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Representation dependence of k-strings in pure Yang-Mills theory via supersymmetry

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We exploit a conjectured continuity between super Yang-Mills on $\mathbb{R}^3 \times \mathbb{S}^1$ and pure Yang-Mills to study k-strings in the latter theory. As expected, we find that Wilson-loop correlation functions depend on the N-ality of a representation $\mathcal{R}$ to the leading order. However, the next-to-leading order correction is not universal and is given by the group characters, in the representation $\mathcal{R}$, of the permutation group. We also study W-bosons in super Yang-Mills. We show that they are deconfined on the string world sheet, and therefore, they can change neither the string N-ality nor its tension. This phenomenon mirrors the fact that soft gluons do not screen probe charges with nonzero N-ality in pure Yang-Mills. Finally, we comment on the scaling law of k-strings in super Yang-Mills and compare our findings with strings in Seiberg-Witten theory, deformed Yang-Mills theory, and holographic studies that were performed in the 't Hooft large-N limit.

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I. INTRODUCTION

Flux tubes, or strings, are among the most fascinating objects in physics. They emerge as long-distance phenomena of various field theories, from the Abelian Higgs model to quantum chromodynamics (QCD). Although we have a good understanding of Abelian strings (Abrikosov-Nielsen-Olesen type [1,2]), QCD strings remain poorly understood [3,4], thanks to the strong coupling of QCD.

One of the important questions in Yang-Mills theories is how the string tension depends on the representation of the probe charges. The general lore, which is based on a pure physical argument, is that the string tension cannot depend on the representation. Instead, it can only depend on its N-ality. The N-ality of a representation $\mathcal{R}$ of $su(N)$ is defined as the number of boxes in the Young tableau of $\mathcal{R}$ modulo N. The physical argument in pure Yang-Mills goes as follows: since one can convert one representation $\mathcal{R}_1$ with N-ality k to another representation $\mathcal{R}_2$ with the same N-ality by emitting soft gluons,[1] the string tension σ_k will depend only on the N-ality k and not on the representation. Unfortunately, it is extremely difficult to provide a direct mathematical proof of such an intuitive argument; the strong coupling nature of QCD hinders the chances to find such a proof.

Lattice field theory provides a nonperturbative definition of strongly coupled theories, and therefore, one hopes that direct simulations of Yang-Mills theory can provide complete nonperturbative pictures of QCD strings. Practical lattice simulations of QCD, however, suffer from lattice artifacts, leading to some dependence of the string tension on the representation [5–7], which is particularly evident in the case of a large number of colors. This is because the relaxation time of higher representation strings can be exponentially large, which mistakenly can signal a dependence of the string tension on the representation rather than its N-ality. Lattice strong coupling expansion, in addition, suffers from the same artifact [8].

Fortunately, the AdS/CFT correspondence can shed some light on the question at hand. In particular, it was shown in [8] (also see [9]) that the expectation value of the Polyakov loop in a representation $\mathcal{R}$ is given by $\langle \mathcal{P}_{\mathcal{R}} \rangle = F(\mathcal{R}) e^{-\sigma_k A}$, where A is the area of the Polyakov loop. Thus, as expected, the string tension depends only on the N-ality k, while there is a nonuniversal representation dependent prefactor $F(\mathcal{R})$. This behavior, however, was shown only in the 't Hooft large-N limit, leaving behind the finite N case with no direct answer.

The lack of a direct proof of the expected universality of string tension, specifically for finite N, calls for a new perspective on the problem. A novel way to approach strongly coupled pure Yang-Mills is to exploit a conjectured continuity that first appeared in [10]. This is a continuity between softly broken (via a mass term) $\mathcal{N} = 1$ super Yang-Mills on $\mathbb{R}^3 \times \mathbb{S}^1$, where $\mathbb{S}^1$ is a spatial rather than a thermal circle, and pure Yang-Mills at finite

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[1]The gluons are in the adjoint representation, and hence they have zero N-ality. Also, remember that in pure Yang-Mills there is no dynamical matter that can screen the probe charges.

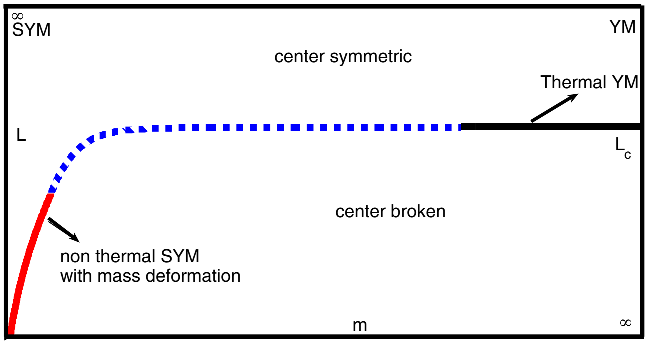

FIG. 1. Continuity between mass deformed $\mathcal{N}=1$ super Yang-Mills on $\mathbb{R}^3\times\mathbb{S}^1$ and pure Yang-Mills at finite temperature. The red thick curve in the lower left corner is the phase separation between the center-symmetric and center-broken phases in super Yang-Mills on $\mathbb{R}^3\times\mathbb{S}^1$. This part of the phase diagram is under analytical control since the theory is in its weakly coupled semiclassical regime. The black curve in the upper right corner is the phase separation between the confined and deconfined phases of the strongly coupled pure Yang-Mills. This part of the curve can be envisaged using lattice Monte Carlo simulations. The dotted curve is conjectured to be smoothly connecting both the weakly coupled and strongly coupled theories.

temperature. According to this continuity, the quantum phase transition in the former theory is continuously connected to the thermal phase transition in the latter one. This is illustrated in Fig. 1. At small circle circumference L and small gaugino mass m (this is the lower left corner, the red curve, of Fig. 1) the theory is confining, is in a weakly coupled regime, has a preserved $\mathbb{Z}_N$ center symmetry, and is under complete analytical control. Therefore, by varying m or L the theory experiences a quantum phase transition and one goes from a center-symmetric phase (at small m and L) to a center-broken phase (larger values of m and L). On the other hand, as $m\to\infty$ the gaugino decouples and the theory flows to a pure Yang-Mills over $\mathbb{S}^1$ (the right side in Fig. 1). This is a pure Yang-Mills theory[2] at finite temperature $T=1/L$. This is a strongly coupled theory whose phase transition can only be inferred from strong coupling calculations, e.g., lattice simulations. According to the continuity conjecture in [10], the quantum phase transition in super Yang-Mills is continuously connected to the thermal phase transition in pure Yang-Mills. This continuity is indicated by the dashed line in the intermediate region in Fig. 1. Despite the fact that a proof of the continuity is still lacking, many checks have shown that various physical observables share the same qualitative behavior in both limits $m\to0$ and $m\to\infty$. This includes the nature of phase transition, i.e., first versus second order [10–13]; the dependence of the critical temperature on the θ angle [14]; and the dependence of the fundamental string on temperature [12].

[2]In the limit $m\to\infty$ there is no dynamical matter. Hence, the fact that we started with a spatial, rather than a thermal, circle does not make any difference, since the gauge fluctuations always obey periodic boundary conditions.

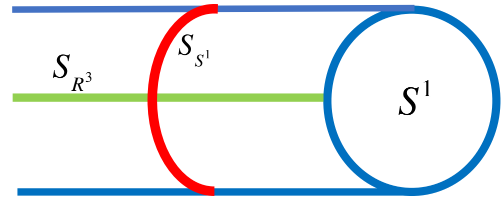

FIG. 2. There are two types of strings in super Yang-Mills on $\mathbb{R}^3\times\mathbb{S}^1$. The first type (green line), which we denote by $\mathcal{S}_{\mathbb{R}^3}$, is the string between two probe charges located on the $\mathbb{R}^2$ plane. The other type of strings (red curve), which we denote by $\mathcal{S}_{\mathbb{S}^1}$, wraps around the $\mathbb{S}^1$ circle. It is this second type of strings that can be interpreted as pure Yang-Mills k-strings in the limit $m\to\infty$.

In the present paper we push the continuity even further: we check whether correlation functions in the mass deformed $\mathcal{N}=1$ super Yang-Mills on $\mathbb{R}^3\times S^1$ and in pure Yang-Mills are continuously connected. This demands that correlation functions do not experience a phase transition as long as we do not cross the phase separation line in Fig. 1. The validity of this conjecture, as well as its limitations, is the main subject of the present work. If this continuity holds, then it can provide a new venue to analytically study various observables, including the strings, which are otherwise very hard to compute directly in the strongly coupled theory.

There are two types of strings in super Yang-Mills on $\mathbb{R}^3\times\mathbb{S}^1$: the strings on $\mathbb{R}^3$ between two probe charges located on the $\mathbb{R}^2$ plane, which we denote[3] by $\mathcal{S}_{\mathbb{R}^3}$, and the strings that wrap around the circle $\mathbb{S}^1$, which we denote by $\mathcal{S}_{\mathbb{S}^1}$. According to the continuity picture, the $\mathcal{S}_{\mathbb{S}^1}$ strings are the "would-be" k-strings in pure Yang-Mills theory in the limit $m\to\infty$; the $\mathbb{S}^1$ circle (which is a spacelike circle) becomes the thermal circle in pure Yang-Mills in the decoupling limit. This picture is depicted in Fig. 2.

In particular, in this work we calculate the tension of these would-be k-strings in pure Yang-Mills theory. This is carried out by computing the Polyakov-loop correlator in super Yang-Mills deep in the weak-coupling confining regime. This is the Polyakov loop that wraps around the spatial $\mathbb{S}^1$ circle: $\mathcal{P}_\mathcal{R}=\mathrm{Tr}_\mathcal{R}\exp[i\oint_{\mathbb{S}^1}\boldsymbol{A}_3]$, where $\boldsymbol{A}_3$ is the gauge field component along the circle and the trace is taken in representation $\mathcal{R}$. Because the theory is in a gapped phase, then for a very large separation between two Polyakov loops one has $\lim_{r\to\infty}\langle\mathcal{P}_\mathcal{R}(\boldsymbol{0})\mathcal{P}^\dagger_\mathcal{R}(\boldsymbol{r})\rangle=\mathcal{F}_\mathcal{R}e^{-\sigma_\mathcal{R}rL}$, where $\sigma_\mathcal{R}$ is a constant that can be exactly determined since the theory is in a calculable regime. According to the conjectured continuity, $\sigma_\mathcal{R}$ should correspond to the string tension in

[3]The $\mathcal{S}_{\mathbb{R}^3}$ strings in deformed Yang-Mills theory are thoroughly studied in [15]. A similar study of $\mathcal{S}_{\mathbb{R}^3}$ in super Yang-Mills is left for a future work.

pure Yang-Mills that also wraps around $\mathbb{S}^1$. Thus, by computing the trace in any representation $\mathcal{R}$, one can infer the dependence of the string tension on $\mathcal{R}$. Our calculations show that for any finite N the string tension $\sigma_{\mathcal{R}}$ depends, to leading order, on the N-ality of the representation $\mathcal{R}$. The precoefficient $\mathcal{F}_{\mathcal{R}}$, however, is found to depend on the representation. Albeit in a weakly coupled regime, this is the first direct proof of the leading-order independence of the string tension of its representation for finite N.

Our work is organized as follows. In Sec. II we review the basics of mass deformed super Yang-Mills on $\mathbb{R}^3 \times \mathbb{S}^1$ and set up the notation and convention. Since this topic has been studied in great detail in the literature, we only provide the necessary formalism that enables the reader to grasp the main ideas. The main results of this section are Eqs. (17)–(19). Experts can skip this section to Sec. III, where we provide a direct proof that the Polyakov-loop correlator depends, to leading order, on the N-ality of a representation. In Sec. IV we study the W-bosons on the string world sheet of super Yang-Mills. We show that these bosons are deconfined on the string, and therefore, they cannot affect the string tension or its N-ality. Finally in Sec. V, we comment on the scaling of the $\mathcal{S}_{\mathbb{S}^1}$ strings and their large-N limit and we compare our findings with strings in the Seiberg-Witten and deformed Yang-Mills theories.

II. MASS DEFORMED SUPER YANG-MILLS

We consider $\mathcal{N}=1$ super Yang-Mills theory on $\mathbb{R}^3 \times \mathbb{S}^1$. This is an $su(N)$ Yang-Mills theory endowed with a single adjoint Weyl fermion (gaugino) obeying periodic boundary conditions along the circle $\mathbb{S}^1$. If we take the circumference of the circle, L, to be much smaller than the strong scale of the theory Λ, i.e., $N\Lambda L \ll 1$, then the theory enters its weakly coupled regime and becomes amenable to semiclassical treatment. Upon dimensionally reducing from $3+1$ to 3 dimensions, the theory generates a scalar field, which is the Wilson line holonomy along the circle: $\boldsymbol{\Phi} = \int_{\mathbb{S}^1} \boldsymbol{A}_3$. Supersymmetry guarantees the vanishing of the perturbative potential $V(\boldsymbol{\Phi})$ that results from integrating out the tower of massive Kaluza-Klein excitations of gauge bosons and gauginos. Thus, the theory has a perturbatively exact flat direction such that turning on any nonzero value of $\boldsymbol{\Phi}$ causes the breaking of $su(N)$ to the maximum Abelian torus $u(1)^{N-1}$. In three dimensions the photons are dual to scalars, and hence, the 3-D long-distance effective field theory contains massless scalars and fermions not charged under $u(1)^{N-1}$. The action of the theory reads

$$S = \frac{1}{L}\int d^3x \left\{ -\frac{1}{g^2}(\partial_\mu \boldsymbol{\Phi})^2 - \frac{g^2}{16\pi^2}(\partial_\mu \boldsymbol{\sigma})^2 - i\frac{2L^2}{g^2}\bar{\boldsymbol{\lambda}}\bar{\sigma}^\mu \partial_\mu \boldsymbol{\lambda} \right\}, \tag{1}$$

where g is the four-dimensional coupling which is kept small, $\boldsymbol{\sigma}$ are the dual photons, and $\boldsymbol{\lambda}$ are the fermions. All light fields have components only along the Cartan generators $\boldsymbol{H} = (H_1, H_2, \ldots, H_{N-1})$, which are denoted by boldface letters, e.g., $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_{N-1})$.

The story does not end at the perturbative sector. The theory, in addition, admits nonperturbative saddles. These are the monopole instantons which lift the flat direction and generate masses for the photons. The details of the story can be found in [10,12,16–18]. In essence, the monopole instantons generate the superpotential,

$$\mathcal{W} \sim \sum_{a=1}^{N} e^{\boldsymbol{\alpha}_a \cdot \boldsymbol{X} + 2\pi i \tau \delta_{a,N}}, \tag{2}$$

where $\boldsymbol{X}$ is the chiral multiplet, $\tau = i\frac{4\pi^2}{g^2} + \frac{\theta}{2\pi}$, and θ is the vacuum angle. The sum is over the simple roots $\{\boldsymbol{\alpha}_a\}$, $a=1,2,\ldots,N-1$ as well as the affine root $\boldsymbol{\alpha}_N = -\sum_{a=1}^{N-1}\boldsymbol{\alpha}_a$. The inclusion of the affine root is a crucial ingredient in order for the theory to have a stable vacuum. In fact, including this root in the sum is how the theory remembers its four-dimensional origin and, as we will see, is responsible in a direct way for the observation that the string tension depends only on the N-ality of the representation to the leading order.

The superpotential will generate the scalar potential (we call it the bion potential[4]) $V_{\text{bion}} = \mathcal{K}^{i\bar{j}} \frac{\partial \mathcal{W}}{\partial X^i} \frac{\partial \bar{\mathcal{W}}}{\partial X^{\dagger j}}$, where $\mathcal{K}^{i\bar{j}}$ is the Kähler potential, which to zeroth order in the coupling constant g is given[5] by $\mathcal{K}^{i\bar{j}} = \delta^{ij}$. As we mentioned in the Introduction, we also turn on a small gaugino mass which breaks the supersymmetry softly and generates a perturbative potential.[6] In addition, the gaugino mass lifts the monopole-instanton zero modes and gives an additional contribution to the scalar potential V_m.

The supersymmetric theory, in the small m and L regime, has a preserved center symmetry and the vacuum expectation value of the Wilson line holonomy is $\boldsymbol{\Phi}_0 = \frac{2\pi}{N}\boldsymbol{\rho}$, where $\boldsymbol{\rho} = \sum_{a=1}^{N-1}\boldsymbol{\omega}_a$ is the Weyl vector, and $\boldsymbol{\omega}_a$ are the fundamental weights. Now we write

$$\boldsymbol{\Phi} = \boldsymbol{\Phi}_0 + \frac{g^2}{4\pi^2}\boldsymbol{b}, \tag{3}$$

such that $\boldsymbol{b}$ are the small fluctuations of the adjoint scalar about the vacuum. After taking the monopoles and gaugino mass into account, we find that the total bosonic Lagrangian in terms of $\boldsymbol{\sigma}$ and $\boldsymbol{b}$ is given by

[4]Magnetic and neutral bions are correlated events made of two monopoles, which appear as a direct sequence of $\mathcal{K}^{i\bar{j}} \frac{\partial \mathcal{W}}{\partial X^i} \frac{\partial \bar{\mathcal{W}}}{\partial X^{\dagger j}}$; see [10,19] for details.

[5]The one-loop correction to the Kähler potential was worked out in [12]. This correction becomes important only if the gauge group does not have a center, e.g., G_2. See [12] for details.

[6]The perturbative potential is the one-loop contribution from the Kaluza-Klein tower of gauge bosons and massive gauginos.

$$\mathcal{L}=\frac{1}{12\pi}\frac{m_W}{\log(m_W/\Lambda)}((\partial_\mu \boldsymbol{b})^2+(\partial_\mu \boldsymbol{\sigma})^2)+V_{\rm np}+V_{\rm pert}, \tag{4}$$

where $V_{\rm np}$ and $V_{\rm pert}$ are, respectively, the nonperturbative and perturbative potentials and $m_W=\frac{2\pi}{NL}$ is the W-boson mass. As shown in [10], $V_{\rm pert}$ is suppressed by three powers of $\log(m_W/\Lambda)$ compared to $V_{\rm np}$, and hence, we neglect it in our analysis. The nonperturbative potential contains contributions from two parts: (1) the monopole part, which is nonvanishing if and only if the gauginos are massive (massless gauginos have two zero modes in the background of monopoles, and hence, the latter cannot contribute to the bosonic potential), and (2) the magnetic and neutral bion potential; see footnote 4. The nonperturbative potential is given by

$$V_{\rm np}=V^0_{\rm bion}\left[\sum_{a=1}^{N}e^{-2\boldsymbol{\alpha}_a\cdot\boldsymbol{b}}-e^{-(\boldsymbol{\alpha}_a+\boldsymbol{\alpha}_{a+1})\cdot\boldsymbol{b}}\cos[(\boldsymbol{\alpha}_a-\boldsymbol{\alpha}_{a+1})\cdot\boldsymbol{\sigma}]\right]$$
$$-V^0_{\rm mon}\left[\sum_{a=1}^{N}e^{-\boldsymbol{\alpha}_a\cdot\boldsymbol{b}}\cos[\boldsymbol{\alpha}_a\cdot\boldsymbol{\sigma}+\psi]\right], \tag{5}$$

where $\psi=\frac{2\pi\ell+\theta}{N}$, and the parameter $\ell=0,1,\ldots,N-1$ labels the vacuum branch, i.e., the branch with minimum ground energy. The bion and monopole coefficients $V^0_{\rm bion}$ and $V^0_{\rm mon}$, expressed in terms of the physical mass m_W and the strong scale Λ, are given by

$$V^0_{\rm bion}=\frac{27}{8\pi}\frac{\Lambda^6}{m_W^3}\log\left(\frac{m_W}{\Lambda}\right),$$
$$V^0_{\rm mon}=\frac{9}{2\pi}\frac{m\Lambda^3}{m_W}\log\left(\frac{m_W}{\Lambda}\right). \tag{6}$$

For convenience, we also introduce the dimensionless gaugino mass parameter

$$c_m=\frac{V^0_{\rm mon}}{V^0_{\rm bion}}=\frac{4mm_W^2}{3\Lambda^3}=\frac{16\pi^2 m}{3\Lambda(\Lambda LN)^2}. \tag{7}$$

To further proceed, one needs to find the masses of the fluctuations $\boldsymbol{b}$. Expanding $V_{\rm np}$ to quadratic order in $\boldsymbol{b}$ and $\boldsymbol{\sigma}$ and rescaling $\boldsymbol{b}$ and $\boldsymbol{\sigma}$ as $\{b_a^2,\sigma_a^2\}\to\frac{6\pi\log(m_W/\Lambda)}{m_W}\{b_a^2,\sigma_a^2\}$ to have a canonically normalized Lagrangian, we obtain

$$\mathcal{L}=\frac{1}{2}(\partial_\mu\boldsymbol{b})^2+\frac{1}{2}(\partial_\mu\boldsymbol{\sigma})^2+\mathcal{V} \tag{8}$$

and

$$\mathcal{V}=-Nc_m\cos\psi+m_0^2\sum_{a=1}^{N}\Big[(\boldsymbol{\alpha}_a\cdot\boldsymbol{b})^2-(\boldsymbol{\alpha}_{a+1}\cdot\boldsymbol{b})(\boldsymbol{\alpha}_a\cdot\boldsymbol{b})$$
$$+(\boldsymbol{\alpha}_a\cdot\boldsymbol{\sigma})^2-(\boldsymbol{\alpha}_{a+1}\cdot\boldsymbol{\sigma})(\boldsymbol{\alpha}_a\cdot\boldsymbol{\sigma})+\frac{c_m}{2}((\boldsymbol{\alpha}_a\cdot\boldsymbol{\sigma})^2$$
$$-(\boldsymbol{\alpha}_a\cdot\boldsymbol{b})^2)\cos\psi-c_m(\boldsymbol{\alpha}_a\cdot\boldsymbol{\sigma})(\boldsymbol{\alpha}_a\cdot\boldsymbol{b})\sin\psi\Big], \tag{9}$$

where

$$m_0^2=\frac{81}{4}\frac{\Lambda^6[\log(m_W/\Lambda)]^2}{m_w^4}. \tag{10}$$

The easiest way to obtain the mass spectra of the quadratic Lagrangian is to go to the $\mathbb{R}^N$ root basis. In this basis the weights of the fundamental representations are given by

$$\boldsymbol{\nu}_a=\boldsymbol{e}_a-\frac{1}{N},\qquad a=1,2,\ldots,N, \tag{11}$$

while the roots are

$$\{\boldsymbol{\alpha}_a=\boldsymbol{e}_a-\boldsymbol{e}_{a+1},1\le a\le N-1,\boldsymbol{\alpha_N}=\boldsymbol{e}_N-\boldsymbol{e}_1\}. \tag{12}$$

Notice the cyclic structure of the roots in these bases. Also, notice that the affine root $\boldsymbol{\alpha_N}$ is the link that completes the cycle.

The cyclic nature of $\{\alpha_a\}$, $a=1,\ldots,N$, enables us to use the discrete Fourier transform defined by

$$\begin{Bmatrix} b_j \\ \sigma_j \end{Bmatrix}=\frac{1}{\sqrt{N}}\sum_{p=0}^{N-1}\begin{Bmatrix} \tilde{b}_p \\ \tilde{\sigma}_p \end{Bmatrix}e^{-2\pi i\frac{pj}{N}}. \tag{13}$$

In doing so, we have introduced the fictitious degree of freedom b_0, the zero mode, which decouples from the rest of the excitations. Had we not included the monopole corresponding to the affine root, we would not be able to use the discrete Fourier transform to simplify our calculations. As we will see in the next section, this transform is pivotal in our proof of the N-ality dependence of the string tension. Now, we substitute the $\mathbb{R}^N$ root vectors into Eq. (9) and use the discrete Fourier transform to find, after straightforward algebra,

$$\mathcal{V}=-Nc_m\cos\psi+m_0^2\sum_p\mathcal{A}_-\tilde{b}_p\tilde{b}_{-p}+\mathcal{A}_+\tilde{\sigma}_p\tilde{\sigma}_{-p}+\mathcal{C}\tilde{\sigma}_p\tilde{b}_{-p}, \tag{14}$$

where

$$\mathcal{A}_{\pm} = 8\sin^4\left(\frac{\pi p}{N}\right) \pm 2c_m \sin^2\left(\frac{\pi p}{N}\right)\cos\psi,$$
$$\mathcal{C} = -4c_m \sin^2\left(\frac{\pi p}{N}\right)\sin\psi. \tag{15}$$

In order to further decouple $\tilde{\sigma}_p$ and $\tilde{b}_p$, we define new fields $\tilde{\sigma}'_p$ and $\tilde{b}'_p$:

$$\tilde{b}'_p = \cos\frac{\psi}{2}\tilde{b}_p + \sin\frac{\psi}{2}\tilde{\sigma}_p,$$
$$\tilde{\sigma}'_p = -\sin\frac{\psi}{2}\tilde{b}_p + \cos\frac{\psi}{2}\tilde{\sigma}_p. \tag{16}$$

The mass square eigenvalues of $\tilde{\sigma}'_p$ and $\tilde{b}'_p$ are

$$\mathcal{M}^2_{\tilde{\sigma}'_p} = 16m_0^2\left[\sin^4\left(\frac{p\pi}{N}\right) + \frac{c_m}{4}\sin^2\left(\frac{p\pi}{N}\right)\right],$$
$$\mathcal{M}^2_{\tilde{b}'_p} = 16m_0^2\left[\sin^4\left(\frac{p\pi}{N}\right) - \frac{c_m}{4}\sin^2\left(\frac{p\pi}{N}\right)\right], \tag{17}$$

where $p = 1, 2, \ldots, N-1$, and we neglected the zero mode $p = 0$.

Now we are in a position to calculate the correlator $\langle\tilde{b}_p(\mathbf{0})\tilde{b}_{-p}(\boldsymbol{r})\rangle$. We consider the Euclidean version of our theory such that $\boldsymbol{r}$ is a three-dimensional vector (the Euclidean time is taken along the third direction). Keeping in mind that the fields $\tilde{\sigma}'_p$ and $\tilde{b}'_p$ do not couple, we find that the propagator $\langle\tilde{b}_p(\mathbf{0})\tilde{b}_{-p}(\boldsymbol{r})\rangle$ is given by

$$\begin{aligned}\langle\tilde{b}_p(\mathbf{0})\tilde{b}_{-p}(\boldsymbol{r})\rangle &= \cos^2\frac{\psi}{2}\langle\tilde{b}'_p(\mathbf{0})\tilde{b}'_{-p}(\boldsymbol{r})\rangle + \sin^2\frac{\psi}{2}\langle\tilde{\sigma}'_p(\mathbf{0})\tilde{\sigma}'_{-p}(\boldsymbol{r})\rangle \\ &= \frac{1}{4\pi r}\left\{\cos^2\frac{\psi}{2}e^{-\mathcal{M}_{\tilde{b}'_p}r} + \sin^2\frac{\psi}{2}e^{-\mathcal{M}_{\tilde{\sigma}'_p}r}\right\}.\end{aligned} \tag{18}$$

In sequence, we use the inverse discrete Fourier transform to obtain the correlator

$$\langle b_j(\mathbf{0})b_l(\boldsymbol{r})\rangle = \frac{1}{N}\sum_{p=0}^{N-1} e^{-\frac{2\pi i p}{N}(j-l)}\langle\tilde{b}_p(\mathbf{0})\tilde{b}_{-p}(\boldsymbol{r})\rangle. \tag{19}$$

The exponents of the correlator $\langle b_j(\mathbf{0})b_k(\boldsymbol{r})\rangle$ are independent of θ.[7] From here on, we set $\theta = 0$ and select the vacuum branch $\ell = 0$. Therefore, the correlator $\langle b_j(\mathbf{0})b_k(\boldsymbol{r})\rangle$ receives a contribution only from the first term in (18). We note that the masses $\mathcal{M}_{\tilde{b}'_p}$ are much lighter than the W-boson mass, $\frac{\pi}{NL}$, as can be checked from (10). The string $\mathcal{S}_{\mathbb{S}^1}$ that wraps around $\mathbb{S}^1$ is made of the light excitations $\mathcal{M}_{\tilde{b}'_p}$, and therefore, the string thickness $\sim\mathcal{M}^{-1}_{\tilde{b}_p}$ is much bigger than the circle $\mathbb{S}^1$. This fact is responsible for the square sine scaling of the $\mathcal{S}_{\mathbb{S}^1}$ string, as we discuss in the conclusion.

[7] Thus, we need to go to the next-to-leading order correction in g to find the dependence of the string tension on θ; see [12,20,21].

III. POLYAKOV-LOOP CORRELATOR AND STRING TENSION

In this section we use the conjectured continuity between mass deformed $\mathcal{N} = 1$ super Yang-Mills and pure Yang-Mills to show that the string tension of the latter theory depends only on the N-ality of the representation to the leading order. In order to show that, we visualize the Polyakov loop along the $\mathbb{S}^1$ circle $\mathrm{Tr}_{\mathcal{R}}\exp[i\oint_{\mathbb{S}^1}\boldsymbol{A}_3]$ as a string wrapping the circle. We can calculate the correlator of two Polyakov loops in the small L and m regime, where the theory is confining, has a preserved center symmetry, is weakly coupled, and is under analytical control. We prove that the correlator $\lim_{r\to\infty}\langle\mathcal{P}_{\mathcal{R}}(\mathbf{0})\mathcal{P}^{\dagger}_{\mathcal{R}}(\boldsymbol{r})\rangle = \mathcal{F}_{\mathcal{R}}e^{-\sigma_{\mathcal{R}}r}$, where $\sigma_{\mathcal{R}}$ is a constant that depends only on the N-ality of the representation $\mathcal{R}$ and the prefactor $\mathcal{F}_{\mathcal{R}}$ depends on the representation $\mathcal{R}$. Then, by continuity (the absence of phase transitions as we take the gaugino mass to infinity), we argue that $\sigma_{\mathcal{R}}$ can be interpreted as the string tension of a pure Yang-Mills theory that depends only on its N-ality, as expected on physical grounds.

A. From the fundamental to any representation of $su(N)$

We first summarize a few important results from group theory concerning traces of $su(N)$ elements in general representations. The following discussion holds for any $N \geq 3$. Let $\mathcal{R} = (y_1, y_2, \ldots, y_{N-1})$ denote the Young tableau with y_i columns of i boxes (where bigger columns are placed on the left as usual), which is associated with a particular tensor representation $\mathcal{R}$ of $su(N)$. Now, let P be an element in $su(N)$. The trace of P in a general representation $\mathcal{R}$ can be written as a sum of products of fundamental traces as is given by the Frobenius formula (see [22] and references therein):

$$\mathrm{Tr}_{\mathcal{R}}P = \frac{1}{n!}\sum_{\vec{j}\in S_n}\chi_{\mathcal{R}}(\vec{j})(\mathrm{Tr}_F P)^{j_1}(\mathrm{Tr}_F P^2)^{j_2}\ldots.(\mathrm{Tr}_F P^n)^{j_n}, \tag{20}$$

where n is the number of boxes in the Young tableau of representation $\mathcal{R}$ (not mod N) and S_n is the permutation group. The permutations $\vec{j} = \{j_1, \ldots, j_n\} \in S_n$ are most easily found as solutions of $\sum_{k=1}^n kj_k = n$. For example, for S_2 we have $\vec{j} = \{(2,0),(0,1)\}$ and for S_3 we have $\vec{j} = \{(3,0,0),(1,1,0),(0,0,1)\}$, etc. $\chi_{\mathcal{R}}(\vec{j})$ is the group character, in the representation $\mathcal{R}$, of the permutation $\vec{j}$. This sets the ground to obtaining the Polyakov-loop correlator in any representation $\mathcal{R}$ in terms of the fundamental representation. We will show that the correlator, to leading order, depends only on the N-ality of the representation and not on the representation itself.

B. Perturbative expansion of the Polyakov-loop correlator

We now turn to the derivation of the Polyakov-loop correlator in a general representation $\mathcal{R}$ of $su(N)$. Since our effective field theory is valid to zeroth-loop order, we shall focus on the correlator expansion up to $\mathcal{O}(g^4)$ in the coupling constant. The Wilson line operator reads

$$\Omega(\boldsymbol{r}) = \exp\left[\oint_{\mathbb{S}^1} i A_3\right] = \mathrm{e}^{i\boldsymbol{H}\cdot\boldsymbol{\Phi}(\boldsymbol{r})}, \tag{21}$$

with $\boldsymbol{r}$ being a three-dimensional Euclidean vector and the Wilson line wraps the $\mathbb{S}^1$ circle. For $su(N)$, the vacuum is given by $\boldsymbol{\Phi}_0 = \frac{2\pi}{N}\boldsymbol{\rho}$. As we pointed out in Sec. II, we write

$$\boldsymbol{\Phi} = \boldsymbol{\Phi}_0 + \frac{g^2}{4\pi}\boldsymbol{b}. \tag{22}$$

We are interested in the Polyakov-loop correlator in representation $\mathcal{R}$:

$$\langle \mathcal{P}_{\mathcal{R}}(\boldsymbol{0})\mathcal{P}^{\dagger}_{\mathcal{R}}(\boldsymbol{r})\rangle \equiv \langle \mathrm{Tr}_{\mathcal{R}}\Omega(0)\mathrm{Tr}_{\mathcal{R}}\Omega^{\dagger}(\boldsymbol{r})\rangle. \tag{23}$$

This is the correlator between two Polyakov loops wrapped around the $\mathbb{S}^1$ circle and located at 0 and $\boldsymbol{r}$. To this end, let us pick any $k \neq 0 (\mathrm{mod}\, N)$ and define $\Omega_0^k \equiv \mathrm{e}^{ik\boldsymbol{H}\cdot\phi_0^{(0)}}$, whose eigenvalues are evenly spread around the unit circle, whence $\mathrm{Tr}_F\Omega_0^k = 0$. We now expand in powers of $\boldsymbol{b}$ the trace in the fundamental of the kth power of the Polyakov operator (recall that the Cartan generators $\boldsymbol{H}$ commute and that $\boldsymbol{b}$ are small fluctuations of the holonomy field about the vacuum):

$$\begin{aligned}\mathrm{Tr}_F\Omega^k(\boldsymbol{r}) &= \mathrm{Tr}_F\left[\Omega_0^k \exp\left(\frac{ikg^2}{4\pi}\boldsymbol{H}\cdot\boldsymbol{b}(\boldsymbol{r})\right)\right] \\ &\cong \mathrm{Tr}_F\left[\Omega_0^k\left\{\mathbf{1} + \frac{ikg^2}{4\pi}\boldsymbol{H}\cdot\tilde{\boldsymbol{b}}(\boldsymbol{r}) + \frac{g^4}{32\pi^2}(ik\boldsymbol{H}\cdot\boldsymbol{b}(\boldsymbol{r}))^2\right\}\right] \\ &\quad + \mathcal{O}(g^6) \\ &= \frac{ikg^2}{4\pi}B_k(\boldsymbol{r}) + \frac{g^4}{32\pi^2}C_k(\boldsymbol{r}) + \mathcal{O}(g^6),\end{aligned} \tag{24}$$

where

$$\begin{aligned} B_k(\boldsymbol{r}) &\equiv \mathrm{Tr}_F[\Omega_0^k \boldsymbol{H}\cdot\boldsymbol{b}(\boldsymbol{r})], \\ C_k(\boldsymbol{r}) &\equiv \mathrm{Tr}_F[\Omega_0^k (ik\boldsymbol{H}\cdot\boldsymbol{b}(\boldsymbol{r}))^2]. \end{aligned} \tag{25}$$

Moreover, since there is no $\mathcal{O}(g^0)$ term, we further obtain

$$(\mathrm{Tr}_F\Omega^k(\boldsymbol{r}))^2 = -\frac{k^2g^4}{16\pi^2}B_k^2(\boldsymbol{r}) + \mathcal{O}(g^6), \tag{26}$$

and $(\mathrm{Tr}_F\Omega^k(\boldsymbol{r}))^a \sim \mathcal{O}(g^6)$ for $a > 2$.

Now, let us make use of the Frobenius formula (20). For a representation $\mathcal{R}$ of $su(N)$, corresponding to a Young tableau of n boxes (not mod N), we express $\mathrm{Tr}_{\mathcal{R}}\Omega(x)$ in terms of Tr_F and expand in g. The only $\mathcal{O}(g^2)$ contribution in this expansion comes from the term with $\vec{j} = (0, 0, 0, \ldots, 1)$, i.e., the term $(\mathrm{Tr}_F P^n)^{j_n}$ with $j_n = 1$. Assuming that $n \neq 0, N, 2N, 3N, \ldots$, then there is no $\mathcal{O}(g^0)$ term, and thus we have

$$\begin{aligned}\langle \mathcal{P}_{\mathcal{R}}(\boldsymbol{0})\mathcal{P}^{\dagger}_{\mathcal{R}}(\boldsymbol{r})\rangle &= \frac{n^2g^4}{16\pi^2}\chi_{\mathcal{R}}^{\mathbb{1}}\langle B_n(\boldsymbol{0})\cdot B_n^{\dagger}(\boldsymbol{r})\rangle + \mathcal{O}(g^6) \\ &= \frac{n^2g^4}{16\pi^2}\chi_{\mathcal{R}}^{\mathbb{1}}\sum_{j,l=1}^{N}\mathrm{Tr}_F[\Omega_0^n H_j]\mathrm{Tr}_F[\Omega_0^{-n}H_l] \\ &\quad \times \langle b_j(\boldsymbol{0})b_l(\boldsymbol{r})\rangle + \mathcal{O}(g^6),\end{aligned} \tag{27}$$

where $\chi_{\mathcal{R}}^{\mathbb{1}} \equiv \chi_{\mathcal{R}}(\vec{j} = (0, 0, 0, \ldots, 1))$ and we have used the $\mathbb{R}^N$ basis in writing the double sum in (27). As we will see next, despite the fact that the prefactor depends on the representation $\mathcal{R}$, the rest of this expression is a function of $\mathrm{Tr}_F[\Omega_0^n H_j]$, which only depends on the N-ality of $\mathcal{R}$ since $\Omega_0^n = \Omega_0^{n (\mathrm{mod}\, N)}$.

The case $n = 0(\mathrm{mod}\, N)$ (e.g., the adjoint) gives a term $\mathcal{O}(g^0)$, which leads to the behavior of the correlator $\langle \mathcal{P}_{\mathcal{R}}(\boldsymbol{0})\mathcal{P}^{\dagger}_{\mathcal{R}}(\boldsymbol{r})\rangle \sim \mathrm{constant} + \mathcal{O}(g^2)$. The $\mathcal{O}(g^0)$ term is interpreted as the breaking of the flux tube. This is the expected behavior of all zero N-ality representations since the probe charges of these representations can be completely screened by soft gluons. The breaking of adjoint strings is extremely difficult to see in lattice simulations; see, e.g., [6,7]. The continuity conjecture, on the other hand, provides a neat way to see the breaking.

C. Combining everything: The string tension

So far we have set the stage to finally obtain a closed-form expression of the Polyakov-loop correlator. First we recall that the Cartan generators H_i are the components of the weights in the fundamental representation (defining representation), i.e., $H_i = \mathrm{diag}((\nu_1)_i, (\nu_2)_i, \ldots, (\nu_N)_i)$. Then, substituting (19) into (27) we obtain

$$\begin{aligned}&\langle \mathcal{P}_{\mathcal{R}}(\boldsymbol{0})\mathcal{P}^{\dagger}_{\mathcal{R}}(\boldsymbol{r})\rangle \\ &\quad = \frac{n^2g^4}{16\pi^2 N}\chi_{\mathcal{R}}^{\mathbb{1}}\sum_{j,l,k,m=1}^{N}\sum_{p=1}^{N} e^{\frac{2\pi i n}{N}(\boldsymbol{\nu}_k - \boldsymbol{\nu}_m)\cdot\boldsymbol{\rho}}(\nu_k)_j(\nu_m)_l \\ &\quad \times e^{-\frac{2\pi i p}{N}(j-l)}\langle \tilde{b}_p(\boldsymbol{0})\tilde{b}_{-p}(\boldsymbol{r})\rangle.\end{aligned} \tag{28}$$

Recalling that in the $\mathbb{R}^N$ basis we have $(\nu_a)_i = \delta_{ai} - \frac{1}{N}$, and that $\boldsymbol{\rho} = \sum_{b=1}^{N-1}\boldsymbol{\omega}_b$, where

$$\boldsymbol{\omega_b} = \sum_{a=1}^{b} \boldsymbol{e}_a - \frac{b}{N}\sum_{a=1}^{N} \boldsymbol{e}_a, \tag{29}$$

we find $\boldsymbol{\rho}\cdot\boldsymbol{\nu}_b = -b + \frac{N+1}{2}$. Using this information in (28) we find three main terms that come from the multiplication $(\nu_k)_j(\nu_m)_l$:

(1) The constant term $\frac{1}{N^2}$, which is the constant part of $(\nu_k)_j(\nu_m)_l$. This term is multiplied by the sum $\sum_{m=1}^{N} e^{\frac{i2\pi nm}{N}}$, which is zero.

(2) The term $\frac{\delta_{kj}}{N}$. Again, this term is multiplied by the sum $\sum_{m=1}^{N} e^{\frac{i2\pi nm}{N}}$, which is zero.

(3) Finally, we have the term $\delta_{kj}\delta_{ml}$, which is the only term contributing a nonzero value to $\langle \mathcal{P}_{\mathcal{R}}(\mathbf{0})\mathcal{P}^{\dagger}_{\mathcal{R}}(\boldsymbol{r})\rangle$:

$$\begin{aligned}
&\sum_{j,l,k,m=1}^{N}\sum_{p=1}^{N} e^{\frac{2\pi i n}{N}(\boldsymbol{\nu}_k-\boldsymbol{\nu}_m)\cdot\boldsymbol{\rho}}\delta_{kj}\delta_{ml}e^{-\frac{2\pi i p}{N}(j-l)}\langle\tilde{b}_p(\mathbf{0})\tilde{b}_{-p}(\boldsymbol{r})\rangle\\
&= \sum_{p,k,l} e^{-\frac{2\pi i k(n+p)}{N}}e^{\frac{2\pi i l(n+p)}{N}}\langle\tilde{b}_p(\mathbf{0})\tilde{b}_{-p}(\boldsymbol{r})\rangle\\
&= N\sum_{p,l}^{N}\delta_{n+p=0}e^{\frac{2\pi i l}{N}(n+p)}\langle\tilde{b}_p(\mathbf{0})\tilde{b}_{-p}(\boldsymbol{r})\rangle\\
&= N^2\langle\tilde{b}_p(\mathbf{0})\tilde{b}_{-p}(\boldsymbol{r})\rangle_{p=-n(\mathrm{mod}\,N)}.
\end{aligned}\tag{30}$$

Therefore, we finally obtain

$$\langle \mathcal{P}_{\mathcal{R}}(\mathbf{0})\mathcal{P}^{\dagger}_{\mathcal{R}}(\boldsymbol{r})\rangle = \frac{Nn^2g^4}{16\pi^2}\chi^{\mathbb{I}}_{\mathcal{R}}\langle\tilde{b}_k(0)\tilde{b}_{-k}(\boldsymbol{r})\rangle_{k=n(\mathrm{mod}\,N)}. \tag{31}$$

Equation (31) is the main result of this work. It shows that apart from a nonuniversal and representation dependent prefactor, the Polyakov-loop correlator can only depend on the N-ality of representation.

We can use Eq. (31) to obtain the string tension as follows. We are interested in a length scale $r > M_{\tilde{b}'_{k=n(\mathrm{mod}N)}}$, which is much bigger than than the compactification length L. Therefore, we take the limit $r\to\infty$ in (18):

$$\lim_{r\to\infty}\log\langle \mathcal{P}_{\mathcal{R}}(\mathbf{0})\mathcal{P}^{\dagger}_{\mathcal{R}}(\boldsymbol{r})\rangle = \text{constant} - \mathcal{M}_{\tilde{b}'_{k=n(\mathrm{mod}N)}}r, \tag{32}$$

from which we read the string tension

$$\sigma_k = L^{-1}\mathcal{M}_{\tilde{b}'_{k=n(\mathrm{mod}\,N)}}. \tag{33}$$

Thus, the string tension of the representation $\mathcal{R}$ will only depend on $n(\mathrm{mod}\,N)\neq 0$, which is the N-ality of the representation.

IV. W-BOSONS ON THE STRING WORLD SHEET

As a corollary of our main result, Eq. (31), one can also examine the effect of W-bosons on the string between two probe charges in representation $\mathcal{R}$. We repeat our previous analysis in super Yang-Mills on a small circle by computing correlators of Polyakov loops wrapping the $\mathbb{S}^1$ circle with W-boson insertions. In the semiclassical limit, the W-bosons are heavy and we can neglect their kinetic energies. They are charged under the moduli fields,[8] $\boldsymbol{b}$ (the charges live in the root system), and hence, they can exchange quanta of $\boldsymbol{b}$ with the probe charges. Therefore, W-bosons can be thought of as adjoint Polyakov loops wrapping the circle and their effect on the string can be inferred by computing higher Polyakov-loop correlators. A typical correlator that is invariant under charge conjugation takes the form

$$\begin{aligned}\mathcal{C}(\boldsymbol{r},\boldsymbol{r}_1,\boldsymbol{r}_2) = \langle &\mathrm{Tr}_{\mathcal{R}}\Omega(\mathbf{0})\mathrm{Tr}_{adj}\Omega_W(\boldsymbol{r}_1)\\ &\times\mathrm{Tr}_{adj}\Omega^{\dagger}_W(\boldsymbol{r}_2)\mathrm{Tr}_{\mathcal{R}}\Omega^{\dagger}(\boldsymbol{r})\rangle,\end{aligned} \tag{34}$$

and we assume that the N-ality of $\mathcal{R}$ is $k\neq 0$. Since the W-bosons are in the adjoint representation, we have

$$\mathrm{Tr}_{adj}\Omega_W(\boldsymbol{r}_1) \cong -1 + i\frac{g^2}{4\pi}\mathrm{Tr}_{adj}[e^{i\boldsymbol{H}\cdot\boldsymbol{\Phi}}\boldsymbol{H}\cdot\boldsymbol{b}] + \mathcal{O}(g^4). \tag{35}$$

By assumption, the N-ality of $\mathcal{R}$ is not zero, and hence, the expansion of $\mathrm{Tr}_{\mathcal{R}}\Omega$ starts at $\mathcal{O}(g^2)$. Then, the leading order contribution to $C(\boldsymbol{r},\boldsymbol{r}_1,\boldsymbol{r}_2)$ comes from the first term in (35) and $\mathcal{O}(g^2)$ term of $\mathrm{Tr}_{\mathcal{R}}\Omega$. Using (31), we find that the correlator, to $\mathcal{O}(g^4)$, is given by

$$\begin{aligned}\mathcal{C}(\boldsymbol{r},\boldsymbol{r}_1,\boldsymbol{r}_2) &= \langle \mathcal{P}_{\mathcal{R}}(\mathbf{0})\mathcal{P}^{\dagger}_{\mathcal{R}}(\boldsymbol{r})\rangle\\ &= \frac{Nn^2g^4}{16\pi^2}\chi^{\mathbb{I}}_{\mathcal{R}}\langle\tilde{b}_k(0)\tilde{b}_{-k}(\boldsymbol{r})\rangle_{k=n(\mathrm{mod}\,N)}.\end{aligned} \tag{36}$$

This shows that the N-ality of the string does not change by placing W-bosons on the string world sheet. Also, the string tension is unaffected, to leading order in g, by the presence of W-bosons. The fact that the string tension does not get a contribution from the W-bosons leads us to conclude that they are deconfined on the string world sheet.

This result was also reached in [24] by analyzing the $\mathcal{S}_{\mathbb{R}^3}$ strings on $\mathbb{R}^3$. Here, we provide a simple explanation of this interesting phenomenon. Let us consider two fundamental probe charges (quarks) of $su(2)$, Q and $\bar{Q}$, with opposite charges, separated a distance r, and ending on the opposite sides of our $\mathcal{S}_{\mathbb{R}^3}$ string. The total energy of the system is $E = 2m_Q + Tr$, where m_Q is the quark mass and T is the $\mathcal{S}_{\mathbb{R}^3}$ string tension. The force between the quarks is $F = -dE/dr = -T$ and hence they experience linear confinement. Now, consider the same situation but with two W-bosons placed on the string world sheet. Since the W-bosons belong to the adjoint representation and hence carry twice the charge of a fundamental quark, it is easy to convince oneself that the only configuration that respects the flux

[8] See [23] for details.

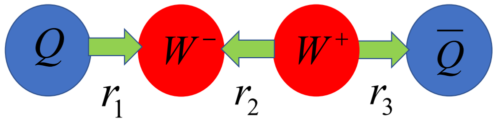

FIG. 3. W-bosons on the world sheet of the $\mathcal{S}_{\mathbb{R}^3}$ string. In this specific example, the quarks, Q, $\bar{Q}$, are taken in the fundamental representation of $su(2)$. Since the W-bosons are in the adjoint representation, they carry twice the charge of the fundamental quark. The shown configuration is the only one that satisfies the conservation of the electric flux. $r_{1,2,3}$ label the W-boson positions on the world sheet.

conservation is that shown in Fig. 3. The total energy of the system is $E = 2m_Q + 2m_W + T(r_1 + r_2 + r_3)$, where m_W is the W-boson mass. Fixing the distance between the probe charges to be $r_1 + r_2 + r_3 = r =$ constant, we find that placing the W-bosons anywhere on the string world sheet cannot change the energy of the system. Hence, the W-bosons do not experience any force on the string world sheet despite the fact that they interact logarithmically off the string.[9]

Therefore, we learn from the above treatment of the W-bosons on $\mathcal{S}_{\mathbb{S}^1}$ and $\mathcal{S}_{\mathbb{R}^3}$ that they are deconfined on the world sheets (experience no force) and they do not affect the string tension. On the pure Yang-Mills side, the W-bosons are the soft gluons that cannot screen the nonzero N-ality probe charges. This is a very intuitive phenomenon that is hardly proven in the strongly coupled regime. Nevertheless, we have shown that this phenomenon can be rigorously proven in the mass deformed super Yang-Mills on $\mathbb{R}^3 \times \mathbb{S}^1$, and by continuity we conclude that the same phenomenon takes place in pure Yang-Mills theory.

V. DISCUSSION

In this work we have shown that the tension of the string wrapping the $\mathbb{S}^1$ circle depends, to leading order, on the N-ality of the representation. The next-to-leading order effect depends on the representation $\mathcal{R}$ and is expressed in terms of the group characters of the permutation group in representation $\mathcal{R}$. These findings exactly match holographic computations that were performed in the 't Hooft large-N limit [8]. It is extremely important to emphasize the role of center symmetry and the affine monopole in arriving at this result. The affine root is the way the theory remembers its four-dimensional origin, and including the corresponding monopole is crucial to link super Yang-Mills to pure Yang-Mills via the conjectured continuity.

In terms of the strong coupling scale and the mass of the W-boson, the k-string tension is given by

$$\sigma_k = \frac{\sqrt{81}}{\pi}\frac{N\Lambda^3}{m_W}\log^2\left(\frac{m_W}{\Lambda}\right) \times \sin^2\left(\frac{\pi k}{N}\right)\sqrt{1 - \frac{4\pi m m_W^2}{3\Lambda^3}\sin^{-2}\left(\frac{\pi k}{N}\right)}, \quad (37)$$

where $k = 1, 2, \ldots, N-1$. At small values of m (this is the regime that is continuously connected to pure Yang-Mills theory), the string tension σ_n follows a square sine law:

$$\frac{\sigma_k}{\sigma_1} = \frac{\sin^2(\frac{\pi k}{N})}{\sin^2(\frac{\pi}{N})}, \quad (38)$$

where σ_1 is the fundamental string tension. This is in contradistinction with the Casimir law, $\sigma_k = (1 - \frac{k-1}{N-1})\sigma_1$, or sine law, $\sigma_k = \frac{\sin(\frac{\pi k}{N})}{\sin(\frac{\pi}{N})}\sigma_1$, which have been advocated in the literature as two possible scalings of k-strings in Yang-Mills theories; see, e.g., [25–28]. The sine law in particular is consistent with the 't Hooft large-N limit, which requires the next-to-leading order correction of σ_k to go as $1/N^2$ instead of $1/N$, as the Casimir law predicts. It is also consistent with various supersymmetric gauge theories and AdS/CFT computations; see, e.g., [29–32].

Another question concerns the large-N limit of (37), which has to be taken with care. In the standard 't Hooft limit one takes $N \to \infty$, keeping Ng^2 fixed. In this limit the W-bosons of super Yang-Mills on $\mathbb{R}^3 \times \mathbb{S}^1$ become very light, $m_W \sim 1/(NL)$, which pushes the theory to strong coupling and invalidates the semiclassical treatment. The proper limit in gauge theories on a circle is the Abelian large-N limit, which amounts to taking $N \to \infty$, keeping the W-boson mass fixed. In this limit we have $\sigma_k = k^2 + \mathcal{O}(\frac{1}{N^2})$, which is different from the expected 't Hooft large-N limit $\frac{\sigma_k}{\sigma_1} = k + \mathcal{O}(\frac{1}{N^2})$ in noncompact Yang-Mills theory. In the latter theory, the linear dependence of the string tension on the N-ality k indicates that the string is made of k independent components that do not interact with each other, which is not the case for the $\mathcal{S}_{\mathbb{S}^1}$ strings in the compactified theory.[10]

The square sine law scaling in super Yang-Mills on $\mathbb{R}^3 \times \mathbb{S}^1$ as well as the unexpected large-N behavior is attributed to the fact that the string $\mathcal{S}_{\mathbb{S}^1}$ is much thicker than the compactification radius, and therefore, one should not expect the string to be composed of N noninteracting components, as in the 4-D 't Hooft large-N case. One expects, however, the string tension to depart from the square sine law and approach the sine law in the $\Lambda^{-1} \ll NL$ limit. Assuming that the continuity between super and pure

[9]Super Yang-Mills on $\mathbb{R}^3 \times \mathbb{S}^1$ is dimensionally reduced to $\mathbb{R}^3$. Electric charges in a three-dimensional theory experience logarithmic interactions.

[10]It was also shown in [33] that super Yang-Mills on $\mathbb{R}^3 \times \mathbb{S}^1$ in the Abelian large-N limit flows to a gapless theory in $\mathbb{R}^4$, which indicates that the large-L and Abelian large-N limits do not commute.

Yang-Mills holds, then this will happen in a way that preserves the N-ality dependence of the representation.

Finally, we compare our findings to other string models in the literature. In particular, we compare our $\mathcal{S}_{\mathbb{S}^1}$ strings to the $\mathcal{S}_{\mathbb{R}^3}$ strings that were studied in deformed Yang-Mills theory[11] on $\mathbb{R}^3 \times \mathbb{S}^1$ in [15] and also to strings in the softly broken Seiberg-Witten (SW) theory [35,36]. We start with SW theory, where the strings are Abelian in nature and of Abrikosov-Nielsen-Olesen type. The Weyl group in SW theory is broken, and therefore, one has $N-1$ different flux tubes corresponding to the $N-1$ fundamental weights $\boldsymbol{\omega}_a$, $a = 1, 2, \ldots, N-1$, as indicated in [31]. The breaking of the Weyl group results in having different string tensions between quarks belonging to the same representation, depending on the specific weights of the quarks. For example, in $su(3)$ we have two nondegenerate strings $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$, corresponding to the two fundamental weights. Hence, fundamental quarks (antiquarks) with weights $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \boldsymbol{\nu}_3$ $(\bar{\boldsymbol{\nu}}_1, \bar{\boldsymbol{\nu}}_2, \bar{\boldsymbol{\nu}}_3)$ will have strings $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2 - \boldsymbol{\mu}_1, \boldsymbol{\mu}_2$, respectively. This is in contradistinction with $\mathcal{S}_{\mathbb{R}^3}$ strings in deformed Yang-Mills (dYM) theory, where we have an unbroken Weyl group. This results in degenerate string tensions among all the fundamental quarks. For higher N-ality, the string tensions of a representation fall into distinct $\mathbb{Z}_N$ orbits, each of which has degenerate string tensions. In this regard, the $\mathcal{S}_{\mathbb{R}^3}$ strings of dYM are closer in nature to the QCD strings than the SW strings. The string tension in dYM, however, will in general depend on the representation, not only on its N-ality. For example, the two-index symmetric and two-index antisymmetric representations have different string tensions. Unlike both types of strings (SW and dYM), we find that $\mathcal{S}_{\mathbb{S}^1}$ strings in super Yang-Mills, even though the theory is still in the Abelian regime, depend only on the N-ality of the representation, making them identical to what one expects for QCD.

This work lends extra support to the continuity picture between a class of deformed Yang-Mills theory on $\mathbb{R}^3 \times \mathbb{S}^1$ and real-world QCD, including the conjectured continuity between super and pure Yang-Mills. Until now, there have been several tests to check the nature of this continuity, its regime of validity, and we were able to extract important lessons about the four-dimensional theory [22,37–40]. It has been found that the deformed theories share a range of characteristics that point to an underlying structure in the four-dimensional Yang-Mills, which is not yet understood but is similar to the structure of the deformed theory.

[11]Deformed Yang-Mills is a theory with massive adjoint fermions or double trace deformation [34].

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