Random quasi-periodic paths and quasi-periodic measures of stochastic differential equations

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Abstract

In this paper, we define random quasi-periodic paths for random dynamical systems and quasi-periodic measures for Markovian semigroups. We give a sufficient condition for the existence and uniqueness of random quasi-periodic paths and quasi-periodic measures for stochastic differential equations and a sufficient condition for the density of the quasi-periodic measure to exist and to satisfy the Fokker-Planck equation. We obtain an invariant measure by considering lifted flow and semigroup on cylinder and the tightness of the average of lifted quasi-periodic measures. We further prove that the invariant measure is unique, and thus ergodic.

Keywords: quasi-periodic measures; invariant measures; random dynamical systems; random quasi-periodic paths; Markovian random dynamical system; Markovian semigroup; Fokker-Planck equation.

1 Introduction

Quasi-periodic oscillation of a dynamical system is a motion given by a quasi-periodic function F such that
$$F(t) = f(t, t, \cdots, t), \tag{1.1}$$
for some continuous function $f(t_1, t_2, \cdots, t_m)$, $(t_1, t_2, \cdots, t_m) \in \mathbb{R}^m$ $(m \geq 2)$ which is periodic in $t_1, t_2, \cdots, t_m$ with periods $\tau_1, \tau_2, \cdots, \tau_m$ respectively, where $\tau_1, \tau_2, \cdots, \tau_m$ are strictly positive and their reciprocals are rationally linearly independent i.e. for any nonzero integer-valued vector $k = (k_1, k_2, \cdots, k_m)$,
$$k_1 \frac{1}{\tau_1} + k_2 \frac{1}{\tau_2} + \cdots + k_m \frac{1}{\tau_m} \neq 0.$$
This topic has been subject to many important studies including Kolmogorov-Arnold-Moser (KAM) theory on Hamiltonian systems ([21],[24],[2]).

Quasi-periodic motion is a common phenomenon in nature, e.g. arising in describing the movement of planets around the sun. The existence of a quasi-periodic motion for the nearly integrable regimes of the three-body problem with some transversality condition is given by the KAM theory. However many problems in nature are mixture of randomness and quasi-periodic motions. For example the temperature process which is random has one year periodicity due to the revolution of the earth around the sun and one day-night periodicity due to the rotation of the earth. Similarly, the energy demands should have similar nature. Thus to provide a rigorous mathematical theory is key in modelling random quasi-periodic phenomena in real world. As far as we know, such a concept still does not exist and the current paper is the first attempt in this direction.

The concepts of random periodic paths and periodic measures were introduced recently ([27],[10],[11],[9],[12]). They are two different indispensable ways to describe random periodicity. The theory has led to progress in the study of bifurcations ([26]), random attractors ([3]), stochastic resonance ([7],[13]), strange attractors ([16]) and modelling the El Nîno phenomenon ([6]).

In this paper, we study random quasi-periodicity of random dynamical systems or semi-flows over a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$. First we define random quasi-periodic path φ of the stochastic-flows $u(t,s) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d, t \geq s$ as a random path satisfying

$$u(t, s, \varphi(s, \omega), \omega) = \varphi(t, \omega), t \geq s, s \in \mathbb{R} \text{ a.s.},$$

and the pull-back random path

$$t \longmapsto \varphi(t, \theta_{-t}\omega)$$

is a quasi-periodic function for almost every sample path $\omega \in \Omega$.

For a Markovian semi-flow, let $p(t, s, x, \cdot), t \geq s$, be its transition probability. Then a measure-valued function $\rho : \mathbb{R} \to \mathcal{P}(\mathbb{R}^d)$ is called a quasi-periodic measure if ρ is an entrance measure i.e.

$$\int_{\mathbb{R}^d} P(t, s, x, \Gamma)\rho_s(dx) = \rho_t(\Gamma) \quad \text{for all } \Gamma \in \mathcal{B}(\mathbb{R}^d),$$

and the measure-valued map

$$s \longmapsto \rho_s$$

is a quasi-periodic function.

We will give a sufficient condition for the existence and uniqueness of random quasi-periodic path for a stochastic differential equation on $\mathbb{R}^d$

$$\begin{cases} dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW_t, & t \geq s, \\ X(s) = \xi, \end{cases} \tag{1.2}$$

where b, σ are quasi-periodic in the time variable t. As this is the first paper in this area, the main purpose here is to establish basic mathematical concepts and useful tools. We do not strike to technical details to try to provide best possible sufficient conditions in the current paper.

We will prove the law of random quasi-periodic path is a quasi-periodic measure. We further give a sufficient condition for the density of the quasi-periodic measure to exist and to satisfy the Fokker-Planck equation.

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For simplicity, we only consider quasi-periodicity with two periods: τ_1 and τ_2 in the current paper. Our results also apply to general cases with any periods $\tau_1, \tau_2, \cdots, \tau_m$ without any extra difficulties.

Solving the reparameterised SDE is a key step in the analysis of finding random quasi-periodic paths. Let $\tilde{b}, \tilde{\sigma}$ be two functions such that

$$\tilde{b}(t, t, x) = b(t, x), \tilde{\sigma}(t, t, x) = \sigma(t, x)$$

where $\tilde{b}(t_1, t_2, x), \tilde{\sigma}(t_1, t_2, x)$ are periodic in t_1, t_2 with periods τ_1 and τ_2 respectively. Define

$$\tilde{b}^{r_1, r_2}(t, x) = \tilde{b}(t + r_1, t + r_2, x)$$

$$\tilde{\sigma}^{r_1, r_2}(t, x) = \tilde{\sigma}(t + r_1, t + r_2, x),$$

then the solution K^{r_1, r_2} of SDE (1.2) when b, σ are replaced by $\tilde{b}^{r_1, r_2}, \tilde{\sigma}^{r_1, r_2}$, where r_1, r_2 are regarded as parameters, satisfies

$$K^{r, r}(t, s, x, \omega) = u(t + r, s + r, x, \theta_{-r}\omega)$$

where $u(t, s, \cdot, \omega)$ is the semi-flow generated by (1.2). Moreover we can prove under a dissipative condition about the drifts b and $\tilde{b}^{r_1, r_2}$,

$$\lim_{s \to -\infty} K^{r_1, r_2}(t, s, x, \omega) = \varphi^{r_1, r_2}(t, \omega) \text{ exists a.s.}$$

and

$$\varphi(r, \omega) = \varphi^{r, r}(0, \theta_{-r}\omega)$$

is a random quasi-periodic path of (1.2).

Note the reparamerterised SDE enjoys the following property: for all $r_1, r_2, r \in \mathbb{R}$, $t \geq s$,

$$K^{r_1, r_2}(t + r, s + r, x, \theta_{-r}\omega) = K^{r_1 + r, r_2 + r}(t, s, x, \omega), \ P - a.s. \text{ on } \omega. \tag{1.3}$$

This is a very useful observation in our analysis, but the original time dependent SDE (1.2) does not have such a convenient relation.

Lifting the semi-flow to $\tilde{\mathbb{X}} = [0, \tau_1) \times [0, \tau_2) \times \mathbb{R}^d$ is key to obtain an invariant measure from the quasi-periodic measure. Define

$$\tilde{\Phi}(t, \omega)(s_1, s_2, x) = (t + s_1 \mod \tau_1, \ t + s_2 \mod \tau_2, \ K^{s_1, s_2}(t, 0, x, \omega))$$

and

$$\tilde{Y}(s, \omega) = (s \mod \tau_1, \ s \mod \tau_2, \ \varphi(s, \omega)).$$

Then $\tilde{Y}$ is a random quasi-periodic path of the cocycle $\tilde{\Phi}$. Moreover we will prove that $\tilde{P}(t, (s_1, s_2, x), \tilde{\Gamma}) = P\{\omega : \tilde{\Phi}(t, \omega)(s_1, s_2, x) \in \tilde{\Gamma}\}, \tilde{\Gamma} \in \mathcal{B}(\tilde{\mathbb{X}})$ is Feller and

$$\tilde{\mu}_s(\tilde{\Gamma}) = P\{\omega : \tilde{Y}(s, \omega) \in \tilde{\Gamma}\} = [\delta_{s \mod \tau_1} \times \delta_{s \mod \tau_2} \times \rho_s](\tilde{\Gamma})$$

is a quasi-periodic measure with respect to $\tilde{P}^*$. We will show that

$$\{\bar{\tilde{\mu}}_T = \frac{1}{T} \int_0^T \tilde{\mu}_s ds : T \in \mathbb{R}^+\}$$

is tight and a weak limit $\bar{\tilde{\mu}}$ is an invariant measure with respect to $\tilde{P}^*$. Moreover, we will further show that the invariant measure is unique and ergodic and is given by the average

$$\frac{1}{\tau_1 \tau_2} \int_0^{\tau_1} \int_0^{\tau_2} \delta_{s_1} \times \delta_{s_2} \times \tilde{\rho}_{s_1, s_2} ds_1 ds_2.$$

2 Random path and entrance measure

2.1 Existence and uniqueness of random path

In the stochastic differential equation (1.2), $b : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$ are continuous functions, W_t is a two-sided $\mathbb{R}^d$-valued Brownian motion on probability space $(\Omega, \mathcal{F}, P)$ with $W_0 = 0$ and ξ is a $\mathbb{R}^d$-valued $\mathcal{F}^s_{-\infty}$-measurable random variable, where $\mathcal{F}^b_a$ is the natural filtration generated by $(W_u - W_v)_{a \leq u, v \leq b}$. Now we consider the following assumptions.

Condition 2.1. *The coefficients b, σ in SDE (1.2) satisfy the following conditions:*

(1) *There exist some $\alpha > 0$ such that for all $x, y \in \mathbb{R}^d$ and $t \in \mathbb{R}$,*

$$(x - y)(b(t, x) - b(t, y)) \leq -\alpha(x - y)^2;$$

(2) *There exists a constant $\beta > 0$ such that for all $x, y \in \mathbb{R}^d$ and $t \in \mathbb{R}$,*

$$\|\sigma(t, x) - \sigma(t, y)\| \leq \beta|x - y|;$$

(3) *There exists $M > 0$ such that*

$$\sup_{t \in \mathbb{R}} |b(t, 0)| + \sup_{t \in \mathbb{R}} \|\sigma(t, 0)\| \leq M;$$

Condition 2.2. *The drift coefficient b in SDE (1.2) is κ-th order growth in x for some $\kappa \geq 1$, i.e. there exist $l > 0$ such that for all $x \in \mathbb{R}^d$ and $t \in \mathbb{R}$,*

$$|b(t, x)| \leq l(1 + |x|^\kappa).$$

Condition 2.2 is needed only for the purpose of perfection. For other results such as (crude) random path and the results in terms of laws including the quasi-periodic measure, the invariant measure and its ergodicity, Condition 2.2 is not needed.

Under Condition 2.1, the solution of (1.2) exists, denoted by $X(t, s, \xi)$, and satisfies for $P - a.e.\ \omega \in \Omega$

$$X(t, s, \xi(\omega), \omega) = X(t, r, \omega) \circ X(r, s, \xi(\omega), \omega), \text{ for all } s \leq r \leq t.$$

We call $u : \Delta \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ with $u(t, s, \omega)x = X(t, s, x, \omega)$ a stochastic semi-flow, where $\Delta = \{(t, s) : t \geq s, t, s \in \mathbb{R}\}$.

Definition 2.3. *A random path of a semi-flow $u : \Delta \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ is a measurable map $\varphi : \mathbb{R} \times \Omega \to \mathbb{R}^d$ such that for any $t \geq s$,*

$$u(t, s, \varphi(s)) = \varphi(t), \ P - a.s.. \tag{2.1}$$

We call φ a perfect random path if equation (2.1) holds for all $t \geq s$, $P - a.s.$ (where the exceptional set N is independent of t and s). In addition, if u is generated by an SDE, we say φ is a (perfect) random path of this SDE.

In the following, we will always use $\|\cdot\|_p$ to denote the norm in the $L^p(\Omega, dP)$ space.

Theorem 2.4. *Assume Condition 2.1 and $\alpha > \frac{(p-1)\beta^2}{2}$ for some $p \geq 2$. Then there exists a unique uniformly L^p-bounded random path φ of SDE (1.2), i.e. $\sup_{t\in\mathbb{R}} \|\varphi(t)\|_p < \infty$. If we further assume Condition 2.2 and $p \geq (4+2d)\kappa$, this unique random path is perfect.*

First we give two lemmas before we prove Theorem 2.4.

Lemma 2.5. *Assume Condition 2.1 and $\alpha > \frac{(p-1)\beta^2}{2}$ for some $p \geq 2$. Let $X_t^{s,\xi}$ be the solution of SDE (1.2) with initial condition (s, ξ), where $\xi \in L^p(\Omega)$. Then there exists a constant $C = C(p, \alpha, \beta, M)$ such that for all $t \geq s$, $\|X_t^{s,\xi}\|_p^p \leq C(1 + \|\xi\|_p^p)$.*

Proof. We only prove this Lemma for $p > 2$, since the case $p = 2$ can be obtained by a similar way. For any fixed λ, applying Itô's formula to $e^{\lambda t}|X_t^{s,\xi}|^p$, we have

$$
e^{\lambda t}|X_t^{s,\xi}|^p = e^{\lambda s}|\xi|^p + \int_s^t e^{\lambda r}|X_r^{s,\xi}|^{(p-2)}\left(\lambda|X_r^{s,\xi}|^2 + pX_r^{s,\xi}\cdot b(r, X_r^{s,\xi}) + \frac{p(p-1)}{2}\|\sigma(r, X_r^{s,\xi})\|^2\right)dr
$$
$$
+ \int_s^t pe^{\lambda r}|X_r^{s,\xi}|^{(p-2)}X_r^{s,\xi}\sigma(r, X_r^{s,\xi})dW_r.
$$

In Condition 2.1, let $y = 0$. Then for arbitrary $\epsilon > 0$, by Young inequality and Condition 2.1

$$
\begin{aligned}
x \cdot b(t, x) &\leq -\alpha|x|^2 + x \cdot b(t, 0) \\
&\leq -(\alpha - \epsilon)|x|^2 + \frac{M^2}{4\epsilon},
\end{aligned}
$$

and

$$
\begin{aligned}
\|\sigma(t, x)\|^2 &\leq (\|\sigma(t, x) - \sigma(t, 0)\| + \|\sigma(t, 0)\|)^2 \\
&\leq (\beta|x| + \|\sigma(t, 0)\|)^2 \\
&\leq (\beta^2 + \epsilon)|x|^2 + (\frac{\beta^2}{\epsilon} + 1)M^2.
\end{aligned}
$$

Since $\alpha > \frac{(p-1)\beta^2}{2}$, we can choose ϵ small enough such that $\alpha > \frac{(p-1)\beta^2}{2} + 2\epsilon$ and $p(\alpha - \frac{(p-1)\beta^2}{2} - 2\epsilon) > \epsilon$. Let $\lambda = p(\alpha - \frac{(p-1)\beta^2}{2} - 2\epsilon) - \epsilon > 0$. Then ϵ, λ are constants depending on p, α, β. Thus there exists a constant $C(p, \alpha, \beta, M)$ depending on p, α, β, M such that

$$
\begin{aligned}
e^{\lambda t}|X_t^{s,\xi}|^p \leq & e^{\lambda s}|\xi|^p + \int_s^t e^{\lambda r}\left(-\epsilon|X_r^{s,\xi}|^p + C(p, \alpha, \beta, M)|X_r^{s,\xi}|^{(p-2)}\right)dr \\
& + \int_s^t pe^{\lambda r}|X_r^{s,\xi}|^{(p-2)}X_r^{s,\xi}\sigma(r, X_r^{s,\xi})dW_r,
\end{aligned}
$$

where $C(p, \alpha, \beta, M) = \frac{pM^2}{4\epsilon} + \frac{p(p-1)}{2}\left(\frac{\beta^2}{\epsilon} + 1\right)M^2$. Since $p - 2 > 0$, by Young inequality

$$
\begin{aligned}
C(p, \alpha, \beta, M)|X_r^{s,\xi}|^{(p-2)} &\leq \epsilon|X_r^{s,\xi}|^p + \frac{2}{p}C(p, \alpha, \beta, M)^{\frac{p}{2}}\left(\frac{\epsilon p}{p-2}\right)^{-\frac{p-2}{2}} \\
&= \epsilon|X_r^{s,\xi}|^p + C(p, \alpha, \beta, M).
\end{aligned}
$$

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Here and in the following, $C(p, \alpha, \beta, M)$ is constant, which may be different from line to line. Then we have

$$e^{\lambda t}|X_t^{s,\xi}|^p \leq e^{\lambda s}|\xi|^p + C(p, \alpha, \beta, M) \int_s^t e^{\lambda r} dr + \int_s^t pe^{\lambda r}|X_r^{s,\xi}|^{(p-2)}X_r^{s,\xi}\sigma(r, X_r^{s,\xi})dW_r$$

$$\leq e^{\lambda s}|\xi|^p + C(p, \alpha, \beta, M)e^{\lambda t} + \int_s^t pe^{\lambda r}|X_r^{s,\xi}|^{(p-2)}X_r^{s,\xi}\sigma(r, X_r^{s,\xi})dW_r.$$

Taking expectation of both sides, we have

$$e^{\lambda t}\|X_t^{s,\xi}\|_p^p \leq e^{\lambda s}\|\xi\|_p^p + C(p, \alpha, \beta, M)e^{\lambda t}.$$

Then

$$\|X_t^{s,\xi}\|_p^p \leq \|\xi\|_p^p + C(p, \alpha, \beta, M),$$

which implies the desired result. $\qquad\square$

Lemma 2.6. *Assume Condition 2.1 holds. Let $X_t^{s,\xi}$ and $X_t^{s,\eta}$ be two solutions of SDE (1.2) with initial values ξ and η respectively, where $\xi, \eta \in L^p(\Omega)$ for some $p > 1$. Then*

$$\|X_t^{s,\xi} - X_t^{s,\eta}\|_p \leq e^{-\left(\alpha - \frac{(p-1)\beta^2}{2}\right)(t-s)}\|\xi - \eta\|_p.$$

Proof. Note

$$X_t^{s,\xi} - X_t^{s,\eta} = \xi - \eta + \int_s^t \left(b(r, X_r^{s,\xi}) - b(r, X_r^{s,\eta})\right)dr + \int_s^t \left(\sigma(r, X_r^{s,\xi}) - \sigma(r, X_r^{s,\eta})\right)dW_r.$$

Let $\hat{X}_t := X_t^{s,\xi} - X_t^{s,\eta}$, $\hat{b}_t = b(t, X_t^{s,\xi}) - b(t, X_t^{s,\eta})$ and $\hat{\sigma}_t := \sigma(t, X_t^{s,\xi}) - \sigma(t, X_t^{s,\eta})$. For any fixed λ, applying Itô's formula to $e^{\lambda t}|\hat{X}_t|^p$, we have

$$e^{\lambda t}|\hat{X}_t|^p = e^{\lambda s}|\xi - \eta|^p + \int_s^t e^{\lambda r}|\hat{X}_r|^{(p-2)}\left(\lambda|\hat{X}_r|^2 + p\hat{X}_r \cdot \hat{b}_r + \frac{p(p-1)}{2}\|\hat{\sigma}\|^2\right)dr$$

$$+ \int_s^t pe^{\lambda r}|\hat{X}_r|^{(p-2)}\hat{X}_r\hat{\sigma}_r dW_r$$

$$\leq e^{\lambda s}|\xi - \eta|^p + \int_s^t e^{\lambda r}|\hat{X}_r|^p\left(\lambda - p\alpha + \frac{p(p-1)\beta^2}{2}\right)dr + \int_s^t pe^{\lambda r}|\hat{X}_r|^{(p-2)}\hat{X}_r\hat{\sigma}_r dW_r.$$

Let $\lambda = p\left(\alpha - \frac{(p-1)\beta^2}{2}\right)$. Taking expectation on both sides, we have

$$e^{\lambda t}\|X_t^{s,\xi} - X_t^{s,\eta}\|_p^p \leq e^{\lambda s}\|\xi - \eta\|_p^p.$$

Thus the lemma follows. $\qquad\square$

Now we give the proof of Theorem 2.4

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Proof of Theorem 2.4. Existence: Let $s_1 < s_2 < t$. Then for any fixed $\xi \in L^p(\Omega)$,

$$X_t^{s_1,\xi} = X_t^{s_2,X_{s_2}^{s_1,\xi}}.$$

Now consider $\|X_t^{s_1,\xi} - X_t^{s_2,\xi}\|_p$. Applying Lemma 2.5 and Lemma 2.6, we have

$$
\begin{aligned}
\|X_t^{s_1,\xi} - X_t^{s_2,\xi}\|_p =& \|X_t^{s_2,X_{s_2}^{s_1,\xi}} - X_t^{s_2,\xi}\|_p \\
\leq& e^{-\left(\alpha - \frac{(p-1)\beta^2}{2}\right)(t-s_2)}\|X_{s_2}^{s_1,\xi} - \xi\|_p \\
\leq& e^{-\left(\alpha - \frac{(p-1)\beta^2}{2}\right)(t-s_2)}\left(\|X_{s_2}^{s_1,\xi}\|_p + \|\xi\|_p\right) \\
\leq& C(p,\alpha,\beta,M,\|\xi\|_p)e^{-\left(\alpha - \frac{(p-1)\beta^2}{2}\right)(t-s_2)}.
\end{aligned}
$$

Thus there exists a L^p-limit of $\left(X_t^{s,\xi}\right)_{s\leq t}$ as $s \to -\infty$. By Lemma 2.6, we know that this limit is independent of ξ. Define

$$\varphi(t) := L^p - \lim_{s\to-\infty} X_t^{s,\xi}, \tag{2.2}$$

then

$$\|\varphi(t)\|_p \leq \limsup_{s\to-\infty}\|X_t^{s,\xi}\|_p \leq C(p,\alpha,\beta,M,\|\xi\|_p) \leq C(p,\alpha,\beta,M).$$

Next we will prove that φ is a random path of SDE (1.2). For any $t \geq s \geq r$, we have

$$u(t,s,X_s^{r,\xi}) = X_t^{r,\xi}, \ P - a.s..$$

By Lemma 2.6, we know that

$$\|u(t,s,X_s^{r,\xi}) - u(t,s,\varphi(s))\|_p \leq e^{-\left(\alpha - \frac{(p-1)\beta^2}{2}\right)(t-s)}\|X_s^{r,\xi} - \varphi(s)\|_p.$$

It follows that for all $t \geq s$

$$L^p - \lim_{r\to-\infty} u(t,s,X_s^{r,\xi}) = u(t,s,\varphi(s)) = \varphi(t) = L^p - \lim_{r\to-\infty} X_t^{r,\xi}, \ P - a.s.. \tag{2.3}$$

Hence φ is a random path of SDE (1.2).

Now under further Condition 2.2 and $p \geq (4+2d)\kappa$, by (i) and (iv) of Lemma 3.7, we know that the solution $u(t,s,x)$ of SDE (1.2) and $\varphi(t)$ are continuous with respect to (t,s,x) and t $P - a.s.$, respectively. Lemma 3.7 contains some key estimates needed for perfection. But in order not to interrupt the main flow of the proof of this theorem, we postpone this Lemma and its proof to the end of Section 3.1. Denote

$$N_{s,t} := \{\omega|u(t,s,\varphi(s,\omega),\omega) \neq \varphi(t,\omega)\}$$

$$N_u := \{\omega|u : (t,s,x) \mapsto u(t,s,x,\omega) \text{ is not continuous}\},$$

$$N_\varphi := \{\omega|\varphi : t \mapsto \varphi(t,\omega) \text{ is not continuous}\},$$

and

$$N = \bigcup_{t,s\in Q, t\geq s} N_{s,t} \bigcup N_u \bigcup N_\varphi$$

where Q is the set of all rational numbers. Since equation (2.3) holds, we know that $P(N) = 0$. Fix $\omega \in N^c$, for any $t \geq s$, we choose $\{t_n, s_n\}_{n \geq 1}$ such that $t_n \geq s_n, t_n, s_n \in Q$ and $t_n \to t, s_n \to s$, by continuity of $u(\cdot, \cdot, \cdot, \omega), \varphi(\cdot, \omega)$, we have

$$u(t, s, \varphi(s, \omega), \omega) = \lim_{n \to \infty} u(t_n, s_n, \varphi(s_n, \omega), \omega) = \lim_{n \to \infty} \varphi(t_n, \omega) = \varphi(t, \omega).$$

Thus φ is a uniformly L^p-bounded perfect random path of SDE (1.2).

Uniqueness: If there are two uniformly L^p-bounded random paths φ_1, φ_2 of SDE (1.2), by Lemma 2.6, we have for any $t \in \mathbb{R}$

$$\begin{aligned}
\|\varphi_1(t) - \varphi_2(t)\|_p &\leq e^{-\left(\alpha - \frac{(p-1)\beta^2}{2}\right)(t-s)} \|\varphi_1(s) - \varphi_2(s)\|_p \\
&\leq e^{-\left(\alpha - \frac{(p-1)\beta^2}{2}\right)(t-s)} (\sup_{r \in \mathbb{R}} \|\varphi_1(r)\|_p + \sup_{r \in \mathbb{R}} \|\varphi_2(r)\|_p) \to 0 \text{ as } s \to -\infty.
\end{aligned}$$

Then $\varphi_1(t) = \varphi_2(t)$, $P - a.s.$.

If there are two uniformly L^p-bounded perfect random paths φ_1, φ_2 of SDE (1.2), denote

$$N_{\varphi_1} := \{\omega | u(t, s, \varphi_1(s, \omega), \omega) \neq \varphi_1(s, \omega), \text{ for some } t \geq s \in \mathbb{R}\},$$

and

$$N_{\varphi_2} := \{\omega | u(t, s, \varphi_2(s, \omega), \omega) \neq \varphi_2(s, \omega), \text{ for some } t \geq s \in \mathbb{R}\}.$$

Since φ_1, φ_2 are random paths of SDE (1.2), by Definition 2.1, we have $P(N_{\varphi_1}) = P(N_{\varphi_2}) = 0$. Let $N_t = \{\omega | \varphi_1(t, \omega) \neq \varphi_2(t, \omega)\}$ and

$$N^0 = \bigcup_{n \geq 1} N_{-n} \bigcup N_{\varphi_1} \bigcup N_{\varphi_2},$$

we obtain $P(N^0) = 0$. Similarly fix $\omega \in (N^0)^c$, then for any $t \in \mathbb{R}$, choose $n \geq t$, we have

$$\varphi_1(t, \omega) = u(t, -n, \varphi_1(-n, \omega), \omega) = u(t, -n, \varphi_2(-n, \omega), \omega) = \varphi_2(t, \omega),$$

which means $P - a.e.$ $\omega \in \Omega$,

$$\varphi_1(t, \omega) = \varphi_2(t, w), \text{ for all } t \in \mathbb{R}.$$

$\square$

2.2 Existence and uniqueness of entrance measure

For a semi-flow $u : \triangle \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ with $u(t, s, x, \omega) = X_t^{s,x}(\omega)$, we define the transition $P : \triangle \times \mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d) \to \mathbb{R}^+$ by $P(t, s, x, \Gamma) = P(X_t^{s,x} \in \Gamma)$ for all $t \geq s$, $x \in \mathbb{R}^d$ and $\Gamma \in \mathcal{B}(\mathbb{R}^d)$. We further define $P^*(t, s) : \mathcal{P}(\mathbb{R}^d) \to \mathcal{P}(\mathbb{R}^d)$ by

$$P^*(t, s)\mu(\Gamma) = \int_{\mathbb{R}^d} P(t, s, x, \Gamma)\mu(dx), \text{ for all } \mu \in \mathcal{P}(\mathbb{R}^d), \Gamma \in \mathcal{B}(\mathbb{R}^d). \tag{2.4}$$

Here

$$\mathcal{P}(\mathbb{R}^d) := \{\text{all probability measures on } (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\}.$$

Definition 2.7. *We say a measure-valued map $\mu : \mathbb{R} \to \mathcal{P}(\mathbb{R}^d)$ is an entrance measure of SDE(1.2) if $P^*(t,s)\mu_s = \mu_t$ for all $t \geq s, s \in \mathbb{R}$.*

Set

$$\mathcal{M}^p := \{\mu : \mathbb{R} \to \mathcal{P}(\mathbb{R}^d) | \sup_{t \in \mathbb{R}} \int_{\mathbb{R}^d} |x|^p \mu_t(dx) < \infty\}.$$

Theorem 2.8. *Assume Condition 2.1 and $\alpha > \frac{(p-1)\beta^2}{2}$ for some $p \geq 2$. Then there exists a unique entrance measure of SDE (1.2) in $\mathcal{M}^p$.*

To prove Theorem 2.8, we need the following lemma.

Lemma 2.9. *Assume μ_1 and μ_2 are two probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, and for any open set $\mathcal{O}$ we have $\mu_1(\mathcal{O}) \leq \mu_2(\mathcal{O})$. Then $\mu_1 = \mu_2$.*

Proof. Let $\mathcal{C} := \{$all open sets on $\mathbb{R}^d\}$. First we know we know that $\mu_1 \leq \mu_2$ on $\mathcal{C}$. We now prove the opposite inequality. For any given $\mathcal{O} \in \mathcal{C}$, $\mathcal{O}^c = \mathbb{R}^d \setminus \mathcal{O}$ is a closed set. Define

$$\mathcal{O}_\delta^c := \{x : dist(x, \mathcal{O}^c) < \delta\},$$

where $dist(x, \mathcal{O}^c) = \inf_{y \in \mathcal{O}^c} |x - y|$. Then we know that $\mathcal{O}_\delta^c$ is open set and $\mathcal{O}_\delta^c \downarrow \mathcal{O}^c$ as $\delta \downarrow 0$. Further more

$$\mu_1(\mathcal{O}^c) = \lim_{\delta \downarrow 0} \mu_1(\mathcal{O}_\delta^c) \leq \lim_{\delta \downarrow 0} \mu_2(\mathcal{O}_\delta^c) = \mu_2(\mathcal{O}^c).$$

Since μ_1 and μ_2 are probability measures, we have

$$1 - \mu_1(\mathcal{O}) \leq 1 - \mu_2(\mathcal{O}),$$

which implies $\mu_1(\mathcal{O}) \geq \mu_2(\mathcal{O})$. Hence $\mu_1 \geq \mu_2$ on $\mathcal{C}$. This leads to $\mu_1 = \mu_2$ on $\mathcal{C}$.

Since $\mathcal{C}$ is a π-system and $\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R}^d)$, thus $\mu_1 = \mu_2$ on $\mathcal{B}(\mathbb{R}^d)$. $\qquad\square$

Now we give the proof of Theorem 2.8.

Proof of Theorem 2.8. Existence: Applying Theorem 2.4, we know that there exists a uniformly L^p-bounded random path φ of SDE (1.2). Let $\rho_t = \mathcal{L}(\varphi(t))$ be the law of $\varphi(t)$. Then for any $\Gamma \in \mathcal{B}(\mathbb{R}^d)$, we have

$$
\begin{aligned}
P^*(t,s)\rho_s(\Gamma) &= \int_{\mathbb{R}^d} P(t,s,x,\Gamma)\rho_s(dx) \\
&= \int_{\mathbb{R}^d} P(X_t^{s,x} \in \Gamma)P(\varphi(s) \in dx) \\
&= P(X_t^{s,\varphi(s)} \in \Gamma) \\
&= P(\varphi(t) \in \Gamma) \\
&= \rho_t(\Gamma).
\end{aligned}
\tag{2.5}
$$

Thus ρ is an entrance measure of SDE (1.2). And since φ is uniformly L^p-bounded, then

$$\sup_{t \in \mathbb{R}} \int_{\mathbb{R}^d} |x|^p \rho_t(dx) = \sup_{t \in \mathbb{R}} \mathbb{E}[|\varphi(t)|^p] < \infty,$$

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which means $\rho \in \mathcal{M}^p$.

Uniqueness: We aim to prove that for any entrance measure μ of SDE (1.2) in $\mathcal{M}^p$, $\mu_t = \rho_t$ for all $t \in \mathbb{R}$. By Lemma 2.9, we just need to prove $\rho_t(\mathcal{O}) \leq \mu_t(\mathcal{O})$ for any open set $\mathcal{O} \subset \mathbb{R}^d$. Since for any $s < t$, we have

$$
\begin{aligned}
\rho_t(\mathcal{O}) - \mu_t(\mathcal{O}) &= \rho_t(\mathcal{O}) - \int_{\mathbb{R}^d} P(t,s,x,\mathcal{O}) \mu_s(dx) \\
&= \int_{\mathbb{R}^d} \left(\rho_t(\mathcal{O}) - P(X_t^{s,x} \in \mathcal{O}) \right) \mu_s(dx) \\
&= \int_{\mathbb{R}^d} \left(P(\varphi(t) \in \mathcal{O}) - P(X_t^{s,x} \in \mathcal{O}) \right) \mu_s(dx).
\end{aligned}
$$

Define

$$
\mathcal{O}_\delta := \{ x : dist(x, \mathcal{O}^c) > \delta \}.
$$

Then $\mathcal{O}_\delta \uparrow \mathcal{O}$ as $\delta \downarrow 0$ and

$$
\begin{aligned}
P(X_t^{s,x} \in \mathcal{O}) &= P(X_t^{s,x} - \varphi(t) + \varphi(t) \in \mathcal{O}) \\
&\geq P(\varphi(t) \in \mathcal{O}_\delta, |X_t^{s,x} - \varphi(t)| < \delta) \\
&\geq P(\varphi(t) \in \mathcal{O}_\delta) - P(|X_t^{s,x} - \varphi(t)| \geq \delta).
\end{aligned}
$$

Thus it turns out from the above and the Chebyshev inequality that

$$
\begin{aligned}
&P(\varphi(t) \in \mathcal{O}) - P(X_t^{s,x} \in \mathcal{O}) \\
&\leq P(\varphi(t) \in \mathcal{O} \setminus \mathcal{O}_\delta) + P(|X_t^{s,x} - \varphi(t)| \geq \delta) \\
&\leq P(\varphi(t) \in \mathcal{O} \setminus \mathcal{O}_\delta) + \frac{1}{\delta^p} \mathbb{E}[|X_t^{s,x} - \varphi(t)|^p].
\end{aligned}
$$

Applying Lemma 2.5 and Lemma 2.6, we have

$$
\begin{aligned}
\mathbb{E}[|X_t^{s,x} - \varphi(t)|^p] &= \lim_{r \to -\infty} \mathbb{E}[|X_t^{s,x} - X_t^{r,x}|^p] \\
&\leq \limsup_{r \to -\infty, r < s} \mathbb{E}[|X_t^{s,x} - X_t^{s,X_s^{r,x}}|^p] \\
&\leq \limsup_{r \to -\infty, r < s} e^{-p\left(\alpha - \frac{(p-1)\beta^2}{2}\right)(t-s)} \mathbb{E}[|x - X_s^{r,x}|^p] \\
&\leq \limsup_{r \to -\infty, r < s} C(1 + |x|^p) e^{-p\left(\alpha - \frac{(p-1)\beta^2}{2}\right)(t-s)} \\
&= C(1 + |x|^p) e^{-p\left(\alpha - \frac{(p-1)\beta^2}{2}\right)(t-s)}.
\end{aligned}
$$

Here $C = C(p, \alpha, \beta, M)$. Then for any $\delta > 0$ and $s < t$, we have

$$
\begin{aligned}
\rho_t(\mathcal{O}) - \mu_t(\mathcal{O}) &= \int_{\mathbb{R}^d} \left(P(\varphi(t) \in \mathcal{O}) - P(X_t^{s,x} \in \mathcal{O}) \right) \mu_s(dx) \\
&\leq \int_{\mathbb{R}^d} \left(P(\varphi(t) \in \mathcal{O} \setminus \mathcal{O}_\delta) + \frac{1}{\delta^p} \mathbb{E}[|X_t^{s,x} - \varphi(t)|^p] \right) \mu_s(dx) \\
&\leq P(\varphi(t) \in \mathcal{O} \setminus \mathcal{O}_\delta) + \frac{C}{\delta^p} e^{-p\left(\alpha - \frac{(p-1)\beta^2}{2}\right)(t-s)} \int_{\mathbb{R}^d} (1 + |x|^p) \mu_s(dx).
\end{aligned}
$$

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Hence for any $\delta > 0$, we have

$$\rho_t(\mathcal{O}) - \mu_t(\mathcal{O}) \leq P(\varphi(t) \in \mathcal{O} \setminus \mathcal{O}_\delta) + \limsup_{s \to -\infty} \frac{C}{\delta^p} \left(1 + \sup_{r \in \mathbb{R}} \int_{\mathbb{R}^d} |x|^p \mu_r(dx) \right) e^{-p\left(\alpha - \frac{(p-1)\beta^2}{2}\right)(t-s)}$$

$$\leq P(\varphi(t) \in \mathcal{O} \setminus \mathcal{O}_\delta) = \rho_t(\mathcal{O} \setminus \mathcal{O}_\delta).$$

Since $\mathcal{O}_\delta \uparrow \mathcal{O}$ as $\delta \downarrow 0$, we have

$$\rho_t(\mathcal{O}) - \mu_t(\mathcal{O}) \leq \lim_{\delta \downarrow 0} \rho_t(\mathcal{O} \setminus \mathcal{O}_\delta) = 0,$$

which implies $\rho_t(\mathcal{O}) \leq \mu_t(\mathcal{O})$. $\qquad\square$

Remark 2.10. *When we consider the entrance measure, we only consider the law of random path and the perfection of φ is not needed, thus the continuity of φ is not needed and hence we do not need Condition 2.2. Then the estimates for $p = 2$ in Section 2.1 is adequate.*

By the proof of Theorem 2.4, we know that $\varphi(t) = L^p - \lim_{s \to -\infty} X_t^{s,x}$. Then we have the following proposition. Denote by $C_b(\mathbb{R}^d)$ the linear space of all continuous and bounded functions on $\mathbb{R}^d$.

Proposition 2.11. *The entrance measure ρ_t is the limit of $P(t, s, x, \cdot)$ in $\mathcal{P}(\mathbb{R}^d)$ with weak topology, i.e. for all $f \in C_b(\mathbb{R}^d)$, we have*

$$\lim_{s \to -\infty} \int_{\mathbb{R}^d} f(y) P(t, s, x, dy) = \int_{\mathbb{R}^d} f(y) \rho_t(dy).$$

Proof. Since $\int_{\mathbb{R}^d} f(y) P(t, s, x, dy) = \mathbb{E} f(X_t^{s,x})$ and $\int_{\mathbb{R}^d} f(y) \rho_t(dy) = \mathbb{E} f(\varphi(t))$, we need to prove that for all $f \in C_b(\mathbb{R}^d)$,

$$\lim_{s \to -\infty} \mathbb{E} f(X_t^{s,x}) = \mathbb{E} f(\varphi(t)).$$

First we prove $\limsup_{s \to -\infty} \mathbb{E} f(X_t^{s,x}) \leq \mathbb{E} f(\varphi(t))$. Otherwise there exists a sequence $s_n \downarrow -\infty$ as $n \to \infty$ and a constant $\lambda = \limsup_{s \to -\infty} \mathbb{E} f(X_t^{s,x}) > \mathbb{E} f(\varphi(t))$ such that $\lim_{n \to \infty} \mathbb{E} f(X_t^{s_n,x}) = \lambda$. Since $\lim_{n \to \infty} \mathbb{E}[|X_t^{s_n,x} - \varphi(t)|^p] = 0$, we know that there exists a subsequence $\{s_{n_k}\} \subseteq \{s_n\}$ such that $X_t^{s_{n_k},x} \xrightarrow{a.s.} \varphi(t)$ as $k \to \infty$. Thus $f(X_t^{s_{n_k},x}) \xrightarrow{a.s.} f(\varphi(t))$. Then by Lebesgue's dominated convergence theorem, we have

$$\lim_{k \to \infty} \mathbb{E} f(X_t^{s_{n_k},x}) = \mathbb{E} f(\varphi(t)),$$

which contradicts that

$$\lim_{k \to \infty} \mathbb{E} f(X_t^{s_{n_k},x}) = \lambda > \mathbb{E} f(\varphi(t)).$$

Hence

$$\limsup_{s \to -\infty} \mathbb{E} f(X_t^{s,x}) \leq \mathbb{E} f(\varphi(t)).$$

Similarly we can also prove that

$$\liminf_{s \to -\infty} \mathbb{E} f(X_t^{s,x}) \geq \mathbb{E} f(\varphi(t)),$$

which completes our proof. $\qquad\square$

3 Random quasi-periodic path, quasi-periodic measure and invariant measure

3.1 Existence and uniqueness of random quasi-periodic path

In SDE (1.2), if we assume the coefficients b, σ are quasi-periodic functions in time t, can we obtain a kind of random quasi-periodic path? What should the "quasi-periodicity" of a random path be defined? We give the following definition.

Definition 3.1. *A measurable path $\varphi : \mathbb{R} \times \Omega \to \mathbb{R}^d$ is called random quasi-periodic path of periods τ_1, τ_2 of a semi-flow u, where the reciprocals of τ_1 and τ_2 are rationally linearly independent, if it is a random path and there exists $\tilde{\varphi} : \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}^d$ such that for any $t, s \in \mathbb{R}$,*

$$\begin{cases} \tilde{\varphi}(t, t) = \varphi(t) \circ \theta_{-t}, \ P - a.s., \\ \tilde{\varphi}(t + \tau_1, s) = \tilde{\varphi}(t, s), \ \tilde{\varphi}(t, s + \tau_2) = \tilde{\varphi}(t, s), \ P - a.s.. \end{cases} \tag{3.1}$$

We call φ a perfect random quasi-periodic path if φ is a perfect random path and equation (3.1) holds for all $t, s \in \mathbb{R}$, $P - a.s.$. We also say φ is a (perfect) random quasi-periodic path of an SDE if u is generated by this SDE.

We give the quasi-periodic condition.

Condition 3.2. *Assume that b, σ in SDE (1.2) are quasi-periodic functions with periods τ_1, τ_2, where the reciprocals of τ_1 and τ_2 are rationally linearly independent, which means there exists $\tilde{b} : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ and $\tilde{\sigma} : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^{d \times n}$ such that $\tilde{b}(t, t, x) = b(t, x)$, $\tilde{\sigma}(t, t, x) = \sigma(t, x)$ for all $t \in \mathbb{R}, x \in \mathbb{R}^d$ satisfying*

$$\tilde{b}(t + \tau_1, s, x) = \tilde{b}(t, s, x), \ \tilde{b}(t, s + \tau_2, x) = \tilde{b}(t, s, x), \tag{3.2}$$

and

$$\tilde{\sigma}(t + \tau_1, s, x) = \tilde{\sigma}(t, s, x), \ \tilde{\sigma}(t, s + \tau_2, x) = \tilde{\sigma}(t, s, x). \tag{3.3}$$

Condition 3.3. *Assume $\tilde{b}, \tilde{\sigma}$ in Condition 3.2 satisfy the following conditions:*

(1) *There exist some $\alpha > 0$ such that for all $x, y \in \mathbb{R}^d$ and $t, s \in \mathbb{R}$,*

$$(x - y) \left(\tilde{b}(t, s, x) - \tilde{b}(t, s, y) \right) \leq -\alpha(x - y)^2;$$

(2) *There exists a constant $\beta > 0$ such that for all $x, y \in \mathbb{R}^d$ and $t, s \in \mathbb{R}$,*

$$\|\tilde{\sigma}(t, s, x) - \tilde{\sigma}(t, s, y)\| \leq \beta |x - y|;$$

(3) *There exists $M > 0$ such that*

$$\sup_{t, s \in \mathbb{R}} |\tilde{b}(t, s, 0)| + \sup_{t, s \in \mathbb{R}} \|\tilde{\sigma}(t, s, 0)\| \leq M;$$

(4) *There exist $C > 0, 0 < \gamma \leq 1$ such that for all $x \in \mathbb{R}^d$, $t_1, t_2, s_1, s_2 \in \mathbb{R}$,*

$$|\tilde{b}(t_1, s_1, x) - \tilde{b}(t_2, s_2, x)| + \|\tilde{\sigma}(t_1, s_1, x) - \tilde{\sigma}(t_2, s_2, x)\| \leq C(|t_1 - t_2|^{\gamma} + |s_1 - s_2|^{\gamma}).$$

Condition 3.4. *We also assume $\tilde{b}$ in Condition 3.2 is κ-th order growth in x for some $\kappa \geq 1$, i.e. there exist $l > 0$ such that for all $x, y \in \mathbb{R}^d$ and $t, s \in \mathbb{R}$,*

$$|\tilde{b}(t, s, x)| \leq l(1 + |x|^{\kappa}).$$

Again Condition 3.3 is only needed for perfection. Note that Conditions 3.2, 3.3 and 3.4 imply Condition 2.1 and 2.2. Now we give the following main theorem.

Theorem 3.5. *Assume Conditions 3.2, 3.3 and $\alpha > \frac{(p-1)\beta^2}{2}$ for some $p \geq 2$. Then there exists a unique uniformly L^p-bounded random quasi-periodic path of SDE (1.2). If we further assume Condition 3.4 and $p \geq (4 + 2d)\kappa \vee \frac{2\kappa}{\gamma}$, this unique random quasi-periodic path is perfect.*

Proof. Uniqueness: Applying Theorem 2.4, we know that if there exists a uniformly L^p-bounded (perfect) random quasi-periodic path, it must be the (perfect) random path φ defined in Theorem 2.4. So uniqueness holds.

Existence: We aim to show that the random path φ in Theorem 2.4 is indeed a random quasi-periodic path. Note that the solution of SDE (1.2) $u(t, s, x)$ can be written as for all $s \leq t, x \in \mathbb{R}^d$

$$u(t, s, x) = x + \int_s^t b(r, u(r, s, x))dr + \int_s^t \sigma(r, u(r, s, x))dW_r, \ P - a.s.$$

Then similar to [1], for a fixed real number r, by the measure preserving property of θ_{-r},

$$
\begin{aligned}
& u(t + r, s + r, x) \circ \theta_{-r} \\
&= \left(x + \int_{s+r}^{t+r} b(v, u(v, s + r, x))dv + \int_{s+r}^{t+r} \sigma(v, u(v, s + r, x))dW_v \right) \circ \theta_{-r} \\
&= x + \left(\int_{s+r}^{t+r} b(v, u(v, s + r, x))dv \right) \circ \theta_{-r} + \left(\int_{s+r}^{t+r} \sigma(v, u(v, s + r, x))dW_v \right) \circ \theta_{-r} \\
&= x + \int_s^t b(v + r, u(v + r, s + r, x) \circ \theta_{-r})dv + \int_s^t \sigma(v + r, u(v + r, s + r, x) \circ \theta_{-r})dW_v \\
&= x + \int_s^t \tilde{b}(v + r, v + r, u(v + r, s + r, x) \circ \theta_{-r})dv \\
&\quad + \int_s^t \tilde{\sigma}(v + r, v + r, u(v + r, s + r, x) \circ \theta_{-r})dW_v, \ P - a.s.
\end{aligned}
\tag{3.4}
$$

Denote $u^r(t, s, x) := u(t + r, s + r, x) \circ \theta_{-r}$, $\tilde{b}^{r_1, r_2}(t, x) := \tilde{b}(t + r_1, t + r_2, x)$ and $\tilde{\sigma}^{r_1, r_2}(t, x) := \tilde{\sigma}(t + r_1, t + r_2, x)$, then for any fixed r, equation (3.4) can be written as

$$u^r(t, s, x) = x + \int_s^t \tilde{b}^{r,r}(v, u^r(v, s, x))dv + \int_s^t \tilde{\sigma}^{r,r}(v, u^r(v, s, x))dW_v, \ P - a.s. \tag{3.5}$$

Note that the null sets in equations (3.4) and (3.5) depend on t, s, r, x. Since Condition 3.3 holds, then for all $r_1, r_2 \in \mathbb{R}$, $\tilde{b}^{r_1, r_2}$ and $\tilde{\sigma}^{r_1, r_2}$ satisfy

$$(x - y)\left(\tilde{b}^{r_1, r_2}(t, x) - \tilde{b}^{r_1, r_2}(t, y) \right) \leq -\alpha(x - y)^2,$$

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and
$$\|\tilde{\sigma}^{r_1,r_2}(t,x) - \tilde{\sigma}^{r_1,r_2}(t,y)\| \le \beta|x-y|,$$

for all $t \in \mathbb{R}$, $x,y \in \mathbb{R}^d$. Thus the following equation

$$K^{r_1,r_2}(t,s,x) = x + \int_s^t \tilde{b}^{r_1,r_2}(v, K^{r_1,r_2}(v,s,x))dv + \int_s^t \tilde{\sigma}^{r_1,r_2}(v, K^{r_1,r_2}(v,s,x))dW_v, \qquad (3.6)$$

has a unique solution, denoted by $K^{r_1,r_2}(t,s,x)$, where $r_1, r_2 \in \mathbb{R}$ are regarded as parameters. Since $\alpha > \frac{(p-1)\beta^2}{2}$, similar to the proof of Theorem 2.4, we know that there exist the random paths $\varphi^r(t), \varphi^{r_1,r_2}(t)$ of u^r, K^{r_1,r_2} respectively such that for all $r, r_1, r_2, t \in \mathbb{R}$

$$\begin{cases} \varphi^r(t) = L^p - \lim_{s\to-\infty} u^r(t,s,x) = L^p - \lim_{s\to-\infty} u(t+r,s+r,x) \circ \theta_{-r} \\ \varphi^{r_1,r_2}(t) = L^p - \lim_{s\to-\infty} K^{r_1,r_2}(t,s,x). \end{cases} \qquad (3.7)$$

Since φ is the unique random path of SDE (1.2), by Theorem 2.4 we have for all $r,t \in \mathbb{R}$

$$\varphi^r(t) = \varphi(t+r) \circ \theta_{-r}, \ P-a.s.. \qquad (3.8)$$

Comparing (3.5) and (3.6), obviously we know that for all $t \ge s, s, r \in \mathbb{R}, x \in \mathbb{R}^d$

$$K^{r,r}(t,s,x) = u^r(t,s,x), \ P-a.s. \qquad (3.9)$$

and thus for all $r,t \in \mathbb{R}$,
$$\varphi^{r,r}(t) = \varphi^r(t), P-a.s.. \qquad (3.10)$$

By quasi-periodicity of $\tilde{b}$ and $\tilde{\sigma}$, we know that $\tilde{b}^{r_1+\tau_1,r_2} = \tilde{b}^{r_1,r_2} = \tilde{b}^{r_1,r_2+\tau_2}, \tilde{\sigma}^{r_1+\tau_1,r_2} = \tilde{\sigma}^{r_1,r_2} = \tilde{\sigma}^{r_1,r_2+\tau_2}$. Thus it turns out that for all $t \ge s, s, r_1, r_2 \in \mathbb{R}, x \in \mathbb{R}^d$

$$K^{r_1+\tau_1,r_2}(t,s,x) = K^{r_1,r_2}(t,s,x) = K^{r_1,r_2+\tau_2}(t,s,x), \ P-a.s..$$

Then for all $t, r_1, r_2 \in \mathbb{R}$

$$\varphi^{r_1+\tau_1,r_2}(t) = \varphi^{r_1,r_2}(t) = \varphi^{r_1,r_2+\tau_2}(t), \ P-a.s.. \qquad (3.11)$$

For all $t, s \in \mathbb{R}$, let
$$\tilde{\varphi}(t,s,\omega) := \varphi^{t,s}(0,\omega). \qquad (3.12)$$

It follows from (3.8), (3.10) and (3.11) that for all $t \in \mathbb{R}$

$$\tilde{\varphi}(t,t) = \varphi(t) \circ \theta_{-t}, \ P-a.s., \qquad (3.13)$$

and for all $t, s \in \mathbb{R}$

$$\tilde{\varphi}(t+\tau_1,s) = \tilde{\varphi}(t,s), \tilde{\varphi}(t,s+\tau_2) = \tilde{\varphi}(t,s), \ P-a.s.. \qquad (3.14)$$

Therefore, this random path φ is a random quasi-periodic path.

Now with the Condition 3.4 and assumption $p\gamma \ge 2\kappa$, by $(v-vi)$ in Lemma 3.7, $\varphi(t) \circ \theta_{-t}$ and $\tilde{\varphi}(t,s)$ are $P-a.s.$ continuous with respect to t and (t,s), respectively. Then use the same perfection argument as in equation (2.3), we know that equations (3.13) and (3.14) hold for all $t \in \mathbb{R}$ and $t,s \in \mathbb{R}$ outside a null set, which ends the proof. $\qquad \square$

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Remark 3.6. *We can conduct similar operations as in (3.4) and (3.5) to re-parameterised equation (3.6). Noticing*

$$\tilde{b}^{r_1,r_2}(v+r,\cdot) = \tilde{b}^{r_1+r,r_2+r}(v,\cdot), \quad \tilde{\sigma}^{r_1,r_2}(v+r,\cdot) = \tilde{\sigma}^{r_1+r,r_2+r}(v,\cdot) \tag{3.15}$$

and using the same argument as in the proof of (3.9), we can conclude important property (1.3). This property is similar to the shift property of the autonomous stochastic differential equations which leads to their cocycle property with a perfection argument. Though there is nothing similar to be said about the original SDEs due to the time dependency of the coefficients, this property holds due to "time-invariance" of the re-parameterised coefficients in the sense of (3.15).

We state the following continuity lemma which was needed in the proof of Theorem 2.4 and Theorem 3.5. It is noted that its proof is independent of Theorem 2.4 and Theorem 3.5 and their proofs.

Lemma 3.7. *Assume Conditions 2.1, 3.2, 3.3 and 3.4 hold. Let $u(t,s,x), u^r(t,s,x), K^{r_1,r_2}(t,s,x)$ are the solutions of SDE (1.2), SDE (3.5) and (3.6), respectively. Assume further that $\alpha > \frac{(p-1)\beta^2}{2}$ for some constant $p \geq 2$, we have*

(i) *If $p \geq (4+2d)\kappa$, $u(t,s,x)$ is continuous with respect to (t,s,x), $P-a.s.$;*

(ii) *If $p\gamma \wedge \frac{p}{2} \geq (3+d)\kappa$, $u^r(t,s,x)$ is continuous with respect to (r,t,s,x), $P-a.s.$;*

(iii) *If $p\gamma \wedge \frac{p}{2} \geq (4+d)\kappa$, $K^{r_1,r_2}(t,s,x)$ is continuous with respect to (r_1,r_2,t,s,x), $P-a.s..$*

(iv) *If $p \geq 2\kappa$, $\varphi(t)$ defined in (2.2) is continuous with respect to t, $P-a.s..$*

(v) *If $p\gamma \geq \kappa$, $\varphi(t) \circ \theta_{-t}$ is continuous with respect to t, $P-a.s..$*

(vi) *If $p\gamma \geq 2\kappa$, $\tilde{\varphi}(t,s)$ defined in (3.12) is continuous with respect to (t,s), $P-a.s..$*

(vii) *If $p\gamma \wedge \frac{p}{2} \geq 2\kappa$, $\varphi^r(t)$ defined in (3.7) is continuous with respect to (r,t), $P-a.s..$*

(viii) *If $p\gamma \wedge \frac{p}{2} \geq 3\kappa$, $\varphi^{r_1,r_2}(t)$ defined in (3.7) is continuous with respect to (r_1,r_2,t), $P-a.s..$*

Proof. (i) Note that

$$u(t,s,x) = x + \int_s^t b(v,u(v,s,x))dv + \int_s^t \sigma(v,u(v,s,x))dW_v.$$

For any $t > t' > s$ with $|t-t'| < 1$, we have

$$\begin{aligned}
&E\left[|u(t,s,x) - u(t',s,x)|^{\frac{p}{\kappa}}\right] \\
&= E\left[\left|\int_{t'}^t b(v,u(v,s,x))dv + \int_{t'}^t \sigma(v,u(v,s,x))dW_v\right|^{\frac{p}{\kappa}}\right] \\
&\leq C(p,\kappa)\left(E\left[\left|\int_{t'}^t b(v,u(v,s,x))dv\right|^{\frac{p}{\kappa}}\right] + E\left[\left|\int_{t'}^t \sigma(v,u(v,s,x))dW_v\right|^{\frac{p}{\kappa}}\right]\right) \\
&=: C(p,\kappa)(I + II).
\end{aligned} \tag{3.16}$$

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Since $\alpha > \frac{(p-1)\beta^2}{2}$, by Lemma 2.5, we know that for all $v \geq s$

$$E[|u(v,s,x)|^p] \leq C(p,\alpha,\beta,M)(1+|x|^p).$$

Then we can calculate

$$\begin{aligned}
I &\leq E\left[\int_{t'}^{t} |b(v,u(v,s,x))|^{\frac{p}{\kappa}} dv \cdot |t-t'|^{(\frac{p}{\kappa}-1)}\right] \\
&\leq C(p,\kappa,l)\int_{t'}^{t} E[1+|u(v,s,x)|^p]dv \cdot |t-t'|^{(\frac{p}{\kappa}-1)} \\
&\leq C(p,\kappa,l,\alpha,\beta,M)(1+|x|^p)|t-t'|^{\frac{p}{\kappa}}.
\end{aligned} \tag{3.17}$$

By B-D-G inequality, we have

$$\begin{aligned}
II &\leq C(p,\kappa)E\left[\left(\int_{t'}^{t}\|\sigma(v,u(v,s,x))\|^2 dv\right)^{\frac{p}{2\kappa}}\right] \\
&\leq C(p,\kappa)\int_{t'}^{t} E[\|\sigma(v,u(v,s,x))\|^{\frac{p}{\kappa}}]dv \cdot |t-t'|^{(\frac{p}{2\kappa}-1)} \\
&\leq C(p,\kappa,\beta,M)\int_{t'}^{t} E\left[1+|u(v,s,x)|^{\frac{p}{\kappa}}\right]dv \cdot |t-t'|^{(\frac{p}{2\kappa}-1)} \\
&\leq C(p,\kappa,\alpha,\beta,M)(1+|x|^{\frac{p}{\kappa}})|t-t'|^{\frac{p}{2\kappa}}.
\end{aligned} \tag{3.18}$$

Then it follows from (3.16)-(3.18) that

$$E\left[|u(t,s,x)-u(t',s,x)|^{\frac{p}{\kappa}}\right] \leq C(p,\kappa,l,\alpha,\beta,M)(1+|x|^p)|t-t'|^{\frac{p}{2\kappa}}. \tag{3.19}$$

Now for any $t > s > s', x, x' \in \mathbb{R}^d$ with $|s-s'| < 1$, let

$$\begin{cases}
\hat{u}(t) := u(t,s,x) - u(t,s',x') \\
\hat{b}(t) := b(t,u(t,s,x)) - b(t,u(t,s',x')) \\
\hat{\sigma}(t) := \sigma(t,u(t,s,x)) - \sigma(t,u(t,s',x')).
\end{cases}$$

Apply Itô formula to $|\hat{u}(t)|^{\frac{p}{\kappa}}$ on $[s,t]$, we have

$|\hat{u}(t)|^{\frac{p}{\kappa}}$

$$\begin{aligned}
&= |\hat{u}(s)|^{\frac{p}{\kappa}} + \frac{p}{\kappa}\int_{s}^{t}|\hat{u}(v)|^{(\frac{p}{\kappa}-2)}\left(\hat{u}(v)\cdot\hat{b}(v) + \frac{p/\kappa-1}{2}\|\hat{\sigma}(v)\|^2\right)dv + \frac{p}{\kappa}\int_{s}^{t}|\hat{u}(v)|^{(\frac{p}{\kappa}-2)}\hat{u}(v)\hat{\sigma}(v)dW_v \\
&\leq |\hat{u}(s)|^{\frac{p}{\kappa}} - \frac{p}{\kappa}\left(\alpha - \frac{(p/\kappa-1)\beta^2}{2}\right)\int_{s}^{t}|\hat{u}(v)|^{\frac{p}{\kappa}}dv + \frac{p}{\kappa}\int_{s}^{t}|\hat{u}(v)|^{(\frac{p}{\kappa}-2)}\hat{u}(v)\hat{\sigma}(v)dW_v \\
&\leq |\hat{u}(s)|^{\frac{p}{\kappa}} + \frac{p}{\kappa}\int_{s}^{t}|\hat{u}(v)|^{(\frac{p}{\kappa}-2)}\hat{u}(v)\hat{\sigma}(v)dW_v.
\end{aligned} \tag{3.20}$$

Taking expectation on both side, we have

$$\begin{aligned}
E\left[|\hat{u}(t)|^{\frac{p}{\kappa}}\right] &\leq E\left[|\hat{u}(s)|^{\frac{p}{\kappa}}\right] = E\left[\left|x - x' - \int_{s'}^{s}b(v,u(v,s',x'))dv - \int_{s'}^{s}\sigma(v,u(v,s',x'))dW_v\right|^{\frac{p}{\kappa}}\right] \\
&\leq C(p,\kappa)\left(|x-x'|^{\frac{p}{\kappa}} + E\left[\left|\int_{s'}^{s}b(v,u(v,s',x'))dv\right|^{\frac{p}{\kappa}}\right] + E\left[\left|\int_{s'}^{s}\sigma(v,u(v,s',x'))dW_v\right|^{\frac{p}{\kappa}}\right]\right).
\end{aligned} \tag{3.21}$$

Similar to (3.16), we have

$$
\begin{aligned}
E\big[|\hat{u}(t)|^{\frac{p}{\kappa}}\big] &= E\big[|u(t,s,x)-u(t,s',x')|^{\frac{p}{\kappa}}\big] \\
&\leq C(p,\kappa,l,\alpha,\beta,M)(1+|x'|^p)\big(|s-s'|^{\frac{p}{2\kappa}}+|x-x'|^{\frac{p}{\kappa}}\big).
\end{aligned}
\tag{3.22}
$$

Comparing (3.19) and (3.22), we conclude that

$$
\begin{aligned}
E\big[|u(t,s,x)-u(t',s',x')|^{\frac{p}{\kappa}}\big] & \\
\leq C(p,\kappa,l,\alpha,\beta,M)(1+|x|^p+|x'|^p)\big(|t-t'|^{\frac{p}{2\kappa}}&+|s-s'|^{\frac{p}{2\kappa}}+|x-x'|^{\frac{p}{\kappa}}\big).
\end{aligned}
\tag{3.23}
$$

Since $\alpha > \frac{(p-1)\beta^2}{2}$ and $p \geq (4+2d)\kappa$, we can choose $p' > p$, i.e. $\frac{p'}{2\kappa} > 2+d$ such that $\alpha > \frac{(p'-1)\beta^2}{2}$ and (3.23) holds for p'. Then by Kolmogorov's continuity criterion, we know that $u(t,s,x)$ is continuous with respect to (t,s,x) $P-a.s.$.

Next we prove (iii), and (ii) can be obtained in a similar way. Similar to the estimation of u in (3.23), we know that for all $t,t',s,s' \in \mathbb{R}, x,x' \in \mathbb{R}^d, r_1,r_2 \in \mathbb{R}$ with $|t-t'| < 1, |s-s'| < 1$,

$$
\begin{aligned}
E\big[|K^{r_1,r_2}(t,s,x)-K^{r_1,r_2}(t',s',x')|^{\frac{p}{\kappa}}\big] & \\
\leq C(p,\kappa,l,\alpha,\beta,M)(1+|x|^p+|x'|^p)(|t-t'|^{\frac{p}{2\kappa}}&+|s-s'|^{\frac{p}{2\kappa}}+|x-x'|^{\frac{p}{\kappa}}).
\end{aligned}
\tag{3.24}
$$

Now for all $r_1,r_1',r_2,r_2' \in \mathbb{R}$, let

$$
\begin{cases}
\hat{K}_t := K^{r_1,r_2}(t,s,x) - K^{r_1',r_2'}(t,s,x) \\
\hat{b}_t^* := \tilde{b}^{r_1,r_2}(t,K^{r_1,r_2}(t,s,x)) - \tilde{b}^{r_1',r_2'}(t,K^{r_1',r_2'}(t,s,x)) \\
\hat{\sigma}_t^* := \tilde{\sigma}^{r_1,r_2}(t,K^{r_1,r_2}(t,s,x)) - \tilde{\sigma}^{r_1',r_2'}(t,K^{r_1',r_2'}(t,s,x)).
\end{cases}
$$

It follows from (3.6) that

$$
\hat{K}_t = \int_s^t \hat{b}_v^* dv + \int_s^t \hat{\sigma}_v^* dW_v.
$$

Now applying Itô's formula to $e^{\lambda t}|\hat{K}_t|^p$ for some $\lambda > 0$ on $[s,t]$, we have

$$
\begin{aligned}
e^{\lambda t}|\hat{K}_t|^p &= \int_s^t e^{\lambda v}\left(\lambda|\hat{K}_v|^p + p|\hat{K}_v|^{(p-2)}\hat{K}_v\cdot\hat{b}_v^* + \frac{p(p-1)}{2}|\hat{K}_v|^{(p-2)}|\hat{\sigma}_v^*|^2\right)dv \\
&\quad + \int_s^t pe^{\lambda v}|\hat{K}_v|^{(p-2)}\hat{K}_v\hat{\sigma}_v^* dW_v.
\end{aligned}
\tag{3.25}
$$

Note that

$$
\begin{aligned}
|\hat{K}_v|^{(p-2)}\hat{K}_v\cdot\hat{b}_v^* =&|\hat{K}_v|^{(p-2)}\hat{K}_v\cdot\big(\tilde{b}^{r_1,r_2}(v,K^{r_1,r_2}(v,s,x)) - \tilde{b}^{r_1',r_2'}(v,K^{r_1,r_2}(v,s,x))\big) \\
&+ |\hat{K}_v|^{(p-2)}\hat{K}_v\cdot\big(\tilde{b}^{r_1',r_2'}(v,K^{r_1,r_2}(v,s,x)) - \tilde{b}^{r_1',r_2'}(v,K^{r_1',r_2'}(v,s,x))\big) \\
\leq& C|\hat{K}_v|^{(p-1)}(|r_1-r_1'|^\gamma + |r_2-r_2'|^\gamma) - \alpha|\hat{K}_v|^p.
\end{aligned}
$$

Then for arbitrary $\epsilon > 0$, by Young inequality, we have

$$
|\hat{K}_v|^{(p-2)}\hat{K}_v\cdot\hat{b}_v^* \leq C(p,\epsilon)(|r_1-r_1'|^{p\gamma} + |r_2-r_2'|^{p\gamma}) - (\alpha-\epsilon)|\hat{K}_v|^p
\tag{3.26}
$$

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Similarly, we have

$$|\hat{K}_v|^{(p-2)}|\hat{\sigma}_v^*|^2 \leq C(p,\beta,\epsilon)(|r_1 - r_1'|^{p\gamma} + |r_2 - r_2'|^{p\gamma}) + (\beta^2 + \epsilon)|\hat{K}_v|^p \qquad (3.27)$$

Then taking expectation on both sides of (3.25), we conclude from (3.26) and (3.27) that

$$e^{\lambda t}E[|\hat{K}_t|^p] \leq \frac{1}{\lambda}C(p,\beta,\epsilon)(|r_1 - r_1'|^{p\gamma} + |r_2 - r_2'|^{p\gamma})e^{\lambda t}$$
$$+ \left(\lambda - p\big(\alpha - \frac{(p-1)\beta^2}{2} - 2\epsilon\big)\right)\int_s^t e^{\lambda v}E[|\hat{K}_v|^p]dv.$$

Since $\alpha > \frac{(p-1)\beta^2}{2}$, we choose $\epsilon = \frac{1}{3}\big(\alpha - \frac{(p-1)\beta^2}{2}\big) > 0$ and $\lambda = p\epsilon > 0$. Then we have

$$E[|K^{r_1,r_2}(t,s,x) - K^{r_1',r_2'}(t,s,x)|^p] \leq C(p,\alpha,\beta)\big(|r_1 - r_1'|^{p\gamma} + |r_2 - r_2'|^{p\gamma}\big). \qquad (3.28)$$

Hence

$$E\big[|K^{r_1,r_2}(t,s,x) - K^{r_1',r_2'}(t,s,x)|^{\frac{p}{\kappa}}\big] \leq \left(E\big[|K^{r_1,r_2}(t,s,x) - K^{r_1',r_2'}(t,s,x)|^p\big]\right)^{\frac{1}{\kappa}}$$
$$\leq C(p,\kappa,\alpha,\beta)\big(|r_1 - r_1'|^{\frac{p\gamma}{\kappa}} + |r_2 - r_2'|^{\frac{p\gamma}{\kappa}}\big). \qquad (3.29)$$

Then together with (3.24), the continuity of $K^{r_1,r_2}(t,s,x)$ can be obtained by the standard argument of using Kolmogorov's continuity criterion.

(iv) For any $t,t' \in \mathbb{R}$, it follows from (2.2) and (3.19) that

$$E\big[|\varphi(t) - \varphi(t')|^{\frac{p}{\kappa}}\big] \leq \liminf_{s \to -\infty} E\big[|u(t,s,0) - u(t',s,0)|^{\frac{p}{\kappa}}\big]$$
$$\leq C(p,\kappa,l,\alpha,\beta,M)|t - t'|^{\frac{p}{2\kappa}}. \qquad (3.30)$$

By Kolmogorov's continuity criterion, we know that $\varphi(t)$ is continuous with respect to t $P-a.s..$

$(v - viii)$ Similarly, Comparing with (3.7), (3.24) and (3.28), we have

$$E\big[|\varphi^{r_1,r_2}(t) - \varphi^{r_1',r_2'}(t')|^{\frac{p}{\kappa}}\big] \leq \liminf_{s \to -\infty} E\big[|K^{r_1,r_2}(t,s,0) - K^{r_1',r_2'}(t',s,0)|^{\frac{p}{\kappa}}\big]$$
$$\leq \liminf_{s \to -\infty} C(p,\kappa)\big\{E\big[|K^{r_1,r_2}(t,s,0) - K^{r_1,r_2}(t',s,0)|^{\frac{p}{\kappa}}\big]$$
$$+ E\big[|K^{r_1,r_2}(t',s,0) - K^{r_1',r_2'}(t',s,0)|^{\frac{p}{\kappa}}\big]\big\} \qquad (3.31)$$
$$\leq C(p,\kappa,l,\alpha,\beta,M)\big(|r_1 - r_1'|^{\frac{p\gamma}{\kappa}} + |r_2 - r_2'|^{\frac{p\gamma}{\kappa}} + |t - t'|^{\frac{p}{2\kappa}}\big).$$

Along with (3.8), (3.10), (3.12) and (3.31), we conclude that

$$E\big[|\varphi(t)\theta_{-t} - \varphi(t')\theta_{-t'}|^{\frac{p}{\kappa}}\big] \leq C(p,\kappa,l,\alpha,\beta,M)|t - t'|^{\frac{p\gamma}{\kappa}}, \qquad (3.32)$$

$$E\big[|\tilde{\varphi}(t,s) - \tilde{\varphi}(t',s')|^{\frac{p}{\kappa}}\big] \leq C(p,\kappa,l,\alpha,\beta,M)\big(|t - t'|^{\frac{p\gamma}{\kappa}} + |s - s'|^{\frac{p\gamma}{\kappa}}\big), \qquad (3.33)$$

$$E\big[|\varphi^r(t) - \varphi^{r'}(t')|^{\frac{p}{\kappa}}\big] \leq C(p,\kappa,l,\alpha,\beta,M)\big(|r - r'|^{\frac{p\gamma}{\kappa}} + |t - t'|^{\frac{p}{2\kappa}}\big). \qquad (3.34)$$

Then we derive $(v), (vi), (vii)$ and $(viii)$ by applying Kolmogorov's continuity criterion respectively. $\qquad \square$

3.2 Existence and uniqueness of quasi-periodic measure

First we give the definition of the quasi-periodic probability measure as follows.

Definition 3.8. *We say a map $\rho : \mathbb{R} \to \mathcal{P}(\mathbb{R}^d)$ is a quasi-periodic probability measure of periods τ_1, τ_2 of SDE (1.2), where the reciprocals of τ_1 and τ_2 are rationally linearly independent, if $P^*(t,s)\rho_s = \rho_t$ for all $t \geq s$, and there exists $\tilde{\rho} : \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R}^d)$ with $\tilde{\rho}_{t,t} = \rho_t$ such that*

$$\tilde{\rho}_{t+\tau_1,s} = \tilde{\rho}_{t,s}, \ \tilde{\rho}_{t,s+\tau_2} = \tilde{\rho}_{t,s}, \tag{3.35}$$

for all $t, s \in \mathbb{R}$.

Theorem 3.9. *Assume Conditions 3.2, 3.3 and $\alpha > \frac{(p-1)\beta^2}{2}$ for some constant $p \geq 2$. Then there exists a unique quasi-periodic probability measure of periods τ_1, τ_2 of SDE (1.2) in $\mathcal{M}^p$.*

Proof. Uniqueness: Applying the proof of Theorem 2.8, we know that if there exists a quasi-periodic probability measure with periods τ_1, τ_2 of SDE (1.2) in $\mathcal{M}^p$, it must be the unique entrance measure of SDE (1.2) defined by the law of the random path.

Existence: Recall equations (3.13) and (3.14) in Theorem 3.5, let

$$\rho_t = \mathcal{L}(\varphi(t)), \ \tilde{\rho}_{t,s} = \mathcal{L}(\tilde{\varphi}(t,s)) \tag{3.36}$$

be the laws of $\varphi(t)$ and $\tilde{\varphi}(t,s)$ respectively. Since φ is the random path of SDE (1.2), then by equation (2.5) we have $P^*(t,s)\rho_s = \rho_t$ for all $t \geq s$. Since θ_{-t} preserves probability measure P, then $\rho_t = \mathcal{L}(\varphi(t)) = \mathcal{L}(\tilde{\varphi}(t,t)) = \tilde{\rho}_{t,t}$. By the construction of $\tilde{\rho}$, we have

$$\tilde{\rho}_{t+\tau_1,s} = \mathcal{L}(\tilde{\varphi}(t + \tau_1, s)) = \mathcal{L}(\tilde{\varphi}(t,s)) = \tilde{\rho}_{t,s}$$

and

$$\tilde{\rho}_{t,s+\tau_2} = \mathcal{L}(\tilde{\varphi}(t, s + \tau_2)) = \mathcal{L}(\tilde{\varphi}(t,s)) = \tilde{\rho}_{t,s}.$$

Also since φ is uniformly L^p-bounded, then

$$\sup_{t \in \mathbb{R}} \int_{\mathbb{R}^d} |x|^p \rho_t(dx) = \sup_{t \in \mathbb{R}} \mathbb{E}[|\varphi(t)|^p] < \infty,$$

which means $\rho \in \mathcal{M}^p$. Moreover, from Theorem 2.4, Lemma 2.5, Lemma 2.6 and the proof of Theorem 3.5, we have

$$\sup_{t,s \in \mathbb{R}} \int_{\mathbb{R}^d} |x|^p \tilde{\rho}_{t,s}(dx) = \sup_{t,s \in \mathbb{R}} \mathbb{E}[|\varphi^{t,s}(0)|^p] < \infty,$$

which shows that $\tilde{\rho} \in \mathcal{M}^p$. □

Remark 3.10. *Similar to Remark 2.10, the continuity of $\tilde{\varphi}$ and Condition 3.4 are not needed when we consider the quasi-periodic measure. From Theorem 3.5 and its proof, we know that the estimates for $p = 2$ in Section 3.1 is also adequate.*

Example 3.11 (Ornstein-Uhlenbeck equation). *We include the following example with a number of reasons. First, O-U process is one of the simplest stochastic process that one would analyse for new concepts. Second, it is instructive and does illustrate clearly the idea of random quasi-periodicity and two kinds of formulations as well as their relation. Third, the formulae for its random quasi-periodic path and quasi-periodic measure can be written down explicitly. Last, but not least, this equation is relevant in various different applications e.g. modelling energy consumptions or temperature variants with two obvious daily and seasonal periodicities.*

The Ornstein-Uhlenbeck process with mean reversion of single-period was used in modelling electricity prices ([5],[23]), daily temperature ([4]), biological neurouns ([18]) etc. The quasi-periodic O-U process we introduce here allows a feature of multiple periods which is natural in many real world situations e.g energy consumptions, temperature, business cycles, economics cycles. While it is not the purpose of this paper to study these interesting applied problems in their specific contexts, our work in this paper provides a mathematical theory of random quasi-periodicity for this purpose.

Here we consider the following mean reversion multidimensional Ornstein-Uhlenbeck equation on $\mathbb{R}^d$

$$dX_t = (S(t) - AX_t)dt + \sigma(t)dW_t \tag{3.37}$$

where $S(t), \sigma(t)$ are deterministic quasi-periodic functions with periods τ_1, τ_2 and $A \in S_d$ with $A > 0$, which means that A is a symmetrical matrix with positive eigenvalues $\{\lambda_n\}_{n=1}^d$. The analysis is given as follows.

Applying Itô's formula to $e^{tA}X_t$, we have

$$X_t = e^{-(t-s)A}X_s + \int_s^t e^{-(t-r)A}S(r)dr + \int_s^t e^{-(t-r)A}\sigma(r)dW_r \quad t \geq s. \tag{3.38}$$

Let

$$\varphi(t) := \int_{-\infty}^t e^{-(t-r)A}S(r)dr + \int_{-\infty}^t e^{-(t-r)A}\sigma(r)dW_r. \tag{3.39}$$

Then we have

$$\varphi(t) = e^{-(t-s)A}\varphi(s) + \int_s^t e^{-(t-r)A}S(r)dr + \int_s^t e^{-(t-r)A}\sigma(r)dW_r, \tag{3.40}$$

which means that φ is a random path of SDE (3.37). Next we will show that φ is also a random quasi-periodic path. We first rewrite $\varphi(t)$ by

$$
\begin{aligned}
\varphi(t,\omega) &= \int_{-\infty}^t e^{-(t-r)A}S(r)dr + \left[\int_{-\infty}^t e^{-(t-r)A}\sigma(r)dW_r\right](\omega) \\
&= \int_{-\infty}^0 e^{rA}S(r+t)dr + \left[\int_{-\infty}^0 e^{rA}\sigma(r+t)dW_r\right](\theta_t\omega).
\end{aligned} \tag{3.41}
$$

Since S, σ are quasi-periodic functions with periods τ_1, τ_2, then there exist $\tilde{S}, \tilde{\sigma}$ such that $S(t) = \tilde{S}(t,t)$ and $\sigma(t) = \tilde{\sigma}(t,t)$ and

$$\begin{cases} \tilde{S}(t+\tau_1, s) = \tilde{S}(t,s) = \tilde{S}(t, s+\tau_2) \\ \tilde{\sigma}(t+\tau_1, s) = \tilde{\sigma}(t,s) = \tilde{\sigma}(t, s+\tau_2). \end{cases} \tag{3.42}$$

Then we have

$$\varphi(t,\omega) = \int_{-\infty}^0 e^{rA}\tilde{S}(r+t,r+t)dr + \left[\int_{-\infty}^0 e^{rA}\tilde{\sigma}(r+t,r+t)dW_r\right](\theta_t\omega).$$

Let

$$\tilde{\varphi}(t,s,\omega) = \int_{-\infty}^0 e^{rA}\tilde{S}(r+t,r+s)dr + \left[\int_{-\infty}^0 e^{rA}\tilde{\sigma}(r+t,r+s)dW_r\right](\omega). \tag{3.43}$$

Then we have $\varphi(t,\theta_{-t}\omega) = \tilde{\varphi}(t,t,\omega)$ and

$$\tilde{\varphi}(t+\tau_1,s,\omega) = \tilde{\varphi}(t,s,\omega), \ \tilde{\varphi}(t,s+\tau_2,\omega) = \tilde{\varphi}(t,s,\omega), \tag{3.44}$$

which shows that φ is a random quasi-periodic path of SDE (3.37) with periods τ_1,τ_2.

Let $\rho_t = \mathcal{L}(\varphi(t))$. By Theorem 3.9, we know that ρ_t is the unique quasi-periodic probability measure with periods τ_1,τ_2 of SDE (3.37). Moreover, from (3.39), we know that

$$\rho_t(\cdot) = \mathcal{N}\left(\int_{-\infty}^t e^{-(t-r)A}S(r)dr, \int_{-\infty}^t e^{-(t-r)A}\sigma(r)\sigma(r)^T e^{-(t-r)A}dr\right)(\cdot),$$

where $\mathcal{N}$ is the multivariate normal distribution. Let $\tilde{\rho}_{t,s} = \mathcal{L}(\tilde{\varphi}(t,s))$. Then from (3.43), we know that

$$\tilde{\rho}_{t,s}(\cdot) = \mathcal{N}\left(\int_{-\infty}^0 e^{rA}\tilde{S}(r+t,r+s)dr, \int_{-\infty}^0 e^{rA}(\tilde{\sigma}\tilde{\sigma}^T)(r+t,r+s)e^{rA}dr\right)(\cdot).$$

It is obvious that $\rho_t = \tilde{\rho}_{t,t}$.

In Subsection 3.3, we will develop a way to lift a quasi-periodic stochastic flow to the cylinder $[0,\tau_1)\times[0,\tau_2)\times\mathbb{R}^d$ and prove $\tilde{\mu}_{t,s} = \delta_t \times \delta_s \times \tilde{\rho}_{t,s}$ is a quasi-periodic measure. This setup will enable us to prove that the average $\frac{1}{\tau_1\tau_2}\int_0^{\tau_1}\int_0^{\tau_2}\tilde{\mu}_{t,s}dtds$ is an ergodic invariant measure on the cylinder. Our result also implies that for this particular case, it is the unique ergodic invariant measure for the lifted quasi-periodic Ornstein-Uhlenbeck process.

3.3 The lift and invariant measure

In Section 3.1, we have the existence and uniqueness of random quasi-periodic path, and in this case, we will lift the semi-flow u and obtain an invariant measure. Consider the cylinder $\tilde{\mathbb{X}} = [0,\tau_1)\times[0,\tau_2)\times\mathbb{R}^d$ with the following metric

$$d(\tilde{x},\tilde{y}) = d_1(t_1,s_1) + d_2(t_2,s_2) + |x-y|, \ \text{for all } \tilde{x} = (t_1,t_2,x), \tilde{y} = (s_1,s_2,y) \in \tilde{\mathbb{X}},$$

where d_1,d_2 are the metrics on $[0,\tau_1),[0,\tau_2)$ defined by

$$d_i(t_i,s_i) = \min(|t_i-s_i|, \tau_i-|t_i-s_i|), \ \text{for all } t_i,s_i \in [0,\tau_i), i = 1,2.$$

Denote by $\mathcal{B}(\tilde{\mathbb{X}})$ the Borel measurable set on $\tilde{\mathbb{X}}$ deduced by metric d. Then we have the following lemma. Note the perfection of $K^{s_1,s_2}(t,0,x,\cdot)$ is needed but that of $\tilde{\varphi}$ is not needed.

Lemma 3.12. *Assume Conditions 3.2, 3.3 hold. We lift the semi-flow $u : \triangle \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ to a random dynamical system on a cylinder $\tilde{\mathbb{X}} = [0, \tau_1) \times [0, \tau_2) \times \mathbb{R}^d$ by the following:*

$$\tilde{\Phi}(t, \omega)(s_1, s_2, x) = (t + s_1 \mod \tau_1, \ t + s_2 \mod \tau_2, \ K^{s_1, s_2}(t, 0, x, \omega)),$$

where K^{r_1, r_2} is the solution of (3.6). Then $\tilde{\Phi} : \mathbb{R}^+ \times \tilde{\mathbb{X}} \times \Omega \to \tilde{\mathbb{X}}$ is a cocycle on $\tilde{\mathbb{X}}$ over the metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$. If we further assume that Condition 3.4 holds and $\alpha > \frac{(p-1)\beta^2}{2}$ for some $p \geq (10 + 2d)\kappa \vee \frac{(5+d)\kappa}{\gamma}$, then $\tilde{\Phi}$ is a perfect cocycle on $\tilde{\mathbb{X}}$.

Moreover, assume $\varphi : \mathbb{R} \times \Omega \to \mathbb{R}^d$ is a random path of the semi-flow u. Then $\tilde{Y} : \mathbb{R} \times \Omega \to \tilde{\mathbb{X}}$ defined by

$$\tilde{Y}(s, \omega) = (s \mod \tau_1, \ s \mod \tau_2, \ \varphi(s, \omega))$$

is a random path of the cocycle $\tilde{\Phi}$ on $\tilde{\mathbb{X}}$.

Proof. We first prove that $\tilde{\Phi}$ is a cocycle on $\tilde{\mathbb{X}}$. Note K^{r_1, r_2} is periodic in r_1, r_2 with periods τ_1, τ_2. It follows that for any $(s_1, s_2, x) \in \tilde{\mathbb{X}}, t, s \in \mathbb{R}^+$, we have

$$\begin{aligned}
&\tilde{\Phi}(t, \theta_s \omega) \circ \tilde{\Phi}(s, \omega)(s_1, s_2, x) \\
=\ &\tilde{\Phi}(t, \theta_s \omega)(s + s_1 \mod \tau_1, \ s + s_2 \mod \tau_2, \ K^{s_1, s_2}(s, 0, x, \omega)) \\
=\ &(t + s + s_1 \mod \tau_1, \ t + s + s_2 \mod \tau_2, \ K^{s+s_1, s+s_2}(t, 0, K^{s_1, s_2}(s, 0, x, \omega), \theta_s \omega)).
\end{aligned}$$

Now we compute the $K^{s+s_1, s+s_2}(t, 0, K^{s_1, s_2}(s, 0, x, \omega), \theta_s \omega)$ term. By Remark 3.6, we know that equation (1.3) holds $P - a.s.$, i.e. for all $r_1, r_2, r \in \mathbb{R}, t \geq s$,

$$K^{r_1, r_2}(t + r, s + r, x, \omega) = K^{r+r_1, r+r_2}(t, s, x, \theta_r \omega), \ P - a.e. \text{ on } \omega.$$

Then we have for all $s_1, s_2 \in \mathbb{R}, t, s \in \mathbb{R}^+$

$$\begin{aligned}
K^{s+s_1, s+s_2}(t, 0, K^{s_1, s_2}(s, 0, x, \omega), \theta_s \omega) &= K^{s_1, s_2}(t + s, s, K^{s_1, s_2}(s, 0, \omega), \omega) \\
&= K^{s_1, s_2}(t + s, 0, \omega), \ P - a.e. \text{ on } \omega,
\end{aligned}$$

i.e. the set $N_{s,t}^{s_1, s_2} := \{\omega | K^{s+s_1, s+s_2}(t, 0, K^{s_1, s_2}(s, 0, x, \omega), \theta_s \omega) \neq K^{s_1, s_2}(t + s, 0, \omega)\}$ is a null set. Then for any fixed $t \geq s, (s_1, s_2, x) \in \tilde{\mathbb{X}}$,

$$\begin{aligned}
&\tilde{\Phi}(t, \theta_s \omega) \circ \tilde{\Phi}(s, \omega)(s_1, s_2, x) \\
&= (t + s + s_1 \mod \tau_1, t + s + s_2 \mod \tau_2, K^{s+s_1, s+s_2}(t, 0, K^{s_1, s_2}(s, 0, x, \omega), \theta_s \omega)) \\
&= (t + s + s_1 \mod \tau_1, t + s + s_2 \mod \tau_2, K^{s_1, s_2}(t + s, 0, x, \omega)) \\
&= \tilde{\Phi}(t + s, \omega)(s_1, s_2, x), P - a.e. \ \omega,
\end{aligned}$$

which means that $\tilde{\Phi}$ is a cocycle on $\tilde{\mathbb{X}}$.

If Condition 3.4 holds and $\alpha > \frac{(p-1)\beta^2}{2}$ for some $p \geq (10 + 2d)\kappa \vee \frac{(5+d)\kappa}{\gamma}$, from (1.3), (3.24) and (3.29) in Lemma 3.7, we know that there exists $C = C(p, \kappa, l, \alpha, \beta, M)(1 + |x|^p + |x'|^p)$ such that

$$E\left[\left| K^{r+r_1, r+r_2}(t, s, x) \circ \theta_r - K^{r'+r'_1, r'+r'_2}(t', s', x') \circ \theta_{r'} \right|^{\frac{p}{\kappa}} \right]$$

$$\leq C \left(|r_1 - r'_1|^{\frac{p\gamma}{\kappa}} + |r_2 - r'_2|^{\frac{p\gamma}{\kappa}} + |r - r'|^{\frac{p}{2\kappa}} + |t - t'|^{\frac{p}{2\kappa}} + |s - s'|^{\frac{p}{2\kappa}} + |x - x'|^{\frac{p}{\kappa}} \right).$$

Then by Kolmogorov's continuity criterion, $K^{r+r_1,r+r_2}(t,s,x,\theta_r\cdot)$ is continuous with respect to (r,r_1,r_2,t,s,x) $P-a.s..$ By (iii) in Lemma 3.7, we also know that $K^{r_1,r_2}(t,s,x)$ is continuous with respect to (r_1,r_2,t,s,x) $P-a.s..$ Denote

$$N^1 := \{\omega | K : (r_1,r_2,t,s,x) \mapsto K^{r_1,r_2}(t,s,x,\omega) \text{ is not continuous}\}$$

$$N^2 := \{\omega | K : (r,r_1,r_2,t,s,x) \mapsto K^{r+r_1,r+r_2}(t,s,x,\theta_r\omega) \text{ is not continuous}\}$$

and

$$N_K := \bigcup_{s_1,s_2\in Q, t,s\in Q^+} N_{s,t}^{s_1,s_2} \bigcup N^1 \bigcup N^2.$$

Then $P(N^1) = P(N^2) = 0$ and hence $P(N_K) = 0$. Fix $\omega \in N_K^c$, for any $s_1, s_2 \in \mathbb{R}, t, s \in \mathbb{R}^+$, we choose $\{s_1^n, s_2^n, t^n, s^n\}_{n\geq 1}$ such that $s_1^n, s_2^n \in Q, t^n, s^n \in Q^+$ and $s_1^n \to s_1, s_2^n \to s_2, t^n \to t, s^n \to s$, then we have

$$\begin{aligned}
&\tilde{\Phi}(t,\theta_s\omega) \circ \tilde{\Phi}(s,\omega)(s_1,s_2,x)\\
&=(t+s+s_1 \mod \tau_1, t+s+s_2 \mod \tau_2, K^{s+s_1,s+s_2}(t,0,K^{s_1,s_2}(s,0,x,\omega),\theta_s\omega))\\
&=(t+s+s_1 \mod \tau_1, t+s+s_2 \mod \tau_2, \lim_{n\to\infty} K^{s^n+s_1^n,s^n+s_2^n}(t^n,0,K^{s_1^n,s_2^n}(s^n,0,x,\omega),\theta_{s^n}\omega))\\
&=(t+s+s_1 \mod \tau_1, t+s+s_2 \mod \tau_2, \lim_{n\to\infty} K^{s_1^n,s_2^n}(t^n+s^n,0,x,\omega))\\
&=(t+s+s_1 \mod \tau_1, t+s+s_2 \mod \tau_2, K^{s_1,s_2}(t+s,0,x,\omega))\\
&=\tilde{\Phi}(t+s,\omega)(s_1,s_2,x),
\end{aligned}$$

which implies the perfect cocycle property of $\tilde{\Phi}$.

Next, assume φ is a random path of the semi-flow u and $\tilde{Y}(s,\omega) = (s \mod \tau_1, s \mod \tau_2, \varphi(s,\omega))$. Denote $N_\varphi := \{\omega | u(t,s,\varphi(s,\omega),\omega) \neq \varphi(t,\omega) \text{ for all } t \geq s\}$, then $P(N_\varphi) = 0$. Fix $\omega \in N_\varphi^c \cap N_K^c$,

$$\begin{aligned}
\tilde{\Phi}(t,\theta_s\omega)\tilde{Y}(s,\omega) &=(t+s \mod \tau_1, t+s \mod \tau_2, K^{s,s}(t,0,\varphi(s,\omega),\theta_s\omega))\\
&=(t+s \mod \tau_1, t+s \mod \tau_2, u^s(t,0,\varphi(s,\omega),\theta_s\omega))\\
&=(t+s \mod \tau_1, t+s \mod \tau_2, u(t+s,s,\varphi(s,\omega),\theta_{-s}\theta_s\omega))\\
&=(t+s \mod \tau_1, t+s \mod \tau_2, \varphi(t+s,\omega))\\
&=\tilde{Y}(t+s,\omega),
\end{aligned}$$

which means $\tilde{Y}$ is a random path of the cocycle $\tilde{\Phi}$ on $\tilde{\mathbb{X}}$. □

Consider the Markovian transition $\tilde{P} : \mathbb{R}^+ \times \tilde{\mathbb{X}} \times \mathcal{B}(\tilde{\mathbb{X}}) \to [0,1]$ generated by the cocycle $\tilde{\Phi}$, i.e.,

$$\tilde{P}(t,(s_1,s_2,x),\tilde{\Gamma}) = P(\omega : \tilde{\Phi}(t,\omega)(s_1,s_2,x) \in \tilde{\Gamma}),$$

for all $t \in \mathbb{R}^+, (s_1,s_2,x) \in \tilde{\mathbb{X}}, \tilde{\Gamma} \in \mathcal{B}(\tilde{\mathbb{X}})$. Similarly, for any $\tilde{\mu} \in \mathcal{P}(\tilde{\mathbb{X}})$, we define

$$\tilde{P}_t^*\tilde{\mu}(\tilde{\Gamma}) = \int_{\tilde{\mathbb{X}}} \tilde{P}(t,(s_1,s_2,x),\tilde{\Gamma})\tilde{\mu}(ds_1 \times ds_2 \times dx).$$

Then we have the following theorem.

Theorem 3.13. *If $\rho : \mathbb{R} \to \mathcal{P}(\mathbb{R}^d)$ is the entrance measure of semi-group P^*, i.e. $P^*(t,s)\rho_s = \rho_t$, then $\tilde{\mu} : \mathbb{R} \to \mathcal{P}(\tilde{\mathbb{X}})$ defined by*

$$\tilde{\mu}_t = \delta_{t \mod \tau_1} \times \delta_{t \mod \tau_2} \times \rho_t$$

is an entrance measure of semi-group $\tilde{P}^*$, i.e.,

$$\tilde{P}_t^* \tilde{\mu}_s = \tilde{\mu}_{t+s}.$$

Moreover, $\tilde{\mu}$ is also a quasi-periodic measure.

Proof. For any $\tilde{\Gamma} \in \mathcal{B}(\tilde{\mathbb{X}})$, let $\tilde{\Gamma}_s := \{ x \in \mathbb{R}^d | (s \mod \tau_1, s \mod \tau_2, x) \in \tilde{\Gamma} \}$. Then we have

$$
\begin{aligned}
\tilde{P}_t^* \tilde{\mu}_s(\tilde{\Gamma}) &= \int_{\tilde{\mathbb{X}}} \tilde{P}(t, (s_1, s_2, x), \tilde{\Gamma}) \tilde{\mu}_s(ds_1 \times ds_2 \times dx) \\
&= \int_{\mathbb{R}^d} \tilde{P}(t, (s \mod \tau_1, \ s \mod \tau_2, x), \tilde{\Gamma}) \rho_s(dx) \\
&= \int_{\mathbb{R}^d} P(\omega : \tilde{\Phi}(t, \omega)(s \mod \tau_1, \ s \mod \tau_2, x) \in \tilde{\Gamma}) \rho_s(dx) \\
&= \int_{\mathbb{R}^d} P(\omega : (t + s \mod \tau_1, \ t + s \mod \tau_2, u^s(t, 0, x, \omega)) \in \tilde{\Gamma}) \rho_s(dx) \\
&= \int_{\mathbb{R}^d} P(\omega : u(t + s, s, x, \theta_{-s}\omega) \in \tilde{\Gamma}_{t+s}) \rho_s(dx) \\
&= \int_{\mathbb{R}^d} P(\omega : u(t + s, s, x, \omega) \in \tilde{\Gamma}_{t+s}) \rho_s(dx) \\
&= \int_{\mathbb{R}^d} P(t + s, s, x, \tilde{\Gamma}_{t+s}) \rho_s(dx) \\
&= P^*(t + s, s) \rho_s(\tilde{\Gamma}_{t+s}) \\
&= \rho_{t+s}(\tilde{\Gamma}_{t+s}) = \tilde{\mu}_{t+s}(\tilde{\Gamma}).
\end{aligned}
$$

Moreover, let

$$\hat{\mu}_{s_1, s_2} = \delta_{s_1 \mod \tau_1} \times \delta_{s_2 \mod \tau_2} \times \tilde{\rho}_{s_1, s_2}, \tag{3.45}$$

we know that $\tilde{\mu}_s = \hat{\mu}_{s_1, s_2}$ and

$$\hat{\mu}_{s_1 + \tau_1, s_2} = \hat{\mu}_{s_1, s_2}, \ \hat{\mu}_{s_1, s_2 + \tau_2} = \hat{\mu}_{s_1, s_2}, \tag{3.46}$$

which completes our proof. $\qquad\square$

For the above entrance measure $\tilde{\mu}$, set

$$\bar{\tilde{\mu}}_T := \frac{1}{T} \int_0^T \tilde{\mu}_s ds$$

and

$$\mathcal{M} := \{ \bar{\tilde{\mu}}_T : T \in \mathbb{R}^+ \}. \tag{3.47}$$

We have the following lemma.

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Lemma 3.14. *Assume Conditions 3.2, 3.3 and $\alpha > \frac{\beta^2}{2}$. Then $\mathcal{M}$ is tight, and hence is weakly compact in $\mathcal{P}(\tilde{\mathbb{X}})$.*

Proof. We just need to prove that for any $\epsilon > 0$, there exists a compact set $\tilde{\Gamma}_\epsilon \in \mathcal{B}(\tilde{\mathbb{X}})$ such that for all $T \in \mathbb{R}^+$, we have

$$\bar{\tilde{\mu}}_T(\tilde{\Gamma}_\epsilon) > 1 - \epsilon.$$

Since the entrance measure ρ_t is the law of the L^2-bounded random path $\varphi(t)$, then $\{\rho_t : t \in \mathbb{R}\}$ is tight because

$$
\begin{aligned}
\rho_t(\bar{B}_N(0)) &= P(|\varphi(t)| \leq N) \\
&= 1 - P(|\varphi(t)| > N) \\
&\geq 1 - \frac{\|\varphi(t)\|_2^2}{N^2} \\
&\geq 1 - \frac{\sup_{t \in \mathbb{R}} \|\varphi(t)\|_2^2}{N^2}.
\end{aligned}
\tag{3.48}
$$

Then for the given $\epsilon > 0$, there exists a compact set $\Gamma_\epsilon \subset \mathbb{R}^d$ such that for all $t \in \mathbb{R}$,

$$\rho_t(\Gamma_\epsilon) > 1 - \epsilon.$$

It is well-known that $[0, \tau_1), [0, \tau_2)$ are both homeomorphic to the circle S^1 under metrics d_1, d_2 respectively. Hence they are compact and $\tilde{\Gamma}_\epsilon = [0, \tau_1) \times [0, \tau_2) \times \Gamma_\epsilon$ is compact on $\tilde{\mathbb{X}}$. Moreover

$$\bar{\tilde{\mu}}_T(\tilde{\Gamma}_\epsilon) = \frac{1}{T}\int_0^T \tilde{\mu}_s(\tilde{\Gamma}_\epsilon)ds = \frac{1}{T}\int_0^T \rho_s(\Gamma_\epsilon)ds > 1 - \epsilon,\tag{3.49}$$

which completes our proof. $\qquad\square$

For any $f \in C^0(\tilde{\mathbb{X}})$, which is defined as the collection of $\mathcal{B}(\tilde{\mathbb{X}})$ measurable functions, we define

$$\tilde{P}_t f(\tilde{x}) = \int_{\tilde{\mathbb{X}}} \tilde{P}(t, \tilde{x}, d\tilde{y})f(\tilde{y}), \text{ for any } \tilde{x} \in \tilde{\mathbb{X}}.\tag{3.50}$$

We have the following Feller property of the semi-group $\tilde{P}_t, t \geq 0$.

Proposition 3.15. *Assume Conditions 3.2, 3.3 and $\alpha > \frac{\beta^2}{2}$. Then the semi-group $\tilde{P}_t, t \geq 0$, defined by (3.50), is Feller, i.e. for all $f \in C_b(\tilde{\mathbb{X}})$, $\tilde{P}_t f \in C_b(\tilde{\mathbb{X}})$.*

Proof. Obviously $\|\tilde{P}_t f\|_\infty \leq \|f\|_\infty$, then we just need to prove that $\tilde{P}_t f$ is continuous. It is sufficient to prove that for any sequence $\tilde{x}_n = (r_1^n, r_2^n, x_n), \tilde{x} = (r_1, r_2, x) \in \tilde{\mathbb{X}}$ with $\tilde{x}_n \xrightarrow{n\to\infty} \tilde{x}$, we have $\tilde{P}_t f(\tilde{x}_n) \xrightarrow{n\to\infty} \tilde{P}_t f(\tilde{x})$. Since

$$
\begin{aligned}
\tilde{P}_t f(\tilde{x}) &= \int_{[0,\tau_1)\times[0,\tau_2)\times\mathbb{R}^d} \tilde{P}(t, (r_1, r_2, x), ds_1 \times ds_2 \times dy)f(s_1, s_2, y) \\
&= \int_{[0,\tau_1)\times[0,\tau_2)\times\mathbb{R}^d} P(\tilde{\Phi}(t, \cdot)(r_1, r_2, x) \in ds_1 \times ds_2 \times dy)f(s_1, s_2, y) \\
&= \int_{\mathbb{R}^d} P(K^{r_1,r_2}(t, 0, x) \in dy)f(t + r_1 \mod \tau_1, t + r_2 \mod \tau_2, y) \\
&= \mathbb{E}f(t + r_1 \mod \tau_1, t + r_2 \mod \tau_2, K^{r_1,r_2}(t, 0, x)).
\end{aligned}
\tag{3.51}
$$

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Let $f_t(r_1, r_2, x) := f(t + r_1 \mod \tau_1, t + r_2 \mod \tau_2, x)$. Then we have

$$
\begin{aligned}
|\tilde{P}_t f(\tilde{x}_n) - \tilde{P}_t f(\tilde{x})| =& |\mathbb{E} f_t(r_1^n, r_2^n, K^{r_1^n, r_2^n}(t, 0, x_n)) - \mathbb{E} f_t(r_1, r_2, K^{r_1, r_2}(t, 0, x))| \\
\leq& |\mathbb{E} f_t(r_1^n, r_2^n, K^{r_1^n, r_2^n}(t, 0, x_n)) - \mathbb{E} f_t(r_1^n, r_2^n, K^{r_1, r_2}(t, 0, x))| \\
&+ |\mathbb{E} f_t(r_1^n, r_2^n, K^{r_1, r_2}(t, 0, x)) - \mathbb{E} f_t(r_1, r_2, K^{r_1, r_2}(t, 0, x))| \\
=:& A_1^n + A_2^n.
\end{aligned}
\tag{3.52}
$$

Since $f \in C_b(\tilde{\mathbb{X}})$, then $f_t \in C_b(\tilde{\mathbb{X}})$ and $f_t(r_1^n, r_2^n, K^{r_1, r_2}(t, 0, x)) \xrightarrow{a.s.} f_t(r_1, r_2, K^{r_1, r_2}(t, 0, x))$ as $n \to \infty$. By Lebesgue's dominated convergence theorem, we have

$$
A_2^n = |\mathbb{E} f_t(r_1^n, r_2^n, K^{r_1, r_2}(t, 0, x)) - \mathbb{E} f_t(r_1, r_2, K^{r_1, r_2}(t, 0, x))| \xrightarrow{n \to \infty} 0.
\tag{3.53}
$$

Furthermore, let

$$
b_n = \sup_{t \in \mathbb{R}, x \in \mathbb{R}^d} |\tilde{b}^{r_1^n, r_2^n}(t, x) - \tilde{b}^{r_1, r_2}(t, x)|
$$

and

$$
\sigma_n = \sup_{t \in \mathbb{R}, x \in \mathbb{R}^d} |\tilde{\sigma}^{r_1^n, r_2^n}(t, x) - \tilde{\sigma}^{r_1, r_2}(t, x)|.
$$

By (4) in Condition 3.3, we know that $b_n + \sigma_n \leq C(|r_1 - r_1^n|^\gamma + |r_2 - r_2^n|^\gamma)$. Then $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \sigma_n = 0$. Similar to the estimation of (3.28), we know that

$$
E\left[\left| K^{r_1^n, r_2^n}(t, 0, x_n) - K^{r_1, r_2}(t, 0, x) \right|^2 \right] \leq C(\alpha, \beta)\left(|x_n - x|^2 + |r_1^n - r_1|^{2\gamma} + |r_2^n - r_2|^{2\gamma} \right).
$$

Then we have $K^{r_1^n, r_2^n}(t, 0, x_n) \xrightarrow[n \to \infty]{L^2} K^{r_1, r_2}(t, 0, x)$. Let

$$
R_N = \{\omega : |K^{r_1, r_2}(t, 0, x, \omega)| \leq N\}
$$

and

$$
R_N^n = \{\omega : |K^{r_1^n, r_2^n}(t, 0, x_n, \omega)| \leq N\}.
$$

Then by the Chebyshev inequality we have $\lim_{N \to \infty}(\inf_{n \in \mathbb{N}} P(R_N^n \cap R_N)) = 1$. Since f is continuous, then it is uniformly continuous on all compact subset of $\tilde{\mathbb{X}}$. Then for arbitrary $\epsilon > 0$, there exists $\delta_N^\epsilon > 0$ such that when $(t_1, t_2, x), (s_1, s_2, y) \in [0, \tau_1) \times [0, \tau_2) \times \bar{B}_N(0)$, where $\bar{B}_N(0)$ is a closed ball centred at 0 with radius N in $\mathbb{R}^d$, and $d_1(t_1, s_1) + d_2(t_2, s_2) + |x - y| < \delta_N^\epsilon$, we have $|f((t_1, t_2, x)) - f((s_1, s_2, y))| < \epsilon$. Set

$$
C_{\delta_N^\epsilon}^n = \{\omega : |K^{r_1^n, r_2^n}(t, 0, x_n) - K^{r_1, r_2}(t, 0, x)| < \delta_N^\epsilon\}.
$$

Then also by the Chebyshev inequality $\lim_{n \to \infty} P(C_{\delta_N^\epsilon}^n) = 1$. Hence for all $\omega \in C_{\delta_N^\epsilon}^n \cap R_N^n \cap R_N$,

$$
|f_t(r_1^n, r_2^n, K^{r_1^n, r_2^n}(t, 0, x_n)) - f_t(r_1^n, r_2^n, K^{r_1, r_2}(t, 0, x))| < \epsilon.
$$

Therefore

$$
\begin{aligned}
\limsup_{n \to \infty} A_1^n =& \limsup_{n \to \infty} |\mathbb{E} f_t(r_1^n, r_2^n, K^{r_1^n, r_2^n}(t, 0, x_n)) - \mathbb{E} f_t(r_1^n, r_2^n, K^{r_1, r_2}(t, 0, x))| \\
\leq& \epsilon + 2\|f\|_\infty \limsup_{n \to \infty}[(1 - P(C_{\delta_N^\epsilon}^n)) + (1 - P(R_N^n \cap R_N))] \\
=& \epsilon.
\end{aligned}
\tag{3.54}
$$

Since $\epsilon > 0$ is arbitrary, we have $A_1^n \xrightarrow{n \to \infty} 0$. We complete the proof of $\tilde{P}_t f(\tilde{x}_n) \xrightarrow{n \to \infty} \tilde{P}_t f(\tilde{x})$.

$\square$

From Lemma 3.14 and Proposition 3.15, we have the existence of invariant measure under $\tilde{P}^*$.

Theorem 3.16. *Assume Conditions 3.2, 3.3 and $\alpha > \frac{\beta^2}{2}$. Then there exists a uniquen invariant probability measure with respect to the semi-group $\tilde{P}^*$ which is given by*

$$\frac{1}{\tau_1 \tau_2} \int_0^{\tau_1} \int_0^{\tau_2} \delta_{s_1} \times \delta_{s_2} \times \tilde{\rho}_{s_1,s_2} ds_1 ds_2.$$

Moreover, this invariant measure is ergodic with respect to the semigroup $\tilde{P}^*$.

Proof. Existence: From Lemma 3.14, we know that $\mathcal{M}$ defined by (3.47) is tight and hence weakly compact. This means that there exists a sequence $\{T_n\}_{n\geq 1}$ with $T_n \uparrow \infty$ as $n \to \infty$ and a probability measure $\bar{\bar{\mu}} \in \mathcal{P}(\tilde{\mathbb{X}})$ such that $\bar{\bar{\mu}}_{T_n} \xrightarrow{w} \bar{\bar{\mu}}$. Moreover, for any fixed $t > 0$, since

$$
\begin{aligned}
\tilde{P}_t^* \bar{\bar{\mu}}_{T_n} - \bar{\bar{\mu}}_{T_n} &= \frac{1}{T_n} \int_0^{T_n} \tilde{P}_t^* \tilde{\mu}_s ds - \frac{1}{T_n} \int_0^{T_n} \tilde{\mu}_s ds \\
&= \frac{1}{T_n} \int_0^{T_n} \tilde{\mu}_{t+s} ds - \frac{1}{T_n} \int_0^{T_n} \tilde{\mu}_s ds \\
&= \frac{1}{T_n} \int_t^{t+T_n} \tilde{\mu}_s ds - \frac{1}{T_n} \int_0^{T_n} \tilde{\mu}_s ds \\
&= \frac{1}{T_n} \int_{T_n}^{t+T_n} \tilde{\mu}_s ds - \frac{1}{T_n} \int_0^t \tilde{\mu}_s ds,
\end{aligned}
\tag{3.55}
$$

so

$$
\begin{aligned}
\limsup_{n\to\infty} \|\tilde{P}_t^* \bar{\bar{\mu}}_{T_n} - \bar{\bar{\mu}}_{T_n}\|_{BV} &\leq \limsup_{n\to\infty} \frac{1}{T_n}\left(\int_0^t \|\tilde{\mu}_s\|_{BV} ds + \int_{T_n}^{T_n+t} \|\tilde{\mu}_s\|_{BV} ds \right) \\
&\leq \limsup_{n\to\infty} \frac{2t}{T_n} = 0.
\end{aligned}
$$

Hence $\tilde{P}_t^* \bar{\bar{\mu}}_{T_n} \xrightarrow{w} \bar{\bar{\mu}}$. On the other hand, for any $f \in C_b(\tilde{\mathbb{X}})$, by Proposition 3.15, we have $\tilde{P}_t f \in C_b(\tilde{\mathbb{X}})$, and therefore

$$
\begin{aligned}
\lim_{n\to\infty} \int_{\tilde{\mathbb{X}}} f(\tilde{y}) \tilde{P}_t^* \bar{\bar{\mu}}_{T_n}(d\tilde{y}) &= \lim_{n\to\infty} \int_{\tilde{\mathbb{X}}} \int_{\tilde{\mathbb{X}}} f(\tilde{y}) \tilde{P}(t,\tilde{x},d\tilde{y}) \bar{\bar{\mu}}_{T_n}(d\tilde{x}) \\
&= \lim_{n\to\infty} \int_{\tilde{\mathbb{X}}} \tilde{P}_t f(\tilde{x}) \bar{\bar{\mu}}_{T_n}(d\tilde{x}) \\
&= \int_{\tilde{\mathbb{X}}} \tilde{P}_t f(\tilde{x}) \bar{\bar{\mu}}(d\tilde{x}) \\
&= \int_{\tilde{\mathbb{X}}} f(\tilde{y}) \tilde{P}_t^* \bar{\bar{\mu}}(d\tilde{y}).
\end{aligned}
\tag{3.56}
$$

This means $\tilde{P}_t^* \bar{\bar{\mu}}_{T_n} \xrightarrow{w} \tilde{P}_t^* \bar{\bar{\mu}}$. Summarising above we have that $\tilde{P}_t^* \bar{\bar{\mu}} = \bar{\bar{\mu}}$.

Moreover, by (3.33) in Lemma 3.7, we know that

$$\lim_{(t,s)\to(t_0,s_0)} \|\tilde{\varphi}(t,s) - \tilde{\varphi}(t_0,s_0)\|_2^2 = 0.$$

Then similar to the proof of Proposition 2.11, we know that $\tilde{\rho}$ is continuous under the weak topology in $\mathcal{P}(\mathbb{R}^d)$, i.e. for all $f \in C_b(\mathbb{R}^d)$,

$$\lim_{(t,s)\to(t_0,s_0)} \int_{\mathbb{R}^d} f(x)\tilde{\rho}_{t,s}(dx) = \int_{\mathbb{R}^d} f(x)\tilde{\rho}_{t_0,s_0}(dx).$$

Let $\hat{\mu}$ defined by (3.45). It is easy to check that $\hat{\mu}$ is also continuous under the weak topology in $\mathcal{P}(\tilde{\mathbb{X}})$. Since $\frac{1}{\tau_1}$ and $\frac{1}{\tau_2}$ are rationally linearly independent, by definition 5.1 in [25], $T_t : [0,\tau_1) \times [0,\tau_2) \to [0,\tau_1) \times [0,\tau_2)$ defined by

$$T_t(s_1, s_2) = (t + s_1 \mod \tau_1,\ t + s_2 \mod \tau_2),\ \text{for all } s_1, s_2 \in [0,\tau_1) \times [0,\tau_2)$$

is a minimal rotation. Then applying Theorem 6.20 in [25], we know that $\frac{1}{\tau_1\tau_2}L$ is a unique ergodic probability measure on $[0,\tau_1) \times [0,\tau_2)$, where L present the Lebesgue measures. Hence by Birkhoff's ergodic theory,

$$\begin{aligned}
\bar{\bar{\mu}}_T &= \frac{1}{T}\int_0^T \tilde{\mu}_t dt \\
&= \frac{1}{T}\int_0^T \hat{\mu}_{T_t(0,0)} dt \\
&\xrightarrow[T\to\infty]{w} \int_{[0,\tau_1)\times[0,\tau_2)} \hat{\mu}_{s_1,s_2} \frac{1}{\tau_1\tau_2} ds_1 ds_2.
\end{aligned}$$

So

$$\bar{\bar{\mu}} = \int_{[0,\tau_1)\times[0,\tau_2)} \hat{\mu}_{s_1,s_2} \frac{1}{\tau_1\tau_2} ds_1 ds_2 = \frac{1}{\tau_1\tau_2}\int_0^{\tau_1}\int_0^{\tau_2} \delta_{s_1} \times \delta_{s_2} \times \tilde{\rho}_{s_1,s_2} ds_1 ds_2$$

is an invariant measure with respect to $\tilde{P}^*$.

Uniqueness: We need to prove that for any invariant probability measure υ, we have $\upsilon = \bar{\bar{\mu}}$. By Lemma 2.9, we only need to prove that for any open set $\tilde{\mathcal{O}} \in \mathcal{B}(\tilde{\mathbb{X}})$, we have $\upsilon(\tilde{\mathcal{O}}) \geq \bar{\bar{\mu}}(\tilde{\mathcal{O}})$. Define

$$\tilde{\mathcal{O}}^{r_1,r_2} = \{x \in \mathbb{R}^d : (r_1 \mod \tau_1,\ r_2 \mod \tau_2, x) \in \tilde{\mathcal{O}}\},$$

$$\tilde{\mathcal{O}}_\delta^{r_1,r_2} = \{x : dist(x, (\tilde{\mathcal{O}}^{r_1,r_2})^c) > \delta\},$$

and

$$\tilde{\mathcal{O}}^\delta = \bigcup_{(s_1,s_2)\in[0,\tau_1)\times[0,\tau_2)} (s_1, s_2) \times \tilde{\mathcal{O}}_\delta^{s_1,s_2}.$$

We know that $\tilde{\mathcal{O}}^{r_1,r_2}, \tilde{\mathcal{O}}_\delta^{r_1,r_2}$ and $\tilde{\mathcal{O}}^\delta$ are open sets, $\tilde{\mathcal{O}}_\delta^{r_1,r_2} \uparrow \tilde{\mathcal{O}}^{r_1,r_2}$ and $\tilde{\mathcal{O}}^\delta \uparrow \tilde{\mathcal{O}}$ as $\delta \downarrow 0$. Then

$$\begin{aligned}
\upsilon\left(\tilde{\mathcal{O}}\right) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T \tilde{P}_t^*\upsilon\left(\tilde{\mathcal{O}}\right) dt \\
&= \lim_{T\to\infty} \frac{1}{T}\int_0^T \int_{\tilde{\mathbb{X}}} \tilde{P}\left(t, (s_1, s_2, x), \tilde{\mathcal{O}}\right) \upsilon(d\tilde{x}) dt \\
&= \lim_{T\to\infty} \int_{\tilde{\mathbb{X}}} \frac{1}{T}\int_0^T P\left(K^{s_1,s_2}(t, 0, x) \in \tilde{\mathcal{O}}^{t+s_1,t+s_2}\right) dt \upsilon(d\tilde{x}).
\end{aligned}$$
(3.57)

Applying Remark 3.6 and measure preserving transformation θ_t, it follows that

$$\upsilon\left(\tilde{\mathcal{O}}\right) = \lim_{T\to\infty} \int_{\tilde{\mathbb{X}}} \frac{1}{T} \int_0^T P\left(K^{t+s_1,t+s_2}(0,-t,x) \in \tilde{\mathcal{O}}^{t+s_1,t+s_2}\right) dt \upsilon(d\tilde{x}).$$

Similar to the proof of Theorem 2.4, Lemma 2.5 and Lemma 2.6, it can be shown that the solution K^{r_1,r_2} of (3.6) has the following estimate

$$\|K^{r_1,r_2}(t,s,x) - \tilde{\varphi}^{r_1,r_2}(t)\|_2 \leq Ce^{-(\alpha-\beta^2/2)(t-s)},$$

for all $r_1, r_2 \in \mathbb{R}, t \geq s$, where $C = C(\alpha,\beta,\tilde{M})$ only depends on $\alpha, \beta, \tilde{M}$ with $\tilde{M} = \sup_{t,s\in\mathbb{R}}(|\tilde{b}(t,s,0)| + \|\tilde{\sigma}(t,s,0)\|)$. Then for all $\delta > 0$, by the Chebyshev inequality, we have

$$\begin{aligned}
&P\left(K^{t+s_1,t+s_2}(0,-t,x) \in \tilde{\mathcal{O}}^{t+s_1,t+s_2}\right) \\
&\geq P\left(\tilde{\varphi}^{t+s_1,t+s_2}(0) \in \tilde{\mathcal{O}}_\delta^{t+s_1,t+s_2}, \ |K^{t+s_1,t+s_2}(0,-t,x) - \tilde{\varphi}^{t+s_1,t+s_2}(0)| < \delta\right) \\
&\geq P\left(\tilde{\varphi}^{t+s_1,t+s_2}(0) \in \tilde{\mathcal{O}}_\delta^{t+s_1,t+s_2}\right) - P\left(|K^{t+s_1,t+s_2}(0,-t,x) - \tilde{\varphi}^{t+s_1,t+s_2}(0)| \geq \delta\right) \\
&\geq \tilde{\rho}_{t+s_1,t+s_2}\left(\tilde{\mathcal{O}}_\delta^{t+s_1,t+s_2}\right) - \frac{C^2}{\delta^2} e^{-2(\alpha-\beta^2/2)t} \\
&= \hat{\mu}_{t+s_1,t+s_2}\left(\tilde{\mathcal{O}}^\delta\right) - \frac{C^2}{\delta^2} e^{-2(\alpha-\beta^2/2)t}.
\end{aligned} \tag{3.58}$$

Thus it turns out from (3.57), (3.58) and Fatou's Lemma that

$$\begin{aligned}
\upsilon\left(\tilde{\mathcal{O}}\right) &\geq \liminf_{T\to\infty} \int_{\tilde{\mathbb{X}}} \frac{1}{T} \int_0^T \left(\hat{\mu}_{t+s_1,t+s_2}\left(\tilde{\mathcal{O}}^\delta\right) - \frac{C^2}{\delta^2} e^{-2(\alpha-\beta^2/2)t}\right) dt \upsilon(d\tilde{x}) \\
&\geq \int_{\tilde{\mathbb{X}}} \left(\liminf_{T\to\infty} \frac{1}{T} \int_0^T \hat{\mu}_{t+s_1,t+s_2}\left(\tilde{\mathcal{O}}^\delta\right) dt - \lim_{T\to\infty} \frac{C^2}{2\delta^2(\alpha-\beta^2/2)T}\right) \upsilon(d\tilde{x}) \\
&\geq \int_{\tilde{\mathbb{X}}} \left(\liminf_{T\to\infty} \frac{1}{T} \int_0^T \hat{\mu}_{t+s_1,t+s_2}\left(\tilde{\mathcal{O}}^\delta\right) dt\right) \upsilon(d\tilde{x}).
\end{aligned} \tag{3.59}$$

Again by Birkhoff's ergodic theory, we know that for all $(s_1, s_2) \in \mathbb{R}^2$

$$\frac{1}{T} \int_0^T \hat{\mu}_{t+s_1,t+s_2} dt \xrightarrow[T\to\infty]{w} \bar{\bar{\mu}}.$$

Then since $\mathcal{O}^\delta$ is open, and by Proposition 2.4 in [17], we have

$$\upsilon\left(\tilde{\mathcal{O}}\right) \geq \bar{\bar{\mu}}\left(\mathcal{O}^\delta\right).$$

Since $\mathcal{O}^\delta \uparrow \mathcal{O}$ as $\delta \downarrow 0$, the desired result follows from the continuity of measures with respect to an increasing sequence of sets.

Moreover, by Theorem 3.2.6 in [8], we know that this unique invariant measure $\bar{\bar{\mu}}$ is ergodic. $\qquad\square$

Remark 3.17. *It is not obvious how to check directly that $\frac{1}{\tau_1\tau_2}\int_0^{\tau_1}\int_0^{\tau_2} \delta_{s_1} \times \delta_{s_2} \times \tilde{\rho}_{s_1,s_2} ds_1 ds_2$ is an invariant measure with respect to $\tilde{P}^*$ without appealing to the tightness argument.*

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By a similar proof of Lemma 3.14, Proposition 3.15 and Theorem 3.16, it is not difficult to derive a general theorem. Here we denote by $\mathbb{X}$ a metric space, $\mathcal{B}(\mathbb{X})$ the Borel σ-algebra on $\mathbb{X}$, $B_b(\mathbb{X})$ the linear space of all $\mathcal{B}(\mathbb{X})$-bounded measurable functions and $\mathcal{P}(\mathbb{X})$ the collection of all probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Assume that $P(t, x, \Gamma), t \geq 0, x \in \mathbb{X}, \Gamma \in \mathcal{B}(\mathbb{X})$, is a Markovian transition function on $\mathbb{X}$. Denote by $P_t, t \geq 0 : B_b(\mathbb{X}) \to B_b(\mathbb{X})$ and $P_t^*, t \geq 0 : \mathcal{P}(\mathbb{X}) \to \mathcal{P}(\mathbb{X})$, the Markovian semi-groups associated with $P(t, x, \cdot)$. We say $\rho : \mathbb{R} \to \mathcal{P}(\mathbb{X})$ is an entrance measure with respect to P^* if $P_t^* \rho_s = \rho_{t+s}$ for all $t \in \mathbb{R}^+, s \in \mathbb{R}$. We say ρ is quasi-periodic if exists a measure-valued function $\tilde{\rho}_{s_1,s_2}$ satisfying the same relation with ρ_s as in Definition 3.8. However we do not have the uniqueness of invariant measure in the general case.

Theorem 3.18. *Assume the entrance measure ρ with respect to $P_t^*, t \geq 0$, is a quasi-periodic measure with periods τ_1 and τ_2, where the reciprocals of τ_1 and τ_2 are rationally linearly independent. If $\{\bar{\rho}_T = \frac{1}{T} \int_0^T \rho_s ds : T \in \mathbb{R}^+\}$ is tight and the Markovian semi-group $P_t, t \geq 0$, is Feller, then there exists one invariant measure given by*

$$\frac{1}{\tau_1 \tau_2} \int_0^{\tau_1} \int_0^{\tau_2} \tilde{\rho}_{s_1,s_2} ds_1 ds_2.$$

4 Density of entrance measure and quasi-periodic measure

In this section, we will give a sufficient condition to guarantee the existence of the density of the entrance measure. We need an extra condition.

Condition 4.1. *The diffusion coefficient σ in SDE (1.2) is invertible with $\sup_{t \in \mathbb{R}, x \in \mathbb{R}^d} \|\sigma^{-1}(t, x)\| < \infty$.*

We now give the definition of the well-known BMO space and some lemmas which will used in this section.

Definition 4.2. *Denote by BMO(s,t) the space of all $(\mathcal{F}_s^r)_{s \leq r \leq t}$-adapted $\mathbb{R}^d$-valued process M with*

$$\|M\|_{BMO(s,t)} := \sup_{T \in \mathcal{T}_s^t} \left\| \left(\mathbb{E}\left[\int_T^t |M_r|^2 dr | \mathcal{F}_s^T \right] \right)^{\frac{1}{2}} \right\|_{L^\infty} < \infty,$$

where $s < t$ and $\mathcal{T}_s^t$ is the set of stopping times taking their values in $[s, t]$.

Then we have the following lemma.

Lemma 4.3. *Let $M \in BMO(s,t)$. Then there exists $p > 1$ such that*

$$\mathbb{E}\left[\left(\mathcal{E}\left(\int_s^t M_r dW_r \right) \right)^p \right] < \infty,$$

where $\mathcal{E}\left(\int_s^t M_r dW_r \right) := \exp\{\int_s^t M_r dW_r - \frac{1}{2} \int_s^t |M_r|^2 dr\}$.

Proof. By Theorem 3.1 in [20], we know that if $\|M\|_{BMO(s,t)} \leq \Phi(p)$ for some $p > 1$, where Φ is a continuous monotone function from $(1, \infty)$ to $\mathbb{R}_+$ with $\Phi(1+) = \infty$ and $\Phi(\infty) = 0$, then $\mathcal{E}\left(\int_s^t M_r dW_r \right)$ is in L^p. $\square$

We also need the following lemma which is almost the same as Lemma 4.1 in [13].

Lemma 4.4. *Assume Conditions 2.1 and 4.1 hold. Let $X_t^{s,x}$ be the solution of SDE (1.2) and $Z_t^{s,x}$ be the solution of the following SDE*

$$\begin{cases} dZ_t = \sigma(t, Z_t)dW_t, \quad t \geq s, \\ Z_s = x \in \mathbb{R}^d. \end{cases} \tag{4.1}$$

Then the laws of $X_t^{s,x}$ and $Z_t^{s,x}$ are equivalent, i.e.

$$P^{X_t^{s,x}}(B) = \tilde{P}^{Z_t^{s,x}}(B), \text{ for all } B \in \mathcal{B}(\mathbb{R}^d),$$

where $\frac{d\tilde{P}}{dP} = \mathcal{E}\left(\int_s^t \sigma^{-1}(r, Z_r^{s,x})b(r, Z_r^{s,x})dW_r\right)$

Proof. This lemma can be proved by almost the same proof as them of Lemma 4.1 in [13]. $\square$

Now we have the following theorem.

Theorem 4.5. *Assume Conditions 2.1 and 4.1 hold. If $\alpha > \frac{\beta^2}{2}$, then $P(t, s, x, \cdot)$ and the entrance measure ρ_t are absolutely continuous with respect to the Lebesgue measure L on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, and hence have the densities $p(t, s, x, y)$ and $q(t, y)$ respectively.*

Proof. First we prove that $P(t, s, x, \cdot)$ is absolutely continuous with respect to L, i.e. for any $\Gamma \in \mathcal{B}(\mathbb{R}^d)$, $L(\Gamma) = 0$ implies $P(t, s, x, \Gamma) = P(X_t^{s,x} \in \Gamma) = 0$. By Lemma 4.4, we know that

$$\begin{aligned} P(X_t^{s,x} \in \Gamma) &= \tilde{P}(Z_t^{s,x} \in \Gamma) = \mathbb{E}_{\tilde{P}}[1_\Gamma(Z_t^{s,x})] \\ &= \mathbb{E}\left[\mathcal{E}\left(\int_s^t \sigma^{-1}(r, Z_r^{s,x})b(r, Z_r^{s,x})dW_r\right)1_\Gamma(Z_t^{s,x})\right], \end{aligned} \tag{4.2}$$

where $Z_t^{s,x}$ is the solution of SDE (4.1). Set $T_n := \inf_{t \geq s}\{|Z_t^{s,x}| \geq n\}$. Since $\mathbb{E}_{\tilde{P}}[\sup_{r \in [s,t]} |Z_r^{s,x}|^2] < \infty$, then we have

$$\tilde{P}(T_n > t) = \tilde{P}(\sup_{r \in [s,t]} |Z_r^{s,x}| \leq n) \to 1 \text{ as } n \to \infty.$$

Thus

$$\begin{aligned} P(X_t^{s,x} \in \Gamma) &= \mathbb{E}_{\tilde{P}}[1_\Gamma(Z_t^{s,x})] \\ &= \mathbb{E}_{\tilde{P}}[1_\Gamma(Z_t^{s,x})1_{[s,T_n]}(t)] + \mathbb{E}_{\tilde{P}}[1_\Gamma(Z_t^{s,x})1_{(T_n,\infty)}(t)] \\ &\leq \lim_{n \to \infty}[\mathbb{E}_{\tilde{P}}[1_\Gamma(Z_t^{s,x})1_{[s,T_n]}(t)] + \tilde{P}(T_n < t)] \\ &= \lim_{n \to \infty} \mathbb{E}_{\tilde{P}}[1_\Gamma(Z_t^{s,x})1_{[s,T_n]}(t)] \\ &= \lim_{n \to \infty} \mathbb{E}\left[1_{[s,T_n]}(t)\mathcal{E}\left(\int_s^t \sigma^{-1}(r, Z_r^{s,x})b(r, Z_r^{s,x})dW_r\right)1_\Gamma(Z_t^{s,x})\right]. \end{aligned} \tag{4.3}$$

Since

$$1_{[s,T_n]}(t)\mathcal{E}\left(\int_s^t \sigma^{-1}(r, Z_r^{s,x})b(r, Z_r^{s,x})dW_r\right) \leq \mathcal{E}\left(\int_s^t 1_{[s,T_n]}(r)\sigma^{-1}(r, Z_r^{s,x})b(r, Z_r^{s,x})dW_r\right),$$

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we have

$$P(X_t^{s,x} \in \Gamma) \le \liminf_{n \to \infty} \mathbb{E}\left[\mathcal{E}\left(\int_s^t 1_{[s,T_n]}(r)\sigma^{-1}(r, Z_r^{s,x})b(r, Z_r^{s,x})dW_r \right) 1_\Gamma(Z_t^{s,x}) \right]. \qquad (4.4)$$

We only need to prove that if $L(\Gamma) = 0$, then for all n

$$\mathbb{E}\left[\mathcal{E}\left(\int_s^t 1_{[s,T_n]}(r)\sigma^{-1}(r, Z_r^{s,x})b(r, Z_r^{s,x})dW_r \right) 1_\Gamma(Z_t^{s,x}) \right] = 0.$$

Let $a_n(r) = 1_{[s,T_n]}(r)\sigma^{-1}(r, Z_r^{s,x})b(r, Z_r^{s,x})$. By Condition 4.1, we know that there exists $C > 0$ such that $\sup_{r \in \mathbb{R}} |a_n(r)| \le C$. Then

$$\sup_{T \in \mathcal{T}_s^t} \left\| \left(\mathbb{E}\left[\int_T^t |a_n(r)|^2 dr | \mathcal{F}_s^T \right] \right)^{\frac{1}{2}} \right\|_{L^\infty} \le C\sqrt{t-s},$$

which means $a_n \in BMO(s,t)$. By Lemma 4.3, there exists $p > 1$ such that

$$\gamma_n := \left(\mathbb{E}\left[\left(\mathcal{E}\left(\int_s^t a_n(r)dW_r \right) \right)^p \right] \right)^{\frac{1}{p}} < \infty.$$

Since $Z_t^{s,x} = x + \int_s^t \sigma(r, Z_r^{s,x})dW_r$, note that $\int_s^t \sigma(r, Z_r^{s,x})dW_r$ is in law a Brownian motion with time $\hat\sigma_t = \int_s^t \|\sigma(r, Z_r^{s,x})\|^2 dr$, i.e. there exists a standard Brownian motion $\tilde W$ such that $\int_s^t \sigma(r, Z_r^{s,x})dW_r \overset{d}{=} \tilde W_{\hat\sigma_t}$. Also notice

$$\sqrt{d} = \|\sigma(t,x)\sigma^{-1}(t,x)\| \le \|\sigma(t,x)\|\|\sigma^{-1}(t,x)\|,$$

thus

$$\|\sigma(t,x)\| \ge \frac{\sqrt{d}}{\|\sigma^{-1}(t,x)\|} \ge \frac{\sqrt{d}}{\sup_{t \in \mathbb{R}, x \in \mathbb{R}^d} \|\sigma^{-1}(t,x)\|} =: \underline{\sigma},$$

which suggests that $\hat\sigma_t \ge \underline{\sigma}(t-s)$. Using Proposition 6.17 in Chapter 2 in [19], we have

$$\begin{aligned}
\mathbb{E}\left[1_\Gamma(Z_t^{s,x})\right] &= \mathbb{E}\left[1_\Gamma(x + \tilde W_{\hat\sigma_t})\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[1_\Gamma(x + \tilde W_{\hat\sigma_t}) | \mathcal{F}_{\hat\sigma_t - \underline{\sigma}(t-s)} \right] \right] \qquad (4.5) \\
&= \mathbb{E}\left[\mathbb{E}\left[1_\Gamma(x + y + \tilde W_{\underline{\sigma}(t-s)}) \right] \Big|_{y = \tilde W_{\hat\sigma_t - \underline{\sigma}(t-s)}} \right].
\end{aligned}$$

Note

$$\begin{aligned}
\mathbb{E}\left[1_\Gamma(x + y + \tilde W_{\underline{\sigma}(t-s)}) \right] &= \frac{1}{(2\pi\underline{\sigma}(t-s))^{d/2}|\det \Sigma|^{1/2}} \int_{\mathbb{R}^d} 1_\Gamma(x + y + z) e^{-(1/2\underline{\sigma}(t-s))|\Sigma^{-1/2}z|^2} dz \\
&\le \frac{1}{(2\pi\underline{\sigma}(t-s))^{d/2}|\det \Sigma|^{1/2}} L(\Gamma),
\end{aligned}$$

where $W_1 \sim \mathcal{N}(0, \Sigma)$. Then

$$\mathbb{E}\left[1_\Gamma(Z_t^{s,x})\right] \le \frac{1}{(2\pi\underline{\sigma}(t-s))^{d/2}|\det \Sigma|^{1/2}} L(\Gamma).$$

Let q be the dual number of p. Then by Cauchy-Schwarz inequality,

$$\mathbb{E}\left[\mathcal{E}\left(\int_s^t 1_{[s,T_n]}(r)\sigma^{-1}(r, Z_r^{s,x})b(r, Z_r^{s,x})dW_r\right)1_\Gamma(Z_t^{s,x})\right] \leq \gamma_n\{\mathbb{E}[1_\Gamma(Z_t^{s,x})]\}^{\frac{1}{q}}$$
$$\leq C_n \cdot L(\Gamma)^{\frac{1}{q}}, \tag{4.6}$$

where $C_n = \gamma_n \cdot \left(\frac{1}{(2\pi\underline{\sigma}(t-s))^{d/2}|\det\Sigma|^{1/2}}\right)^{\frac{1}{q}}$.

So if $L(\Gamma) = 0$, then $\mathbb{E}\left[\mathcal{E}\left(\int_s^t 1_{[s,T_n]}(r)\sigma^{-1}(r, Z_r^{s,x})b(r, Z_r^{s,x})dW_r\right)1_\Gamma(Z_t^{s,x})\right] = 0$, and hence $P(t, s, x, \Gamma) = P(X_t^{s,x} \in \Gamma) = 0$. Thus $P(t, s, x, \cdot)$ is absolutely continuous with respect to the Lebesgue measure and by Radon-Nikodym theorem, the density of $P(t, s, x, \cdot)$ with respect to the Lebesgue measure exists.

For the entrance measure ρ_t, since

$$\rho_t(\Gamma) = P^*(t, s)\rho_s(\Gamma) = \int_{\mathbb{R}^d} P(t, s, x, \Gamma)\rho_s(dx), \tag{4.7}$$

then if $L(\Gamma) = 0$, we have $\rho_t(\Gamma) = 0$. This also suggests that ρ_t is absolutely continuous with respect to L and thus its density exists. $\qquad\square$

We already know the conditions to guarantee the existence of the densities $p(t, s, x, y)$ and $q(t, y)$ of the two- parameter Markov transition kernel $P(t, s, x, \cdot)$ and entrance measure ρ_t respectively. By Fubini theorem, we know that

$$\rho_t(\Gamma) = \int_{\mathbb{R}^d} P(t, s, x, \Gamma)\rho_s(dx) = \int_\Gamma \int_{\mathbb{R}^d} p(t, s, x, y)\rho_s(dx)dy = \int_\Gamma \int_{\mathbb{R}^d} p(t, s, x, y)q(s, x)(dx)dy.$$

Then it is obvious that

$$q(t, y) = \int_{\mathbb{R}^d} p(t, s, x, y)q(s, x)(dx). \tag{4.8}$$

In addition, we give the following condition:

Condition 4.6. *Assume $b = (b_i)_{i=1}^d$, $\sigma = (\sigma_{ij})_{i,j=1}^d$ in SDE (1.2) satisfy the following conditions:*

(1) *The functions $b(t, x), \sigma(t, x)$ are globally bounded and uniformly Hölder-continuous in (t, x).*

(2) *The functions $b(t, \cdot) \in C^1(\mathbb{R}^d; \mathbb{R}^d)$, $\sigma(t, \cdot) \in C^2(\mathbb{R}^d; \mathbb{R}^{d\times d})$ such that $\partial_{x_i}b_i, \partial_{x_ix_j}^2\sigma_{ij}$ are bounded and Hölder-continuous.*

Then it is well-known that (see [14, 15, 19] for more details) under Condition 4.6, $p(\cdot, s, x, \cdot) \in C^{1,2}(\mathbb{R} \times \mathbb{R}^d)$ satisfies the following Fokker-Planck equation

$$\partial_t p(t, s, x, y) = \mathcal{L}^*(t)p(t, s, x, y), \ t \geq s, \tag{4.9}$$

with initial conditions $p(s, s, x, y) = \delta_x(y)$, where $\mathcal{L}^*(t)p$ is the Fokker-Planck operator given by

$$\mathcal{L}^*(t)p = -\sum_{i=1}^d \partial_{x_i}(b_i(t, y)p) + \frac{1}{2}\sum_{i,j=1}^d \partial_{x_ix_j}^2\left(\sigma\sigma_{ij}^T(t, y)p\right). \tag{4.10}$$

Now we have the following theorem.

Theorem 4.7. *Assume Conditions 2.1, 4.1, 4.6 hold and $\alpha > \frac{\beta^2}{2}$. Let $q \in C_+^{1,2}(\mathbb{R} \times \mathbb{R}^d) \bigcap L^1(\mathbb{R}^d)$ with $\|q(t, \cdot)\|_{L^1(\mathbb{R}^d)} = 1$ for all t, and define $\rho : \mathbb{R} \to \mathcal{P}(\mathbb{R}^d)$ by*

$$\rho_t(\Gamma) = \int_\Gamma q(t, y) dy, \text{ for all } t \in \mathbb{R}.$$

Then ρ is an entrance measure if and only if q satisfies the infinite horizon Fokker-Planck equation problem:

$$\partial_t q = \mathcal{L}^*(t) q, \ t \geq s \tag{4.11}$$

for any $s \in \mathbb{R}$, and the additional condition

$$\|q(t, \cdot)\|_{L^1(\mathbb{R}^d)} = 1, \ t \in \mathbb{R}, \tag{4.12}$$

and the solution to infinite horizon Fokker-Planck equation problem is unique. Hence the solution of (4.11) and the entrance measure have one to one correspondence.

Proof. Assume first that ρ is an entrance measure. We already know that p, q satisfy (4.8) and $p(t, s, x, y)$ satisfies Fokker-Planck equation (4.9). We take the derivative with respect to t on both sides of (4.8) to have

$$
\begin{aligned}
\partial_t q(t, x) &= \int_{\mathbb{R}^d} \partial_t p(t, s, y, x) q(s, y) dy \\
&= \int_{\mathbb{R}^d} \mathcal{L}^*(t) p(t, s, y, x) q(s, y) dy \\
&= \int_{\mathbb{R}^d} \left(-\sum_{i=1}^d \partial_{x_i}(b_i(t, x) p(t, s, y, x)) q(s, y) \right) dy \\
&\quad + \int_{\mathbb{R}^d} \frac{1}{2} \sum_{i,j=1}^d \partial_{x_i x_j}^2 \left(\sigma \sigma_{ij}^T(t, x) p(t, s, y, x) \right) q(s, y) dy \\
&=: I + II.
\end{aligned} \tag{4.13}
$$

For the first part, we have

$$
\begin{aligned}
I &= -\sum_{i=1}^d \int_{\mathbb{R}^d} [\partial_{x_i}(b_i(t, x)) p(t, s, y, x) + b_i(t, x) \partial_{x_i}(p(t, s, y, x))] q(s, y) dy \\
&= -\sum_{i=1}^d \partial_{x_i}(b_i(t, x)) \int_{\mathbb{R}^d} p(t, s, y, x) q(s, y) dy - \sum_{i=1}^d b_i(t, x) \partial_{x_i} \int_{\mathbb{R}^d} p(t, s, y, x) q(s, y) dy \\
&= -\sum_{i=1}^d \partial_{x_i}(b_i(t, x)) q(t, x) - \sum_{i=1}^d b_i(t, x) \partial_{x_i} q(t, x) \\
&= -\sum_{i=1}^d \partial_{x_i}(b_i(t, x) q(t, x)).
\end{aligned} \tag{4.14}
$$

Similarly, for the second part, we have

$$II = \frac{1}{2} \sum_{i,j=1}^d \partial_{x_i x_j}^2 \left(\sigma \sigma_{ij}^T(t, x) q(t, x) \right).$$

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Hence the density function $q(t,x)$ of entrance measure ρ_t satisfies

$$\partial_t q = \mathcal{L}^*(t)q.$$

Conversely, if q is the solution of (4.11) satisfying (4.12). First from the heat kernel $p(t,s,x,y)$ of Fokker-Planck equation, we have (4.8). Then by Fubini's theorem, we have for all $\Gamma \in \mathcal{B}(\mathbb{R}^d)$, any $s \in \mathbb{R}$ and $t \geq s$

$$\begin{aligned}
P^*(t,s)\rho_s(\Gamma) &= \int_{\mathbb{R}^d} P(t,s,y,\Gamma)\rho_s(dy) \\
&= \int_{\mathbb{R}^d} \int_{\Gamma} p(t,s,y,x)dx q(s,y)dy \\
&= \int_{\Gamma} \int_{\mathbb{R}^d} p(t,s,y,x)q(s,y)dy dx \\
&= \int_{\Gamma} q(t,x)dx \\
&= \rho_t(\Gamma)
\end{aligned}$$

which means ρ is an entrance measure. With the correspondence of the periodic measure and the solution of infinite horizon Fokker-Planck equation, since the entrance measure is unique in this theorem, we know that the solution of the infinite horizon Fokker-Planck equation problem is unique. $\qquad\square$

Now assume that $u^r(t,s,x)$ and $K^{r_1,r_2}(t,s,x)$ are the solutions of equation (3.5) and (3.6) respectively, and the corresponding semi-groups P^r, P^{r_1,r_2} defined as

$$\begin{cases} P^r(t,s,x,\Gamma) := P(u^r(t,s,x) \in \Gamma) \\ P^{r_1,r_2}(t,s,x,\Gamma) := P(K^{r_1,r_2}(t,s,x) \in \Gamma). \end{cases} \tag{4.15}$$

We can also define $P^{r,*}(t,s)$ (resp. $P^{r_1,r_2,*}(t,s)$) as in (2.4) when we replace $\{P^*(t,s), P(t,s,x,\Gamma)\}$ by $\{P^{r,*}(t,s), P^r(t,s,x,\Gamma)\}$ (resp. $\{P^{r_1,r_2,*}(t,s), P^{r_1,r_2}(t,s,x,\Gamma)\}$). Let $\varphi^r(t), \varphi^{r_1,r_2}(t)$ be defined as in (3.7), and $\rho_t^r, \rho_t^{r_1,r_2}$ be the laws of $\varphi^r(t), \varphi^{r_1,r_2}(t)$ respectively. Then we have

$$P^{r,*}(t,s)\rho_s^r = \rho_t^r, \;\; P^{r_1,r_2,*}(t,s)\rho_s^{r_1,r_2} = \rho_t^{r_1,r_2}.$$

Similar to Condition 4.1 and 4.6, we give the following condition.

Condition 4.8. *The functions* $\tilde{b} = (\tilde{b}_i)_{i=1}^d, \tilde{\sigma} = (\tilde{\sigma}_{ij})_{i,j=1}^d$ *in Condition 3.2 satisfy the following conditions:*

(1) *The functions* $\tilde{b}(t,s,x), \tilde{\sigma}(t,s,x)$ *are globally bounded and uniformly Hölder-continuous in* (t,s,x).

(2) *The functions* $\tilde{b}(t,s,\cdot) \in C^1(\mathbb{R}^d; \mathbb{R}^d)$, $\tilde{\sigma}(t,s,\cdot) \in C^2(\mathbb{R}^d; \mathbb{R}^{d\times d})$ *such that* $\partial_{x_i}\tilde{b}_i, \partial^2_{x_ix_j}\tilde{\sigma}_{ij}$ *are bounded and Hölder-continuous.*

(3) *The function* $\tilde{\sigma}(t,s,x)$ *is invertible with* $\sup_{t,s\in\mathbb{R}, x\in\mathbb{R}^d} \|\tilde{\sigma}^{-1}(t,s,x)\| < \infty$.

Then by Theorem 2.8 and Theorem 4.5, we can directly deduce the following theorem

Theorem 4.9. *Assume Conditions 3.2, 3.3 and 4.8 hold. If $\alpha > \frac{\beta^2}{2}$, then ρ^r, ρ^{r_1,r_2} are the entrance measures of equation (3.5) and (3.6) respectively. Moreover $P^r(t,s,x,\cdot), P^{r_1,r_2}(t,s,x,\cdot)$ and the entrance measures $\rho_t^r, \rho_t^{r_1,r_2}$ are absolutely continuous with respect to the Lebesgue measure L on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, and hence have the density $p^r(t,s,x,y)$, $p^{r_1,r_2}(t,s,x,y)$, $q^r(t,y)$, $q^{r_1,r_2}(t,y)$ respectively.*

Similarly, we know that

$$q^r(t,x) = \int_{\mathbb{R}^d} p^r(t,s,y,x) q^r(s,y)(dy)$$

and

$$q^{r_1,r_2}(t,x) = \int_{\mathbb{R}^d} p^{r_1,r_2}(t,s,y,x) q^{r_1,r_2}(s,y)(dy).$$

Moreover, q^{r_1,r_2} (resp. q^r) satisfies the following quasi-periodic Fokker-Planck equation problem:

$$\partial_t q^{r_1,r_2} = \mathcal{L}^{r_1,r_2,*}(t) q^{r_1,r_2}, t \geq s \quad (resp.\ \partial_t q^r = \mathcal{L}^{r,*}(t) q^r,\ t \geq s)$$

where $\mathcal{L}^{r_1,r_2,*}$ (resp. $\mathcal{L}^{r,*}$) is given in (4.10) where b, σ is replaced by $\tilde{b}^{r_1,r_2}, \tilde{\sigma}^{r_1,r_2}$ (resp. $\tilde{b}^r, \tilde{\sigma}^r$).

By the proof of Theorem 3.5, we know that $u^r(t,s,x,\cdot) = u(t+r,s+r,x,\theta_{-r}\cdot)$ and $\varphi^r(t,\cdot) = \varphi(t+r,\theta_{-r}\cdot)$. Since θ_{-r} preserves the probability measure P, then $P^r(t,s,x,\cdot) = P(t+r,s+r,x,\cdot)$ and $\rho_t^r = \rho_{t+r}$. Hence their densities have the following relations

$$p^r(t,s,x,y) = p(t+r,s+r,x,y), \quad q^r(t,x) = q(t+r,x).$$

Acknowledgements

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