Bounding the Mim-Width of Hereditary Graph Classes

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Abstract

A large number of NP-hard graph problems are solvable in XP time when parameterized by some width parameter. Hence, when solving problems on special graph classes, it is helpful to know if the graph class under consideration has bounded width. In this paper we consider mim-width, a particularly general width parameter that has a number of algorithmic applications whenever a decomposition is "quickly computable" for the graph class under consideration.

We start by extending the toolkit for proving (un)boundedness of mim-width of graph classes. By combining our new techniques with known ones we then initiate a systematic study into bounding mim-width from the perspective of hereditary graph classes, and make a comparison with clique-width, a more restrictive width parameter that has been well studied.

We prove that for a given graph H, the class of H-free graphs has bounded mim-width if and only if it has bounded clique-width. We show that the same is not true for (H_1, H_2)-free graphs.

We identify several general classes of (H_1, H_2)-free graphs having unbounded clique-width, but bounded mim-width; moreover, we show that a branch decomposition of constant mim-width can be found in polynomial time for these classes. Hence, these results have algorithmic implications: when the input is restricted to such a class of (H_1, H_2)-free graphs, many problems become polynomial-time solvable, including classical problems such as k-Colouring and Independent Set, domination-type problems known as LC-VSVP problems, and distance versions of LC-VSVP problems, to name just a few. We also prove a number of new results showing that, for certain H_1 and H_2, the class of (H_1, H_2)-free graphs has unbounded mim-width.

Boundedness of clique-width implies boundedness of mim-width. By combining our results with the known bounded cases for clique-width, we present summary theorems of the current state of the art for the boundedness of mim-width for (H_1, H_2)-free graphs. In particular, we classify the mim-width of (H_1, H_2)-free graphs for all pairs (H_1, H_2) with $|V(H_1)| + |V(H_2)| \leq 8$. When H_1 and H_2 are connected graphs, we classify all pairs (H_1, H_2) except for one remaining infinite family and a few isolated cases.

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1 Introduction

Many computationally hard graph problems can be solved efficiently after placing appropriate restrictions on the input graph. Instead of trying to solve individual problems in an ad hoc way, one may aim to find the underlying reasons why some sets of problems behave better on certain graph classes than other sets of problems. The ultimate goal in this type of research is to obtain complexity dichotomies for large families of graph problems. Such dichotomies tell us for which graph classes a certain problem or set of problems can or cannot be solved efficiently (under standard complexity assumptions).

One reason that might explain the jump from computational hardness to tractability after restricting the input to some graph class $\mathcal{G}$ is that $\mathcal{G}$ has bounded "width", that is, every graph in $\mathcal{G}$ has width at most c for some constant c. One can define the notion of "width" in many different ways (see the surveys [31, 32, 38, 49]). As such, the various width parameters differ in strength. To explain this, we say that a width parameter p *dominates* a width parameter q if there is a function f such that $p(G) \leq f(q(G))$ for all graphs G. If p dominates q but q does not dominate p, then p is said to be *more powerful* than q. As a consequence, proving that a problem is polynomial-time solvable for graph classes for which p is bounded yields more tractable graph classes than doing this for graph classes for which q is bounded. If both p and q dominate each other, then p and q are *equivalent*. For instance, the width parameters boolean-width, clique-width, module-width, NLC-width and rank-width are all equivalent [15, 37, 44, 46], but more powerful than the equivalent parameters branch-width and treewidth [19, 47, 49]. In this paper we focus on an even more powerful width parameter called *mim-width* (maximum induced matching width). Vatshelle [49] introduced mim-width, which we define in Section 3, and proved that mim-width is more powerful than boolean-width, and consequently, clique-width, module-width, NLC-width and rank-width.

1.1 Algorithmic Implications

One trade-off of a more powerful width parameter is the difficulty in obtaining a branch decomposition of bounded width. In general, computing mim-width is NP-hard; deciding if the mim-width is at most k is $\mathsf{W}[1]$-hard when parameterized by k; and there is no polynomial-time algorithm for approximating the mim-width of a graph to within a constant factor of the optimal, unless $\mathsf{NP} = \mathsf{ZPP}$ [48]. Moreover, in contrast to algorithms for graphs of bounded treewidth [6] or clique-width [44], it is an open problem whether we can compute a branch decomposition of constant mim-width for graphs of bounded mim-width. Hence, algorithms for graphs of bounded mim-width still require a branch decomposition of constant mim-width as part of the input. However, there are many interesting graph classes for which mim-width is bounded and *quickly computable*, that is, the class admits a polynomial-time algorithm for obtaining a branch decomposition of constant mim-width. We give examples of such graph classes known in the literature in Section 1.2 before discussing the new graph classes we found in Section 1.4. Below we briefly discuss known algorithms for problems on graph classes of bounded mim-width.

Belmonte and Vatshelle [1] and Bui-Xuan, Telle and Vatshelle [16] proved that a large set of problems, known as Locally Checkable Vertex Subset and Vertex Partitioning (LC-VSVP) problems [45], can be solved in polynomial time for graph classes where mim-width is bounded and quickly computable. Well-known examples of such problems include (Total)

Dominating Set, Independent Set and k-Colouring for every fixed positive integer k.[1] Later, Fomin, Golovach and Raymond [27] proved that the XP algorithms for Independent Set and Dominating Set are in a sense best possible, showing that these two problems are W[1]-hard when parameterized by mim-width.

On the positive side, XP algorithms parameterized by mim-width are now also known for problems outside the LC-VSVP framework. In particular, Jaffke, Kwon, Strømme and Telle [34] proved that the distance versions of LC-VSVP problems can be solved in polynomial time for graph classes where mim-width is bounded and quickly computable. Jaffke, Kwon and Telle [35, 36] proved similar results for Longest Induced Path, Induced Disjoint Paths, H-Induced Topological Minor and Feedback Vertex Set. The latter result has recently been generalized to Subset Feedback Vertex Set and Node Multiway Cut, by Bergougnoux, Papadopoulos and Telle [3].

Bergougnoux and Kanté [2] gave a meta-algorithm for problems with a global constraint, providing unifying XP algorithms in mim-width for several of the aforementioned problems, as well as Connected Dominating Set, Node Weighted Steiner Tree, and Maximum Induced Tree. Galby, Munaro and Ries [29] proved that Semitotal Dominating Set is polynomial-time solvable for graph classes where mim-width is bounded and quickly computable.

1.2 Mim-width of Special Graph Classes

Belmonte and Vatshelle [1] proved that the mim-width of the following graph classes is bounded and quickly computable: permutation graphs, convex graphs and their complements, interval graphs and their complements, circular k-trapezoid graphs, circular permutation graphs, Dilworth-k graphs, k-polygon graphs, circular-arc graphs and complements of d-degenerate graphs.

Some of the results of Belmonte and Vatshelle [1] have been extended. Let $K_r \boxminus K_r$ be the graph obtained from $2K_r$ by adding a perfect matching, and let $K_r \boxminus rP_1$ be the graph obtained from $K_r \boxminus K_r$ by removing all the edges in one of the complete graphs (see Section 2 for undefined notation). Kang et al. [39] showed that for any integer $r \geq 2$, there is a polynomial-time algorithm for computing a branch decomposition of mim-width at most $r-1$ when the input is restricted to $(K_r \boxminus rP_1)$-free chordal graphs, which generalize interval graphs, or $(K_r \boxminus K_r)$-free co-comparability graphs, which generalize permutation graphs. Hence, in particular, all these classes have bounded mim-width.

Kang et al. [39] also proved that the classes of chordal graphs, circle graphs and co-comparability graphs have unbounded mim-width; for the latter two classes, this was shown independently by Mengel [43]. Vatshelle [49] and Brault-Baron et al. [13] showed the same for grids and chordal bipartite graphs, respectively, whereas Mengel [43] proved that strongly chordal split graphs have unbounded mim-width.

Let $K_{1,s}^1$ be the graph obtained from the $(s+1)$-vertex star $K_{1,s}$ after subdividing each edge once. Brettell et al. [14] showed that the mim-width of $(K_r, K_{1,s}^1, P_t)$-free graphs is bounded and quickly computable for every $r \geq 1$, $s \geq 1$ and $t \geq 1$. As $sP_1 + P_5$ is an induced subgraph of $K_{1,s+2}^1$, this yielded an alternative proof of a result of Couturier et al. [20] who showed that List k-Colouring is polynomially solvable for $(sP_1 + P_5)$-free graphs for all $k \geq 1$ and $s \geq 0$. As another consequence, for all $k \geq 3$, $s \geq 1$ and $t \geq 1$, List k-Colouring

[1] In contrast to clique-width [41], Colouring (where k is part of the input) is NP-complete for graphs of bounded mim-width, as it is NP-complete for circular-arc graphs [30], which have mim-width at most 2 [1].

is polynomial-time solvable for $(K_{1,s}^1, P_t)$-free graphs; previously this was shown for $k = 3$ by Chudnovsky et al. [18].

Bonomo-Braberman et al. [8] considered the following generalisation of convex graphs. A bipartite graph $G = (A, B, E)$ is $\mathcal{H}$-convex, for some family of graphs $\mathcal{H}$, if there exists a graph $H \in \mathcal{H}$ with $V(H) = A$ such that the set of neighbours in A of each $b \in B$ induces a connected subgraph of H (when $\mathcal{H}$ is the set of paths, we obtain exactly convex graphs). They showed that the mim-width of the class of $\mathcal{H}$-convex graphs is bounded and quickly computable if $\mathcal{H}$ is the set of cycles, or $\mathcal{H}$ is the set of trees with bounded maximum degree and bounded number of vertices of degree at least 3.

1.3 Our Focus

We continue the study on boundedness of mim-width and aim to identify more graph classes of bounded or unbounded mim-width. Our motivation is both algorithmic and structural. As discussed above, there are clear algorithmic benefits if a graph class has bounded mim-width. From a structural point of view, we aim to initiate a *systematic* study of the boundedness of mim-width, comparable to a similar, long-standing study of the boundedness of clique-width (see [23, 32, 38] for some surveys on clique-width).

The framework of hereditary graph classes is highly suitable for such a study. A graph class $\mathcal{G}$ is *hereditary* if it is closed under vertex deletion. A class $\mathcal{G}$ is hereditary if and only if there exists a (unique) set of graphs $\mathcal{F}$ of (minimal) forbidden induced subgraphs for $\mathcal{G}$. That is, a graph G belongs to $\mathcal{G}$ if and only if G does not contain any graph from $\mathcal{F}$ as an induced subgraph. We also say that G is *$\mathcal{F}$-free*. Note that $\mathcal{F}$ may have infinite size. For example, if $\mathcal{G}$ is the class of bipartite graphs, then $\mathcal{F}$ is the set of all odd cycles.

As a natural starting point we consider the case where $|\mathcal{F}| = 1$, say $\mathcal{F} = \{H\}$. It is not difficult to verify that a class of H-free graphs has bounded mim-width if and only if it has bounded clique-width if and only if H is an induced subgraph of the 4-vertex path P_4; see Section 3 for details. On the other hand, there exist hereditary graph classes, such as interval graphs and permutation graphs, that have bounded mim-width, even mim-width 1 [49], but unbounded clique-width [33]. However, these graph classes have an *infinite* set of forbidden induced subgraphs. Hence, questions we aim to address in this paper are: Does there exist a hereditary graph class characterized by a finite set $\mathcal{F}$ that has bounded mim-width but unbounded clique-width? Can we use the same techniques as when dealing with clique-width? In particular we focus on the case where $|\mathcal{F}| = 2$. Such classes are called *bigenic*.

1.4 Our Results and Methodology

In order to work with width parameters it is useful to have a set of graph operations that *preserve* boundedness or unboundedness of the width parameter. That is, if we apply such a width-preserving operation, or only apply it a constant number of times, the width of the graph does not change by too much. In this way one might be able to modify an arbitrary graph from a given "unknown" class $\mathcal{G}_1$ into a graph from a class $\mathcal{G}_2$ known to have bounded or unbounded width. This would then imply that $\mathcal{G}_1$ also has bounded or unbounded width, respectively. Two useful operations preserving clique-width are vertex deletion [42] and subgraph complementation [38]. The latter operation replaces every edge in some subgraph of the graph by a non-edge, and vice versa. As we will see in Section 6, subgraph complementation does not preserve boundedness or unboundedness of mim-width[2].

To work around this limitation, we collect and generalize known mim-width preserving graph operations from the literature in Section 3 (some of these operations only show that the mim-width cannot decrease after applying them). In the same section we also state some known useful results on mim-width and prove that elementary graph classes, such as walls and net-walls, have unbounded mim-width.

In Sections 4 and 5 we use the results from Section 3. In Section 4 we present new bigenic classes of bounded mim-width. These graph classes are all known to have unbounded clique-width. Hence, our results show that the dichotomy for boundedness of mim-width no longer coincides with the one for clique-width when $|\mathcal{F}| = 2$ instead of $|\mathcal{F}| = 1$. Moreover, for each of these classes, a branch decomposition of constant mim-width is easily computable for any graph in the class. This immediately implies that there are polynomial-time algorithms for many problems when restricted to these classes, as described in Section 1.1. In Section 5 we present new bigenic classes of unbounded mim-width; these graph classes are known to have unbounded clique-width.

In Section 6 we give a state-of-the-art summary of our new results combined with known results. The known results include the bigenic graph classes of bounded clique-width (as bounded clique-width implies bounded mim-width). In the same section we compare our results for the mim-width of bigenic graph classes with the ones for clique-width. We also state a number of open problems.

2 Preliminaries

We consider only finite graphs $G = (V, E)$ with no loops and no multiple edges. For a vertex $v \in V$, the *neighbourhood* $N(v)$ is the set of vertices adjacent to v in G. The *degree* $d(v)$ of a vertex $v \in V$ is the size $|N(v)|$ of its neighbourhood. A graph is *subcubic* if every vertex has degree at most 3. For disjoint $S, T \subseteq V$, we say that S is *complete to* T if every vertex of S is adjacent to every vertex of T, and S is *anticomplete to* T if there are no edges between S and T. The *distance* from a vertex u to a vertex v in G is the length of a shortest path between u and v. A set $S \subseteq V$ *induces* the subgraph $G[S] = (S, \{uv : u, v \in S, uv \in E\})$. If G' is an induced subgraph of G we write $G' \subseteq_i G$. The *complement* of G is the graph $\overline{G}$ with vertex set $V(G)$, such that $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

Given a graph G and a degree-k vertex v of G with $N(v) = \{u_1, \ldots, u_k\}$, the *clique implant on* v is the operation of deleting v, adding k new vertices $v_1, \ldots, v_k$ forming a clique, and adding edges $v_i u_i$ for each $i \in \{1, \ldots, k\}$. The *k-subdivision* of an edge uv in a graph replaces uv by k new vertices $w_1, \ldots, w_k$ with edges $uw_1, w_k v$ and $w_i w_{i+1}$ for each $i \in \{1, \ldots, k-1\}$, i.e. the edge is replaced by a path of length $k + 1$. The *disjoint union* $G + H$ of graphs G and H has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. We denote the disjoint union of k copies of G by kG. For a graph H, a graph G is *H-free* if G has no induced subgraph isomorphic to H. For a set of graphs $\{H_1, \ldots, H_k\}$, a graph G is *$(H_1, \ldots, H_k)$-free* if G is H_i-free for every $i \in \{1, \ldots, k\}$.

An *independent set* of a graph is a set of pairwise non-adjacent vertices. A *clique* of a graph is a set of pairwise adjacent vertices. A *matching* of a graph is a set of pairwise non-adjacent edges. A matching M of a graph G is *induced* if there are no edges of G between vertices incident to distinct edges of M.

[2] The situation is different for mim-width 1; Vatshelle [49] showed that if $\mathrm{mimw}(G) = 1$ then $\mathrm{mimw}(\overline{G}) = 1$.

The path, cycle and complete graph on n vertices are denoted by P_n, C_n and K_n, respectively. The graph K_3 is also called the *triangle*. A graph is *r-partite*, for $r \geq 2$, if its vertex set admits a partition into r classes such that every edge has its endpoints in different classes. An r-partite graph in which every two vertices from different partition classes are adjacent is a *complete r-partite graph* and a 2-partite graph is also called *bipartite*. A graph is *co-bipartite* if it is the complement of a bipartite graph. A *split graph* is a graph G that admits a *split partition* (C, I), that is, $V(G)$ can be partitioned into a clique C and an independent set I. Equivalently, a graph is split if and only if it is $(2P_2, C_4, C_5)$-free. The *subdivided claw* $S_{h,i,j}$, for $1 \leq h \leq i \leq j$ is the tree with one vertex x of degree 3 and exactly three leaves, which are of distance h, i and j from x, respectively. Note that $S_{1,1,1} = K_{1,3}$. For $t \geq 3$, sun_t denotes the graph on $2t$ vertices obtained from a complete graph on t vertices $u_1, \ldots, u_t$ by adding t vertices $v_1, \ldots, v_t$ such that v_i is adjacent to u_i and u_{i+1} for each $i \in \{1, \ldots, t-1\}$ and v_t is adjacent to u_1 and u_t. See Figure 1 for a picture of sun_5.

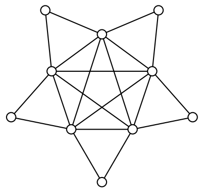

Figure 1 The graph sun_5.

3 Mim-Width: Definition and Basic Results

A *branch decomposition* for a graph G is a pair (T, δ), where T is a subcubic tree and δ is a bijection from $V(G)$ to the leaves of T. Each edge $e \in E(T)$ naturally partitions the leaves of T into two classes, depending on which component they belong to when e is removed. In this way, each edge $e \in E(T)$ corresponds to a partition L_e and $\overline{L_e}$ of the set of leaves of T, depending on which component of $T - e$ the leaves of T belong to. Consequently, each edge e induces a partition $(A_e, \overline{A_e})$ of $V(G)$, where $\delta(A_e) = L_e$ and $\delta(\overline{A_e}) = \overline{L_e}$. For two disjoint sets X and Y, let $G[X, Y]$ denote the bipartite subgraph of G induced by the edges with one endpoint in X and the other in Y. For each edge $e \in E(T)$ and corresponding partition $(A_e, \overline{A_e})$ of $V(G)$, we denote by $\text{cutmim}_G(A_e, \overline{A_e})$ the size of a maximum induced matching in $G[A_e, \overline{A_e}]$. The *mim-width* of the branch decomposition (T, δ) is the quantity $\text{mimw}_G(T, \delta) = \max_{e \in E(T)} \text{cutmim}_G(A_e, \overline{A_e})$. The *mim-width* of the graph G, denoted $\text{mimw}(G)$, is the minimum value of $\text{mimw}_G(T, \delta)$ over all possible branch decompositions (T, δ) for G. See Figure 2 for an example.

3.1 Mim-Width Preserving Operations

The next lemma, which is due to Vatshelle, shows that deleting a vertex from a graph has only a small effect on the mim-width. In particular, any class of graphs with mim-width bounded by some constant is closed under vertex deletion, so it is indeed natural to study which hereditary classes have (un)bounded mim-width.

▶ **Lemma 1** ([49]). *Let G be a graph and $v \in V(G)$. Then* $\text{mimw}(G) - 1 \leq \text{mimw}(G - v) \leq \text{mimw}(G)$.

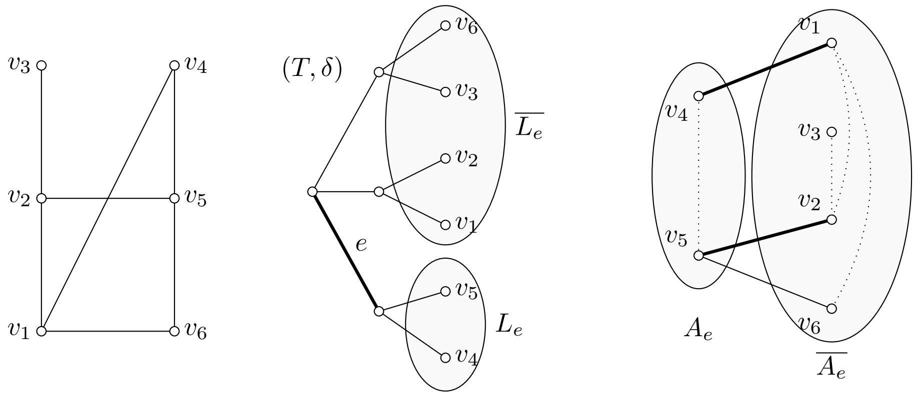

Figure 2 An example of a graph G with a branch decomposition (T, δ). It can be easily seen that $\text{mimw}_G(T, \delta) \leq 2$. The partition $(A_e, \overline{A_e})$ of $V(G)$ in the rightmost figure witnesses that $\text{mimw}_G(T, \delta) \geq 2$. Hence, $\text{mimw}_G(T, \delta) = 2$. It can be checked that the branch decomposition (T', δ') obtained from (T, δ) by swapping v_2 and v_5 and swapping v_3 and v_4 shows that $\text{mimw}(G) = 1$.

The next two lemmas show that edge subdivision and clique implantation do not change the mim-width of a graph by too much.

▶ **Lemma 2.** *Let G be a graph and let G' be the graph obtained by* 1*-subdividing an edge of G. Then* $\text{mimw}(G) \leq \text{mimw}(G') \leq \text{mimw}(G) + 1$.

Proof. Let uv be the subdivided edge of G, and let $w \in V(G') \setminus V(G)$ such that $\{uw, wv\} \subseteq E(G')$. We first prove that $\text{mimw}(G) \leq \text{mimw}(G')$. Given a branch decomposition (T', δ') for G', we construct a branch decomposition (T, δ) for G such that $\text{mimw}_G(T, \delta) \leq \text{mimw}_{G'}(T', \delta')$. Since $V(G') = V(G) \cup \{w\}$, we simply let T be the tree obtained from T' by deleting the leaf $\delta'(w)$, and let δ be the restriction of δ' to $V(G)$. Clearly, (T, δ) is a branch decomposition for G.

We claim that $\text{mimw}_G(T, \delta) \leq \text{mimw}_{G'}(T', \delta')$. Suppose, to the contrary, that there exists $e \in E(T)$ such that $\text{cutmim}_G(A_e, \overline{A_e}) > \text{mimw}_{G'}(T', \delta')$, and let M be a maximum induced matching in $G[A_e, \overline{A_e}]$. By construction, e is also an edge of T' and the partition $(B_e, \overline{B_e})$ of $V(G')$ corresponding to e is either $(A_e \cup \{w\}, \overline{A_e})$ or $(A_e, \overline{A_e} \cup \{w\})$. If $uv \notin M$, then M is also an induced matching in $G'[B_e, \overline{B_e}]$. On the other hand, if $uv \in M$, then either $M \setminus \{uv\} \cup \{uw\}$ or $M \setminus \{uv\} \cup \{wv\}$ is an induced matching in $G'[B_e, \overline{B_e}]$. In all cases, we find an induced matching in $G'[B_e, \overline{B_e}]$ of size $|M| = \text{cutmim}_G(A_e, \overline{A_e}) > \text{mimw}_{G'}(T', \delta')$, a contradiction.

We now prove that $\text{mimw}(G') \leq \text{mimw}(G) + 1$. Given a branch decomposition (T, δ) for G, we construct a branch decomposition (T', δ') for G' such that $\text{mimw}_{G'}(T', \delta') \leq \text{mimw}_G(T, \delta) + 1$. Let T' be the subcubic tree obtained by attaching two pendant vertices x_1 and x_2 to the leaf $\delta(u)$ of T, and let $\delta'(x) = \delta(x)$, for each $x \in V(G) \setminus \{u\}$, and $\delta'(u) = x_1$ and $\delta'(w) = x_2$. Clearly, (T', δ') is a branch decomposition for G'.

We claim that $\text{mimw}_{G'}(T', \delta') \leq \text{mimw}_G(T, \delta) + 1$. Suppose, to the contrary, that there exists $e \in E(T')$ such that $\text{cutmim}_{G'}(A_e, \overline{A_e}) > \text{mimw}_G(T, \delta) + 1$. Clearly, $e \in E(T)$, for otherwise $\text{cutmim}_{G'}(A_e, \overline{A_e}) \leq 1$. As e is an edge of T, u and w belong to the same partition class of $V(G')$ and the partition $(B_e, \overline{B_e})$ of $V(G)$ corresponding to e is obtained from $(A_e, \overline{A_e})$ by removing w. Let M' be a maximum induced matching in $G'[A_e, \overline{A_e}]$. If w is matched in M', then it must be $wv \in M'$ and we remove this edge. If both u and v are matched in M', we remove the matching edge incident to u. In all the other cases, we keep the matching edges. In this way we obtain an induced matching in $G[B_e, \overline{B_e}]$ of size at least $|M'| - 1 = \text{cutmim}_{G'}(A_e, \overline{A_e}) - 1 > \text{mimw}_G(T, \delta)$, a contradiction. ◀

▶ **Lemma 3.** *Let G be a graph and let G' be the graph obtained from G by a clique implant on $v \in V(G)$. Then* $\text{mimw}(G) \leq \text{mimw}(G') \leq \text{mimw}(G) + d(v)$.

Proof. We first prove that $\text{mimw}(G) \leq \text{mimw}(G')$. Suppose that v is a degree-k vertex of G with $N(v) = \{u_1, \dots, u_k\}$ and let $\{v_1, \dots, v_k\}$ be the clique implanted on v. Given a branch decomposition (T', δ') for G', we construct a branch decomposition (T, δ) for G such that $\text{mimw}_G(T, \delta) \leq \text{mimw}_{G'}(T', \delta')$. Since $V(G') = V(G) \setminus \{v\} \cup \{v_1, \dots, v_k\}$, we build a tree T as follows. We delete the leaves $\delta'(v_2), \dots, \delta'(v_k)$ from T' and let $\delta(x) = \delta'(x)$ if $x \in V(G) \setminus \{v\}$ and $\delta(v) = \delta'(v_1)$. Clearly, (T, δ) is a branch decomposition for G.

We claim that $\text{mimw}_G(T, \delta) \leq \text{mimw}_{G'}(T', \delta')$. Suppose, to the contrary, that there exists $e \in E(T)$ such that $\text{cutmim}_G(A_e, \overline{A_e}) > \text{mimw}_{G'}(T', \delta')$ and let M be a maximum induced matching in $G[A_e, \overline{A_e}]$. Suppose, without loss of generality, that $v \in A_e$. By construction, e is also an edge of T' and the partition $(B_e, \overline{B_e})$ of $V(G')$ corresponding to e is of the form $((A_e \setminus \{v\}) \cup \{v_1\} \cup X, \overline{A_e} \cup Y)$, where $X \subseteq \{v_2, \dots, v_k\}$ and $Y = \{v_2, \dots, v_k\} \setminus X$. If v is not matched in M, then M is also an induced matching in $G'[B_e, \overline{B_e}]$ of size $|M| = \text{cutmim}_G(A_e, \overline{A_e}) > \text{mimw}_{G'}(T', \delta')$, a contradiction. Therefore, suppose that v is matched in M. We have that $vu_i \in M$, for some $i \in \{1, \dots, k\}$. If $i = 1$, then M is an induced matching in $G'[B_e, \overline{B_e}]$. Otherwise, $i > 1$ and we proceed as follows. If v_i belongs to the partition class of v_1, we replace M with $M \setminus \{vu_i\} \cup \{v_i u_i\}$. If v_i does not belong to the partition class of v_1, we replace M with $M \setminus \{vu_i\} \cup \{v_1 v_i\}$. It is easy to see that in all cases we find an induced matching in $G'[B_e, \overline{B_e}]$ of size $|M| > \text{mimw}_{G'}(T', \delta')$, a contradiction.

We now prove that $\text{mimw}(G') \leq \text{mimw}(G) + d(v)$. Suppose that v is a degree-k vertex of G, and let $\{v_1, \dots, v_k\}$ be the clique implanted on v. Given a branch decomposition (T, δ) for G, we construct a branch decomposition (T', δ') for G' such that $\text{mimw}_{G'}(T', \delta') \leq \text{mimw}_G(T, \delta) + k$. We $(k-1)$-subdivide the edge of T incident to $\delta(v)$ with new vertices $x_1, \dots, x_{k-1}$, attach a pendant vertex y_i to each x_i, let $\delta'(v_k) = \delta(v)$ and $\delta'(v_i) = y_i$, for each $i \in \{1, \dots, k-1\}$, and finally let $\delta'(u) = \delta(u)$ for each $u \in V(G') \setminus \{v_1, \dots, v_k\}$. Clearly, (T', δ') is a branch decomposition for G'.

We claim that $\text{mimw}_{G'}(T', \delta') \leq \text{mimw}_G(T, \delta) + k$. Suppose, to the contrary, that there exists $e \in E(T')$ such that $\text{cutmim}_{G'}(A_e, \overline{A_e}) > \text{mimw}_G(T, \delta) + k$. We have that $e \in E(T)$, for otherwise $\text{cutmim}_{G'}(A_e, \overline{A_e}) \leq k$. But since e is an edge of T, the vertices $v_1, \dots, v_k$ all belong to the same partition class of $V(G')$, say A_e, and the partition $(B_e, \overline{B_e})$ of $V(G)$ corresponding to e is obtained from $(A_e, \overline{A_e})$ by removing $\{v_1, \dots, v_k\}$ and adding v to A_e. Let M' be a maximum induced matching in $G'[A_e, \overline{A_e}]$. By possibly removing the at most k edges in M' incident to vertices in $\{v_1, \dots, v_k\}$, we obtain an induced matching in $G[B_e, \overline{B_e}]$ of size at least $|M'| - k = \text{cutmim}_{G'}(A_e, \overline{A_e}) - k > \text{mimw}_G(T, \delta)$, a contradiction. ◀

Mengel [43] showed that adding edges inside the partition classes of a bipartite graph does not decrease mim-width by much. This result can be generalized to k-partite graphs in the following way.

▶ **Lemma 4.** *Let G be a k-partite graph with partition classes $V_1, \dots, V_k$, and let G' be a graph obtained from G by adding edges where for each added edge, there exists some i such that both endpoints are in V_i. Then* $\text{mimw}(G') \geq \frac{1}{k} \cdot \text{mimw}(G)$.

Proof. Let (T, δ) be a branch decomposition for G'. Since G and G' have the same vertex set, (T, δ) is a branch decomposition for G as well. It is enough to show that $\text{mimw}_G(T, \delta) \leq k \cdot \text{mimw}_{G'}(T, \delta)$. Therefore, let $e \in E(T)$ be such that $\text{mimw}_G(T, \delta) = \text{cutmim}_G(A_e, \overline{A_e})$, and let M be a maximum induced matching in $G[A_e, \overline{A_e}]$. For each i, consider the set $M_i = \{uv \in M : u \in A_e \cap V_i\}$. These k sets partition M. Let M' be a partition class of size at least $|M|/k$. Clearly, M' is an induced matching in $G'[A_e, \overline{A_e}]$ and so $k \cdot \mathrm{mimw}_{G'}(T, \delta) \geq k \cdot |M'| \geq |M| = \mathrm{mimw}_G(T, \delta)$. ◀

The next lemma shows that to bound the mim-width of a class of graphs, we may restrict our attention to 2-connected graphs in the class. We note that this property is not specific to mim-width: Gottlob et al. [31] observed it for rank-width, and this argument also applies for any appropriate width parameter defined using branch decompositions. A *block* is a maximal connected subgraph with no cut-vertex.

▶ **Lemma 5.** *Let G be a graph. Then* $\mathrm{mimw}(G) = \max\{\mathrm{mimw}(H) : H$ *is a block of* $G\}$*. Moreover, given branch decompositions (T_H, δ_H) of each block H of G, with* $\mathrm{mimw}_H(T_H, \delta_H) \leq k$*, we can compute a branch decomposition of G with mim-width at most k in polynomial time.*

Proof. By Lemma 1, $\mathrm{mimw}(G) \geq \max\{\mathrm{mimw}(H) : H \text{ is a block of } G\}$. We describe how to compute a branch decomposition (T, δ) of G such that $\mathrm{mimw}_G(T, \delta) \leq \max\{\mathrm{mimw}_H(T_H, \delta_H) : H \text{ is a block of } G\}$, in polynomial time. It suffices to describe a polynomial-time procedure when G consists of two blocks H_1 and H_2 joined at a vertex v (we can repeat this procedure $O(n)$ times, thereby constructing a branch decomposition for G block-by-block). To construct T, join T_{H_1} and T_{H_2} by identifying the leaf $t_1 \in T_{H_1}$ and the leaf $t_2 \in T_{H_2}$ such that $\delta_{H_1}(v) = t_1$ and $\delta_{H_2}(v) = t_2$, and then create a new leaf t incident to the identified vertex. Let δ inherit the mappings from δ_{H_1} and δ_{H_2}, and set $\delta(v) = t$. If $e \in E(T)$ is incident to t, then $\mathrm{cutmim}_G(A_e, \overline{A_e}) \leq 1$, since one of A_e and $\overline{A_e}$ has size one. For any other edge of T, either A_e or $\overline{A_e}$ contains $V(H_1)$ or $V(H_2)$. The result follows. ◀

The following lemma is due to Galby and Munaro, who used it to prove that Dominating Set admits a PTAS for a subclass of VPG graphs when the representation is given.

▶ **Lemma 6** ([28])**.** *Let G be a graph and let $S \subseteq V$. Let $G' = (V', E')$ denote the graph with $V' = V$ and $E' = E \cup \{uv : u, v \in S\}$. Then* $\mathrm{mimw}(G') \leq \mathrm{mimw}(G) + 1$.

The final structural lemma is used to prove that $(sP_1 + P_5, K_t)$-free graphs have bounded mim-width for every $s \geq 0$ and $t \geq 1$. It shows how we can bound the mim-width of a graph in terms of the mim-width of the graphs induced by blocks of a partition of the vertex set and the mim-width between any two of the parts. We include it here as it might be useful for bounding the mim-width of other graph classes.

▶ **Lemma 7** ([14])**.** *Let G be a graph and $(X_1, \ldots, X_p)$ be a partition of $V(G)$ such that* $\mathrm{cutmim}_G(X_i, X_j) \leq c$ *for all distinct $i, j \in \{1, \ldots, p\}$, and $p \geq 2$. Then*

$$\mathrm{mimw}(G) \leq \max\left\{c\left\lfloor\left(\frac{p}{2}\right)^2\right\rfloor, \max_{i\in\{1,\ldots,p\}}\{\mathrm{mimw}(G[X_i])\} + c(p-1)\right\}.$$

Moreover, if (T_i, δ_i) is a branch decomposition of $G[X_i]$ for each i, then we can construct, in $O(1)$ time, a branch decomposition (T, δ) of G with

$$\mathrm{mimw}_G(T, \delta) \leq \max\left\{c\left\lfloor\left(\frac{p}{2}\right)^2\right\rfloor, \max_{i\in\{1,\ldots,p\}}\{\mathrm{mimw}_G(T_i, \delta_i)\} + c(p-1)\right\}.$$

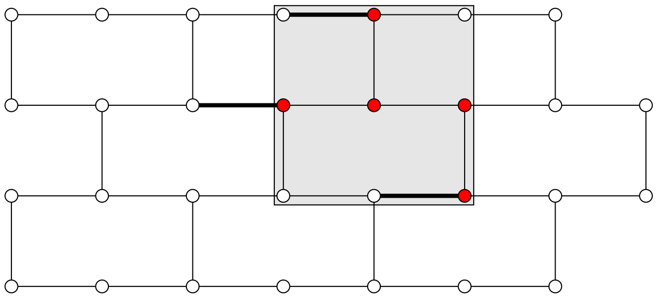

Figure 3 An elementary (4×4)-wall. We illustrate an example of the case where $h \geq \sqrt[4]{n(W)/3}$ and $r < 2n$ in the proof of Theorem 8: Q consists of the red vertices, B is the the grey box, and the thick edges are a matching in $W[A_e, \overline{A_e}]$.

3.2 Mim-width of Some Basic Classes

Recall that Vatshelle [49] showed that the class of grids has unbounded mim-width. We next prove that the same holds for the class of walls, which we define momentarily. Thus, we obtain a class of graphs with maximum degree 3 having unbounded mim-width, and we will use this result in order to prove Lemma 11. Note that it also gives us a dichotomy, as graphs with maximum degree 2 have bounded clique-width and hence bounded mim-width.

A *wall of height h and width r* (an *$(h \times r)$-wall* for short) is the graph obtained from the grid of height h and width $2r$ as follows. Let $C_1, \ldots, C_{2r}$ be the set of vertices in each of the $2r$ columns of the grid, in their natural left-to-right order. For each column C_j, let $e_1^j, e_2^j, \ldots, e_{h-1}^j$ be the edges between two vertices of C_j, in their natural top-to-bottom order. If j is odd, we delete all edges e_i^j with i even. If j is even, we delete all edges e_i^j with i odd. We then remove all vertices of the resulting graph whose degree is 1. This final graph is an *elementary $(h \times r)$-wall* and any subdivision of the elementary $(h \times r)$-wall is an *$(h \times r)$-wall*. For an example, see Figure 3.

▶ **Theorem 8.** *Let W be an elementary $(n \times n)$-wall with $n \geq 7$. Then $\mathrm{mimw}(W) \geq \frac{\sqrt{n}}{50}$. In particular, the class of walls has unbounded mim-width.*

Proof. We let $n(W) = |V(W)| = 2n^2 - 2$. Consider now a branch decomposition (T, δ) for W. Kang et al. [39, Lemma 2.3] showed that there exists an edge $e \in E(T)$ such that both partition classes A_e and $\overline{A_e}$ of $V(W)$ contain at least $n(W)/3$ vertices. Kanj et al. [40, Lemma 4.10] showed that if G is a graph such that each of its subgraphs has average degree at most d, then any matching M in G contains an induced matching in G of size at least $|M|/(2d-1)$. Since W is subcubic, it is sufficient to show that $W[A_e, \overline{A_e}]$ has a matching of size $\sqrt{n}/10$. We distinguish two cases, according to whether or not one of $W[A_e]$ and $W[\overline{A_e}]$ has a component of size at least $\sqrt{n(W)/3}$.

Suppose first that $W[A_e]$ has a component Q of size at least $\sqrt{n(W)/3}$. The component Q is contained in a rectangle of the underlying $n \times 2n$ grid. Consider the smallest such rectangle B, i.e., the rectangle whose horizontal sides contain the uppermost and lowermost vertex in Q and whose vertical sides contain the leftmost and rightmost vertex in Q. Let h and r be the height and width of B, respectively. Since $|V(Q)| \geq \sqrt{n(W)/3}$, one of h and r is at least $\sqrt[4]{n(W)/3}$.

Suppose first that $h \geq \sqrt[4]{n(W)/3}$. If $r < 2n$, say without loss of generality B does not intersect column C_1, we do the following. For each row of B, consider the leftmost vertex of Q in that row (since Q is connected, each row contains at least one vertex of Q). Clearly, the left neighbours of each such vertex belongs to $\overline{A_e}$, and so we have a matching in $W[A_e, \overline{A_e}]$ of size $h - 2 \geq \sqrt[4]{n(W)/3} - 2$, which is at least $\sqrt{n}/10$ when $n \geq 7$. If $r = 2n$, we distinguish two cases according to whether $h = n$ or not. In the first case (i.e., $r = 2n$ and $h = n$) we argue as follows. Since Q is connected, each row of B contains a vertex of $Q \subseteq A_e$. Moreover, there are at most $2n/3$ rows of B with all vertices contained in A_e, for otherwise $|A_e| > (2n/3) \cdot 2n \geq 2n(W)/3$. So there are at least $n/3$ rows of B containing a vertex of A_e and a vertex of $\overline{A_e}$. We can therefore find a matching in $W[A_e, \overline{A_e}]$ of size at least $n/3$. In the second case (i.e., $r = 2n$ and $h < n$), we proceed as follows. We assume, without loss of generality, that B does not intersect the uppermost row of the grid. We partition the columns of B into disjoint layers containing two consecutive columns each. For each layer, we consider its left column and the uppermost vertex $v \in A_e$ therein (since Q is connected, such a vertex exists). Let v_1 be the vertex on the grid above v, let v_2 be the vertex to the right of v and let v_3 be the vertex above v_2. By construction, $v_1 \in \overline{A_e}$ and if $vv_1 \in E(W)$, we select this edge. Otherwise, $vv_1 \notin E(W)$ and so $v_2v_3 \in E(W)$ and we have a path $vv_2v_3v_1$ in W with $v \in A_e$ and $v_1 \in \overline{A_e}$. We then select an edge of this path which belongs to $W[A_e, \overline{A_e}]$. Proceeding similarly for each layer, we obtain a matching in $W[A_e, \overline{A_e}]$ of size at least $r/2 = n$. Suppose finally that $h < \sqrt[4]{n(W)/3}$. We have that $r \geq \sqrt[4]{n(W)/3}$ and we proceed exactly as in the case $r = 2n$ and $h < n$ to obtain a matching in $W[A_e, \overline{A_e}]$ of size at least $r/2 \geq \sqrt[4]{n(W)/3}/2$.

It remains to consider the situation in which all components of $W[A_e]$ and $W[\overline{A_e}]$ have size less than $\sqrt{n(W)/3}$. In particular, since $W[A_e]$ has more than $n(W)/3$ vertices, it has more than $\sqrt{n(W)/3}$ components. Let $Q_1, \ldots, Q_k$ be these components. For each $i \in \{1, \ldots, k\}$, there exists a vertex $u_i \in Q_i$ with a neighbour $v_i \in \overline{A_e}$, as W is connected. Let H be the subgraph of $W[A_e, \overline{A_e}]$ induced by $\{u_1, \ldots, u_k\} \cup \{v_1, \ldots, v_k\}$ (notice that we might have $v_i = v_j$ for some $i \neq j$). Let $H_1, \ldots, H_\ell$ be the components of H and let $n_i = |V(H_i)|$, for each $i \in \{1, \ldots, \ell\}$. By construction, $n_i \geq 2$, for each i. Moreover, since H_i is a connected subcubic graph, it has a matching of size at least $(n_i - 1)/3 \geq n_i/6$ [4]. But then H has a matching of size

$$\sum_{i=1}^{\ell} \frac{n_i}{6} = \frac{|V(H)|}{6} \geq \frac{k}{6} \geq \frac{1}{6} \cdot \sqrt{\frac{n(W)}{3}}.$$

As in all cases we find a matching in $W[A_e, \overline{A_e}]$ of size at least $\frac{\sqrt{n}}{10}$, this concludes the proof. ◀

▶ **Corollary 9.** *For an integer Δ, let $\mathcal{G}_\Delta$ be the class of graphs of maximum degree at most Δ. Then the mim-width of $\mathcal{G}_\Delta$ is bounded if and only if $\Delta \leq 2$.*

A *net-wall* is a graph that can be obtained from a wall G by performing a clique implant on each vertex of G having degree three. An example of part of a net-wall is given in Figure 7.

The following lemma is a straightforward consequence of Theorem 8 and Lemma 3.

▶ **Lemma 10.** *The class of net-walls has unbounded mim-width.*

Mengel [43] showed that strongly chordal split graphs, or equivalently $(\mathsf{sun}_3, \mathsf{sun}_4, \ldots)$-free split graphs, have unbounded mim-width. Recall that the class of split graphs coincides with the class of $(C_4, C_5, 2P_2)$-free graphs. We find two more subclasses of split graphs with unbounded mim-width by using Lemmas 2 and 4.

▶ **Lemma 11.** *The following subclasses of split graphs have unbounded mim-width:*

(i) *the class in which every graph G has a split partition (C, I) where each vertex in I has degree 2 and each vertex in C has at most three neighbours in I,*

(ii) *the class in which every graph G has a split partition (C, I) where each vertex in I has degree at most 3, and each vertex in C has two neighbours in I, and*

(iii) *the class in which every graph G is sun_t-free for all $t \geq 3$.*

Proof. Case (iii) is due to Mengel [43]. To prove (i) and (ii), let G be a wall, and let G' be the graph obtained by 1-subdividing each edge of G. Partition $V(G')$ into (A, B), where B consists of the vertices of degree two introduced by the 1-subdivisions. Observe that G' is bipartite, with vertex bipartition (A, B). Let G'' be the graph obtained by making one of A or B a clique. By Lemmas 2 and 4, $\text{mimw}(G'') \geq \text{mimw}(G)/2$. The result now follows from Theorem 8. ◀

A graph is *chordal bipartite* if it is bipartite and every induced cycle has four vertices. Brault-Baron et al. [13] showed that the class of chordal bipartite graphs has unbounded mim-width (we describe their construction in Section 5). Combining their result with Lemma 4, after adding all edges between any two vertices in the same colour class, yields the following:

▶ **Lemma 12.** *The class of co-bipartite graphs, or equivalently $(3P_1, C_5, \overline{C_7}, \overline{C_9}, \ldots)$-free graphs, has unbounded mim-width.*

As the last result in this section we consider hereditary classes defined by one forbidden induced subgraph. It is folklore that the class of H-free graphs has bounded clique-width if and only if $H \subseteq_i P_4$ (see [26] for a proof). It turns out that the same dichotomy holds for mim-width.

▶ **Theorem 13.** *The class of H-free graphs has bounded mim-width if and only if $H \subseteq_i P_4$.*

Proof. If $H \subseteq_i P_4$, then H-free graphs form a subclass of P_4-free graphs. Every P_4-free graph has clique-width at most 2 [19] and so mim-width at most 2 [49]. Suppose now that H is a graph such that the class of H-free graphs has bounded mim-width. Recall that chordal bipartite graphs have unbounded mim-width [13] (see also Section 5). Hence, H is C_3-free. As co-bipartite graphs, and thus $3P_1$-free graphs, and split graphs, or equivalently, $(C_4, C_5, 2P_2)$-free graphs, have unbounded mim-width by Lemmas 11 and 12, this means that H is a $(3P_1, 2P_2)$-free forest. It follows that $H \subseteq_i P_4$. ◀

4 New Bounded Cases

In this section, we present three general classes and two further specific classes, of (H_1, H_2)-free graphs having bounded mim-width, but unbounded clique-width. First, we present the three infinite families of classes of (H_1, H_2)-free graphs. We show that for a class in one of these three families, there exists a constant k such that for every graph G in the class, and every $X \subseteq V(G)$, we have that $\text{cutmim}_G(X, \overline{X}) \leq k$. This implies that every branch decomposition of G has mim-width at most k. Thus, *for a graph in one of these classes, a branch decomposition of constant mim-width is quickly computable*: any branch decomposition will suffice. Finally, we present two more classes of (H_1, H_2)-free graphs having bounded mim-width, which do not have this property, but for which we prove that a branch decomposition of constant width can be computed in polynomial-time.

We make use of Ramsey theory. By Ramsey's Theorem, for all positive integers a and b, there exists an integer $R(a, b)$ such that if G is a graph on at least $R(a, b)$ vertices, then G has either a clique of size a, or an independent set of size b.

Recall that $K_r \boxminus K_r$ is the graph obtained from $2K_r$ by adding a perfect matching and that $K_r \boxminus rP_1$ is the graph obtained from $K_r \boxminus K_r$ by removing all the edges in one of the complete graphs. We let $K_r \boxminus P_1$ denote the graph obtained from K_r by adding a single vertex, attached to K_r by a single pendant edge. We also denote $\overline{C_4 + P_1}$ as bowtie. Examples of these graphs are given in Figure 4.

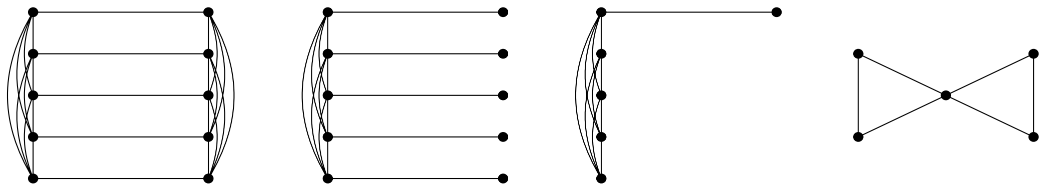

Figure 4 The graphs $K_5 \boxminus K_5$, $K_5 \boxminus 5P_1$, $K_5 \boxminus P_1$, and bowtie $= \overline{C_4 + P_1}$.

▶ **Theorem 14.** *Let G be a $(K_r \boxminus rP_1, 2P_2)$-free graph for $r \geq 3$. Then* $\text{cutmim}_G(X, \overline{X}) < \max\{6, r\}$ *for every $X \subseteq V(G)$. In particular,* $\text{mimw}(G) < \max\{6, r\}$.

Proof. Let $k = \max\{6, r\}$ and let (T, δ) be a branch decomposition of G. Towards a contradiction, suppose that there exists $X \subseteq V(G)$ such that $G[X, \overline{X}]$ has an induced matching of size at least k. Let $X' = \{x_1, x_2, \ldots, x_k\} \subseteq X$ and $Y' = \{y_1, y_2, \ldots, y_k\} \subseteq \overline{X}$ such that $x_i y_i$ is an edge of the induced matching for each $i \in \{1, 2, \ldots, k\}$.

First, observe that for any distinct $i, j \in \{1, 2, \ldots, k\}$, either $x_i x_j$ or $y_i y_j$ is an edge, otherwise $G[\{x_i, x_j, y_i, y_j\}] \cong 2P_2$. We claim that X' or Y' contains a clique of size 3. Since $|X'| = k \geq 6 = R(3, 3)$, the set X' contains either a clique on 3 vertices, or an independent set on 3 vertices. So we may assume that X' contains an independent set on 3 vertices, $\{x_i, x_j, x_\ell\}$ say. Then $\{y_i, y_j, y_\ell\}$ is a clique of size 3 contained in Y', proving the claim.

Without loss of generality, we may now assume that X' contains a clique of size 3. Suppose X' is not a clique. Then there exist distinct $i, j \in \{1, 2, \ldots, k\}$ such that x_i is not adjacent to x_j. Now $y_i y_j$ is an edge, since G is $2P_2$-free. Let X'' be a maximum-sized clique contained in X', so $|X''| \geq 3$. Note that $\{x_i, x_j\} \nsubseteq X''$, since X'' is a clique, so we may assume that $x_j \notin X''$. As any pair in $X'' \setminus \{x_i\}$ induces an edge that is anticomplete to the edge $y_i y_j$, we see that G contains an induced $2P_2$, a contradiction. We deduce that X' is a clique of size k. Now, since G is $(K_r \boxminus rP_1)$-free, there exist distinct $i, j \in \{1, 2, \ldots, k\}$ such that $y_i y_j$ is an edge. Note that since $k \geq 6$, there exist distinct $s, t \in \{1, 2, \ldots, k\} \setminus \{i, j\}$. But now $x_s x_t$ is anticomplete to $y_i y_j$, contradicting that G is $2P_2$-free. ◀

The class of $(K_r \boxminus rP_1, 2P_2)$-free graphs for $r \in \{1, 2\}$ is a subclass of P_4-free graphs, and thus has bounded clique-width and mim-width. However, for $r \geq 3$, the class of $(K_r \boxminus rP_1, 2P_2)$-free graphs has unbounded clique-width [23, Theorem 4.18], whereas Theorem 14 shows it has bounded mim-width. In particular, $(\text{net}, 2P_2)$-free graphs and $(\text{bull}, 2P_2)$-free graphs have bounded mim-width but unbounded clique-width.

In our next two results, we present two other new classes of bounded mim-width.

▶ **Theorem 15.** *Let G be a $(K_r \boxminus P_1, tP_2)$-free graph for $r \geq 1$ and $t \geq 1$. Then* $\text{cutmim}_G(X, \overline{X}) < R(r, R(r, t))$ *for every $X \subseteq V(G)$. In particular,* $\text{mimw}(G) < R(r, R(r, t))$.

Proof. Let $k = R(r, R(r, t))$ and let (T, δ) be a branch decomposition of G. Towards a contradiction, suppose that there exists $X \subseteq V(G)$ such that $G[X, \overline{X}]$ has an induced matching of size at least k. Let $X' = \{x_1, x_2, \ldots, x_k\} \subseteq X$ and $Y' = \{y_1, y_2, \ldots, y_k\} \subseteq \overline{X}$ such that $x_i y_i$ is an edge of the induced matching for each $i \in \{1, 2, \ldots, k\}$.

Since $|X'| = k = R(r, R(r,t))$, the set X' contains either a clique of size r, or an independent set of size $R(r,t)$. Suppose there is some $J \subseteq \{1,2,\ldots,k\}$ such that $X_J = \{x_i : i \in J\}$ is a clique of size r. Then, for an arbitrarily chosen $j \in J$, the vertices $X_J \cup \{y_j\}$ induce a $K_r \boxminus P_1$, a contradiction. So X' contains an independent set of size $R(r,t)$. Let $I \subseteq \{1,2,\ldots,k\}$ such that $X_I = \{x_i : i \in I\}$ is an independent set of size $R(r,t)$, and consider the set $Y_I = \{y_i : i \in I\}$. Since $|Y_I| = R(r,t)$, the set Y_I either contains a clique of size r, or an independent set of size t. In the former case, G contains an induced $K_r \boxminus P_1$, while in the latter case, G contains an induced tP_2, a contradiction. ◀

▶ **Theorem 16.** *Let G be a $(K_r \boxminus K_r, sP_1 + P_2)$-free graph for $r \geq 1$ and $s \geq 0$. Then* $\mathrm{cutmim}_G(X, \overline{X}) < R(R(r,s+1),s+1)$ *for every $X \subseteq V(G)$. In particular,* $\mathrm{mimw}(G) < R(R(r,s+1),s+1)$.

Proof. Let $k = R(R(r,s+1),s+1)$ and let (T,δ) be a branch decomposition of G. Towards a contradiction, suppose that there exists $X \subseteq V(G)$ such that $G[X,\overline{X}]$ has an induced matching of size at least k. Let $X' = \{x_1,x_2,\ldots,x_k\} \subseteq X$ and $Y' = \{y_1,y_2,\ldots,y_k\} \subseteq \overline{X}$ such that x_iy_i is an edge of the induced matching for each $i \in \{1,2,\ldots,k\}$.

Since $|X'| = k = R(R(r,s+1),s+1)$, the set X' contains either a clique of size $R(r,s+1)$, or an independent set of size $s+1$. But the latter implies that G has an induced $sP_1 + P_2$ subgraph, a contradiction. So X' contains a clique of size $R(r,s+1)$. Let $I \subseteq \{1,2,\ldots,k\}$ such that $X_I = \{x_i : i \in I\}$ is an clique of size $R(r,s+1)$, and consider the set $Y_I = \{y_i : i \in I\}$. Since $|Y_I| = R(r,s+1)$, the set Y_I either contains a clique of size r, or an independent set of size $s+1$. In the former case, G contains an induced $K_r \boxminus K_r$, while in the latter case, G contains an induced $sP_1 + P_2$, a contradiction. ◀

Note that $(K_r \boxminus P_1, tP_2)$-free graphs have unbounded clique-width if and only if $r \geq 3$ and $t \geq 3$, or $r \geq 4$ and $t \geq 2$ [23, Theorem 4.18]. Note also that $(K_r \boxminus K_r, sP_1 + P_2)$-free graphs have unbounded clique-width if and only if $r = 2$ and $s \geq 3$, or $r \geq 3$ and $s \geq 2$ [23, Theorem 4.18].

Our final results of the section are used to resolve the remaining cases where $|V(H_1)| + |V(H_2)| \leq 8$.[3] For these results, we employ the following approach. Suppose we wish to show that the class of (H_1', H_2')-free graphs is bounded, where $H_1' \subseteq_i H_1$ for one of the pairs (H_1, H_2) appearing in Theorems 14 to 16. If G is a H_2-free graph in the class, then we can compute a branch decomposition of constant mim-width by one of Theorems 14 to 16. So it remains only to show that we can compute a branch decomposition of constant mim-width for (H_1', H_2')-free graphs having an induced subgraph isomorphic to H_2. When $H_1' = 2P_2$ and $H_2' = K_{1,3}$, we exploit the structure of $(2P_2, K_{1,3})$-free graphs having an induced $K_3 \boxminus 3P_1$ to prove Lemma 17. Then, by combining this lemma with Theorem 14, we obtain Theorem 18. Similarly, when $H_1' = 2P_1 + P_2$ and $H_2' =$ bowtie (see Figure 4), we use Lemma 19 and Theorem 16 to obtain Theorem 20.

For the proofs of Lemmas 17 and 19, we require the following definition. For an integer $l \geq 1$, an *l-caterpillar* is a subcubic tree T on $2l$ vertices with $V(T) = \{s_1,\ldots,s_l,t_1,\ldots,t_l\}$, such that $E(T) = \{s_it_i \ : \ 1 \leq i \leq l\} \cup \{s_is_{i+1} \ : \ 1 \leq i \leq l-1\}$. Note that we label the leaves of an l-caterpillar $t_1,t_2,\ldots,t_l$, in this order. See Figure 5 for an example.

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[3] In Corollary 34 in Section 6 we prove that we determined all pairs (H_1,H_2) with $|V(H_1)|+|V(H_2)| \leq 8$ for which the mim-width of (H_1,H_2)-free graphs is bounded, and in fact also quickly computable.

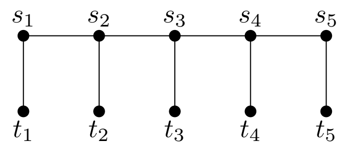

Figure 5 The 5-caterpillar.

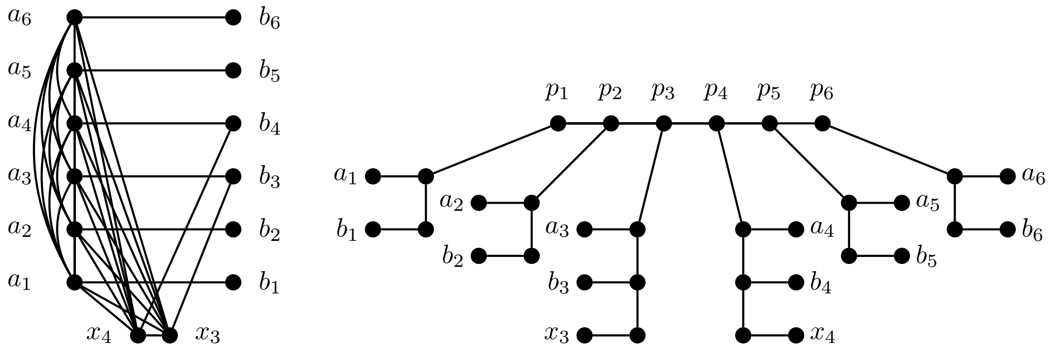

Figure 6 On the left, a $(2P_2, K_{1,3})$-free graph G, and on the right the branch decomposition (T, δ) of G as constructed in the proof of Lemma 17.

▶ **Lemma 17.** *Let G be a connected $(2P_2, K_{1,3})$-free graph. Given $X \subseteq V(G)$ such that $G[X] \cong K_r \boxminus rP_1$ for some $r \geq 3$, where X is maximal, we can construct, in $O(n)$ time, a branch decomposition (T, δ) of G such that $\mathrm{mimw}_G(T, \delta) = 1$.*

Proof. Let $A = \{a_1, \ldots, a_r\}$ and $B = \{b_1, \ldots, b_r\}$ such that A is a clique, B is an independent set, and (A, B) is a partition of X. Note that $G[X] \cong K_r \boxminus rP_1$, but for every $S \subseteq V(G) \setminus X$, we have that $G[X \cup S] \ncong K_{r'} \boxminus r'P_1$ for each integer $r' > r$. We assume that $a_ib_i \in E(G)$ for each $i \in \{1, \ldots, r\}$. Let N_1 be the set of vertices from $V(G) \setminus X$ that have a neighbour in X, and let $N_2 = V(G) \setminus (X \cup N_1)$.

Let $v \in N_1$. Suppose that $N(v) \cap B = \varnothing$. Since G is connected, v has a neighbour in A; by symmetry, we may assume that $va_1 \in E(G)$. Let $i \in \{2, \ldots, r\}$, and suppose that $va_i \notin E(G)$. But then $G[\{a_1, b_1, v, a_i\}] \cong K_{1,3}$, a contradiction. Therefore $N(v) \cap X = A$. Suppose now that $N(v) \cap B \neq \varnothing$; without loss of generality we may assume that $vb_1 \in E(G)$. If v is complete to B, then any three vertices of B together with v induces a $K_{1,3}$, a contradiction. Therefore, without loss of generality we assume that $vb_2 \notin E(G)$. Since G is $2P_2$-free, $va_2 \in E(G)$. Now suppose that $va_i \notin E(G)$ for some $i \in \{1, \ldots, r\} \setminus \{2\}$. But then $G[\{a_2, b_2, v, a_i\}] \cong K_{1,3}$, a contradiction. Therefore v is complete to A. Now suppose that $|N(v) \cap B| \geq 2$; without loss of generality we may assume that $b_1, b_3 \in N(v)$. Recall that $b_2 \notin N(v)$. But then $G[\{v, b_1, b_3, a_2\}] \cong K_{1,3}$, a contradiction. Therefore $N(v) \cap B = \{b_1\}$. Hence, for every vertex $v \in N_1$, either $N(v) \cap X = A$ or $N(v) \cap X = A \cup \{b\}$ for some $b \in B$.

Suppose that there exist vertices $v, v' \in N_1$ such that $vv' \notin E(G)$. Since vertices of N_1 have at most one neighbour in B, we may assume without loss of generality that $b_1 \notin N(v) \cup N(v')$. But then $G[\{a_1, b_1, v, v'\}] \cong K_{1,3}$, a contradiction. Therefore $vv' \in E(G)$, and hence N_1 is a clique.

We now prove that $N_2 = \varnothing$. Towards a contradiction, suppose that there exists a vertex $w \in N_2$. Since G is connected, there exists a vertex $v \in N(w) \cap N_1$. By what we have already proved, either $N(v) \cap X = A$ or $N(v) \cap X = A \cup \{b\}$ for some $b \in B$. Suppose that $N(v) \cap B \neq \varnothing$; without loss of generality, we may assume that $N(v) \cap B = \{b_1\}$. But then $G[\{v, b_1, w, a_2\}] \cong K_{1,3}$, a contradiction. Therefore v is anticomplete to B. It now follows that $G[X \cup \{v, w\}] \cong K_{r+1} \boxminus (r+1)P_1$, contradicting the maximality of X. Therefore $N_2 = \varnothing$.

For $i \in \{1, \dots, r\}$, let B_i denote the set of vertices from N_1 that are adjacent to b_i and let B_0 denote the set of vertices from N_1 that have no neighbour in B. Note that $(A, B, B_0, B_1, \dots, B_r)$ is a partition of $V(G)$ (into possibly empty sets), and we can construct this partition in $O(n)$ time. Consider the branch decomposition (T, δ) of G defined as follows; see also Figure 6. For each $i \in \{1, \dots, r\}$, let T_i be a $(|B_i| + 2)$-caterpillar and let t_i be a vertex of T_i of degree 2. If $B_0 \neq \varnothing$, let T_0 be a $|B_0|$-caterpillar and t_0 a vertex of T_0 of degree 2, or of degree 1 if $|B_0| = 1$. Let $P = p_1, \dots, p_r$ be a path on r vertices. Let T' be the tree with $V(T') = V(P) \cup \bigcup_{i=1}^{r} V(T_i)$ and $E(T') = E(P) \cup \bigcup_{i=1}^{r} E(T_i) \cup \{t_i p_i \;:\; 1 \leq i \leq r\}$. If $B_0 = \varnothing$ then let $T = T'$, and otherwise let T be the tree obtained from T' by adding an additional vertex p_{r+1} together with all vertices of $V(T_0)$, and adding edges $p_r p_{r+1}$ and $p_{r+1} t_0$ together with all edges of T_0. Finally, let δ be any bijection from $V(G)$ to the leaves of T such that for all $i \in \{1, \dots, r\}$ and for all $v \in V(G)$, $\delta(v) \in V(T_i)$ if $v \in \{a_i, b_i\} \cup B_i$, and $\delta(v) \in V(T_0)$ if $v \in B_0$.

We now prove that $\text{mimw}_G(T, \delta) = 1$. Let e be an edge of T and let M be a maximum induced matching of $G[A_e, \overline{A_e}]$. We begin by claiming that at most one edge of M has one endpoint in B and the other in $A \cup N_1$. On the contrary, suppose without loss of generality that $b_1 x$ and $b_2 y$ are distinct edges of M, where $b_1, b_2 \in B \cap A_e$ and $x, y \in (A \cup N_1) \cap \overline{A_e}$. Observe that if $x \in N_1$ (respectively $y \in N_1$), then $x \in B_1$ (respectively $y \in B_2$); and if $x \in A$ (respectively $y \in A$), then $x = a_1$ (respectively $y = a_2$). Since $b_1, b_2 \in A_e$, we have that $e \notin E(T_1) \cup E(T_2) \cup \{p_1 p_2, p_1 t_1, p_2 t_2\}$, and therefore $\{a_1, a_2\} \cup B_1 \cup B_2 \subseteq A_e$. But $N(b_1) \cup N(b_2) \subseteq \{a_1, a_2\} \cup B_1 \cup B_2$, a contradiction. Therefore at most one edge of M has one endpoint in B and the other in $A \cup N_1$. Since $A \cup N_1$ is a clique, at most one edge of M has both endpoints in $A \cup N_1$, and since B is an independent set, no edge of M has both endpoints in B. Suppose that $|M| \geq 2$. Then $M = \{uv, xy\}$, where, without loss of generality, $u, x \in A_e$, $u, v, x \in A \cup N_1$ and $y \in B$. But since $A \cup N_1$ is a clique, xv is an edge, contradicting M being an induced matching. Therefore $|M| \leq 1$ and hence $\text{mimw}_G(T, \delta) = 1$, as required. ◀

▶ **Theorem 18.** *Let G be a $(2P_2, K_{1,3})$-free graph. Then $\text{mimw}(G) < 6$, and one can construct, in polynomial time, a branch decomposition (T, δ) of G with $\text{mimw}_G(T, \delta) < 6$.*

Proof. If G is not connected, we may consider each component in turn, by Lemma 5. If G is $(K_3 \boxminus 3P_1)$-free, then $\text{mimw}(G) < 6$ by Theorem 14. On the other hand, if G has an induced subgraph isomorphic to $K_3 \boxminus 3P_1$, then $\text{mimw}(G) = 1$ by Lemma 17.

We now show how to compute a branch decomposition (T, δ) of G, with $\text{mimw}_G(T, \delta) < 6$, in polynomial time. Consider the following algorithm, which takes as input a connected $(2P_2, K_{1,3})$-free graph G.

Step 1 Enumerate all subsets $S \subseteq V(G)$ such that $|S| = 6$ and check whether $G[S] \cong K_3 \boxminus 3P_1$. If no such set S exists, then return an arbitrary branch decomposition of G.

Step 2 Let $S \subseteq V(G)$ such that $G[S] \cong K_3 \boxminus 3P_1$ and let (A, B) be a partition of S such that A is a clique and B is an independent set.

Step 3 Set $E = E(G) \setminus E(G[S])$. While $E \neq \varnothing$:

- Choose an edge $e \in E$.

- If one endpoint of e (say a) is complete to A and anticomplete to B, and the other endpoint of e (say b) is anticomplete to $A \cup B$, then set $A \leftarrow A \cup \{a\}$ and $B \leftarrow B \cup \{b\}$.
- Set $E \leftarrow E \setminus \{e\}$.

Step 4 Using Lemma 17, with $X = A \cup B$, compute a branch decomposition (T, δ) of G and return it.

It is easily checked that Steps 1–4 of this algorithm can be performed in polynomial time. If the algorithm returns a branch decomposition in Step 1, then by Theorem 14 it has mim-width less than 6. Otherwise, the branch decomposition has mim-width 1 by Lemma 17. ◀

▶ **Lemma 19.** *Let G be a $(2P_1 + P_2, \textsf{bowtie})$-free graph. Given $X \subseteq V(G)$ such that $G[X] \cong K_r \boxminus K_r$ for some $r \geq 5$, where X is maximal, we can construct, in $O(n)$ time, a branch decomposition (T, δ) of G such that* $\text{mimw}_G(T, \delta) = 2$.

Proof. Let $A = \{a_1, \ldots, a_r\}$ and $B = \{b_1, \ldots, b_r\}$ be cliques that partition X, with $a_ib_i \in E(G)$ for all $i \in \{1, \ldots, r\}$. Let N_1 be the set of vertices of $V(G) \setminus X$ with a neighbour in X. Suppose there exists a vertex $v \in V(G) \setminus (X \cup N_1)$. Then $G[\{v, a_1, b_2, b_3\}] \cong 2P_1 + P_2$, a contradiction. So $X \cup N_1 = V(G)$.

We claim that each vertex in N_1 is either complete or anticomplete to A. Suppose $v \in N_1$ has a neighbour and a non-neighbour in A. Without loss of generality, let a_r be the neighbour and let a_1 be the non-neighbour. If there is a pair of distinct vertices b_i, b_j non-adjacent to v for $i, j \in \{2, 3, \ldots, r\}$, then $G[\{v, a_1, b_i, b_j\}] \cong 2P_1 + P_2$. So v has at most one non-neighbour in $\{b_2, b_3, \ldots, b_r\}$. In particular, as $r \geq 5$, we may assume without loss of generality that b_3 and b_4 are neighbours of v. If v is adjacent to a_2, then $G[\{a_2, a_r, v, b_3, b_4\}] \cong \textsf{bowtie}$, a contradiction. So a_2 is a non-neighbour of v. Now, if b_r is adjacent to v, then $G[\{a_1, a_2, a_r, v, b_r\}] \cong \textsf{bowtie}$; whereas if b_r is non-adjacent to v, then $G[\{a_1, a_2, v, b_r\}] \cong 2P_1 + P_2$. From this contradiction, we deduce that v is either complete or anticomplete to A. By symmetry, each $v \in N_1$ is complete or anticomplete to B.

If $v \in N_1$ is complete to both A and B, then $G[\{a_1, a_2, v, b_3, b_4\}] \cong \textsf{bowtie}$, a contradiction. If $v \in N_1$ is anticomplete to both A and B, then $G[\{a_1, a_2, v, b_3\}] \cong 2P_1 + P_2$, a contradiction. So each vertex in N_1 is either complete to A and anticomplete to B, or complete to B and anticomplete to A. Call these two sets A' and B' respectively. If a vertex $a \in A'$ has a neighbour $b \in B'$, then $G[X \cup \{a, b\}] \cong K_{r+1} \boxminus K_{r+1}$, contradicting the maximality of X. So A' and B' are anticomplete. Moreover, if $a, a' \in A'$ are distinct and non-adjacent, then $G[\{a, a', b_1, b_2\}] \cong 2P_1 + P_2$, a contradiction. So $A' \cup A$ and, similarly, $B' \cup B$ are cliques.

Now let $a'_1, a'_2, \ldots, a'_{|A'|}$ be an arbitrary ordering of A', and let $b'_1, b'_2, \ldots, b'_{|B'|}$ be an arbitrary ordering of B'. Let (T, δ) be the branch decomposition with linear ordering

$$(a'_1, a'_2, \ldots, a'_{|A'|}, a_1, b_1, a_2, b_2, \ldots, a_r, b_r, b'_1, b'_2, \ldots, b'_{|B'|});$$

that is, let T be a $|V(G)|$-caterpillar where δ respects this ordering, so $\delta(a'_1) = t_1$, $\delta(a'_2) = t_2$, $\ldots$, $\delta(b'_{|B'|}) = t_{|V(G)|}$. Note that, given X, we can find A and B, together with the labelling of a_i's and b_i's, as well as A' and B', in $O(n)$ time, so we can compute (T, δ) in $O(n)$ time. We claim that $\text{mimw}_G(T, \delta) = 2$. Let $e \in E(T)$ and consider the corresponding cut $(A_e, \overline{A_e})$. First, observe that when $A_e = A' \cup \{a_1, b_1\}$, the graph $G[A_e, \overline{A_e}]$ has an induced matching of size 2, with edges a_1a_2 and b_1b_2, so $\text{mimw}_G(T, \delta) \geq 2$.

Let M be an induced matching in $G[A_e, \overline{A_e}]$. Let $V(M)$ denote the vertices incident to an edge of M. Suppose $V(M) \cap A_e$ contains at least two vertices of A. Then there exist

$i,j \in \{1,2,\ldots,r\}$ such that $a_i, a_j \in V(M) \cap A_e$, with $i < j$. Observe that $b_i \in A_e$, since b_i is between a_i and a_j in the linear ordering, and $a_i, a_j \in A_e$. Let $v, v' \in \overline{A_e}$ such that $a_i v, a_j v' \in M$. If $v \in A \cup A'$, then $a_j v$ is an edge of G, so M is not induced. Moreover, $v \notin B'$, since B' is anticomplete to A. So $v \in B$, and hence $v = b_i$. But then $v \in A_e$, a contradiction. So $|V(M) \cap A_e \cap A| \leq 1$. Similarly, $V(M) \cap A_e$ contains at most one vertex of B.

Now suppose $V(M) \cap A_e$ contains a vertex $a' \in A'$. Suppose $a'v \in M$, where $a' \in A' \cap A_e$ and $v \in \overline{A_e}$. Then $v \in A \cup A'$, since A' is anticomplete to $B \cup B'$. Hence a' is the only vertex of $A \cup A'$ in $A_e \cap V(M)$, for otherwise v has two neighbours in $V(M) \cap A_e$. So $|V(M) \cap A_e \cap (A' \cup A)| \leq 1$. Similarly, $V(M) \cap A_e$ contains at most one vertex of $B' \cup B$. So $|M| \leq 2$, and hence $\mathrm{mimw}_G(T,\delta) = 2$. ◀

▶ **Theorem 20.** *Let G be a $(2P_1+P_2, \mathsf{bowtie})$-free graph. Then $\mathrm{mimw}(G) < R(14,3)$, and one can construct, in polynomial time, a branch decomposition (T,δ) of G with $\mathrm{mimw}_G(T,\delta) < R(14,3)$.*

Proof. If G is $(K_5 \boxminus K_5)$-free, then $\mathrm{mimw}(G) < R(R(5,3),3) = R(14,3)$ by Theorem 16. On the other hand, if G has an induced subgraph isomorphic to $K_5 \boxminus K_5$, then $\mathrm{mimw}(G) = 2$ by Lemma 19.

We now show how to compute a branch decomposition (T,δ) of G, with $\mathrm{mimw}_G(T,\delta) < R(14,3)$, in polynomial time. Consider the following algorithm, which takes as input a connected $(2P_1+P_2, \mathsf{bowtie})$-free graph G.

Step 1 Enumerate all subsets $S \subseteq V(G)$ such that $|S| = 10$ and check whether $G[S] \cong K_5 \boxminus K_5$. If no such set S exists, then return an arbitrary branch decomposition of G.

Step 2 Let $S \subseteq V(G)$ such that $G[S] \cong K_5 \boxminus K_5$ and let (A,B) be a partition of S such that A is a clique and B is an independent set.

Step 3 Set $E = E(G) \setminus E(G[S])$. While $E \neq \varnothing$:
- Choose an edge $e \in E$.
- If one endpoint of e (say a) is complete to A and anticomplete to B, and the other endpoint of e (say b) is complete to B and anticomplete to A, then set $A \leftarrow A \cup \{a\}$ and $B \leftarrow B \cup \{b\}$.
- Set $E \leftarrow E \setminus \{e\}$.

Step 4 Using Lemma 19, with $X = A \cup B$, compute a branch decomposition (T,δ) of G and return it.

It is easily checked that Steps 1–4 of this algorithm can be performed in polynomial time. If the algorithm returns a branch decomposition in Step 1, then by Theorem 16 it has mim-width less than R(14,3). Otherwise, the branch decomposition has mim-width 2 by Lemma 19. ◀

5 New Unbounded Cases

We present a number of graph classes of unbounded mim-width, starting with the following two theorems.

▶ **Theorem 21.** *The class of $(\mathsf{diamond}, 5P_1)$-free graphs has unbounded mim-width.*

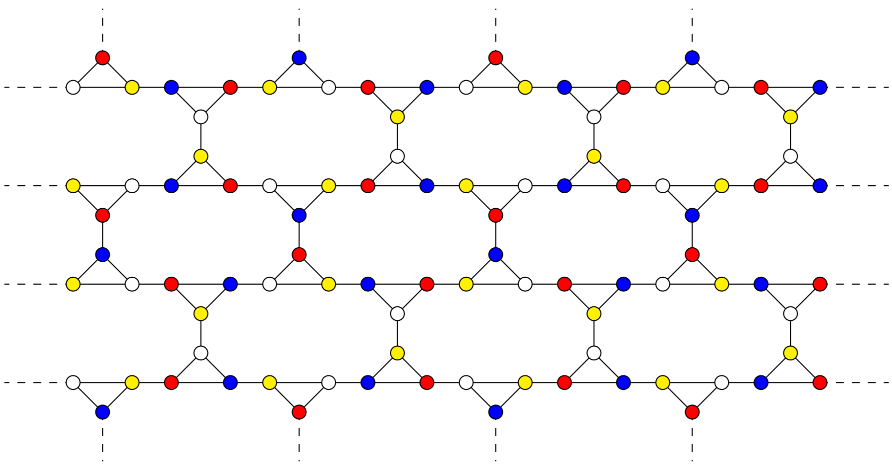

Figure 7 A particular 4-colouring of a net-wall, used in the proof of Theorem 21.

Proof. For every integer k, we will construct a $(\mathsf{diamond}, 5P_1)$-free graph G such that $\mathrm{mimw}(G) > k$. By Lemma 10, for any integer k there exists a net-wall W such that $\mathrm{mimw}(W) > 4k$. We partition the vertex set $V(W)$ into four colour classes (V_1, V_2, V_3, V_4) as illustrated in Figure 7. Observe that, for each $i \in \{1, 2, 3, 4\}$, the set V_i is independent, and no two distinct vertices $v, v' \in V_i$ have a common neighbour; that is, $N_W(v) \cap N_W(v') = \varnothing$.

Let G be the graph obtained from W by making each of V_1, V_2, V_3 and V_4 into a clique. By Lemma 4, $\mathrm{mimw}(G) \geq \mathrm{mimw}(W)/4 > k$. Since any set of five vertices of G contains at least two vertices in one of V_1, V_2, V_3, and V_4, and each of these four sets is a clique, G is $5P_1$-free.

It remains to show that G is $\mathsf{diamond}$-free. First, observe that if $G[X] \cong K_3$ for some $X \subseteq V(G)$ with $|X \cap V_i| \geq 2$ for some $i \in \{1, 2, 3, 4\}$, then, since no two vertices in V_i have a common neighbour in W, it follows that $X \subseteq V_i$. Now, towards a contradiction, suppose $G[Y] \cong \mathsf{diamond}$ for some $Y \subseteq V(G)$. Then Y is the union of two sets X' and X'' that induce triangles in G, and $|X' \cap X''| = 2$. Since W is $\mathsf{diamond}$-free, we may assume that $W[X']$ is not a triangle. Then X' contains at least two vertices of V_i for some $i \in \{1, 2, 3, 4\}$. By the earlier observation, $X' \subseteq V_i$. Since $|X' \cap X''| = 2$, we then have $|X'' \cap V_i| \geq 2$, so $X'' \subseteq V_i$, and hence $Y \subseteq V_i$. But this implies that Y is a clique in G; a contradiction. So G is $\mathsf{diamond}$-free. ◀

▶ **Theorem 22.** *The class of $(4P_1, \overline{3P_1 + P_2}, \overline{P_1 + 2P_2})$-free graphs has unbounded mim-width.*

Proof. For every integer k, we will construct a $(4P_1, \overline{3P_1 + P_2}, \overline{P_1 + 2P_2})$-free graph G such that $\mathrm{mimw}(G) > k$. By Lemma 10, for any integer k there exists a net-wall W such that $\mathrm{mimw}(W) > 3k$. We partition the vertex set $V(W)$ into three colour classes (V_1, V_2, V_3) such that V_i is an independent set for each $i \in \{1, 2, 3\}$ as illustrated in Figure 8. Since W has maximum degree 3 and each vertex belongs to a triangle, a vertex has at most two neighbours in each colour class; that is, for each $i \in \{1, 2, 3\}$ and $v \in V_i$, we have $|N(v) \cap V_j| \leq 2$ for $j \in \{1, 2, 3\}$. Note that these colour classes are chosen to satisfy the following properties. Firstly, W does not contain a bichromatic induced P_5; that is, if $W[X] \cong P_5$ for some $X \subseteq V(P_5)$, then $X \cap V_i \neq \varnothing$ for each $i \in \{1, 2, 3\}$. Secondly, if $W[X] \cong \mathsf{bull}$, then $|X \cap V_i| \leq 2$ for each $i \in \{1, 2, 3\}$.

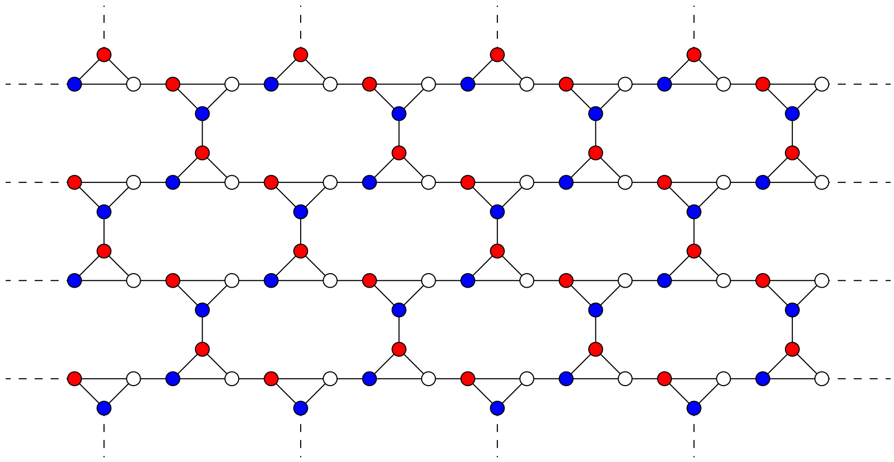

Figure 8 The 3-colouring of a net-wall used in the proof of Theorem 22.

Let G be the graph obtained from W by making each of V_1, V_2, and V_3 into a clique. By Lemma 4, $\mathrm{mimw}(G) \geq \mathrm{mimw}(W)/3 > k$. As any set of 4 vertices of G contains at least two vertices in one of the cliques V_1, V_2, or V_3, we deduce that G is $4P_1$-free.

We now show that G is $\left(\overline{3P_1 + P_2}\right)$-free. To the contrary, suppose $G[X] \cong \overline{3P_1 + P_2}$ for some $X \subseteq V(G)$. Then X is not contained in V_i for any $i \in \{1,2,3\}$. Moreover, $|X \cap V_i| \leq 2$ for each $i \in \{1,2,3\}$, for otherwise there is a vertex with at least three neighbours in a different colour class. So, assume without loss of generality that $X \cap V_1 = \{v_1, v_1'\}$, $X \cap V_2 = \{v_2, v_2'\}$, and $X \cap V_3 = \{v_3\}$. Then at least two of $\{v_1, v_2, v_3\}$, $\{v_1', v_2, v_3\}$, $\{v_1, v_2', v_3\}$, $\{v_1', v_2', v_3\}$ induce triangles in G. These triangles consist of one vertex in each colour class, so they correspond to induced triangles in W. This is contradictory, as W has no two distinct triangles that share a vertex.

It remains to show that G is $\left(\overline{P_1 + 2P_2}\right)$-free. Towards a contradiction, suppose $G[X] \cong \overline{P_1 + 2P_2}$. Note that $G[X]$ has a dominating vertex h. Without loss of generality, let $h \in V_1$. Since h has degree 4 in $G[X]$, we have $|X \cap V_1| \geq 2$. In fact, as W has no cycles of length 4, no two vertices in $V_2 \cup V_3$ share a pair of common neighbours in V_1, so $|X \cap V_1| \geq 3$. Since $G[X]$ is K_4-free, we have $|X \cap V_1| = 3$. Let $X \cap V_1 = \{x, x', h\}$ and $X \setminus V_1 = \{y, z\}$. We may assume without loss of generality that $y \in V_2$. Now there is a 5-vertex path $xyhzx'$ in W, up to the labels of x and x'. If $z \in V_2$, then the four edges of this path are the only edges in $G[\{x, y, h, z, x'\}]$ where the two endpoints are in different colour classes, so $W[\{x, y, h, z, x'\}] \cong P_5$. Since W has no bichromatic induced P_5, we deduce that $z \in V_3$. But then $W[X] \cong$ bull and $|X \cap V_1| = 3$, a contradiction. ◀

Next we use the construction of a chordal bipartite graph G' from a graph G, given by Brault-Baron, Capelli and Mengel [13][4]. Let $G = (V, E)$ be a graph. We take two copies of V labelled as follows: $X = \{x_v : v \in V\}$ and $Y = \{y_v : v \in V\}$. To construct G', start with a complete bipartite graph with vertex bipartition (X, Y), and add, for each edge $e \in E$ with endpoints u and v, two paths: an $x_u y_v$-path $x_u q_e t_e y_v$, and an $x_v y_u$-path $x_v q_e' t_e' y_u$. For

[4] Alternatively, we could take a wall, which has bipartition classes A and B; 2-subdivide all of its edges; and make A complete to B. The resulting graph has the same structure as G' and can have arbitrarily large mim-width due to Theorem 8 and Lemmas 2 and 4.

convenience, we let $Q = \bigcup_{e\in E(G)}\{q_e, q'_e\}$ and $T = \bigcup_{e\in E(G)}\{t_e, t'_e\}$. Observe that (X, Y, Q, T) partitions $V(G')$; see also Figure 9.

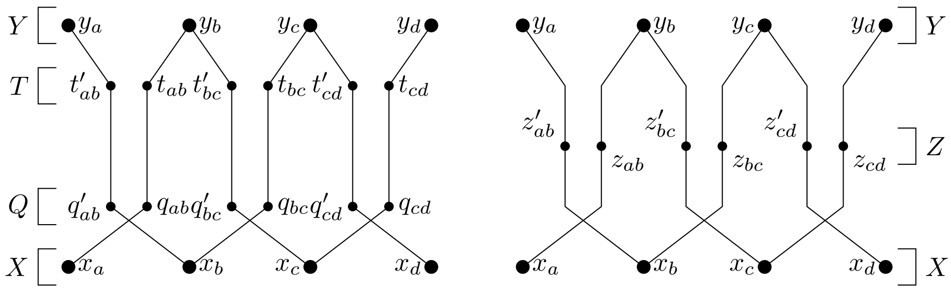

Figure 9 The graphs G' and G'', where we did not draw the edges between X and Y.

We need two lemmas. The first one is due to Baron, Capelli and Mengel.

▶ **Lemma 23** ([13, Lemmas 15 and 16]). *For any graph G, the graph G' as constructed above is chordal bipartite. Moreover, if G is bipartite, then* $\text{mimw}(G') \geq \text{tw}(G)/6$, *where* $\text{tw}(G)$ *denotes the treewidth of G.*

▶ **Lemma 24.** *For any graph G, the chordal bipartite graph G' is $(P_8, P_3 + P_6, S_{1,1,5})$-free.*

Proof. We label the vertices of G' as described in the construction, so (X, Y, Q, T) is a partition of $V(G')$. We first claim that if some $A \subseteq V(G')$ induces a path in G', with $|A| \geq 6$, then $X \cap A$ and $Y \cap A$ are non-empty. Suppose $G'[A] \cong P_{|A|}$ and $Y \cap A = \varnothing$. In $G'[X \cup Q \cup T]$, each vertex in T has degree 1, and each vertex in Q has two neighbours: one in X and one in T. If a vertex of T is in A, then it is an end of the path $G'[A]$; so $|T \cap A| \leq 2$. If a vertex of Q is in A, then either it is an end of the path $G'[A]$, or it is adjacent to a vertex of T that is an end of the path $G'[A]$. So $|Q \cap A| \leq 2$. Since X is independent, $|A| \leq 5$. The claim now follows by symmetry.

Now suppose some $A \subseteq V(G')$ induces a path in G' where $A \cap X \neq \varnothing$ and $A \cap Y \neq \varnothing$. Since $G'[X \cup Y]$ is complete bipartite, we may also assume that $|X \cap A| \in \{1, 2\}$ and $|Y \cap A| = 1$. For each vertex $v \in Q \cap A$ (respectively, $v \in T \cap A$), either v is the end of the path $G'[A]$, or v has a neighbour in $X \cap A$ (respectively, $Y \cap A$). Suppose $|(Q \cup T) \cap A| \geq 5$. Let A' be the vertices in $(Q \cup T) \cap A$ that are not ends of the path $G'[A]$. Then $|A'| \geq 3$, and each vertex in A' has a neighbour in $(X \cup Y) \cap A$. Since $A \cap X$ and $A \cap Y$ are non-empty, no two vertices in $(Q \cup T) \cap A$ share a neighbour in $(X \cup Y) \cap A$. So $|N_{G'[A]}(A') \cap (X \cup Y)| \geq 3$, implying $|X \cap A| = 2$. However, then the vertex in the singleton set $Y \cap A$ has degree 3 in $G'[A]$, a contradiction. So $|(Q \cup T) \cap A| < 5$, and $|A| < 8$. It now follows that G' is P_8-free.

Next we suppose, for some $F \subseteq V(G')$, that $G'[F]$ is a linear forest, one component of which is a P_6. Let $A \subseteq F$ such that $G'[A] \cong P_6$. By the foregoing claim, $X \cap A$ and $Y \cap A$ are non-empty. Since $G'[X \cup Y]$ is complete bipartite, it follows that $F \setminus A \subseteq Q \cup T$. Hence $G'[F \setminus A] \cong sP_1 + tP_2$ for some $s, t \geq 0$, implying G' is $(P_3 + P_6)$-free.

Finally, suppose $G'[S] \cong S_{1,1,5}$ for some $S \subseteq V(G)$. Let $A \subseteq S$ such that $G'[A] \cong P_7$. By the foregoing, $X \cap A$ and $Y \cap A$ are non-empty, and $|(Q \cup T) \cap A| < 5$. Hence $\{|X \cap A|, |Y \cap A|\} = \{1, 2\}$. Observe now that both ends of the path $G'[A]$ are in either Q or T, and the vertices of the path adjacent to the ends are in either T or Q, respectively. But then some vertex in T or Q has degree 3 in $G'[S]$ and hence in G', a contradiction. Hence G' is $S_{1,1,5}$-free. ◀

Lemma 24 is tight in the following sense: for some graph G, the graph G' can contain, as an induced subgraph, $tP_2 + P_7$ or tP_5 for any non-negative integer t, or $S_{2,2,4}$.

Theorem 25 now follows from Lemmas 23 and 24 and the fact that bipartite graphs can have arbitrarily large treewidth (see, e.g., [49]). We use Lemma 4 to obtain Theorems 26 and 27.

▶ **Theorem 25.** *The class of chordal bipartite* $(P_8, P_3 + P_6, S_{1,1,5})$*-free graphs has unbounded mim-width.*

▶ **Theorem 26.** *The class of* $(4P_1, \mathsf{gem}, \overline{P_1 + 2P_2})$*-free graphs has unbounded mim-width.*

Proof. For every integer k, we will construct a $(4P_1, \mathsf{gem})$-free graph G such that $\text{mimw}(G) > k$. Let B be a bipartite graph with $\text{tw}(B) > 24k$. Then, $\text{mimw}(B') > 4k$ by Lemma 23. Observe that B' is 4-partite, where $V(B')$ has a partition (X, Y, T, Q) into independent colour classes, using the labelling described in the construction. Let G be the graph obtained from B' by making X, Y, T, and Q into cliques. By Lemma 4, $\text{mimw}(G) \geq \text{mimw}(B')/4 > k$.

Observe that $X \cup Y$, T, and Q are cliques that partition $V(G)$, so G is $4P_1$-free. Note also that each vertex in Q has exactly one neighbour in T, exactly one neighbour in X, and no neighbours in Y. By symmetry, each vertex in T has exactly one neighbour in Q, exactly one neighbour in Y, and no neighbours in X. In particular, each vertex in $Q \cup T$ has at most one neighbour in $X \cup Y$. It remains to show that G is $(\mathsf{gem}, \overline{P_1 + 2P_2})$-free.

Suppose $G[D] \cong \mathsf{diamond}$ for some $D \subseteq V(G)$. Since $X \cup Y$ is a clique, $|D \cap (X \cup Y)| \leq 3$. In fact, $|D \cap (X \cup Y)| \leq 1$, since each vertex in $Q \cup T$ has at most one neighbour in $X \cup Y$. Note also that $D \nsubseteq T \cup Q$, since a vertex in T has at most one neighbour in Q (and vice versa). It follows, without loss of generality, that $|D \cap Q| = 3$ and $|D \cap X| = 1$.

Now suppose $G[D']$ is isomorphic to gem or $\overline{P_1 + 2P_2}$ for some $D' = D \cup z$ with $z \in V(G) \setminus D$. Note that a gem or a $\overline{P_1 + 2P_2}$ has a dominating vertex h, and $h \in D \cap Q$. If $z \in X$, then hz is not an edge, since the only neighbour of h in X is the vertex in $D \cap X$. If $z \in Y \cup T$, then z has degree 1 in $G[D']$. If $z \in Q$, then $G[D']$ contains a K_4. From this contradiction we deduce that G is $(\mathsf{gem}, \overline{P_1 + 2P_2})$-free. ◀

▶ **Theorem 27.** *The class of* $(\mathsf{diamond}, 2P_3)$*-free graphs has unbounded mim-width.*

Proof. For every integer k, we will construct a $(\mathsf{diamond}, 2P_3)$-free graph G such that $\text{mimw}(G) > k$. Let B be a bipartite graph with $\text{tw}(B) > 12k$. Then, $\text{mimw}(B') > 2k$ by Lemma 23. Observe that B' is bipartite, where $(X \cup T, Y \cup Q)$ is a bipartition of $V(B')$. Let G be the graph obtained from B' by making X and Y into cliques. By Lemma 4, $\text{mimw}(G) \geq \text{mimw}(B')/2 > k$.

Observe now that $X \cup Y$ is a clique of G. Moreover, G can be obtained starting from $G[X \cup Y]$ by adding 3-edge xy-paths for some $x \in X$ and $y \in Y$. It follows that each induced P_3 subgraph of G contains some vertex of $X \cup Y$. Since $X \cup Y$ is a clique, any two disjoint induced P_3 subgraphs of G have an edge between them. So G is $2P_3$-free.

Finally, observe that for each induced K_3 subgraph of G we have $V(K_3) \subseteq X \cup Y$. Hence, if $G[A] \cong \mathsf{diamond}$ for some $A \subseteq V(G)$, then $A \subseteq X \cup Y$, but then A is a clique, a contradiction. So G is $\mathsf{diamond}$-free. ◀

We now describe the construction of a graph G'' from a graph $G = (V, E)$. This construction is similar to the construction of G'; we adapt the approach taken by Brault-Baron et al. [13] to construct graphs with arbitrarily large mim-width. Take two copies of V labelled as follows: $X = \{x_v : v \in V\}$ and $Y = \{y_v : v \in V\}$. Construct a graph G'' on vertex set $X \cup Y \cup Z$ where $Z = \bigcup_{e \in E(G)} \{z_e, z'_e\}$. Start with a complete bipartite graph with vertex bipartition (X, Y), and add, for each edge $e \in E$ with endpoints u and v, two paths $x_u z_e y_v$ and $x_v z'_e y_u$. Observe that G'' is 3-partite, with colour classes (X, Y, Z); see also Figure 9.

The following lemma is proven by modifying the proof of Lemma 23 given by Brault-Baron et al. [13]. Alternatively, we could take the $n \times n$ wall W, which has bipartition classes A and B; 1-subdivide each edge of W; and make A complete to B. By applying Theorem 8 and Lemmas 2 and 4, we obtain a lower bound on the mim-width in terms of n.

▶ **Lemma 28.** *If G is a bipartite graph, then* $\text{mimw}(G'') \geq \text{tw}(G)/6$.

Proof. Let G be a bipartite graph with vertex bipartition (A, B), and let (T'', δ'') be an arbitrary branch decomposition of G''. We will show that $\text{mimw}_{G''}(T'', \delta'') \geq \text{tw}(G)/6$.

We first construct a branch decomposition (T, δ) of G such that $E(T) \subseteq E(T'')$, as follows. Let T be the tree obtained from T'' by deleting the leaves $t \in V(T'')$ such that $\delta''(t) = x_v$ for some $v \in B$, or $\delta''(t) = y_u$ for some $u \in A$, or $\delta''(t) \in Q \cup T$. In the resulting tree T, for each leaf $t \in T$ we define $\delta(t) = v$ if $\delta''(t) = x_v$ for some $v \in A$; and $\delta(t) = u$ if $\delta''(t) = x_u$ for some $u \in B$.

Suppose $e \in E(T)$. Recall that $(A_e, \overline{A_e})$ denotes the partition of $V(G)$ induced by the two components of $T \backslash e$, and let $(A''_e, \overline{A''_e})$ denote the partition of $V(G'')$ induced by the two components of $T'' \backslash e$. Let uv be an edge in the cut $G[A_e, \overline{A_e}]$. Since G is bipartite, we may assume $u \in A$ and $v \in B$. Then x_u and y_v are on different sides of the cut $G''[A''_e, \overline{A''_e}]$; we may assume that $x_u \in A''_e$ and $y_v \in \overline{A''_e}$. Since there is a path $x_u z_{uv} y_v$ in G'', either the edge $x_u z_{uv}$ or the edge $z_{uv} y_v$ is in $G''[A''_e, \overline{A''_e}]$.

Let M be a matching of $G[A_e, \overline{A_e}]$. We obtain a matching M' of $G[A''_e, \overline{A''_e}]$ of size $|M|$ as follows: for each edge uv in M, choose the edge $x_u z_{uv}$ or $z_{uv} y_v$ that is in $G[A''_e, \overline{A''_e}]$. We partition M' into (M'_X, M'_Y) where M'_X consists of the edges incident to a vertex of X and M'_Y consists of the edges incident to a vertex of Y. Let M'' be the larger of M'_X and M'_Y; then $|M''| \geq |M|/2$. Note that M'' is a matching of $G''[A''_e, \overline{A''_e}]$ since $M'' \subseteq M'$.

By [13, Lemma 9], there exists some edge $e \in E(T)$ such that $G[A_e, \overline{A_e}]$ has a (not necessarily induced) matching M of size at least $\text{tw}(G)/3$. By the previous paragraph, $G''[A''_e, \overline{A''_e}]$ has a matching M'' of size at least $|M|/2 \geq \text{tw}(G)/6$, which consists of edges between a vertex in Z and a vertex in either X or Y.

We claim that M'' is an induced matching. Suppose not. Then we may assume (up to swapping X and Y) that M'' has edges $x_u z_{uv}$ and $x_{u'} z_{u'v'}$, for some distinct $u, u' \in V(G)$, and G'' also has an edge $x_u z_{u'v'}$ or $x_{u'} z_{uv}$. But, by construction, the vertices $z_{uv}, z_{u'v'} \in Z$ have only one neighbour in X, so neither $x_u z_{u'v'}$ nor $x_{u'} z_{uv}$ is an edge of G''. Thus M'' is induced, and hence $\text{mimw}_{G''}(T'', \delta'') \geq \text{tw}(G)/6$, as required. ◀

We use Lemma 28 to show the following theorem.

▶ **Theorem 29.** *The class of $(K_4, \textsf{diamond}, P_6, P_2+P_4)$-free graphs has unbounded mim-width.*

Proof. We show that for every integer k, there is a $(K_4, \textsf{diamond}, P_6, P_2 + P_4)$-free graph G such that $\text{mimw}(G) > k$. Let B be a (simple) bipartite graph with $\text{tw}(B) > 6k$ and let $G = B''$. Then $\text{mimw}(G) > k$ by Lemma 28. Observe that X, Y and Z are independent sets.

First we claim that G is K_4-free. Suppose $G[A] \cong K_4$ for some $A \subseteq V(G)$. Since each vertex in Z has degree 2, $A \subseteq X \cup Y$. But then $|A \cap X| \geq 2$ or $|A \cap Y| \geq 2$, a contradiction.

Next we claim that G is $\textsf{diamond}$-free. Suppose $G[A] \cong \textsf{diamond}$ for some $A \subseteq V(G)$. Since each vertex in Z has degree 2, the degree-3 vertices of the diamond must be in X or Y. Since these vertices are adjacent, one is in X and one is in Y. As the other two vertices of the diamond are complete to these two vertices, these vertices are in Z. Let $A \cap X = \{x_u\}$,

$A \cap Y = \{y_v\}$, and $A \cap Z = \{z_e, z_{e'}\}$. Now $x_u z_e y_v$ and $x_u z_{e'} y_v$ are paths in G, corresponding to multiple edges $e = uv$ and $e' = uv$ in B, but this contradicts that B is simple.

Next we claim that G is $P_2 + P_4$-free. Suppose $G[A] \cong P_2 + P_4$ for some $A \subseteq V(G)$ and $G[A'] \cong P_4$ for some $A' \subseteq A$.If $A' \subseteq Y \cup Z$, then one end of $G[A']$ is in Y, and the other end is in Z. But each vertex in Z has one neighbour in X and one neighbour in Y, so $A' \cap X \neq \varnothing$ and, by symmetry, $A' \cap Y \neq \varnothing$. Now each vertex in X or Y is adjacent to a vertex of $G[A']$. So $A \setminus A' \subseteq Z$, but then $G[A \setminus A'] \cong 2P_1$, a contradiction.

It remains to show that G is P_6-free. Suppose $G[A] \cong P_6$ for some $A \subseteq V(G)$. If $A \subseteq X \cup Z$, then each vertex of $A \cap Z$ has degree at most 1 in $G[A]$, so there are at most two such vertices. But then $|A \cap X| \geq 4$, and this set is independent in $G[A]$, a contradiction. So $A \cap Y \neq \varnothing$ and, by symmetry, $A \cap X \neq \varnothing$. Since X is complete to Y, we also have $|A \cap (X \cup Y)| \leq 3$. Without loss of generality we may assume $A \cap X$ is a singleton $\{x\}$. Then x has two neighbours in $A \cap Y$, so $A \cap X$ and $A \cap Z$ are anticomplete. But then $A \cap (X \cup Z)$ is an independent set of size at least 4, a contradiction. ◀

6 State of the Art

In this section, we show the consequences of the results from Sections 3–5 for the boundedness and unboundedness of mim-width of classes of (H_1, H_2)-free graphs. We will also make a comparison between the results for mim-width and clique-width. In contrast to the situation where only one induced subgraph is forbidden, we note many differences when two induced subgraphs H_1 and H_2 are forbidden. Figure 10 illustrates a number of graphs that we use throughout the section.

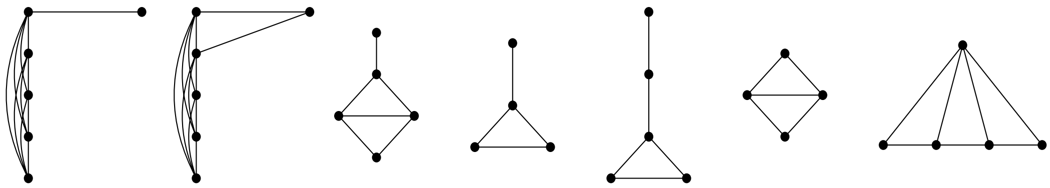

Figure 10 The graphs $K_5 \boxminus P_1 = \overline{K_{1,4} + P_1}$, $\overline{K_{1,3} + 2P_1}$, $\overline{S_{1,1,2}}$, paw, hammer, diamond and gem.

6.1 Two Summary Theorems

In our first summary theorem we give all pairs (H_1, H_2) for which the mim-width of the class of (H_1, H_2)-free graphs is bounded. This theorem gives more bounded cases than the corresponding summary theorem for boundedness of *clique-width* of classes of (H_1, H_2)-free graphs, which can be found in the survey of Dabrowski, Johnson and Paulusma [23] and which we need for our proof. To get the summary theorem for clique-width, replace Cases (x)–(xv) of Theorem 30 by the more restricted case where $H_1 = K_r$ and $H_2 = sP_1$ for some $s, t \geq 1$. Note that for Cases (xii)–(xv), the obtained bound on mim-width depends on the constants r, s and/or t.

▶ **Theorem 30.** *For graphs H_1 and H_2, the mim-width of the class of (H_1, H_2)-free graphs is* bounded *and* quickly computable *if one of the following holds:*

(i) H_1 *or* $H_2 \subseteq_i P_4$*,*

(ii) $H_1 \subseteq_i$ paw *and* $H_2 \subseteq_i K_{1,3} + 3P_1,\ K_{1,3} + P_2,\ P_1 + P_2 + P_3,\ P_1 + P_5,\ P_1 + S_{1,1,2},\ P_2 + P_4,\ P_6,\ S_{1,1,3}$ *or* $S_{1,2,2}$*,*

(iii) $H_1 \subseteq_i P_1 + P_3$ *and* $H_2 \subseteq_i \overline{K_{1,3} + 3P_1},\ \overline{K_{1,3} + P_2},\ \overline{P_1 + P_2 + P_3},\ \overline{P_1 + P_5},\ \overline{P_1 + S_{1,1,2}},\ \overline{P_2 + P_4},\ \overline{P_6},\ \overline{S_{1,1,3}}$ *or* $\overline{S_{1,2,2}}$*,*
(iv) $H_1 \subseteq_i$ diamond *and* $H_2 \subseteq_i P_1 + 2P_2,\ 3P_1 + P_2$ *or* $P_2 + P_3$*,*
(v) $H_1 \subseteq_i 2P_1 + P_2$ *and* $H_2 \subseteq_i \overline{P_1 + 2P_2},\ \overline{3P_1 + P_2}$ *or* $\overline{P_2 + P_3}$*,*
(vi) $H_1 \subseteq_i$ gem *and* $H_2 \subseteq_i P_1 + P_4$ *or* P_5*,*
(vii) $H_1 \subseteq_i P_1 + P_4$ *and* $H_2 \subseteq_i \overline{P_5}$*,*
(viii) $H_1 \subseteq_i K_3 + P_1$ *and* $H_2 \subseteq_i K_{1,3}$*,*
(ix) $H_1 \subseteq_i 2P_1 + P_3$ *and* $H_2 \subseteq_i \overline{2P_1 + P_3}$*,*
(x) $H_1 \subseteq_i 2P_1 + P_2$ *and* $H_2 \subseteq_i$ bowtie*,*
(xi) $H_1 \subseteq_i K_{1,3}$ *and* $H_2 \subseteq_i 2P_2$*,*
(xii) $H_1 \subseteq_i K_r$ *for* $r \geq 1$ *and* $H_2 \subseteq_i sP_1 + P_5$ *for* $s \geq 0$*,*
(xiii) $H_1 \subseteq_i K_r \boxminus rP_1$ *for* $r \geq 1$ *and* $H_2 \subseteq_i 2P_2$*,*
(xiv) $H_1 \subseteq_i K_r \boxminus P_1$ *for* $r \geq 1$ *and* $H_2 \subseteq_i tP_2$ *for* $t \geq 1$*, or*
(xv) $H_1 \subseteq_i K_r \boxminus K_r$ *for* $r \geq 1$ *and* $H_2 \subseteq_i sP_1 + P_2$ *for* $s \geq 0$*.*

Proof. Cases (i)–(ix) follows from the fact that each of the classes of (H_1, H_2)-free graphs in these cases has bounded clique-width and that clique-width is quickly computable for general graphs [44]. For Case (i) we also refer to Theorem 13. Boundedness of clique-width has been proven for Case (ii) as follows: in [26] for $K_{1,3} + 3P_1$; in [25] for $K_{1,3} + P_2$; in [21] for $P_1 + P_2 + P_3$ and $P_1 + P_5$; in [26] for $P_1 + S_{1,1,2}$; in [24] for $P_2 + P_4$; in [9] for P_6; in [25] for $S_{1,1,3}$; and in [21] for $S_{1,2,2}$. It has been proven for Case (iv) as follows: in [21] for $P_1 + 2P_2$; and in [22] for $3P_1 + P_2$ and $P_2 + P_3$. It has been been proven for Case (vi) as follows: in [10] for $P_1 + P_4$; and in [11] for P_5. It has been proven for Case (viii) and (ix) in [7, 12] and [5], respectively. Cases (iii), (v), (vii) follow from Cases (ii), (iv) and (vi), respectively, after recalling that the clique-width of a class of (H_1, H_2)-free graphs is bounded if and only if the clique-width of the class of $(\overline{H_1}, \overline{H_2})$-free graphs is bounded [38]. Cases (x) and (xi) follow from Theorems 20 and 18 respectively. Case (xii) has been proven in [14]. Cases (xiii)–(xv) follow from Theorems 14–16, respectively. ◀

For our second summary theorem, we turn to the unbounded cases. We let $\mathcal{S}$ be the class of graphs every connected component of which is either a subdivided claw or a path. We let $\mathcal{N}$ denote the class of graphs that contain a connected component with either a cycle of length at least 4 or at least two (not necessarily vertex-disjoint) triangles; note, for example, that $\mathcal{N}$ contains C_4, diamond, and K_4.

▶ **Theorem 31.** *For graphs H_1 and H_2, the class of (H_1, H_2)-free graphs has* unbounded *mim-width if one of the following holds:*
(i) $H_1 \notin \mathcal{S}$ *and* $H_2 \notin \mathcal{S}$*,*
(ii) $H_1 \supseteq_i C_3$ *and* $H_2 \supseteq_i P_3 + P_6,\ P_8$ *or* $S_{1,1,5}$*,*
(iii) $H_1 \supseteq_i K_{1,3}$ *and* $H_2 \in \mathcal{N}$*,*
(iv) $H_1 \supseteq_i$ diamond *and* $H_2 \supseteq_i 5P_1,\ P_2 + P_4,\ 2P_3$ *or* P_6*,*
(v) $H_1 \supseteq_i 3P_1$ *and* $H_2 \supseteq_i 3P_1,\ C_5$ *or* $\overline{C_{2s+1}}$ *for* $s \geq 3$*,*
(vi) $H_1 \supseteq_i 4P_1$ *and* $H_2 \supseteq_i$ gem*,* $\overline{3P_1 + P_2}$ *or* $\overline{P_1 + 2P_2}$*,*
(vii) $H_1 \supseteq_i 2P_2$ *and* $H_2 \supseteq_i C_4,\ C_5,\ K_{1,4},\ 2P_2,\ \overline{3P_1 + P_2}$ *or* sun_t *for* $t \geq 3$*, or*
(viii) $H_1 \supseteq_i K_4$ *and* $H_2 \supseteq_i P_2 + P_4$ *or* P_6*.*

Proof. Cases (i) and (iii) follow from Theorem 8 and Lemma 10, respectively, possibly after applying Lemma 2 a sufficient number of times. All three subcases of Case (ii) follows from Theorem 25. The first subcase of Case (iv) follows from Theorem 21, the second one follows from Theorem 29, the third one follows from Theorem 27 and the fourth one follows from

Theorem 29. All three subcases of Case (v) follow from Lemma 12. Case (vi) follows from Theorems 22 and 26. All subcases of Case (vii) follow from Lemma 11. Case (viii) follows from Theorem 29. ◀

We note that the situation for the unbounded cases is again different from the situation for the unbounded cases of clique-width. For example, (H_1, H_2)-free graphs have unbounded clique-width if both $\overline{H_1} \notin \mathcal{S}$ and $\overline{H_2} \notin \mathcal{S}$ (see, for example, [26]). Take, for instance, $H_1 = 4P_1$ and $H_2 = 2P_2$. Then $\overline{H_1} = K_4$ and $\overline{H_2} = C_4$, and thus $\overline{H_1} \notin \mathcal{S}$ and $\overline{H_2} \notin \mathcal{S}$, so (H_1, H_2)-free graphs have unbounded clique-width. However, by Theorem 30-(xiii), (H_1, H_2)-free graphs have bounded mim-width. As $(\overline{H_1}, \overline{H_2})$-free graphs have unbounded mim-width by Theorem 31-(i), this example also shows that the complementation operation, a standard tool for working with clique-width, does not preserve mim-width. Consequently, for mim-width there are many more open cases than the only five open cases for clique-width [23].

6.2 Three Consequences of the Summary Theorems

In order to get a handle on the open cases for mim-width, we now present some consequences of Theorems 30 and 31. We first consider the case where H_1 and H_2 are forests.

▶ **Corollary 32.** *Let H_1 and H_2 be* forests*. Either the pair (H_1, H_2) satisfies Theorem 30 or Theorem 31, or one of the following holds:*
1. *$H_1 = 2P_2$ and $H_2 = K_{1,3} + sP_1$ for $s \geq 1$, or*
2. *$H_1 = 2P_2$ and $H_2 = S_{1,1,2} + sP_1$ for $s \geq 0$.*

Proof. Throughout the proof we assume that H_1 and H_2 are not induced subgraphs of P_4, as otherwise we can apply Theorem 30-(i). This means that H_1 contains an induced $3P_1$ or an induced $2P_2$ and the same holds for H_2. If both contain an induced $3P_1$, then we can apply Theorem 31-(v). If both contain an induced $2P_2$, then we can apply Theorem 31-(vii). Suppose neither of these two cases apply. Then we may assume without loss of generality that $2P_2 \subseteq_i H_1$ while $3P_1 \not\subseteq_i H_1$, and $3P_1 \subseteq_i H_2$ while $2P_2 \not\subseteq_i H_2$. The above implies that $H_1 = 2P_2$ and H_2 has at most one connected component with an edge.

First suppose that H_2 is a linear forest. Then $H_2 = sP_1 + P_3$ or $H_2 = sP_1 + P_4$ for some $s \geq 1$, and we apply Theorem 30-(xiii). Now suppose that H_2 is not a linear forest, so $K_{1,3} \subseteq_i H_2$. If $K_{1,4} \subseteq_i H_2$, then we apply Theorem 31-(vii). If $H_2 = K_{1,3}$, then we apply Theorem 30-(xi). Hence $H_2 = K_{1,3} + sP_1$ for some $s \geq 1$ or $H_2 = S_{1,1,2} + tP_1$ for some $t \geq 0$. ◀

▶ **Open Problem 1.** *Determine the (un)boundedness of mim-width of (H_1, H_2)-free graphs when*
1. *$H_1 = 2P_2$ and $H_2 = K_{1,3} + sP_1$ for $s \geq 1$, or*
2. *$H_1 = 2P_2$ and $H_2 = S_{1,1,2} + sP_1$ for $s \geq 0$.*

Next we consider the case where H_1 and H_2 are connected.

▶ **Corollary 33.** *Let H_1 and H_2 be* connected *graphs. Either the pair (H_1, H_2) satisfies Theorem 30 or Theorem 31, or one of the following holds:*
1. *$H_1 = P_5$ and $H_2 = \overline{S_{1,1,2}}$ or $\overline{K_{1,r} + sP_1}$ for $r \geq 3$ and $s \in \{1, 2\}$,*
2. *$H_1 = P_7$ or $S_{h,i,j}$ for $h \leq i \leq j \leq 4$ with $i + j \leq 6 \leq h + i + j$ and $H_2 = C_3$ or* paw*, or*
3. *$H_1 = K_{1,3}$ or $S_{1,1,2}$ and $H_2 =$* hammer*.*

Proof. If $H_1 \notin \mathcal{S}$ and $H_2 \notin \mathcal{S}$, then we apply Theorem 31-(i). Hence, we may assume without loss of generality that $H_1 \in \mathcal{S}$. As H_1 is connected, this means that H_1 is a subdivided claw or a path. If H_1 is $3P_1$-free, then $H_1 \subseteq_i P_4$, and we apply Theorem 30-(i). Assume that $3P_1 \subseteq_i H_1$. Then H_2 must be co-bipartite, as otherwise we can apply Theorem 31-(v).

First suppose H_1 is a path. If $H_1 \subseteq_i P_4$, then we apply Theorem 30-(i). Now suppose $P_5 \subseteq_i H_1$. Then both $3P_1 \subseteq_i H_1$ and $2P_2 \subseteq_i H_1$. Then H_2 must be a co-bipartite $\overline{3P_1 + P_2}$-free split graph, as otherwise we can apply Theorem 31-(vii). Suppose $H_1 = P_5$. If $H_2 = \mathsf{gem}$, then we apply Theorem 30-(vi). If $H_2 = K_r$ for any $r \geq 1$, then we apply Theorem 30-(xii). Otherwise we find that $H_2 = \overline{S_{1,1,2}}$ or $H_2 = \overline{K_{1,r} + sP_1}$ for some $r \geq 3$ and $s \in \{1, 2\}$, which correspond to Case 1. Now suppose $H_1 = P_6$. If $K_4 \subseteq_i H_2$, then we apply Theorem 31-(viii). Suppose H_2 is K_4-free. If $H_2 \subseteq_i \mathsf{paw}$, then we apply Theorem 30-(ii). Otherwise $\mathsf{diamond} \subseteq_i H_2$ and we apply Theorem 31-(iv). Now suppose $H_1 = P_7$. If $K_4 \subseteq_i H_2$ or $\mathsf{diamond} \subseteq_i H_2$, then we apply Theorem 31-(viii) or Theorem 31-(iv), respectively. Otherwise we find that $H_2 = C_3$ or paw; this falls under Case 2. Finally suppose $P_8 \subseteq_i H_1$. If $C_3 \subseteq_i H_2$, then we apply Theorem 31-(ii). Otherwise we find that $H_2 \subseteq_i P_4$ and we apply Theorem 30-(i).

Now suppose H_1 is a subdivided claw. If C_4, K_4, or $\mathsf{diamond} \subseteq_i H_2$, then we apply Theorem 31-(iii). From now on assume that H_2 is $(C_4, K_4, \mathsf{diamond})$-free. Recall that H_2 is co-bipartite. If H_2 is C_3-free, this means that $H_2 \subseteq_i P_4$ and we apply Theorem 30-(i). Hence, we may assume that $C_3 \subseteq_i H_2$. This means that $H_2 \in \{C_3, \mathsf{paw}, \mathsf{bowtie}, \mathsf{hammer}, 2C_3 + e\}$, where the graph $2C_3 + e$ is obtained from $2C_3$ by inserting an edge between the two triangles. First suppose $H_1 \in \{K_{1,3}, S_{1,1,2}\}$. If $H_2 \subseteq_i \mathsf{paw}$, then we apply Theorem 30-(ii). Otherwise we find that $H_2 \in \{\mathsf{bowtie}, \mathsf{hammer}, 2C_3 + e\}$. If $H_2 \in \{\mathsf{bowtie}, 2C_3 + e\}$, then we apply Theorem 31-(iii). The two remaining cases correspond to Case 3. Now suppose that $H_1 \notin \{K_{1,3}, S_{1,1,2}\}$. Then $2P_2 \subseteq_i H_1$. If $H_2 \in \{\mathsf{bowtie}, \mathsf{hammer}, 2C_3 + e\}$, then $2P_2 \subseteq_i H_2$, which means that we can apply Theorem 31-(vii). Hence, we may assume that $H_2 \in \{C_3, \mathsf{paw}\}$. If $H_1 \in \{S_{1,2,2}, S_{1,1,3}\}$, then we apply Theorem 30-(ii). If H_1 is not $(P_3 + P_6, P_8, S_{1,1,5})$-free, then we apply Theorem 31-(ii). Otherwise we obtain the remaining cases of Case 2. ◀

▶ **Open Problem 2.** *Determine the (un)boundedness of mim-width of (H_1, H_2)-free graphs when*
1. $H_1 = P_5$ *and* $H_2 = \overline{S_{1,1,2}}$ *or* $\overline{K_{1,r} + sP_1}$ *for* $r \geq 3$ *and* $s \in \{1, 2\}$,
2. $H_1 = P_7$ *or* $S_{h,i,j}$ *for* $h \leq i \leq j \leq 4$ *with* $i + j \leq 6 \leq h + i + j$ *and* $H_2 = C_3$ *or* paw, *or*
3. $H_1 = K_{1,3}$ *or* $S_{1,1,2}$ *and* $H_2 = \mathsf{hammer}$.

Finally, we note that Theorems 30 and 31 cover all pairs (H_1, H_2) with $|V(H_1)| + |V(H_2)| \leq 8$.

▶ **Corollary 34.** *Let H_1 and H_2 be graphs with $|V(H_1)| + |V(H_2)| \leq 8$. Then the pair (H_1, H_2) satisfies Theorem 30 or Theorem 31.*

Proof. Recall that $\mathcal{S}$ is the class consisting of graphs where every connected component is either a subdivided claw or a path. If $H_1 \notin \mathcal{S}$ and $H_2 \notin \mathcal{S}$, then we apply Theorem 31-(i). Hence, we may assume without loss of generality that $H_1 \in \mathcal{S}$. As each of the pairs (H_1, H_2) in Open Problem 1 (Corollary 32) has $|V(H_1)| + |V(H_2)| \geq 9$, we deduce that H_2 contains a cycle. As each of the pairs (H_1, H_2) in Open Problem 2 (Corollary 33) has $|V(H_1)| + |V(H_2)| \geq 9$, we deduce that at least one of H_1, H_2 is disconnected.

Case 1. H_1 is disconnected.
First suppose that H_1 is $3P_1$-free. Then either $H_1 \subseteq_i P_4$ or $H_1 = 2P_2$. In the first case we apply Theorem 30-(i), Assume the latter case. Then H_2 is C_4-free, as otherwise we apply

Theorem 31-(vii). Hence H_2 contains a C_3. If $H_2 \in \{C_3, K_3 + P_1, K_3 \boxminus P_1, K_4\}$, then we apply Theorem 30-(xiii). Otherwise, $H_2 = \mathsf{diamond}$ and we apply Theorem 30-(iv).

Now suppose H_1 contains an induced $3P_1$. Then H_2 must be $3P_1$-free, as otherwise we can apply Theorem 31-(v). First consider when $|V(H_1)| \leq 4$ and $|V(H_2)| \leq 4$. Then $H_1 \in \{3P_1, 4P_1, 2P_1 + P_2, P_1 + P_3\}$ and $H_2 \in \{C_3, C_4, \mathsf{diamond}, \mathsf{paw}, K_3 + P_1, K_4\}$. If $H_1 = P_1 + P_3$, then we apply Theorem 30-(iii). So $H_1 \in \{3P_1, 4P_1, 2P_1 + P_2\}$. If $H_2 \in \{C_3, C_4, \mathsf{paw}, K_3 + P_1, K_4\}$, then we apply Theorem 30-(xiv) or Theorem 30-(xv); whereas if $H_2 = \mathsf{diamond}$, then we apply Theorem 30-(iv).

It remains to consider when $H_1 = 3P_1$ and $|V(H_2)| = 5$, or $H_2 = C_3$ and $|V(H_1)| = 5$. In the latter case, $H_2 = C_3$ and H_1 is a linear forest on 5 vertices, in which case we apply Theorem 30-(ii). In the former case, if $H_2 \in \{K_3 + P_2, \mathsf{hammer}, \overline{P_5}, K_4 + P_1, K_4 \boxminus P_1, K_5\}$, then $H_2 \subseteq_i K_5 \boxminus K_5$, so we apply Theorem 30-(xv); whereas if H_2 belongs to $\{\overline{S_{1,1,2}}, \overline{P_2 + P_3}, \mathsf{gem}, \overline{P_1 + 2P_2}, \overline{2P_1 + P_3}, \overline{3P_1 + P_2}\}$, then we apply Theorem 30-(iii). The only possibility that remains is $H_2 = \mathsf{bowtie}$, for which we apply Theorem 30-(x).

Case 2. H_1 is connected.
Then H_2 is disconnected. As H_2 contains a cycle, $|V(H_2)| \geq 4$, so $|V(H_1)| \leq 4$. As H_1 is connected and belongs to $\mathcal{S}$, we find that $H_1 \subseteq_i P_4$ or $H_1 = K_{1,3}$. In the first case we apply Theorem 30-(i). In the second case, $|V(H_1)| = 4$, so $|V(H_2)| = 4$. As H_2 is disconnected and contains a cycle, $H_2 = K_3 + P_1$, so we apply Theorem 30-(viii). ◀

6.3 When $\mathbf{H_1}$ is Complete or Edgeless

We first consider the (un)boundedness of mim-width for the class of (K_r, H_2)-free graphs for a positive integer r and a graph H_2. Such classes are interesting for the following reason. For any H_2 such that mim-width is bounded and quickly computable for the class of (K_r, H_2)-free graphs, k-Colouring is polynomial-time solvable for all $k < r$; for example, see [14] for the case where $H_2 \subseteq_i sP_1 + P_5$. More generally, for problems having polynomial-time algorithms when mim-width is bounded and quickly computable, we obtain $n^{f(\omega(G))}$-time algorithms, for some function f, when restricted to H_2-free graphs; that is, XP algorithms parameterized by $\omega(G)$ (the size of the largest clique in G). Recently, Chudnovsky et al. [17] showed that for P_5-free graphs, there exists an $n^{O(\omega(G))}$-time algorithm for Max Partial H-Colouring, a problem generalizing Maximum Independent Set and Odd Cycle Transversal, and which is polynomial-time solvable when mim-width is bounded and quickly computable.

For $r \geq 4$, Theorems 30 and 31 imply that the mim-width of the class of (K_r, H_2)-free graphs is bounded and quickly computable when $H_2 \subseteq_i sP_1 + P_5$ or tP_2, and unbounded when $H_2 \supseteq_i K_{1,3}$, $P_2 + P_4$, or P_6, or $H_2 \notin \mathcal{S}$. In the following theorem we prove that all remaining cases belong to one infinite family: when $H_2 = tP_2 + uP_3$ for $u \geq 1$ and $t + u \geq 2$. Note that Theorem 35 just concerns the case that $r \geq 4$. When $r = 3$, further open cases arise; for example, see Open Problem 2.

▶ **Theorem 35.** *Let H be a graph and let $r \geq 4$ be an integer. Then exactly one of the following holds:*

- *$H \subseteq_i sP_1 + P_5$ or tP_2, and the mim-width of the class of (K_r, H)-free graphs is bounded and quickly computable;*
- *$H \notin \mathcal{S}$, or $H \supseteq_i K_{1,3}$, $P_2 + P_4$, or P_6, and the mim-width of the class of (K_r, H)-free graphs is unbounded; or*
- *$H = tP_2 + uP_3$ where $u \geq 1$ and $t + u \geq 2$.*

Proof. By Theorem 31-(i), if $H \notin \mathcal{S}$, then the mim-width of the class of (K_r, H)-free graphs is unbounded. So we may assume that H is a forest of paths and subdivided claws. By Theorem 31-(iii), if H contains a $K_{1,3}$, then the mim-width is again unbounded. So we may assume that H is a linear forest. If $H \subseteq_i sP_1 + P_5$ or $H \subseteq_i tP_2$, then mim-width is bounded and quickly computable by parts (xii) and (xiv) of Theorem 30. So we may assume that H is a linear forest containing $P_2 + P_3$. By Theorem 31-(viii), we may also assume H contains neither $P_2 + P_4$ nor P_6, otherwise the mim-width is again unbounded. It now follows that $H \subseteq_i tP_2 + uP_3$ for some u, t such that $u \geq 1$ and $t + u \geq 2$. ◀

▶ **Open Problem 3.** *For an integer $r \geq 4$, and for each integer $t \geq 0$ and $u \geq 1$ such that $t + u \geq 2$, determine the (un)boundedness of the class of $(K_r, tP_2 + uP_3)$-free graphs.*

We note that this is also open when $r = 3$, except when $u = t = 1$ (so $H_2 = P_2 + P_3$) in which case we can apply Theorem 30-(ii).

We now consider the class of (rP_1, H_2)-free graphs, for an integer r and a graph H_2. If the mim-width of such a class of graphs is bounded and quickly computable, we obtain, for many problems, XP algorithms parameterized by $\alpha(G)$ for the class of H_2-free graphs, where $\alpha(G)$ is the size of the largest independent set in G. For $r \geq 5$, Theorems 30 and 31 imply that the mim-width of the class of (rP_1, H_2)-free graphs is bounded and quickly computable when $H_2 \subseteq_i K_t \boxminus K_t$ for some t, and unbounded when H_2 is not co-bipartite, or $H_2 \supseteq_i$ diamond. Below we show that all unresolved cases belong to the infinite family $H_2 = \overline{K_{s,t} + P_1}$ for $s, t \geq 2$ (we observe that if $s = t = 2$, then $H_2 =$ bowtie). Note that Theorem 36 just concerns the case that $r \geq 5$. When $r \in \{3, 4\}$, further open cases arise, and there are more cases where the class of (rP_1, H)-free graphs has bounded mim-width, by cases (iii) and (x) of Theorem 30.

▶ **Theorem 36.** *Let H be a graph and let $r \geq 5$ be an integer. Then exactly one of the following holds:*

- *$H \subseteq_i K_t \boxminus K_t$ for some integer $t \geq 1$, and the mim-width of the class of (rP_1, H)-free graphs is bounded and quickly computable;*
- *H is not co-bipartite or $H \supseteq_i$ diamond, and the mim-width of the class of (rP_1, H)-free graphs is unbounded; or*
- *$H = \overline{K_{s,t} + P_1}$ for some $s, t \geq 2$.*

Proof. By Theorem 31-(v), if H is not co-bipartite, then the mim-width of the class of (rP_1, H)-free graphs is unbounded. So we may assume that H is co-bipartite. In particular, H is $3P_1$-free, and hence if H is a forest, we have that $H \subseteq_i P_4$ or $H \subseteq_i 2P_2$. In either case, $H \subseteq K_4 \boxminus K_4$, so the mim-width is bounded and quickly computable by Theorem 30-(i). So we may assume that H contains a cycle. In particular, since H is $(C_5, 3P_1)$-free, H contains no induced cycle of length at least 5. By Theorem 31-(iv) we may assume that H contains no diamond, otherwise the class has unbounded mim-width.

Suppose that H contains an induced C_4. It follows from H being co-bipartite and diamond-free that $H \subseteq_i K_t \boxminus K_t$ for some t, in which case mim-width is bounded and quickly computable by Theorem 30-(xv). So we may assume that H does not contain an induced C_4, and hence H is chordal.

It remains to show that H is a block graph consisting of two blocks each being complete and having at least 3 vertices. Let K be a maximum clique of H. So K has size at least 3. By Theorem 30-(xv) we may assume that $V(H) \setminus K \neq \varnothing$. Since H is diamond-free and by the maximality of K, any vertex of H not in K has at most one neighbour in K. Then since

H is $3P_1$-free, $V(H) \setminus K$ is a clique. Now, if at most one vertex of $V(H) \setminus K$ has a neighbour in K, then H is an induced subgraph of $K_r \boxminus K_r$, so we can apply Theorem 30-(xv). So we may assume there are distinct vertices $u, v \in V(H) \setminus K$ each with a single neighbour in K. Suppose that $N(u) \cap K = \{k_u\}$ and $N(v) \cap K = \{k_v\}$ for distinct $k_u, k_v \in K$. Since H is $3P_1$-free, $uv \in E(H)$. But then $\{u, v, k_u, k_v\}$ induces a C_4 in H, a contradiction. Without loss of generality, $N(V(H) \setminus K) \cap K \subseteq \{k_u\}$. Now, since H is diamond-free and $V(H) \setminus K$ is a clique, $V(H) \setminus K$ is complete to $\{k_u\}$. It follows that $H = \overline{K_{s,t} + P_1}$ for some $s, t \geq 2$. ◀

▶ **Open Problem 4.** *For each integer $r \geq 4$, and for each integer $s, t \geq 2$, determine the (un)boundedness of the class of $(rP_1, \overline{K_{s,t} + P_1})$-free graphs.*

We note that Open Problem 4 includes the case $r = 4$, in contrast to Theorem 36, since the (un)boundedness of $(4P_1, \overline{K_{s,t} + P_1})$-free graphs is also open for $s \geq 2$ and $t \geq 2$. In fact, when $r = 3$, the (un)boundedness of $(3P_1, \overline{K_{s,t} + P_1})$-free graphs is also open except when $s = t = 2$, in which case we have the class of $(3P_1, \text{bowtie})$-free graphs, and so we can apply Theorem 20.

7 Conclusion

We extended the toolkit for proving (un)boundedness of mim-width of hereditary graph classes. Using the extended toolkit, we found new classes of (H_1, H_2)-free graphs of bounded width and unbounded mim-width. We showed that the situation for mim-width of hereditary graph classes is different from the situation for clique-width, even when only two induced subgraphs H_1 and H_2 are forbidden. For future work, Open Problems 1–4 deserve attention. In particular, the class of $(P_5, \overline{K_{1,r} + sP_1})$-free graphs, for $r \geq 3$ and $s \in \{1, 2\}$ (Case 1 of Open Problem 2), is the only remaining infinite family of pairs (H_1, H_2) where both H_1 and H_2 are connected. Moreover, for Open Problem 1, a similar approach to Theorem 18 might be conducive to resolving further open cases where $H_1 = 2P_2$.

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