb{E}$ is regular and has the representation as upper integrals. Then for any $\xi \in L_b$ and $\epsilon > 0$,*

$$\bar{\xi}(\omega) := \limsup_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} \xi(\theta^m\omega) \leq \bar{\xi}^*(\omega) + \epsilon, \ \ v\text{-a.s.} \tag{2.11}$$

and

$$\underline{\xi}(\omega) := \liminf_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} \xi(\theta^m\omega) \geq \underline{\xi}^*(\omega) - \epsilon, \ \ v\text{-a.s.} \tag{2.12}$$

and $\bar{\xi}(\omega)$ and $\underline{\xi}(\omega)$ satisfy $\bar{\xi}(\theta\omega) = \bar{\xi}(\omega)$ and $\underline{\xi}(\theta\omega) = \underline{\xi}(\omega)$.

We use the notation of capacities from Chen [5] and Denis, Hu, and Peng [11]. Under the assumption of the representation as upper integrals, we define a pair $(\mathbb{V}, v)$ of capacities by

$$\mathbb{V}(A) := \sup_{P\in\mathcal{P}} P(A) = \mathbb{E}[I_A],\ v(A) := \inf_{P\in\mathcal{P}} P(A) = -\mathbb{E}[-I_A] \text{ for any } A \in \mathcal{F},$$

which are called upper probability and lower probability. A set function $\mu : \mathcal{F} \to [0, 1]$ is called continuous if $\lim_{n\to\infty} \mu(A_n) = \mu(A)$ when either $A_n \uparrow A$ or $A_n \downarrow A$ (Cerreia-Vioglio, Maccheroni, and Marinacci [4]).

We need the following lemma from Cerreia-Vioglio, Maccheroni, and Marinacci [4].

Lemma 2.13. *Let v be a continuous lower probability on $(\Omega, \mathcal{F})$. If v is θ-invariant, then for any bounded $\mathcal{F}$-measurable random variable ξ,*

$$v\left(\left\{\omega :\ \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \xi(\theta^k(\omega)) \ \text{exists}\ \right\}\right) = 1.$$

This tells us that if $\mathbb{E}$ is regular and θ-invariant, then $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \xi(\theta^k(\omega))$ exists quasi-surely for any bounded measurable random variable ξ.

Theorem 2.14. *Assume $\mathbb{E}$ is regular, is θ-invariant, and has the representation as upper integrals. If the dynamical system S is ergodic, then the SLLN holds and the constant in* (2.7) *satisfies $c \in [-\mathbb{E}(-\xi), \mathbb{E}(\xi)]$.*

Proof. As θ is $\mathbb{E}$ preserving, then it is v-preserving. Moreover, if $\mathbb{E}$ is regular, by Lemma 2.13, we know that $\bar{\xi}(\omega) := \lim_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n-1} \xi(\theta^m\omega)$ exists quasi-surely for any bounded measurable random variable ξ and $\bar{\xi}(\omega)$ satisfies $\bar{\xi}(\theta\omega) = \bar{\xi}(\omega)$. As the dynamical system S is ergodic, so $\bar{\xi} = c$ is a constant. The SLLN is proved.

On the other hand, for any $P \in \mathcal{P}$, $E_P[\xi|\mathcal{I}]$ is $\mathcal{I}$-measurable, so $E_P[\xi|\mathcal{I}](\omega) = E_P[\xi|\mathcal{I}](\theta\omega)$ quasi-surely. As S is ergodic, by Theorem 2.6, $E_P[\xi|\mathcal{I}]$ is constant quasi-surely. Thus for any $P \in \mathcal{P}$ and any bounded measurable random variable ξ,

$$E_P[\xi|\mathcal{I}] = E_P(\xi) \le \mathbb{E}(\xi)$$

and

$$-E_P[-\xi|\mathcal{I}] = -E_P(-\xi) \ge -\mathbb{E}(-\xi)$$

quasi-surely. Thus we can take $\bar{\xi}^* = \mathbb{E}[\xi]$ and $\underline{\xi}^* = -\mathbb{E}[-\xi]$ and use Lemma 2.12 to get

$$-\mathbb{E}[-\xi] - \epsilon \le \bar{\xi}(\omega) := \lim_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n-1} \xi(\theta^m\omega) \le \mathbb{E}[\xi] + \epsilon,\ v\text{-a.s.}$$

As $\bar{\xi} = c$ is a constant, it is obvious that $c \in [-\mathbb{E}(-\xi), \mathbb{E}(\xi)]$. ∎

Remark 2.15. An SLLN with $c \in \left[\int_\Omega \xi \mathrm{d}v, \int_\Omega \xi \mathrm{d}V\right]$ in the case of an upper and lower probability setup was obtained in Feng, Wu, and Zhao [20], where $\int_\Omega \xi \mathrm{d}v, \int_\Omega \xi \mathrm{d}V$ are Choquet integrals. It is noted that the bound obtained in Theorem 2.14 is better as $[-\mathbb{E}(-\xi), \mathbb{E}(\xi)] \subset \left[\int_\Omega \xi \mathrm{d}v, \int_\Omega \xi \mathrm{d}V\right]$. This can be easily seen due to the well-known fact that from the definition of the Choquet integral,

$$\begin{aligned}\int_\Omega \xi \mathrm{d}V &= \int_0^\infty V(\omega:\xi(\omega)\geq t)dt + \int_{-\infty}^0 \Big(V(\omega:\xi(\omega)\geq t)-1\Big)dt\\ &= \int_0^\infty \sup_{P\in\mathcal{P}} P(\omega:\xi(\omega)\geq t)dt + \int_{-\infty}^0 \Big(\sup_{P\in\mathcal{P}} P(\omega:\xi(\omega)\geq t)-1\Big)dt\\ &\geq \sup_{P\in\mathcal{P}}\int_0^\infty P(\omega:\xi(\omega)\geq t)dt + \sup_{P\in\mathcal{P}}\int_{-\infty}^0 \Big(P(\omega:\xi(\omega)\geq t)-1\Big)dt\\ &\geq \sup_{P\in\mathcal{P}}\Big[\int_0^\infty P(\omega:\xi(\omega)\geq t)dt + \int_{-\infty}^0 \Big(P(\omega:\xi(\omega)\geq t)-1\Big)dt\Big]\\ &= \sup_{P\in\mathcal{P}}\int_\Omega \xi dP\\ &= \mathbb{E}(\xi).\end{aligned}$$

Similarly one can prove that

$$\int_\Omega \xi \mathrm{d}v \leq -\mathbb{E}(-\xi).$$

3. Sublinear Markovian systems and their ergodicity: The general setting.

Consider a measurable space $(\Omega,\mathcal{F})$ with similar notation such as $\mathcal{D}=L_b(\mathcal{F})$ in section 2. Let $(\Omega,\mathcal{D},\mathbb{E})$ be a sublinear expectation space where $\mathbb{E}[\cdot]$ is a sublinear expectation on $L_b(\mathcal{F})$. Denote by $C_{b,lip}(\mathbb{R}^d)$ the space of real-valued bounded Lipschitz continuous functions on $\mathbb{R}^d$, and by $C_b(\mathbb{R}^d)$ the space of real-valued bounded continuous functions on $\mathbb{R}^d$. We denote by $L_b(\mathcal{B}(\mathbb{R}^d))$ the space of $\mathcal{B}(\mathbb{R}^d)$-measurable real-valued functions defined on $\mathbb{R}^d$ such that $\sup_{x\in\mathbb{R}^d}|\varphi(x)|<\infty$. Let $\xi\in(L_b(\mathcal{F}))^{\otimes d}$ be given. The sublinear distribution of ξ under $\mathbb{E}[\cdot]$ is defined by

$$T[\varphi]:=\mathbb{E}[\varphi(\xi)],\ \varphi\in L_b(\mathcal{B}(\mathbb{R}^d)).$$

This distribution $T[\cdot]$ is again a sublinear expectation defined on $L_b(\mathcal{B}(\mathbb{R}^d))$. Denote by $S(d)$ the collection of symmetric $d\times d$ matrices and by $S_+(d)$ the collection of positive definite symmetric $d\times d$ matrices.

Consider a family of sublinear expectations parameterized by $t\in\mathbb{R}^+$:

$$T_t: L_b(\mathcal{B}(\mathbb{R}^d))\to L_b(\mathcal{B}(\mathbb{R}^d)),\ t\geq 0.$$

DEFINITION 3.1 (Peng [34]). *The operator T_t is called a sublinear Markov semigroup if it satisfies the following:*

(m1) *For each fixed $(t,x)\in\mathbb{R}^+\times\mathbb{R}^d$, $T_t[\varphi](x)$ is a sublinear expectation defined on $L_b(\mathcal{B}(\mathbb{R}^d))$.*
(m2) $T_0[\varphi](x)=\varphi(x)$.
(m3) *$T_t[\varphi](x)$ satisfies the following Chapman semigroup formula:*

$$(T_t\circ T_s)[\varphi]=T_{t+s}[\varphi],\ t,s\geq 0.$$

There are many examples of sublinear Markov semigroups. We list some of them here, though they are already known, for completeness and to aid in understanding the problem we address here.

Example 3.2 (Lions [32, 33]). Consider the Hamilton–Jacobi–Bellman equation:

$$\begin{cases} \dfrac{\partial}{\partial t}u = \sup\limits_{v\in V}\left\{\sum\limits_{i,j=1}^{d} a_{ij}(x,v)\dfrac{\partial^2}{\partial x_i \partial x_j}u + \sum\limits_{i=1}^{d} b_i(x,v)\dfrac{\partial}{\partial x_i}u\right\}, \\ u(0,\cdot) = \varphi(\cdot) \in C_b(\mathbb{R}^d). \end{cases} \tag{3.1}$$

Here $a : \mathbb{R}^d \times \mathbb{R}^k \to S(d)$ and $b : \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}^d$ are bounded and uniformly continuous functions, uniformly Lipschitz in x, and V is a closed and bounded subset of $\mathbb{R}^k$. Under the notion of viscosity solutions, this equation has a unique solution $u(t,x)$ in $C_b(\mathbb{R}^d)$ with initial value φ. Set

$$(T_t\varphi)(x) := u(t,x), \quad x \in \mathbb{R}^d.$$

This defines a sublinear Markov semigroup.

Example 3.3 (Peng [36]). Let $G : S(d) \to \mathbb{R}$ be a given sublinear function which is monotonic on $S(d)$. Then there exists a bounded, convex, and closed subset $\sum \subset S_+(d)$ such that

$$G(A) = \sup_{B\in\sum}\left[\frac{1}{2}\mathrm{tr}(AB)\right] \text{ for } A \in S(d).$$

Define $\Omega = C_0(\mathbb{R}^+, \mathbb{R}^d)$, the space of all $\mathbb{R}^d$-valued continuous functions $(\omega_t)_{t\in\mathbb{R}^+}$, with $\omega_0 = 0$, equipped with the distance

$$\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i}\left[\max_{t\in[0,i]} |\omega_t^1 - \omega_t^2| \wedge 1\right]$$

with $\mathcal{F} = \mathcal{B}(C_0(\mathbb{R}^+, \mathbb{R}^d))$. Let

$$L_{ip}(\Omega) := \{\varphi(\omega_{t_1}, \omega_{t_2}, \ldots, \omega_{t_m}) \text{ for any } m \geq 1,\ t_1, t_2, \ldots, t_m \in \mathbb{R}^+, \varphi \in C_{b,Lip}((\mathbb{R}^d)^m)\}.$$

Then the G-normal distribution $N(\{0\} \times \sum)$ on $(\Omega, L_{ip}(\Omega))$ exists, i.e., there exists a d-dimensional random vector X on a sublinear expectation space $(\Omega, \mathcal{D}, \mathbb{E})$ satisfying

$$aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2}X \text{ for } a, b \geq 0,$$

where $\bar{X}$ is an independent copy of X and $G(A) = \mathbb{E}[\frac{1}{2}\langle AX, X\rangle]$. It was proved in Theorem 2.5 in Chapter VI in Peng [36] that there exists a weakly compact family of probability measures $\mathcal{P}$ on $(\Omega, \mathcal{B}(\Omega))$ such that

$$\mathbb{E}[X] = \max_{P\in\mathcal{P}} E_P[X] \text{ for } X \in L_{ip}(\Omega).$$

Its canonical path is G-Brownian motion $\{B_t\}_{t\geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{D}, \mathbb{E})$ satisfying the following:

(i) $B_0(\omega) = 0$.

(ii) For each $t, s \geq 0$, the increment $B_{t+s} - B_t$ is $N(\{0\} \times s\sum)$ distributed and independent of $(B_{t_1}, B_{t_2}, \ldots, B_{t_n})$ for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq t$.

For each fixed $\varphi \in C_{b,Lip}(\mathbb{R}^d)$, the function

$$u(t,x) := \mathbb{E}\varphi(x + B_t),\ (t,x) \in [0,\infty) \times \mathbb{R}^d, \tag{3.2}$$

is the viscosity solution of the G-heat equation

$$\frac{\partial}{\partial t}u = G(D^2u),\ u(0,\cdot) = \varphi(\cdot). \tag{3.3}$$

Then $(T_t\varphi)(x) = u(t,x)$ defines a semilinear Markovian semigroup.

Example 3.4 (Peng [36]). Let $\{B_t\}_{t\geq 0}$ be a k-dimensional G-Brownian motion on the sublinear expectation space $(\Omega, \mathcal{D}, \mathbb{E})$, and let $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times k}$, and $h : \mathbb{R}^d \to \mathbb{R}^{d\times k\times k}$ be global Lipschitz functions. Here $G : S(d) \to \mathbb{R}$ is a given sublinear function which is monotonic on $S(d)$. Consider the stochastic differential equations on $\mathbb{R}^d$ driven by the G-Brownian motion B,

$$dX_t = b(X_t)dt + \sum_{i,j=1}^{k} h_{ij}(X_t)d\langle B^i, B^j\rangle_t + \sum_{i=1}^{k}\sigma_j(X_s)dB_t^j, \tag{3.4}$$

with initial condition $X_t = x$. Here $\langle\cdot,\cdot\rangle_\cdot$ is the mutual variation process. Define $F : S(d)\times\mathbb{R}^d\times\mathbb{R}^d \to S(d)$ with

$$F_{ij}(A,p,x) = \frac{1}{2}\langle A\sigma_i(x), \sigma_j(x)\rangle + \langle p, h_{ij}(x) + h_{ji}(x)\rangle. \tag{3.5}$$

Then $T_t\varphi(x) = \mathbb{E}\varphi(X_t) =: u(t,x)$ satisfies

$$\frac{\partial}{\partial t}u = G(F(D^2u, Du, x)) + bDu \tag{3.6}$$

and defines a sublinear Markovian semigroup for $\varphi \in C_{b,lip}(\mathbb{R}^d)$.

In this section, we will give the construction of a canonical dynamical system on a path space under the assumption of the existence of invariant sublinear expectations of Markovian semigroups. Then we follow the standard philosophy in the literature to define the ergodicity of the canonical dynamical system as the ergodicity of the stochastic dynamical systems (cf. Da Prato and Zabczyk [8]). The invariant sublinear expectation has not been studied much in the literature. As far as we know, so far there is only one work (Hu et al. [27]) on the existence of invariant sublinear expectation for G-diffusion processes if the system is sufficiently dissipative.

First, we give the definition of an invariant expectation of sublinear Markovian semigroups as a natural extension of invariant measures.

Definition 3.5. *An invariant sublinear expectation* $\tilde{T} : L_b(\mathcal{B}(\mathbb{R}^d)) \to \mathbb{R}$ *is a sublinear expectation satisfying*

$$(\tilde{T}T_s)(\varphi) = \tilde{T}(\varphi) \quad \textit{for any } \varphi \in L_b(\mathcal{B}(\mathbb{R}^d)),$$

where $T_s, s \geq 0$ *is a sublinear Markov semigroup.*

Define $\Omega^* = C(\mathbb{R}, \mathbb{R}^d)$, the space of all $\mathbb{R}^d$-valued continuous functions $(\omega_t^*)_{t\in\mathbb{R}}$ equipped with the distance

$$\rho(\omega^{*1}, \omega^{*2}) := \sum_{i=1}^{\infty} 2^{-i}\left[\max_{t\in[-i,i]} |\omega_t^{*1} - \omega_t^{*2}| \wedge 1\right] \tag{3.7}$$

with $\mathcal{F}^* = \mathcal{B}(C(\mathbb{R},\mathbb{R}^d))$. Moreover, set $\hat{\Omega} = (\mathbb{R}^d)^{(-\infty,+\infty)}$ as the space of all $\mathbb{R}^d$-valued functions on $(-\infty,+\infty)$, $\hat{\mathcal{F}} = \mathcal{B}(\hat{\Omega})$ as the smallest σ-field containing all cylindrical sets of $\hat{\Omega}$, and $\hat{\mathcal{D}}$ as the linear space of all $\hat{\mathcal{F}}$-measurable real-valued functions.

Given a sublinear Markov semigroup $T_t, t \geq 0$, and the invariant sublinear expectation $\tilde{T}[\cdot]$, we can define the family of finite-dimensional sublinear distributions of the canonical process $(\omega_t)_{t\in\mathbb{R}} \in \Omega$ under a sublinear expectation $\mathbb{E}^{\tilde{T}}[\cdot]$ on $((\mathbb{R}^d)^m, L_b(\mathcal{B}[(\mathbb{R}^d)^m]))$ as follows. For each integer $m \geq 1$, $\varphi \in L_b(\mathcal{B}[(\mathbb{R}^d)^m])$ and $t_1 < t_2 < \cdots < t_m$, we successively define functions $\varphi_i \in L_b(\mathcal{B}[(\mathbb{R}^d)^{(m-i)}])$, $i = 1, \ldots, m$, by

$$\begin{aligned}\varphi_1(x_1, \ldots, x_{m-1}) &:= T_{t_m - t_{m-1}}[\varphi(x_1, \ldots, x_{m-1}, \cdot)](x_{m-1}),\\ \varphi_2(x_1, \ldots, x_{m-2}) &:= T_{t_{m-1} - t_{m-2}}[\varphi_1(x_1, \ldots, x_{m-2}, \cdot)](x_{m-2}),\\ &\vdots\\ \varphi_{m-1}(x_1) &:= T_{t_2 - t_1}[\varphi_{m-2}(x_1, \cdot)](x_1).\end{aligned}$$

We now consider two different setups. The first one is to consider $\varphi_m := \tilde{T}[\varphi_{m-1}(\cdot)]$ and

$$\mathbb{E}^{\tilde{T}}[\varphi(\hat{\omega}_{t_1}, \hat{\omega}_{t_2}, \ldots, \hat{\omega}_{t_m})] := T^{\tilde{T}}_{t_1,t_2,\ldots,t_m}[\varphi(\cdot)] := \varphi_m.$$

In fact, $T^{\tilde{T}}_t = \tilde{T}$ for $t \geq 0$, and $T^{\tilde{T}}_{t_1,t_2,\ldots,t_m}[\varphi(\cdot)]$ is a sublinear expectation defined on $L_b(\mathcal{B}[(\mathbb{R}^d)^m])$. Denote

$$\tilde{\mathcal{E}}(\varphi(\hat{\omega}_0)) = \tilde{T}[\varphi] \text{ for any } \varphi \in L_b(\mathcal{B}(\mathbb{R}^d));$$

then

$$\tilde{\mathcal{E}}(\varphi(\hat{\omega}_t)) = \tilde{\mathcal{E}}(\varphi(\hat{\omega}_0)) = \tilde{T}[\varphi] \text{ for any } \varphi \in L_b(\mathcal{B}(\mathbb{R}^d)).$$

For an ordered set of distinct real numbers $\mathbb{I} = \{t_1, t_2, \ldots, t_m\}$, let $\mathbb{I}' = \{t_{\pi_1}, t_{\pi_2}, \ldots, t_{\pi_m}\}$ be a permutation of $\mathbb{I}$ so that $t_{\pi_1} < t_{\pi_2} \cdots < t_{\pi_m}$. Define

$$T^{\tilde{T}}_{t_1,t_2,\ldots,t_m}\varphi(x_1, x_2, \ldots, x_m) = T^{\tilde{T}}_{t_{\pi_1},t_{\pi_2},\ldots,t_{\pi_m}}\varphi(x_{\pi_1}, x_{\pi_2}, \ldots, x_{\pi_m}).$$

The second setup is to set $\varphi_m(x) := T_{t_1}[\varphi_{m-1}(\cdot)](x)$ for $t_1 \geq 0$ following Peng [34]. Then

$$\mathbb{E}^x[\varphi(\hat{\omega}_{t_1}, \hat{\omega}_{t_2}, \ldots, \hat{\omega}_{t_m})] := T^x_{t_1,t_2,\ldots,t_m}[\varphi(\cdot)] := \varphi_m(x),$$

and $T^x_{t_1,t_2,\ldots,t_m}[\cdot]$ defines a sublinear expectation.

Set

$$L_0(\hat{\mathcal{F}}) := \{\varphi(\hat{\omega}_{t_1}, \hat{\omega}_{t_2}, \ldots, \hat{\omega}_{t_m}) \text{ for any } m \geq 1,\ t_1, t_2, \ldots, t_m \in \mathbb{R}, \varphi \in L_b(\mathcal{B}[(\mathbb{R}^d)^m])\}.$$

It is clear that $L_0(\hat{\mathcal{F}})$ is a linear subspace of $L_b(\hat{\mathcal{F}})$. Denote by $L^p_0(\hat{\Omega})$ the completion of $L_0(\hat{\mathcal{F}})$ under the norm $(\mathbb{E}^{\tilde{T}}[|\cdot|^p])^{\frac{1}{p}}$, $p \geq 1$. Define the space

$$\begin{aligned}&\mathrm{Lip}_{b,cyl}(\hat{\Omega})\\ &:= \{\varphi(\hat{\omega}_{t_1}, \hat{\omega}_{t_2}, \ldots, \hat{\omega}_{t_m}) \text{ for any } m \geq 1,\ t_1, t_2, \ldots, t_m \in \mathbb{R}, \varphi \in C_{b,\mathrm{Lip}}((\mathbb{R}^d)^m)\},\end{aligned}$$

and define $L^p_G(\hat{\Omega})$ as the completion of $\mathrm{Lip}_{b,cyl}(\hat{\Omega})$ under the norm $||\cdot||_{L^p_G} = (\mathbb{E}^{\tilde{T}}[|\cdot|^p])^{\frac{1}{p}}$. From Denis, Hu, and Peng [11], we know that the completions of $C_b(\hat{\Omega})$ and $\mathrm{Lip}_{b,cyl}(\hat{\Omega})$ under the norm $||\cdot||_{L^p_G}$ are the same, and $L^p_G(\hat{\Omega}) \subset L^p_0(\hat{\Omega})$. Here $C_b(\hat{\Omega})$ is defined in a similar way to $\mathrm{Lip}_{b,cyl}(\hat{\Omega})$, but replacing $C_{b,Lip}((\mathbb{R}^d)^m)$ by $C_b((\mathbb{R}^d)^m)$.

It was already known that there exists a unique sublinear expectation $\mathbb{E}^x$ with finite-dimensional expectation $\mathbb{E}^x = T^x_{t_1,t_2,\ldots,t_m}$, $m \in \mathbb{N}$, by applying the nonlinear

Kolmogorov extension theorem [34]. For our purposes, by applying Kolmogorov's theorem again, there exists a unique sublinear expectation $\mathbb{E}^{\tilde{T}}$ on $L_0^1(\hat{\Omega})$ such that

$$\mathbb{E}^{\tilde{T}}[Y] = T^{\tilde{T}}_{t_1,t_2,\ldots,t_m}[\varphi(\cdot)]$$

for any $m \geq 1$, $t_1, t_2, \ldots, t_m \in \mathbb{R}$, $Y \in L_0(\hat{\mathcal{F}})$ with $Y(\hat{\omega}) = \varphi(\hat{\omega}_{t_1}, \hat{\omega}_{t_2}, \ldots, \hat{\omega}_{t_m})$, $\varphi \in L_b(\mathcal{B}[(\mathbb{R}^d)^m])$.

Following the idea in [34], we can also define the conditional expectation. Define $\hat{\Omega}_t := \{\hat{\omega} \in \hat{\Omega} : \hat{\omega}_s \equiv \hat{\omega}_t \text{ for any } s \geq t\}$ and $\hat{\mathcal{F}}_t := \mathcal{B}(\hat{\Omega}_t)$. Let $X \in L_0(\hat{\mathcal{F}})$ be given as

$$X = \varphi(\hat{\omega}_{t_1}, \ldots, \hat{\omega}_{t_n}, \hat{\omega}_{t_{n+1}}, \ldots, \hat{\omega}_{t_{n+m}}),\ t_1 < \cdots < t_n < \cdots < t_{n+m},$$

where $\varphi \in L_b(\mathcal{B}[(\mathbb{R}^d)^{n+m}])$. Without loss of generality, we may assume $t_n = t$. The conditional expectation under $\hat{\mathcal{F}}_t$ denoted by $\mathbb{E}^{\tilde{T}}[\cdot|\hat{\mathcal{F}}_t] : L_0(\hat{\mathcal{F}}) \to L_0(\hat{\mathcal{F}}_t)$ is defined by

$$\mathbb{E}^{\tilde{T}}[X|\hat{\mathcal{F}}_t] := \Phi(\hat{\omega}_{t_1}, \hat{\omega}_{t_2}, \ldots, \hat{\omega}_{t_n}), \tag{3.8}$$

where $\Phi(x_1, \ldots, x_n) := T^{x_n}_{t_{n+1}-t_n,\ldots,t_{n+m}-t_n}[\varphi(x_1, \ldots, x_n, \cdot)]$. Similar to Proposition 5.1 in [34], this can also be extended to $L_0^p(\hat{\Omega})$.

Now we write the canonical process and associated σ-field as

$$\hat{X}_t(\hat{\omega}) = \hat{\omega}_t,\ \hat{\omega} \in \hat{\Omega},\ t \in \mathbb{R}. \tag{3.9}$$

The process $\hat{X}_t$, $t \in \mathbb{R}$, is Markovian in the sense that for $h > 0$

$$\mathbb{E}^{\tilde{T}}[\varphi(\hat{X}(t+h))|\hat{\mathcal{F}}_t] = T_h^{\hat{X}(t)}\varphi. \tag{3.10}$$

Now we introduce a group of invertible measurable transformations

$$\hat{\theta}_t\hat{\omega}(s) = \hat{\omega}(t+s),\ t, s \in \mathbb{R}.$$

Then it is easy to see that for any $\varphi \in L_0^1(\hat{\Omega})$,

$$\mathbb{E}^{\tilde{T}}[\varphi(\hat{X})] = \mathbb{E}^{\tilde{T}}[\varphi(\hat{\theta}_t\hat{X})],$$

i.e.,

$$\hat{\theta}_t\mathbb{E}^{\tilde{T}} = \mathbb{E}^{\tilde{T}}.$$

Thus $\hat{\theta}_t$ is an expectation preserving (or distribution preserving) transformation. Thus $S^{\tilde{T}} = (\hat{\Omega}, \hat{\mathcal{D}}, (\hat{\theta}_t)_{t\in\mathbb{R}}, \mathbb{E}^{\tilde{T}})$ defines a dynamical system, called the *canonical dynamical system* associated with $T_t, t \geq 0$, and $\tilde{T}$, $\hat{\theta}_t$ preserving the expectation $\mathbb{E}^{\tilde{T}}$ for any function $\varphi \in L_0^1(\hat{\Omega})$. The group $\hat{\theta}_t, t \in \mathbb{R}$, induces a group of linear transformations $U_t, t \in \mathbb{R}$, either on the real space $L_0^2(\hat{\Omega}, \hat{\mathcal{D}}, \mathbb{E}^{\tilde{T}})$ or the complex-valued function space $L^2_{0,\mathbb{C}}(\hat{\Omega}, \hat{\mathcal{D}}, \mathbb{E}^{\tilde{T}})$, by formula

$$U_t\xi(\hat{\omega}) = \xi(\hat{\theta}_t\hat{\omega}),\ \xi \in L_0^2(\hat{\Omega})\ (\text{or } L^2_{0,\mathbb{C}}(\hat{\Omega})),\ \hat{\omega} \in \hat{\Omega},\ t \in \mathbb{R}.$$

DEFINITION 3.6. *A dynamical system $S^{\tilde{T}} = (\hat{\Omega}, \hat{\mathcal{D}}, \hat{\theta}_t, \mathbb{E}^{\tilde{T}})$ is said to be continuous if for any $\xi \in L_0^2(\hat{\Omega})$ (or $L^2_{0,\mathbb{C}}(\hat{\Omega})$),*

$$\lim_{t\to 0} U_t\xi = \xi \ \textit{in}\ L_0^2(\hat{\Omega})\ (\textit{or}\ L^2_{0,\mathbb{C}}(\hat{\Omega})).$$

Denote

$$B(x,\delta)=\{y\in\mathbb{R}^d : |y-x|<\delta\}.$$

DEFINITION 3.7. *A stochastic process $\hat X(t)$, $t\in\mathbb{R}$, on $(\hat\Omega,\hat{\mathcal{D}},\mathbb{E}^{\tilde T})$ is said to be stochastically continuous if, for any $\delta>0$,*

$$\lim_{t\downarrow s}\mathbb{E}^{\tilde T}[\mathrm{I}_{\{|\hat X(t)-\hat X(s)|\geq\delta\}}]=0.$$

DEFINITION 3.8. *A sublinear Markov semigroup $T_t, t\geq 0$, is said to be stochastically continuous if*

$$T_t(x,B^c(x,\delta)):=\mathbb{E}^x[\mathrm{I}_{B^c(x,\delta)}(\hat X_t)]\downarrow 0, \quad \textit{as } t\to 0 \ \ \textit{for any } x\in\mathbb{R}^d,\ \delta>0.$$

THEOREM 3.9. *If a Markov semigroup $T_t, t>0$, is stochastically continuous, then*

$$\lim_{t\to 0}T_tf(x)=f(x) \ \ \textit{for all } f\in C_b(\mathbb{R}^d),\ x\in\mathbb{R}^d.$$

Proof. For any $f\in C_b(\mathbb{R}^d)$, let $\epsilon>0$, $\delta>0$ be such that

$$|f(x)-f(y)|<\epsilon, \quad \text{provided } |x-y|<\delta.$$

So

$$\begin{aligned}
&|T_tf(x)-f(x)|\\
&=|\mathbb{E}^x[f(\hat X(t))]-\mathbb{E}^x[f(\hat X(0))]|\\
&\leq\mathbb{E}^x|f(\hat X(t))-f(\hat X(0))|\\
&=\mathbb{E}^x|(f(\hat X(t))-f(\hat X(0)))\mathrm{I}_{\{|\hat X(t)-\hat X(0)|<\delta\}}|+\mathbb{E}^x|(f(\hat X(t))-f(\hat X(0)))\mathrm{I}_{\{|\hat X(t)-\hat X(0)|\geq\delta\}}|\\
&\leq\epsilon+2||f||_\infty\mathbb{E}^x[\mathrm{I}_{\{|\hat X(t)-\hat X(0)|\geq\delta\}}].
\end{aligned}$$

Since T_t is stochastically continuous, we have $\lim_{t\to 0}T_tf(x)=f(x)$. ∎

PROPOSITION 3.10. *Let $T_t, t\geq 0$, be a sublinear Markov semigroup, and let $\tilde T$ be the invariant expectation. If the corresponding canonical process $\hat X(t), t\in\mathbb{R}$, on $(\hat\Omega,\hat{\mathcal{D}},\mathbb{E}^{\tilde T})$ is stochastically continuous, then the dynamical system $S^{\tilde T}$ is continuous, i.e.,*

$$\lim_{s\to t}U_s\xi=U_t\xi,\ \xi\in L_G^2(\hat\Omega). \tag{3.11}$$

Proof. First we check (3.11) for all $\xi\in\mathrm{Lip}_{b,cyl}(\hat\Omega)$, i.e., for all ξ of the form

$$\xi=f(\hat\omega_{t_1},\hat\omega_{t_2},\ldots,\hat\omega_{t_m}),$$

where $f\in C_{b,Lip}(\mathcal{B}[(\mathbb{R}^d)^m])$, $t_1<t_2<\cdots<t_m$. Let $\epsilon>0$, $\delta>0$ be such that

$$|f(x_1,\ldots,x_m)-f(y_1,\ldots,y_m)|<\epsilon,\ \text{provided } |x_i-y_i|<\delta,\ i=1,\ldots,m.$$

Then

$$
\begin{aligned}
&\mathbb{E}^{\tilde{T}}|U_t\xi - U_s\xi|^2 \\
&= \mathbb{E}^{\tilde{T}}|f(\hat{\omega}(t_1+t),\dots,\hat{\omega}(t_m+t)) - f(\hat{\omega}(t_1+s),\dots,\hat{\omega}(t_m+s))|^2 \\
&= \mathbb{E}^{\tilde{T}}|f(\hat{X}(t_1+t),\dots,\hat{X}(t_m+t)) - f(\hat{X}(t_1+s),\dots,\hat{X}(t_m+s))|^2 \\
&\le \mathbb{E}^{\tilde{T}}\Big[|f(\hat{X}(t_1+t),\dots,\hat{X}(t_m+t)) - f(\hat{X}(t_1+s),\dots,\hat{X}(t_m+s))|^2 \\
&\qquad\qquad \mathrm{I}_{\{|\hat{X}(t_i+t)-\hat{X}(t_i+s)|<\delta \text{ for any } i=1,\dots,m\}}\Big] \\
&\quad + \mathbb{E}^{\tilde{T}}\Big[|f(\hat{X}(t_1+t),\dots,\hat{X}(t_m+t)) - f(\hat{X}(t_1+s),\dots,\hat{X}(t_m+s))|^2 \\
&\qquad\qquad \mathrm{I}_{\{|\hat{X}(t_i+t)-\hat{X}(t_i+s)|\ge\delta \text{ for some } i=1,\dots,m\}}\Big] \\
&\le \epsilon + 2||f||_\infty^2 \sum_{i=1}^m \mathbb{E}^{\tilde{T}}\Big[\mathrm{I}_{\{|\hat{X}(t_i+t)-\hat{X}(t_i+s)|\ge\delta\}}\Big].
\end{aligned}
$$

Since $\hat{X}_t$ is stochastically continuous, (3.11) follows for all $\xi \in \mathrm{Lip}_{b,cyl}(\hat{\Omega})$.

For any $\xi \in L_G^2(\hat{\Omega})$, there exists $\xi_n \in \mathrm{Lip}_{b,cyl}(\hat{\Omega})$ such that for any $\epsilon > 0$, there exists $N > 0$ such that for any $n \ge N$, we have

$$\mathbb{E}^{\tilde{T}}|\xi_n - \xi|^2 < \frac{\epsilon}{9}.$$

Now for the fixed N, there exists a $\delta > 0$,

$$\mathbb{E}^{\tilde{T}}|U_t\xi_N - U_s\xi_N|^2 < \frac{\epsilon}{9}, \text{ when } |t-s| < \delta.$$

Therefore

$$
\begin{aligned}
\mathbb{E}^{\tilde{T}}|U_t\xi - U_s\xi|^2 &\le 3\left[\mathbb{E}^{\tilde{T}}|U_t\xi - U_t\xi_N|^2 + \mathbb{E}^{\tilde{T}}|U_t\xi_N - U_s\xi_N|^2 + \mathbb{E}^{\tilde{T}}|U_s\xi_N - U_s\xi|^2\right] \\
&\le 3\left[\mathbb{E}^{\tilde{T}}|\xi - \xi_N|^2 + \mathbb{E}^{\tilde{T}}|U_t\xi_N - U_s\xi_N|^2 + \mathbb{E}^{\tilde{T}}|\xi_N - \xi|^2\right] \\
&< \epsilon.
\end{aligned}
$$

The proposition is proved. ∎

Remark 3.11. When we discuss the ergodicity of G-Brownian motion on S^1, we can show that $\hat{X}$ has a continuous modification which is also stochastically continuous in Proposition 4.5.

PROPOSITION 3.12. *Let $T_t, t \ge 0$, be a stochastically continuous Markov semigroup, and let $\tilde{\mathcal{E}}$ satisfy* (2.2)*. Then the corresponding canonical process $\hat{X}(t), t \in \mathbb{R}$, on $(\hat{\Omega}, \hat{\mathcal{D}}, \mathbb{E}^{\tilde{T}})$ is stochastically continuous.*

Proof. Assume that $T_t, t \ge 0$, is stochastically continuous; then for any $t > s$ and $\delta > 0$, we have

$$
\begin{aligned}
\mathbb{E}^{\tilde{T}}[\mathrm{I}_{\{|\hat{X}(t)-\hat{X}(s)|\ge\delta\}}] &= \mathbb{E}^{\tilde{T}}\left[\mathbb{E}^{\tilde{T}}[\mathrm{I}_{\{|\hat{X}(t)-\hat{X}(s)|\ge\delta\}}|\mathcal{F}_s]\right] \\
&= \mathbb{E}^{\tilde{T}}[T_{t-s}(\hat{X}(s), B^c(\hat{X}(s),\delta))] \\
&= \tilde{\mathcal{E}}[T_{t-s}(\hat{X}(s), B^c(\hat{X}(s),\delta))]
\end{aligned}
$$

by the Markov property. Note here that the conditional expectation can be defined in the Markovian case, as we already explained in (3.8) (Peng [34]). Since $T_t, t \geq 0$, is stochastically continuous and $\tilde{\mathcal{E}}$ satisfies (2.2), we have

$$\lim_{t \downarrow s} \mathbb{E}^{\tilde{T}}[\mathrm{I}_{\{|\hat{X}(t)-\hat{X}(s)| \geq \delta\}}] = 0.$$

Mirrored by the discrete case discussed in section 2, we can give the following definitions.

Definition 3.13. *A set $A \in \hat{\mathcal{F}}$ is said to be invariant with respect to $S^{\tilde{T}} = (\hat{\Omega}, \hat{\mathcal{D}}, \hat{\theta}_t, \mathbb{E}^{\tilde{T}})$ if, for any $t \in \mathbb{R}$, $\hat{\theta}_t^{-1} A = A$.*

Definition 3.14. *The invariant expectation $\tilde{T}$ is said to be ergodic with respect to the Markov semigroup $T_t, t \geq 0$, if its associated canonical dynamical system $S^{\tilde{T}} = (\hat{\Omega}, \hat{\mathcal{D}}, \hat{\theta}_t, \mathbb{E}^{\tilde{T}})$ is ergodic, i.e., any invariant set A satisfies either $\mathbb{E}^{\tilde{T}}[\mathrm{I}_A] = 0$ or $\mathbb{E}^{\tilde{T}}[\mathrm{I}_{A^c}] = 0$.*

Since $U_t 1 = 1$ by the definition of U_t, it is obvious that 1 is an eigenvalue of $U_t : L_b(\hat{\mathcal{F}}) \to L_b(\hat{\mathcal{F}})$. Similarly to the proof of Theorem 2.6, we can prove the following theorem.

Theorem 3.15. *The continuous dynamical system $S^{\tilde{T}}$ is ergodic if and only if the eigenvalue 1 of U_t on $L_b(\hat{\mathcal{F}})$ is simple.*

Definition 3.16. *A dynamical system $S^{\tilde{T}} = (\hat{\Omega}, \hat{\mathcal{D}}, (\hat{\theta}_t)_{t\in\mathbb{R}}, \mathbb{E}^{\tilde{T}})$ is said to satisfy the SLLN if, for any bounded measurable function ξ, there exists a constant c such that*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T U_t \xi dt = c \quad \textit{quasi-surely.} \tag{3.12}$$

Theorem 3.17. *If $S^{\tilde{T}}$ satisfies the SLLN, then the eigenvalue 1 of U_t on $L_b(\hat{\mathcal{F}})$ is simple and $S^{\tilde{T}}$ is ergodic.*

Proof. The proof is similar to that of Theorem 2.9. ∎

Now let us prove the converse part of Theorem 3.17 under the regularity assumption.

Theorem 3.18. *Assume the eigenvalue 1 of U_t on $L_b(\hat{\mathcal{F}})$ is simple and $\mathbb{E}^{\tilde{T}}$ is regular. Then the dynamical system $S^{\tilde{T}}$ satisfies the SLLN, and the constant in* (3.12) *satisfies $c \in [-\mathbb{E}^{\tilde{T}}(-\int_0^1 U_t\xi dt), \mathbb{E}^{\tilde{T}}(\int_0^1 U_t \xi dt)]$.*

Proof. Assume 1 is a simple eigenvalue of U_t on $L_b(\hat{\mathcal{F}})$. For an arbitrary $h > 0$, $\xi \in L_b(\hat{\mathcal{F}})$, $\xi \geq 0$, define

$$\xi_h = \int_0^h U_s \xi ds$$

and consider $\hat{\theta}_h$ a fixed expectation preserving transformation on $\hat{\Omega}$. Then

$$\frac{1}{n} \sum_{k=0}^{n-1} \xi_h(\hat{\theta}_h^k(\hat{\omega})) = \frac{1}{n} \int_0^{nh} U_s \xi(\hat{\omega}) ds,$$

and as $\mathbb{E}^{\tilde{T}}$ is regular, by Theorem 2.14 we have

$$-\mathbb{E}^{\tilde{T}}[-\xi_h] \leq \lim_{n\to\infty} \frac{1}{n} \int_0^{nh} U_s \xi ds =: \bar{\xi}_h^* \leq \mathbb{E}^{\tilde{T}}[\xi_h] \text{ quasi-surely.}$$

For arbitrary $T \geq 0$, let $n_T = [\frac{T}{h}]$ be the maximal nonnegative integer less than or equal to $\frac{T}{h}$. Then $n_T h \leq T \leq (n_T+1)h$ and quasi-surely

$$\frac{n_T}{(n_T+1)h}\frac{1}{n_T}\int_0^{n_T h} U_s\xi ds \leq \frac{1}{T}\int_0^T U_s\xi ds \leq \frac{n_T+1}{n_T h}\frac{1}{n_T+1}\int_0^{(n_T+1)h} U_s\xi ds.$$

Thus,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T U_s\xi ds = \frac{1}{h}\bar{\xi}_h^* \text{ quasi-surely.}$$

In particular, it follows that $\bar{\xi}_h^* = h\bar{\xi}_1^*$. But it is easy to see that

$$U_h\bar{\xi}_h^* = \bar{\xi}_h^*.$$

Thus

$$U_h\bar{\xi}_1^* = \bar{\xi}_1^* \text{ for all } h \geq 0.$$

However, from the assumption, $\bar{\xi}_1^*$ should be a constant quasi-surely. So

$$-\mathbb{E}^{\tilde{T}}\left[-\int_0^1 U_t\xi dt\right] = -\mathbb{E}^{\tilde{T}}[-\bar{\xi}_1^*] \leq \bar{\xi}_1^* = \mathbb{E}^{\tilde{T}}[\bar{\xi}_1^*] \leq \mathbb{E}^{\tilde{T}}[\xi_1] = \mathbb{E}^{\tilde{T}}\left[\int_0^1 U_t\xi dt\right].$$

This proves that the dynamical system $S^{\tilde{T}}$ satisfies the SLLN. ∎

PROPOSITION 3.19. *If $\varphi \in L_b(\mathcal{B}(\mathbb{R}^d))$ satisfies $T_t\varphi = \varphi$, $T_t(-\varphi) = -\varphi$ and $|\varphi(\hat{\omega}(0))|^2$ has no mean-uncertainty, then $\xi \in L_0^2$ given by*

$$\xi(\hat{\omega}) = \varphi(\hat{\omega}(0)),\ \hat{\omega} \in \hat{\Omega},$$

satisfies $U_t\xi = \xi$ quasi-surely.

Proof. Note that

$$U_t\xi(\hat{\omega}) = \xi(\hat{\theta}_t\hat{\omega}) = \varphi(\hat{\theta}_t\hat{\omega}(0)) = \varphi(\hat{\omega}(t)).$$

So the condition that $U_t\xi = \xi$, quasi-surely, is equivalent to

$$\varphi(\hat{\omega}(t)) = \varphi(\hat{\omega}(0)) \text{ quasi-surely}$$

and therefore

$$\varphi(\hat{X}(t)) = \varphi(\hat{X}(0)) \text{ quasi-surely,} \tag{3.13}$$

where $\hat{X}(t),\ t \in \mathbb{R}$, is the canonical process. To prove (3.13), note that

$$\begin{aligned}&\mathbb{E}^{\tilde{T}}|\varphi(\hat{X}(t)) - \varphi(\hat{X}(0))|^2\\ &\leq 2\mathbb{E}^{\tilde{T}}\big[-\varphi(\hat{X}(t))\varphi(\hat{X}(0))\big] + \mathbb{E}^{\tilde{T}}|\varphi(\hat{X}(t))|^2 + \mathbb{E}^{\tilde{T}}|\varphi(\hat{X}(0))|^2.\end{aligned}$$

By the Markovian property and the assumptions that $T_t\varphi = \varphi$, $T_t(-\varphi) = -\varphi$, and

$|\varphi(\hat\omega(0))|^2$ has no mean-uncertainty, we have

$$\begin{aligned}
&\mathbb{E}^{\tilde T}\Big[-\varphi(\hat X(t))\varphi(\hat X(0))\Big]\\
&=\mathbb{E}^{\tilde T}\Big[\mathbb{E}^{\tilde T}\Big[-\varphi(\hat X(t))\varphi(\hat X(0))|\hat{\mathcal{F}}_0\Big]\Big]\\
&\le \mathbb{E}^{\tilde T}\Big[\big(-\varphi(\hat X(0))\big)^+\mathbb{E}^{\tilde T}\Big[\varphi(\hat X(t))|\hat{\mathcal{F}}_0\Big]+\big(-\varphi(\hat X(0))\big)^-\mathbb{E}^{\tilde T}\Big[-\varphi(\hat X(t))|\hat{\mathcal{F}}_0\Big]\Big]\\
&=\mathbb{E}^{\tilde T}\Big[\big(-\varphi(\hat X(0))\big)^+(T_t\varphi)(\hat X(0))+\big(-\varphi(\hat X(0))\big)^-(T_t\big(-\varphi)\big)(\hat X(0))\Big]\\
&=\mathbb{E}^{\tilde T}\Big[\big(-\varphi(\hat X(0))\big)^+\varphi(\hat X(0))+\big(-\varphi(\hat X(0))\big)^-\big(-\varphi(\hat X(0))\big)\Big]\\
&=\mathbb{E}^{\tilde T}\big[-|\varphi(\hat X(0))|^2\big]\\
&=-\mathbb{E}^{\tilde T}|\varphi(\hat X(0))|^2.
\end{aligned}$$

Note also that

$$\mathbb{E}^{\tilde T}|\varphi(\hat X(t))|^2=\mathbb{E}^{\tilde T}|\varphi(\hat X(0))|^2.$$

So

$$\mathbb{E}^{\tilde T}|\varphi(\hat X(t))-\varphi(\hat X(0))|^2\le -2\mathbb{E}^{\tilde T}|\varphi(\hat X(0))|^2+2\mathbb{E}^{\tilde T}|\varphi(\hat X(0))|^2=0.$$

Thus

$$\mathbb{E}^{\tilde T}|\varphi(\hat X(t))-\varphi(\hat X(0))|^2=0.$$

It follows that

$$|\varphi(\hat X(t))-\varphi(\hat X(0))|=0 \text{ quasi-surely.}$$

The result is proved. ∎

LEMMA 3.20. *Assume that $\xi\in L_0^2$ satisfies $U_t\xi=\xi$ quasi-surely. Then for an arbitrary random variable $\tilde\xi\in L_0^2$ which is $\hat{\mathcal{F}}_{[-t,t]}$-measurable, $t\ge 0$, we have*

$$\mathbb{E}^{\tilde T}\Big|\mathbb{E}^{\tilde T}\big[U_t\tilde\xi|\hat{\mathcal{F}}_{[0,0]}\big]-\xi\Big|^2\le 10\mathbb{E}^{\tilde T}|\xi-\tilde\xi|^2.$$

Proof. First we have for the sublinear expectation, for $t\ge 0$, that

$$\begin{aligned}
&\mathbb{E}^{\tilde T}\Big|\mathbb{E}^{\tilde T}\big[U_t\tilde\xi|\hat{\mathcal{F}}_{[0,0]}\big]-\xi\Big|^2\\
&\le 2\mathbb{E}^{\tilde T}\Big|\mathbb{E}^{\tilde T}\big[U_t\tilde\xi|\hat{\mathcal{F}}_{[0,0]}\big]-U_{-t}\tilde\xi\Big|^2+2\mathbb{E}^{\tilde T}|U_{-t}\tilde\xi-\xi|^2\\
&=2\mathbb{E}^{\tilde T}\Big|\mathbb{E}^{\tilde T}\big[U_t\tilde\xi|\hat{\mathcal{F}}_0\big]-\mathbb{E}^{\tilde T}\big[U_{-t}\tilde\xi|\hat{\mathcal{F}}_0\big]\Big|^2+2\mathbb{E}^{\tilde T}|U_{-t}\tilde\xi-U_{-t}\xi|^2\\
&=2\mathbb{E}^{\tilde T}\Big|\mathbb{E}^{\tilde T}\big[U_t\tilde\xi|\hat{\mathcal{F}}_0\big]-\mathbb{E}^{\tilde T}\big[U_{-t}\tilde\xi|\hat{\mathcal{F}}_0\big]\Big|^2+2\mathbb{E}^{\tilde T}|\tilde\xi-\xi|^2,
\end{aligned}$$

where we have used that $\hat X$ is a Markov process, that $U_t\tilde\xi$ and $U_{-t}\tilde\xi$ are, respectively, $\hat{\mathcal{F}}_{[0,2t]}$- and $\hat{\mathcal{F}}_0$-measurable, and that U_t is the $\mathbb{E}^{\tilde T}$-preserving transformation.

By Jensen's inequality and sublinearity of $\mathbb{E}^{\tilde T}$, we have

$$\begin{aligned}
\Big|\mathbb{E}^{\tilde T}\big[U_t\tilde\xi|\hat{\mathcal{F}}_0\big]-\mathbb{E}^{\tilde T}\big[U_{-t}\tilde\xi|\hat{\mathcal{F}}_0\big]\Big|^2&\le\Big|\mathbb{E}^{\tilde T}\Big[|U_t\tilde\xi-U_{-t}\tilde\xi|\big|\hat{\mathcal{F}}_0\Big]\Big|^2\\
&\le\mathbb{E}^{\tilde T}\Big[|U_t\tilde\xi-U_{-t}\tilde\xi|^2\big|\hat{\mathcal{F}}_0\Big].
\end{aligned}$$

Moreover, it follows from the $\mathbb{E}^{\tilde{T}}$-preserving property of U_t that

$$\begin{aligned}\mathbb{E}^{\tilde{T}}\left[\mathbb{E}^{\tilde{T}}\left[|U_t\tilde{\xi}-U_{-t}\tilde{\xi}|^2\big|\hat{\mathcal{F}}_0\right]\right] &= \mathbb{E}^{\tilde{T}}\left[\left|U_t\tilde{\xi}-U_{-t}\tilde{\xi}\right|^2\right]\\ &= \mathbb{E}^{\tilde{T}}\left[\left|U_{2t}\tilde{\xi}-\tilde{\xi}\right|^2\right]\\ &\le 2\mathbb{E}^{\tilde{T}}\left[\left|U_{2t}\tilde{\xi}-U_{2t}\xi\right|^2\right]+2\mathbb{E}^{\tilde{T}}\left[\left|U_{2t}\xi-\tilde{\xi}\right|^2\right]\\ &= 2\mathbb{E}^{\tilde{T}}\left[\left|\tilde{\xi}-\xi\right|^2\right]+2\mathbb{E}^{\tilde{T}}\left[\left|\xi-\tilde{\xi}\right|^2\right]\\ &\le 4\mathbb{E}^{\tilde{T}}|\tilde{\xi}-\xi|^2.\end{aligned}$$

The result follows. ∎

Now we are ready to prove the converse part of Proposition 3.19.

PROPOSITION 3.21. *If $\xi\in L_0^2(\hat{\Omega})$ and $U_t\xi=\xi$, then there exists $\varphi\in L_b(\mathcal{B}(\mathbb{R}^d))$ such that $T_t\varphi=\varphi$, $T_t(-\varphi)=-\varphi$, and $\xi(\hat{\omega})=\varphi(\hat{\omega}(0))$ quasi-surely.*

Proof. For $\xi\in L_0^2(\hat{\Omega})$, by definition of $L_0^2(\hat{\Omega})$, there exists a sequence $\{\tilde{\xi}_n\}$ of $\mathcal{F}_{[-nt,nt]}$-measurable elements of $L_b(\hat{\mathcal{F}})$ such that

$$\mathbb{E}^{\tilde{T}}|\tilde{\xi}_n-\xi|^2\to 0 \text{ as } n\to\infty.$$

Thus by Lemma 3.20,

$$\lim_{n\to\infty}\mathbb{E}^{\tilde{T}}[U_{nt}\tilde{\xi}_n|\mathcal{F}_{[0,0]}]=\xi \text{ in } L_0^2.$$

Moreover, there exists $\varphi_n\in L^2_{\mathbb{C}}(\mathbb{R}^d,\tilde{T})$ such that

$$\mathbb{E}^{\tilde{T}}[U_{nt}\tilde{\xi}_n|\mathcal{F}_{[0,0]}]=\varphi_n(\hat{X}(0)) \text{ quasi-surely.}$$

Thus

$$\lim_{n\to\infty}\varphi_n(\hat{X}(0))=\xi \text{ in } L_0^2(\hat{\Omega}).$$

By the Borel–Cantelli lemma (Denis, Hu, and Peng [11]), we can choose a quasi-surely convergent subsequence, still denoted by $\varphi_n(\hat{X}(0))$. Now we define

$$\varphi(x)=\begin{cases}\lim_{n\to\infty}\varphi_n(x) & \text{if the limit exists,}\\ 0 & \text{otherwise.}\end{cases}$$

Then $\xi=\varphi(\hat{X}(0))$. It follows from $U_t\xi=\xi$ that

$$\varphi(\hat{X}(t))=U_t\varphi(\hat{X}(0))=\varphi(\hat{X}(0)).$$

By using conditional expectations, we have

$$(T_t\varphi)(\hat{X}(0))=\mathbb{E}^{\tilde{T}}[\varphi(\hat{X}(t))|\mathcal{F}_0]=\mathbb{E}^{\tilde{T}}[\varphi(\hat{X}(0))|\mathcal{F}_0]=\varphi(\hat{X}(0))$$

and

$$(T_t(-\varphi))(\hat{X}(0))=\mathbb{E}^{\tilde{T}}[-\varphi(\hat{X}(t))|\mathcal{F}_0]=\mathbb{E}^{\tilde{T}}[-\varphi(\hat{X}(0))|\mathcal{F}_0]=-\varphi(\hat{X}(0)).$$

The proof is complete. ∎

THEOREM 3.22. *Assume the Markov chain T_t has an invariant expectation $\tilde{T}$. Let $\hat{X}$ be the canonical processes on the canonical dynamical system $(\hat{\Omega}, \hat{\mathcal{D}}, \hat{\theta}_t, \mathbb{E}^{\tilde{T}})$ and stochastically continuous. Then the following two statements have the relation that* (i) *implies* (ii)*:*

(i) *If $T_t\varphi = \varphi$, $T_t(-\varphi) = -\varphi$, $\varphi \in L_b(\mathcal{B}(\mathbb{R}^d))$ for any $t \geq 0$, then φ is constant, $\tilde{T}$-quasi-surely.*

(ii) *$\tilde{T}$ is ergodic.*

Moreover, if we assume further that for any $\varphi \in L_b(\mathcal{B}(\mathbb{R}^d))$, $|\varphi(\hat{X}(0))|^2$ has no mean-uncertainty, then (i) *and* (ii) *are equivalent.*

Proof. The theorem can be proved easily by Theorem 3.15 and Propositions 3.19 and 3.21. □

4. Ergodicity of *G*-Brownian motion on the unit circle. As an example, we consider a G-Brownian motion on the unit circle $S^1 = [0, 2\pi]$ defined by $X(t) = x + B_t \bmod 2\pi$, where B is a one-dimensional G-Brownian motion such that B_1 has normal distribution $N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$. Here $\overline{\sigma}^2 \geq \underline{\sigma}^2$ are constants. See Example 3.3 for the definition of the G-Brownian motions. For $\varphi \in C_{b,lip}(S^1)$, set

$$T_t\varphi(x) = u(t, x) = \mathbb{E}\varphi(X(t)). \tag{4.1}$$

Then u is a viscosity solution of the following fully nonlinear PDE (Peng [35, 36]):

$$\frac{\partial}{\partial t} u = \frac{1}{2}\overline{\sigma}^2 u_{xx}^+ - \frac{1}{2}\underline{\sigma}^2 u_{xx}^-, \; u|_{t=0} = \varphi, \; x \in S^1. \tag{4.2}$$

If we assume $\underline{\sigma}^2 > 0$, according to Krylov [30, 31], or Peng [36], when $t > 0$, $u(t, x)$ is $C^{1,2}$ in (t, x), thus a classical solution for any $t > 0$. In fact, we can extend the solution to the case when φ is bounded and measurable and obtain a classical solution for any $t > 0$. Before we give this result, we need the following lemma about the regularity of T_t.

LEMMA 4.1. *Assume $\underline{\sigma}^2 > 0$. For T_t defined in* (4.1) *we have, for any $t > 0$, $A_n \in \mathcal{B}(S^1)$ such that for $A_n \downarrow \emptyset$, we have $(T_t I_{A_n})(x) \downarrow 0$.*

Proof. From Denis, Hu, and Peng [11], we know that for any function $\varphi \in L_b(\mathcal{B}(S^1))$,

$$\begin{aligned} T_t\varphi(x) &= \mathbb{E}\varphi(X(t)) \\ &= \sup_{\theta^2_\cdot \in \{\text{adapted processes with values in } [\underline{\sigma}^2, \overline{\sigma}^2]\}} E\left[\varphi\left(x + \int_0^t \theta_s dW_s \bmod 2\pi\right)\right], \end{aligned} \tag{4.3}$$

where $W_\cdot$ is the classical Brownian motion on $\mathbb{R}^1$, $W_0 = 0$, and E is the linear expectation with respect to $W_\cdot$. Denote $\mathcal{F}_t = \sigma\{W_s : 0 \leq s \leq t\}$. Note that, by Theorem 3.4.6 in Karatzas and Shreve [29], $\int_0^t \theta_s dW_s$ is in law a Brownian motion with time $\tilde{\theta}_t^2 = \int_0^t \theta_s^2 ds$, i.e., there exists a standard Brownian motion $\tilde{W}$ such that $\int_0^t \theta_s dW_s = \tilde{W}_{\tilde{\theta}_t^2}$, where $\tilde{\theta}_t^2$ is a stopping time with respect to the filtration $\mathcal{G}_s = \mathcal{F}_{\tau(s)}$, where $\tau(s) = \inf\{t \geq 0 : \tilde{\theta}_t^2 > s\}$. Note that $\underline{\sigma}^2 \leq \theta^2_\cdot \leq \bar{\sigma}^2$ and $\underline{\sigma}^2 > 0$, so $\tilde{\theta}_t^2$ is strictly increasing in t, and we have $\tau(s) = \inf\{t \geq 0 : \tilde{\theta}_t^2 \geq s\}$. Define $\tilde{\tau} := \tau(\underline{\sigma}^2 t)$. It is easy to see that $\tilde{\tau} \leq t$ and $\int_0^{\tilde{\tau}} \theta_s dW_s = \tilde{W}_{\tilde{\theta}^2_{\tilde{\tau}}} = \tilde{W}_{\underline{\sigma}^2 t}$, which is a Brownian motion with respect to $\mathcal{G}_{\underline{\sigma}^2 t} = \mathcal{F}_{\tilde{\tau}}$. Therefore

$$\begin{aligned}
&E\left[\varphi\left(x+\int_0^t \theta_s dW_s \bmod 2\pi\right)\right]\\
&= E\left[E\left[\varphi\left(x+\int_0^{\tilde{\tau}} \theta_s dW_s + \int_{\tilde{\tau}}^t \theta_s dW_s \bmod 2\pi\right)\middle|\mathcal{F}_{\tilde{\tau}}\right]\right]\\
&= E\left[E\left[\varphi\left(x+y+\int_{\tilde{\tau}}^t \theta_s dW_s \bmod 2\pi\right)\right]\Big|_{y=\int_0^{\tilde{\tau}}\theta_s dW_s}\right]\\
&= E\left[E\left[\varphi\left(x+y+\int_{\tilde{\tau}}^t \theta_s dW_s \bmod 2\pi\right)\right]\Big|_{y=\tilde{W}_{\underline{\sigma}^2 t}}\right]\\
&= \int_0^{2\pi} p(\underline{\sigma}^2 t, y) E\left[\varphi\left(x+y+\int_{\tilde{\tau}}^t \theta_s dW_s \bmod 2\pi\right)\right] dy\\
&= E\left[\int_0^{2\pi} p(\underline{\sigma}^2 t, y)\varphi\left(x+y+\int_{\tilde{\tau}}^t \theta_s dW_s \bmod 2\pi\right) dy\right]\\
&= E\left[\int_0^{2\pi} p(\underline{\sigma}^2 t, y)\varphi(x+y+z \bmod 2\pi) dy|_{z=\int_{\tilde{\tau}}^t \theta_s dW_s}\right]\\
&= E\left[E[\varphi(x+z+\tilde{W}_{\underline{\sigma}^2 t} \bmod 2\pi)]|_{z=\int_{\tilde{\tau}}^t \theta_s dW_s}\right], \qquad (4.4)
\end{aligned}$$

where $p(\cdot,\cdot)$ is the heat kernel of Brownian motion $\tilde{W}_\cdot$ on S^1 starting at position 0 at time 0. In fact,

$$\begin{aligned}
&E[\varphi(x+z+\tilde{W}_{\underline{\sigma}^2 t} \bmod 2\pi)]\\
&= \int_{S^1} p(\underline{\sigma}^2 t, y-x-z \bmod 2\pi)\varphi(y)dy\\
&= \sum_{k\in\mathbb{Z}} \int_0^{2\pi} \frac{1}{\sqrt{2\pi\underline{\sigma}^2 t}} \mathrm{e}^{-\frac{(x+z \bmod 2\pi - y - 2k\pi)^2}{2\underline{\sigma}^2 t}} \varphi(y)dy.
\end{aligned}$$

So for any $A_n \in \mathcal{B}(S^1)$, using inequality $(a-b)^2 \geq \frac{1}{2}a^2 - b^2$, we have

$$\begin{aligned}
&E[I_{A_n}(x+z+\tilde{W}_{\underline{\sigma}^2 t} \bmod 2\pi)]\\
&= \sum_{k\in\mathbb{Z}} \int_0^{2\pi} \frac{1}{\sqrt{2\pi\underline{\sigma}^2 t}} \mathrm{e}^{-\frac{(x+z \bmod 2\pi - y - 2k\pi)^2}{2\underline{\sigma}^2 t}} I_{A_n}(y)dy\\
&\leq \int_0^{2\pi} I_{A_n}(y) \frac{1}{\sqrt{2\pi\underline{\sigma}^2 t}} \mathrm{e}^{\frac{(x+z \bmod 2\pi - y)^2}{2\underline{\sigma}^2 t}} \sum_{k\in\mathbb{Z}} \mathrm{e}^{-\frac{(2k\pi)^2}{4\underline{\sigma}^2 t}} dy\\
&\leq \mathrm{Leb}(A_n) \frac{1}{\sqrt{2\pi\underline{\sigma}^2 t}} \mathrm{e}^{\frac{(2\pi)^2}{2\underline{\sigma}^2 t}} \frac{1}{1-e^{-\frac{\pi^2}{\underline{\sigma}^2 t}}}. \qquad (4.5)
\end{aligned}$$

Note the upper bound of (4.5) is independent of x, z, and $\theta_\cdot$, so it follows from (4.3) and (4.4) that

$$
\begin{aligned}
&(T_tI_{A_n})(x)\\
&= \sup_{\theta^2_\cdot \in \{\text{adapted processes with values in } [\underline{\sigma}^2, \overline{\sigma}^2]\}} E\left[E[I_{A_n}(x+z+\tilde{W}_{\underline{\sigma}^2 t} \bmod 2\pi)]|_{z=\int_\tau^t \theta_s dW_s}\right]\\
&\le \mathrm{Leb}(A_n)\frac{1}{\sqrt{2\pi\underline{\sigma}^2 t}} \mathrm{e}^{\frac{(2\pi)^2}{2\underline{\sigma}^2 t}} \frac{1}{1-e^{-\frac{\pi^2}{\underline{\sigma}^2 t}}}\\
&\to 0,
\end{aligned}
$$

since $\mathrm{Leb}(A_n) \to 0$ as $n \to \infty$. ∎

The following lemma is vitally important. It is the strong Feller property in the classical case of linear probability space. But in the sublinear setting, it is not clear whether or not this holds in general. The proof of this result is quite involved where the regularity of T_t (Lemma 4.1) plays an important role.

LEMMA 4.2. *Assume $\underline{\sigma}^2 > 0$ and $\varphi \in L_b(\mathcal{B}(S^1))$. Then for any $t > 0$, $u(t,x) = T_t\varphi(x)$ given by (4.1) is $C^{1,2}$ and a classical solution of (4.2).*

Proof. Consider $\varphi \in L_b(\mathcal{B}(S^1))$. First note there exists an increasing sequence of simple functions $\varphi_n^{(1)} \uparrow \varphi$ with $||\varphi_n^{(1)}||_\infty \le ||\varphi||_\infty$. Thus by the monotone convergence of sublinear expectation we know that

$$u_n^{(1)}(t,x) = \mathbb{E}\varphi_n^{(1)}(x+B_t) \uparrow \mathbb{E}\varphi(x+B_t) = u(t,x).$$

Denote

$$\varphi_n^{(1)} = \sum_{i=1}^{2^n} x_i I_{A_i^1},$$

where $\{A_i^1\}$ are Borel sets on S^1. By a standard result (cf. Taylor [38]), there exists a finite number of open intervals whose union is denoted by B_i^0 such that $A_i^1 \triangle B_i^0$ can be sufficiently small. Define

$$\varphi_n^{(2)} = \sum_{i=1}^{2^n} x_i I_{B_i^0}.$$

Then

$$|\mathbb{E}\varphi_n^{(2)}(x+B_t) - \mathbb{E}\varphi_n^{(1)}(x+B_t)| \le \sum_{i=1}^{2^n} |x_i| \mathbb{E} I_{A_i^1 \triangle B_i^0}(x+B_t).$$

As the Brownian motion is nondegenerate ($\underline{\sigma}^2 > 0$), so by Lemma 4.1, the expectation $\mathbb{E}I_{A_i^1\triangle B_i^0}(x+B_t)$ can be sufficiently small since the Lebesgue measure of $A_i^1 \triangle B_i^0$ is sufficiently small. Thus $u_n^{(2)}(t,x) = \mathbb{E}\varphi_n^{(2)}(x+B_t)$ is sufficiently close to $u_n^{(1)}(t,x)$.

Now note that one can easily find an increasing (or decreasing) sequence of continuous functions to approximate $I_{B_i^0}$. Thus there exists an increasing sequence of continuous functions $\varphi_{nm}^{(3)} \uparrow \varphi_n^{(2)}$ as $m \to \infty$ with $||\varphi_{nm}^{(3)}||_\infty \le ||\varphi_n^{(2)}||_\infty$. By the monotone convergence theorem,

$$u_{nm}^{(3)}(t,x) = \mathbb{E}\varphi_{nm}^{(3)}(x+B_t) \uparrow u_n^{(2)}(t,x).$$

Summarizing above, we conclude there exists a sequence of continuous functions φ_n such that

$$u_n(t,x) = \mathbb{E}\varphi_n(x+B_t) \to u(t,x) = \mathbb{E}\varphi(x+B_t).$$

For any given $\delta > 0$, by Krylov's result of the regularity of fully nonlinear parabolic PDEs of nondegenerate type (Krylov [30, 31]), we know that

$$|D_t u_n(\delta, x)| + |D_x u_n(\delta, x)| \leq M$$

for a constant $M > 0$ independent of n and x. Thus the sequence $u_n(\delta, x) = (T_\delta \varphi_n)(x)$ of continuous functions is equicontinuous. Thus its limit $u(\delta, x) = (T_\delta \varphi)(x)$ is continuous in x. As $T_t\varphi = T_{t-\delta}T_\delta\varphi$, by Krylov's result again, we can see that $u(t, x) = T_t\varphi(x)$ given by (4.1) is $C^{1,2}$ in (t, x) for any $t > 0$. ∎

THEOREM 4.3. *Let T_t be the Markovian semigroup defined by (4.1) with the G-Brownian motion on the unit circle $S^1 = [0, 2\pi]$ with normal distribution $N(0, [\underline{\sigma}^2 t, \overline{\sigma}^2 t])$, where $\overline{\sigma}^2 \geq \underline{\sigma}^2 > 0$ are constant. Then*

$$\tilde{T}\varphi = \frac{1}{2\pi}\int_0^{2\pi} (T_\delta\varphi)(x)dx,\ \ \varphi \in L_b(\mathcal{B}(S^1)),\ \ \delta > 0, \tag{4.6}$$

is independent of $\delta > 0$ and is the unique invariant expectation of $T_t,\ t \geq 0$. Moreover, $T_t\varphi \to \tilde{T}\varphi$ as $t \to \infty$.

Proof. For each $\varphi \in L_b(\mathcal{B}(S^1))$, define $m(\varphi)$ as integral of φ with respect to the Lebesgue measure (normalized)

$$m(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(x)dx. \tag{4.7}$$

Set

$$T_t^{\overline{\sigma}}\varphi(x) = \int_0^{2\pi} p^{\overline{\sigma}}(t, x, y)\varphi(y)dy$$

and

$$T_t^{\underline{\sigma}}\varphi(x) = \int_0^{2\pi} p^{\underline{\sigma}}(t, x, y)\varphi(y)dy,$$

where $p^{\overline{\sigma}}$ and $p^{\underline{\sigma}}$, the density of the transition probabilities of Brownian motions $\overline{\sigma}W_.$ and $\underline{\sigma}W_.$, respectively, are given by

$$p^{\overline{\sigma}}(t, x, y) = \sum_{k\in\mathbb{Z}} \frac{1}{\sqrt{2\pi\overline{\sigma}^2 t}} \mathrm{e}^{-\frac{(x-y-2k\pi)^2}{2\overline{\sigma}^2 t}} \tag{4.8}$$

and

$$p^{\underline{\sigma}}(t, x, y) = \sum_{k\in\mathbb{Z}} \frac{1}{\sqrt{2\pi\underline{\sigma}^2 t}} \mathrm{e}^{-\frac{(x-y-2k\pi)^2}{2\underline{\sigma}^2 t}}. \tag{4.9}$$

Here $W_.$ is the classical Brownian motion on S^1. These standard Poisson summation formulae of heat kernels can be obtained using Fourier analysis or stochastic methods (cf. Elworthy [16]). It is easy to see that if φ is convex, then $T_t^{\overline{\sigma}}\varphi(x)$ is a convex function of x for each t and $T_t\varphi(x) = T_t^{\overline{\sigma}}\varphi(x)$. If φ is concave, then $T_t\varphi(x) = T_t^{\underline{\sigma}}\varphi(x)$, which is a concave function of x for each t. Moreover, it is well known that the Lebesgue measure is the invariant measure of Brownian motion on S^1 (cf. Proposition 4.5 and the corollary of Theorem 4.6 in Chapter V in [28]), so

$$mT_t^{\underline{\sigma}}\varphi = m\varphi,\quad mT_t^{\overline{\sigma}}\varphi = m\varphi \ \text{ for } t \geq 0,$$

and as $t \to \infty$, for any $x \in [0, 2\pi]$

$$T_t^{\underline{\sigma}}\varphi(x) \to m\varphi, \quad T_t^{\overline{\sigma}}\varphi(x) \to m\varphi.$$

Thus if φ is convex or concave,

$$mT_t\varphi = m\varphi, \tag{4.10}$$

and as $t \to \infty$, for any $x \in [0, 2\pi]$

$$T_t\varphi(x) \to m\varphi. \tag{4.11}$$

Now we consider φ being a polynomial function defined in $[0, 2\pi]$. It is well known that there exist a convex function φ_1 and a concave function φ_2 such that $\varphi = \varphi_1 + \varphi_2$. By the sublinearity of T_t, we have

$$T_t\varphi_1(x) - T_t(-\varphi_2)(x) \leq T_t\varphi(x) \leq T_t\varphi_1(x) + T_t\varphi_2(x). \tag{4.12}$$

It follows from the linearity of m that

$$mT_t\varphi \leq mT_t\varphi_1 + mT_t\varphi_2 = m\varphi_1 + m\varphi_2 = m(\varphi_1 + \varphi_2) = m\varphi$$

and

$$mT_t\varphi \geq mT_t\varphi_1 - mT_t(-\varphi_2) = m\varphi_1 - m(-\varphi_2) = m(\varphi_1 + \varphi_2) = m\varphi.$$

So (4.10) holds true for any polynomial function φ. It then follows from an approximation argument using the Weierstrass theorem that (4.10) is also true for $\varphi \in C([0, 2\pi])$.

Moreover, for any polynomial function φ, as above $\varphi = \varphi_1 + \varphi_2$, φ_1 is convex, and φ_2 is concave, we have that when $t \to \infty$,

$$T_t\varphi_1(x) + T_t\varphi_2(x) \to m\varphi_1 + m\varphi_2 = m(\varphi_1 + \varphi_2) = m\varphi$$

and

$$T_t\varphi_1(x) - T_t(-\varphi_2(x)) \to m\varphi_1 - m(-\varphi_2) = m(\varphi_1 + \varphi_2) = m\varphi.$$

Thus (4.11) holds for any polynomial φ.

Now we consider $\varphi \in C([0, 2\pi])$. First note that by the Weierstrass approximation theorem, for any $\epsilon > 0$, there exists a polynomial $\tilde{\varphi}$ such that $\sup_{x \in [0,2\pi]} |\tilde{\varphi}(x) - \varphi(x)| < \frac{1}{3}\epsilon$. So $|T_t\tilde{\varphi}(x) - T_t\varphi(x)| < \frac{1}{3}\epsilon$ for any x, t and $|m\tilde{\varphi}(x) - m\varphi(x)| < \frac{1}{3}\epsilon$. On the other hand, for such $\tilde{\varphi}$, there exists $R > 0$ such that for any $t \geq R$, $|T_t\tilde{\varphi}(x) - m\tilde{\varphi}| < \frac{1}{3}\epsilon$. Thus for $t \geq R$,

$$|T_t\varphi(x) - m\varphi| \leq |T_t\varphi(x) - T_t\tilde{\varphi}(x)| + |T_t\tilde{\varphi}(x) - m\tilde{\varphi}| + |m\tilde{\varphi} - m\varphi| < \epsilon. \tag{4.13}$$

This leads to (4.11) for any $\varphi \in C([0, 2\pi])$.

Now consider $\varphi \in L_b(\mathcal{B}(S^1))$. By Lemma 4.2, for any $\delta > 0$, $(T_\delta\varphi)(x)$ is continuous in x. Applying (4.11) for continuous function, we have

$$T_t\varphi = T_{t-\delta}T_\delta\varphi \to m(T_\delta\varphi) = (mT_\delta)\varphi \text{ as } t \to \infty.$$

So the last statement of the theorem is verified. But $T_t\varphi$ is independent of δ, and then $m(T_\delta\varphi)$ is independent of $\delta > 0$, which means $m(T_{\delta_1}) = m(T_{\delta_2})$ for any $\delta_1, \delta_2 > 0$. Define $\tilde{T} : L_b(\mathcal{B}(S^1)) \to \mathbb{R}^1$:

$$\tilde{T}\varphi = (mT_\delta)\varphi, \ \delta > 0.$$

Then for any $t \geq 0$,

$$\tilde{T}T_t\varphi = mT_\delta T_t\varphi = mT_{t+\delta}\varphi = \tilde{T}\varphi.$$

Thus $\tilde{T}$ is an invariant expectation. The uniqueness follows from the convergence of $T_t\varphi$. ∎

Remark 4.4. (i) From the proof, we can see that when $\varphi \in C([0,2\pi])$, $\tilde{T}\varphi = \frac{1}{2\pi}\int_0^{2\pi}\varphi(x)dx$.

(ii) We do not attempt to give the result in Theorem 4.3 in broad terms, e.g., of Brownian motions on a compact manifold. Here we only show such a result as an example. More general cases will be treated in future publications.

As we have proved the invariant expectation $\tilde{T}$ of G-Brownian motion on S^1 exists, we can follow the procedure in section 3 to construct the canonical process $\hat{X}$ and the canonical dynamical system on the path space.

Applying Theorem 3.22, in the following we prove that the G-Brownian motion on the unit circle is ergodic. First, we need the following proposition, where the no mean-uncertainty condition needed in Theorem 3.22 is proved in (ii) below. Recall $\Omega^* = C(\mathbb{R}, \mathbb{R}^d)$ with the topology given in (3.7).

PROPOSITION 4.5. *Consider the G-Brownian motion on the unit circle $S^1 = [0, 2\pi]$ with normal distribution $N(0, [\underline{\sigma}^2 t, \overline{\sigma}^2 t])$, where $\overline{\sigma}^2 \geq \underline{\sigma}^2 > 0$. The following results hold:*

(i) *The stationary process $\hat{X}$ defined in* (3.9) *has a continuous modification $\tilde{X}$ and is stochastically continuous.*

(ii) *For each $\varphi \in L_b(\mathcal{B}(S^1))$, $\varphi(\tilde{X}(0))$ has no mean-uncertainty with respect to the invariant expectation $\tilde{\mathcal{E}}$.*

(iii) *There exists a weakly compact family of probability measures $\mathcal{P}$ on $(\Omega^*, \mathcal{B}(\Omega^*))$ such that*

$$\hat{\mathbb{E}}^{\tilde{T}}[\xi] = \sup_{P\in\mathcal{P}} E_P[\xi], \quad \xi \in Lip_{b,cyl}(\Omega^*).$$

(iv) *The invariant expectation $\tilde{T}$ is regular.*

(v) *Define, for each $\xi \in \mathcal{B}(\Omega^*)$, the upper expectation*

$$\mathbb{E}^*[\xi] = \sup_{P\in\mathcal{P}} E_P[\xi]. \tag{4.14}$$

For any $F_n \in \mathcal{B}(\Omega^*)$ such that $I_{F_n} \downarrow 0$, then $\mathbb{E}^*[I_{F_n}] \downarrow 0$. Thus $\mathbb{E}^*$ is regular.

Proof. (i) Note that by the sublinear expectation representation theorem, for the sublinear expectation $\mathbb{E}^{\tilde{T}}$ on $(\hat{\Omega}, L_0^1(\hat{\Omega}))$, there exists a family of linear expectations $\{E_\theta : \theta \in \Theta\}$ such that

$$\mathbb{E}^{\tilde{T}}[X] = \sup_{\theta\in\Theta} E_\theta[X], \quad X \in L_0^1(\hat{\Omega}). \tag{4.15}$$

Note further that if $\{\varphi_n\}_{n=1}^\infty \subset C_{b,Lip}((S^1)^m)$ satisfies $\varphi_n \downarrow 0$, then by an argument similar to that in the proof of Lemma 3.3 of Chapter I in Peng [36],

$$\mathbb{E}^{\tilde{T}}[\varphi_n(\hat{\omega}_{t_1}, \hat{\omega}_{t_2}, \ldots, \hat{\omega}_{t_m})] \downarrow 0 \quad \text{as } n \to \infty,$$

and it follows from (4.15) that

$$\mathbb{E}^{\tilde{T}}[\varphi_n(\hat{\omega}_{t_1}, \hat{\omega}_{t_2}, \ldots, \hat{\omega}_{t_m})] = \sup_{\theta\in\Theta} E_\theta[\varphi_n(\hat{\omega}_{t_1}, \hat{\omega}_{t_2}, \ldots, \hat{\omega}_{t_m})].$$

But for each $\theta \in \Theta$, E_θ is controlled by $\mathbb{E}^{\tilde{T}}$. Thus $E_\theta[\varphi_n(\hat{\omega}_{t_1}, \hat{\omega}_{t_2}, \dots, \hat{\omega}_{t_m})] \downarrow 0$ as $n \to \infty$. So by the Daniell–Stone theorem (cf. Peng [36]), there is a unique probability measure $Q_{\{t_1,t_2,\dots,t_m\}}$ on $((S^1)^m, \mathcal{B}((S^1)^m))$ such that

$$E_\theta[\varphi_n(\hat{\omega}_{t_1}, \hat{\omega}_{t_2}, \dots, \hat{\omega}_{t_m})] = E_{Q_{\{t_1,t_2,\dots,t_m\}}}[\varphi_n(\hat{\omega}_{t_1}, \hat{\omega}_{t_2}, \dots, \hat{\omega}_{t_m})].$$

Denote $\mathcal{T} = \{\underline{\mathrm{t}} = \{t_1, t_2, \dots, t_m\} : t_1 < t_2 < \cdots < t_m, m \in \mathbb{N}\}$. Thus we have a family of finite-dimensional distributions $\{Q_{\underline{\mathrm{t}}}, \underline{\mathrm{t}} \in \mathcal{T}\}$. It is easy to check that $\{Q_{\underline{\mathrm{t}}}, \underline{\mathrm{t}} \in \mathcal{T}\}$ is consistent. By Kolmogorov's consistency theorem, there is a probability measure Q on $(\hat{\Omega}, \hat{\mathcal{F}})$ such that $\{Q_{\underline{\mathrm{t}}}, \underline{\mathrm{t}} \in \mathcal{T}\}$ is the finite-dimensional distribution of Q. The probability distribution Q is unique since by the Daniell–Stone theorem its finite-dimensional distribution is unique, so the uniqueness of Q follows from the monotone class theorem. It is now clear that $E_\theta[X] = E_Q[X]$ for any $X \in Lip_{b,cyl}(\hat{\Omega})$. Thus it follows from (4.15) that

$$\mathbb{E}^{\tilde{T}}[X] = \sup_{Q \in \mathcal{P}_e} E_Q[X], \quad X \in Lip_{b,cyl}(\hat{\Omega}),$$

where $\mathcal{P}_e$ is a family of probability measures on $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}))$. Define the associated capacity

$$\hat{c}(A) := \sup_{Q \in \mathcal{P}_e} Q(A), \; A \in \mathcal{B}(\hat{\Omega}),$$

and the upper expectation of each $\mathcal{B}(\hat{\Omega})$-measurable real-valued function X which makes the following definition meaningful:

$$\hat{\mathbb{E}}^{\tilde{T}}[X] = \sup_{Q \in \mathcal{P}_e} E_Q[X].$$

On the space $\mathrm{Lip}_{b,cyl}(\hat{\Omega})$, $\mathbb{E}^{\tilde{T}} = \hat{\mathbb{E}}^{\tilde{T}}$. Consider the canonical process $\hat{X}$ on $(\hat{\Omega}, L_0^1(\hat{\Omega}), \mathbb{E}^{\tilde{T}}, \hat{\theta}_t)$. For $t \geq s$, by the conditional expectation,

$$\begin{aligned} &\hat{\mathbb{E}}^{\tilde{T}}(\hat{X}(t) - \hat{X}(s))^4 \\ &= \mathbb{E}^{\tilde{T}}(\hat{X}(t) - \hat{X}(s))^4 \\ &= \mathbb{E}^{\tilde{T}}[\mathbb{E}^{\tilde{T}}[(\hat{X}(t) - \hat{X}(s))^4 | \mathcal{F}_s]] \\ &= \mathbb{E}^{\tilde{T}}[T_{t-s}\varphi(y)|_{y=\hat{X}(s)}] \\ &\leq c|t-s|^2, \end{aligned} \tag{4.16}$$

where $\varphi(y) = (y - \hat{X}(s))^4$, and $c > 0$ is a constant independent of t and s. Then by the Kolmogorov continuity theorem for sublinear expectations (Theorem 1.36, Chapter VI, Peng [36]), the process $\hat{X}$ has a continuous modification, denoted by $\tilde{X}$, such that $\hat{c}(\tilde{X}_t \neq \hat{X}_t) = 0$. Note that for any $\delta > 0$,

$$\hat{\mathbb{E}}^{\tilde{T}}(\hat{X}(t) - \hat{X}(s))^4 \geq \hat{\mathbb{E}}^{\tilde{T}}[(\hat{X}(t) - \hat{X}(s))^4 I_{\{|\hat{X}(t) - \hat{X}(s)| > \delta\}}] \geq \delta^4 \hat{\mathbb{E}}^{\tilde{T}} I_{\{|\hat{X}(t) - \hat{X}(s)| > \delta\}},$$

so

$$\hat{\mathbb{E}}^{\tilde{T}} I_{\{|\hat{X}(t) - \hat{X}(s)| > \delta\}} \leq \delta^{-4} \hat{\mathbb{E}}^{\tilde{T}}(\hat{X}(t) - \hat{X}(s))^4.$$

Thus the stochastic continuity follows from (4.16).

(ii) Now we prove for any $\varphi \in L_b(\mathcal{B}(S^1))$ that $\varphi(\tilde{X}(0))$ has no mean-uncertainty. We follow the 3-step approximation procedure of using a sequence of continuous functions to approximate φ. Note the no mean-uncertainty of $\varphi(\tilde{X}(0))$ when $\varphi \in C_b(S^1)$ follows from (4.6) and the fact that $\tilde{T}$ is a Lebesgue integral in this case automatically. Adopting the same notation as in the proof of Lemma 4.2, consider the increasing sequence of continuous functions $\varphi_{nm}^{(3)} \uparrow \varphi_n^{(2)}$, when $m \to \infty$. First note by Remark 4.4(i) that

$$\tilde{\mathcal{E}}(-\varphi_{nm}^{(3)}(\tilde{X}(0))) = -\tilde{\mathcal{E}}(\varphi_{nm}^{(3)}(\tilde{X}(0))). \tag{4.17}$$

By Lemma 2.11, we have that $\varphi_n^{(2)}(\tilde{X}(0))$ has no mean-uncertainty,

$$\tilde{\mathcal{E}}(-\varphi_n^{(2)}(\tilde{X}(0))) = -\tilde{\mathcal{E}}(\varphi_n^{(2)}(\tilde{X}(0))). \tag{4.18}$$

But

$$|\tilde{\mathcal{E}}(\varphi_n^{(2)}(\tilde{X}(0))) - \tilde{\mathcal{E}}(\varphi_n^{(1)}(\tilde{X}(0)))| \le \sum_{i=1}^{r} |x_i| \tilde{\mathcal{E}}(I_{A_i^1 \triangle B_i^0}(\tilde{X}(0))) \tag{4.19}$$

and

$$|\tilde{\mathcal{E}}(-\varphi_n^{(2)}(\tilde{X}(0))) - \tilde{\mathcal{E}}(-\varphi_n^{(1)}(\tilde{X}(0)))| \le \sum_{i=1}^{r} |x_i| \tilde{\mathcal{E}}(I_{A_i^1 \triangle B_i^0}(\tilde{X}(0))), \tag{4.20}$$

so $\varphi_n^{(1)}(\tilde{X}(0))$ has no mean-uncertainty. As $\varphi_n^{(1)} \uparrow \varphi$, by Lemma 2.11 again, $\varphi(\tilde{X}(0))$ has no mean-uncertainty,

$$\tilde{\mathcal{E}}(-\varphi(\tilde{X}(0))) = -\tilde{\mathcal{E}}(\varphi(\tilde{X}(0))).$$

(iii) In the following we will find a weakly compact family of probability measures $\mathcal{P}$ on $(\Omega^*, \mathcal{B}(\Omega^*))$ such that the upper expectation (4.14) gives a sublinear expectation on $\mathcal{P}$ on $(\Omega^*, \mathcal{B}(\Omega^*))$ with finite-dimensional expectation of $\varphi(\omega^*_{t_1}, \omega^*_{t_2}, \dots, \omega^*_{t_m})$, $t_1 < t_2 < \cdots < t_m$, to be $T^{\tilde{T}}_{t_1,t_2,\dots,t_m}\varphi$ for $\varphi \in L_b(\mathcal{B}((S^1)^m))$.

For each $Q \in \mathcal{P}_e$, let $Q \circ \tilde{X}^{-1}$, which is a probability measure on $(\Omega^*, \mathcal{B}(\Omega^*))$ induced by $\tilde{X}$ from Q, and set $\mathcal{P}_1 = \{Q \circ \tilde{X}^{-1} : Q \in \mathcal{P}_e\}$. Then, similarly to (4.16), we have

$$\hat{\mathbb{E}}^{\tilde{T}}(\tilde{X}(t) - \tilde{X}(s))^4 = \hat{\mathbb{E}}^{\tilde{T}}(\hat{X}(t) - \hat{X}(s))^4 \le c|t-s|^2, \quad t, s \in \mathbb{R}.$$

Applying the moment criterion for the tightness of Kolmogorov–Chentsov type, we conclude that $\mathcal{P}_1$ as a family of probability measures on $(\Omega^*, \mathcal{B}(\Omega^*))$ is tight. Denote by $\mathcal{P} = \bar{\mathcal{P}}_1$ the closure of $\mathcal{P}_1$ under the topology of weak convergence. Then $\mathcal{P}$ is weakly compact. Note that

$$\hat{\mathbb{E}}^{\tilde{T}}[\xi] = \sup_{P \in \mathcal{P}_1} E_P[\xi], \quad \xi \in \mathrm{Lip}_{b,cyl}(\Omega^*).$$

For each $\xi \in \mathrm{Lip}_{b,cyl}(\Omega^*)$, from Lemma 3.3 of Chapter I in [36], we get $\hat{\mathbb{E}}^{\tilde{T}}[|\xi - (\xi \wedge N) \vee (-N)|] \downarrow 0$ as $N \to \infty$. So

$$\hat{\mathbb{E}}^{\tilde{T}}[\xi] = \sup_{P \in \mathcal{P}} E_P[\xi], \quad \xi \in \mathrm{Lip}_{b,cyl}(\Omega^*).$$

(iv) For any $A_n \in \mathcal{B}(S^1)$ such that $I_{A_n} \downarrow 0$, then by (4.6) and Lemma 4.1, we have $\tilde{T}[I_{A_n}] \downarrow 0$, i.e., $\tilde{T}$ is regular.

(v) For $\mathcal{P}$ given in (iii), we define the associated G-capacity

$$c^*(F) := \sup_{P \in \mathcal{P}} P(F),\ F \in \mathcal{B}(\Omega^*),$$

and upper expectation for each $\mathcal{B}(\Omega^*)$-measurable real-valued function ξ which makes the following definition meaningful:

$$\mathbb{E}^*[\xi] := \sup_{P \in \mathcal{P}} E_P[\xi].$$

On $\mathrm{Lip}_{b,cyl}(\Omega^*)$, $\mathbb{E}^* = \hat{\mathbb{E}}^{\tilde{T}}$ and as $\mathcal{P}$ is a weakly compact family of probability measures on $(\Omega^*, \mathcal{B}(\Omega^*))$, we have for any continuous ξ_n and $\xi_n \downarrow 0$, $\mathbb{E}^*[\xi_n] \downarrow 0$ as $n \to \infty$. Now consider for any $F_n \in \mathcal{B}(\Omega^*)$, such that $I_{F_n} \downarrow 0$. Define

$$C_n = \left\{ \omega \in \Omega^* : \rho(\omega, F_n) \le \frac{1}{n} \right\}, \qquad D_n = \left\{ \omega \in \Omega^* : \rho(\omega, F_n) < \frac{2}{n} \right\}.$$

Moreover, define

$$\xi_n(\omega) = n[\min\{\rho(\omega, D_n^c), \rho(C_n, D_n^c)\}].$$

Then it is easy to see that $\xi_n(\omega)$ is continuous in $\omega \in \Omega^*$ and $I_{F_n} \le \xi_n$. As when $\xi_n \downarrow 0$, we have that $\mathbb{E}^*[\xi_n] \downarrow 0$; thus as $n \to \infty$, it follows that $\mathbb{E}^*[I_{F_n}] \downarrow 0$. ◻

From the result of Proposition 4.5 and Proposition 3.10, we can conclude that the canonical dynamical system generated by the semigroup of the G-Brownian motion on the unit circle is continuous.

THEOREM 4.6. *The invariant expectation of the G-Brownian motion on the unit circle $S^1 = [0, 2\pi]$ with normal distribution $N(0, [\underline{\sigma}^2 t, \overline{\sigma}^2 t])$, where $\overline{\sigma}^2 \ge \underline{\sigma}^2 > 0$ are constant, is ergodic.*

Proof. Consider $\varphi \in L_b(\mathcal{B}(S^1))$ with $T_t\varphi = \varphi$ and $T_t(-\varphi) = -\varphi, t \ge 0$. From the convergence result that as $t \to \infty$, $T_t\varphi \to \tilde{T}\varphi$ in Theorem 4.3, it is easy to see that $\varphi = \tilde{T}\varphi$, so φ is constant. By Theorem 3.22, the invariant expectation is ergodic. ◻

Remark 4.7. Following the regularity result of $\mathbb{E}^*$ in Proposition 4.5, and the ergodicity results for the G-Brownian motion on the unit circle, it follows that the SLLN holds by Theorem 3.18.

Inspired by Theorem 3.22, we observe that the study of the ergodicity of the invariant expectation $\tilde{T}$ is equivalent to the study of the spectrum of the semigroup T_t on the space of $L_b(\mathcal{B}(\mathbb{R}^d))$. It is noted that due to the constant preserving property of the sublinear expectation, the sublinear semigroup T_t on $L_b(\mathcal{B}(\mathbb{R}^d))$ has eigenvalue 1. Theorem 3.22 says that ergodicity is equivalent to 1 being a simple eigenvalue of T_t on $L_b(\mathcal{B}(\mathbb{R}^d))$ as $|\varphi(X(0))|^2$ has no mean-uncertainty.

Now we consider the relation of the eigenvalues of T_t and its infinitesimal generator $\mathbb{G}$. First assume 1 is a simple eigenvalue of T_t. Recall $\mathbb{G}(u) = \frac{1}{2}\overline{\sigma}^2 u_{xx}^+ - \frac{1}{2}\underline{\sigma}^2 u_{xx}^-$ and $u(t,x) = T_t\varphi(x)$ satisfying (4.2). It is easy to see that $\mathbb{G}(c) = 0$ for any constant c. This suggests that 0 is an eigenvalue of the generator $\mathbb{G}$ in the space of twice differentiable functions. However, if φ is continuous and a viscosity solution of $\mathbb{G}(\varphi) = 0$, it is easy to see that $T_t\varphi = \varphi$. So φ is constant. This means 0 is a simple eigenvalue of $\mathbb{G}$. Conversely, now assume 0 is a simple eigenvalue of $\mathbb{G}$. Consider φ as a continuous function satisfying $T_t\varphi = \varphi$. As $T_t\varphi$ is a solution of (4.2), so $\mathbb{G}(\varphi) = 0$. Thus φ is a constant due to the spectrum assumption of $\mathbb{G}$. This correspondence is also true for sublinear Markovian semigroups and their infinitesimal generators in general cases.

From the above discussions, our result shows that as the G-Brownian motion on the unit circle is ergodic, so 0 is a simple eigenvalue of the corresponding infinitesimal generator $\mathbb{G}(\cdot)$. In fact, we can prove this result analytically without referring to the result of ergodicity.

PROPOSITION 4.8. *Let a continuous function φ be a viscosity solution of*

$$\frac{1}{2}\overline{\sigma}^2\varphi_{xx}^+ - \frac{1}{2}\underline{\sigma}^2\varphi_{xx}^- = 0, \quad x \in [0, 2\pi],\ \varphi(0) = \varphi(2\pi). \tag{4.21}$$

If $\underline{\sigma}^2 > 0$, then φ is constant.

Proof. Let ψ be a C^2 function on $[0, 2\pi]$ such that $\psi \geq \varphi$ and $\psi(x) = \varphi(x)$ at certain $x \in [0, 2\pi]$ with $\psi''(x) \neq 0$. Then $\frac{1}{2}\overline{\sigma}^2\psi''(x)^+ - \frac{1}{2}\underline{\sigma}^2\psi''(x)^- \geq 0$. It is then obvious that

$$\underline{\sigma}^2\psi''(x)^- \leq \overline{\sigma}^2\psi''(x)^+. \tag{4.22}$$

If $\psi''(x) < 0$, then $\psi''(x)^- > 0$ and $\psi''(x)^+ = 0$. This contradicts (4.22). Thus $\psi''(x) \geq 0$, so ψ is locally a convex function near x.

Similarly, let $\tilde{\psi}$ be a C^2 function on $[0, 2\pi]$ such that $\tilde{\psi} \leq \varphi$ and $\tilde{\psi}(x) = \varphi(x)$ at certain $x \in [0, 2\pi]$ with $\tilde{\psi}''(x) \neq 0$. Then $\frac{1}{2}\overline{\sigma}^2\tilde{\psi}''(x)^+ - \frac{1}{2}\underline{\sigma}^2\tilde{\psi}''(x)^- \leq 0$. It is then obvious that

$$\overline{\sigma}^2\tilde{\psi}''(x)^+ \leq \underline{\sigma}^2\tilde{\psi}''(x)^-. \tag{4.23}$$

If $\tilde{\psi}''(x) > 0$, then $\tilde{\psi}''(x)^+ > 0$ and $\tilde{\psi}''(x)^- = 0$. This contradicts (4.23). Thus $\tilde{\psi}''(x) \leq 0$ and $\tilde{\psi}$ is locally a concave function near x.

A function φ that satisfies the above two properties must be a linear function. Now from the periodic boundary of φ, we conclude easily that φ is a constant. ☐

Remark 4.9. The condition $\underline{\sigma}^2 > 0$ is crucial for Proposition 4.8. Otherwise, any smooth concave periodic function φ with period 2π satisfies (4.21) since $\varphi_{xx}^+ = 0$. In that case, Brownian motion (degenerate) on S^1 fails to be ergodic. So Theorem 4.6 can be stated as follows.

THEOREM 4.10. *The invariant expectation of the G-Brownian motion on the unit circle $S^1 = [0, 2\pi]$ with normal distribution $N(0, [\underline{\sigma}^2 t, \overline{\sigma}^2 t])$, where $\overline{\sigma}^2 \geq \underline{\sigma}^2$ are constant, is ergodic if and only if $\underline{\sigma}^2 > 0$.*

Appendix A. Proofs of Theorem 2.4 and Lemma 2.12.

Proof of Theorem 2.4. (i)⇒(ii) Assume $B \in \mathcal{F}$ and $\mathbb{E}I_{\theta^{-1}B\Delta B} = 0$. Define

$$B_\infty = \bigcap_{n=0}^{\infty}\bigcup_{i=n}^{\infty}\theta^{-i}B. \tag{A.1}$$

Then it is easy to see that

$$\theta^{-1}B_\infty = \bigcap_{n=0}^{\infty}\bigcup_{i=n+1}^{\infty}\theta^{-i}B = B_\infty.$$

Thus B_∞ is an invariant set. By the assumption, we have

$$\mathbb{E}\mathrm{I}_{B_\infty} = 0 \quad \text{or} \quad \mathbb{E}\mathrm{I}_{B_\infty^c} = 0. \tag{A.2}$$

Note that for any $n \in N$

$$\begin{aligned}\theta^{-n}B \triangle B &\subset \bigcup_{i=0}^{n-1}(\theta^{-(i+1)}B \triangle \theta^{-i}B)\\ &= \bigcup_{i=0}^{n-1}\theta^{-i}(\theta^{-1}B \triangle B).\end{aligned}$$

So by the monotonicity and subadditivity of $\mathbb{E}$ and the expectation preserving property of θ,

$$\begin{aligned}\mathbb{E}\mathrm{I}_{\theta^{-n}B\triangle B} &\le \mathbb{E}\mathrm{I}_{\bigcup_{i=0}^{n-1}\theta^{-i}(\theta^{-1}B\triangle B)}\\ &\le \mathbb{E}\left[\sum_{i=0}^{n-1}\mathrm{I}_{\theta^{-i}(\theta^{-1}B\triangle B)}\right]\\ &\le \sum_{i=0}^{n-1}\mathbb{E}\mathrm{I}_{\theta^{-i}(\theta^{-1}B\triangle B)}\\ &= \sum_{i=0}^{n-1}\mathbb{E}\mathrm{I}_{\theta^{-1}B\triangle B}\\ &= 0.\end{aligned} \tag{A.3}$$

Moreover

$$\left(\bigcup_{i=1}^{\infty}\theta^{-i}B\right) \triangle B \subset \bigcup_{i=1}^{\infty}(\theta^{-i}B \triangle B). \tag{A.4}$$

Thus it follows from (A.3) and (A.4) that

$$\begin{aligned}\mathbb{E}\mathrm{I}_{(\bigcup_{i=n}^{\infty}\theta^{-i}B)\triangle B} &\le \mathbb{E}\mathrm{I}_{\bigcup_{i=0}^{\infty}(\theta^{-i}B\triangle B)}\\ &\le \sum_{i=0}^{\infty}\mathbb{E}\mathrm{I}_{\theta^{-i}B\triangle B}\\ &= 0.\end{aligned}$$

From the above we have

$$\mathbb{E}\mathrm{I}_{(\bigcup_{i=n}^{\infty}\theta^{-i}B)\setminus B} = 0 \tag{A.5}$$

and

$$\mathbb{E}\mathrm{I}_{B\setminus\bigcup_{i=n}^{\infty}\theta^{-i}B} = 0. \tag{A.6}$$

But note that as $n \to \infty$,

$$\mathrm{I}_{(B\setminus\bigcup_{i=n}^{\infty}\theta^{-i}B)} \uparrow \mathrm{I}_{(B\setminus\bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty}\theta^{-i}B)} = I_{B\setminus B_\infty}.$$

So by the monotone (increasing) convergence of sublinear expectation (see [34] and [11]), we have that as $n \to +\infty$,

$$\mathbb{E}\mathrm{I}_{B\setminus\bigcup_{i=n}^{\infty}\theta^{-i}B} \to \mathbb{E}\mathrm{I}_{B\setminus B_\infty}.$$

Thus it follows from (A.6) that

$$\mathbb{E}\mathrm{I}_{B\setminus B_\infty} = 0. \tag{A.7}$$

Moreover

$$\mathrm{I}_{(\bigcup_{i=n}^{\infty}\theta^{-i}B)\setminus B} \downarrow \mathrm{I}_{B_\infty\setminus B}.$$

It then follows by applying the monotonicity of sublinear expectation and (A.5) that

$$\mathbb{E}\mathrm{I}_{B_\infty\setminus B} = 0.$$

Note that the regularity condition is not needed here. Thus

$$\mathbb{E}\mathrm{I}_{B_\infty\triangle B} = 0.$$

Now recall (A.2). Consider the case that $\mathbb{E}\mathrm{I}_{B_\infty} = 0$. Note that

$$\begin{aligned} 0 = \mathbb{E}\mathrm{I}_{B\setminus B_\infty} &= \mathbb{E}\mathrm{I}_{B\setminus(B\cap B_\infty)} \\ &= \mathbb{E}[\mathrm{I}_B - \mathrm{I}_{(B\cap B_\infty)}] \\ &\geq \mathbb{E}[\mathrm{I}_B] - \mathbb{E}[\mathrm{I}_{(B\cap B_\infty)}] \\ &\geq \mathbb{E}[\mathrm{I}_B] - \mathbb{E}[\mathrm{I}_{B_\infty}] \\ &= \mathbb{E}[\mathrm{I}_B]. \end{aligned}$$

Hence

$$\mathbb{E}[\mathrm{I}_B] = 0.$$

Now consider the case that $\mathbb{E}\mathrm{I}_{B_\infty^c} = 0$. Note that

$$\begin{aligned} 0 = \mathbb{E}\mathrm{I}_{B_\infty\setminus B} &= \mathbb{E}\mathrm{I}_{B^c\setminus(B^c\cap B_\infty^c)} \\ &= \mathbb{E}[\mathrm{I}_{B^c} - \mathrm{I}_{B^c\cap B_\infty^c}] \\ &\geq \mathbb{E}[\mathrm{I}_{B^c}] - \mathbb{E}[\mathrm{I}_{B^c\cap B_\infty^c}] \\ &\geq \mathbb{E}[\mathrm{I}_{B^c}] - \mathbb{E}[\mathrm{I}_{B_\infty^c}] \\ &= \mathbb{E}[\mathrm{I}_{B^c}]. \end{aligned}$$

Thus

$$\mathbb{E}[\mathrm{I}_{B^c}] = 0.$$

Therefore the assertion (ii) is proved.

(iii)$\Rightarrow$(iv) Let $\mathbb{E}\mathrm{I}_A > 0$ and $\mathbb{E}\mathrm{I}_B > 0$. From (iii), we know that $\mathbb{E}\mathrm{I}_{(\bigcup_{n=1}^{\infty}\theta^{-n}A)^c} = 0$. It then follows, together with applying subadditivity and monotonicity of $\mathbb{E}$, that

$$\begin{aligned} 0 < \mathbb{E}\mathrm{I}_B &= \mathbb{E}[\mathrm{I}_{B\bigcap(\bigcup_{n=1}^{\infty}\theta^{-n}A)} + \mathrm{I}_{B\bigcap(\bigcup_{n=1}^{\infty}\theta^{-n}A)^c}] \\ &\leq \mathbb{E}[\mathrm{I}_{B\bigcap(\bigcup_{n=1}^{\infty}\theta^{-n}A)}] + \mathbb{E}[\mathrm{I}_{B\bigcap(\bigcup_{n=1}^{\infty}\theta^{-n}A)^c}] \\ &\leq \mathbb{E}[\mathrm{I}_{\bigcup_{n=1}^{\infty}(B\bigcap\theta^{-n}A)}] + \mathbb{E}[\mathrm{I}_{(\bigcup_{n=1}^{\infty}\theta^{-n}A)^c}] \\ &= \mathbb{E}[\mathrm{I}_{\bigcup_{n=1}^{\infty}(B\bigcap\theta^{-n}A)}] \\ &\leq \sum_{n=1}^{\infty}\mathbb{E}[\mathrm{I}_{B\bigcap\theta^{-n}A}]. \end{aligned}$$

Thus it is obvious that there must exist $n \in \mathbb{N}$ such that $\mathbb{E}[\mathrm{I}_{(B \bigcap \theta^{-n}A)}] > 0$. So (iv) is proved.

(iv)⇒(i) Suppose that $B \in \mathcal{F}$ and $\theta^{-1}B = B$. If $\mathbb{E}\mathrm{I}_B > 0$ and $\mathbb{E}\mathrm{I}_{B^c} > 0$, then by assumption (iv) and invariant assumption of B,

$$0 < \mathbb{E}[\mathrm{I}_{B^c \bigcap \theta^{-n}B}] = \mathbb{E}[\mathrm{I}_{B^c \bigcap B}] = 0.$$

This is a contradiction, and thus $\mathbb{E}\mathrm{I}_B = 0$ or $\mathbb{E}\mathrm{I}_{B^c} = 0$. So (i) is proved.

(ii)⇒(iii) This follows under the regularity assumption. Assume $A \in \mathcal{F}$ and $\mathbb{E}\mathrm{I}_A > 0$. Set

$$A_1 = \bigcup_{n=1}^{\infty} \theta^{-n}A.$$

It is easy to see that $\theta^{-1}A_1 \subset A_1$ and $\theta^{-n}A_1 = \bigcup_{i=n+1}^{\infty} \theta^{-i}A$. So $\{\theta^{-n}A_1\}_{n\in\mathbb{N}}$ forms a decreasing sequence of sets with limit

$$\theta^{-n}A_1 \downarrow A_\infty = \limsup_n (\theta^{-n}A), \tag{A.8}$$

where the notation A_∞ is used in the same fashion as in the proof of (i)⇒(ii). It is easy to see that

$$\theta^{-1}A_\infty = A_\infty.$$

Thus

$$\mathbb{E}\mathrm{I}_{\theta^{-1}A_\infty \triangle A_\infty} = 0.$$

According to assumption (ii), we know that either $\mathbb{E}\mathrm{I}_{A_\infty} = 0$ or $\mathbb{E}\mathrm{I}_{A_\infty^c} = 0$. We claim the case that $\mathbb{E}\mathrm{I}_{A_\infty} = 0$ is impossible. Otherwise, $\mathrm{I}_{A_\infty} = 0$ quasi-surely. It then follows that $\mathrm{I}_{\theta^{-n}A_1} \downarrow \mathrm{I}_{A_\infty} = 0$ quasi-surely. So as $\mathbb{E}$ is regular, therefore $\mathbb{E}\mathrm{I}_{\theta^{-n}A_1} \to 0$ as $n \to \infty$. However, by the expectation preserving property of θ, the definition of A_1, and the monotonicity of $\mathbb{E}$,

$$\mathbb{E}\mathrm{I}_{\theta^{-n}A_1} = \mathbb{E}\mathrm{I}_{A_1} \geq \mathbb{E}\mathrm{I}_{\theta^{-1}A} = \mathbb{E}\mathrm{I}_A > 0.$$

We have a contraction. Thus $\mathbb{E}\mathrm{I}_{A_\infty^c} = 0$ holds. Then it follows that $\mathbb{E}\mathrm{I}_{A_1^c} = 0$ as $A_\infty \subset A_1$, so (iii) is proved. It is then obvious that all four statements are equivalent under the regularity condition. ∎

Proof of Lemma 2.12. Recall that S_n is defined by (2.8). Let

$$\bar{\xi} = \limsup_{n\to\infty} \frac{S_n}{n},$$

$\epsilon > 0$, and

$$D = \{\omega : \bar{\xi}(\omega) > \bar{\xi}^*(\omega) + \epsilon\}.$$

Our goal is to prove $\mathbb{E}[-\mathrm{I}_D] = 0$. Note that $\bar{\xi}(\theta\omega) = \bar{\xi}(\omega)$, and $\bar{\xi}^*(\theta\omega) = \bar{\xi}^*(\omega)$ quasi-surely, so $D \in \mathcal{I}$.

Define

$$\begin{aligned} \xi^*(\omega) &= (\xi(\omega) - \bar{\xi}^*(\omega) - \epsilon)\mathrm{I}_D(\omega), \\ S_n^*(\omega) &= \xi^*(\omega) + \cdots + \xi^*(\theta_{n-1}^*\omega), \end{aligned}$$

$$M_n^*(\omega) = \sup\{0, S_1^*(\omega), \dots, S_n^*(\omega)\},$$
$$F_n = \{\omega : M_n^*(\omega) > 0\},$$

and

$$F = \cup_n F_n = \left\{\omega : \sup_{k\geq 1} \frac{S_k^*}{k} > 0\right\}.$$

Since $\xi^*(\omega) = (\xi(\omega) - \bar{\xi}^*(\omega) - \epsilon)\mathrm{I}_D(\omega)$ and $D = \{\omega : \limsup_{k\to\infty} \frac{S_k}{k} > \bar{\xi}^* + \epsilon\}$, it follows that $F = D$. In fact, if $\omega \in D$, then $\sup_{k\geq 1} \frac{S_k}{k} > \bar{\xi}^* + \epsilon$, and by definition of ξ^*, $\frac{S_k^*}{k} = \frac{S_k}{k} - \epsilon - \bar{\xi}^*$. So $\sup_{k\geq 1} \frac{S_k^*}{k} > 0$, i.e., $\omega \in F$. Therefore $D \subset F$. If $\omega \notin D$, then $\xi^*(\omega) = 0$. Note that $D \in \mathcal{I}$, so $\xi^*(\theta_k \omega) = 0$ quasi-surely for all k. Therefore $S_k^*(\omega) = 0$ for all k, and so $\omega \notin F$. This tells us that $F \subset D$. Thus $F = D$.

Now applying the maximal ergodic theorem, we know that $\mathbb{E}[\xi^* \mathrm{I}_{F_n}] \geq 0$. But

$$\begin{aligned}\mathbb{E}[\xi^* \mathrm{I}_{F_n}] &= \mathbb{E}[(\xi^*)^+ \mathrm{I}_{F_n} - (\xi^*)^- \mathrm{I}_{F_n}] \\ &\leq \mathbb{E}[(\xi^*)^+ \mathrm{I}_F - (\xi^*)^- \mathrm{I}_F + (\xi^*)^- \mathrm{I}_{F\setminus F_n}] \\ &\leq \mathbb{E}[\xi^* \mathrm{I}_F] + \mathbb{E}[(\xi^*)^- \mathrm{I}_{F\setminus F_n}].\end{aligned}$$

But $\mathbb{E}[(\xi^*)^- \mathrm{I}_{F\setminus F_n}] \downarrow 0$ as $n \to \infty$ because $\mathrm{I}_{F\setminus F_n} \downarrow 0$ and $\mathbb{E}$ is regular. Thus

$$\mathbb{E}[\xi^* \mathrm{I}_F] \geq 0.$$

However, it follows that

$$\begin{aligned}0 \leq \mathbb{E}[(\bar{\xi} - \bar{\xi}^* - \epsilon)\mathrm{I}_D] &\leq \mathbb{E}[(\bar{\xi} - \bar{\xi}^*)\mathrm{I}_D] + \mathbb{E}[-\epsilon \mathrm{I}_D] \\ &= \sup_{P\in\mathcal{P}} E_P[(\bar{\xi} - \bar{\xi}^*)\mathrm{I}_D] + \mathbb{E}[-\epsilon \mathrm{I}_D] \\ &= \sup_{P\in\mathcal{P}} E_P[E_P[(\bar{\xi} - \bar{\xi}^*)\mathrm{I}_D|\mathcal{I}]] + \mathbb{E}[-\epsilon \mathrm{I}_D] \\ &= \sup_{P\in\mathcal{P}} E_P[E_P[(\bar{\xi} - \bar{\xi}^*)|\mathcal{I}]\mathrm{I}_D] + \mathbb{E}[-\epsilon \mathrm{I}_D] \\ &= \sup_{P\in\mathcal{P}} E_P[[E_P[\bar{\xi}|\mathcal{I}] - \bar{\xi}^*]\mathrm{I}_D] + \epsilon\mathbb{E}[-\mathrm{I}_D] \\ &\leq \epsilon\mathbb{E}[-\mathrm{I}_D].\end{aligned}$$

Thus $\mathbb{E}[-\mathrm{I}_D] \geq 0$. On the other hand, $\mathbb{E}[-\mathrm{I}_D] \leq 0$. So $\mathbb{E}[-\mathrm{I}_D] = 0$, which is equivalent to $v(D) = 0$. Thus we get (2.11). Define

$$\tilde{D} = \left\{\omega : -\liminf_{n\to\infty} \frac{S_n}{n} > -\underline{\xi}^* + \epsilon\right\}.$$

Applying the above result to $-\xi$, we can get $v(\tilde{D}) = 0$. Therefore (2.12) holds. ☐

Acknowledgment. We are grateful to the anonymous referees for their constructive comments, which helped us to improve the paper substantially.

REFERENCES

[1] P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath, *Coherent measures of risk*, Math. Finance, 9 (1999), pp. 203–228.

[2] G. D. Birkhoff, *Proof of the ergodic theorem*, Proc. Natl. Acad. Sci. USA, 17 (1931), pp. 656–660.

[3] L. A. Cafarelli and X. Cabre, *Fully Nonlinear Elliptic Equations*, Amer. Math. Soc. Colloq. Publ. 43, AMS, Providence, RI, 1995.

[4] S. Cerreia-Vioglio, F. Maccheroni, and M. Marinacci, *Ergodic theorems for lower probabilities*, Proc. Amer. Math. Soc., 144 (2016), pp. 3381–3396.

[5] Z. J. Chen, *Strong laws of large numbers for sub-linear expectations*, Sci. China Math., 59 (2016), pp. 945–954.

[6] Z. J. Chen and L. G. Epstein, *Ambiguity, risk and asset returns in continuous time*, Econometrica, 70 (2002), pp. 1403–1443.

[7] D. M. Cutler, J. M. Poterba, and L. H. Summers, *Speculative dynamics and the role of feedback traders*, Amer. Econom. Rev., 80 (1990), pp. 63–68.

[8] G. Da Prato and J. Zabczyk, *Ergodicity for Infinite Dimensional Systems*, London Math. Soc. Lecture Note Ser. 229, Cambridge University Press, 1996.

[9] F. Delbaen, *Coherent measures of risk on general probability space*, in Advances in Finance and Stochastics, Essays in Honour of Dieter Sondermann, K. Sandmann and P. J. Schonbucher, eds., Springer-Verlag, Berlin, 2002, pp. 1–37.

[10] J. B. De Long, A. Shleifer, L. Summers, and R. Waldmann, *Noise trader risk in financial markets*, J. Political Econ., 98 (1990), pp. 703–738.

[11] L. Denis, M. S. Hu, and S. G. Peng, *Function spaces and capacities related to a sublinear expectation: Applications to G-Brownian motion paths*, Potential Anal., 34 (2011), pp. 139–161.

[12] R. Durrett, *Probability: Theory and Examples*, 3rd ed., Duxbury Press, 2004.

[13] I. Ekren, N. Touzi, and J. Zhang, *Viscosity solutions of fully nonlinear parabolic path dependent PDEs: Part* I, Ann. Probab., 44 (2016), pp. 1212–1253.

[14] I. Ekren, N. Touzi, and J. Zhang, *Viscosity solutions of fully nonlinear parabolic path dependent PDEs: Part* II, Ann. Probab., 44 (2016), pp. 2507–2553.

[15] N. El Karoui, S. G. Peng, and M. C. Quenez, *Backward stochastic differential equations in finance*, Math. Finance, 7 (1997), pp. 1–71.

[16] K. D. Elworthy, *The method of images for the heat kernel of* S^3, in Proceedings of International Conference on Stochastic Processes, Physics and Geometry, Ascona-Locarno, 1988, S. Albeverio et al., eds., World Scientific, 1990, pp. 434–438.

[17] L. G. Epstein and S. Ji, *Ambiguous volatility and asset pricing in continuous time*, Rev. Financ. Stud., 26 (2013), pp. 1740–1786.

[18] C. R. Feng, B. Y. Qu, and H. Z. Zhao, *A sufficient and necessary condition of PS-ergodicity of periodic measures and generated ergodic upper expectations*, Nonlinearity, 33 (2020), pp. 5324–5354.

[19] C. R. Feng, B. Y. Qu, and H. Z. Zhao, *Random quasi-periodic paths and quasi-periodic measures of stochastic differential equations*, J. Differential Equations, 286 (2021), pp. 119–163.

[20] C. R. Feng, P. Y. Wu, and H. Z. Zhao, *Ergodicity of invariant capacities*, Stochastic Process. Appl., 130 (2020), pp. 5037–5059.

[21] C. R. Feng and H. Z. Zhao, *Random periodic processes, periodic measures and ergodicity*, J. Differential Equations, 269 (2020), pp. 7382–7428.

[22] H. Follmer and A. Schied, *Convex measures of risk and trading constraints*, Finance Stoch., 6 (2002), pp. 429–447.

[23] H. Follmer and A. Schied, *Stochastic Finance, an Introduction in Discrete Time*, Walter de Gruyter, 2004.

[24] J. Frankel and K. Froot, *Understanding the U.S. dollar in the eighties: The expectations of chartists and fundamentalists*, Econom. Rec. Special Issue, (1986), pp. 24–38.

[25] A. Garsia, *A simple proof of E. Hopf's maximal ergodic theorem*, J. Math. Mech., 14 (1965), pp. 381–382.

[26] R. M. Greenwood and A. Shleifer, *Expectations of returns and expected returns*, Rev. Financial Stud., 27 (2014), pp. 714–746.

[27] M. S. Hu, H. W. Li, F. L. Wang, and G. Zheng, *Invariant and ergodic nonlinear expectations for G-diffusion processes*, Electron. Commun. Probab., 20 (2015), pp. 1–15.

[28] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, 2nd ed., Elsevier, 1992.

[29] I. Karatzas and S. E. Shreve, *Brownian Motions and Stochastic Calculus*, 2nd ed., Springer-Verlag, New York, 1991.

[30] N. V. Krylov, *Some new results in the theory of nonlinear elliptic and parabolic equations*, in Proceedings of the International Congress of Mathematicians, Berkeley, CA, 1986, pp. 1101–1109.

[31] N. V. Krylov, *Nonlinear Elliptic and Parabolic Equations of the Second Order*, Reidel, 1987.

[32] P.-L. Lions, *Optimal control of diffusion processes and Hamilton–Jacobi–Bellman equations. Part 1: The dynamic programming principle and applications*, Comm. Partial Differential Equations, 8 (1983), pp. 1101–1174.

[33] P.-L. Lions, *Optimal control of diffusion processes and Hamilton–Jacobi–Bellman equations. Part 2: Viscosity solutions and uniqueness*, Comm. Partial Differential Equations, 8 (1983), pp. 1229–1276.

[34] S. G. Peng, *Nonlinear expectations and nonlinear Markov chains*, Chin. Ann. Math., 26 (2005), pp. 159–184.

[35] S. G. Peng, *G-expectation, G-Brownian motion and related stochastic calculus of Itô type*, in The Abel Symposium 2005, Abel Symposia 2, Stochastic Analysis and Applications, F. E. Benth et. al., eds., Springer-Verlag, 2007, pp. 541–567.

[36] S. G. Peng, *Nonlinear Expectations and Stochastic Calculus under Uncertainty. With Robust Central Limit Theorem and G-Brownian Motion*, Springer, 2019.

[37] S. G. Peng, *Note on Viscosity Solution of Path-Dependent PDE and G-Martingales*, preprint, https://arxiv.org/abs/1106.1144, 2011.

[38] S. J. Taylor, *Introduction to Measure and Integration*, Cambridge University Press, 1973.

[39] J. von Neumann, *Proof of the quasi-ergodic hypothesis*, Proc. Natl. Acad. Sci. USA, 18 (1932), pp. 70–82.

[40] J. von Neumann, *Physical applications of the ergodic hypothesis*, Proc. Natl. Acad. Sci. USA, 18 (1932), pp. 263–266.

[41] P. Walters, *An Introduction to Ergodic Theory*, Grad. Texts in Math. 79, Springer-Verlag, New York, 1982.

[42] J. C. Williams, *Bubbles Tomorrow and Bubbles Yesterday, but Never Bubbles Today*, Presentation to the National Association for Business Economics, San Francisco, CA, September 9, 2013, http://citeseerx.ist.psu.edu/viewdoc/citations?doi=10.1.1.398.6106.