Sequential Search with Adaptive Intensity*

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Abstract

This paper studies sequential search problems, where a searcher chooses search intensity adaptively in each period. We fully characterize the optimal search rule and value, decomposing the inter-temporal change of search intensity into the fall-back value effect and the deadline effect. We show that the optimal search intensity (value) is submodular (supermodular) in fall-back value and time. It follows that the fall-back value effect increases when the deadline approaches, and the deadline effect decreases when a searcher's fall-back value gets higher. We further identify the connection between search with *full* and *no* recall to quantify the *value of recall*.

Keywords: Compound search; Optimal search intensity; Search value; Fall-back value effect; Deadline effect; Value of time; Value of recall; Encouragement effect; Discouragement effect

JEL Classification: D83, C61

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1 Introduction

Consider a compound search problem with a deadline, where a searcher searches not only sequentially but chooses her search intensity *adaptively* in each period. There are many such examples in the real world. For instance, consumer searches usually involve a deadline, such as gifts for anniversary and holiday hotel bookings, and consumers may adjust their search intensity from time to time. A football club, if having not signed the ideal players, may increase its recruitment intensity before the transfer window closes. In academia, a junior researcher needs to choose her research effort from time to time, depending on her research outputs and the remaining probation periods. Empirical studies have also suggested that economic agents commonly conduct compound, rather than purely sequential, searches in practice (Honka and Chintagunta, 2016; Gavazza et al., 2018), and that the majority of online consumers appear to be searching under deadline pressures (Coey et al., 2019).

In this paper, we develop a tractable framework for analyzing this category of compound search problems. We fully characterize the optimal search rule and value, in both cases of search with full and no recall, and derive some previously undiscovered properties of the optimal search intensity and value. We also discuss some extensions and potential applications of our results.

The theoretical literature on search has mainly concentrated on purely sequential search problems, where a searcher explores random prizes one-by-one sequentially. For example, the classic paper of Weitzman (1979) studied a so-called Pandora's problem: Pandora faces a number of closed boxes, which are heterogeneous in both search cost and prize distribution; in each period, she decides whether to open just a box or not; and her objective is to maximize the expected search value. Weitzman fully characterized the optimal search rule in this case. In contrast, the literature on compound search problems is relatively small, which seems unusual as compound search models are more general and would have more applications.

A strand of literature has adopted a reduced-form approach to model search intensity in compound search problems. In a seminal paper, Lippman and McCall (1976) considered search intensity in job searches, and assumed an *exogenous* relationship between search intensity and success probability. Likewise, continuous-time job search models usually assume that search intensity affects only the arrival rate of job offers, yet not their values (Rogerson et al., 2005, for a survey). Due to its tractability, the reduced-form approach has been widely applied to various settings, yet it does not reveal the intrinsic mechanism through which search intensity affects search outcomes.

By applying sampling theory, another strand of literature investigates compound search problems with a more solid micro-foundation, where the choice of search intensity endogenously determines the distribution of search outcomes. This literature can be further divided into two groups. The first group investigates infinite-horizon compound search problems, and a common conclusion is that constant search intensity is optimal. Vishwanath (1992) re-examined Pandora's problem, yet allowing a searcher to adopt a compound search strategy. She provided a sufficient condition of prize distributions, e.g., second-order stochastic dominance, under which the optimal compound search order remains the same as the sequential one, but she did not fully characterize the optimal search rule. Poblete and Spulber (2017) considered compound search strategies in a sequential innovation problem without a deadline, and showed the optimality of constant search intensity with full recall. Benkert et al. (2018) instead interpreted the choice of search intensity as the choice of prize distribution, and again showed that constant search intensity is optimal, as a searcher will sample from the same distribution in every period until a constant reservation value is reached.[1]

The second group introduces a finite deadline to Pandora's problem *à la* Weitzman (1979). The resulted finite sequential search model necessitates a searcher's adoption of a compound search strategy, as a purely sequential one is generally not optimal. Solving this problem is technically challenging, as it is no longer a stationary search process. For this finite sequential search model, Gal et al. (1981) and Benhabib and Bull (1983) considered the case of search with no recall, and Morgan (1983) and Morgan and Manning (1985) further studied the case of search with full recall.[2] These papers identified some monotonic properties of optimal search intensity, yet did not provide complete analyses to some key questions, such as optimal search value, the interaction between different effects determining the dynamics of optimal search intensity, and the role of recall. Moreover, the complexity of their analyses may have also impeded follow-up theoretical explorations and applications of their model.

This paper contributes to the literature by developing a simple framework for analyzing the

[1] The denumerable-armed bandit model of Banks and Sundaram (1992) includes a similar interpretation of search intensity, as the prize distributions are allowed to vary across different arms.

[2] With full recall, a searcher can reclaim a previously declined outcome at no extra search cost. With no recall, in contrast, a previously declined outcome is not reclaimable in later periods.

finite sequential search problems. With its tractability, we not only provide richer characterizations but deliver new insights of the optimal search rules and outcomes. Our results can be applied to a large set of related problems.

First, we fully characterize the optimal search rule and value in the case of search with full recall (Theorem 1). Based on the optimal search rule, we investigate the interaction between the *deadline effect* and the *fall-back value effect* that jointly determine the inter-temporal change of optimal search intensity.[3] The deadline effect suggests that a searcher will search more intensively when the deadline approaches, as there are fewer search opportunities left. On the other hand, the fall-back value effect suggests that a searcher will search less intensively when her fall-back value increases, as it becomes less likely for an additional search to increase her search outcome. We provide a decomposition result of the two effects (Lemma 1 and Equation (8)). Notably, we show that the optimal search intensity is submodular in fall-back value and time (Proposition 1). It then implies that the fall-back value effect gets stronger when the deadline approaches, and the deadline effect becomes weaker when the fall-back value increases.

We further show that the search value is supermodular in fall-back value and time (Proposition 2). Define the *value of time* as the additional value of conducting an optimal search in a certain period, which also measures a searcher's willingness-to-pay for such a search opportunity. The supermodularity result then suggests that the value of time is decreasing in a searcher's fall-back value and is increasing over time.

There is an intrinsic connection between search with *full* and *no* recall. Specifically, a searcher with no recall behaves as if she is searching with full recall in the last period, with her continuation value being the fall-back value. With this property, we can easily characterize the optimal search rule and value in the case of no recall (Proposition 3). Define the *value of recall* as the difference in search values across the two cases of search with full and no recall. We show that it is single-peaked, i.e., the value of recall first increases until the fall-back value reaches a threshold, decreases thereafter, and eventually turns to zero when the fall-back value becomes sufficiently large (Proposition 4).

Moreover, we find Morgan (1983)'s conjecture, that a searcher with full recall will search less intensively than one with no recall, is not always true. His conjecture is true only when

[3] These two effects were reported in Gal et al. (1981) and Morgan (1983), though not under the same terminology. Yet, the interaction between the two effects has not been investigated in the literature.

the fall-back value is relatively small, such that it is optimal to continue searching in both cases of full and no recall. When the fall-back value increases, a searcher with no recall may stop searching, while one with full recall may still continue and hence induce higher search intensity. We term this as the *encouragement* and *discouragement effect of recall*, and provide relevant characterizations (Proposition 5).

Finally, we introduce some potential applications and extensions of our model. First, we can model an agent's endogenous and non-stationary time preference within our framework, without introducing any behavioral assumption (Section 5.1). There is also a natural connection between our model and contests.

Our approach can extend standard contest models in various directions, especially dynamic ones (Section 5.2). We further re-investigate our model without imposing a finite deadline, and characterize the optimal search outcomes (Section 5.3). Moreover, we also explore other possible applications of our results in various dynamic decision-making problems (Section 5.4).

We relegate all the proofs to Appendix A.1, and provide additional numerical examples that help to illustrate the theoretical results in Appendix A.2.

2 The Model

A risk-neutral searcher, endowed with an initial fall-back value $y_1 \in \mathbb{R}_+$, has T periods to search. Her search technology is characterized by (C, F), where $C : \mathbb{R}_+ \to \mathbb{R}_+$ is a search cost function and F is the distribution of a unit search outcome defined on a real interval $[0, B]$. F is absolutely continuous with full support, i.e., $F(x) > 0$ for any $0 < x \leq B$. In each period, the searcher chooses a search intensity $m \in \mathbb{R}_+$ at the cost of $C(m)$, and the search outcome X_m is a random draw from the distribution $F^m := F(\cdot)^m$. When m is an integer, X_m is just the first order statistic, i.e., the largest outcome of m independent draws from F. We allow m to take real values, and hence the search intensity in our analysis can be interpreted as search effort, work hour, expenditure, and so on. We assume the search cost function $C(m)$ satisfies $C(0) = 0$, $C' \geq 0$, $C'' \geq 0$, and $C''' \leq 0$.[4] The marginal search cost function is denoted by $c(m) := C'(m)$.

When the search is with full recall, a searcher can reclaim a previously discovered value at no extra search cost. The fall-back value y_t (i.e., the highest value she has discovered till the

[4] This technical assumption of C''' is required only for the submodularity of optimal intensity (Proposition 1). All other results in this paper do not depend on this assumption.

beginning of period t) then evolves according to: for $t = 2, \cdots, T$,

(Fall-Back Value with Full Recall) $$y_t := \max\left\{y_{t-1}, x_{m_{t-1}}\right\},$$

where $x_{m_{t-1}}$ is the realized search outcome in period $t-1$ with search intensity m_{t-1}. Our analyses in Section 2 and 3 focus on the case of full recall, and we will explore the other case of no recall in Section 4. For simplicity, we do not consider time discounting in our model.[5] Without causing confusion, we may suppress the subscript t in m_t and y_t sometimes.

Given a fall-back value y at the beginning of period t, define the search value $W_t(y)$ as the searcher's expected search payoff by following an optimal search rule from period t till the deadline. The Bellman equation for the search problem is: for $t = 1, \cdots, T$,

$$W_t(y) = \max_{m\geq 0}\left\{\mathbb{E}W_{t+1}\left(\max\left\{y, X_m\right\}\right) - C(m)\right\}, \tag{1}$$

and $W_{T+1}(y) = y$, as the searcher keeps whatever she has after the deadline. For any t, note that $W_t(y)$ is convex, strictly increasing in y, and therefore, its derivative $W_t'(y)$ is well defined almost everywhere.[6]

For given y at the beginning of period t, we define the expected revenue of choosing intensity m by $R_t(m;y) := \mathbb{E}W_{t+1}\left(\max\left\{y, X_m\right\}\right)$. It then follows from the monotonicity of W_{t+1} that

$$\begin{aligned} R_t(m;y) &= \mathbb{E}\max\left\{W_{t+1}(y), W_{t+1}(X_m)\right\} \\ &= W_{t+1}(y) + \mathbb{E}\max\{W_{t+1}(X_m) - W_{t+1}(y), 0\}, \end{aligned} \tag{2}$$

where the first term $W_{t+1}(y)$ is the minimum search value guaranteed by choosing $m = 0$, and the second term is the expected surplus $W_{t+1}(X_m)$ above $W_{t+1}(y)$ by continuing searching with $m \geq 0$.

[5]The introduction of time discounting would not change our results qualitatively. Moreover, our model can be easily extended to allow stationary time discounting. For example, after introducing a constant discounting factor δ, the Bellman equation (1) becomes

$$W_t(y) = \max\left\{y, \delta \max_{m\geq 0}\left\{\mathbb{E}W_{t+1}\left(\max\left\{y, X_m\right\}\right) - C(m)\right\}\right\}.$$

[6]We can prove it by mathematical induction. Observe that $W_{T+1}(y) = y$ is convex and strictly increasing in y, and its derivative is well defined. As an induction hypothesis, suppose $W_{\tau+1}(y)$ is convex and strictly increasing in y for any $t \leq \tau \leq T$. We next show the properties also hold for $W_t(y)$. From (1), $W_t(y)$ is strictly increasing in y, as $\mathbb{E}W_{t+1}(\max\{y, X_m\}) = \mathbb{E}\max\{W_{t+1}(y), W_{t+1}(X_m)\}$ is strictly increasing in y for any $m \geq 0$. Moreover, $W_t(y)$ is convex, as a maximum of a family of convex functions. As a convex function, $W_t(y)$ is absolutely continuous and $W_t'(y)$ is well defined almost everywhere. See Proof of Theorem 1 for more details and related technical references.

Alternatively, we have the following equivalent expression[7]

$$R_t(m;y) \quad = \quad y + \int_y^B [1 - W'_{t+1}(x)F(x)^m]dx. \tag{3}$$

As (3) is differentiable in m, the (expected) marginal revenue $r_t(m;y)$ is given by

$$r_t(m;y) := \frac{\partial R_t(m;y)}{\partial m} = \int_y^B W'_{t+1}(x)F(x)^m |\ln F(x)| dx. \tag{4}$$

A search rule is a sequence of contingent plans of search intensity, denoted by $\{m_t\}_{t=1}^T$. To be specific, at the beginning of each period t, contingent on the fall-back value y, a searcher chooses her search intensity $m_t : [0,B] \to \mathbb{R}_+$ of that period. In principle, she can skip searching in that period by choosing $m_t = 0$, or stop searching permanently by choosing $\{m_\tau = 0\}_{\tau=t}^T$. In our setting, however, it is never optimal to resume a previously stopped search, as search cost does not depend on past search decisions, and search outcomes are determined by independent draws from the same distribution. We will formally show this result in the next section.

3 The Optimal Search Rule and Value

In this section, we characterize the optimal search rule and value in the case of full recall in Theorem 1 and further explore their properties in Proposition 1 and Proposition 2.

First, we derive the optimal search rule and value by backward induction. To be specific, let $m_t^*(y)$ denote the optimal search intensity in period t with a fall-back value y. In the last period T, a searcher with a fall-back value y would stop searching, i.e., $m_T^*(y) = 0$, if the marginal revenue $r_T(0;y)$ is greater than the marginal cost $c(0)$. Otherwise, she would continue searching with an intensity $m_T^*(y) > 0$ that equates the marginal revenue and the marginal cost. To ease notation, denote $X_T^* := X_{m_T^*(y)}$, the best search outcome by choosing $m_T^*(y)$. In the case of full recall, the search value in period T is thus

$$W_T(y) = \mathbb{E}\max\{y, X_T^*\} = y + \mathbb{E}\max\{X_T^* - y, 0\},$$

which consists of the fall-back value y and the expected surplus in search outcome X_T^* above y.

[7] Integrating by parts and noting that $W_{t+1}(B) = B$, the definition of $R_t(m;y)$ yields

$$\begin{aligned}\mathbb{E}W_{t+1}\left(\max\{y, X_m\}\right) &= W_{t+1}(y)F(y)^m + \textstyle\int_y^B W_{t+1}(x)dF(x)^m \\ &= W_{t+1}(y)F(y)^m + [W_{t+1}(B) - W_{t+1}(y)F(y)^m] - \textstyle\int_y^B W'_{t+1}(x)F(x)^m dx \\ &= B - \textstyle\int_y^B W'_{t+1}(x)F(x)^m dx = y + \textstyle\int_y^B [1 - W'_{t+1}(x)F(x)^m]dx.\end{aligned}$$

Let ξ denote the optimal cutoff value for stopping in period T. As $r_T(0,y)$ is decreasing in y by (4), it is evident that ξ is the unique solution to $r_T(0,\xi) = c(0)$ if $r_T(0,0) > c(0)$; and $\xi = 0$ otherwise. Therefore, a searcher with a fall-back value y stops searching in period T if and only if $y \geq \xi$. Interestingly, it turns out that this cutoff value is constant over time.[8] For instance, in period $T-1$, if she has a fall-back value $y \geq \xi$, then it would be her last period of search if she intends to, as she will certainly stop searching in period T given that her fall-back value will increase. Therefore, in period $T-1$, the cutoff value for optimal stopping is also ξ. In any period t, as long as $y < \xi$, the searcher will continue searching, and the optimal search intensity $m_t^*(y)$ is determined by the unique solution to $r_t(m;y) = c(m)$.

Theorem 1 formally states the optimal search rule and the search value in the case of full recall.

Theorem 1 (Search with Full Recall)**.** *Let ξ be the unique solution to $\int_\xi^B |\ln F(x)| dx = c(0)$ if $\int_0^B |\ln F(x)| dx > c(0)$; and $\xi = 0$ otherwise. The optimal search rule $\{m_t^*\}_{t=1}^T$ is determined as follows*

i. if $y \geq \xi$, then $m_t^*(y) = 0$;

ii. if $y < \xi$, then $m_t^*(y) > 0$ is the unique solution to $r_t(m;y) = c(m)$.

Moreover, the search value is

$$\begin{aligned} W_t(y) &= \mathbb{E}\max\{y, X_t^*\} \\ &= y + \mathbb{E}\max\{X_t^* - y, 0\} \\ &= y + \int_y^{\max\{y,\xi\}} \left(1 - \prod_{\tau=t}^T F(x)^{m_\tau^*(x)}\right) dx, \end{aligned} \tag{5}$$

where X_t^* is the best search outcome by following the optimal search rule $\{m_\tau^*\}_{\tau=t}^T$ from period t till the last period T, which has a distribution $\Pr[X_t^* \leq x] = \prod_{\tau=t}^T F(x)^{m_\tau^*(x)}$.

Remark. If the marginal search cost $c(0)$ is zero, then $\xi = B$, and hence the searcher will continue searching as long as possible unless she obtains the highest possible value B. On the other hand, when the marginal search cost is large enough (i.e., $c(0) \geq \int_0^B |\ln F(x)| dx$), then $\xi = 0$ and the searcher never conducts a search. It is noteworthy the cost function does not explicitly appear in (5), as it is already eliminated through the optimal condition $r_t(m;y) = c(m)$.

[8] In the case of search with *no* recall, Proposition 3 and Corollary 2 show the optimal cutoff value for stopping is strictly decreasing over time.

The best future search outcome X_t^* is drawn from the distribution $\prod_{\tau=t}^{T} F^{m_\tau^*}$ which is *endogenously* determined by the optimal plan $\{m_\tau^*\}_{\tau=t}^{T}$.[9] This is different from the distribution F, which is *exogenously* given as the search technology. Importantly, the distribution of the best future search outcome X_t^* is just equal to the marginal search value, as (5) yields the following corollary.

Corollary 1. *The marginal search value satisfies*

$$W_t'(y) = \prod_{\tau=t}^{T} F(y)^{m_\tau^*(y)}. \tag{6}$$

By introducing the new random variable X_t^*, we simplify the presentation and calculation of search value $W_t(y)$ in (5), if compared with that using the Bellman equation. To be specific, formula (5) embeds the solution to the Bellman equation (1) into a single random variable X_t^*, which has a distribution W_t' as shown in (6). It further yields the recurrence relationship between two adjacent periods t and $t+1$,

$$W_t'(y) = F(y)^{m_t^*(y)} W_{t+1}'(y), \tag{7}$$

which enables to solve the search value $W_t(y)$ recursively.

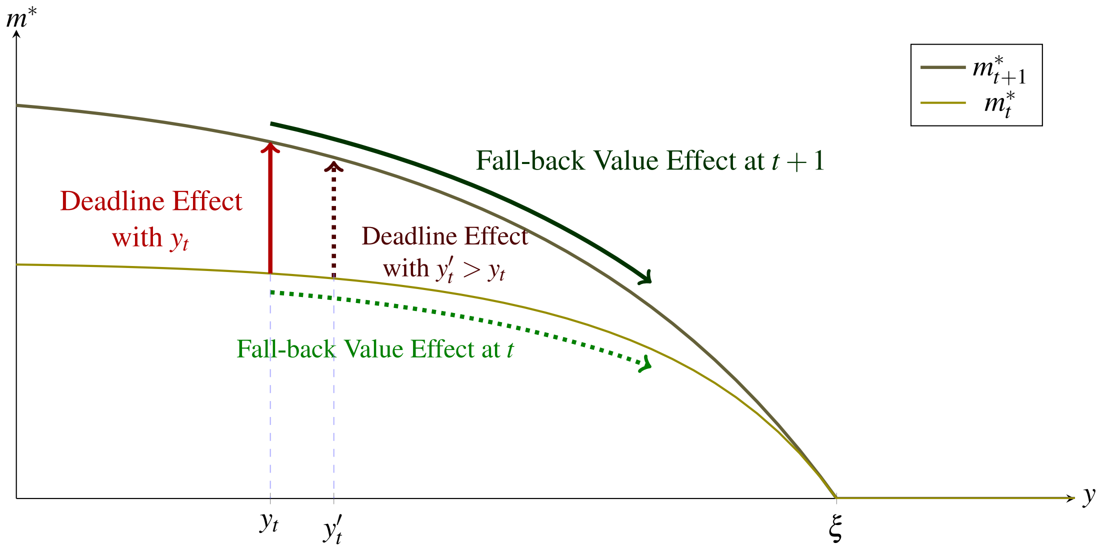
3.1 Dynamics of Optimal Search Intensity

When the fall-back value y increases, a searcher intends to search less intensively, because it becomes less likely that a further search would generate a better outcome. On the other hand, for a given y, the searcher will search more intensively when the deadline approaches, as she has fewer opportunities to improve her search outcome. We term the former as the *fall-back value effect* and the latter as the *deadline effect* on the optimal choice of search intensity. The two effects are formally summarizes in Lemma 1 below and also illustrated in Figure 4 and Figure 5 in Appendix A.2.[10]

Lemma 1. *The optimal intensity $m_t^*(y)$ is decreasing in y and increasing in t (strictly if and only if $y < \xi$).*

[9] Proof of Theorem 1 confirms that $\prod_{\tau=t}^{T} F^{m_\tau^*}$ indeed defines a distribution function.

[10] Lemma 1 corresponds to Morgan (1983, Proposition 3 and 4), yet under different terminology. Proof of Lemma 1 provides a simple derivation of these results based on Theorem 1.



NOTES: When the fall-back value increases from y_t to y'_t, the deadline effect, $m^*_{t+1}(y) - m^*_t(y)$, decreases. Moreover, $m^*_{t+1}(y)$ drops at a faster rate in y than $m^*_t(y)$.

Figure 1: Submodularity of Optimal Intensity

As the two effects are opposite to each other, their relative magnitudes determine whether the optimal intensity will increase or decrease in the next period. For an inter-temporal change of optimal intensity, e.g., from $m^*_t(y_t)$ to $m^*_{t+1}(y_{t+1})$, we can decompose it into the fall-back value effect and the deadline effect, as follows

$$m^*_{t+1}(y_{t+1}) - m^*_t(y_t) = \underbrace{\left[m^*_{t+1}(y_{t+1}) - m^*_{t+1}(y_t)\right]}_{\text{Fall-back value effect}} + \underbrace{\left[m^*_{t+1}(y_t) - m^*_t(y_t)\right]}_{\text{Deadline effect}}. \tag{8}$$

If the deadline effect dominates, the optimal intensity will increase, i.e., $m^*_t(y_t) < m^*_{t+1}(y_{t+1})$; otherwise, it will decrease. As the search outcome is *ex-ante* random, the sequence of $\{m^*_t(y_t)\}_{1\leq t\leq T}$ is necessarily a stochastic process, which is in general non-monotonic. Figure 6 in Appendix A.2 presents some sample paths.

We now establish an important property on the interactions between the two effects. Proposition 1 below shows that the optimal intensity $m^*_t(y)$ is submodular in y and t.

Proposition 1 (Submodularity of Optimal Intensity)**.** *For any t and $y < y'$,*

$$m^*_{t+1}(y) - m^*_t(y) \geq m^*_{t+1}(y') - m^*_t(y'), \tag{9}$$

where the inequality holds strictly if and only if $y < \xi$.

The submodularity of optimal intensity implies that the deadline effect, $m^*_{t+1}(y) - m^*_t(y)$, becomes smaller when the fall-back value y increases, or equivalently, the fall-back value effect, $m_t^*(y) - m_t^*(y')$, becomes stronger when the deadline approaches. Figure 1 illustrates the submodularity of optimal search intensity.

3.2 Properties of Search Value

Based on Theorem 1, we next provide some further properties of the search value $W_t(y)$. Denote $D_tW(y) := W_{t+1}(y) - W_t(y)$ as the difference of the search value between two adjacent periods. Proposition 2 below summarizes the first-order and the second-order properties of $W_t(y)$, which are also illustrated in Figure 2 (a).

Proposition 2 (Properties of Search Value)**.** *The search value $W_t(y)$ is*

i. increasing and convex in y, that is, $W_t'(y) \geq 0$ is increasing in y;

ii. decreasing and concave in t, that is, $D_tW(y) \leq 0$ is decreasing in t; and

iii. supermodular in y and t, that is, $W_t'(y)$ is increasing in t and $D_tW(y)$ is increasing in y.

The results hold strictly if and only if $y < \xi$.

We define the absolute value $|D_tW(y)| = W_t(y) - W_{t+1}(y)$ as the *value of time* in period t, which measures the value of an optimal search in that period. For instance, given a fall-back value y, the search value at the beginning of period t is $W_t(y)$. If the searcher has one less period for search, then the search value decreases to $W_{t+1}(y)$. Therefore, $|D_tW(y)|$ measures the value of an optimal search in that period, which is also a searcher's "willingness-to-pay" for such a search opportunity.[11]

Proposition 2 suggests some interesting properties of the value of time. First, the supermodularity of $W_t(y)$ implies that, when y increases, it becomes less likely to achieve a better search outcome through an extra search, and therefore, the value of time decreases. Second, the concavity of $W_t(y)$ in t further implies that the value of time is increasing in t. The intuition is that, when the deadline gets closer, the searcher will have fewer opportunities to improve her search value. Therefore, she is willing to pay more for an extra search opportunity.

Example 1 below considers an oil firm that needs a license to explore a new oil field for a finite period of time.[12] The numerical results illustrate how the value of time, or the value of a

[11] It is evident that the value of time is 0 in an infinite horizon search problem without time discounting, if all the objects are homogeneous in terms of F and unit search cost.

[12] This example is inspired by Morgan (1983).

	$y=0$		$y=0.2$		$y=0.4$		$y=0.5$		$y=0.55$	
	W_t	$\lvert D_tW\rvert$	W_t	$\lvert D_tW\rvert$	W_t	$\lvert D_tW\rvert$	W_t	$\lvert D_tW\rvert$	W_t	$\lvert D_tW\rvert$
$t=1$	0.54738	0.01074	0.54739	0.01071	0.54849	0.00954	0.55413	0.00615	0.56487	0.00250
$t=2$	0.53664	0.01963	0.53668	0.01947	0.53895	0.01599	0.54798	0.00893	0.56236	0.00312
$t=3$	0.51701	0.04946	0.51721	0.04759	0.52296	0.03204	0.53906	0.01405	0.55925	0.00399
$t=4$	0.46745	0.46745	0.46962	0.26962	0.49093	0.09093	0.52501	0.02501	0.55527	0.00527

Table 1: The Search Value (W_t) and the Value of Time ($|D_tW| = W_t - W_{t+1}$): $C(m) = 0.1m$, $F(x) = x \cdot \mathbb{1}(0 \leq x \leq 1)$, $T = 4$

license, changes with t and y.

Example 1 (Value of License)**.** An oil firm wishes to explore a new oil field for four months. The firm can choose the monthly exploration intensity m at the cost of $C(m) = 0.1m$. For an exploration intensity m, the outcome is a random draw from F^m, where the unit exploration technology is given by a uniform distribution $F(x) = x \cdot \mathbb{1}(0 \leq x \leq 1)$. The value of time $|D_tW(y)|$ measures the firm's willingness-to-pay for an exploration opportunity in period t, when its fall-back value is y. Table 1 provides the numerical results on how the firm values the license over time and across different current fall-back value y's.[13] For instance, when the firm has zero fall-back value, the firm's willingness-to-pay for the license for the first month is $|D_1W(0)| = W_1(0) - W_2(0) = 0.54738 - 0.53664 = 0.01074$. The willingness-to-pay for the license increases when the deadline approaches, and it increases dramatically in the last period (e.g., $|D_3W(0)| = 0.04946$ and $|D_4W(0)| = 0.46745$). However, when the firm has a high fall-back value y, the value of time is marginal even in the last period (e.g., $|D_3W(0.55)| = 0.00399$ and $|D_4W(0.55)| = 0.00527$).

4 The Role of Recall

This section considers the other important case of search with no recall, where a previous discovered outcome is not reclaimable in later periods. In the case of no recall, it is evident that the fall-back value y_t is equal to the best search outcome in period $t-1$, that is, $y_1 = 0$ and for $t = 2, \cdots, T$,

(Fall-Back Value with No Recall) $$y_t := x_{m_{t-1}}.$$

[13] See Figure 7 for its Mathematica code.

Given a fall-back value y at the beginning of period t, in the case of no recall, denote $\hat{W}_t(y)$ as the search value of following an optimal search rule from period t on. The Bellman equation is thus: for $t = 1, \cdots, T$,

$$(10) \qquad \hat{W}_t(y) = \max_{m \geq 0} \left\{ y, \mathbb{E}\hat{W}_{t+1}(X_m) - C(m) \right\},$$

and $\hat{W}_{T+1}(y) = y$, as a searcher keeps whatever she has in period $T+1$.

In the case of no recall, we first derive the optimal cutoff for stopping in period t, denoted by $\hat{\xi}_t$. In any period $t \leq T$, the searcher can either opt for the fall-back value y or continue searching by discarding y. In the latter case, she expects the search value of $\hat{W}_t(0)$ because her fall-back value turns to zero. Therefore, the search value with a fall-back value y is, for any $t \leq T$,

$$(11) \qquad \hat{W}_t(y) = \max\{y, \hat{W}_t(0)\}.$$

Clearly, it is optimal to stop searching if $y \geq \hat{W}_t(0)$.[14] Therefore, the optimal cutoff for stopping is determined by $\hat{\xi}_t = \hat{W}_t(0)$. It then follows from (10) that

$$\begin{aligned} \hat{\xi}_t = \hat{W}_t(0) &= \max_{m \geq 0} \left\{ \mathbb{E}\hat{W}_{t+1}(X_m) - C(m) \right\} \\ &= \max_{m \geq 0} \left\{ \mathbb{E}\max\{X_m, \hat{W}_{t+1}(0)\} - C(m) \right\} \\ (12) \qquad &= W_T(\hat{W}_{t+1}(0)), \end{aligned}$$

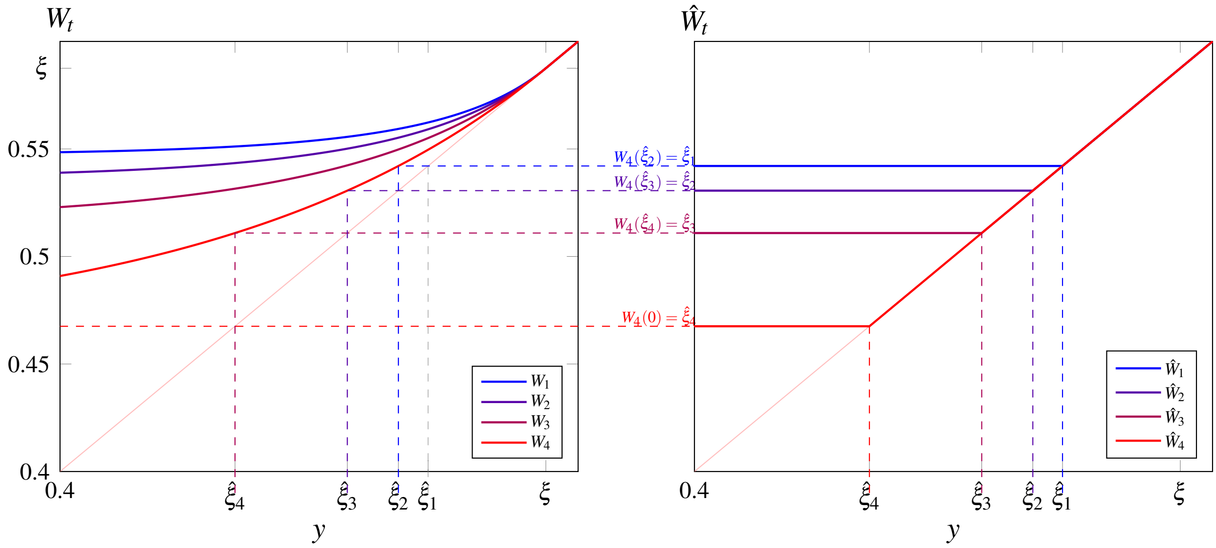
where the second line is due to (11) and the last one is from (1). Substituting $\hat{W}_{t+1}(0) = \hat{\xi}_{t+1}$ into (12), we have a recurrence relation between the optimal cutoffs of two adjacent periods, that is, for any $t < T$,

$$(13) \qquad \hat{\xi}_t = W_T(\hat{\xi}_{t+1}).$$

It is interesting to observe that $\hat{\xi}_t$ and $\hat{\xi}_{t+1}$, the optimal cutoffs in the case of no recall, are linked through W_T, the last-period search value in the case of full recall. By recursively applying (13) with the initial condition of $\hat{\xi}_T = W_T(0)$, we can solve for $\hat{\xi}_t$ as an iterated function value of $W_T(0)$:

$$(14) \qquad \hat{\xi}_t = W_T^{T-t+1}(0) := \underbrace{W_T(W_T(\cdots(W_T(0))))}_{T-t+1 \text{ times}}.$$

[14] At $y = \hat{\xi}_t$, both searching with a positive intensity and stopping are optimal, because they yield the same search value $\hat{W}_t(y)$. For convenience, we assume that a searcher stops at the cutoff $\hat{\xi}_t$.



(a) Search Value with Full Recall

(b) Search Values with No Recall

NOTES: With full recall, the search value $W_t(y)$ is strictly increasing and convex for $y < \xi$, with a slope smaller than 1. With no recall, in contrast, the search value $\hat{W}_t(y)$ is constant for $y < \hat{\xi}_t$. Moreover, the cutoff value $\hat{\xi}_t$ with no recall is determined by the search value $W_T(\hat{\xi}_{t+1})$ with full recall in period T having the next period cutoff value $\hat{\xi}_{t+1}$ as a fall-back value.

Figure 2: The Search Value: $C(m) = 0.1m$, $F(x) = x \cdot \mathbb{1}(0 \leq x \leq 1)$, $T = 4$

We next derive the optimal search intensity, denoted by $\hat{m}_t^*(y)$, in the case of no recall. Given a fall-back value y at the beginning of period t, if a searcher decides to search, the expected revenue of choosing intensity m is, from (10),

$$\hat{R}_t(m;y) := \mathbb{E}\hat{W}_{t+1}(X_m) = \mathbb{E}\max\{X_m, \hat{\xi}_{t+1}\} = R_T(m;\hat{\xi}_{t+1}), \tag{15}$$

where the second equation is implied by (11) with $\hat{\xi}_{t+1} = \hat{W}_{t+1}(0)$, and the third by (2) with $W_{T+1}(y) = y$. That is, $\hat{R}_t(m;y)$, the expected revenue with no recall, is equal to $R_T(m;\hat{\xi}_{t+1})$, the last-period search revenue with full recall with $\hat{\xi}_{t+1}$ as her fall-back value instead.[15] As a result, if a searcher with no recall continues searching (i.e., $y < \hat{\xi}_t$), the optimal search intensity is determined by $\hat{m}_t^*(y) = m_T^*(\hat{\xi}_{t+1})$, which does not depend on y. In the case of no recall, therefore, the *fall-back value effect* vanishes, and the *deadline effect* solely determines the dynamics of optimal intensity. Proposition 3 characterizes the optimal search rule and search value with no

[15] Using (2), we can decompose the expected revenue with no recall

$$\hat{R}_t(m;y) = R_T(m;\hat{\xi}_{t+1}) = \hat{\xi}_{t+1} + \mathbb{E}\max\{X_m - \hat{\xi}_{t+1}, 0\}$$

into the next-period cutoff $\hat{\xi}_{t+1}$ for stopping and the expected surplus X_m above $\hat{\xi}_{t+1}$ by searching with $m \geq 0$ in a single-period problem.

recall.[16]

Proposition 3 (Search with No Recall)**.** *For any $t \leq T$, the optimal cutoff for stopping with no recall is determined by $\hat{\xi}_t = W_T^{T-t+1}(0)$, as given in (14), and the optimal search rule $\{\hat{m}_t^*\}_{t=1}^T$ is given by*

$$\hat{m}_t^*(y) = \begin{cases} m_T^*(\hat{\xi}_{t+1}) & \text{if } y < \hat{\xi}_t \\ 0 & \text{if } y \geq \hat{\xi}_t, \end{cases} \tag{16}$$

with $\hat{\xi}_{T+1} = 0$. Furthermore, the search value by following the optimal search rule is

$$\hat{W}_t(y) = \max\{y, \hat{\xi}_t\}. \tag{17}$$

Theorem 1 and Proposition 3 enable us to examine the role of recall in affecting the optimal search rules and the search values. First, Corollary 2 compares the optimal cutoffs for stopping between the two cases of no and full recall.

Corollary 2. *If $\int_0^B |\ln F(x)|dx > c(0)$, then it holds*

$$\xi > \hat{\xi}_1 > \hat{\xi}_2 > \cdots > \hat{\xi}_T > 0. \tag{18}$$

Otherwise, $\hat{\xi}_t = \xi = 0$ for all t.

Note that $\int_0^B |\ln F(x)|dx$ is the largest possible marginal revenue $r_T(0;0)$ with full recall, as suggested by (4) and Corollary 1. If it is greater than the smallest possible marginal search cost $c(0)$, then the optimal cutoff $\xi > 0$ and searching is desirable when $y < \xi$. Otherwise, the expected revenue is always dominated by the search cost. In this case, a searcher with full recall never initiates a search (i.e., $\xi = 0$), neither does a searcher with no recall (i.e., $\hat{\xi}_t = 0$).

In the non-trivial case of $\int_0^B |\ln F(x)|dx > c(0)$, Corollary 2 shows that the optimal cutoff $\hat{\xi}_t$ with no recall is decreasing over time, and is strictly bounded above by ξ. The intuition of this monotonicity is as follows. As shown in (17), for a searcher with no recall, $\hat{\xi}_t$ represents her continuation value of search in period t, which naturally gets smaller over time, as the searcher has fewer opportunities to increase her payoff when approaching T. Moreover, by (14), the highest cutoff $\hat{\xi}_1 = W_T(0)$ is strictly smaller than $\xi = W_T(\xi)$. Therefore, when $\xi > 0$, all the cutoff values with no recall are strictly smaller than that with full recall.

[16] Gal et al. (1981) and Morgan (1983) studied the optimal search rule with no recall. However, our characterization reveals the connection between the two cases of search with full and no recall.

We next examine the impacts of a recall option on the search values. Specifically, we define the *value of recall* as the difference in search values between the two cases of full and no recall, as follows

$$\text{(Value of Recall)} \qquad VR_t(y) := W_t(y) - \hat{W}_t(y).$$

It is clear that $VR_t(y)$ is non-negative, as a searcher always benefits from the option of recalling a fall-back value. Furthermore, we are interested in how $VR_t(y)$ changes with y. We answer this question by comparing the shapes of $W_t(y)$ and $\hat{W}_t(y)$.

For a search with full recall, if it is optimal to continue searching, a searcher has a positive probability that the best search outcome X_t^* is greater than her fall-back value y (i.e., $\Pr[X_t^* > y] > 0$). On the other hand for any $y > 0$, X_t^* may not exceed y with a positive probability (i.e., $\Pr[X_t^* \leq y] > 0$). From Theorem 1 and Corollary 1, the marginal search value is then, for any $0 < y < \xi$,

$$0 < W_t'(y) = \Pr[X_t^* \leq y] < 1. \tag{19}$$

For a search with no recall, it is clear from (17) in Proposition 3 that

$$\hat{W}_t'(y) = \begin{cases} 0 & \text{if } y < \hat{\xi}_t \\ 1 & \text{if } y > \hat{\xi}_t. \end{cases} \tag{20}$$

Figure 2 (a) and (b) illustrate the shapes of $W_t(y)$ and $\hat{W}_t(y)$ respectively.

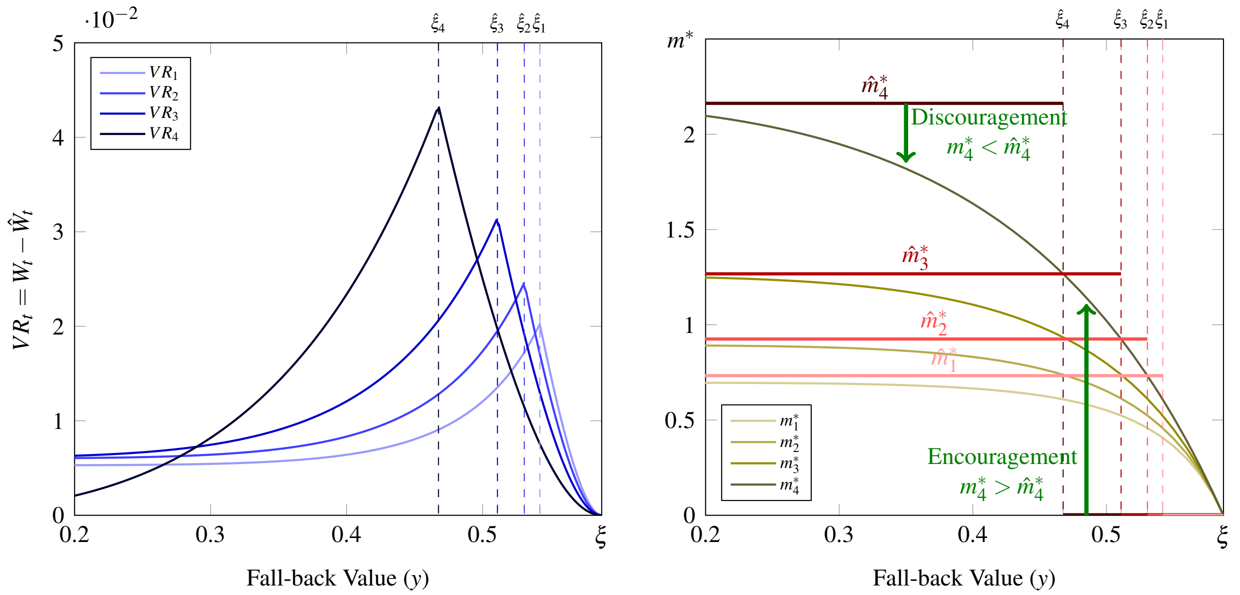
When $\xi > 0$, we know from Corollary 2 that $0 < \hat{\xi}_t < \xi$ for all $t \leq T$. The simple comparison between (19) and (20) then shows that $VR_t(y)$ first increases in y for $y \leq \hat{\xi}_t$, then decreases, and eventually turns to zero when $y \geq \xi$. Proposition 4 below summarizes the properties of $VR_t(y)$, as also illustrated in Figure 3 (a).

Proposition 4 (Value of Recall)**.** *Suppose* $\int_0^B |\ln F(x)| dx > c(0)$*. For any t and y, the value of recall* $VR_t(y) \geq 0$*. Moreover, for any t:*

i. $VR_t(y)$ *is strictly increasing in y on* $[0, \hat{\xi}_t]$*;*

ii. $VR_t(y)$ *is strictly decreasing in y on* $[\hat{\xi}_t, \xi]$*; and*

iii. $VR_t(y) = 0$ *for any* $y \geq \xi$*.*



(a) Value of Recall, $VR_t(y)$: it is single-peaked in y – It increases until reaching $\hat{\xi}_t$, and decreases thereafter. It is not monotone in t either. For example, $VR_1 < VR_2 < VR_3 < VR_4$ at $y = \hat{\xi}_4$ and $VR_1 > VR_2 > VR_3 > VR_4$ at $y = \hat{\xi}_1$; while $VR_2 > VR_3 > VR_1 > VR_4$ at $y = \hat{\xi}_2$.

(b) Optimal Intensity m^* with Full and $\hat{m}^*$ with No Recall: With full recall, $m^*(y)$ decreases continuously in y; yet with no recall, $\hat{m}^*(y)$ is constant when $y < \hat{\xi}_t$ and drops to 0 thereafter. Therefore, a searcher with full recall search more intensively for any $y \in [\hat{\xi}_t, \xi)$.

Figure 3: Role of Recall in Search: $C(m) = 0.1m$, $F(x) = x \cdot \mathbb{1}(0 \leq x \leq 1)$, $T = 4$

The next question is how the presence of a recall option would affect optimal search intensity. Morgan (1983) conjectured that a search with full recall induces lower search intensity than one with no recall *ceteris paribus*. We find that the conjecture is not always true. Theorem 1 shows that, for a search with full recall, the optimal search intensity $m_t^*(y)$ decreases continuously in y and turns to zero when $y \geq \xi$. Proposition 3 shows that, for a search with no recall, the optimal search intensity $\hat{m}_t^*$ is constant when $y < \hat{\xi}_t$ and drops to zero when $y \geq \hat{\xi}_t$. Furthermore, in the non-trivial case of $\int_0^B |\ln F(x)| dx > c(0)$, Corollary 2 shows that $0 < \hat{\xi}_t < \xi$ for all $t \leq T$.

As a result, when the fall-back value is small, i.e., $0 < y < \hat{\xi}_t$, a recall option discourages search intensity as Morgan conjectured, where $\hat{m}_t^*(y) > m_t^*(y) > 0$. When $\hat{\xi}_t \leq y < \xi$, however, a searcher with no recall stops searching while one with full recall still continues. In this case, a recall option encourages search intensity instead, as $m_t^*(y) > \hat{m}_t^*(y) = 0$. Finally, when $y \geq \xi$, it is optimal to stop searching in both cases. Proposition 5 below summarizes the en/discouragment effect of a recall option on optimal search intensity, as also illustrated in Figure 3 (b).[17]

Proposition 5 (En/Dis-couragement Effect of Recall)**.** *Suppose* $\int_0^B |\ln F(x)| dx > c(0)$*. For any* $t \leq T$*:*

i. *for any* $0 < y < \hat{\xi}_t$*, recall discourages search intensity, that is,* $m_t^*(y) < \hat{m}_t^*(y)$*;*

ii. *for any* $\hat{\xi}_t \leq y < \xi$*, recall encourages search intensity, that is,* $m_t^*(y) > \hat{m}_t^*(y)$*; and*

iii. *for any* $y \geq \xi$*,* $m_t^*(y) = \hat{m}_t^*(y) = 0$*.*

In addition, for any $t < T$*,* $\hat{m}_t^*(0) > m_t^*(0)$*; and* $\hat{m}_T^*(0) = m_T^*(0)$*.*

5 Discussion

5.1 Endogenous Time Preferences

We do not consider time discounting in this paper, although its introduction will not change our results qualitatively. An interesting observation is that we can define an agent's endogenous time preference based on the framework of our model. Our results would suggest that an agent's time preference is non-stationary, with no need to introduce any behavioral assumption.

To be specific, we can think of $W_t(y)$ as an agent's *perceived utility* of her fall-back value y in period t. Suppose she is indifferent between having a fall-back value y in period t and another value z in a later period $t' > t$. It then follows that $W_t(y) = W_{t'}(z)$, or equivalently $y = W_t^{-1}(W_{t'}(z))$, which implies that a future value of z in period t' is equivalent to the value of $W_t^{-1}(W_{t'}(z))$ at the present period t.[18] Thus, following the same spirit of discounted utility theory, we can define a time discounting function

$$\delta_{t,t'}(z) := W_t^{-1}(W_{t'}(z))/z, \tag{21}$$

which depends on the wealth level z as well as the time periods of t and t'. Note that the discounting function in (21) incorporates the agent's optimal search plan as well as the exogenous search environment.

[17] On the role of recall, Morgan (1983, Proposition 6) only derived the result for the last period T. By implicitly assuming $y \leq \hat{\xi}_T$, he showed that $m_T^*(y) \leq m_T^*(0) = \hat{m}_T^*(y)$. We formally prove his conjecture for any period t, provided that it is optimal to continue searching in both cases. In addition, we find this comparative result is not true for $\hat{\xi}_T < y < \xi$.

[18] The inverse function W_t^{-1} exists as Theorem 1 confirms W_t is strictly increasing.

In Lee and Li (2020), we formally derive the non-stationary and context-dependent time preference using this finite sequential search model, which provides a rational foundation of empirical/behavioral evidence such as procrastination and wealth effect in inter-temporal decision making.

5.2 A Search Contest Model

There is a natural connection between our search model and contest models. Consider a contest among n homogeneous searchers, who have a common search technology (C,F) and an initial fall-back value $y \geq 0$. Each searcher i simultaneously chooses her search intensity m_i, and the one with the highest search outcome wins. With full recall, the search outcome $Y_i := \max\{y, X_i\}$ where X_i is a random draw from F^{m_i}. Let $M_{-i} := \sum_{j \neq i} m_j$ and $Y_{-i} := \max_{j \neq i}\{Y_j\}$. A searcher i's winning probability by choosing m_i is

$$\Pr(Y_i \geq Y_{-i}) = F(y)^{M_{-i}} \left[\frac{F(y)^{m_i}}{n} + (1 - F(y)^{m_i}) \right] + \int_y^B F(x)^{M_{-i}} dF(x)^{m_i}.$$

When $y = 0$, it is clear that the winning probability is $m_i/(m_i + M_{-i})$, which coincides with the Tullock contest success function (Tullock, 1980).[19]

Our model can be applied to the extensions of standard contest models in various directions: for example, *asymmetric contests* where searchers are heterogeneous in y_i; and contests under complete or incomplete information, depending on whether y_i is private information or not. To be concrete, consider the above example in a case where there is a single leader 1 in the sense that $y_1 > \max_{j \neq 1}\{y_j\}$. The other setting remains unchanged, and a searcher i's winning probability by choosing m_i is thus

$$\Pr(Y_i \geq Y_{-i}) = \int_{y_1}^B F(x)^{M_{-i}} dF(x)^{m_i} + F(y_1)^{M_{-1}} F(y_1)^{m_1} \cdot \mathbb{1}(i = 1).$$

When all the searchers choose the same intensity level, the single leader 1 has a strictly higher winning probability than others.

Moreover, there is a close link between our single-agent sequential search model and dynamic contest models that involve dynamic competition among multiple players. For example,

[19] In fact, Baye and Hoppe (2003, Theorem 1) have shown the strategic equivalence between a Tullock contest and an innovation tournament, where they modeled innovation the same way as we did for a search problem, except that they implicitly assumed $y = 0$. Moreover, there has been other literature trying to provide a solid theoretical foundation for contest success functions. Jia et al. (2013, Section 3) provided a clear discussion of the related literature.

one can apply our results to dynamic R&D tournaments, which are usually bounded by a deadline, and innovations are modeled as sequential search processes. The extant literature on R&D tournaments usually assumes a binary choice of search effort (i.e., $m = 0$ or 1), while ignores the possibility that agents may adjust their effort inputs adaptively and continuously over time.[20]

5.3 Infinite Horizon Search

We re-investigate our model without a finite deadline, i.e., $T \to \infty$. To ease discussion, we explicitly specify the deadline T in notations. For a search with no recall, by (14), we have that for any finite t, the cutoff for stopping $\hat{\xi}_{t,T} = W_T^{T-t+1}(0)$, which is the $(T-t+1)$-th iterate of $W_T(0)$. As $W_T(y)$ is the search value of the last-period search problem, it does not depend on T in fact. Furthermore, from Corollary 1, a self-mapping W_T on $[0,\xi]$ is *contractive*, i.e, for all distinct $y, y' \in [0,\xi]$, $|W_T(y) - W_T(y')| < |y - y'|$. By Edelstein (1962)'s Fixed Point Theorem, the iterate of $W_T(y)$ converges to the fixed point ξ.

As a result, when $T \to \infty$, the optimal cutoff $\hat{\xi}_{t,T} = W_T^{T-t+1}(0)$ for stopping with no recall converges to the optimal cutoff ξ with full recall, which is constant over time. The intuition is that, in an infinite horizon problem, a searcher with no recall faces the same decision problem at the beginning of each period, and therefore, the optimal cutoff value for stopping does not change over time. This result is in sharp contrast to the result of strictly decreasing cutoffs when T is finite. Applying the results of Theorem 1 and Proposition 3, we then have, for any y

$$\lim_{T \to \infty} \hat{W}_{t,T}(y) = \max\{y, \xi\}.$$

For a search with full recall, as $\hat{W}_{t,T}(y) \leq W_{t,T}(y) \leq \max\{y, \xi\}$, it also follows that for any y

$$\lim_{T \to \infty} W_{t,T}(y) = \max\{y, \xi\}.$$

In summary, in an infinite horizon search problem, i.e., $T \to \infty$, the search values of full and no recall will converge to the same value.

5.4 Further Applications

There are potentials to apply our results to a large set of dynamic decision-making problems that involve the adaptive decisions of economic agents. For example, in judicial decision-making,

[20] For example, Taylor (1995) studied a dynamic R&D tournament model, yet imposed a restrictive assumption that contestants can only make a binary choice of effort level in each period. Benkert and Letina (2020) studied the design of optimal R&D tournaments by introducing interim transfers in every period, yet they did not consider agents' adaptive choice of effort either.

Chen and Eraslan (2020) considered an infinite-horizon problem in which a decision-maker chooses between investigation or not in each period. Applying our model, one can study a more general situation where the decision-maker may decide her "investigation intensity" (e.g., investigation hour, amount of document read, and so on) adaptively in each period.

In scheduling interviews for a job vacancy, an employer usually conducts "compound" searches for potential job candidates. For example, the employer may invite candidates for interviews batch by batch; depending on past interview outcomes, the employer may either stop searching and make job offers, or continue searching with adjusted intensity in the next round. Morgan (2017) studied a related problem of optimal interviewing, yet the employer does not directly select the sample size in each period. Our results can be applied to this kind of optimal scheduling problem.

In search-bargaining literature, such as Baucells and Lippman (2004) and Gantner (2008), players can search for an outside option while bargaining with each other. Applying our results, one can develop a dynamic search-bargaining model in which players can choose their search intensity for outside options. In particular, it would be interesting to investigate the case where players can observe other players' search intensity, while the realized outside options are each player's private information.

Our results can also be applied to the studies of auditing. Since Simunic (1980) and Dye (1993), it is a standard assumption in auditing literature that auditors can directly choose the quality or the informativeness of their audit reports, which is measured by the probability of the "correct attestation." In practice, however, auditors choose their audit efforts instead, which then stochastically affects the quality of their reports. The distinction between audit effort as a control variable and the quality of the audit report as a stochastic outcome is particularly important in modeling dynamics audit problems.

For instance, Schwartz (1997) and Pae and Yoo (2001) consider the strategic interaction between an auditee and an auditor, where the auditee and the auditor sequentially choose the quality of the internal control system and the audit effort level. By applying our results, one can develop a dynamic audit model, in which auditors can adjust their effort level adaptively in each period after assessing the internal control system and updating information from previous auditing outcomes.

A Appendix

A.1 Proofs

Proof of Theorem 1.

We prove it by backward induction, starting from the last period T.

Step 1: Let $t = T$. As $W_{T+1}(y) = y$, by definition of $R_T(m;y)$, integrating by parts yields the expected revenue

$$R_T(m;y) = \mathbb{E}\max\{y, X_m\} = yF(y)^m + \int_y^B x dF(x)^m = y + \int_y^B (1 - F(x)^m) dx, \tag{22}$$

which is differentiable with respect to m. The marginal revenue is thus well-defined and it follows

$$r_T(m;y) := \frac{\partial R_T(m;y)}{\partial m} = \int_y^B F(x)^m |\ln F(x)| dx, \tag{23}$$

which is non-negative and continuous in m and y, and $\lim_{m\to\infty} r_T(m;y) = 0$. Moreover,

$$\frac{\partial r_T}{\partial m} = -\int_y^B F(x)^m \ln^2 F(x) dx \leq 0 \text{ and } \frac{\partial r_T}{\partial y} = -F(y)^m |\ln F(y)| \leq 0, \tag{24}$$

where both inequalities hold strictly if and only if $y < B$. Recall also that the marginal search cost $c(m)$ is continuous and increasing in m.

Solving the Kuhn-Tucker condition that

$$r_T(m;y) - c(m) \leq 0, \quad m \geq 0, \text{ and } \quad (r_T(m;y) - c(m))m = 0,$$

we get the solution $m_T^*(y)$ as follows.

(i) If $r_T(0;0) = \int_0^B |\ln F(x)| dx \leq c(0)$, then for any $y \geq 0$ and $m > 0$, $r_T(m;y) < r_T(0;0) \leq c(0) \leq c(m)$. The Kuhn-Tucker condition then implies that $m_T^*(y) = 0$ for any y. Hence, the optimal cutoff for stopping is $\xi = 0$.

(ii) If $r_T(0;0) > c(0)$, as r_T is continuous and strictly decreasing in y with $r_T(0;B) = 0$, there exists a unique $\xi \in (0, B]$ that satisfies $r_T(0;\xi) = \int_\xi^B |\ln F(x)| dx = c(0)$. For any $y \geq \xi$ and $m > 0$, as $r_T(m;y) < r_T(0;\xi) \leq c(0) \leq c(m)$, from the Kuhn-Tucker condition again, we have $m_T^*(y) = 0$. For any $y < \xi$, we have $r_T(0;y) > c(0)$. As $c(m)$ is increasing and $r_T(m;y)$ is strictly decreasing in m, with $\lim_{m\to\infty} r_T(m;y) = 0$, there then exists a unique $m_T^*(y)$ that solves $r_T(m;y) = c(m)$.

Next we derive $W_T(y)$. For any $y \geq \xi$, we have from (1) that $W_T(y) = W_{T+1}(y) = y$ as $m_T^*(y) = 0$. From (22), $\partial R_T(m;y)/\partial y = F(y)^m$, which is increasing in y. It then follows that $R_T(m;y) - C(m)$ is convex in y, and $W_T(y)$, as the maximum of a family of convex functions, is also convex in y.[21] As a convex function, $W_T(y)$ is absolutely continuous, and $W_T'(y)$ is defined almost everywhere.[22] As such, $W_T(y)$ can be represented by a definite integral of its derivative,[23] i.e., for any $y \leq \xi$,

$$W_T(y) = W_T(\xi) - \int_y^\xi W_T'(x)dx = \xi - \int_y^\xi W_T'(x)dx, \tag{25}$$

where we use the boundary condition $W_T(\xi) = \xi$. Applying envelop theorem to (1) and (22),[24] we then have

$$W_T'(x) = \frac{\partial R_T(m_T^*(x);x)}{\partial x} = F(x)^{m_T^*(x)}.$$

Substituting it back into (25) gives

$$W_T(y) = \xi - \int_y^\xi F(x)^{m_T^*(x)}dx = y + \int_y^\xi \left(1 - F(x)^{m_T^*(x)}\right) dx. \tag{26}$$

Combining (26) for $y < \xi$ and $W_T(y) = y$ for $y \geq \xi$, we have (5) for the last period T.

Finally, we show that $F(y)^{m_t^*(y)}$ is continuous and increasing in y, in order to verify it indeed defines a distribution function on $[0,\xi]$. For any $y \in (0,\xi)$, recall $m_T^*(y)$ satisfies

$$r_T(m;y) - c(m) = 0, \tag{27}$$

and by (23) we have

$$\frac{\partial[r_T(m;y) - c(m)]}{\partial m} = -\int_y^B F(x)^m \ln^2 F(x)dx - c'(m) < 0,$$

as $c'(m) \geq 0$. Applying implicit function theorem to (27),[25] $m_T^*(y)$ is continuously differentiable on $(0,\xi)$ with

$$\begin{aligned} \frac{dm_T^*(y)}{dy} &= -\left.\frac{\partial[r_T(m;y) - c(m)]/\partial y}{\partial[r_T(m;y) - c(m)]/\partial m}\right|_{m=m_T^*(y)} \\ &= -\frac{F(y)^{m_T^*(y)}|\ln F(y)|}{\int_y^B F(x)^{m_T^*(y)} \ln^2 F(x)dx + c'(m_T^*(y))} < 0. \end{aligned} \tag{28}$$

[21] See Aliprantis and Border (2006, p.187).
[22] See Varberg and Roberts (1973, p.4,5) and Royden and Fitzpatrick (2010, p.124,131,132).
[23] See Royden and Fitzpatrick (2010, p.125).
[24] See Simon and Blume (1994, Theorem 19.4).
[25] See Simon and Blume (1994, Theorem 15.2).

By taking a derivative of $F(y)^{m_T^*(y)}$, it follows

$$\frac{dF(y)^{m_T^*(y)}}{dy} = F(y)^{m_T^*(y)} \left[\frac{dm_T^*(y)}{dy} \ln F(y) + m_T^*(y) \frac{f(y)}{F(y)} \right] > 0,$$

where the density function f exists as F is absolutely continuous. Therefore, $F(y)^{m_T^*(y)} = W_T'(y)$ is a distribution function with a strictly positive density.

Step 2: $t < T$**.** Consider any $t < T$. As an induction hypothesis, assume

$$W_{t+1}(y) = y + \int_y^{\max\{y,\xi\}} \left(1 - \prod_{\tau=t+1}^{T} F(x)^{m_\tau^*(x)}\right) dx, \tag{29}$$

where, for any $\tau = t+1, \cdots, T$, $m_\tau^*(y) = 0$ if $y \geq \xi$; and $m_\tau^*(y) > 0$ is the unique solution to $r_\tau(m; y) = c(m)$ otherwise. We also assume that

$$r_{t+1}(m; y) = \int_y^B \prod_{\tau=t+2}^{T} F(x)^{m_\tau^*(x)} F(x)^m |\ln F(x)| dx. \tag{30}$$

Following the same process of integrating by parts as (22), we have

$$\begin{aligned}
R_t(m; y) &= \mathbb{E} W_{t+1}\big(\max\{y, X_m\}\big) \\
&= W_{t+1}(y) F(y)^m + \int_y^B W_{t+1}(x) dF(x)^m \\
&= y + \int_y^B (1 - W_{t+1}'(x) F(x)^m) dx \\
&= y + \int_y^B (1 - \prod_{\tau=t+1}^{T} F(x)^{m_\tau^*(x)} F(x)^m) dx,
\end{aligned} \tag{31}$$

where the last equality is from (29). R_t is differentiable with respect to m and the marginal revenue is

$$\begin{aligned}
r_t(m; y) := \frac{\partial R_t(m; y)}{\partial m} &= \int_y^B W_{t+1}'(x) F(x)^m |\ln F(x)| dx \\
&= \int_y^B \prod_{\tau=t+1}^{T} F(x)^{m_\tau^*(x)} F(x)^m |\ln F(x)| dx,
\end{aligned} \tag{32}$$

which is non-negative and continuous in m and y, with $\lim_{m \to \infty} r_t(m; y) = 0$. Moreover, we have

$$\begin{aligned}
\frac{\partial r_t(m; y)}{\partial m} &= -\int_y^B \prod_{\tau=t+1}^{T} F(x)^{m_\tau^*(x)} F(x)^m \ln^2 F(x) dx \leq 0 \quad \text{and} \\
\frac{\partial r_t(m; y)}{\partial y} &= -\prod_{\tau=t+1}^{T} F(y)^{m_\tau^*(y)} F(y)^m |\ln F(y)| \leq 0,
\end{aligned}$$

where both inequalities hold strictly if and only if $y < B$. From (32), we also observe that $r_t(m; y) \leq r_T(m; y) = \int_y^B F(x)^m |\ln F(x)| dx$.

With the monotonicity and the continuity of r_t and c, again by solving the Kuhn-Tucker condition that

$$r_t(m;y) - c(m) \leq 0, \quad m \geq 0, \text{and} \quad (r_t(m;y) - c(m))m = 0,$$

we get the solution $m_t^*(y)$ as follows.

(i) If $r_T(0;0) = \int_0^B |\ln F(x)| dx \leq c(0)$, then for any $y \geq 0$ and $m > 0$, $r_t(m;y) \leq r_T(m;y) < r_T(0;0) \leq c(0) \leq c(m)$. The Kuhn-Tucker condition then implies that $m_t^*(y) = 0$ for any y. Hence, the cutoff for stopping is $\xi = 0$.

(ii) If $r_T(0;0) > c(0)$, then for any $y \geq \xi$ and $m > 0$, as $r_t(m;y) \leq r_t(m;\xi) \leq r_t(0;\xi) \leq r_T(0;\xi) = c(0) \leq c(m)$, from the Kuhn-Tucker condition again, we have $m_T^*(y) = 0$. For any $y < \xi$, we have

$$r_t(0;y) = r_{t+1}(m_{t+1}^*(y);y) = c(m_{t+1}^*(y)) \geq c(0),$$

where the first equality is from (30) and (32), and the second equality is due to the induction hypothesis that $m_{t+1}^*(y)$ solves $r_{t+1}(m;y) = c(m)$. The monotonicity and continuity of r_t and c ensure the uniqueness of $m_t^*(y)$, as a solution of $r_t(m;y) = c(m)$, which maximizes $R_t(m;y) - C(m)$.

Next we derive $W_t(y)$. For any $y \geq \xi$, we have $W_t(y) = y$ as $m_\tau^*(y) = 0$ for any $\tau \geq t$. As $R_t(m;y)$ is differentiable with respect to y, from (31) we have

$$\frac{\partial R_t(m;y)}{\partial y} = \prod_{\tau=t+1}^{T} F(y)^{m_\tau^*(y)} F(y)^m \geq 0,$$

$R_t(m;y) - C(m)$ is convex in y for any $m \geq 0$. As in Step 1, $W_t(y)$ is convex and absolutely continuous, and $W_t'(y)$ is defined almost everywhere. Similarly, for any $y \leq \xi$, we can represent $W_t(y)$ by

$$W_t(y) = W_t(\xi) - \int_y^\xi W_t'(x)dx = \xi - \int_y^\xi W_t'(x)dx, \tag{33}$$

where we use the boundary condition $W_t(\xi) = \xi$. Applying envelop theorem to (1) and (31), we then have

$$W_t'(x) = \frac{\partial R_t(m_t^*(x);x)}{\partial x} = \prod_{\tau=t+1}^{T} F(x)^{m_\tau^*(x)} F(x)^{m_t^*(x)} = \prod_{\tau=t}^{T} F(x)^{m_\tau^*(x)}. \tag{34}$$

Substituting it back into (33) gives

$$W_t(y) = \xi - \int_y^{\xi} \prod_{\tau=t}^{T} F(x)^{m_\tau^*(x)} dx = y + \int_y^{\xi} \left(1 - \prod_{\tau=t}^{T} F(x)^{m_\tau^*(x)}\right) dx. \tag{35}$$

Combining (35) for $y < \xi$ and $W_t(y) = y$ for $y \geq \xi$, we have (5) for period t.

Finally, we show that $\prod_{\tau=t}^{T} F(y)^{m_\tau^*(y)}$ is continuous and increasing in y, in order to verify it indeed defines a distribution function on $[0,\xi]$. For any $y \in (0,\xi)$, recall $m_t^*(y)$ satisfies

$$r_t(m;y) - c(m) = 0, \tag{36}$$

and

$$\frac{\partial[r_t(m;y) - c(m)]}{\partial m} = -\int_y^B \prod_{\tau=t+1}^{T} F(x)^{m_\tau^*(x)} F(x)^m \ln^2 F(x) dx - c'(m) < 0.$$

Applying implicit function theorem to (36), $m_t^*(y)$ is continuously differentiable on $(0,\xi)$ with

$$\begin{aligned} \frac{dm_t^*(y)}{dy} &= -\frac{\partial[r_t(m;y) - c(m)]/\partial y}{\partial[r_t(m;y) - c(m)]/\partial m}\bigg|_{m=m_t^*(y)} \\ &= -\frac{\prod_{\tau=t}^{T} F(y)^{m_\tau^*(y)} |\ln F(y)|}{\int_y^B \prod_{\tau=t+1}^{T} F(x)^{m_\tau^*(x)} F(x)^{m_t^*(y)} \ln^2 F(x) dx + c'(m_t^*(y))} < 0. \end{aligned} \tag{37}$$

By taking a derivative of $\prod_{\tau=t}^{T} F(y)^{m_\tau^*(y)}$, it follows that

$$\frac{d\prod_{\tau=t}^{T} F(y)^{m_\tau^*(y)}}{dy} = \prod_{\tau=t}^{T} F(y)^{m_\tau^*(y)} \left[\left(\sum_{\tau\geq t} \frac{dm_\tau^*(y)}{dy}\right) \cdot \ln F(y) + \left(\sum_{\tau\geq t} m_\tau^*(y)\right) \frac{f(y)}{F(y)}\right] > 0.$$

Therefore, $\prod_{\tau=t}^{T} F(y)^{m_\tau^*(y)} = W_t'(y)$ is a distribution function with a strictly positive density. □

Proof of Corollary 1

The proof is embedded in Proof of Theorem 1, with the result shown in (34). □

Proof of Lemma 1

If $y \geq \xi$, the results hold as $m_t^*(y) = 0$ for any t. Suppose $y < \xi$. By (28) and (37) in Proof of Theorem 1, for any t, we have $dm_t^*(y)/dy < 0$. By (23) and (32) in Proof of Theorem 1, for any t, we have

$$r_t(m;y) = \int_y^B \prod_{\tau=t+1}^{T} F(x)^{m_\tau^*(x)} F(x)^m |\ln F(x)| dx, \tag{38}$$

which implies that $r_t(m;y)$ is strictly increasing in t. As a solution to $r_t(m;y) = c(m)$, $m_t^*(y)$ is also strictly increasing in t. □

Proof of Proposition 1.

If $y \geq \xi$, it is clear that (9) hold with equality as $m_t^*(y) = 0$ for any t. For $y < \xi$, it follows from (37) that

$$
\begin{aligned}
\frac{dm_t^*(y)}{dy} &= -\frac{\prod_{\tau=t}^{T} F(y)^{m_\tau^*(y)} |\ln F(y)|}{\int_y^B \prod_{\tau=t+1}^{T} F(x)^{m_\tau^*(x)} F(x)^{m_t^*(y)} \ln^2 F(x) dx + c'(m_t^*(y))} \\
&= -\frac{\prod_{\tau=t+1}^{T} F(y)^{m_\tau^*(y)} |\ln F(y)|}{\int_y^B \left[\prod_{\tau=t+2}^{T} F(x)^{m_\tau^*(x)}\right] \frac{F(x)^{m_{t+1}^*(x)} F(x)^{m_t^*(y)}}{F(y)^{m_t^*(y)}} \ln^2 F(x) dx + \frac{c'(m_t^*(y))}{F(y)^{m_t^*(y)}}} \\
&> -\frac{\prod_{\tau=t+1}^{T} F(y)^{m_\tau^*(y)} |\ln F(y)|}{\int_y^B \left[\prod_{\tau=t+2}^{T} F(x)^{m_\tau^*(x)}\right] F(x)^{m_{t+1}^*(y)} \ln^2 F(x) dx + \frac{c'(m_t^*(y))}{F(y)^{m_t^*(y)}}} \\
&> -\frac{\prod_{\tau=t+1}^{T} F(y)^{m_\tau^*(y)} |\ln F(y)|}{\int_y^B \left[\prod_{\tau=t+2}^{T} F(x)^{m_\tau^*(x)}\right] F(x)^{m_{t+1}^*(y)} \ln^2 F(x) dx + c'(m_{t+1}^*(y))} \\
&= \frac{dm_{t+1}^*(y)}{dy},
\end{aligned}
$$

where the first inequality comes from $F(x)^{m_{t+1}^*(x)} F(x)^{m_t^*(y)} > F(x)^{m_{t+1}^*(y)} F(y)^{m_t^*(y)}$ for $y < x$, and the second comes from $F(y)^{m_t^*(y)} < 1$, $m_{t+1}^*(y) > m_t^*(y)$, and $c'(m)$ is non-increasing in m (i.e., $C''' \leq 0$). Thus, we have $dm_{t+1}^*(y)/dy < dm_t^*(y)/dy < 0$. Furthermore, for any $y < y'$, it follows that

$$
m_{t+1}^*(y') - m_{t+1}^*(y) = \int_y^{y'} dm_{t+1}^*(x) < \int_y^{y'} dm_t^*(x) = m_t^*(y') - m_t^*(y),
$$

which implies (9), by arranging the terms. □

Proof of Proposition 2.

For part i, Proof of Theorem 1 has shown that $W_t(y)$ is increasing and convex in y.

For part ii, from (5), if $y \geq \xi$, then $W_t(y) = W_{t+1}(y) = y$ and hence $D_t W(y) = 0$. If $y < \xi$, then

$$
\begin{aligned}
D_t W(y) &= \left[y + \int_y^{\xi} \left(1 - \prod_{\tau=t+1}^{T} F(x)^{m_\tau^*(x)}\right) dx\right] - \left[y + \int_y^{\xi} \left(1 - \prod_{\tau=t}^{T} F(x)^{m_\tau^*(x)}\right) dx\right] \\
&= \int_y^{\xi} \left(\prod_{\tau=t}^{T} F(x)^{m_\tau^*(x)} - \prod_{\tau=t+1}^{T} F(x)^{m_\tau^*(x)}\right) dx \\
&= -\int_y^{\xi} \left(1 - F(x)^{m_t^*(x)}\right) \prod_{\tau=t+1}^{T} F(x)^{m_\tau^*(x)} dx < 0. \qquad (39)
\end{aligned}
$$

Moreover, for any $y < \xi$, we have

$$
\begin{aligned}
& D_t W(y) - D_{t-1} W(y) \\
= & \int_y^{\xi} \left(1 - F(x)^{m_{t-1}^*(x)}\right) \prod_{\tau=t}^{T} F(x)^{m_\tau^*(x)} dx - \int_y^{\xi} \left(1 - F(x)^{m_t^*(x)}\right) \prod_{\tau=t+1}^{T} F(x)^{m_\tau^*(x)} dx \\
< & \int_y^{\xi} \left(1 - F(x)^{m_{t-1}^*(x)}\right) \prod_{\tau=t}^{T} F(x)^{m_\tau^*(x)} dx - \int_y^{\xi} \left(1 - F(x)^{m_{t-1}^*(x)}\right) \prod_{\tau=t+1}^{T} F(x)^{m_\tau^*(x)} dx \\
= & -\int_y^{\xi} \left(1 - F(x)^{m_{t-1}^*(x)}\right) \left(1 - F(x)^{m_t^*(x)}\right) \prod_{\tau=t+1}^{T} F(x)^{m_\tau^*(x)} dx < 0,
\end{aligned}
$$

where the first inequality is from $m_t^*(x) > m_{t-1}^*(x)$ due to the deadline effect (Lemma 1). Thus, $D_t W(y)$ is decreasing in t (strictly if and only if $y < \xi$, as $m_t^*(y) > 0$).

For part iii, from Corollary 1, we have

$$
W_t'(y) = \prod_{\tau=t}^{T} F(y)^{m_\tau^*(y)} = F(y)^{m_t^*(y)} \prod_{\tau=t+1}^{T} F(y)^{m_\tau^*(y)} \leq \prod_{\tau=t+1}^{T} F(y)^{m_\tau^*(y)} = W_{t+1}'(y).
$$

Therefore, $W_t'(y)$ is increasing in t (strictly if and only if $y < \xi$). Again from (39),

$$
\frac{dD_t W(y)}{dy} = -(1 - F(y)^{m_t^*(y)}) \prod_{\tau=t+1}^{T} F(y)^{m_\tau^*(y)} \leq 0.
$$

we then have $D_t W(y)$ is decreasing in y (strictly if and only if $y < \xi$, as $m_t^*(y) > 0$). □

Proof of Proposition 3.

From (11) and (14), the optimal cutoff for stopping is determined by $\hat{\xi}_t = \hat{W}_t(0) = W_T^{T-t+1}(0)$ and the search value is $\hat{W}_t(y) = \max\{y, \hat{\xi}_t\}$. To complete the proof, we derive the optimal intensity $\hat{m}_t^*(y)$ for any $y < \hat{\xi}_t$. In period T, it is clear that $\hat{m}_T^*(y) = m_T^*(0)$ for any $y < \hat{\xi}_T$. In any period $t < T$, by (15), we have $\hat{R}_t(m; y) = R_T(m; \hat{\xi}_{t+1})$. The marginal search revenue $\hat{r}_t(m, y)$ is then

$$
\hat{r}_t(m; y) := \frac{\partial \hat{R}_t(m, y)}{\partial m} = \frac{\partial R_T(m, \hat{\xi}_{t+1})}{\partial m} = r_T(m; \hat{\xi}_{t+1}). \tag{40}
$$

It follows that $\hat{m}_t^*(y) = m_T^*(\hat{\xi}_{t+1})$ as a solution to $\hat{r}_t(m; y) = c(m)$, for any y such that $\hat{m}_t^*(y) > 0$. □

Proof of Corollary 2.

When $\int_0^B |\ln F(x)| dx > c(0)$, Theorem 1 implies $\xi > 0$, and Corollary 1 implies that, for any $0 < y < \xi$, $0 < W_T'(y) < 1$ and $y < W_T(y) < \xi$. Hence, the iterated function value $W_T^{T-t+1}(0)$

is strictly decreasing in t. As $\hat{\xi}_t = W_T^{T-t+1}(0)$ from (14), $\hat{\xi}_t$ is then strictly decreasing in t. Furthermore, as $W_T^{T-t+1}(0) < \xi$ for any t, $\hat{\xi}_t$ is always strictly smaller than ξ. If $\int_0^B |\ln F(x)|dx < c(0)$, Theorem 1 defines $\xi = 0$. Hence, it follows $W_T(0) = 0$ and $\hat{\xi}_t = W_T^{T-t+1}(0) = 0$ for any t. □

Proof of Proposition 4.

Comparing (1) and (10) recursively, it is clear that $W_t(y) - \hat{W}_t(y) \geq 0$ for any y. It is also clear that $W_t(y) - \hat{W}_t(y) = 0$ for $y \geq \xi$, due to Theorem 1 and Proposition 3. Now focus on $0 < y < \xi$. Recalling $0 < W'(y) < 1$ from Theorem 1, due to Proposition 3, if $y < \hat{\xi}_t$ then $\hat{W}_t'(y) = 0$ and hence $W_t(y) - \hat{W}_t(y)$ is strictly increasing on $[0, \hat{\xi}_t]$. Similarly, if $y > \hat{\xi}_t$ then $\hat{W}_t'(y) = 1$ and hence $W_t(y) - \hat{W}_t(y)$ is strictly decreasing on $[\hat{\xi}_t, \xi]$. □

Proof of Proposition 5.

By Theorem 1 and Proposition 3, part ii is straightforward as

$$\hat{\xi}_t \leq y < \xi \implies \hat{m}_t^*(y) = 0 \text{ and } m_t^*(y) > 0.$$

For other parts, recalling R_t and $\hat{R}_t$ defined in (3) and (15), we have

$$R_t(m,y) \geq R_t(m,0) \geq \hat{R}_t(m;0) = R_T(m,\hat{\xi}_{t+1}), \tag{41}$$

where the first inequality is from the monotonicity of $R_t(m;y)$ in y, the second one is from the fact that the revenue with no recall can also be obtained by recall. For any $m > 0$ and $y < \xi$, there exists a unique ζ that solves $R_t(m,y) = R_T(m,\zeta)$, as $R_T(m;\zeta)$ is continuous and strictly decreasing in ζ for $\zeta < \xi$. Let $\zeta_t(m,y)$ denote the unique solution. By definition, we then have

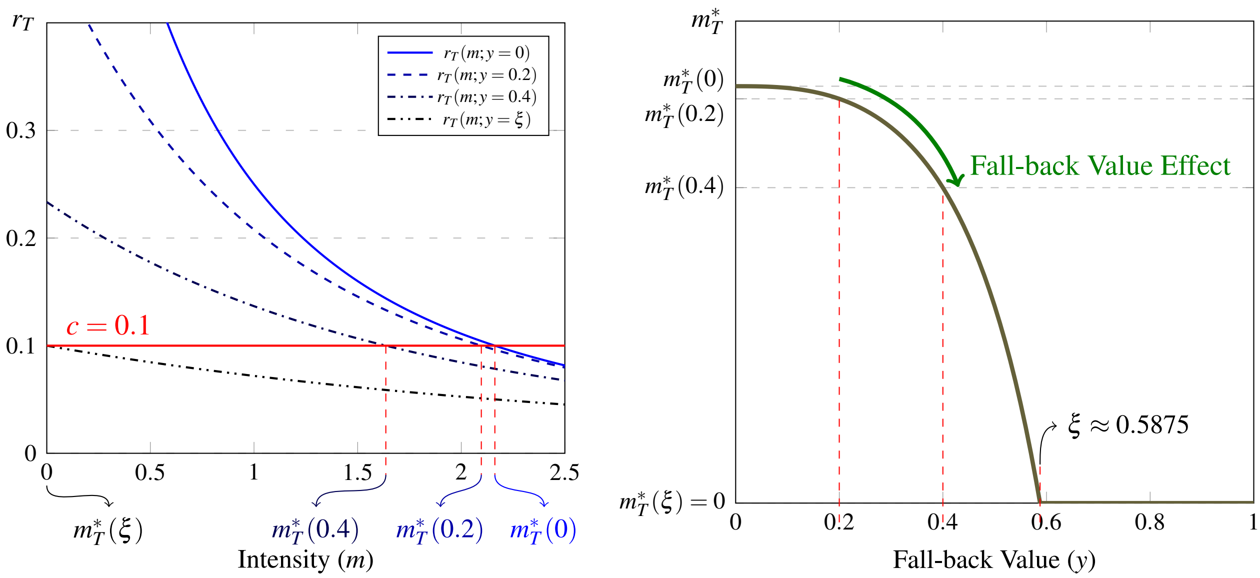
$$y + \int_y^B \left(1 - W_{t+1}'(x)F(x)^m\right) dx = \zeta_t(m,y) + \int_{\zeta_t(m,y)}^B \left(1 - F(x)^m\right) dx.$$

Furthermore, by inequality (41), we have $R_T(m,\zeta_t(m,y)) = R_t(m,y) \geq R_T(m,\hat{\xi}_{t+1})$. It then follows that, for any $m > 0$ and any $y < \xi$,

$$\zeta_t(m,y) \geq \hat{\xi}_{t+1}. \tag{42}$$

Note that $m_t^*(y)$ maximizes $R_t(m,y) - C(m)$, and for given value of $\zeta_t(m,y)$, $m_t^*(y)$ also maximizes $R_T(m,\zeta_t(m,y)) - C(m)$. Therefore, it follows that

$$r_t(m_t^*(y),y) = r_T(m_t^*(y),\zeta_t(m_t^*(y),y)) = c(m_t^*(y)) \leq r_T(m_t^*(y),\hat{\xi}_{t+1}) = \hat{r}_t(m_t^*(y)), \tag{43}$$



(a) Optimal Intensity Determined

(b) Optimal Intensity

NOTES: The optimal intensity $m_t^*(y)$ is strictly decreasing in y, provided that $y < \xi$. The cutoff value ξ for stopping solves $c(m) = r_t(0; \xi)$. That is, if $y \geq \xi$ then $m_t^*(y) = 0$; otherwise, $m_t^*(y) > 0$.

Figure 4: Fall-back Value Effect: $C(m) = 0.1m$, $F(x) = x \cdot \mathbb{1}(0 \leq x \leq 1)$, $T = 4$

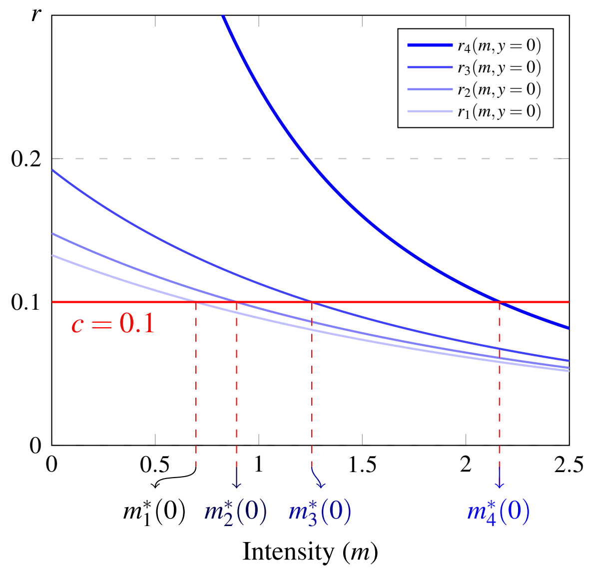
where the inequality is due to the monotonicity of $r_T(m; y)$ in y and (42) and the last equality is from (40). As $\hat{r}_t(m_t^*(y)) \geq c$, for any $y < \hat{\xi}_t$, it must be $\hat{m}_t^*(y) \geq m_t^*(y)$ to maximize $\hat{R}_t(m) - C(m)$. Recalling $\hat{\xi}_{T+1} = 0$, the inequality (42) holds strictly if and only if $t = T$ and $y = 0$. Thus, for $t = T$ and $y = 0$, we have $\hat{m}_t^*(y) = m_t^*(y)$, otherwise $\hat{m}_t^*(y) > m_t^*(y)$, as desired. □

A.2 Numerical Examples

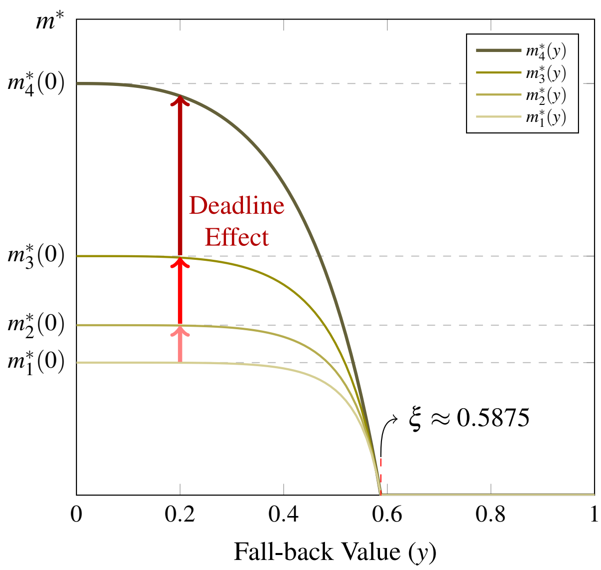
This section provides a few figures of numerical examples, which help to illustrate some results of this paper. In these examples, we consider a search problem with a linear search cost $C(m) = 0.1m$ and a uniform distribution $F(x) = x \cdot \mathbb{1}(0 \leq x \leq 1)$.[26]

Figure 4 illustrates the fall-back value effect in the last period T. In panel (a), the horizontal axes is the search intensity m, and the downward sloping curves plot the marginal search revenue $r_T(m; y)$ across different y's. Their crossing points with the constant marginal search cost $c = 0.1$ determine the optimal intensity $m_T^*(y)$ (Theorem 1). With increasing y, the $r_T(m; y)$ curve shifts downwards, as shown in (24). Therefore, $m_T^*(y)$ is decreasing in y. Panel (b) also plots the fall-back value effect, i.e., the optimal search intensity keeps on decreasing in y, and equals 0 when $y \geq \xi$.

[26]The Mathematica code for computing r_t, m_t^*, and W_t can be found in Figure 7.



(a) Marginal Revenue r_t (with $y = 0$)



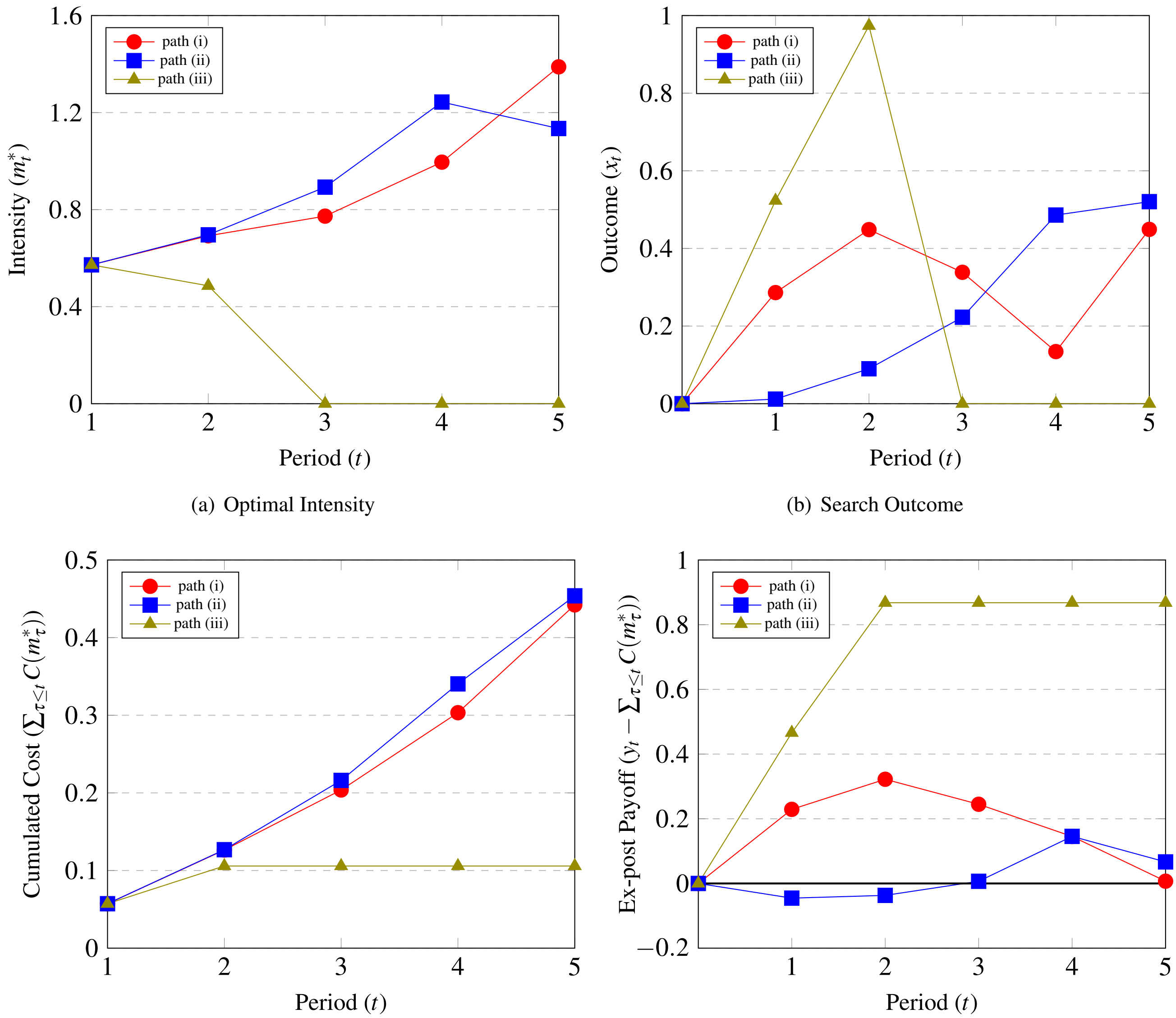
(b) Optimal Intensity m_t^*

NOTES: For any $y < \xi$, $m_{t+1}^*(y)$ is greater than $m_t^*(y)$ as the marginal revenue r_t of search shifts upwards. Note that, the difference of $m_{t+1}^*(y) - m_t^*(y)$, namely the deadline effect, is diminishing as the fall-back value y increases.

Figure 5: Deadline Effect: $C(m) = 0.1m$, $F(x) = x \cdot \mathbb{1}(0 \leq x \leq 1)$, $T = 4$

Figure 5 illustrates the deadline effect with $T = 4$. In panel (a), for given $y = 0$, the downward sloping curves plot the marginal search revenue $r_t(m;0)$ across different period t. With increasing t, the $r_t(m;0)$ curve shifts upwards, as shown in (38). Panel (b) then shows the deadline effect, i.e., for given y, the optimal intensity $m_t^*(y)$ is increasing over time.

Figure 6 shows some sample paths of the sequences of optimal search intensity, the search outcomes, the accumulated search costs, and the ex-post payoff with $T = 5$. For example, along sample path (iii), the searcher stops searching from period 3 on, as her realized fall-back value surpasses the cutoff for stopping at the end of period 2, as shown in panel (a) and (b) respectively.



(a) Optimal Intensity (b) Search Outcome

(c) Cumulative Cost (d) Ex-Post Payoff

Figure 6: Some Sample Paths: $y_1 = 0$, $C(m) = 0.1m$, $F(x) = x \cdot \mathbb{1}(0 \leq x \leq 1)$, $T = 5$

```
(* The marginal cost and the search value after the deadline (T=4) are given *)
c = 0.1;
W5[y_] := y;
(* The marginal revenue r4 at T=4 is obtained from Equation (2) *)
r4[m_, y_] := -((-1 + y^(1 + m) (1 - (1 + m) Log[y]))/(1 + m)^2);
(* The optimal cutoff, xi, for stopping is constant*)
xi = FindRoot[r4[0, y] == c, {y, 0.5}][[1, 2]];
(* The optimal intensity equates the marginal revenue and the marginal cost *)
mP4[y_?NumericQ] := FindRoot[r4[m, y] == c, {m, 1}][[1, 2]];
(* The optimal intensity, m4, is obtained by interpolation of sample points *)
mI4 = Interpolation[Table[{y, mP4[y]}, {y, 0.01, xi, 0.01}]];
m4[y_] := Piecewise[{{mI4[y], y < xi}, {0, y >= xi}}];
(* The search value is obtained with interpolation of the sample points *)
W4[y_] := y + NIntegrate[1 - x^m4[x], {x, y, 1}];
(* Using m4, recursively solve r3, m3, and W3 *)
r3[m_, y_] := -Integrate[x^(m + m4[x])*Log[x], {x, y, 1}];
mP3[y_?NumericQ] := FindRoot[r3[m, y] == c, {m, 1}][[1, 2]];
mI3 = Interpolation[Table[{y, mP3[y]}, {y, 0.01, xi, 0.01}]];
m3[y_] := Piecewise[{{mI3[y], y < xi}, {0, y >= xi}}];
W3[y_] := y + NIntegrate[1 - x^(m3[x] + m4[x]), {x, y, 1}];
(* Using m4 and m3, recursively solve r2, m2, and W2 *)
r2[m_, y_] := -Integrate[x^(m + m3[x] + m4[x])*Log[x], {x, y, 1}];
mP2[y_?NumericQ] := FindRoot[r2[m, y] == c, {m, 1}][[1, 2]];
mI2 = Interpolation[Table[{y, mP2[y]}, {y, 0.01, xi, 0.01}]];
m2[y_] := Piecewise[{{mI2[y], y < xi}, {0, y >= xi}}];
W2[y_] := y + NIntegrate[1 - x^(m2[x] + m3[x] + m4[x]), {x, y, 1}];
(* Using m4, m3, and m2, recursively solve r1, m1, and W1 *)
r1[m_, y_] := -Integrate[x^(m + m2[x] + m3[x] + m4[x])*Log[x], {x, y, 1}];
mP1[y_?NumericQ] := FindRoot[r1[m, y] == c, {m, 1}][[1, 2]];
mI1 = Interpolation[Table[{y, mP1[y]}, {y, 0.01, xi, 0.01}]];
m1[y_] := Piecewise[{{mI1[y], y < xi}, {0, y >= xi}}];
W1[y_] := y + NIntegrate[1 - x^(m1[x] + m2[x] + m3[x] + m4[x]), {x, y, 1}];
```

NOTES: Based on Theorem 1, the code recursively solves the marginal revenue functions $r_t(m;y)$, the optimal search intensity $m_t^*(y)$, and the search value $W_t(y)$.

Figure 7: Mathematica Code for Numerical Example

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