Survival signature for reliability evaluation of a multi-state system with multi-state components

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Abstract: Survival signature technology has recently attracted increasing attention for its merits on quantifying reliability of systems with multiple types of components. In order to implement reliability evaluation of multi-state system (MSS), computing methods of survival signature are studied for reliability analysis of several different systems in this paper. For an MSS consisting of multi-state components, its survival signature can be developed based on the different state definition of system. For the binary-state system with multi-state components, its survival signature is based on the generalization of survival signature for multi-state components. For a real life engineering MSS consisting of subsystems, the computing method of survival signature of system has also been derived based on the survival signature of subsystems and mapping of subsystems' states to system's states. This enables consecutive application of the new method to substantial realistic MSS, with no theoretical limit on the size of the systems. Examples illustrate the applicability of the analysis approach for systems reliability.

Keywords: multi-state system; multi-state component; survival signature; reliability evaluation; subsystem

1. Introduction

With the rapid progress in science and technology, today's engineering systems are designed to be with more and more powerful and complex functions in recent years. At the same time, system risks and failures such as breakdown of a nuclear plant, miscommunication in internet and malfunction of an air traffic control, are also resulting in a greater economic losses and more significant effects on society than before. So reducing risks and improving performance and reliability of systems become the concerns of researchers. A controllable immune algorithm is developed to minimize the network risk in an integral routing risk model [1]. In wind power investment, risk preference problem is solved by multi-criteria decision-making and the feasibility is also verified by example [2]. For the risk assessment problem involving customer demands and power supply risk, a risk assessment model is proposed [3]. In order to improve the reliability of industrial wireless network, a routing algorithm based on redundant mechanism is proposed [4]. Computing method based on moment generation function is proposed and its reliability is also verified through simulation for the wireless communication networks [5]. A new second-order reliability analysis method is presented using saddle point approximating for system reliability assessment and comparison has been done with Monte Carlo simulation [6].

Nowadays, a lucid review of system signature has been presented to quantify the reliability of systems and networks, and some applications and theories of signatures have been developed in engineering systems [7-10]. However, it is not really possible when attempting to generalize it to systems with more than one component type, as it requires the computation of the probabilities of different orderings of order statistics of the different failure times distributions, which tends to intractable. In order to overcome the limitations of the system signature, survival signature has been developed as an effective tool for analyzing complex systems consisting of multiple component types[11].Therefore, with regard to reliability quantification for complex systems and networks, survival signature is often to be calculated once providing a massive reduction of the computational cost required by the analysis. Non-parametric predictive inference method has been developed for the reliability of system with multiple types of components using survival signature [12]. From a Bayesian perspective, both non-parametric and parametric methods using survival signature are presented for the reliability quantification of systems and networks [13]. Simulated method based on survival signature has been introduced to deal with upper and lower bounds of reliability function of system with uncertainty about parameters of assumed component failure times distributions [14]. Several algorithms for survival signature-based simulation are presented to answer the following question in the affirmative, whether or not the survival signature provides sufficient information for efficient simulation to derive the system's failure times distribution [15]. Expressions for marginal and joint reliability importance measures of a coherent system consisting of multiple types of dependent components are presented by utilizing the concept of survival signature [16]. An alternative to the existing limited approaches in the literature is provided by survival signature for reliability analysis of phased mission system with similar types of component in each phase [17]. The optimization of reliability-redundancy allocation is modeled by using the theory of survival signature and the information of the structure of a system is summarized by the survival signature [18].

In recent years, multi-state system (MSS) becomes focus because some intermediate states between "complete failure" and "perfect functioning" need to be considered in reliability analysis. Some concepts and applications have been applied in industrial system and management [19, 20]. An implicit two-stage approach is suggested for evaluating reliability functions of non-repairable series-parallel MSS with common cause failure (CCF) [21]. A method based on universal generating function (UGF) is put forward to evaluate the reliability and sensitivity of MSS with CCF [22]. And the UGF technology is also suitable for the MSS with protection mechanism and imperfect cover failure[23, 24]. Considering the maintenance action, Kalman filter and Markov process are employed to estimate state of system and component respectively [25, 26]. System reliability and life cycle cost analysis are assessed considering stochastic multiple degradation process [27]. Composite importance measures are evaluated and implemented via Monte-Carlo simulation for MSS with multi-state components [28]. Furthermore, a simulated algorithm has been put forward for reliability evaluation and prediction in multi-state coherent system with multi-state components [29]. For the MSS consisting of multi-state components with minor failure and minor reparation, a combined method has been studied based on UGF technology [30]. The theory of survival signature has been extended to allow for MSS models now, so as to represent MSS in system life reliability. Survival signature for a certain class of MSS, viz., multi-state consecutive-k-out-of-n: G system, is defined and studied [31]. The stress-strength reliability of MSS is also defined based on generalized survival signature for a certain class of MSS with multi-state components in both discrete and continuous cases [32]. The generalization of survival signature for MSS with typical unrepairable components is presented [31] and generalization for stress-strength reliability model with multi-state components is also developed [32]. Although the survival signature has been well studied for several typical MSS, no work has appeared in the literature to address MSS with binary-state components, binary-state system with multi-state components, MSS with multi-state component. In addition, there are few literatures on the calculation of survival signature of MSS consisting of subsystems in complicated configuration.

In this article, the MSS consisting of multi-state and/or binary-state components is considered to calculate its reliability function based on survival signature. Compared to the existed literatures, this paper develops a new method for reliability evaluation of MSS under some assumptions, e.g., structure of system is known and additional information such as survival signature of subsystems is needed. With regard to this advantage of this new method, it includes at least three aspects as follows. First, deriving the survival signature is a major challenge for general system, but only needs to be done once. But for multi-state systems where it can be build up from several subsystems, as presented in this paper, its survival signature is a computationally trivial exercise and can be well resolved by this method. Secondly, when the survival signature of system is available, statistical inference or simulation et cetera are computationally faster than when only the structure function is available. The advantage depends on the level of reduction achieved, on the structure of system, the number of components types and the number of components of the same types. The last is about the storage and calculation. Because of the combinational problem of different types and states of components, the computing cost will soar rapidly with the increase of number of components. For example, for a system with 3 types of components and each type of components have 3 ones. Its states are 3, 4 and 5 respectively. For each type of component, state combinations are 10, 20 and 35. So the state combinations of system will reach 7000 which is a large number and even is impossible if one directly computes the survival signature of each state combination and stores it by manual. Similar, complex computations are often involved in the computation of signature for coherent systems of large order and the computation of signature of system in terms of ones of subsystems or modules is also considered [33, 34]. Inspired by the derivation of survival signature of system consisting of subsystems [12, 35], this idea is also developed furthermore for the MSS. For the simplifying computation of reliability analysis using survival signature, a new computing method for MSS consisting of several subsystems and mapping from subsystems' states to system's states is also developed. The rest of this paper is organized as follows. In Section 2, the basic definitions, formulas and notations will be presented. Section 3 provides a state definition of MSS with binary-state components and method for reliability analysis based on survival signature. The generalization of survival signature for binary-state system with multi-state components is derived in Section 4. Section 5 presents the probabilities of the MSS with multi-state components at different states. Section 6 derives the computing approach of survival signature of MSS consisting of subsystems for reliability analysis. And its applicability is illustrated by a larger system. Finally, the concluding remarks are given in Section 7.

2. Survival signature

Consider a binary-state system with $K \geq 2$ types of binary-state components, the total number of component is m, with m_k components of type $k \in \{1, 2, \cdots, K\}$ and $\sum_{k=1}^{K} m_k = m$. We assume that the random failure times of components of the same type are independent and identity distribution (*iid*), while full independence is assumed for the random times of components of different types. Due to the arbitrary ordering of the components in the state vector, components of the same type can be grouped together, so state vector $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^k, \cdots, \mathbf{x}^K)$ can be expressed with the sub-vector $\mathbf{x}^k = (x_1^k, x_2^k, \cdots, x_{m_k}^k)$ representing the states of the components of type k. The structure function $\phi(\mathbf{x}) : \{0,1\}^m \to \{0,1\}$ can be defined for all possible combinations of $\mathbf{x}$.

Let $\Phi(l_1, l_2, \cdots, l_k, \cdots, l_K)$, for $l_k = 0, 1, \cdots, m_k$, denote the survival signature that a system functions given that exact l_k of its components of type k function [11]. There are $\binom{m_k}{l_k}$ state vectors $\mathbf{x}^k$ with exact l_k of its m_k components $x_i^k = 1$, so with $\sum_{i=1}^{m_k} x_i^k = l_k$. The set of these state vectors for components of type k can be denoted by S_l^k. Furthermore, let $S_{l_1, l_2, \cdots, l_k, \cdots, l_K}$ denote the set of all state vectors for the whole system for which $\sum_{i=1}^{m_k} x_i^k = l_k$. Due to the *iid* assumption for the failure times of m_k components of type k, all the state vectors $\mathbf{x}^k \in S_l^k$ are equally likely to occur, hence

$$\Phi(l_1, l_2, \cdots, l_k, \cdots, l_K) = \left(\prod_{k=1}^{K} \binom{m_k}{l_k} \right)^{-1} \times \sum_{\mathbf{x} \in S_{l_1, l_2, \cdots, l_k, \cdots, l_K}} \phi(\mathbf{x}) \qquad (1)$$

Let $C_t^k \in \{0, 1, \cdots, m_k\}$ denote the number of components of type k in the system that function at time $t > 0$. If we assume that the failure times of components of different types are independent and that the failure times of components of the same type are *iid* with cumulative distribution function (CDF) $F_k(t)$ for components of type k, then for $l_k \in \{0, 1, \cdots, m_k\}$, $k = 1, 2, \cdots, K$,

$$P\left(\bigcap_{k=1,2,\cdots,K} \{C_t^k = l_k\}\right) = \prod_{k=1}^{K} P(C_t^k = l_k) = \prod_{k=1}^{K} \left(\binom{m_k}{l_k} (F_k(t))^{m_k - l_k} (1 - F_k(t))^{l_k} \right) \qquad (2)$$

Let T be the random failure times of the system, then the probability that the system functions at time $t > 0$ is

$$
\begin{aligned}
P(T>t) &= \sum_{l_1=0}^{m_1}\cdots\sum_{l_K=0}^{m_K}\Phi(l_1,l_2,\cdots,l_k,\cdots,l_K)P(\bigcap_{k=1}^{K}\{C_t^k=l_k\}) \\
&= \sum_{l_1=0}^{m_1}\cdots\sum_{l_K=0}^{m_K}\left[\Phi(l_1,l_2,\cdots,l_k,\cdots,l_K)\prod_{k=1}^{K}P(C_t^k=l_k)\right] \\
&= \sum_{l_1=0}^{m_1}\cdots\sum_{l_K=0}^{m_K}\left[\Phi(l_1,l_2,\cdots,l_k,\cdots,l_K)\prod_{k=1}^{K}\left(\binom{m_k}{l_k}(F_k(t))^{m_k-l_k}(1-F_k(t))^{l_k}\right)\right]
\end{aligned} \tag{3}
$$

The main advantage of Eq. (3) is that the information about system structure is fully separated from the information about the components' failure times, and the inclusion of the failure times distribution is straightforward due to the assumed independence of failure times of components of different types. At the same time, it can be also applied if there is dependences between failures of components of different types [16]. For the special case where a system consists of only a single type of components, the survival signature is equivalent to system signature [7, 11]. When evaluating the reliability of system, the key procedure is to obtain the survival signature in terms of system structure. The following three examples illustrate this point.

3. MSS with binary-state components

In this case, an MSS can be modeled as follows. Because of binary-state characteristics of components, corresponding multi-state properties of system need to be defined firstly. The state definition of MSS can be introduced according to the numbers of component in one mini-path. Obviously, the more numbers of components in one mini-path, the higher failure probability in this mini-path is of. Let C_p denote the numbers of component in the min-path set of a system while it functions. This gives a definition of system state denoted by random variable $H \in \{0,1,\cdots,h_p,\cdots,h_{\max}\}$, $h_{\max}$ is the maximum value for the best state, h_p for any intermediate state and 0 for "complete failure" state of the considered system. Based on the survival signature, it can be grouped by C_p corresponding to the state H. When the survival function of MSS at different state $H = h$ needs to be calculated, the survival signature for the whole system at the state $H = h$ needs to be obtained firstly. For the MSS, we can define the survival signature as the probability that the system is in state $H = h$ given that l_k components of type k function. We denote this probability by $\Phi_{H=h}(l_1, l_2, \cdots, l_k, \cdots, l_K)$. Here we introduce H_t as the system state at time $t > 0$, then based on Eq. (3), the probability of system at state h can be denoted by

$$\begin{aligned}
P(H_t = h) &= \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi_{H=h}(l_1, l_2, \cdots, l_k, \cdots, l_K) P(\bigcap_{k=1}^{K} \{C_t^k = l_k\}) \\
&= \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \left[\Phi_{H=h}(l_1, l_2, \cdots, l_k, \cdots, l_K)) \prod_{k=1}^{K} P(C_t^k = l_k) \right] \\
&= \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \left[\Phi_{H=h}(l_1, l_2, \cdots, l_k, \cdots, l_K)) \prod_{k=1}^{K} \left(\binom{m_k}{l_k} (F_k(t))^{m_k - l_k} (1 - F_k(t))^{l_k} \right) \right]
\end{aligned} \tag{4}$$

Furthermore, the reliability function $R_{h_p}(t)$ of multi-state system at a state h_p and better state can be

$$R_{h_p}(t) = \sum_{h=h_p}^{h_{\max}} P(H_t = h) \tag{5}$$

If there are three different values of C_p for the system, then $H \in \{0,1,2\}$. Obviously, complete failure state $H = 0$ is corresponding to the $C_p = 0$, perfect function state $H = 2$ is the minimum value of C_p in all its mini-path set and the others for the intermediate state $H = 1$.

Example 1.

The use of the survival signature for a system with $K = 2$ types of components is illustrated as follows. First, the system structure is presented in Fig. 1, there are three same components for each type component. They are labeled with the number 1, 2, 3 for type 1 and 4, 5, 6 for type 2. Component 1 is connected in series with the others which form a bridge structure.

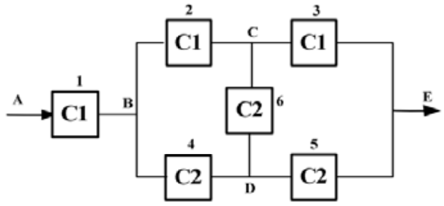

Fig. 1. System with two types of components.

In this example, the state 0 is for $C_p = 0$. For more specification, there is no a mini-path from A to E. So, the number of C_p is zero. The perfect state 2 is for $C_p = 3$. For example, the mini-path ABCE functions when the components of type 1 function. And the intermediate state 1 is for $C_p = 4$. In this state, one mini-path ABCDE contains four components with two types. More components connected in series leads to the risk of system failure.

Secondly, survival signatures need to be calculated. With $m_1 = m_2 = 3$ components of each type, the survival signature $\Phi_{H=h}(l_1, l_2)$ for three states of the system is listed in Table 1.

Orders	l_1	l_2	$\Phi_{H=h}(l_1,l_2)$ $h=0$	$h=1$	$h=2$
1	0	0	1	0	0
2	0	1	1	0	0
3	0	2	1	0	0
4	0	3	1	0	0
5	1	0	1	0	0
6	1	1	1	0	0
7	1	2	8/9	0	1/9
8	1	3	2/3	0	1/3
9	2	0	1	0	0
10	2	1	1	0	0
11	2	2	5/9	2/9	2/9
12	2	3	1/3	0	2/3
13	3	0	0	0	1
14	3	1	0	0	1
15	3	2	0	0	1
16	3	3	0	0	1

Table 1 Survival signatures for different states of system.

From the above table, it can be found that only survival signature of $\Phi_{H=1}(2,2)$ is not equal to zero in state 1. It can be obtained from Table 2 according to the value of C_p. The labels of components are as represented in Fig. 1.

Orders	State combination of components						Min-path	C_p	State H
	1	2	3	4	5	6			
1	1	1	0	1	1	0	ABDE	3	2
2	1	1	0	1	0	1	-	0	0
3	1	1	0	0	1	1	ABCDE	4	1
4	1	0	1	1	1	0	ABDE	3	2
5	1	0	1	1	0	1	ABDCE	4	1
6	1	0	1	0	1	1	-	0	0
7	0	1	1	1	1	0	-	0	0
8	0	1	1	1	0	1	-	0	0
9	0	1	1	0	1	1	-	0	0

Table 2 Different states for survival signature $\Phi_{H=1}(2,2)$.

Finally, we consider two different cases according to their failure distributions. In Case A, the failure distribution of type 1 components C1 is an Exponential distribution with expected value 1, so with

$$f_1(t) = e^{-t} \quad \text{and} \quad F_1(t) = 1 - e^{-t} \tag{6}$$

and the failure distribution of type 2 components C2 is a Weibull distribution with shape parameter 2 and scale parameter 1, so with

$$f_2(t) = 2te^{-t^2} \quad \text{and} \quad F_2(t) = 1 - e^{-t^2} \tag{7}$$

According to Eq. (4), the probabilities of system at different state can be plotted as Fig. 2.

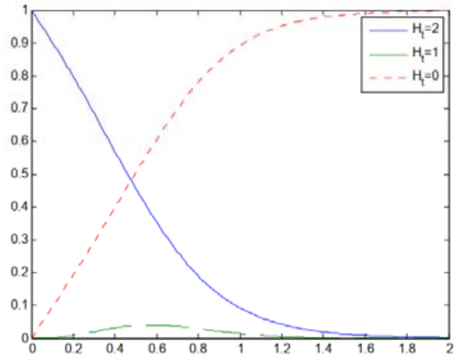

Fig. 2. Probabilities that the system is in state h at time t for $h=0,1,2$.

For comparison of reliability function at different states, here two types of reliability functions at the perfect state $R_2(t)$ and intermediate state $R_1(t)$ are depicted respectively as follows.

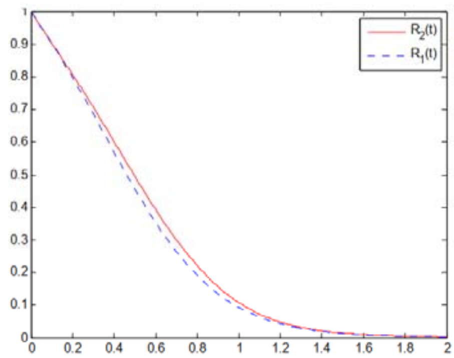

Fig. 3. Comparison for reliability functions at different states.

Another case is Case B. In this case, these same failure times distributions are used but for the other components types than in Case A. So the failure distribution of C1 has the above Weibull distribution, while the failure distribution of C2 has the above Exponential distribution. In Case B, its probability distribution at different state can be also calculated like the plot in Case A. For the purpose of contrast, the reliability function in Case A and B at its perfect states are depicted as Fig. 4.

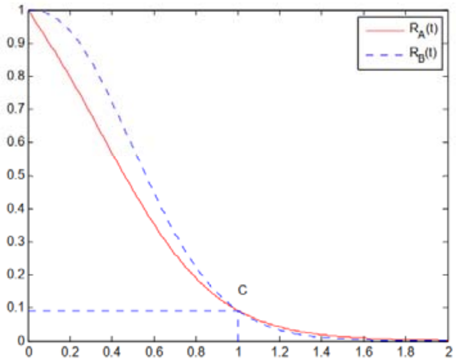

Fig. 4. Reliability functions of Case A and B.

From the Fig. 4, the reliability function of Case A is lower than that of Case B before the point C=(1,0.0928). It shows that Type 1 components are a bit more critical in this system because of the left-most component. Meanwhile, it shows that the Exponential distribution makes early failures more likely than the Weibull distribution used in this example. However, it is interesting that the reliability function of Case A, after the point C, has a higher value than that of Case B. This also shows it would have been difficult to predict exactly which case can make system run at a higher reliability level without the detailed computations.

4. Binary-state system with multi-state components

Unlike the model in Section 3, the components here are of multiple states. For the k^{th} type of multi-state component with $s_k \in \{0,1,\cdots,s_k^{\max}\}$ states, where the state 0 is for the complete failure and $s_k^{\max}$ for the perfect function, the first thing is to determine the probability distribution $P_{s_k}^k(t)$ at different states. Due to the multi-state of component, survival signature of system also needs to be generalized. According to the combinations of different types of component's states, the survival signature of the system $\Phi(l_1,l_2,\cdots,l_k,\cdots,l_K)$ can be generalized to $\Phi(l_{1,0},l_{1,1},\cdots,l_{1,s_1^{\max}},\cdots,l_{k,i},\cdots,l_{K,0},l_{K,1},\cdots,l_{K,s_K^{\max}}), k=1,\cdots,K, i=0,\cdots,s_k^{\max}$, where $l_{k,i}$ is the number of components of type k which are in state i. At the same time, we need to introduce notation $C_{t,i}^{k}$ to express the exact number of component of type k which are in state i at time $t > 0$. Here the state of system can be defined as only two states, one is for the complete failure and the other is for the functioning state. Based on the Eq. (3), the generalized survival signature will be adopted and the survival function can be derived by

$$\begin{aligned} P(T>t) &= \sum_{l_{1,0}=0}^{m_1} \cdots \sum_{l_{K,s_K^{\max}}=0}^{m_K} \left[\Phi(l_{1,0}, l_{1,1}, \cdots, l_{1,s_1^{\max}}, \cdots, l_{k,i}, \cdots, l_{K,0}, l_{K,1}, \cdots, l_{K,s_K^{\max}}) P\left(\bigcap_{k=1}^{K} \bigcap_{i=0}^{s_k^{\max}} \{C_{t,i}^{k} = l_{k,i}\}\right) \right] \\ &= \sum_{l_{1,0}=0}^{m_1} \cdots \sum_{l_{K,s_K^{\max}}=0}^{m_K} \left[\Phi(l_{1,0}, l_{1,1}, \cdots, l_{1,s_1^{\max}}, \cdots, l_{k,i}, \cdots, l_{K,0}, l_{K,1}, \cdots, l_{K,s_K^{\max}}) \prod_{k=1}^{K} \prod_{i=0}^{s_k^{\max}} P(C_{t,i}^{k} = l_{k,i}) \right] \\ &= \sum_{l_{1,0}=0}^{m_1} \cdots \sum_{l_{K,s_K^{\max}}=0}^{m_K} \left[\Phi(l_{1,0}, l_{1,1}, \cdots, l_{1,s_1^{\max}}, \cdots, l_{k,i}, \cdots, l_{K,0}, l_{K,1}, \cdots, l_{K,s_K^{\max}}) \prod_{k=1}^{K} \left(\frac{m_k!}{l_{k,0}!\, l_{k,1}! \cdots l_{k,s_k^{\max}}!} (P_0^k(t))^{l_{k,0}} (P_1^k(t))^{l_{k,1}} \cdots (P_{s_k^{\max}}^k(t))^{l_{k,s_k^{\max}}} \right) \right] \end{aligned} \quad (8)$$

where $\dfrac{m_k!}{l_{k,0}!\, l_{k,1}! \cdots l_{k,s_k^{\max}}!} = \binom{m_k}{l_{k,0}} \binom{m_k - l_{k,0}}{l_{k,1}} \cdots \binom{m_k - l_{k,0} - \cdots - l_{k,s_k^{\max}-1}}{l_{k,s_k^{\max}}}$.

Example 2.

Here, we still take the Fig. 1 for an example. The failure distributions of components are calculated as follows. Let component's state transition times follow the exponential distribution and comply with the Markov process, its probability distribution can be solved by corresponding differential equations built by its state transmission process. For the illustration of this process, we can assume that the component $C1$ in Fig. 1 is a multi-state component with 3 states. State 2 is for the perfect function, 1 for intermediate state and 0 for the complete failure. Here, repair actions are not included, viz. the components are not repairable. Its state transmission can be depicted by Fig. 5 and parameters such as transmission rates from a higher state to a lower one are also noted in it. Here, consider the case when only minor failures occur.

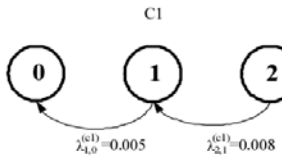

Fig. 5. State transmission of component $C1$.

Based on the Fig. 5, the differential equations can be built as follows

$$\begin{cases} \dfrac{dP_0^1(t)}{dt} = \lambda_{10} P_1^1(t) \\ \dfrac{dP_1^1(t)}{dt} = \lambda_{21} P_2^1(t) - \lambda_{10} P_1^1(t), \\ \dfrac{dP_2^1(t)}{dt} = -\lambda_{2,1} P_2^1(t) \end{cases} \tag{9}$$

with the initial conditions

$$\begin{cases} P_0^1(0) = P_1^1(0) = 0 \\ P_2^1(0) = 1 \end{cases}. \tag{10}$$

Here, After the solving of the equation (9) and (10), its probability distribution can be obtained as

$$\begin{cases} P_0^1(t) = \dfrac{\lambda_{21} \times e^{-\lambda_{10}t} - \lambda_{10} \times e^{-\lambda_{21}t}}{\lambda_{10} - \lambda_{21}} + 1 \\ P_1^1(t) = \dfrac{\lambda_{21}}{\lambda_{10} - \lambda_{21}} \times (e^{-\lambda_{21}t} - e^{-\lambda_{10}t}) \\ P_2^1(t) = e^{-\lambda_{21}t} \end{cases}. \tag{11}$$

For the binary state component C2 of type 2, its probability distribution is assigned same as Weibull distribution. The differences are the changing of parameters, so its probability distribution is expressed as

$$\begin{cases} P_0^2(t) = 1 - \mathrm{e}^{(-(\frac{t}{300})^3)} \\ P_1^2(t) = 1 - P_0^2(t) \end{cases}. \tag{12}$$

Next is to calculate survival signature for this case. To generalize the survival signature for the multi-state component, denote the exact number of components in the specified state by l_{k,m_k}, where k is for the type of component and m_k for the state of component. For the purpose of illustration, the different state combination of component C1 in Fig. 1 is listed in the following Table 3.

Index_1	l_1		
	$l_{1,0}$	$l_{1,1}$	$l_{1,2}$
1	3	0	0
2	2	1	0
3	2	0	1
4	1	2	0
5	1	1	1
6	1	0	2
7	0	3	0
8	0	2	1
9	0	1	2
10	0	0	3

Table 3 Combinations of component C1 at different states.

Similarly, the combinations of component C2 in Fig. 1 at two states are listed in Table 4.

Index_2	l_2	
	$l_{2,0}$	$l_{2,1}$
1	3	0
2	2	1
3	1	2
4	0	3

Table 4 Combinations of component C2 at two states.

In total, there are 10*4 combinations for the two types of components at different states. Survival signature for the system at different states can be figured out as Table 5. In this table, some combinations have the same values. For example, the combinations of Index_1, where ordered value is equal to 10, there are totally 4 items, have the vector (0 1) for the system states. Some combinations of Index_1, where ordered value is between 1 and 9, there are totally 26 items, have the same values (1 0) for the system states. Here, the principle of system function is based on the state of component 1. If state of component 1 is not in state 2, the system will not function.

Order of Combination	$Index_1$	$Index_2$	$\Phi_{H=h}(l_{1,0},l_{1,1},l_{1,2},l_{2,0},l_{2,1})$	
			$h=0$	$h=1$
1	1	1	1	0
2	1	2	1	0
3	1	3	1	0
4	1	4	1	0
5	2	1	1	0
6	2	2	1	0
7	2	3	1	0
8	2	4	1	0
9	3	1	1	0
10	3	2	1	0
11	3	3	8/9	1/9
12	3	4	2/3	1/3
13	4	1	1	0
14	4	2	1	0
15	4	3	1	0
16	4	4	1	0
17	5	1	1	0
18	5	2	1	0
19	5	3	8/9	1/9
20	5	4	2/3	1/3
21	6	1	1	0
22	6	2	1	0
23	6	3	5/9	4/9
24	6	4	1/3	2/3
25	7	1	1	0
26	7	2	1	0
27	7	3	1	0
28	7	4	1	0
29	8	1	1	0
30	8	2	1	0
31	8	3	2/3	1/3
32	8	4	2/3	1/3
33	9	1	1	0
34	9	2	1	0
35	9	3	1/3	2/3
36	9	4	1/3	2/3
37	10	1	0	1
38	10	2	0	1
39	10	3	0	1
40	10	4	0	1

Table 5 Survival signatures for the system with two states.

To explain the calculating procedure of survival signatures in Table 5, an example is illustrated as follows. For the 19th order of combination, viz. the tuple (Index_1, Index_2) = (5, 3), so with $(l_{1,0}, l_{1,1}, l_{1,2}, l_{2,0}, l_{2,1}) = (1,1,1,1,2)$. Under this combination of numbers of components at different states, the concrete state of each component in the system and its corresponding state of system is listed in the following Table 6. As shown in this table, there are totally 18 combinations of component's state under this combination $(l_{1,0}, l_{1,1}, l_{1,2}, l_{2,0}, l_{2,1}) = (1,1,1,1,2)$. However, there is only two combinations which make the system be in the function state. So, its survival signature is equal to 1/9 for function state and 8/9 for failure state.

Order of component state combination	$(l_{1,0}, l_{1,1}, l_{1,2}) = (1,1,1)$			$(l_{2,0}, l_{2,1}) = (1,2)$			State of system
	Label of component						
	1	2	3	4	5	6	
1	0	1	2	0	1	1	0
2	0	1	2	1	1	0	0
3	0	1	2	0	1	1	0
4	0	2	1	0	1	1	0
5	0	2	1	1	1	0	0
6	0	2	1	0	1	1	0
7	1	0	2	0	1	1	0
8	1	0	2	1	1	0	0
9	1	0	2	0	1	1	0
10	1	2	0	0	1	1	0
11	1	2	0	1	1	0	0
12	1	2	0	0	1	1	0
13	2	1	0	0	1	1	0
14	2	1	0	1	1	0	1
15	2	1	0	0	1	1	0
16	2	0	1	0	1	1	0
17	2	0	1	1	1	0	1
18	2	0	1	0	1	1	0

Table 6 Explanation of survival signature of $(l_{1,0}, l_{1,1}, l_{1,2}, l_{2,0}, l_{2,1}) = (1,1,1,1,2)$.

Based on the Eq. (8), (11) and (12), the reliability function can be plotted as Fig. 6. When using the Eq. (8) in this example, the survival signature of system is taken from the last column of Table 5, viz. the survival signature at state $h = 1$.

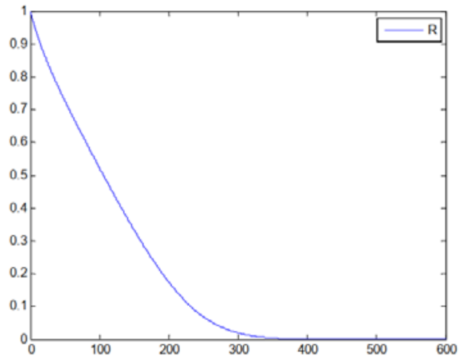

Fig. 6. Reliability function of binary-state system with multi-state components.

5. MSS with multi-state components

In this case, the state of MSS needs to be defined firstly according to the physical structural of system. Then, combining Eq. (4), (5) and (8), we can calculate the probability of system at different state $H_t = h$ as follows

$$\begin{aligned}\tilde{P}(H_t = h) &= \sum_{l_{1,0}=0}^{m_1} \cdots \sum_{l_{K,s_K^{\max}}=0}^{m_K} \left[\Phi_{H=h}(l_{1,0}, l_{1,1}, \cdots, l_{1,s_1^{\max}}, \cdots, l_{k,i}, \cdots, l_{K,0}, l_{K,1}, \cdots, l_{K,s_K^{\max}}) P(\bigcap_{k=1}^{K} \bigcap_{i=0}^{s_k^{\max}} \{C_{t,i}^k = l_{k,i}\}) \right] \\ &= \sum_{l_{1,0}=0}^{m_1} \cdots \sum_{l_{K,s_K^{\max}}=0}^{m_K} \left[\Phi_{H=h}(l_{1,0}, l_{1,1}, \cdots, l_{1,s_1^{\max}}, \cdots, l_{k,i}, \cdots, l_{K,0}, l_{K,1}, \cdots, l_{K,s_K^{\max}}) \prod_{k=1}^{K} \prod_{i=0}^{s_k^{\max}} P(C_{t,i}^k = l_{k,i}) \right] \\ &= \sum_{l_{1,0}=0}^{m_1} \cdots \sum_{l_{K,s_K^{\max}}=0}^{m_K} \left[\Phi_{H=h}(l_{1,0}, l_{1,1}, \cdots, l_{1,s_1^{\max}}, \cdots, l_{k,i}, \cdots, l_{K,0}, l_{K,1}, \cdots, l_{K,s_K^{\max}}) \prod_{k=1}^{K} \left(\frac{m_k!}{l_{k,0}! l_{k,1}! \cdots l_{k,s_k^{\max}}!} (P_0^k(t))^{l_{k,0}} (P_1^k(t))^{l_{k,1}} \cdots (P_{s_k^{\max}}^k(t))^{l_{k,s_k^{\max}}} \right) \right]\end{aligned} \tag{13}$$

Example 3.

Considering the system structure in Fig. 1, the failure distributions of components are same with those in Example 2. Based on the state of component labeled 1, the state of system can be defined as follows. Because component of type 1, viz. C1, has three states, and the location of component labeled 1 is crucial for the function of overall system. The state of system function can be defined based on the component labeled 1. That is to say, component labeled 1 works at the state of 1, the system state can be defined as intermediate state 1. Component labeled 1 works at the state of 2, the system state can be defined as the perfect state 2. When the component labeled 1 cannot work, the system will not function too. So, the system can be defined as failure state 0. Combining with the definition of system state, we can calculate the survival signature of system.

According to the above analyzing, survival signatures of system working at different states are listed in Table 7.

Order of Combination	Index_1	Index_2	$\Phi_{H=h}(l_{1,0},l_{1,1},l_{1,2},l_{2,0},l_{2,1})$		
			$h=0$	$h=1$	$h=2$
1	1	1	1	0	0
2	1	2	1	0	0
3	1	3	1	0	0
4	1	4	1	0	0
5	2	1	1	0	0
6	2	2	1	0	0
7	2	3	8/9	1/9	0
8	2	4	2/3	1/3	0
9	3	1	1	0	0
10	3	2	1	0	0
11	3	3	8/9	0	1/9
12	3	4	2/3	0	1/3
13	4	1	1	0	0
14	4	2	1	0	0
15	4	3	5/9	4/9	0
16	4	4	1/3	2/3	0
17	5	1	1	0	0
18	5	2	1	0	0
19	5	3	5/9	2/9	2/9
20	5	4	1/3	1/3	1/3
21	6	1	1	0	0
22	6	2	1	0	0
23	6	3	5/9	0	4/9
24	6	4	1/3	0	2/3
25	7	1	0	1	0
26	7	2	0	1	0
27	7	3	0	1	0
28	7	4	0	1	0
29	8	1	0	2/3	1/3
30	8	2	0	2/3	1/3
31	8	3	0	2/3	1/3
32	8	4	0	2/3	1/3

33	9	1	0	1/3	2/3
34	9	2	0	1/3	2/3
35	9	3	0	1/3	2/3
36	9	4	0	1/3	2/3
37	10	1	0	0	1
38	10	2	0	0	1
39	10	3	0	0	1
40	10	4	0	0	1

Table 7 Survival signatures for system at different states.

Based on the above Eq. (13), the probability of system at states 0, 1 and 2 can be depicted as Fig. 7.

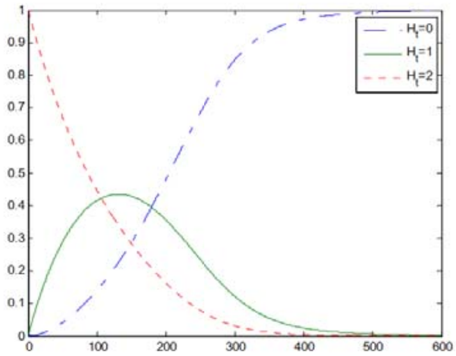

Fig. 7. Probabilities that the system is in state h at time t for $h=0,1,2$ ..

Then, the reliability function of system at state h_p can be calculated by

$$\tilde{R}_{h_p}(t)=\sum_{h=h_p}^{h_{\max}}\tilde{P}(H_t=h) \tag{14}$$

According to this formula, reliability functions at different states are depicted respectively as follows

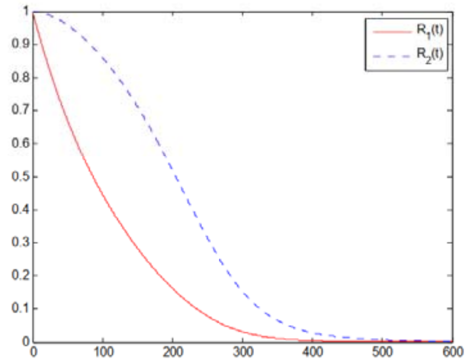

Fig. 8. Reliability functions of system at different states.

From the above figure, it is also to validate the effectiveness of this analyzing method for multi-state system reliability by survival signature.

6. Computing the survival signature of a system consisting of subsystems

In practical industrial system, a whole system is often composed of several subsystems. So, the survival signatures of subsystem and states mapping from subsystems to system play an important role in the reliability evaluation of multi-state system. Considering a system consisting of two subsystems where survival signatures are known, and the state mapping from subsystems to system can be determined by its configuration. Let the system consist of $K \geq 1$ types of components, with m_k components of type k, for $k = 1, \cdots, K$, of which $m_k^r \geq 0$ are in subsystem r, for $r = 1, 2$. Let subsystem r consist in total of m^r components, so $m^r = \sum_{k=1}^{K} m_k^r$. The survival signature for subsystem r can be denoted by $\Phi^r(l_1^r, l_2^r, \cdots, l_k^r, \cdots, l_K^r)$, for $l_k^r = 0, 1, \cdots, m_k^r$.

Example 4.

Suppose that a MSS with the following configuration shown as Fig. 9, there are three types of components C1, C2 and C3. C1 is a multi-state component with three states 0, 1 and 2. C2 and C3 are only of two states 0 and 1. If state of each type of component k can be expressed by $s_k \in \{0,1,\cdots,s_k^{\max}\}$, of which $s_k^{\max}$ is for the perfect function state, 0 for the complete failure state and others for intermediate states, then $s_1 \in \{0,1,2\}$, $s_2 \in \{0,1\}$ and $s_3 \in \{0,1\}$. Along each component, there is a label with number in order to use conveniently. Component 1 and 2 constitute a subsystem SubSys1. The left components make up another subsystem SubSys2.

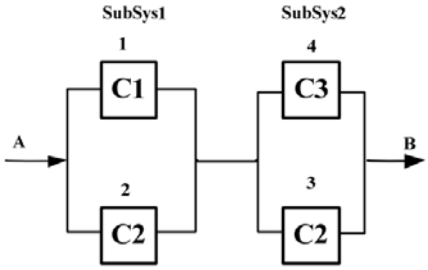

Fig. 9. A system with two subsystems.

Denote the exact number of component of type k at state s_k in subsystem r by l_{k,s_k}^r, subsystem's survival signature can be rewritten by $\Phi_{H=h}^{s1}(l_{1,0}^1, l_{1,1}^1, \cdots, l_{1,s_1^{\max}}^1, l_{2,0}^1, l_{2,1}^1, \cdots, l_{2,s_2^{\max}}^1)$, of which h is the state of subsystem SubSys1 denoted by $s1$. The state of subsystem is defined as follows. State 0 is for the complete failure. State 2 is for the perfect function that is defined as each component of path is in the best state. State 1 is for the others. Based on the above definition, the analyzing of survival signature of subsystem SubSys1 can be shown as Table 8. The second line of table indicates the states of components and subsystem.

Order of combination	$l_1^1 = l_{1,0}^1 + l_{1,1}^1 + l_{1,2}^1 = 1$			$l_2^1 = l_{2,0}^1 + l_{2,1}^1 = 1$		$l_3^1 = l_{3,0}^1 + l_{3,1}^1 = 0$		$\Phi_{H=h}^{s1}(l_{1,0}^1, l_{1,1}^1, l_{1,2}^1, l_{2,0}^1, l_{2,1}^1, l_{3,0}^1, l_{3,1}^1)$		
	$l_{1,0}^1$	$l_{1,1}^1$	$l_{1,2}^1$	$l_{2,0}^1$	$l_{2,1}^1$	$l_{3,0}^1$	$l_{3,1}^1$	$h=0$	$h=1$	$h=2$
1	1	0	0	1	0	0	0	1	0	0
2	1	0	0	0	1	0	0	0	1	0
3	0	1	0	1	0	0	0	0	1	0
4	0	1	0	0	1	0	0	0	1	0
5	0	0	1	1	0	0	0	0	1	0
6	0	0	1	0	1	0	0	0	0	1

Table 8 Survival signatures of subsystem SubSys1.

Similarly, survival signatures of $s2$ can be easily calculated as follows. Survival signatures of SubSys2 are listed in Table 9.

Order of combination	$l_1^2 = l_{1,0}^2 + l_{1,1}^2 + l_{1,2}^2 = 0$			$l_2^2 = l_{2,0}^2 + l_{2,1}^2 = 1$		$l_3^2 = l_{3,0}^2 + l_{3,1}^2 = 1$		$\Phi_{H=h}^{s2}(l_{1,0}^2, l_{1,1}^2, l_{1,2}^2, l_{2,0}^2, l_{2,1}^2, l_{3,0}^2, l_{3,1}^2)$		
	$l_{1,0}^2$	$l_{1,1}^2$	$l_{1,2}^2$	$l_{2,0}^2$	$l_{2,1}^2$	$l_{3,0}^2$	$l_{3,1}^2$	$h=0$	$h=1$	$h=2$
1	0	0	0	1	0	1	0	1	0	0
2	0	0	0	0	1	1	0	0	1	0
3	0	0	0	1	0	0	1	0	1	0
4	0	0	0	0	1	0	1	0	0	1

Table 9 Survival signature of SubSys2.

Before calculating the survival signature of system, the mapping rules to define the system state need to be presented firstly. For two subsystem connected in this example, the rules are listed in Table 10.

Order of rules	States		
	SubSys1	SubSys2	System
1	0	>=0	0
2	>=0	0	0
3	1	>=1	1
4	>=1	1	1
5	2	2	2
6	at different probability at each state	at different probability at each state	Sum the corresponding probability and divide by the number of combination evenly

Table 10 Mapping rules of system's state for two subsystems.

After the survival signatures of two subsystems are obtained, then survival signature of whole system can be further achieved by a programming method. The following is a developed approach for the calculating of survival signature of two subsystems according to the mapping rules.

Assume the survival signature of subsystem 1 and subsystem 2 is stored in matrix A and B respectively. The survival signatures of the system composed by those two subsystems are going to be stored in matrix C. Here, the structures of matrix A, B and C, have the same dimensions with the Table 8, Table 9 and Table 11. That is to say, the former 7 columns are the combination of different states for different types of components and the last 3 columns are the survival signature of system at different states. However, the rows of each matrix can vary according to the state number and component number in each system. Then, according to the following steps, we will procure the survival signature of the system.

Step 1. Define a variable k to express the ordinal row of matrix C and assign the initial value k=1;

Step 2. Initiate several variables such as c=0, s=[0 0 0] and temp1 to store the state combination of system at the k row. Here, it can be defined to temp1=C(k,1:7);

Step 3. Define a variables i to store the ordinal number of row in matrix A. The initial value of this variable is i=1;

Step 4. Similarly, define another variable j to store the ordinal number of row in matrix B. The initial value of this variable is j=1;

Step 5. Assemble the variable temp2 with the state combination of components in two subsystems, viz., temp2=A(i,1:7)+B(j,1:7);

Step 6. If temp1=temp2, then increase c by 1. Whilst according to the state of two subsystems, deduce the state of system and store it to the variable s. Here, s should be operated by cumulative adding and return to Step 5 again with increasing j by 1;

Step 7. Else, also return to the Step 5 with increasing j by 1;

Step 8. When the variable j arrives at the maximum, viz., the total number of row in B, return to the Step 3 with adding i by 1;

Step 9. When variable i reaches its maximum value, viz., the total number of row in A, divide s with c and assemble s to the state of system. Meantime, s should be placed in the k row of matrix C, viz., C(k,8:10)=s/c;

Step 10. Return to the Step 1 and increase variable k by 1. Repeat the left steps until the k reaches its maximum.

For the purpose of more clarity about the above steps, the flow chart Fig. 10 can be depicted as follows.

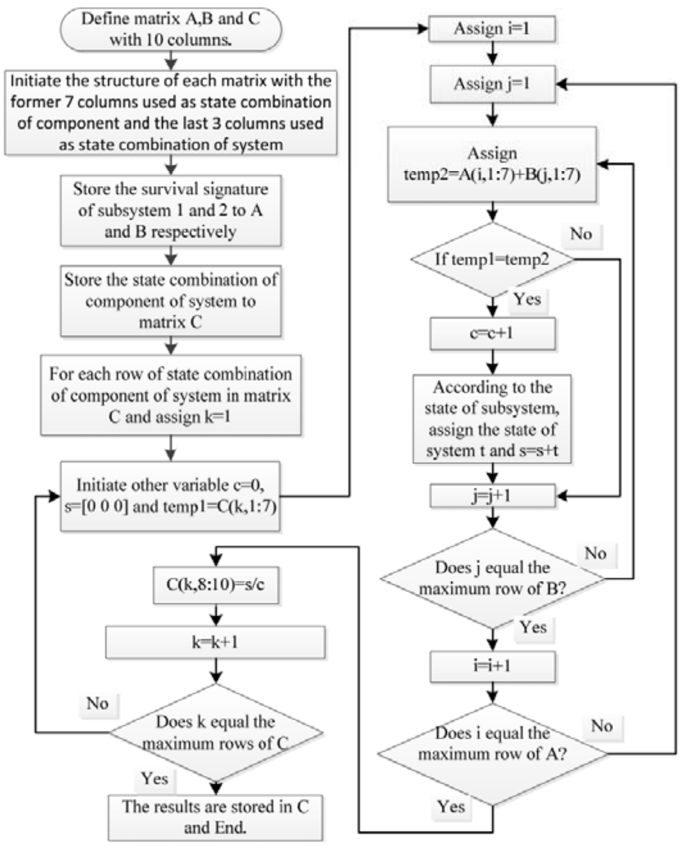

Fig. 10. Flow chart for calculating survival signatures of system consisting of two sub systems.

After running for the above program, the survival signature of system can be shown in Table 11.

Order of combination	$l_1 = l_{1,0}+l_{1,1}+l_{1,2} = 1$			$l_2 = l_{2,0}+l_{2,1} = 2$		$l_3 = l_{3,0}+l_{3,1} = 1$		$\Phi^s_{H=h}(l_{1,0},l_{1,1},l_{1,2},l_{2,0},l_{2,1},l_{3,0},l_{3,1})$		
	$l_{1,0}$	$l_{1,1}$	$l_{1,2}$	$l_{2,0}$	$l_{2,1}$	$l_{3,0}$	$l_{3,1}$	$h=0$	$h=1$	$h=2$
1	1	0	0	2	0	1	0	1	0	0
2	1	0	0	2	0	0	1	1	0	0
3	1	0	0	1	1	1	0	1	0	0
4	1	0	0	1	1	0	1	1/2	1/2	0
5	1	0	0	0	2	1	0	0	1	0
6	1	0	0	0	2	0	1	0	1	0
7	0	1	0	2	0	1	0	1	0	0
8	0	1	0	2	0	0	1	0	1	0
9	0	1	0	1	1	1	0	1/2	1/2	0
10	0	1	0	1	1	0	1	0	1	0
11	0	1	0	0	2	1	0	0	1	0
12	0	1	0	0	2	0	1	0	1	0
13	0	0	1	2	0	1	0	1	0	0
14	0	0	1	2	0	0	1	0	1	0
15	0	0	1	1	1	1	0	1/2	1/2	0
16	0	0	1	1	1	0	1	0	1	0
17	0	0	1	0	2	1	0	0	1	0
18	0	0	1	0	2	0	1	0	0	1

Table 11 Survival signatures of the whole system.

For the purpose of verifying the validating of above program, we can take the 9th combination for an example. Under this situation, the state of component and system is listed in Table 12. The number of system state in state 0 and 1 is both equal to 1. So, the survival signature of system state at state 0 and state 1 is both equal to 1/2. These values are same as the values obtained from the above program.

Orders	State of component				State of subsys1	State of subsys2	State of system		
	1	2	3	4			$h=0$	$h=1$	$h=2$
1	1	0	1	0	1	1	0	1	0
2	1	1	0	0	1	0	1	0	0

Table 12 Details of the 9th combination in Table 11.

Here, the probability distribution of component type 1 is same as Eq. (11). As for component C2, which is of two states, its Weibull probability distribution is same as Eq. (12). The component C3 is of the same Weibull type but different parameters, it can be expressed by

$$\begin{cases} P_0^3(t) = 1 - \mathrm{e}^{(-(\frac{t}{500})^5)} \\ P_1^3(t) = 1 - P_0^3(t) \end{cases} \tag{15}$$

The probability of system at different state can be calculated by Eq. (13). Combining Eq. (13) and survival signature from Table 11, the probability of system at three states can be depicted as Fig. 11.

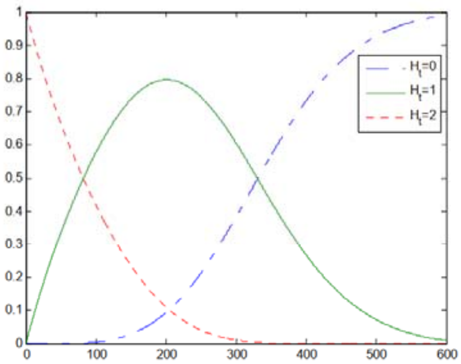

Fig. 11. Probabilities that the system is in state h at time t for $h = 0,1,2$.

Then the reliability function of system at h_p state can be calculated by Eq. (14). Based on Eq. (14), the reliability function of system at state 1 and 2 can be depicted as Fig. 12.

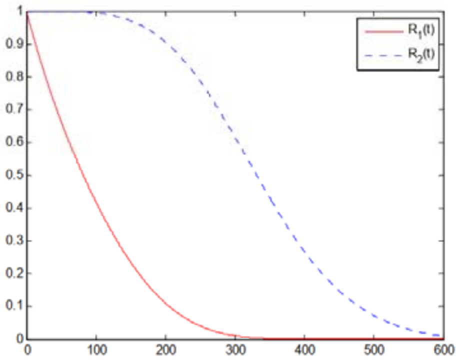

Fig. 12. Reliability functions of system at different states.

Example 5.

Here is another example of MSS consisting of subsystems connected in different configuration. Taking the following Fig. 13 as an example where the system consists of two subsystems, SubSys1 and SubSys2. Survival signature of system can be obtained similarly.

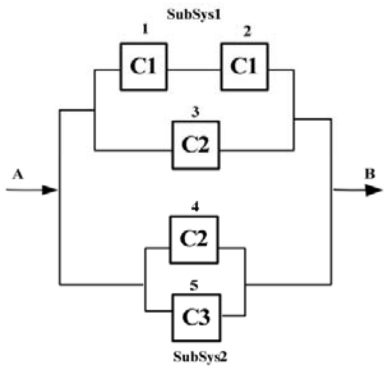

Fig. 13. Subsystems connected in different configuration.

The survival signature of SubSys1 is listed in Table 13.

Order of combination	$l_1^1 = l_{1,0}^1 + l_{1,1}^1 + l_{1,2}^1 = 2$			$l_2^1 = l_{2,0}^1 + l_{2,1}^1 = 1$		$l_3^1 = l_{3,0}^1 + l_{3,1}^1 = 0$		$\Phi_{H=h}^{s1}(l_{1,0}^1, l_{1,1}^1, l_{1,2}^1, l_{2,0}^1, l_{2,1}^1, l_{3,0}^1, l_{3,1}^1)$		
	$l_{1,0}^1$	$l_{1,1}^1$	$l_{1,2}^1$	$l_{2,0}^1$	$l_{2,1}^1$	$l_{3,0}^1$	$l_{3,1}^1$	$h=0$	$h=1$	$h=2$
1	2	0	0	1	0	0	0	1	0	0
2	1	1	0	1	0	0	0	1	0	0
3	1	0	1	1	0	0	0	1	0	0
4	0	2	0	1	0	0	0	0	1	0

5	0	1	1	1	0	0	0	0	1	0
6	0	0	2	1	0	0	0	0	1	0
7	2	0	0	0	1	0	0	0	1	0
8	1	1	0	0	1	0	0	0	1	0
9	1	0	1	0	1	0	0	0	1	0
10	0	2	0	0	1	0	0	0	1	0
11	0	1	1	0	1	0	0	0	1	0
12	0	0	2	0	1	0	0	0	0	1

Table 13 Survival signatures of SubSys1.

Similarly, the survival signature of SubSys2 is listed in Table 14.

Order of combination	$l_1^2 = l_{1,0}^2 + l_{1,1}^2 + l_{1,2}^2 = 0$			$l_2^2 = l_{2,0}^2 + l_{2,1}^2 = 1$		$l_3^2 = l_{3,0}^2 + l_{3,1}^2 = 1$		$\Phi_{H=h}^{s2}(l_{1,0}^2, l_{1,1}^2, l_{1,2}^2, l_{2,0}^2, l_{2,1}^2, l_{3,0}^2, l_{3,1}^2)$		
	$l_{1,0}^2$	$l_{1,1}^2$	$l_{1,2}^2$	$l_{2,0}^2$	$l_{2,1}^2$	$l_{3,0}^2$	$l_{3,1}^2$	$h=0$	$h=1$	$h=2$
1	0	0	0	1	0	1	0	1	0	0
2	0	0	0	0	1	1	0	0	1	0
3	0	0	0	1	0	0	1	0	1	0
4	0	0	0	0	1	0	1	0	0	1

Table 14 Survival signatures of SubSys2.

According to the following rules, the system's state can be determined based on the states of subsystems. Those rules are listed in Table 15.

Order of rules	Probability at different states	States		
		SubSys1	SubSys2	System
1	0 or 1	0	0	0
2		1	<=1	1
3		<=1	1	1
4		2	<=2	2
5		<=2	2	2
6	Other situations: including probability distribution at different states of one or two subsystems is between the interval (0,1)	Probability is a numeric at any state	Probability is a numeric at any state	Sum the corresponding probability and divide evenly by the number of combination

Table 15 Rules for the state of system consisting of two subsystems in different configuration.

Similarly, the above rules can be used to calculate the state of system by programming. The results can be shown in Table 16.

Order of combination	$l_1 = l_{1,0}+l_{1,1}+l_{1,2} = 2$			$l_2 = l_{2,0}+l_{2,1} = 2$		$l_3 = l_{3,0}+l_{3,1} = 1$		$\Phi^s_{H=h}(l_{1,0},l_{1,1},l_{1,2},l_{2,0},l_{2,1},l_{3,0},l_{3,1})$		
	$l_{1,0}$	$l_{1,1}$	$l_{1,2}$	$l_{2,0}$	$l_{2,1}$	$l_{3,0}$	$l_{3,1}$	$h=0$	$h=1$	$h=2$
1	2	0	0	2	0	1	0	1	0	0
2	2	0	0	2	0	0	1	0	1	0
3	2	0	0	1	1	1	0	0	1	0
4	2	0	0	1	1	0	1	0	1/2	1/2
5	2	0	0	0	2	1	0	0	1	0
6	2	0	0	0	2	0	1	0	0	1
7	1	1	0	2	0	1	0	1	0	0
8	1	1	0	2	0	0	1	0	1	0
9	1	1	0	1	1	1	0	0	1	0
10	1	1	0	1	1	0	1	0	1/2	1/2
11	1	1	0	0	2	1	0	0	1	0
12	1	1	0	0	2	0	1	0	0	1
13	1	0	1	2	0	1	0	1	0	0
14	1	0	1	2	0	0	1	0	1	0
15	1	0	1	1	1	1	0	0	1	0
16	1	0	1	1	1	0	1	0	1/2	1/2
17	1	0	1	0	2	1	0	0	1	0
18	1	0	1	0	2	0	1	0	0	1
19	0	2	0	2	0	1	0	0	1	0
20	0	2	0	2	0	0	1	0	1	0
21	0	2	0	1	1	1	0	0	1	0
22	0	2	0	1	1	0	1	0	1/2	1/2
23	0	2	0	0	2	1	0	0	1	0
24	0	2	0	0	2	0	1	0	0	1
25	0	1	1	2	0	1	0	0	1	0
26	0	1	1	2	0	0	1	0	1	0
27	0	1	1	1	1	1	0	0	1	0
28	0	1	1	1	1	0	1	0	1/2	1/2
29	0	1	1	0	2	1	0	0	1	0
30	0	1	1	0	2	0	1	0	0	1
31	0	0	2	2	0	1	0	0	1	0
32	0	0	2	2	0	0	1	0	1	0
33	0	0	2	1	1	1	0	0	1/2	1/2
34	0	0	2	1	1	0	1	0	0	1
35	0	0	2	0	2	1	0	0	0	1
36	0	0	2	0	2	0	1	0	0	1

Table 16 Survival signatures of system consisting of two subsystems in different configuration.

To verify these results, the 28th combination can be taken as an example. The combination of system and component is listed in Table 17. From this table, there are 4 different combinations for the 28th combination. According to the states of components, the state of system can be determined. As shown in this table, there are 2 combinations in state 1 and 2 respectively. So, the survival signature of system at the corresponding state is equal to 2/4, viz. 1/2.

Orders	State of component					State of	State of	State of system		
	1	2	3	4	5	subsys1	subsys2	$h=0$	$h=1$	$h=2$
1	1	2	0	1	1	1	2	0	0	1
2	1	2	1	0	1	1	1	0	1	0
3	2	1	0	1	1	1	2	0	0	1
4	2	1	1	0	1	1	1	0	1	0

Table 17 System and component state of 28th combination of system.

Using formulae (11), (12), (13), (15)and the Table 16, the probability of system at different state can be depicted in Fig. 14.

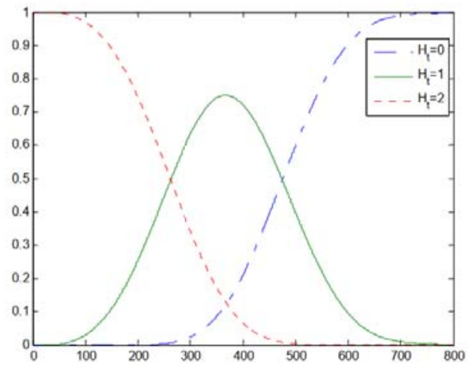

Fig. 14. Probabilities that the system is in state h at time t for $h=0,1,2$.

Similarly, the reliability curve can be expressed as Fig. 15 by using Eq. (14).

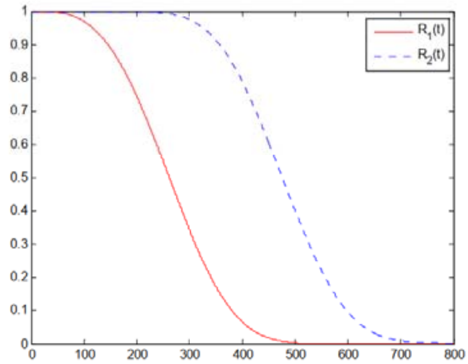

Fig. 15. Reliability curves of system at different states.

Example 6.

A more complicated example with more components and subsystems is presented here. Its configuration is shown as Fig. 16.

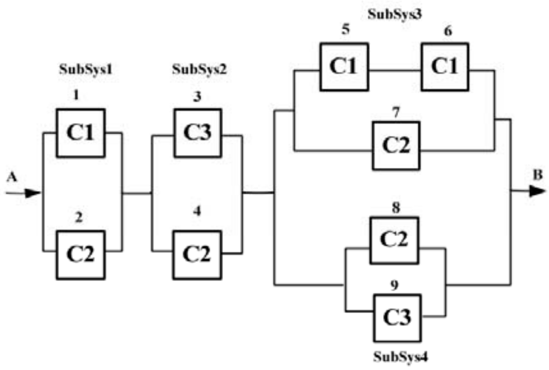

Fig. 16. System with more components and subsystems.

According to the survival signature in Table 11 and Table 16, the survival signature of system in Fig. 16 can be calculated by the programming method mentioned above. Results are listed in Table 18 which can be founded in the Appendix. The states mapping from subsystems to system can be finished by consecutive applying of Table 10 and Table 15.

Similarly, the probability of system at different state can be plotted as shown in Fig. 17 by Eq. (11), (12), (13), (15) and Table 18.

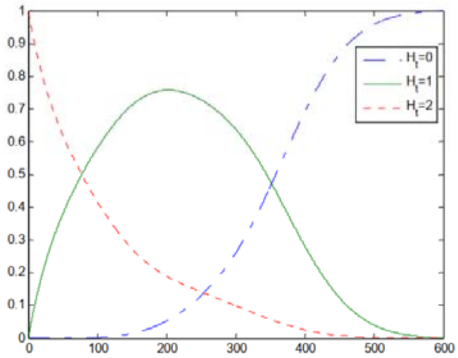

Fig. 17. Probabilities that the system is in state h at time t for $h = 0,1,2$.

According to the Eq. (14), its reliability function can be depicted in Fig. 18.

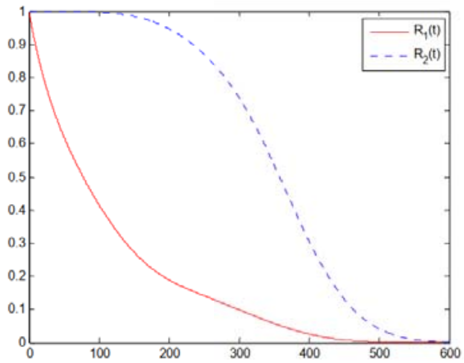

Fig. 18. Reliability functions of system.

7. Concluding remarks

In this article, the concept of survival signature has been extended from binary state system to the mixture of multi-state and binary-state. For the MSS with binary-state components, the definition of system state is based on the number of components in its mini-path. For the multi-state components, its state transition is based on the Markov stochastic process without considering the repair and maintenance tactics. The survival function and reliability function for binary-state and multi-state system are derived respectively.

For a practical system consisting of several subsystems connected in different configuration, the computing method for the survival signature of the system has been developed based on the survival signature of subsystems. It provides a feasible path to implement the reliability analysis for a real industrial system where survival signature can be obtained by iterating program on the survival signatures of subsystems. A larger example illustrates its validity. Other multi-state models such as competitive and cascade model will be considered in the future research.

As reliability of MSS is an important topic for research and application, there are a large number of research topics building on the new survival signature method presented in this paper. Some topics are further computational methods, statistical inference methods, detailed modelling of component state change processes and support of decisions with regard to inspection and maintenance. Development of general methods for such topics is of important, it is particularly useful to do this related to real world applications, to ensure that the future developments are of most practical relevance.

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References

1. Fan, B., et al., *A controllable chaotic immune algorithm for risk-aware routing in DiffServ networks.* Computer Communications, 2016. **76**: p. 67-76.

2. Li, C.B., J.H. Yuan, and X. Feng, *Linguistic assessment information risky multi-criteria decision-making about wind power investment.* Journal of Intelligent & Fuzzy Systems, 2016. **30**(5): p. 3017-3023.
3. Li, C.B., et al., *Risk group assessment based on the demands of important consumers with standby electric source.* Journal of Intelligent & Fuzzy Systems, 2017. **33**(4): p. 2177-2188.
4. Ju, Y., J.Y. Yan, and H. Xu, *Performance Optimization Based on Compressive Sensing for Wireless Sensor Networks.* Wireless Personal Communications, 2017. **95**(3): p. 1927-1941.
5. Chen, Z.X., et al., *Performance analysis of dual-media cooperative communication based on wireless and power line under hybrid fading.* International Journal of Distributed Sensor Networks, 2019. **15**(5): p. 11.
6. Huang, X., et al., *A new direct second-order reliability analysis method.* Applied Mathematical Modelling, 2018. **55**(2018): p. 68-80.
7. Samaniego, F.J., *System signatures and their applications in engineering reliability*. 2007, New York: Springer.
8. Eryilmaz, S., *Estimation in coherent reliability systems through copulas.* Reliability Engineering & System Safety, 2011. **96**(5): p. 564-568.
9. Navarro, J., et al., *On the application and extension of system signatures in engineering reliability.* Naval Research Logistics (NRL), 2008. **55**(4): p. 313-327.
10. Coolen, F.P.A. and A.H. Al-nefaiee, *Nonparametric predictive inference for failure times of systems with exchangeable components.* Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability, 2012. **226**(3): p. 262-273.
11. Coolen, F.P.A. and T. Coolen-Maturi, *Generalizing the Signature to Systems with Multiple Types of Components*. 2012, Springer Berlin Heidelberg: Berlin, Heidelberg. p. 115-130.
12. Coolen, F.P., T. Coolen-Maturi, and A.H. Al-nefaiee, *Nonparametric predictive inference for system reliability using the survival signature.* Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability, 2014. **228**(5): p. 437-448.
13. Aslett, L.J.M., F.P.A. Coolen, and S.P. Wilson, *Bayesian Inference for Reliability of Systems and Networks Using the Survival Signature.* Risk Analysis, 2015. **35**(9): p. 1640-1651.
14. Feng, G., et al., *Imprecise system reliability and component importance based on survival signature.* Reliability Engineering & System Safety, 2016. **150**: p. 116-125.
15. Patelli, E., et al., *Simulation methods for system reliability using the survival signature.* Reliability Engineering & System Safety, 2017. **167**: p. 327-337.
16. Eryilmaz, S., F.P.A. Coolen, and T. Coolen-Maturi, *Marginal and joint reliability importance based on survival signature.* Reliability Engineering & System Safety, 2018. **172**: p. 118-128.
17. Huang, X., L.J.M. Aslett, and F.P.A. Coolen, *Reliability analysis of general phased mission systems with a new survival signature.* Reliability Engineering & System Safety, 2019. **189**(2019): p. 416-422.
18. Huang, X., F.P.A. Coolen, and T. Coolen-Maturi, *A heuristic survival signature based approach for reliability-redundancy allocation.* Reliability Engineering & System Safety, 2019. **185**(2019): p. 511-517.
19. Lisnianski, A. and G. Levitin, *Multi-state system reliability: assessment, optimization and applications*. 2003: World scientific.
20. Anatoly, L., F. Ilia, and D. Yi *Multi-state System Reliability Analysis and Optimization for Engineers and Industrial Managers* 2010, London: Springer-Verlag.

21. Levitin, G., *Incorporating common-cause failures into nonrepairable multistate series-parallel system analysis.* Reliability, IEEE Transactions on, 2001. **50**(4): p. 380-388.
22. Qin, J.L. and Z. Li, *Reliability and Sensitivity Analysis Method for a Multistate System with Common Cause Failure.* Complexity, 2019(Article ID 6535726,): p. 8.
23. Li, Z. and J. Qin, *An Efficient Universal Generating Function-Based Analyzing Approach for Multistate System with Imperfect Coverage Failure.* Complexity, 2021. **Artical ID:5560555**: p. 10.
24. Qin, J.L., Z. Li, and Y.C. Zhu, *Protection Mechanism in Reliability Evaluation Approach to Multistate System with Common Cause Failure.* Complexity, 2021. **Article ID 6691362**: p. 12.
25. Liu, B., et al., *Stochastic Filtering Approach for Condition-Based Maintenance Considering Sensor Degradation.* Ieee Transactions on Automation Science and Engineering, 2020. **17**(1): p. 177-190.
26. Qin, J. and Z. Li, *Reliability Modeling for Multistate System with Preventive Maintenance under Customer Demand.* Complexity, 2020. **Artical ID:3165230**: p. 9.
27. Liu, B., et al., *Life cycle cost analysis considering multiple dependent degradation processes and environmental influence.* Reliability Engineering & System Safety, 2020. **197**: p. 10.
28. Ramirez-Marquez, J.E. and D.W. Coit, *Multi-state component criticality analysis for reliability improvement in multi-state systems.* Reliability Engineering & System Safety, 2007. **92**(12): p. 1608-1619.
29. Qin, J.L., et al., *Simulated Method for Reliability Evaluation of Multi-State Coherent System.* Iranian Journal of Science and Technology Transaction a-Science, 2018. **42**(A3): p. 1363-1371.
30. Qin, J., Y. Niu, and Z. Li, *A COMBINED METHOD FOR RELIABILITY ANALYSIS OF MULTI-STATE SYSTEM OF MINOR-REPAIRABLE COMPONENTS.* Eksploatacja I Niezawodnosc-Maintenance and Reliability, 2016. **18**(1): p. 80-88.
31. Eryilmaz, S. and A. Tuncel, *Generalizing the survival signature to unrepairable homogeneous multi-state systems.* Naval Research Logistics (NRL), 2016. **63**(8): p. 593-599.
32. Liu, Y., et al., *Stress–strength reliability analysis of multi-state system based on generalized survival signature.* Journal of Computational and Applied Mathematics, 2018. **342**: p. 274-291.
33. Gertsbakh, I., Y. Shpungin, and F. Spizzichino, *Signatures of Coherent Systems Built with Separate Modules.* Journal of Applied Probability, 2011. **48**(3): p. 843-855.
34. Da, G., B. Zheng, and T. Hu, *On computing signatures of coherent systems.* Journal of Multivariate Analysis, 2012. **103**(1): p. 142-150.
35. Coolen, F.P.A. and T. Coolen-Maturi, *On the structure function and survival signature for system reliability.* Safety and Reliability, 2016. **36**(2): p. 77-87.

Appendix

Order of combination	$l_1 = l_{1,0}+l_{1,1}+l_{1,2} = 3$	$l_2 = l_{2,0}+l_{2,1} = 4$	$l_3 = l_{3,0}+l_{3,1} = 2$	$\Phi^s_{H=h}(l_{1,0},l_{1,1},l_{1,2},l_{2,0},l_{2,1},l_{3,0},l_{3,1})$

	$l_{1,0}$	$l_{1,1}$	$l_{1,2}$	$l_{2,0}$	$l_{2,1}$	$l_{3,0}$	$l_{3,1}$	$h=0$	$h=1$	$h=2$
1	3	0	0	4	0	2	0	1	0	0
2	3	0	0	4	0	1	1	1	0	0
3	3	0	0	4	0	0	2	1	0	0
4	3	0	0	3	1	2	0	1	0	0
5	3	0	0	3	1	1	1	0.8125	0.125	0.0625
6	3	0	0	3	1	0	2	0.375	0.5	0.125
7	3	0	0	2	2	2	0	1	0	0
8	3	0	0	2	2	1	1	0.5625	0.375	0.0625
9	3	0	0	2	2	0	2	0.375	0.5	0.125
10	3	0	0	1	3	2	0	0.5	0.5	0
11	3	0	0	1	3	1	1	0.3125	0.625	0.0625
12	3	0	0	1	3	0	2	0.125	0.5	0.375
13	3	0	0	0	4	2	0	0	1	0
14	3	0	0	0	4	1	1	0	1	0
15	3	0	0	0	4	0	2	0	1	0
16	2	1	0	4	0	2	0	1	0	0
17	2	1	0	4	0	1	1	1	0	0
18	2	1	0	4	0	0	2	0.5	0.5	0
19	2	1	0	3	1	2	0	0.875	0.125	0
20	2	1	0	3	1	1	1	0.625	0.3125	0.0625
21	2	1	0	3	1	0	2	0.125	0.7917	0.0833
22	2	1	0	2	2	2	0	0.625	0.375	0
23	2	1	0	2	2	1	1	0.4445	0.4722	0.0833
24	2	1	0	2	2	0	2	0.1875	0.6875	0.125
25	2	1	0	1	3	2	0	0.2917	0.7083	0
26	2	1	0	1	3	1	1	0.1875	0.6875	0.125
27	2	1	0	1	3	0	2	0.0416	0.7917	0.1667
28	2	1	0	0	4	2	0	0	1	0
29	2	1	0	0	4	1	1	0	1	0
30	2	1	0	0	4	0	2	0	1	0
31	2	0	1	4	0	2	0	1	0	0
32	2	0	1	4	0	1	1	1	0	0
33	2	0	1	4	0	0	2	0.5	0.5	0
34	2	0	1	3	1	2	0	0.875	0.125	0
35	2	0	1	3	1	1	1	0.625	0.3125	0.0625
36	2	0	1	3	1	0	2	0.125	0.7917	0.0833
37	2	0	1	2	2	2	0	0.625	0.375	0
38	2	0	1	2	2	1	1	0.4445	0.4722	0.0833
39	2	0	1	2	2	0	2	0.1875	0.6875	0.125
40	2	0	1	1	3	2	0	0.2917	0.7083	0
41	2	0	1	1	3	1	1	0.1875	0.6875	0.125
42	2	0	1	1	3	0	2	0.0417	0.7083	0.25

43	2	0	1	0	4	2	0	0	1	0
44	2	0	1	0	4	1	1	0	1	0
45	2	0	1	0	4	0	2	0	0.5	0.5
46	1	2	0	4	0	2	0	1	0	0
47	1	2	0	4	0	1	1	1	0	0
48	1	2	0	4	0	0	2	0.5	0.5	0
49	1	2	0	3	1	2	0	0.875	0.125	0
50	1	2	0	3	1	1	1	0.5625	0.375	0.0625
51	1	2	0	3	1	0	2	0.125	0.7917	0.0833
52	1	2	0	2	2	2	0	0.525	0.475	0
53	1	2	0	2	2	1	1	0.3889	0.5278	0.0833
54	1	2	0	2	2	0	2	0.1875	0.6875	0.125
55	1	2	0	1	3	2	0	0.2917	0.7083	0
56	1	2	0	1	3	1	1	0.1875	0.6875	0.125
57	1	2	0	1	3	0	2	0.0417	0.7917	0.1666
58	1	2	0	0	4	2	0	0	1	0
59	1	2	0	0	4	1	1	0	1	0
60	1	2	0	0	4	0	2	0	1	0
61	1	1	1	4	0	2	0	1	0	0
62	1	1	1	4	0	1	1	1	0	0
63	1	1	1	4	0	0	2	0.3333	0.6667	0
64	1	1	1	3	1	2	0	0.875	0.125	0
65	1	1	1	3	1	1	1	0.5208	0.4167	0.0625
66	1	1	1	3	1	0	2	0.0937	0.8125	0.0938
67	1	1	1	2	2	2	0	0.5536	0.4464	0
68	1	1	1	2	2	1	1	0.3705	0.5357	0.0938
69	1	1	1	2	2	0	2	0.125	0.75	0.125
70	1	1	1	1	3	2	0	0.25	0.75	0
71	1	1	1	1	3	1	1	0.1458	0.7084	0.1458
72	1	1	1	1	3	0	2	0.0312	0.75	0.2188
73	1	1	1	0	4	2	0	0	1	0
74	1	1	1	0	4	1	1	0	1	0
75	1	1	1	0	4	0	2	0	0.6667	0.3333
76	1	0	2	4	0	2	0	1	0	0
77	1	0	2	4	0	1	1	1	0	0
78	1	0	2	4	0	0	2	0.5	0.5	0
79	1	0	2	3	1	2	0	0.8125	0.125	0.0625
80	1	0	2	3	1	1	1	0.5625	0.375	0.0625
81	1	0	2	3	1	0	2	0.3125	0.625	0.0625
82	1	0	2	2	2	2	0	0.5625	0.375	0.0625
83	1	0	2	2	2	1	1	0.375	0.5	0.125
84	1	0	2	2	2	0	2	0.1875	0.625	0.1875
85	1	0	2	1	3	2	0	0.3125	0.625	0.0625
86	1	0	2	1	3	1	1	0.1875	0.625	0.1875

87	1	0	2	1	3	0	2	0.0625	0.625	0.3125
88	1	0	2	0	4	2	0	0	1	0
89	1	0	2	0	4	1	1	0	1	0
90	1	0	2	0	4	0	2	0	0.5	0.5
91	0	3	0	4	0	2	0	1	0	0
92	0	3	0	4	0	1	1	0.5	0.5	0
93	0	3	0	4	0	0	2	0	1	0
94	0	3	0	3	1	2	0	0.625	0.375	0
95	0	3	0	3	1	1	1	0.1875	0.75	0.0625
96	0	3	0	3	1	0	2	0	0.875	0.125
97	0	3	0	2	2	2	0	0.375	0.625	0
98	0	3	0	2	2	1	1	0.225	0.65	0.125
99	0	3	0	2	2	0	2	0	0.875	0.125
100	0	3	0	1	3	2	0	0.125	0.875	0
101	0	3	0	1	3	1	1	0.0625	0.75	0.1875
102	0	3	0	1	3	0	2	0	0.875	0.125
103	0	3	0	0	4	2	0	0	1	0
104	0	3	0	0	4	1	1	0	1	0
105	0	3	0	0	4	0	2	0	1	0
106	0	2	1	4	0	2	0	1	0	0
107	0	2	1	4	0	1	1	0.5	0.5	0
108	0	2	1	4	0	0	2	0	1	0
109	0	2	1	3	1	2	0	0.625	0.375	0
110	0	2	1	3	1	1	1	0.1875	0.75	0.0625
111	0	2	1	3	1	0	2	0	0.875	0.125
112	0	2	1	2	2	2	0	0.375	0.625	0
113	0	2	1	2	2	1	1	0.225	0.65	0.125
114	0	2	1	2	2	0	2	0	0.875	0.125
115	0	2	1	1	3	2	0	0.125	0.875	0
116	0	2	1	1	3	1	1	0.0625	0.75	0.1875
117	0	2	1	1	3	0	2	0	0.75	0.25
118	0	2	1	0	4	2	0	0	1	0
119	0	2	1	0	4	1	1	0	1	0
120	0	2	1	0	4	0	2	0	0.5	0.5
121	0	1	2	4	0	2	0	1	0	0
122	0	1	2	4	0	1	1	0.5	0.5	0
123	0	1	2	4	0	0	2	0	1	0
124	0	1	2	3	1	2	0	0.6667	0.2917	0.0416
125	0	1	2	3	1	1	1	0.25	0.6875	0.0625
126	0	1	2	3	1	0	2	0	0.875	0.125
127	0	1	2	2	2	2	0	0.375	0.5625	0.0625
128	0	1	2	2	2	1	1	0.1944	0.6389	0.1667
129	0	1	2	2	2	0	2	0	0.875	0.125
130	0	1	2	1	3	2	0	0.0833	0.7917	0.125

131	0	1	2	1	3	1	1	0.0625	0.75	0.1875
132	0	1	2	1	3	0	2	0	0.625	0.375
133	0	1	2	0	4	2	0	0	1	0
134	0	1	2	0	4	1	1	0	1	0
135	0	1	2	0	4	0	2	0	0.5	0.5
136	0	0	3	4	0	2	0	1	0	0
137	0	0	3	4	0	1	1	0.5	0.5	0
138	0	0	3	4	0	0	2	0	1	0
139	0	0	3	3	1	2	0	0.375	0.5	0.125
140	0	0	3	3	1	1	1	0.3125	0.625	0.0625
141	0	0	3	3	1	0	2	0	1	0
142	0	0	3	2	2	2	0	0.375	0.5	0.125
143	0	0	3	2	2	1	1	0.1875	0.625	0.1875
144	0	0	3	2	2	0	2	0	1	0
145	0	0	3	1	3	2	0	0.125	0.5	0.375
146	0	0	3	1	3	1	1	0.0625	0.625	0.3125
147	0	0	3	1	3	0	2	0	0.5	0.5
148	0	0	3	0	4	2	0	0	1	0
149	0	0	3	0	4	1	1	0	0.5	0.5
150	0	0	3	0	4	0	2	0	0	1

Table 18 Survival signatures of the system with more components and subsystems.