Seiberg-like dualities for orthogonal and symplectic 3d $\mathcal{N} = 2$ gauge theories with boundaries

Tadashi Okazaki[1]

and

Douglas J Smith[2]

Department of Mathematical Sciences, Durham University,
Upper Mountjoy Campus, Stockton Road, Durham DH1 3LE, UK

Abstract

We propose dualities of $\mathcal{N} = (0,2)$ supersymmetric boundary conditions for 3d $\mathcal{N} = 2$ gauge theories with orthogonal and symplectic gauge groups. We show that the boundary 't Hooft anomalies and half-indices perfectly match for each pair of the proposed dual boundary conditions.

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[1]tadashi.okazaki@durham.ac.uk
[2]douglas.smith@durham.ac.uk

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1   Introduction and conclusions

Seiberg duality [1] of 4d $\mathcal{N} = 1$ gauge theories is the duality in the IR which relates an 'electric' $SU(N_c)$ gauge theories with N_f flavours of quarks and antiquarks to a 'magnetic' $SU(N_f - N_c)$ gauge theories with N_f flavours of quarks and antiquarks

together with a gauge singlet field coupled through a superpotential (See e.g. [2, 3, 4, 5, 6, 7, 8, 9] for various generalisations.).

There exists a three-dimensional analogue, *aka* Seiberg-like duality in 3d $\mathcal{N} = 2$ gauge theories. For the unitary gauge group, the IR duality [10] relates $\mathcal{N} = 2$ gauge theory with gauge group $U(N_c)$, N_f fundamental chiral multiplets and N_f anti-fundamental chiral multiplets to $\mathcal{N} = 2$ gauge theory with gauge group $U(N_f - N_c)$, N_f fundamental chiral multiplets and N_f anti-fundamental chiral multiplets as well as additional gauge singlet chiral multiplets and a superpotential. For the symplectic gauge group, the IR duality [10] relates $USp(2N_c)$ gauge theory with $2N_f$ fundamental chiral multiplets to $USp(2N_f - 2N_c - 2)$ gauge theory with $2N_f$ fundamental chiral multiplets along with gauge singlets and a superpotential. The dualities are generalised to the theories with a Chern-Simons term [11, 12], which can be achieved by adding real masses to flavours, leading to an effective Chern-Simons term at low energy. The 3d Seiberg-like dualities are extended to the $SU(N_c)$ gauge theories [13, 14, 15, 16, 17, 18, 19], the orthogonal gauge theories [20, 21, 22, 23, 24, 25, 26], the G_2 gauge theory [25], the quiver gauge theories [27, 28], the gauge theories with arbitrary numbers of fundamental and anti-fundamental matter fields obeying the $\mathbb{Z}_2$ anomaly constraint $k + \frac{N_f - N_a}{2} \in \mathbb{Z}$ [21], with adjoint matter fields [29, 30, 14, 31, 16, 32, 17, 19], with other tensor matter fields [33, 34, 35, 36] and with a monopole superpotential [37, 38].

In the presence of a boundary, the dualities become more elaborate as the bulk fields are subject to certain boundary conditions and they can further couple to additional boundary degrees of freedom. The half-BPS boundary conditions preserving $\mathcal{N} = (0,2)$ supersymmetry in 3d $\mathcal{N} = 2$ gauge theories have been studied in [39, 40, 41, 42, 43, 44, 45, 46]. Simple dualities of $\mathcal{N} = (0,2)$ half-BPS boundary conditions of 3d $\mathcal{N} = 2$ gauge theories for Abelian gauge theories were proposed in [40] and various dualities of $\mathcal{N} = (0,2)$ half-BPS boundary conditions for 3d $\mathcal{N} = 2$ gauge theories with unitary gauge groups were proposed in [43]. [3]

Other approaches to describing Chern-Simons theories with boundary have focussed on the relation to CFTs[4] and specifically WZW models [51, 52, 53]. This has been used to infer possible boundary matter in many examples including CS with various amounts of supersymmetry, including ABJM models [54, 55, 56, 57, 58, 59], while boundary conditions have been analysed for multiple M2-branes in [60, 61, 62]. In the context of Seiberg-like dualities, several level-rank dualities of gauged WZW models have been proposed [63, 64] using the relation to Chern-Simons theories and

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[3]See [47, 48] for the study of dualities of $\mathcal{N} = (0,4)$ boundary conditions and [49, 50] for $\mathcal{N} = (2,2)$ boundary conditions in 3d $\mathcal{N} = 4$ supersymmetric gauge theories.

[4]In fact, Chern-Simons theories on compact manifolds without boundary are also related to CFTs [51, 52].

their dualities.

In this paper, we study $\mathcal{N} = (0, 2)$ half-BPS boundary conditions for 3d $\mathcal{N} = 2$ gauge theories with symplectic and orthogonal gauge groups and propose dualities of these boundary conditions. We support our claims by computing boundary 't Hooft anomalies and supersymmetric half-indices which perfectly match for the proposed dual pairs of boundary conditions. For orthogonal gauge groups the global structure of the group is important and has been discussed for 3d Chern-Simons theories in the context of Seiberg-like duality [24] and level-rank duality [65]. We show that with boundary conditions there is a similar set of dualities relating the following groups under Seiberg-like duality: $SO \leftrightarrow SO$, $O_+ \leftrightarrow O_+$, $O_- \leftrightarrow Spin$ and $Pin_\pm \leftrightarrow Pin_\pm$.

The organisation of this article is as follows. In section 2 we discuss the duality we suggest for $\mathcal{N} = (0, 2)$ half-BPS boundary conditions in 3d $\mathcal{N} = 2$ $USp(2N_c)$ gauge theories and test it by computing supersymmetric half-indices. In section 3 we propose the dualities of $\mathcal{N} = (0, 2)$ half-BPS boundary conditions for 3d $\mathcal{N} = 2$ $SO(N_c)$ gauge theories. We find more extensive identities of half-indices parameterised by two parameters ζ and χ corresponding to the global $\mathbb{Z}_2$ symmetries [24]. In section 4 we discuss the dualities of $\mathcal{N} = (0, 2)$ half-BPS boundary conditions for other orthogonal gauge groups. In Appendix A we present the notation and convention of our tools, including boundary anomalies and supersymmetric indices. In Appendix B we show numerical results obtained from Mathematica.

2 $USp(2N_c)$ gauge theories

The quantum dynamics of 3d $\mathcal{N} = 2$ supersymmetric gauge theories with gauge groups $G = USp(2N_c)$ has been studied in [66, 67, 10]. $USp(2N_c)$ is the subgroup of $SU(2N_c)$ that keeps an antisymmetric tensor $J^{ab} = (\mathbb{I}_{N_c} \otimes i\sigma_2)^{ab}$ invariant. Since the antisymmetric tensor $\epsilon^{i_1 \cdots i_{2N_c}}$ breaks up into sums of products of the J^{ab}, the $USp(2N_c)$ gauge theory has no baryons. Also there is no topological current as the gauge group is simple.

For the symplectic gauge groups, the IR dualities are proposed in [10, 67]. We will refer to these dual theories as theory A and theory B, but they are also commonly referred to as the electric and the magnetic theories:

- Theory A: $USp(2N_c)$ gauge theory with $2N_f$ chiral multiplets Q in the fundamental representation. It contains gauge invariant operators as the meson $M = QQ$ and the monopole operator V.

- Theory B: $USp(2(N_f - N_c - 1))$ gauge theory with $2N_f$ chiral multiplets q in

the fundamental representation, $N_f(2N_f - 1)$ neutral chiral multiplets M in the rank-2 antisymmetric representation of $SU(2N_f)$ and a chiral multiplet V which has the superpotential

$$\mathcal{W} = Mqq + V\tilde{V} \tag{2.1}$$

where $\tilde{V}$ is the monopole operator.

The charges of the chiral multiplets are given by

	$G = USp(2N_c)$	$\tilde{G} = USp(2(N_f - N_c - 1))$	$SU(2N_f)$	$U(1)_a$	$U(1)_R$
Q	$\mathbf{N_c}$	$\mathbf{1}$	$\mathbf{2N_f}$	$+$	r
q	$\mathbf{1}$	$\mathbf{2(N_f - N_c - 1)}$	$\overline{\mathbf{2N_f}}$	$-$	$1 - r$
M	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{N_f(2N_f - 1)}$	2	$2r$
V	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$-2N_f$	$2N_f - 2N_c - 2rN_f$

$$\tag{2.2}$$

The quantum numbers crucially depend on the rank of gauge group and the number of flavours.

2.1 $\mathcal{N} = (0, 2)$ half-BPS boundary conditions

We introduce a boundary to the 3d $\mathcal{N} = 2$ theories in such a way as to preserve $\mathcal{N} = (0, 2)$ supersymmetry in 2d. For the 3d bulk fields we must impose boundary conditions, and basic $\mathcal{N} = (0, 2)$ boundary conditions [40] impose either Neumann or Dirichlet boundary conditions on chiral multiplet (which we denote by N or by D) and either Neumann or Dirichlet boundary conditions on vector multiplet (VM) (which we denote by $\mathcal{N}$ or by $\mathcal{D}$) [5] which is compatible with the decomposition of the 3d $\mathcal{N} = 2$ supermultiplets into 2d $\mathcal{N} = (0, 2)$ supermultiplets on the boundary. The choice of Neumann or Dirichlet boundary conditions projects out specific $\mathcal{N} = (0, 2)$ supermultiplets.

On the 2d boundary we can have anomalies and these give two important constraints. If we have a gauge symmetry, which will be the case for Neumann boundary conditions for bulk vector multiplets, we require gauge anomaly cancellation. On the other hand, for global symmetries, including gauge symmetry broken by Dirichlet boundary conditions for the vector multiplet, we do not require cancellation of the anomalies. Instead, we get constraint on any proposed duality that the 't Hooft

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[5]One can also consider singular boundary conditions. Although it is intriguing to reduce them to the B-type boundary conditions found in [68], we leave it to future work.

anomalies must match, i.e. the anomaly polynomials for the two theories must be equal.

The method to calculate the 2d boundary anomaly polynomial was given in [43]. There are three types of contribution to the anomalies: 3d bulk supermultiplets projected onto the boundary with Neumann of Dirichlet boundary conditions; 2d supermultiplets introduced on the boundary; background Chern-Simons or FI terms. We summarise the general results presented in [43] in appendix A.2. The specific results for the fields in the $USp(2N_c) - [2N_f]$ theory A and $USp(2(N_c - N_f - 1)) - [2N_f]$ theory B we consider are

$$\mathcal{A}^{\mathrm{VM}} = - (N_c + 1) \operatorname{Tr}(\mathbf{s}^2) - \frac{N_c}{2}(2N_c + 1)\mathbf{r}^2, \tag{2.3}$$

$$\mathcal{A}^Q = N_f \operatorname{Tr}(\mathbf{s}^2) + N_c \operatorname{Tr}(\mathbf{x}^2) + 2N_c N_f(\mathbf{a} + (r - 1)\mathbf{r})^2, \tag{2.4}$$

$$\mathcal{A}^q = N_f \operatorname{Tr}(\tilde{\mathbf{s}}^2) + (N_f - N_c - 1) \operatorname{Tr}(\mathbf{x}^2) + 2(N_f - N_c - 1)N_f(-\mathbf{a} - r\mathbf{r})^2, \tag{2.5}$$

$$\mathcal{A}^M = (N_f - 1) \operatorname{Tr}(\mathbf{x}^2) + \frac{1}{2}N_f(2N_f - 1)(2\mathbf{a} + (2r - 1)\mathbf{r})^2, \tag{2.6}$$

$$\mathcal{A}^V = \frac{1}{2}(-2N_f\mathbf{a} + (2(1 - r)N_f - 2N_c - 1)\mathbf{r})^2 \tag{2.7}$$

for Dirichlet boundary conditions. For Neumann boundary conditions the contributions are the same but with opposite sign. The notation $\mathbf{s}$ ($\tilde{\mathbf{s}}$) refers to the gauge field strength in theory A (B), $\mathbf{x}$ to the field strength for the global $SU(2N_f)$ flavour symmetry, and $\mathbf{a}$ and $\mathbf{r}$ to the field strengths for the global $U(1)_a$ and $U(1)_R$ symmetries.

We also need 2d boundary matter and the only multiplet required for the examples we consider is a $USp(2N_c) \times USp(2(N_f - N_c - 1))$ bifundamental Fermi multiplet, Ψ. Here $USp(2N_c)$ is the 2d gauge group inherited from the bulk vector multiplet with Neumann boundary conditions while $USp(2(N_f - N_c - 1))$ is a global flavour symmetry which is identified with the dual gauge group broken to a global symmetry by the Dirichlet boundary conditions for the dual vector multiplet. This gives anomaly contribution

$$\mathcal{A}^\Psi = (N_f - N_c - 1) \operatorname{Tr}(\mathbf{s}^2) + N_c \operatorname{Tr}(\tilde{\mathbf{s}}^2) . \tag{2.8}$$

Note that this contribution is actually only half of what might be expected for such a Fermi. However, this is precisely the contribution required for anomaly cancellation and we interpret this as due to a reality condition on the Fermi. We will comment again on this when discussing the Fermi contribution to the half-index.

For now we consider the 3d dualities with vanishing Chern-Simons level, in which case we have no background Chern-Simons terms.

The boundary conditions we consider are $(\mathcal{N}, \mathrm{N})$ in theory A, referring to the choice of Neumann boundary conditions for (VM, Q) together with $(\mathcal{D}, \mathrm{D}, \mathrm{N}, \mathrm{D})$ in theory B

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for (VM, q, M, V). With this choice we need to cancel the gauge anomaly in theory A which we can do by including the bifundamental Fermi in theory A. This also leads to anomaly matching with theory B without any further 2d matter. In particular we have

$$\mathcal{A}^{A,\mathrm{bulk}}_{\mathcal{N},\mathrm{N}} = -(N_f - N_c - 1)\operatorname{Tr}(\mathbf{s}^2) - N_c\operatorname{Tr}(\mathbf{x}^2) - \frac{N_c}{2}(4N_f(1-r)^2 - 2N_c - 1)\mathbf{r}^2$$
$$+ 4(1-r)N_cN_f\mathbf{ar} - 2N_cN_f\mathbf{a}^2, \tag{2.9}$$

$$\mathcal{A}^{A,\mathrm{bdry}}_{\mathcal{N},\mathrm{N}} = \mathcal{A}^{\Psi} = (N_f - N_c - 1)\operatorname{Tr}(\mathbf{s}^2) + N_c\operatorname{Tr}(\tilde{\mathbf{s}}^2), \tag{2.10}$$

$$\mathcal{A}^{B,\mathrm{bulk}}_{\mathcal{D},\mathrm{D,N,D}} = N_c\operatorname{Tr}(\tilde{\mathbf{s}}^2) - N_c\operatorname{Tr}(\mathbf{x}^2) - \frac{N_c}{2}(4N_f(1-r)^2 - 2N_c - 1)\mathbf{r}^2$$
$$+ 4(1-r)N_cN_f\mathbf{ar} - 2N_cN_f\mathbf{a}^2 \tag{2.11}$$

and it is easy to see that all dependence on $\mathbf{s}$ is cancelled and

$$\mathcal{A}^{A}_{\mathcal{N},\mathrm{N}} = \mathcal{A}^{A,\mathrm{bulk}}_{\mathcal{N},\mathrm{N}} + \mathcal{A}^{A,\mathrm{bdry}}_{\mathcal{N},\mathrm{N}} = \mathcal{A}^{B,\mathrm{bulk}}_{\mathcal{D},\mathrm{D,N,D}} = \mathcal{A}^{B}_{\mathcal{D},\mathrm{D,N,D}} . \tag{2.12}$$

Taking the opposite choice of boundary conditions for all fields we also get anomaly matching if we add the bifundamental Fermi to theory B instead of theory A. This also cancels the gauge anomaly in theory B.

We therefore have the following proposed dualities

- Theory A with $(\mathcal{N}, \mathrm{N})$ boundary conditions plus Fermi $\Psi \leftrightarrow$ Theory B with $(\mathcal{D}, \mathrm{D}, \mathrm{N}, \mathrm{D})$ boundary conditions.

- Theory A with $(\mathcal{D}, \mathrm{D})$ boundary conditions $\leftrightarrow$ Theory B with $(\mathcal{N}, \mathrm{N}, \mathrm{D}, \mathrm{N})$ boundary conditions plus Fermi Ψ.

For these proposed dualities we have only considered two-dimensional charged Fermi multiplets which cancel the gauge anomaly as minimal choices of boundary degrees of freedom. However, it would be interesting to allow two-dimensional vector and charged chiral multiplets to investigate other dual boundary conditions. We leave this to future work.

2.2 Supersymmetric indices

The supersymmetric "full-index" of 3d $\mathcal{N} = 2$ supersymmetric theories can be defined as a trace over the states on $S^2 \times \mathbb{R}$ [69, 70, 71, 72, 73]. It can be evaluated as a partition function on $S^2 \times S^1$ from the UV description and the UV formula was obtained in [70] for the theory with canonical conformal dimension and in [71] for the theory with any conformal dimension via the localisation technique.

The supersymmetric "half-index" of 3d $\mathcal{N} = 2$ supersymmetric theory $\mathcal{T}$ obeying the $\mathcal{N} = (0,2)$ half-BPS boundary condition $\mathcal{B}$ can similarly be defined as a trace over the states that correspond to half-BPS local operators on the boundary [39, 41, 42, 43]

$$\mathbb{II}_{\mathcal{B}}^{\mathcal{T}} = \text{Tr}_{\text{Op}}(-1)^F q^{J+\frac{R}{2}} x^f \tag{2.13}$$

where F is the Fermion number operator, J is the generator of the $Spin(2) \cong U(1)_J$ rotational symmetry of the two-dimensional plane where the boundary local operators are supported, R is the R-charge and f is the Cartan generators of the other global symmetry group.

One can compute the half-index as a partition function on $HS^2 \times S^1$ which encodes the $\mathcal{N} = (0,2)$ half-BPS boundary conditions on $\partial(HS^2 \times S^1) = S^1 \times S^1$ where HS^2 is a hemisphere. The UV formula for the half-index of the Neumann boundary condition for the gauge multiplet was derived in [39, 41, 42] and the formula for the Dirichlet boundary conditions for the gauge multiplet was proposed in [43]. The half-index has applications in the context of quantum K-theory [?] and can also realise the holomorphic block [74, 75], the q-series 3-manifold invariant $\hat{Z}$ [76] which has a number of applications to topology and number theory (see e.g. [77, 78, 79, 80, 81, 82]) and the 4-manifold invariant [83] (see e.g. [84]).

We summarise the general results here for the full-indices and half-indices, using notation $USp(2N_c) - [2N_f]^A$ for theory A with gauge group $USp(2N_c)$ and $2N_f$ fundamental chiral multiplets, and $USp(2\tilde{N}_c) - [2N_f]^B$ for the dual theory B.

2.2.1 Full-index

The test of the IR duality for the symplectic gauge group were performed by computing supersymmetric indices in [85, 22].

The full-index of theory A takes the form

$$I^{USp(2N_c)-[2N_f]^A}$$

$$= \frac{1}{2^{N_c} N_c!} \sum_{m_1, \cdots, m_{N_c} \in \mathbb{Z}} \oint \prod_{i=1}^{N_c} \frac{ds_i}{2\pi i s_i} (1 - q^{|m_i|} s_i^{\pm 2}) \prod_{1 \leq i < j \leq N_c} (1 - q^{\frac{|m_i \pm m_j|}{2}} s_i s_j^{\pm})(1 - q^{\frac{|-m_i \pm m_j|}{2}} s_i^{-1} s_j^{\pm})$$

$$\times \prod_{i=1}^{N_c} \prod_{\alpha=1}^{2N_f} \frac{(q^{1-\frac{r}{2}+\frac{|m_i|}{2}} a^{-1} s_i^{\pm} x_\alpha^{-1}; q)_\infty}{(q^{\frac{r}{2}+\frac{|m_i|}{2}} a s_i^{\pm} x_\alpha; q)_\infty} q^{N_f(1-r) \sum_{i=1}^{N_c} |m_i| - \sum_{i=1}^{N_c} |m_i| - \frac{1}{2} \sum_{i<j} |m_i \pm m_j|} a^{-2N_f \sum_{i=1}^{N_c} |m_i|}.$$

$$\tag{2.14}$$

where we note that a is the fugacity for the $U(1)_a$ symmetry which can be combined with the $SU(2N_f)$ flavour symmetry to form a $U(2N_f)$ flavour symmetry. Here we have set the $SU(2N_f)$ flavour fugacities $x_\alpha \to 1$ for simplicity.

The full-index of theory B, which is the dual of theory A with gauge group $USp(2N_c)$ so has gauge group $USp(2\tilde{N}_c = 2N_f - 2N_c - 2)$, is

$$I^{USp(2\tilde{N}_c) - [2N_f]^B}$$

$$= \frac{1}{2^{\tilde{N}_c}(\tilde{N}_c)!} \sum_{m_1, \cdots, m_{\tilde{N}_c} \in \mathbb{Z}} \oint \prod_{i=1}^{\tilde{N}_c} \frac{ds_i}{2\pi i s_i}$$

$$\times (1 - q^{|m_i|} s_i^{\pm 2}) \prod_{1 \le i < j \le \tilde{N}_c} (1 - q^{\frac{|m_i \pm m_j|}{2}} s_i s_j^\pm)(1 - q^{\frac{|-m_i \pm m_j|}{2}} s_i^{-1} s_j^\pm)$$

$$\times \prod_{i=1}^{\tilde{N}_c} \prod_{\alpha=1}^{2N_f} \frac{(q^{\frac{1+r+|m_i|}{2}} a s_i^\pm x_\alpha; q)_\infty}{(q^{\frac{1-r+|m_i|}{2}} a^{-1} s_i^\pm x_\alpha^{-1}; q)_\infty} q^{N_f r \sum_{i=1}^{\tilde{N}_c} |m_i| - \sum_{i=1}^{\tilde{N}_c} |m_i| - \frac{1}{2} \sum_{i<j} |m_i \pm m_j|} a^{2N_f \sum_{i=1}^{\tilde{N}_c} |m_i|}$$

$$\times \prod_{\alpha < \beta} \frac{(q^{1-r} a^{-2} x_\alpha^{-1} x_\beta^{-1}; q)_\infty}{(q^r a^2 x_\alpha x_\beta; q)_\infty} \frac{(q^{-\tilde{N}_c + rN_f} a^{2N_f}; q)_\infty}{(q^{\tilde{N}_c + 1 - rN_f} a^{-2N_f}; q)_\infty}. \tag{2.15}$$

2.2.2 Half-index

We can construct the half-indices for the 3d $\mathcal{N} = 2$ gauge theories with Dirichlet boundary condition $\mathcal{D}$ for the vector multiplet using the expressions in [43]. We briefly review the construction for both Neumann and Dirichlet boundary conditions, and describe some details for symplectic gauge groups in appendix A.3. We find for Dirichlet gauge field boundary conditions in theory A

$$\mathbb{III}_{\mathcal{D}}^{USp(2N_c)^A}$$

$$= \frac{1}{(q)_\infty^{N_c}} \sum_{m_i \in \mathbb{Z}} \frac{(-q^{1/2})^{k_{eff}[m,m]} u^{k_{eff}[m,-]}}{\prod_{i \ne j}^{N_c}(q^{1+m_i-m_j} u_i u_j^{-1}; q)_\infty \prod_{i \le j}(q^{1 \pm (m_i+m_j)} u_i^\pm u_j^\pm; q)_\infty} \mathbb{III}^{matter}(s_i \to q^{m_i} u_i) \tag{2.16}$$

where the effective CS coupling k_{eff} is determined by the 't Hooft anomaly. As we are not turning on background fluxes, the only term relevant is the gauge field contribution which is $\tilde{N}_c \operatorname{tr}(\mathbf{s}^2)$ for Dirichlet boundary conditions in theory A. This gives

$$k_{eff}[m, m] = 2\tilde{N}_c \sum_{i=1}^{N_c} m_i^2, \tag{2.17}$$

$$(-q^{1/2})^{k_{eff}[m,m]} = q^{\tilde{N}_c \sum_{i=1}^{N_c} m_i^2}, \tag{2.18}$$

$$u^{k_{eff}[m,-]} = \prod_{i=1}^{N_c} u_i^{2\tilde{N}_c m_i} \tag{2.19}$$

where we note that relative to the unitary cases [43] and the orthogonal cases we discuss later, there is a factor of 2 when mapping the anomaly polynomial terms to

the effective Chen-Simons coupling contribution. This factor arises from the conversion of the magnetic charge $m \in cochar(G)$ to the electric charge $km \in weight(G)$, noting that Cartan-Killing form for $G = USp(2N_c)$ differs by a factor of 2 from that for the unitary or orthogonal groups – in particular, the weights of $USp(2N_c)$ have length squared $\frac{1}{2}$, not 1.

For Neumann gauge field boundary conditions in theory A we have

$$\mathbb{III}_{\mathcal{N}}^{USp(2N_c)^A} = \frac{(q)_\infty^{N_c}}{2^{N_c}N_c!}\left(\prod_{i=1}^{N_c}\oint\frac{ds_i}{2\pi is_i}\right)\left(\prod_{i\neq j}(s_is_j^{-1};q)_\infty\right)\left(\prod_{i\leq j}(s_i^\pm s_j^\pm;q)_\infty\right)\mathbb{III}^{\mathrm{matter}}I^\Psi.$$

(2.20)

In both cases the matter contributions are given by the product of contributions for each chiral multiplet in the theory, depending on the chosen boundary conditions. For the bulk chiral multiplets in theories A and B we find the following contributions to $\mathbb{III}^{\mathrm{matter}}$:

$$\mathbb{III}_{\mathrm{N}}^Q = \prod_{\alpha=1}^{2N_f}\frac{1}{\prod_{i=1}^{N_c}(q^{\frac{r}{2}}s_iax_\alpha;q)_\infty(q^{\frac{r}{2}}s_i^{-1}ax_\alpha;q)_\infty},$$

(2.21)

$$\mathbb{III}_{\mathrm{D}}^Q = \prod_{\alpha=1}^{2N_f}\prod_{i=1}^{N_c}(q^{1-\frac{r}{2}}s_ia^{-1}x_\alpha^{-1};q)_\infty(q^{1-\frac{r}{2}}s_i^{-1}a^{-1}x_\alpha^{-1};q)_\infty,$$

(2.22)

$$\mathbb{III}_{\mathrm{N}}^q = \prod_{\alpha=1}^{2N_f}\frac{1}{\prod_{i=1}^{\tilde{N}_c}(q^{\frac{1-r}{2}}s_ia^{-1}x_\alpha^{-1};q)_\infty(s_i^{-1}a^{-1}x_\alpha^{-1}q^{(1-r)/2};q)_\infty},$$

(2.23)

$$\mathbb{III}_{\mathrm{D}}^q = \prod_{\alpha=1}^{2N_f}\prod_{i=1}^{\tilde{N}_c}(s_iax_\alpha q^{(1+r)/2};q)_\infty(q^{\frac{1-r}{2}}s_i^{-1}ax_\alpha;q)_\infty,$$

(2.24)

$$\mathbb{III}_{\mathrm{N}}^M = \prod_{\alpha<\beta}\frac{1}{(q^rx_\alpha x_\beta a^2;q)_\infty},$$

(2.25)

$$\mathbb{III}_{\mathrm{D}}^M = \prod_{\alpha<\beta}(x_\alpha^{-1}x_\beta^{-1}a^{-2}q^{1-r};q)_\infty,$$

(2.26)

$$\mathbb{III}_{\mathrm{N}}^V = \frac{1}{(q^{(1-r)N_f-N_c}a^{-2N_f};q)_\infty},$$

(2.27)

$$\mathbb{III}_{\mathrm{D}}^V = (q^{N_c-(1-r)N_f+1}a^{2N_f};q)_\infty.$$

(2.28)

The bifundamental Fermi would be expected to give contribution

$$I^{2d} = \prod_{i=1}^{N_c}\prod_{j=1}^{\tilde{N}_c}F(s_i^\pm u_j^\pm q^{1/2};q)F(s_i^\pm u_j^\mp q^{1/2};q),$$

(2.29)

$$F(s_i^\pm u_j^\pm q^{1/2};q)F(s_i^\pm u_j^\mp q^{1/2};q) = \left((s_iu_jq^{1/2};q)_\infty(s_iu_j^{-1}q^{1/2};q)_\infty(s_i^{-1}u_jq^{1/2};q)_\infty(s_i^{-1}u_j^{-1}q^{1/2};q)_\infty\right)^2,$$

(2.30)

but as noted for the anomaly contribution we should impose a reality condition. This will half the number of degrees of freedom so the contribution to the 2d index will be the square root of this quantity[6]. As it is a perfect square this gives a consistent contribution to the index

$$I^{\Psi} = \prod_{i=1}^{N_c} \prod_{j=1}^{\tilde{N}_c} (s_i u_j^{\pm} q^{1/2}; q)_{\infty} (s_i^{-1} u_j^{\pm} q^{1/2}; q)_{\infty}. \tag{2.31}$$

The half-indices for theory B are constructed in exactly the same way but exchanging $N_c \leftrightarrow \tilde{N}_c$ except for the matter contributions which are already written specifically for each theory. Note that since the dualities we consider are between theories with opposite boundary conditions for the vector multiplets it is consistent to use fugacities s_i for the preserved gauge symmetry in whichever theory has Neumann boundary conditions, and fugacities u_i in the other theory with Dirichlet boundary conditions breaking the gauge symmetry to a global symmetry.

We now present some examples which we have checked numerically to find matching indices. For calculational purposes these are all checked with flavour fugacities $x_{\alpha} = 1$. After presenting these examples we will construct the half-indices with non-zero Chern-Simons level.

2.3 $N_c = 1, N_f = 3$ $(USp(2) - [6])$

We start with the simplest example with $N_c = 1, N_f = 3$, where Theory A is $USp(2) = SU(2)$ gauge theory with six fundamental chiral multiplets Q.

The full-index is

$$I^{USp(2)-[6]^A} = \frac{1}{2} \sum_{m \in \mathbb{Z}} \oint \frac{ds}{2\pi i s} (1 - q^{|m|} s^{\pm 2}) \prod_{\alpha=1}^{6} \frac{(q^{1-\frac{r}{2}+\frac{|m|}{2}} a^{-1} s^{\pm} x_{\alpha}^{-1}; q)_{\infty}}{(q^{\frac{r}{2}+\frac{|m|}{2}} a s^{\pm} x_{\alpha}; q)_{\infty}} q^{3(1-r)|m|-|m|} a^{-6|m|}. \tag{2.32}$$

Theory B is $USp(2) = SU(2)$ gauge theory with six fundamental chiral multiplets q as well as gauge singlets M and V. The index (2.32) agrees with the index

$$I^{USp(2)-[6]^B} = \frac{1}{2} \sum_{m \in \mathbb{Z}} \oint \frac{ds}{2\pi i s} (1 - q^{|m|} s^{\pm 2}) \prod_{\alpha=1}^{6} \frac{(q^{\frac{1+r+|m|}{2}} a s^{\pm} x_{\alpha}; q)_{\infty}}{(q^{\frac{1-r+|m|}{2}} a^{-1} s^{\pm} x_{\alpha}^{-1}; q)_{\infty}} q^{3r|m|-|m|} a^{6|m|}$$
$$\times \prod_{\alpha<\beta} \frac{(q^{1-r} a^{-2} x_{\alpha}^{-1} x_{\beta}^{-1}; q)_{\infty}}{(q^r a^2 x_{\alpha} x_{\beta}; q)_{\infty}} \frac{(q^{3r-1} a^6; q)_{\infty}}{(q^{2-3r} a^{-6}; q)_{\infty}} \tag{2.33}$$

of Theory B. See [85] [7] and Appendix B.1 for the expansions of indices.

---

[6]There is no sign ambiguity as the contribution must be a q-series starting with $+1$ rather than -1.

[7]There is a typo in eq.(6) of [85].

2.3.1 $USp(2) - [6]$ **with** $\mathcal{N}, (\text{N}, \text{N}, \text{N}; \text{N}, \text{N}, \text{N}) + \text{Fermis}$

The half-index of Neumann b.c. $(\mathcal{N}, \text{N})$ together with charged Fermi multiplets for Theory A reads

$$
\mathbb{II}^{USp(2)-[6]^A}_{\mathcal{N},\text{N}+\Psi} = \frac{1}{2}(q)_\infty \oint \frac{ds}{2\pi i s}(s^2;q)_\infty(s^{-2};q)_\infty \frac{(q^{\frac{1}{2}}s^\pm u^\pm;q)_\infty(q^{\frac{1}{2}}s^\pm u^\mp;q)_\infty}{(q^{\frac{r}{2}}as^\pm;q)^6_\infty}. \tag{2.34}
$$

We have checked that the half-index (2.34) agrees with the half-index

$$
\mathbb{II}^{USp(2)-[6]^B}_{\mathcal{D},\text{D},\text{N},\text{D}} = \frac{1}{(q)_\infty} \sum_{m\in\mathbb{Z}} \frac{1}{(q^{1+2m}u^2;q)_\infty(q^{1-2m}u^{-2};q)_\infty}(q^{\frac{1+r\pm m}{2}}au^\pm;q)^6_\infty q^{m^2}u^{2m}
$$
$$
\times \frac{1}{(q^r a^2;q)^{15}_\infty}(q^{3r-1}a^6;q)_\infty \tag{2.35}
$$

of Dirichlet b.c. $(\mathcal{D}, \text{D}, \text{N}, \text{D})$ for Theory B. See Appendix B.1.2 for the expansion of indices.

2.3.2 $USp(2) - [6]$ **with** $\mathcal{D}, (\text{D}, \text{D}, \text{D}; \text{D}, \text{D}, \text{D})$

The half-index of Dirichlet b.c. $(\mathcal{D}, \text{D})$ for Theory A takes the form

$$
\mathbb{II}^{USp(2)-[6]^A}_{\mathcal{D},\text{D}} = \frac{1}{(q)_\infty} \sum_{m\in\mathbb{Z}} \frac{1}{(q^{1+2m}u^2;q)_\infty(q^{1-2m}u^{-2};q)_\infty}(q^{1-\frac{r}{2}\pm m}a^{-1}u^\pm;q)^6_\infty q^{m^2}u^{2m}. \tag{2.36}
$$

On the other hand, the half-index of Neumann b.c. $(\mathcal{N}, \text{N}, \text{D}, \text{N})$ plus charged Fermi multiplets for Theory B is

$$
\mathbb{II}^{USp(2)-[6]^B}_{\mathcal{N},\text{N},\text{D},\text{N}+\Psi} = \frac{1}{2}(q)_\infty \oint \frac{ds}{2\pi i s}(s^2;q)_\infty(s^{-2};q)_\infty
$$
$$
\times \frac{1}{(q^{\frac{1-r}{2}}a^{-1}s^\pm;q)^6_\infty}(q^{\frac{1}{2}}s^\pm u^\pm;q)_\infty(q^{\frac{1}{2}}s^\pm u^\mp;q)_\infty(q^{1-r}a^{-2};q)^{15}_\infty \frac{1}{(q^{2-3r}a^{-6};q)_\infty}. \tag{2.37}
$$

We have confirmed that the half-indices (2.36) and (2.37) perfectly match. See Appendix B.1.3 for the expansion of indices.

2.4 $N_c = 2, N_f = 5$ $(USp(4) - [10])$

Next example is the case with $N_c = 2, N_f = 5$, where Theory A has gauge group $USp(4)$ and ten fundamental flavours.

The full-index of Theory A is

$$I^{USp(4)-[10]^A} = \frac{1}{2^2 \cdot 2} \sum_{m_1,m_2 \in \mathbb{Z}} \oint \prod_{i=1}^{2} \frac{ds_i}{2\pi i s_i} (1 - q^{|m_i|} s_i^{\pm 2})(1 - q^{\frac{|m_1 \pm m_2|}{2}} s_1 s_2^{\pm})(1 - q^{\frac{|-m_1 \pm m_2|}{2}} s_1^{-1} s_2^{\pm})$$

$$\times \prod_{i=1}^{2} \prod_{\alpha=1}^{10} \frac{(q^{1-\frac{r}{2}+\frac{|m_i|}{2}} a^{-1} s_i^{\pm} x_\alpha^{-1}; q)_\infty}{(q^{\frac{r}{2}+\frac{|m_i|}{2}} a s_i^{\pm} x_\alpha; q)_\infty} q^{5(1-r) \sum_{i=1}^{2} |m_i| - \sum_{i=1}^{2} |m_i| - \frac{1}{2}|m_1 \pm m_2|} a^{-10 \sum_{i=1}^{2} |m_i|}.$$

$$(2.38)$$

Unlike the previous example, Theory B has a different gauge group, i.e. $USp(4)$ as well as ten fundamental chiral multiplets and neutral chiral multiplets.

We have the full-index

$$I^{USp(4)-[10]^B} = \frac{1}{2^2 \cdot 2} \sum_{m_1,m_2 \in \mathbb{Z}} \oint \prod_{i=1}^{2} \frac{ds_i}{2\pi i s_i} (1 - q^{|m_i|} s_i^{\pm 2})(1 - q^{\frac{|m_1 \pm m_2|}{2}} s_1 s_2^{\pm})(1 - q^{\frac{|-m_1 \pm m_2|}{2}} s_1^{-1} s_2^{\pm})$$

$$\times \prod_{i=1}^{2} \prod_{\alpha=1}^{10} \frac{(q^{\frac{1+r+|m|}{2}} a s_i^{\pm} x_\alpha; q)_\infty}{(q^{\frac{1-r+|m|}{2}} a^{-1} s_i^{\pm} x_\alpha^{-1}; q)_\infty} q^{5r \sum_{i=1}^{2} |m_i| - \sum_{i=1}^{2} |m_i| - \frac{1}{2}|m_1 \pm m_2|} a^{10 \sum_{i=1}^{2} |m_i|}$$

$$\times \prod_{\alpha<\beta} \frac{(q^{1-r} a^{-2} x_\alpha^{-1} x_\beta^{-1}; q)_\infty}{(q^r a^2 x_\alpha x_\beta; q)_\infty} \frac{(q^{5r-2} a^{10}; q)_\infty}{(q^{3-5r} a^{-10}; q)_\infty} \qquad (2.39)$$

of Theory B. One finds the agreement of indices (2.38) and (2.39). See [85] and Appendix B.1 for the expansion of indices.

2.4.1 $USp(4) - [10]$ **with** $\mathcal{N}, (N, N, N, N, N; N, N, N, N, N) + $ Fermis

The half-index of Neumann b.c. $(\mathcal{N}, N)$ together with the charged Fermi multiplets for Theory A is calculated as

$$\mathbb{III}_{\mathcal{N},N+\Psi}^{USp(4)-[10]^A} = \frac{1}{2^2 \cdot 2} (q)_\infty^2 \oint \frac{ds_1}{2\pi i s_1} \frac{ds_2}{2\pi i s_2} (s_1^{\pm 2}; q)_\infty (s_2^{\pm 2}; q)_\infty (s_1 s_2^{\pm}; q)_\infty (s_1^{-1} s_2^{\pm}; q)_\infty$$

$$\times \prod_{i=1}^{2} \frac{1}{(q^{\frac{r}{2}} a s_i^{\pm}; q)_\infty^{10}} \prod_{i,j=1}^{2} (q^{\frac{1}{2}} s_i^{\pm} u_j^{\pm}; q)_\infty (q^{\frac{1}{2}} s_i^{\pm} u_j^{\mp}; q)_\infty. \qquad (2.40)$$

The half-index of Dirichlet b.c. $(\mathcal{D}, D, N, D)$ for Theory B is

$$\mathbb{III}_{\mathcal{D},D,N,D}^{USp(4)-[10]^B}$$

$$= \frac{1}{(q)_\infty^2} \sum_{m_1,m_2 \in \mathbb{Z}} \frac{1}{\prod_{i=1}^{2} (q^{1 \pm 2m_i} u_i^{\pm 2}; q)_\infty (q^{1+m_1 \pm m_2} u_1 u_2^{\pm}; q)_\infty (q^{1-m_1 \pm m_2} u_1^{-1} u_2^{\pm}; q)_\infty}$$

$$\times \prod_{i=1}^{2} (q^{\frac{1+r \pm m_i}{2}} a u_i^{\pm}; q)_\infty^{10} q^{2 \sum_{i=1}^{2} m_i^2} u^{4 \sum_{i=1}^{2} m_i} \frac{1}{(q^r a^2; q)_\infty^{45}} (q^{5r-2} a^{10}; q)_\infty. \qquad (2.41)$$

As shown in Appendix B.1.2, we find that the half-indices (2.40) and (2.41) precisely match.

2.4.2 $USp(4) - [10]$ **with** $\mathcal{D}, (D, D, D, D, D; D, D, D, D, D)$

For Dirichlet b.c. $(\mathcal{D}, D)$, we have the half-index of Theory A:

$$
\mathbb{III}^{USp(4)-[10]^A}_{\mathcal{D},D}
$$

$$
= \frac{1}{(q)^2_\infty} \sum_{m_1,m_2\in\mathbb{Z}} \frac{1}{\prod_{i=1}^2 (q^{1\pm 2m_i}u_i^{\pm 2};q)_\infty (q^{1+m_1\pm m_2}u_1 u_2^\pm;q)_\infty (q^{1-m_1\pm m_2}u_1^{-1}u_2^\pm;q)_\infty}
$$

$$
\times \prod_{i=1}^2 (q^{1-\frac{r}{2}\pm m_i}a^{-1}u_i^\pm;q)^{10}_\infty q^{2\sum_{i=1}^2 m_i^2} u^{4\sum_{i=1}^2 m_i}. \tag{2.42}
$$

We have confirmed that the half-index (2.40) agrees with the half-index

$$
\mathbb{III}^{USp(4)-[10]^B}_{\mathcal{N},N,D,N+\Psi} = \frac{1}{2^2\cdot 2}(q)^2_\infty \oint \frac{ds_1}{2\pi i s_1}\frac{ds_2}{2\pi i s_2}(s_1^{\pm 2};q)_\infty (s_2^{\pm 2};q)_\infty (s_1 s_2^\pm;q)_\infty (s_1^{-1}s_2^\pm;q)_\infty
$$

$$
\times \prod_{i=1}^2 \frac{1}{(q^{\frac{1-r}{2}}a^{-1}s^\pm;q)^{10}_\infty} \prod_{i,j=1}^2 (q^{\frac{1}{2}}s_i^\pm u_j^\pm;q)_\infty (q^{\frac{1}{2}}s_i^\pm u_j^\mp;q)_\infty
$$

$$
\times (q^{1-r}a^{-2};q)^{45}_\infty \frac{1}{(q^{3-5r}a^{-10};q)_\infty} \tag{2.43}
$$

of Neumann b.c. $(\mathcal{N}, N, D, N)$ for Theory B which involves the charged Fermi multiplets. See B.1.3 for the expansion of indices.

2.5 Chern-Simons level $k \neq 0$

As shown in [21] we can induce a non-zero Chern-Simons coupling starting from the case of vanishing Chern-Simons level with $N_F + 2|k|$ fundamental chirals and giving masses to $2|k|$ of them. Taking the masses to $\pm\infty$ we integrate out these $2|k|$ chirals and are left with N_F flavours and Chern-Simons coupling $k = \pm|k|$. Here the $\pm$ signs are the same for the masses and the sign of k. Note that we only require $2|k| \in \mathbb{Z}$ so although $N_F + 2|k|$ is an even integer (which we can label $2N_f$ as in the case of $k = 0$), N_F is only constrained to be integer. In the dual theory B the corresponding chirals get masses of the opposite sign and consequently theory B gets a Chern-Simons level $-k$. This produces a duality $USp(2N_c)_k \leftrightarrow USp(2|k| + N_F - 2N_c - 2)_{-k}$ with N_F fundamental chirals in both theories.

As shown in [43], chiral edge modes are present for bulk chirals with Neumann boundary conditions and positive masses, and with Dirichlet boundary conditions and negative masses. These edge modes introduce additional 2d chiral or Fermi multiplets. On the other hand, with negative masses for Neumann and positive masses for Dirichlet boundary conditions there are no edge modes and the dualities follow through without additional 2d multiplets. We focus on the latter cases for simplicity.

To derive the half-indices with $k \neq 0$ we can take limits of the fugacities following the procedure described in [21] for the full indices, and somewhat similar to the process outlined in [43] for unitary half-indices. This will result in the matching of half-indices

$$\mathbb{III}_{\mathcal{D},\mathrm{D}}^{USp(2N_c)_k - [N_F]^A} = \mathbb{III}_{\mathcal{N},\mathrm{N},\mathrm{D}+\Psi}^{USp(2\tilde{N}_c)_{-k} - [N_F]^B} \qquad\qquad k > 0 \qquad\qquad (2.44)$$

$$\mathbb{III}_{\mathcal{N},\mathrm{N}+\Psi}^{USp(2N_c)_k - [N_F]^A} = \mathbb{III}_{\mathcal{D},\mathrm{D},\mathrm{N}}^{USp(2\tilde{N}_c)_{-k} - [N_F]^B} \qquad\qquad k < 0 \qquad\qquad (2.45)$$

where we note the singlet V is also integrated out for $k \neq 0$.

These half-indices are given by a simple prescription of removing the $\mathbb{III}_V$ contribution which is present in the half-indices for the $k = 0$ cases and noting that $\tilde{N}_c = |k| + N_F/2 - N_c - 1$.

A derivation follows from the limit taking masses to infinity as explained in [21]. First write $SU(2N_f) \times U(1)_a$ as $U(2N_f)$ with fugacities $\hat{x}_\alpha = a x_\alpha$ and in $\mathbb{III}^V$ replace $a^{2N_f} \to \prod_{\alpha=1}^{2N_f} \hat{x}_\alpha$. Note that while $\prod_{\alpha=1}^{2N_f} x_\alpha = 1$ for the $SU(2N_f)$ fugacities x_α, there is no such constraint on the $U(2N_f)$ fugacities $\hat{x}_\alpha$. Now, for the $2|k|$ values of α corresponding to massive chirals we take the limit $\hat{x}_\alpha \to 0$ for negative mass in theory A and $\hat{x}_\alpha \to \infty$ for positive mass in theory A. For the simple choice of boundary conditions we are considering it is easy to see that this simply removes (set to 1) the contributions to the half-indices from those $2|k|$ fundamental chirals in theories A and B and at the same time removes the contribution from the corresponding parts of M and the contribution of V. Hence, in the cases we consider, this is a well-defined limit and the matching of half-indices follows for the matching for $k = 0$, assuming that holds for arbitrary flavour fugacities. We can then write the remaining $U(N_F)$ flavour symmetry as $SU(N_F) \times U(1)_a$ and replace the remaining N_F fugacities $\hat{x}_\alpha$ with $a x_\alpha$, now with the constraint on the $SU(N_F)$ fugacities $\prod_{\alpha=1}^{N_F} x_\alpha = 1$. This gives precisely the half-indices following the simple prescription above of simply omitting the contributions from the massive fields. Note that if we took the limits with the 'wrong' boundary conditions, the individual contributions would not have well-defined limits, and indeed we would need to introduce appropriate additional 2d chirals or Fermis to get well-defined matching half-indices.

One additional point to note is that if we calculate the anomaly polynomials for the duals with $k \neq 0$ simply by including the CS contribution $\pm k \operatorname{Tr}(\mathbf{s}^2)$ for theory A and $\mp k \operatorname{Tr}(\tilde{\mathbf{s}}^2)$ for theory B, we would naively not get matching results. This is because there are background Chern-Simons terms generated by integrating out the massive fields and these contributions only taken into account those associated to the gauge groups. Including these background Chern-Simons levels we do have matching anomaly polynomials and indeed we can see these Chern-Simons levels by simply taking the difference in the anomaly polynomial before and after integrating out the

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massive fields in each theory. As we have not turned on background fluxes, our half-indices are not sensitive to these other background Chern-Simons levels. In particular we have the following contributions from background Chern-Simons levels, in addition to the $\pm k \, \mathrm{Tr}(\mathbf{s}^2)$ or $\mp k \, \mathrm{Tr}(\tilde{\mathbf{s}}^2)$ contributions

$$
\begin{aligned}
\mathcal{A}_{CS\,\mathcal{N},\mathrm{N}}^{USp(2N_c)_{k<0}-[N_F]^A} &= \mathcal{A}_{\mathcal{N},\mathrm{N}}^{USp(2N_c)-[N_F+2|k|]^A} - \mathcal{A}_{\mathcal{N},\mathrm{N}}^{USp(2N_c)_{k<0}-[N_F]^A} \\
&= 2kN_c \mathbf{a}^2 + 4kN_c(r-1)\mathbf{ar} + 2kN(r-1)^2\mathbf{r}^2,
\end{aligned} \tag{2.46}
$$

$$
\begin{aligned}
\mathcal{A}_{CS\,\mathcal{D},\mathrm{D,N}}^{USp(2\tilde{N}_c)_{k<0}-[N_F]^B} &= \mathcal{A}_{\mathcal{D},\mathrm{D,N,D}}^{USp(2\tilde{N}_c)-[N_F+2|k|]^B} - \mathcal{A}_{\mathcal{D},\mathrm{D,N}}^{USp(2\tilde{N}_c)_{k<0}-[N_F]^B} \\
&= k\,\mathrm{Tr}(\mathbf{x}^2) + 2\left(k(N_c+N_f)+N_f^2\right)\mathbf{a}^2 \\
&\quad + \left(4k((r-1)N_c + rN_f) + 2N_f(2N_c + 2N_f(r-1)+1)\right)\mathbf{ar} \\
&\quad + \frac{1}{2}\left(2k^2 + k\left(4N\left(r^2-2r+2\right)+4N_f\left(r^2-1\right)+3\right)+(2N+2N_f(r-1)+1)^2\right)\mathbf{r}^2,
\end{aligned} \tag{2.47}
$$

$$
\begin{aligned}
\mathcal{A}_{CS\,\mathcal{D},\mathrm{D}}^{USp(2N_c)_{k>0}-[N_F]^A} &= \mathcal{A}_{\mathcal{D},\mathrm{D}}^{USp(2N_c)-[N_F+2|k|]^A} - \mathcal{A}_{\mathcal{D},\mathrm{D}}^{USp(2N_c)_{k>0}-[N_F]^A} \\
&= 2kN_c\mathbf{a}^2 + 4kN_c(r-1)\mathbf{ar} + 2kN(r-1)^2\mathbf{r}^2,
\end{aligned} \tag{2.48}
$$

$$
\begin{aligned}
\mathcal{A}_{CS\,\mathcal{N},\mathrm{N,D}}^{USp(2\tilde{N}_c)_{k>0}-[N_F]^B} &= \mathcal{A}_{\mathcal{N},\mathrm{N,D,N}}^{USp(2\tilde{N}_c)-[N_F+2|k|]^B} - \mathcal{A}_{\mathcal{N},\mathrm{N,D}}^{USp(2\tilde{N}_c)_{k>0}-[N_F]^B} \\
&= k\,\mathrm{Tr}(\mathbf{x}^2) + 2\left(k(N_c+N_f)-N_f^2\right)\mathbf{a}^2 \\
&\quad + \left(4k((r-1)N_c + rN_f) - 2N_f(2N_c + 2N_f(r-1)+1)\right)\mathbf{ar} \\
&\quad - \frac{1}{2}\left(2k^2 - k\left(4N\left(r^2-2r+2\right)+4N_f\left(r^2-1\right)+3\right)+(2N+2N_f(r-1)+1)^2\right)\mathbf{r}^2.
\end{aligned} \tag{2.49}
$$

In the following, we explicitly show examples of the half-indices with CS level $k \neq 0$ for which we have checked the precise agreement.

2.5.1 $\quad USp(2)_{\pm k} - [6-2k]$

For $k \in \{\frac{1}{2}, 1, \frac{3}{2}, \dots 3\}$ we have the matching pairs of half-indices

$$
\mathbb{III}_{\mathcal{N},\mathrm{N}+\Psi}^{USp(2)_{-k}-[6-2k]^A} = \frac{1}{2}(q)_\infty \oint \frac{ds}{2\pi i s}(s^{\pm 2};q)_\infty \frac{(q^{1/2}s^{\pm}u^{\pm};q)_\infty(q^{1/2}s^{\pm}u^{\mp};q)_\infty}{(q^{r/2}as^{\pm};q)_\infty^{6-2k}}, \tag{2.50}
$$

$$
\mathbb{III}_{\mathcal{D},\mathrm{D,N,D}}^{USp(2)_k-[6-2k]^B} = \frac{1}{(q)_\infty} \sum_{m \in \mathbb{Z}} \frac{q^{m^2}u^2}{(q^{1\pm 2m}u^{\pm 2};q)_\infty}(q^{(1+r)/2\pm m}au^{\pm};q)_\infty^{6-2k} \frac{1}{(q^r a^2;q)_\infty^{(6-2k)(5-2k)/2}}, \tag{2.51}
$$

and

$$\mathbb{III}^{USp(2)_k-[6-2k]^A}_{\mathcal{D},\mathrm{D}} = \frac{1}{(q)_\infty} \sum_{m\in\mathbb{Z}} \frac{q^{m^2}u^2}{(q^{1\pm2m}u^{\pm2};q)_\infty}(q^{1-r/2\pm m}a^{-1}u^\pm;q)^{6-2k}_\infty, \tag{2.52}$$

$$\mathbb{III}^{USp(2)_{-k}-[6-2k]^B}_{\mathcal{N},\mathrm{N},\mathrm{D},\mathrm{N}+\Psi} = \frac{1}{2}(q)_\infty \oint \frac{ds}{2\pi is}(s^{\pm2};q)_\infty \frac{(q^{1/2}s^\pm u^\pm;q)_\infty(q^{1/2}s^\pm u^\mp;q)_\infty}{(q^{(1-r)/2}a^{-1}s^\pm;q)^{6-2k}_\infty}(q^{1-r}a^{-2};q)^{(6-2k)(5-2k)/2}_\infty . \tag{2.53}$$

3 $SO(N_c)$ gauge theories

The quantum dynamics of 3d $\mathcal{N}=2$ supersymmetric gauge theories with orthogonal gauge groups which is more subtle than with the unitary or symplectic gauge groups has been studied in [23, 24].

The IR dualities associated with the Lie algebra $\mathfrak{g} = \mathfrak{so}(N_c)$ of gauge symmetry are derived in [24] by taking an appropriate limit of the 4d $\mathcal{N}=1$ gauge theories on a circle. There are three distinct 4d gauge theories for $\mathfrak{g} = \mathfrak{so}(N_c)$ with gauge group $SO(N_c)_+$, $SO(N_c)_-$ or $Spin(N_c)$ where $SO(N_c)_\pm$ are distinguished by $\mathbb{Z}_2^{\mathcal{M}}$, a $\mathbb{Z}_2$ global symmetry that changes the sign of the non-trivial line operator [86].

The 4d duality between $SO(N_c)_+$ gauge theories leads to the 3d IR duality [24]:

- Theory A: $SO(N_c)$ gauge theory with N_f chiral multiplets Q in the vector representation.

- Theory B: $SO(\tilde{N}_c = N_f - N_c + 2)$ gauge theory with N_f chiral multiplets q in the vector representation, $N_f(N_f + 1)/2$ neutral chiral multiplets M in the rank-2 symmetric representation of $SU(N_f)$ and a chiral multiplet V which has the superpotential

$$\mathcal{W} = \frac{1}{2}Mqq + \frac{i^{N_f-N_c}}{4}V\tilde{V}. \tag{3.1}$$

The charges of the chiral multiplets are

	$G = SO(N_c)$	$\tilde{G} = SO(\tilde{N}_c)$	$SU(N_f)$	$U(1)_a$	$U(1)_R$	$\mathbb{Z}_2^{\mathcal{C}}$	$\mathbb{Z}_2^{\mathcal{M}}$	$\mathbb{Z}_2^{\tilde{\mathcal{C}}}$	$\mathbb{Z}_2^{\tilde{\mathcal{M}}}$
Q	$\mathbf{N_c}$	$\mathbf{1}$	$\mathbf{N_f}$	$+$	r	0	0	0	0
q	1	$\mathbf{N_f - N_c + 2}$	$\mathbf{N_f}$	$-$	$1-r$	0	0	0	0
M	1	1	$\frac{\mathbf{N_f(N_f+1)}}{\mathbf{2}}$	2	$2r$	0	0	$+$	$+$
V	1	1	1	$-N_f$	$\tilde{N}_c - rN_f$	0	0	$+$	$-$

$$\tag{3.2}$$

Here $\mathbb{Z}_2^{\mathcal{C}}$ (resp. $\mathbb{Z}_2^{\tilde{\mathcal{C}}}$) is the charge conjugation symmetry in theory A (resp. theory B). $\mathbb{Z}_2^{\mathcal{M}}$ (resp. $\mathbb{Z}_2^{\tilde{\mathcal{M}}}$) is the magnetic symmetry in theory A (resp. theory B) [65].

3.1 $\mathcal{N} = (0,2)$ half-BPS boundary conditions

The requirement of gauge anomaly cancellation and 't Hooft anomaly matching are the same as for symplectic gauge theories. Again, using the general results presented in [43] and summarised in appendix A.2 we find the following specific results for the fields in the $SO(N_c) - [N_f]$ theory A and $SO(N_f - N_c + 2) - [N_f]$ theory B

$$\mathcal{A}^{\mathrm{VM}} = -(N_c - 2)\operatorname{Tr}(\mathbf{s}^2) - \frac{N_c}{4}(N_c - 1)\mathbf{r}^2, \tag{3.3}$$

$$\mathcal{A}^Q = N_f \operatorname{Tr}(\mathbf{s}^2) + \frac{N_c}{2}\operatorname{Tr}(\mathbf{x}^2) + \frac{1}{2}N_c N_f(\mathbf{a} + (r-1)\mathbf{r})^2, \tag{3.4}$$

$$\mathcal{A}^q = N_f \operatorname{Tr}(\tilde{\mathbf{s}}^2) + \frac{1}{2}(N_f - N_c + 2)\operatorname{Tr}(\mathbf{x}^2) + \frac{1}{2}(N_f - N_c + 2)N_f(-\mathbf{a} - r\mathbf{r})^2, \tag{3.5}$$

$$\mathcal{A}^M = \frac{1}{2}(N_f + 2)\operatorname{Tr}(\mathbf{x}^2) + \frac{1}{4}N_f(N_f + 1)(2\mathbf{a} + (2r - 1)\mathbf{r})^2, \tag{3.6}$$

$$\mathcal{A}^V = \frac{1}{2}(-N_f \mathbf{a} + ((1 - r)N_f - N_c + 2)\mathbf{r})^2 \tag{3.7}$$

for Dirichlet boundary conditions. For Neumann boundary conditions the contributions are the same but with opposite sign. As for the symplectic case, the notation $\mathbf{s}$ ($\tilde{\mathbf{s}}$) refers to the gauge field strength in theory A (B), $\mathbf{x}$ to the field strength for the global $SU(N_f)$ flavour symmetry, and $\mathbf{a}$ and $\mathbf{r}$ to the field strengths for the global $U(1)_a$ and $U(1)_R$ symmetries.

We also need 2d boundary matter and the only multiplet required for the examples we consider is an $SO(N_c) \times SO(N_f - N_c + 2)$ bifundamental Fermi. Analogously to the symplectic case, here $SO(N_c)$ is the 2d gauge group inherited from the bulk vector multiplet with Neumann boundary conditions while $SO(N_f - N_c + 2)$ is a global flavour symmetry which is identified with the dual gauge group broken to a global symmetry by the Dirichlet boundary conditions for the dual vector multiplet. This gives anomaly contribution

$$\mathcal{A}^\Psi = (N_f - N_c + 2)\operatorname{Tr}(\mathbf{s}^2) + N_c \operatorname{Tr}(\tilde{\mathbf{s}}^2). \tag{3.8}$$

Note that as for the symplectic case this contribution is actually only half of what might be expected for such a Fermi. However, this is precisely the contribution required for anomaly cancellation and we interpret this as due to a reality condition on the Fermi. We will comment again on this when discussing the Fermi contribution to the half-index.

For now we consider the 3d dualities with vanishing Chern-Simons level, in which case we have no background Chern-Simons terms.

The boundary conditions we consider are $(\mathcal{N}, \mathrm{N})$ in theory A, referring to the choice of Neumann boundary conditions for (VM, Q) together with $(\mathcal{D}, \mathrm{D}, \mathrm{N}, \mathrm{D})$ in theory B

for (VM, q, M, V). With this choice we need to cancel the gauge anomaly in theory A which we can do by including the bifundamental Fermi in theory A. This also leads to anomaly matching with theory B without any further 2d matter. In particular we have

$$\mathcal{A}^{A,\text{bulk}}_{\mathcal{N},\text{N}} = -(N_f - N_c + 2)\,\text{Tr}(\mathbf{s}^2) - \frac{N_c}{2}\,\text{Tr}(\mathbf{x}^2) - \frac{N_c}{4}(2N_f(1-r)^2 - N_c + 1)\mathbf{r}^2$$
$$+ (1-r)N_c N_f \mathbf{a}\mathbf{r} - \frac{1}{2}N_c N_f \mathbf{a}^2, \tag{3.9}$$

$$\mathcal{A}^{A,\text{bdry}}_{\mathcal{N},\text{N}} = \mathcal{A}^{\Psi} = (N_f - N_c + 2)\,\text{Tr}(\mathbf{s}^2) + N_c\,\text{Tr}(\tilde{\mathbf{s}}^2), \tag{3.10}$$

$$\mathcal{A}^{B,\text{bulk}}_{\mathcal{D},\text{D,N,D}} = N_c\,\text{Tr}(\tilde{\mathbf{s}}^2) - \frac{N_c}{2}\,\text{Tr}(\mathbf{x}^2) - \frac{N_c}{4}(2N_f(1-r)^2 - N_c + 1)\mathbf{r}^2$$
$$+ (1-r)N_c N_f \mathbf{a}\mathbf{r} - \frac{1}{2}N_c N_f \mathbf{a}^2, \tag{3.11}$$

and it is easy to see that all dependence on $\mathbf{s}$ is cancelled and

$$\mathcal{A}^{A}_{\mathcal{N},\text{N}} = \mathcal{A}^{A,\text{bulk}}_{\mathcal{N},\text{N}} + \mathcal{A}^{A,\text{bdry}}_{\mathcal{N},\text{N}} = \mathcal{A}^{B,\text{bulk}}_{\mathcal{D},\text{D,N,D}} = \mathcal{A}^{B}_{\mathcal{D},\text{D,N,D}}\,. \tag{3.12}$$

Taking the opposite choice of boundary conditions for all fields we also get anomaly matching if we add the bifundamental Fermi to theory B instead of theory A. This also cancels the gauge anomaly in theory B.

We therefore have the following proposed dualities

- Theory A with $(\mathcal{N}, \text{N})$ boundary conditions plus Fermi $\Psi \leftrightarrow$ Theory B with $(\mathcal{D}, \text{D}, \text{N}, \text{D})$ boundary conditions.

- Theory A with $(\mathcal{D}, \text{D})$ boundary conditions $\leftrightarrow$ Theory B with $(\mathcal{N}, \text{N}, \text{D}, \text{N})$ boundary conditions plus Fermi Ψ.

and in all cases we have an identification of discrete fugacities (defined in the following section) $\tilde{\zeta}$ and $\tilde{\chi}$ in theory B in terms of those in theory A, $\tilde{\zeta} = \zeta$ and $\tilde{\chi} = \zeta\chi$.

3.2 Supersymmetric indices

3.2.1 Full-index

For the orthogonal gauge groups the Seiberg-like dualities have been tested from the evaluation of supersymmetric indices in [22, 87, 24]. The index depends on the global structure of the gauge group. As shown in [24] all the indices can be constructed from the $SO(N_c)$ indices provided we include discrete fugacities ζ and χ for the $\mathbb{Z}_2^{\mathcal{M}}$ and $\mathbb{Z}_2^{\mathcal{C}}$ groups.

For $\chi = +1$ or N_c being odd the full-index of theory A with gauge group $G = SO(N_c = 2N + \epsilon)$ for $\epsilon \in \{0,1\}$ reads

$$I^{SO(N_c=2N+\epsilon)-[N_f]^A_{\zeta,\chi}}$$

$$= \frac{1}{2^{N+\epsilon-1}N!} \sum_{m_1,\cdots,m_N} \oint \prod_{i=1}^{N} \frac{ds_i}{2\pi i s_i} \left[\prod_{i=1}^{N} (1 - \chi q^{\frac{|m_i|}{2}} s_i)(1 - \chi q^{\frac{|m_i|}{2}} s_i^{-1}) \right]^{\epsilon}$$

$$\times \prod_{i<j} (1 - q^{\frac{|m_i-m_j|}{2}} s_i s_j^{-1})(1 - q^{\frac{|m_i-m_j|}{2}} s_i^{-1} s_j)(1 - q^{\frac{|m_i+m_j|}{2}} s_i s_j)(1 - q^{\frac{|m_i+m_j|}{2}} s_i^{-1} s_j^{-1})$$

$$\times \left[\prod_{\alpha=1}^{N_f} \frac{(\chi^{-1} q^{1-\frac{r}{2}} a^{-1} x_\alpha^{-1}; q)_\infty}{(\chi q^{\frac{r}{2}} a x_\alpha; q)_\infty} \right]^{\epsilon} \prod_{i=1}^{N} \prod_{\alpha=1}^{N_f} \frac{(q^{1-\frac{r+|m|}{2}} a^{-1} s_i^{\mp} x_\alpha^{-1}; q)_\infty}{(q^{\frac{r+|m|}{2}} a s_i^{\pm} x_\alpha; q)_\infty}$$

$$\times q^{N_f \sum_{i=1}^{N} \frac{1-r}{2} |m_i| - \sum_{i=1}^{N} \frac{|m_i|}{2} \epsilon - \sum_{i<j} \frac{|m_i-m_j|}{2} - \sum_{i<j} \frac{|m_i+m_j|}{2}} a^{-\sum_{i=1}^{N} |m_i|} \zeta^{\sum_{i=1}^{N} m_i}. \qquad (3.13)$$

For $\chi = -1$ and N_c even the full-index of theory A is

$$I^{SO(N_c=2N)-[N_f]^A_{\zeta,\chi=-}}$$

$$= \frac{1}{2^{N-1}(N-1)!} \sum_{m_1,\cdots,m_{N-1}} \oint \prod_{i=1}^{N-1} \frac{ds_i}{2\pi i s_i} \prod_{i=1}^{N-1} (1 - q^{|m_i|} s_i^2)(1 - q^{|m_i|} s_i^{-2})$$

$$\times \prod_{i<j} (1 - q^{\frac{|m_i-m_j|}{2}} s_i s_j^{-1})(1 - q^{\frac{|m_i-m_j|}{2}} s_i^{-1} s_j)(1 - q^{\frac{|m_i+m_j|}{2}} s_i s_j)(1 - q^{\frac{|m_i+m_j|}{2}} s_i^{-1} s_j^{-1})$$

$$\times \prod_{\alpha=1}^{N_f} \frac{(\pm q^{1-\frac{r}{2}} a^{-1} x_\alpha^{-1}; q)_\infty}{(\pm q^{\frac{r}{2}} a x_\alpha; q)_\infty} \prod_{i=1}^{N-1} \prod_{\alpha=1}^{N_f} \frac{(q^{1-\frac{r+|m|}{2}} a^{-1} s_i^{\mp} x_\alpha^{-1}; q)_\infty}{(q^{\frac{r+|m|}{2}} a s_i^{\pm} x_\alpha; q)_\infty}$$

$$\times q^{N_f \sum_{i=1}^{N-1} \frac{1-r}{2} |m_i| - \sum_{i=1}^{N} |m_i| - \sum_{i<j} \frac{|m_i-m_j|}{2} - \sum_{i<j} \frac{|m_i+m_j|}{2}} a^{-\sum_{i=1}^{N-1} |m_i|} \zeta^{\sum_{i=1}^{N-1} m_i}. \qquad (3.14)$$

In theory B there are additional gauge singlets M and V. As shown in (3.2), the chiral superfield V is odd under $\mathbb{Z}_2^{\mathcal{M}}$ so that the index of V depends on the fugacity

ζ. For $\chi = +1$ or N_c being odd the full-index of theory B is

$$I^{SO(\tilde{N}_c=2\tilde{N}+\tilde{\epsilon})-[N_f]^B_{\zeta,\chi}}$$

$$= \frac{1}{2^{\tilde{N}+\tilde{\epsilon}-1}\tilde{N}!} \sum_{m_1,\cdots,m_{\tilde{N}}} \oint \prod_{i=1}^{\tilde{N}} \frac{ds_i}{2\pi i s_i} \left[\prod_{i=1}^{\tilde{N}} (1-\chi q^{\frac{|m_i|}{2}} s_i)(1-\chi q^{\frac{|m_i|}{2}} s_i^{-1}) \right]^{\tilde{\epsilon}}$$

$$\times \prod_{i<j} (1-q^{\frac{|m_i-m_j|}{2}} s_i s_j^{-1})(1-q^{\frac{|m_i-m_j|}{2}} s_i^{-1} s_j)(1-q^{\frac{|m_i+m_j|}{2}} s_i s_j)(1-q^{\frac{|m_i+m_j|}{2}} s_i^{-1} s_j^{-1})$$

$$\times \left[\prod_{\alpha=1}^{\tilde{N}_f} \frac{(\chi^{-1} q^{\frac{1+r}{2}} a x_\alpha^{-1}; q)_\infty}{(\chi q^{\frac{1-r}{2}} a^{-1} x_\alpha; q)_\infty} \right]^{\tilde{\epsilon}} \prod_{i=1}^{\tilde{N}} \prod_{\alpha=1}^{N_f} \frac{(q^{\frac{1+r+|m|}{2}} a s_i^{\mp} x_\alpha^{-1}; q)_\infty}{(q^{\frac{1-r+|m|}{2}} a^{-1} s_i^{\pm} x_\alpha; q)_\infty}$$

$$\times q^{N_f \sum_{i=1}^{\tilde{N}} \frac{1-r}{2}|m_i| - \sum_{i=1}^{\tilde{N}} \frac{|m_i|}{2}\tilde{\epsilon} - \sum_{i<j} \frac{|m_i-m_j|}{2} - \sum_{i<j} \frac{|m_i+m_j|}{2}} a^{\sum_{i=1}^{N} |m_i|} \zeta^{\sum_{i=1}^{N} m_i}$$

$$\times \prod_{\alpha\leq\beta} \frac{(q^{1-r} a^{-2} x_\alpha^{-1} x_\beta^{-1}; q)_\infty}{(q^r a^2 x_\alpha x_\beta; q)_\infty} \frac{(\zeta q^{\frac{N_c+rN_f-N_f}{2}} a^{N_f}; q)_\infty}{(\zeta^{-1} q^{1+\frac{N_f-rN_f-N_c}{2}} a^{-N_f}; q)_\infty}. \tag{3.15}$$

For $\chi = -1$ and N_c even the full-index of theory B is [8]

$$I^{SO(\tilde{N}_c=2\tilde{N})-[N_f]^B_{\zeta,\chi=-}}$$

$$= \frac{1}{2^{\tilde{N}-1}(\tilde{N}-1)!} \sum_{m_1,\cdots,m_{\tilde{N}-1}} \oint \prod_{i=1}^{\tilde{N}-1} \frac{ds_i}{2\pi i s_i} \prod_{i=1}^{\tilde{N}-1} (1-q^{|m_i|} s_i^2)(1-q^{|m_i|} s_i^{-2})$$

$$\times \prod_{i<j} (1-q^{\frac{|m_i-m_j|}{2}} s_i s_j^{-1})(1-q^{\frac{|m_i-m_j|}{2}} s_i^{-1} s_j)(1-q^{\frac{|m_i+m_j|}{2}} s_i s_j)(1-q^{\frac{|m_i+m_j|}{2}} s_i^{-1} s_j^{-1})$$

$$\times \prod_{\alpha=1}^{N_f} \frac{(\pm q^{\frac{1+r}{2}} a x_\alpha^{-1}; q)_\infty}{(\pm q^{\frac{1-r}{2}} a^{-1} x_\alpha; q)_\infty} \prod_{i=1}^{\tilde{N}-1} \prod_{\alpha=1}^{N_f} \frac{(q^{\frac{1+r+|m|}{2}} a s_i^{\mp} x_\alpha^{-1}; q)_\infty}{(q^{\frac{1-r+|m|}{2}} a^{-1} s_i^{\pm} x_\alpha; q)_\infty}$$

$$\times q^{N_f \sum_{i=1}^{\tilde{N}-1} \frac{1-r}{2}|m_i| - \sum_{i=1}^{\tilde{N}} |m_i| - \sum_{i<j} \frac{|m_i-m_j|}{2} - \sum_{i<j} \frac{|m_i+m_j|}{2}} a^{\sum_{i=1}^{\tilde{N}-1} |m_i|} \zeta^{\sum_{i=1}^{\tilde{N}-1} m_i}$$

$$\times \prod_{\alpha\leq\beta} \frac{(q^{1-r} a^{-2} x_\alpha^{-1} x_\beta^{-1}; q)_\infty}{(q^r a^2 x_\alpha x_\beta; q)_\infty} \frac{(\zeta q^{\frac{N_c+rN_f-N_f}{2}} a^{N_f}; q)_\infty}{(\zeta^{-1} q^{1+\frac{N_f-rN_f-N_c}{2}} a^{-N_f}; q)_\infty}. \tag{3.16}$$

The dualities lead to the identities of full-indices (3.13)-(3.16) [24]:

$$I^{SO(N_c)-[N_f]^A_{\zeta,\chi}} = I^{SO(\tilde{N}_c)-[N_f]^B_{\zeta,\zeta\chi}}. \tag{3.17}$$

3.2.2 Half-index

There are four half-indices for each choice of boundary conditions, labelled by the values of ζ and χ. The standard $SO(N_c)$ index is given by $\zeta = \chi = +1$. The half-indices for the theory A $\mathrm{III}^{SO(N_c)-[N_f]^A_{\zeta,\chi}}$ match the half-indices for the dual theory B $\mathrm{III}^{SO(\tilde{N}_c)-[N_f]^B_{\tilde{\zeta}\tilde{\chi}}}$ with $\tilde{\zeta} = \zeta$ and $\tilde{\chi} = \zeta\chi$.

---

[8]We have an extra overall factor 2 compared to the formula in [24].

Using the expressions in [43] for general gauge group and choosing group $SO(N_c)$ (see appendix A.3 for further details) where $N_c = 2N + \epsilon$ for $\epsilon \in \{0, 1\}$ since we often have to distinguish between N_c even or odd, we have for Dirichlet gauge field boundary conditions in the cases with N_c being even with $\chi = +1$ or N_c being odd with $\chi = \pm 1$

$$
\mathbb{III}_{\mathcal{D}}^{SO(N_c)_{\zeta\chi}}
$$

$$
= \frac{1}{(q)_\infty^N} \sum_{m_i \in \mathbb{Z}} \frac{(\zeta\chi)^{\sum_i m_i} (-q^{1/2})^{k_{eff}[m,m]} u^{k_{eff}[m,-]}}{\prod_{i=1}^N (\chi q^{1 \pm m_i} u_i^\pm; q)_\infty^\epsilon \prod_{i \neq j} (q^{1+m_i-m_j} u_i u_j^{-1}; q)_\infty \prod_{i<j} (q^{1 \pm m_i \pm m_j} u_i^\pm u_j^\pm; q)_\infty}
$$

$$
\times \mathbb{III}^{\text{matter}}(s_i \to q^{m_i} u_i) \tag{3.18}
$$

where the effective CS coupling k_{eff} is determined by the 't Hooft anomaly and the indices i and j take values from 1 to N. The relevant anomaly contribution is $\tilde{N}_c \operatorname{Tr}(\mathbf{s}^2)$ giving

$$
k_{eff}[m, m] = \tilde{N}_c \sum_{i=1}^N m_i^2, \tag{3.19}
$$

$$
(-q^{1/2})^{k_{eff}[m,m]} = (-1)^{\tilde{N}_c \sum_{i=1}^N m_i} q^{\tilde{N}_c \sum_{i=1}^N m_i^2}, \tag{3.20}
$$

$$
u^{k_{eff}[m,-]} = \prod_{i=1}^N u_i^{\tilde{N}_c m_i}. \tag{3.21}
$$

In the case $\chi = -1$ with N_c even we have instead

$$
\mathbb{III}_{\mathcal{D}}^{SO(N_c)_{\zeta-}}
$$

$$
= \frac{1}{(q)_\infty^{N-1}(-q; q)_\infty} \sum_{m_i \in \mathbb{Z}} \frac{(-\zeta)^{\sum_i m_i} (-q^{1/2})^{k_{eff}[m,m]} u^{k_{eff}[m,-]}}{\prod_i (u_i^\pm q^{1 \pm m_i}; q)_\infty \prod_i (-u_i^\pm q^{1 \pm m_i}; q)_\infty}
$$

$$
\times \frac{1}{\prod_{i \neq j} (q^{1+m_i-m_j} u_i u_j^{-1}; q)_\infty \prod_{i<j} (q^{1 \pm m_i \pm m_j} u_i^\pm u_j^\pm; q)_\infty} \mathbb{III}^{\text{matter}}(s_i \to q^{m_i} u_i) \tag{3.22}
$$

where now i and j take values from 1 to $N - 1$.

For Neumann gauge field boundary conditions we have, excluding the case of N_c even with $\chi = -1$

$$
\mathbb{III}_{\mathcal{N}}^{SO(N_c)_{\zeta\chi}}
$$

$$
= \frac{(q)_\infty^N}{2^{N-1+\epsilon} N!} \left(\prod_{i=1}^N \oint \frac{ds_i}{2\pi i s_i} \right) \left(\prod_{i \neq j} (s_i s_j^{-1}; q)_\infty \right) \left(\prod_{i<j} (s_i^\pm s_j^\pm; q)_\infty \right) \left(\prod_i (\chi s_i^\pm; q)_\infty \right)^\epsilon
$$

$$
\times \mathbb{III}^{\text{matter}} I^\Psi \tag{3.23}
$$

while for N_c even with $\chi = -1$ we have

$$\mathbb{III}_{\mathcal{N}}^{SO(N_c)\zeta-} = \frac{(q)_\infty^{N-1}(-q;q)_\infty}{2^{N-1}(N-1)!} \left(\prod_{i=1}^{N-1} \oint \frac{ds_i}{2\pi i s_i}\right) \left(\prod_{i=1}(s_i^\pm;q)_\infty(-s_i^\pm;q)_\infty\right)$$

$$\times \left(\prod_{i \neq j}(s_i s_j^{-1};q)_\infty\right) \left(\prod_{i<j}(s_i^\pm s_j^\pm;q)_\infty\right) \mathbb{III}^{\text{matter}} I^\Psi. \tag{3.24}$$

Note that the symmetry factor is $2^{N-1+\epsilon}N!$, the order of the Weyl group for $SO(N_c)$, except for the case of N_c even and $\chi = -1$. In the latter case it is $2^{N-1}(N-1)!$ which is twice the order of the Weyl group of $SO(N_c - 2)$. This result can be found in [88], although note that they refer to the 'Weyl group' of $O(N_c)$ which have an order twice that of the Weyl groups for $SO(N_c)$.

The half-indices for theory B are given by the same expressions after replacing $N_c = 2N + \epsilon$ with $\tilde{N}_c = 2\tilde{N} + \tilde{\epsilon}$.

For the 3d fields in theory A with gauge group $SO(N_c = 2N+\epsilon)$ and theory B with gauge group $SO(\tilde{N}_c = 2\tilde{N} + \tilde{\epsilon} = N_f - N_c + 2)$ with Neumann or Dirichlet boundary conditions we have the following contributions to the half-indices

$$\mathbb{III}_{\text{N}}^Q = \prod_{\alpha=1}^{N_f} \frac{1}{(\chi q^{\frac{r}{2}} a x_\alpha; q)_\infty^\epsilon \prod_{i=1}^N (q^{\frac{r}{2}} s_i^\pm a x_\alpha; q)_\infty}, \tag{3.25}$$

$$\mathbb{III}_{\text{D}}^Q = \prod_{\alpha=1}^{N_f} (\chi q^{1-\frac{r}{2}} a^{-1} x_\alpha^{-1}; q)_\infty^\epsilon \prod_{i=1}^N (q^{1-\frac{r}{2}} s_i^\pm a^{-1} x_\alpha^{-1}; q)_\infty, \tag{3.26}$$

$$\mathbb{III}_{\text{N}}^q = \prod_{\alpha=1}^{N_f} \frac{1}{(\tilde{\chi} q^{\frac{1-r}{2}} a^{-1} x_\alpha^{-1}; q)_\infty^{\tilde{\epsilon}} \prod_{i=1}^{\tilde{N}} (q^{\frac{1-r}{2}} s_i^\pm a^{-1} x_\alpha^{-1}; q)_\infty}, \tag{3.27}$$

$$\mathbb{III}_{\text{D}}^q = \prod_{\alpha=1}^{N_f} (\tilde{\chi} q^{\frac{1+r}{2}} a x_\alpha; q)_\infty^{\tilde{\epsilon}} \prod_{i=1}^{\tilde{N}} (q^{\frac{1+r}{2}} s_i^\pm a x_\alpha; q)_\infty, \tag{3.28}$$

$$\mathbb{III}_{\text{N}}^M = \prod_{\alpha \leq \beta} \frac{1}{(q^r x_\alpha x_\beta a^2; q)_\infty}, \tag{3.29}$$

$$\mathbb{III}_{\text{D}}^M = \prod_{\alpha \leq \beta} (q^{1-r} x_\alpha^{-1} x_\beta^{-1} a^{-2}; q)_\infty, \tag{3.30}$$

$$\mathbb{III}_{\text{N}}^V = \frac{1}{\left(\zeta q^{\frac{(1-r)N_f - N_c + 2}{2}} a^{-N_f}; q\right)_\infty}, \tag{3.31}$$

$$\mathbb{III}_{\text{D}}^V = \left(\zeta q^{\frac{N_c - (1-r)N_f}{2}} a^{N_f}; q\right)_\infty, \tag{3.32}$$

where for $\chi = -1$ and N_c even we send $s_N \to 1$ and $s_N^{-1} \to -1$ in the contributions from Q, while for $\tilde{\chi} = -1$ and $\tilde{N}_c$ even we send $s_{\tilde{N}} \to 1$ and $s_{\tilde{N}}^{-1} \to -1$ in the contributions from q.

The bifundamental Fermi, again with a reality condition as in the symplectic case, gives contribution to the 2d index

$$I^\Psi = \left(\prod_{i=1}^{N}\prod_{j=1}^{\tilde{N}} F_{00}\right) \left(\prod_{j=1}^{\tilde{N}} F_{10}\right)^\epsilon \left(\prod_{i=1}^{N} F_{01}\right)^{\tilde{\epsilon}} F_{11}^{\epsilon\tilde{\epsilon}}, \tag{3.33}$$

$$F_{00} = (z_i\tilde{z}_j q^{1/2}; q)_\infty (z_i\tilde{z}_j^{-1} q^{1/2}; q)_\infty (z_i^{-1}\tilde{z}_j q^{1/2}; q)_\infty (z_i^{-1}\tilde{z}_j^{-1} q^{1/2}; q)_\infty, \tag{3.34}$$

$$F_{10} = (\chi\tilde{z}_j q^{1/2}; q)_\infty (\chi\tilde{z}_j^{-1} q^{1/2}; q)_\infty, \tag{3.35}$$

$$F_{01} = (\tilde{\chi}z_i q^{1/2}; q)_\infty (\tilde{\chi}z_i^{-1} q^{1/2}; q)_\infty, \tag{3.36}$$

$$F_{11} = (\chi\tilde{\chi} q^{1/2}; q)_\infty, \tag{3.37}$$

where F_{10} is included only if N_c is odd, F_{01} is included only if $\tilde{N}_c$ is odd and F_{11} is included only if both N_c and $\tilde{N}_c$ are odd. In the case of N_c even and $\chi = -1$ we replace $z_N \to 1$ and $z_N^{-1} \to -1$. Similarly in the case of even $\tilde{N}_c$ and $\tilde{\chi} = -1$ we replace $\tilde{z}_N \to 1$ and $\tilde{z}_N^{-1} \to -1$.

Then in the half-index for theory A with $\mathcal{N}$ boundary conditions we include I^Ψ with the replacements $z_i \to s_i$ and $\tilde{z}_i \to u_i$. If instead we have $\mathcal{N}$ boundary conditions in theory B we replace $z_i \to u_i$ and $\tilde{z}_i \to s_i$.

We now list some explicit examples using notation $SO(N_c) - [N_f]_{\zeta\chi}^A$ to refer to theory A with gauge group $SO(N_c)$ with N_f fundamental chirals and discrete fugacities ζ and χ. This is dual to theory B with gauge group $SO(\tilde{N}_c)$ with N_f fundamental chirals and discrete fugacities $\tilde{\zeta} = \zeta$ and $\tilde{\chi} = \zeta\chi$ which we label $SO(\tilde{N}_c) - [N_f]_{\tilde{\zeta}\tilde{\chi}}^B$.

We first consider some examples with gauge group $SO(1)$ in theory A. These are free theories with N_f chirals but they are dual to interacting $SO(N_f + 1) - [N_f]$ theories. The indices and half-indices for N_f free chirals are very simple so this leads to an interesting set of q-series identities (assuming the duality holds). In some examples the (half-)index of the dual theory also takes a simple form and the matching of (half-)indices can easily be checked analytically. Following these $SO(1)$ examples we consider some cases with gauge group $SO(2)$, equivalent to $U(1)$ with N_f chirals of charge $+1$ and N_f of charge -1, or gauge group $SO(3)$. The latter differs from $SU(2)$ examples in that the matter is in the triplet representation corresponding to adjoint, not fundamental, of $SU(2)$. As there are significant differences between $SO(N_c)$ with N_c odd or even, our examples cover all combinations with $(N_c, \tilde{N}_c)$ both even, both odd or one even and one odd.

3.3 $N_c = 1, N_f = 1$ $(SO(1) - [1])$

For theory A we have the full-indices

$$I^{SO(1)-[1]^A_{\pm+}} = \frac{(q^{1-\frac{r}{2}}a^{-1}; q)_\infty}{(q^{\frac{r}{2}}a; q)_\infty}, \tag{3.38}$$

$$I^{SO(1)-[1]^A_{\pm-}} = \frac{(-q^{1-\frac{r}{2}}a^{-1}; q)_\infty}{(-q^{\frac{r}{2}}a; q)_\infty}. \tag{3.39}$$

These indices agree with the indices of theory B

$$I^{SO(2)-[1]^B_{++}} = \sum_{m\in\mathbb{Z}} \oint \frac{ds}{2\pi i s} \frac{(q^{\frac{1+r+|m|}{2}}as^{\pm}; q)_\infty}{(q^{\frac{1-r+|m|}{2}}a^{-1}s^{\pm}; q)_\infty} q^{\frac{r|m|}{2}} a^{|m|} \frac{(q^{1-r}a^{-2}; q)_\infty}{(q^r a^2; q)_\infty} \frac{(q^{\frac{r}{2}}a; q)_\infty}{(q^{1-\frac{r}{2}}a^{-1}; q)_\infty}, \tag{3.40}$$

$$I^{SO(2)-[1]^B_{--}} = \frac{(\pm q^{\frac{1+r}{2}}a; q)_\infty}{(\pm q^{\frac{1-r}{2}}a^{-1}; q)_\infty} \frac{(q^{1-r}a^{-2}; q)_\infty}{(q^r a^2; q)_\infty} \frac{(-q^{\frac{r}{2}}a; q)_\infty}{(-q^{1-\frac{r}{2}}a^{-1}; q)_\infty}, \tag{3.41}$$

$$I^{SO(2)-[1]^B_{+-}} = \frac{(\pm q^{\frac{1+r}{2}}a; q)_\infty}{(\pm q^{\frac{1-r}{2}}a^{-1}; q)_\infty} \frac{(q^{1-r}a^{-2}; q)_\infty}{(q^r a^2; q)_\infty} \frac{(q^{\frac{r}{2}}a; q)_\infty}{(q^{1-\frac{r}{2}}a^{-1}; q)_\infty}, \tag{3.42}$$

$$I^{SO(2)-[1]^B_{-+}} = \sum_{m\in\mathbb{Z}} \oint \frac{ds}{2\pi i s} \frac{(q^{\frac{1+r+|m|}{2}}as^{\pm}; q)_\infty}{(q^{\frac{1-r+|m|}{2}}a^{-1}s^{\pm}; q)_\infty} q^{\frac{r|m|}{2}} a^{|m|} (-1)^m \frac{(q^{1-r}a^{-2}; q)_\infty}{(q^r a^2; q)_\infty} \frac{(-q^{\frac{r}{2}}a; q)_\infty}{(-q^{1-\frac{r}{2}}a^{-1}; q)_\infty}. \tag{3.43}$$

Note that in these examples, since the half-index of theory A is not sensitive to the value of ζ we have

$$I^{SO(2)-[1]^B_{++}} = I^{SO(1)-[1]^A_{\pm+}} = I^{SO(2)-[1]^B_{--}}, \tag{3.44}$$

$$I^{SO(2)-[1]^B_{+-}} = I^{SO(1)-[1]^A_{\pm-}} = I^{SO(2)-[1]^B_{-+}}. \tag{3.45}$$

This is a particular example of the more general case

$$I^{SO(N_f-1)-[N_f]^B_{++}} = I^{SO(1)-[N_f]^A_{\pm+}} = I^{SO(N_f-1)-[N_f]^B_{--}}, \tag{3.46}$$

$$I^{SO(N_f-1)-[N_f]^B_{+-}} = I^{SO(1)-[N_f]^A_{\pm-}} = I^{SO(N_f-1)-[N_f]^B_{-+}}. \tag{3.47}$$

We will see the cases of $N_f = 2, 3$ below but not specifically comment again on this point.

3.3.1 $SO(1) - [1]$ **with** $\mathcal{N}, \mathrm{N} + \mathrm{Fermi}$

We now present the half-indices with Neumann boundary conditions in theory A

$$\mathbb{II}_{\mathcal{N},\mathrm{N}+\Psi}^{SO(1)-[1]_{++}^A} = \frac{1}{(q^{\frac{r}{2}}a;q)_\infty}(q^{\frac{1}{2}}u^\pm;q)_\infty, \tag{3.48}$$

$$\mathbb{II}_{\mathcal{N},\mathrm{N}+\Psi}^{SO(1)-[1]_{-+}^A} = \frac{1}{(q^{\frac{r}{2}}a;q)_\infty}(\pm q^{\frac{1}{2}};q)_\infty, \tag{3.49}$$

$$\mathbb{II}_{\mathcal{N},\mathrm{N}+\Psi}^{SO(1)-[1]_{+-}^A} = \frac{1}{(-q^{\frac{r}{2}}a;q)_\infty}(\pm q^{\frac{1}{2}};q)_\infty, \tag{3.50}$$

$$\mathbb{II}_{\mathcal{N},\mathrm{N}+\Psi}^{SO(1)-[1]_{--}^A} = \frac{1}{(-q^{\frac{r}{2}}a;q)_\infty}(-q^{\frac{1}{2}}u^\pm;q)_\infty . \tag{3.51}$$

These half-indices agree with the theory B half-indices

$$\mathbb{II}_{\mathcal{D},\mathrm{D},\mathrm{N},\mathrm{D}}^{SO(2)-[1]_{++}^B} = \frac{1}{(q)_\infty}\sum_{m\in\mathbb{Z}}(q^{\frac{1+r}{2}\pm m}au^\pm;q)_\infty(-1)^m q^{\frac{m^2}{2}}u^m\frac{1}{(q^r a^2;q)_\infty}(q^{\frac{r}{2}}a;q)_\infty, \tag{3.52}$$

$$\mathbb{II}_{\mathcal{D},\mathrm{D},\mathrm{N},\mathrm{D}}^{SO(2)-[1]_{--}^B} = \frac{1}{(-q;q)_\infty}(\pm q^{\frac{1+r}{2}}a;q)_\infty\frac{1}{(q^r a^2;q)_\infty}(-q^{\frac{r}{2}}a;q)_\infty, \tag{3.53}$$

$$\mathbb{II}_{\mathcal{D},\mathrm{D},\mathrm{N},\mathrm{D}}^{SO(2)-[1]_{+-}^B} = \frac{1}{(-q;q)_\infty}(\pm q^{\frac{1+r}{2}}a;q)_\infty\frac{1}{(q^r a^2;q)_\infty}(q^{\frac{r}{2}}a;q)_\infty, \tag{3.54}$$

$$\mathbb{II}_{\mathcal{D},\mathrm{D},\mathrm{N},\mathrm{D}}^{SO(2)-[1]_{-+}^B} = \frac{1}{(q)_\infty}\sum_{m\in\mathbb{Z}}(q^{\frac{1+r}{2}\pm m}au^\pm;q)_\infty q^{\frac{m^2}{2}}u^m\frac{1}{(q^r a^2;q)_\infty}(-q^{\frac{r}{2}}a;q)_\infty. \tag{3.55}$$

Note that, unlike for the full-indices, these theory A half-indices are sensitive to the value of ζ due to the bifundamental Fermi multiplet.

3.3.2 $SO(1) - [1]$ **with** $\mathcal{D}, \mathrm{D}$

Instead, taking Dirichlet boundary conditions in theory A we have

$$\mathbb{II}_{\mathcal{D},\mathrm{D}}^{SO(1)-[1]_{\pm+}^A} = (q^{1-\frac{r}{2}}a^{-1};q)_\infty, \tag{3.56}$$

$$\mathbb{II}_{\mathcal{D},\mathrm{D}}^{SO(1)-[1]_{\pm-}^A} = (-q^{1-\frac{r}{2}}a^{-1};q)_\infty. \tag{3.57}$$

These agree with the theory B half-indices

$$\mathbb{III}^{SO(2)-[1]^B_{++}}_{\mathcal{N},N,D,N+\Psi} = (q)_\infty \oint \frac{ds}{2\pi i s} \frac{1}{(q^{\frac{1-r}{2}}a^{-1}s^\pm;q)_\infty}(q^{\frac{1}{2}}s^\pm;q)_\infty(q^{1-r}a^{-2};q)_\infty \frac{1}{(q^{1-\frac{r}{2}}a^{-1};q)_\infty},$$

(3.58)

$$\mathbb{III}^{SO(2)-[1]^B_{--}}_{\mathcal{N},N,D,N+\Psi} = (-q;q)_\infty \frac{1}{(\pm q^{\frac{1-r}{2}}a^{-1};q)_\infty}(\pm q^{\frac{1}{2}};q)_\infty(q^{1-r}a^{-2};q)_\infty \frac{1}{(-q^{1-\frac{r}{2}}a^{-1};q)_\infty},$$

(3.59)

$$\mathbb{III}^{SO(2)-[1]^B_{+-}}_{\mathcal{N},N,D,N+\Psi} = (-q;q)_\infty \frac{1}{(\pm q^{\frac{1-r}{2}}a^{-1};q)_\infty}(\pm q^{\frac{1}{2}};q)_\infty(q^{1-r}a^{-2};q)_\infty \frac{1}{(q^{1-\frac{r}{2}}a^{-1};q)_\infty}, \quad (3.60)$$

$$\mathbb{III}^{SO(2)-[1]^B_{-+}}_{\mathcal{N},N,D,N+\Psi} = (q)_\infty \oint \frac{ds}{2\pi i s} \frac{1}{(q^{\frac{1-r}{2}}a^{-1}s^\pm;q)_\infty}(-q^{\frac{1}{2}}s^\pm;q)_\infty(q^{1-r}a^{-2};q)_\infty \frac{1}{(-q^{1-\frac{r}{2}}a^{-1};q)_\infty}.$$

(3.61)

Note that, as for the full-indices, the half-indices of theory A are not sensitive to the value of ζ so we have

$$\mathbb{III}^{SO(2)-[1]^B_{++}}_{\mathcal{N},N,D,N+\Psi} = \mathbb{III}^{SO(1)-[1]^A_{\pm+}}_{\mathcal{D},D} = \mathbb{III}^{SO(2)-[1]^B_{--}}_{\mathcal{N},N,D,N+\Psi}, \tag{3.62}$$

$$\mathbb{III}^{SO(2)-[1]^B_{+-}}_{\mathcal{N},N,D,N+\Psi} = \mathbb{III}^{SO(1)-[1]^A_{\pm-}}_{\mathcal{D},D} = \mathbb{III}^{SO(2)-[1]^B_{-+}}_{\mathcal{N},N,D,N+\Psi}. \tag{3.63}$$

This is a particular example of the more general case

$$\mathbb{III}^{SO(N_f+1)-[N_f]^B_{++}}_{\mathcal{N},N,D,N+\Psi} = \mathbb{III}^{SO(1)-[N_f]^A_{\pm+}}_{\mathcal{D},D} = \mathbb{III}^{SO(N_f+1)-[N_f]^B_{--}}_{\mathcal{N},N,D,N+\Psi}, \tag{3.64}$$

$$\mathbb{III}^{SO(N_f+1)-[N_f]^B_{+-}}_{\mathcal{N},N,D,N+\Psi} = \mathbb{III}^{SO(1)-[N_f]^A_{\pm-}}_{\mathcal{D},D} = \mathbb{III}^{SO(N_f+1)-[N_f]^B_{-+}}_{\mathcal{N},N,D,N+\Psi}. \tag{3.65}$$

We will see the cases for $N_f = 2, 3$ below.

3.4 $N_c = 1, N_f = 2 \ (SO(1) - [2])$

In theory A we have the full-indices

$$I^{SO(1)-[2]^A_{\pm+}} = \frac{(q^{1-\frac{r}{2}}a^{-1};q)^2_\infty}{(q^{\frac{r}{2}}a;q)^2_\infty}, \tag{3.66}$$

$$I^{SO(1)-[2]^A_{\pm-}} = \frac{(-q^{1-\frac{r}{2}}a^{-1};q)^2_\infty}{(-q^{\frac{r}{2}}a;q)^2_\infty}, \tag{3.67}$$

which agree with the indices of theory B

$$I^{SO(3)-[2]^B_{++}} = \frac{1}{2} \sum_{m \in \mathbb{Z}} \oint \frac{ds}{2\pi i s} (1 - q^{\frac{|m|}{2}} s)(1 - q^{\frac{|m|}{2}} s^{-1})$$

$$\times \frac{(q^{\frac{1+r+|m|}{2}} a s^{\pm}; q)^2_{\infty}}{(q^{\frac{1-r+|m|}{2}} a^{-1} s^{\pm}; q)^2_{\infty}} \frac{(q^{\frac{1+r}{2}} a; q)^2_{\infty}}{(q^{\frac{1-r}{2}} a^{-1}; q)^2_{\infty}} q^{r|m| - \frac{|m|}{2}} a^{2|m|} \frac{(q^{1-r} a^{-2}; q)^3_{\infty}}{(q^r a^2; q)^3_{\infty}} \frac{(q^{-\frac{1}{2}+r} a^2; q)_{\infty}}{(q^{\frac{3}{2}-r} a^{-2}; q)_{\infty}},$$

$$(3.68)$$

$$I^{SO(3)-[2]^B_{--}} = \frac{1}{2} \sum_{m \in \mathbb{Z}} \oint \frac{ds}{2\pi i s} (1 + q^{\frac{|m|}{2}} s)(1 + q^{\frac{|m|}{2}} s^{-1})$$

$$\times \frac{(q^{\frac{1+r+|m|}{2}} a s^{\pm}; q)^2_{\infty}}{(q^{\frac{1-r+|m|}{2}} a^{-1} s^{\pm}; q)^2_{\infty}} \frac{(-q^{\frac{1+r}{2}} a; q)^2_{\infty}}{(-q^{\frac{1-r}{2}} a^{-1}; q)^2_{\infty}} q^{r|m| - \frac{|m|}{2}} a^{2|m|} (-1)^m \frac{(q^{1-r} a^{-2}; q)^3_{\infty}}{(q^r a^2; q)^3_{\infty}} \frac{(-q^{-\frac{1}{2}+r} a^2; q)_{\infty}}{(-q^{\frac{3}{2}-r} a^{-2}; q)_{\infty}},$$

$$(3.69)$$

$$I^{SO(3)-[2]^B_{+-}} = \frac{1}{2} \sum_{m \in \mathbb{Z}} \oint \frac{ds}{2\pi i s} (1 + q^{\frac{|m|}{2}} s)(1 + q^{\frac{|m|}{2}} s^{-1})$$

$$\times \frac{(q^{\frac{1+r+|m|}{2}} a s^{\pm}; q)^2_{\infty}}{(q^{\frac{1-r+|m|}{2}} a^{-1} s^{\pm}; q)^2_{\infty}} \frac{(-q^{\frac{1+r}{2}} a; q)^2_{\infty}}{(-q^{\frac{1-r}{2}} a^{-1}; q)^2_{\infty}} q^{r|m| - \frac{|m|}{2}} a^{2|m|} \frac{(q^{1-r} a^{-2}; q)^3_{\infty}}{(q^r a^2; q)^3_{\infty}} \frac{(q^{-\frac{1}{2}+r} a^2; q)_{\infty}}{(q^{\frac{3}{2}-r} a^{-2}; q)_{\infty}},$$

$$(3.70)$$

$$I^{SO(3)-[2]^B_{-+}} = \frac{1}{2} \sum_{m \in \mathbb{Z}} \oint \frac{ds}{2\pi i s} (1 - q^{\frac{|m|}{2}} s)(1 - q^{\frac{|m|}{2}} s^{-1})$$

$$\times \frac{(q^{\frac{1+r+|m|}{2}} a s^{\pm}; q)^2_{\infty}}{(q^{\frac{1-r+|m|}{2}} a^{-1} s^{\pm}; q)^2_{\infty}} \frac{(q^{\frac{1+r}{2}} a; q)^2_{\infty}}{(q^{\frac{1-r}{2}} a^{-1}; q)^2_{\infty}} q^{r|m| - \frac{|m|}{2}} a^{2|m|} (-1)^m \frac{(q^{1-r} a^{-2}; q)^3_{\infty}}{(q^r a^2; q)^3_{\infty}} \frac{(-q^{-\frac{1}{2}+r} a^2; q)_{\infty}}{(-q^{\frac{3}{2}-r} a^{-2}; q)_{\infty}} .$$

$$(3.71)$$

3.4.1 $SO(1) - [2]$ with $\mathcal{N}, (\mathrm{N}, \mathrm{N}) +$ Fermis

For the Neumann half-indices in theory A we have

$$\mathbb{III}^{SO(1)-[2]^A_{++}}_{\mathcal{N},\mathrm{N}+\Psi} = \frac{1}{(q^{\frac{r}{2}} a; q)^2_{\infty}} (q^{\frac{1}{2}} u^{\pm}; q)_{\infty} (q^{\frac{1}{2}}; q)_{\infty}, \tag{3.72}$$

$$\mathbb{III}^{SO(1)-[2]^A_{-+}}_{\mathcal{N},\mathrm{N}+\Psi} = \frac{1}{(q^{\frac{r}{2}} a; q)^2_{\infty}} (q^{\frac{1}{2}} u^{\pm}; q)_{\infty} (-q^{\frac{1}{2}}; q)_{\infty}, \tag{3.73}$$

$$\mathbb{III}^{SO(1)-[2]^A_{+-}}_{\mathcal{N},\mathrm{N}+\Psi} = \frac{1}{(-q^{\frac{r}{2}} a; q)^2_{\infty}} (-q^{\frac{1}{2}} u^{\pm}; q)_{\infty} (q^{\frac{1}{2}}; q)_{\infty}, \tag{3.74}$$

$$\mathbb{III}^{SO(1)-[2]^A_{--}}_{\mathcal{N},\mathrm{N}+\Psi} = \frac{1}{(-q^{\frac{r}{2}} a; q)^2_{\infty}} (-q^{\frac{1}{2}} u^{\pm}; q)_{\infty} (-q^{\frac{1}{2}}; q)_{\infty}. \tag{3.75}$$

These agrees with the theory B half-indices

$$\mathbb{III}_{\mathcal{D},\mathrm{D},\mathrm{N},\mathrm{D}}^{SO(3)-[2]_{++}^{B}} = \frac{1}{(q)_\infty} \sum_{m\in\mathbb{Z}} \frac{1}{(q^{1\pm m}u^\pm;q)_\infty}(q^{\frac{1+r}{2}\pm m}au^\pm;q)_\infty^2(q^{\frac{1+r}{2}}a;q)_\infty^2(-1)^m q^{\frac{m^2}{2}}u^m$$

$$\times \frac{1}{(q^r a^2;q)_\infty^3}(q^{-\frac12+r}a^2;q)_\infty, \tag{3.76}$$

$$\mathbb{III}_{\mathcal{D},\mathrm{D},\mathrm{N},\mathrm{D}}^{SO(3)-[2]_{--}^{B}} = \frac{1}{(q)_\infty} \sum_{m\in\mathbb{Z}} \frac{1}{(-q^{1\pm m}u^\pm;q)_\infty}(q^{\frac{1+r}{2}\pm m}au^\pm;q)_\infty^2(-q^{\frac{1+r}{2}}a;q)_\infty^2(-1)^m q^{\frac{m^2}{2}}u^m$$

$$\times \frac{1}{(q^r a^2;q)_\infty^3}(-q^{-\frac12+r}a^2;q)_\infty, \tag{3.77}$$

$$\mathbb{III}_{\mathcal{D},\mathrm{D},\mathrm{N},\mathrm{D}}^{SO(3)-[2]_{+-}^{B}} = \frac{1}{(q)_\infty} \sum_{m\in\mathbb{Z}} \frac{1}{(-q^{1\pm m}u^\pm;q)_\infty}(q^{\frac{1+r}{2}\pm m}au^\pm;q)_\infty^2(-q^{\frac{1+r}{2}}a;q)_\infty^2 q^{\frac{m^2}{2}}u^m$$

$$\times \frac{1}{(q^r a^2;q)_\infty^3}(q^{-\frac12+r}a^2;q)_\infty, \tag{3.78}$$

$$\mathbb{III}_{\mathcal{D},\mathrm{D},\mathrm{N},\mathrm{D}}^{SO(3)-[2]_{-+}^{B}} = \frac{1}{(q)_\infty} \sum_{m\in\mathbb{Z}} \frac{1}{(q^{1\pm m}u^\pm;q)_\infty}(q^{\frac{1+r}{2}\pm m}au^\pm;q)_\infty^2(q^{\frac{1+r}{2}}a;q)_\infty^2 q^{\frac{m^2}{2}}u^m$$

$$\times \frac{1}{(q^r a^2;q)_\infty^3}(-q^{-\frac12+r}a^2;q)_\infty. \tag{3.79}$$

3.4.2 $SO(1) - [2]$ with $\mathcal{D}, (\mathrm{D},\mathrm{D})$

The theory A Dirichlet half-indices are

$$\mathbb{III}_{\mathcal{D},\mathrm{D}}^{SO(1)-[2]_{\pm+}^{A}} = (q^{1-\frac{r}{2}}a^{-1};q)_\infty^2, \tag{3.80}$$

$$\mathbb{III}_{\mathcal{D},\mathrm{D}}^{SO(1)-[2]_{\pm-}^{A}} = (-q^{1-\frac{r}{2}}a^{-1};q)_\infty^2, \tag{3.81}$$

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and the matching theory B half-indices are

$$\mathbb{III}^{SO(3)-[2]^B_{++}}_{\mathcal{N},N,D,N+\Psi} = \frac{1}{2}(q)_\infty \oint \frac{ds}{2\pi i s}(s^\pm;q)_\infty \frac{1}{(q^{\frac{1-r}{2}}a^{-1}s^\pm;q)^2_\infty(q^{\frac{1-r}{2}}a^{-1};q)^2_\infty}(q^{\frac{1}{2}}s^\pm;q)_\infty(q^{\frac{1}{2}};q)_\infty$$

$$\times (q^{1-r}a^{-2};q)^3_\infty \frac{1}{(q^{\frac{3}{2}-r}a^{-2};q)_\infty}, \tag{3.82}$$

$$\mathbb{III}^{SO(3)-[2]^B_{--}}_{\mathcal{N},N,D,N+\Psi} = \frac{1}{2}(q)_\infty \oint \frac{ds}{2\pi i s}(-s^\pm;q)_\infty \frac{1}{(q^{\frac{1-r}{2}}a^{-1}s^\pm;q)^2_\infty(-q^{\frac{1-r}{2}}a^{-1};q)^2_\infty}(q^{\frac{1}{2}}s^\pm;q)_\infty(-q^{\frac{1}{2}};q)_\infty$$

$$\times (q^{1-r}a^{-2};q)^3_\infty \frac{1}{(-q^{\frac{3}{2}-r}a^{-2};q)_\infty}, \tag{3.83}$$

$$\mathbb{III}^{SO(3)-[2]^B_{+-}}_{\mathcal{N},N,D,N+\Psi} = \frac{1}{2}(q)_\infty \oint \frac{ds}{2\pi i s}(-s^\pm;q)_\infty \frac{1}{(q^{\frac{1-r}{2}}a^{-1}s^\pm;q)^2_\infty(-q^{\frac{1-r}{2}}a^{-1};q)^2_\infty}(-q^{\frac{1}{2}}s^\pm;q)_\infty(q^{\frac{1}{2}};q)_\infty$$

$$\times (q^{1-r}a^{-2};q)^3_\infty \frac{1}{(q^{\frac{3}{2}-r}a^{-2};q)_\infty}, \tag{3.84}$$

$$\mathbb{III}^{SO(3)-[2]^B_{-+}}_{\mathcal{N},N,D,N+\Psi} = \frac{1}{2}(q)_\infty \oint \frac{ds}{2\pi i s}(s^\pm;q)_\infty \frac{1}{(q^{\frac{1-r}{2}}a^{-1}s^\pm;q)^2_\infty(q^{\frac{1-r}{2}}a^{-1};q)^2_\infty}(-q^{\frac{1}{2}}s^\pm;q)_\infty(-q^{\frac{1}{2}};q)_\infty$$

$$\times (q^{1-r}a^{-2};q)^3_\infty \frac{1}{(-q^{\frac{3}{2}-r}a^{-2};q)_\infty}. \tag{3.85}$$

3.5 $N_c = 1, N_f = 3 \ (SO(1) - [3])$

For theory A the full-indices

$$I^{SO(1)-[3]^A_{\pm+}} = \frac{(q^{1-\frac{r}{2}}a^{-1};q)^3_\infty}{(q^{\frac{r}{2}}a;q)^3_\infty}, \tag{3.86}$$

$$I^{SO(1)-[3]^A_{\pm-}} = \frac{(-q^{1-\frac{r}{2}}a^{-1};q)^3_\infty}{(-q^{\frac{r}{2}}a;q)^3_\infty} \tag{3.87}$$

agree with the full-indices in theory B given by

$$
I^{SO(4)-[3]^B_{++}}
$$

$$
= \frac{1}{2 \cdot 2} \sum_{m_1, m_2 \in \mathbb{Z}} \oint \frac{ds_1}{2\pi i s_1} \frac{ds_2}{2\pi i s_2} (1 - q^{\frac{|m_1-m_2|}{2}} s_1^{\pm} s_2^{\mp})(1 - q^{\frac{|m_1-m_2|}{2}} s_1^{\pm} s_2^{\pm})
$$

$$
\times \prod_{i=1}^{3} \frac{(q^{\frac{1+r+|m|}{2}} a s_i^{\pm}; q)^3_{\infty}}{(q^{\frac{1-r+|m|}{2}} a^{-1} s_i^{\pm}; q)^3_{\infty}} q^{\sum_{i=1}^{3} \frac{3r}{2}|m_i| - \sum_{i<j} \frac{|m_i-m_j|}{2} - \sum_{i<j} \frac{|m_i+m_j|}{2}} a^{3\sum_{i=1}^{3}|m|}
$$

$$
\times \frac{(q^{1-r} a^{-2}; q)^6_{\infty}}{(q^r a^2; q)^6_{\infty}} \frac{(q^{-1+\frac{3r}{2}} a^3; q)_{\infty}}{(q^{2-\frac{3r}{2}} a^{-3}; q)_{\infty}}, \tag{3.88}
$$

$$
I^{SO(4)-[3]^B_{--}} = \frac{1}{2} \sum_{m \in \mathbb{Z}} \oint \frac{ds}{2\pi i s} (1 - q^{|m|} s^2)(1 - q^{|m|} s^{-2})
$$

$$
\times \frac{(q^{\frac{1+r+|m|}{2}} a s^{\pm}; q)^3_{\infty}}{(q^{\frac{1-r+|m|}{2}} a^{-1} s^{\pm}; q)^3_{\infty}} \frac{(\pm q^{\frac{1+r}{2}} a; q)^3_{\infty}}{(\pm q^{\frac{1-r}{2}} a^{-1}; q)^3_{\infty}} q^{\frac{3r}{2}|m| - |m|} a^{3|m|} (-1)^m
$$

$$
\times \frac{(q^{1-r} a^{-2}; q)^6_{\infty}}{(q^r a^2; q)^6_{\infty}} \frac{(-q^{-1+\frac{3r}{2}} a^3; q)_{\infty}}{(-q^{2-\frac{3r}{2}} a^{-3}; q)_{\infty}}, \tag{3.89}
$$

$$
I^{SO(4)-[3]^B_{+-}} = \frac{1}{2} \sum_{m \in \mathbb{Z}} \oint \frac{ds}{2\pi i s} (1 - q^{|m|} s^2)(1 - q^{|m|} s^{-2})
$$

$$
\times \frac{(q^{\frac{1+r+|m|}{2}} a s^{\pm}; q)^3_{\infty}}{(q^{\frac{1-r+|m|}{2}} a^{-1} s^{\pm}; q)^3_{\infty}} \frac{(\pm q^{\frac{1+r}{2}} a; q)^3_{\infty}}{(\pm q^{\frac{1-r}{2}} a^{-1}; q)^3_{\infty}} q^{\frac{3r}{2}|m| - |m|} a^{3|m|}
$$

$$
\times \frac{(q^{1-r} a^{-2}; q)^6_{\infty}}{(q^r a^2; q)^6_{\infty}} \frac{(q^{-1+\frac{3r}{2}} a^3; q)_{\infty}}{(q^{2-\frac{3r}{2}} a^{-3}; q)_{\infty}}, \tag{3.90}
$$

$$
I^{SO(4)-[3]^B_{-+}}
$$

$$
= \frac{1}{2 \cdot 2} \sum_{m_1, m_2 \in \mathbb{Z}} \oint \frac{ds_1}{2\pi i s_1} \frac{ds_2}{2\pi i s_2} (1 - q^{\frac{|m_1-m_2|}{2}} s_1^{\pm} s_2^{\mp})(1 - q^{\frac{|m_1-m_2|}{2}} s_1^{\pm} s_2^{\pm})
$$

$$
\times \prod_{i=1}^{3} \frac{(q^{\frac{1+r+|m|}{2}} a s_i^{\pm}; q)^3_{\infty}}{(q^{\frac{1-r+|m|}{2}} a^{-1} s_i^{\pm}; q)^3_{\infty}} q^{\sum_{i=1}^{3} \frac{3r}{2}|m_i| - \sum_{i<j} \frac{|m_i-m_j|}{2} - \sum_{i<j} \frac{|m_i+m_j|}{2}} a^{3\sum_{i=1}^{3}|m|} (-1)^{\sum_{i=1}^{3} m_i}
$$

$$
\times \frac{(q^{1-r} a^{-2}; q)^6_{\infty}}{(q^r a^2; q)^6_{\infty}} \frac{(-q^{-1+\frac{3r}{2}} a^3; q)_{\infty}}{(-q^{2-\frac{3r}{2}} a^{-3}; q)_{\infty}} . \tag{3.91}
$$

3.5.1 $SO(1) - [3]$ with $\mathcal{N}, (\mathrm{N,N,N}) + $ Fermis

The half-indices with Neumann boundary conditions in theory A are

$$\mathrm{III}^{SO(1)-[3]^A_{++}}_{\mathcal{N},\mathrm{N}+\Psi} = \frac{1}{(q^{\frac{r}{2}}a;q)^3_\infty}(q^{\frac{1}{2}}u^\pm_1;q)_\infty(q^{\frac{1}{2}}u^\pm_2;q)_\infty, \tag{3.92}$$

$$\mathrm{III}^{SO(1)-[3]^A_{-+}}_{\mathcal{N},\mathrm{N}+\Psi} = \frac{1}{(q^{\frac{r}{2}}a;q)^3_\infty}(q^{\frac{1}{2}}u^\pm;q)_\infty(\pm q^{\frac{1}{2}};q)_\infty, \tag{3.93}$$

$$\mathrm{III}^{SO(1)-[3]^A_{+-}}_{\mathcal{N},\mathrm{N}+\Psi} = \frac{1}{(-q^{\frac{r}{2}}a;q)^3_\infty}(-q^{\frac{1}{2}}u^\pm;q)_\infty(\pm q^{\frac{1}{2}};q)_\infty, \tag{3.94}$$

$$\mathrm{III}^{SO(1)-[3]^A_{--}}_{\mathcal{N},\mathrm{N}+\Psi} = \frac{1}{(-q^{\frac{r}{2}}a;q)^3_\infty}(-q^{\frac{1}{2}}u^\pm_1;q)_\infty(-q^{\frac{1}{2}}u^\pm_2;q)_\infty \tag{3.95}$$

and the matching theory B half-indices are

$$\mathrm{III}^{SO(4)-[3]^B_{++}}_{\mathcal{D},\mathrm{D,N,D}} = \frac{1}{(q)^2_\infty}\sum_{m_1,m_2\in\mathbb{Z}}\frac{1}{(q^{1\pm m_1\mp m_2}u^\pm_1 u^\mp_2;q)_\infty(q^{1\pm m_1\pm m_2}u^\pm_1 u^\pm_2;q)_\infty}$$
$$\times\prod_{i=1}^2(q^{\frac{1+r}{2}\pm m_i}au^\pm_i;q)^3_\infty(-1)^{m_1+m_2}q^{\frac{m_1^2+m_2^2}{2}}u^{m_1}_1 u^{m_2}_2$$
$$\times\frac{1}{(q^r a^2;q)^6_\infty}(q^{-1+\frac{3r}{2}}a^3;q)_\infty, \tag{3.96}$$

$$\mathrm{III}^{SO(4)-[3]^B_{--}}_{\mathcal{D},\mathrm{D,N,D}} = \frac{1}{(q)_\infty(-q;q)_\infty}\sum_{m\in\mathbb{Z}}\frac{1}{(q^{1\pm m}u^\pm;q)_\infty(-q^{1\pm m}u^\pm;q)_\infty}$$
$$\times(q^{\frac{1+r}{2}\pm m}au^\pm;q)^3_\infty(\pm q^{\frac{1+r}{2}}a;q)^3_\infty(-1)^m q^{\frac{m^2}{2}}u^m$$
$$\times\frac{1}{(q^r a^2;q)^6_\infty}(-q^{-1+\frac{3r}{2}}a^3;q)_\infty, \tag{3.97}$$

$$\mathrm{III}^{SO(4)-[3]^B_{+-}}_{\mathcal{D},\mathrm{D,N,D}} = \frac{1}{(q)_\infty(-q;q)_\infty}\sum_{m\in\mathbb{Z}}\frac{1}{(q^{1\pm m}u^\pm;q)_\infty(-q^{1\pm m}u^\pm;q)_\infty}$$
$$\times(q^{\frac{1+r}{2}\pm m}au^\pm;q)^3_\infty(\pm q^{\frac{1+r}{2}}a;q)^3_\infty q^{\frac{m^2}{2}}u^m$$
$$\times\frac{1}{(q^r a^2;q)^6_\infty}(q^{-1+\frac{3r}{2}}a^3;q)_\infty, \tag{3.98}$$

$$\mathrm{III}^{SO(4)-[3]^B_{-+}}_{\mathcal{D},\mathrm{D,N,D}} = \frac{1}{(q)^2_\infty}\sum_{m_1,m_2\in\mathbb{Z}}\frac{1}{(q^{1\pm m_1\mp m_2}u^\pm_1 u^\mp_2;q)_\infty(q^{1\pm m_1\pm m_2}u^\pm_1 u^\pm_2;q)_\infty}$$
$$\times\prod_{i=1}^2(q^{\frac{1+r}{2}\pm m_i}au^\pm_i;q)^3_\infty q^{\frac{m_1^2+m_2^2}{2}}u^{m_1}_1 u^{m_2}_2$$
$$\times\frac{1}{(q^r a^2;q)^6_\infty}(-q^{-1+\frac{3r}{2}}a^3;q)_\infty. \tag{3.99}$$

3.5.2 $SO(1) - [3]$ with $\mathcal{D}, (\mathrm{D}, \mathrm{D}, \mathrm{D})$

With Dirichlet conditions the theory A half-indices are

$$\mathrm{III}_{\mathcal{D},\mathrm{D}}^{SO(1)-[3]_{++}^{A}} = (q^{1-\frac{r}{2}}a^{-1};q)_{\infty}^{3}, \tag{3.100}$$

$$\mathrm{III}_{\mathcal{D},\mathrm{D}}^{SO(1)-[3]_{\pm-}^{A}} = (-q^{1-\frac{r}{2}}a^{-1};q)_{\infty}^{3}. \tag{3.101}$$

The matching theory B half-indices are

$$\mathrm{III}_{\mathcal{N},\mathrm{N},\mathrm{D},\mathrm{N}+\Psi}^{SO(4)-[3]_{++}^{B}} = \frac{1}{2\cdot 2}\oint \frac{ds_1}{2\pi i s_1}\frac{ds_2}{2\pi i s_2}(s_1^{\pm}s_2^{\mp};q)_{\infty}(s_1^{\pm}s_2^{\pm};q)_{\infty}$$

$$\times \prod_{i=1}^{2}\frac{1}{(q^{\frac{1-r}{2}}a^{-1}s_i^{\pm};q)_{\infty}^{3}}(q^{\frac{1}{2}}s_i^{\pm};q)_{\infty}(q^{1-r}a^{-2};q)_{\infty}^{6}\frac{1}{(q^{2-\frac{3r}{2}}a^{-3};q)_{\infty}}, \tag{3.102}$$

$$\mathrm{III}_{\mathcal{N},\mathrm{N},\mathrm{D},\mathrm{N}+\Psi}^{SO(4)-[3]_{--}^{B}} = \frac{1}{2}(q)_{\infty}(-q;q)_{\infty}\oint \frac{ds}{2\pi i s}(s^{\pm};q)_{\infty}(-s^{\pm};q)_{\infty}$$

$$\times \frac{1}{(q^{\frac{1-r}{2}}a^{-1}s^{\pm};q)_{\infty}^{3}(\pm q^{\frac{1-r}{2}}a^{-1};q)_{\infty}^{3}}(q^{\frac{1}{2}}s^{\pm};q)_{\infty}(\pm q^{\frac{1}{2}};q)_{\infty}$$

$$\times (q^{1-r}a^{-2};q)_{\infty}^{6}\frac{1}{(-q^{2-\frac{3r}{2}}a^{-3};q)_{\infty}}, \tag{3.103}$$

$$\mathrm{III}_{\mathcal{N},\mathrm{N},\mathrm{D},\mathrm{N}+\Psi}^{SO(4)-[3]_{+-}^{B}} = \frac{1}{2}(q)_{\infty}(-q;q)_{\infty}\oint \frac{ds}{2\pi i s}(s^{\pm};q)_{\infty}(-s^{\pm};q)_{\infty}$$

$$\times \frac{1}{(q^{\frac{1-r}{2}}a^{-1}s^{\pm};q)_{\infty}^{3}(\pm q^{\frac{1-r}{2}}a^{-1};q)_{\infty}^{3}}(-q^{\frac{1}{2}}s^{\pm};q)_{\infty}(\pm q^{\frac{1}{2}};q)_{\infty}$$

$$\times (q^{1-r}a^{-2};q)_{\infty}^{6}\frac{1}{(q^{2-\frac{3r}{2}}a^{-3};q)_{\infty}}, \tag{3.104}$$

$$\mathrm{III}_{\mathcal{N},\mathrm{N},\mathrm{D},\mathrm{N}+\Psi}^{SO(4)-[3]_{-+}^{B}} = \frac{1}{2\cdot 2}\oint \frac{ds_1}{2\pi i s_1}\frac{ds_2}{2\pi i s_2}(s_1^{\pm}s_2^{\mp};q)_{\infty}(s_1^{\pm}s_2^{\pm};q)_{\infty}$$

$$\times \prod_{i=1}^{2}\frac{1}{(q^{\frac{1-r}{2}}a^{-1}s_i^{\pm};q)_{\infty}^{3}}(-q^{\frac{1}{2}}s_i^{\pm};q)_{\infty}(q^{1-r}a^{-2};q)_{\infty}^{6}\frac{1}{(-q^{2-\frac{3r}{2}}a^{-3};q)_{\infty}}. \tag{3.105}$$

3.6 $N_c = 2, N_f = 2$ $(SO(2) - [2])$

We now consider an example with $SO(2)$ gauge group in theory A. This is equivalent to $U(1)$ with each fundamental chiral of $SO(2)$ corresponding to two chirals of $U(1)$ with charges ± 1. Note that the cases with $\chi = -1$ in theory A are not sensitive to the value of ζ so we have

$$I^{SO(N_f)-[N_f]_{+-}^{B}} = I^{SO(2)-[N_f]_{\pm-}^{A}} = I^{SO(N_f)-[N_f]_{--}^{B}}. \tag{3.106}$$

Here we present only the cases with $N_f = 2$ which gives gauge group $SO(2)$ also in theory B.

The theory A full-indices are

$$I^{SO(2)-[2]^A_{++}} = \sum_{m\in\mathbb{Z}} \oint \frac{ds}{2\pi i s} \frac{(q^{1-\frac{r+|m|}{2}}a^{-1}s^{-1};q)^2_\infty (q^{1-\frac{r+|m|}{2}}a^{-1}s;q)^2_\infty}{(q^{\frac{r+|m|}{2}}as;q)^2_\infty (q^{\frac{r+|m|}{2}}as^{-1};q)^2_\infty} q^{(1-r)|m|}a^{-2|m|},$$

(3.107)

$$I^{SO(2)-[2]^A_{-+}} = \sum_{m\in\mathbb{Z}} \oint \frac{ds}{2\pi i s} \frac{(q^{1-\frac{r+|m|}{2}}a^{-1}s^{-1};q)^2_\infty (q^{1-\frac{r+|m|}{2}}a^{-1}s;q)^2_\infty}{(q^{\frac{r+|m|}{2}}as;q)^2_\infty (q^{\frac{r+|m|}{2}}as^{-1};q)^2_\infty} q^{(1-r)|m|}a^{-2|m|}(-1)^m,$$

(3.108)

$$I^{SO(2)-[2]^A_{\pm-}} = \frac{(q^{1-\frac{r}{2}}a^{-1};q)^2_\infty (-q^{1-\frac{r}{2}}a^{-1};q)^2_\infty}{(q^{\frac{r}{2}}a;q)^2_\infty (-q^{\frac{r}{2}}a;q)^2_\infty}.$$

(3.109)

The matching theory B indices are

$$I^{SO(2)-[2]^B_{++}} = \sum_{m\in\mathbb{Z}} \oint \frac{ds}{2\pi i s} \frac{(q^{\frac{1+r+|m|}{2}}as^{-1};q)^2_\infty (q^{\frac{1+r+|m|}{2}}as;q)^2_\infty}{(q^{\frac{1-r+|m|}{2}}a^{-1}s;q)^2_\infty (q^{\frac{1-r+|m|}{2}}a^{-1}s^{-1};q)^2_\infty} q^{r|m|}a^{2|m|}$$
$$\times \frac{(q^{1-r}a^{-2};q)^3_\infty}{(q^r a^2;q)^3_\infty} \frac{(q^r a^2;q)_\infty}{(q^{1-r}a^{-2};q)_\infty},$$

(3.110)

$$I^{SO(2)-[2]^B_{--}} = \frac{(q^{\frac{1+r}{2}}a;q)^2_\infty (-q^{\frac{1+r}{2}}a;q)^2_\infty}{(q^{\frac{1-r}{2}}a^{-1};q)^2_\infty (-q^{\frac{1-r}{2}}a^{-1};q)^2_\infty} \frac{(q^{1-r}a^{-2};q)^3_\infty}{(q^r a^2;q)^3_\infty} \frac{(-q^r a^2;q)_\infty}{(-q^{1-r}a^{-2};q)_\infty}, \quad (3.111)$$

$$I^{SO(2)-[2]^B_{+-}} = \frac{(q^{\frac{1+r}{2}}a;q)^2_\infty (-q^{\frac{1+r}{2}}a;q)^2_\infty}{(q^{\frac{1-r}{2}}a^{-1};q)^2_\infty (-q^{\frac{1-r}{2}}a^{-1};q)^2_\infty} \frac{(q^{1-r}a^{-2};q)^3_\infty}{(q^r a^2;q)^3_\infty} \frac{(q^r a^2;q)_\infty}{(q^{1-r}a^{-2};q)_\infty}, \quad (3.112)$$

$$I^{SO(2)-[2]^B_{-+}} = \sum_{m\in\mathbb{Z}} \oint \frac{ds}{2\pi i s} \frac{(q^{\frac{1+r+|m|}{2}}as^{-1};q)^2_\infty (q^{\frac{1+r+|m|}{2}}as;q)^2_\infty}{(q^{\frac{1-r+|m|}{2}}a^{-1}s;q)^2_\infty (q^{\frac{1-r+|m|}{2}}a^{-1}s^{-1};q)^2_\infty} q^{r|m|}a^{2|m|}(-1)^m$$
$$\times \frac{(q^{1-r}a^{-2};q)^3_\infty}{(q^r a^2;q)^3_\infty} \frac{(-q^r a^2;q)_\infty}{(-q^{1-r}a^{-2};q)_\infty}.$$

(3.113)

In the indices (3.110) and (3.112) the contribution from the gauge singlet V is cancelled by that from the meson M. Accordingly, the index (3.110) can be alternatively interpreted as the index of the mirror theory [89, 90, 71] for the SQED$_2$, i.e. the $U(1)$ gauge theory with two pairs of chiral multiplets of charge ± 1 as well as gauge singlets. Note that such a cancellation does not occur and the indices of theory B and that of the mirror theory are distinguished when one turns on the fugacities for the flavour and topological symmetries.

3.6.1 $SO(2) - [2]$ **with** $\mathcal{N}, (\mathrm{N}, \mathrm{N}) + \mathrm{Fermis}$

The theory A half-indices are

$$\mathbb{III}_{\mathcal{N},\mathrm{N}+\Psi}^{SO(2)-[2]_{++}^A} = (q)_\infty \oint \frac{ds}{2\pi i s} \frac{1}{(q^{\frac{r}{2}}as;q)_\infty^2(q^{\frac{r}{2}}as^{-1};q)_\infty^2}(q^{\frac{1}{2}}s^\pm u^\pm;q)_\infty(q^{\frac{1}{2}}s^\pm u^\mp;q)_\infty, \tag{3.114}$$

$$\mathbb{III}_{\mathcal{N},\mathrm{N}+\Psi}^{SO(2)-[2]_{-+}^A} = (q)_\infty \oint \frac{ds}{2\pi i s} \frac{1}{(q^{\frac{r}{2}}as;q)_\infty^2(q^{\frac{r}{2}}as^{-1};q)_\infty^2}(\pm q^{\frac{1}{2}}s;q)_\infty(\pm q^{\frac{1}{2}}s^{-1};q)_\infty, \tag{3.115}$$

$$\mathbb{III}_{\mathcal{N},\mathrm{N}+\Psi}^{SO(2)-[2]_{+-}^A} = (-q;q)_\infty \frac{1}{(q^{\frac{r}{2}}a;q)_\infty^2(-q^{\frac{r}{2}}a;q)_\infty^2}(q^{\frac{1}{2}};q)_\infty^2(-q^{\frac{1}{2}};q)_\infty^2, \tag{3.116}$$

$$\mathbb{III}_{\mathcal{N},\mathrm{N}+\Psi}^{SO(2)-[2]_{--}^A} = (-q;q)_\infty \frac{1}{(q^{\frac{r}{2}}a;q)_\infty^2(-q^{\frac{r}{2}}a;q)_\infty^2}(q^{\frac{1}{2}}u^\pm;q)_\infty^2(-q^{\frac{1}{2}}u^\pm;q)_\infty^2 \tag{3.117}$$

They agree with theory B indices

$$\mathbb{III}_{\mathcal{D},\mathrm{D},\mathrm{N},\mathrm{D}}^{SO(2)-[2]_{++}^B} = \frac{1}{(q)_\infty}\sum_{m\in\mathbb{Z}}(q^{\frac{1+r}{2}\pm m}au^\pm;q)_\infty^2 q^{m^2}u^{2m}\frac{1}{(q^r a^2;q)_\infty^3}(q^r a^2;q)_\infty, \tag{3.118}$$

$$\mathbb{III}_{\mathcal{D},\mathrm{D},\mathrm{N},\mathrm{D}}^{SO(2)-[2]_{--}^B} = \frac{1}{(-q;q)_\infty}(q^{\frac{1+r}{2}}a;q)_\infty^2(-q^{\frac{1+r}{2}}a;q)_\infty^2\frac{1}{(q^r a^2;q)_\infty^3}(-q^r a^2;q)_\infty, \tag{3.119}$$

$$\mathbb{III}_{\mathcal{D},\mathrm{D},\mathrm{N},\mathrm{D}}^{SO(2)-[2]_{+-}^B} = \frac{1}{(-q;q)_\infty}(q^{\frac{1+r}{2}}a;q)_\infty^2(-q^{\frac{1+r}{2}}a;q)_\infty^2\frac{1}{(q^r a^2;q)_\infty^3}(q^r a^2;q)_\infty, \tag{3.120}$$

$$\mathbb{III}_{\mathcal{D},\mathrm{D},\mathrm{N},\mathrm{D}}^{SO(2)-[2]_{-+}^B} = \frac{1}{(q)_\infty}\sum_{m\in\mathbb{Z}}(q^{\frac{1+r}{2}\pm m}au^\pm;q)_\infty^2(-1)^m q^{m^2}u^{2m}\frac{1}{(q^r a^2;q)_\infty^3}(-q^r a^2;q)_\infty. \tag{3.121}$$

3.6.2 $SO(2) - [2]$ **with** $\mathcal{D}, (\mathrm{D}, \mathrm{D})$

As for the full-indices, for $\chi = -1$ the theory A Dirichlet index is not sensitive to the value of ζ so we have identities

$$\mathbb{III}_{\mathcal{N},\mathrm{N},\mathrm{D},\mathrm{N}}^{SO(N_f)-[N_f]_{+-}^B} = \mathbb{III}_{\mathcal{D},\mathrm{D}}^{SO(2)-[N_f]_{\pm-}^A} = \mathbb{III}_{\mathcal{N},\mathrm{N},\mathrm{D},\mathrm{N}}^{SO(N_f)-[N_f]_{--}^B}. \tag{3.122}$$

For $N_f = 2$ we have the half-indices in theory A

$$\mathbb{III}_{\mathcal{D},\mathrm{D}}^{SO(2)-[2]_{++}^A} = \frac{1}{(q)_\infty}\sum_{m\in\mathbb{Z}}(q^{1-\frac{r}{2}\pm m}a^{-1}u^\pm;q)_\infty^2 q^{m^2}u^{2m}, \tag{3.123}$$

$$\mathbb{III}_{\mathcal{D},\mathrm{D}}^{SO(2)-[2]_{-+}^A} = \frac{1}{(q)_\infty}\sum_{m\in\mathbb{Z}}(q^{1-\frac{r}{2}\pm m}a^{-1}u^\pm;q)_\infty^2 (-1)^m q^{m^2}u^{2m}, \tag{3.124}$$

$$\mathbb{III}_{\mathcal{D},\mathrm{D}}^{SO(2)-[2]_{\pm-}^A} = \frac{1}{(-q;q)_\infty}(q^{1-\frac{r}{2}}a^{-1};q)_\infty^2(-q^{1-\frac{r}{2}}a^{-1};q)_\infty^2. \tag{3.125}$$

The matching theory B half-indices are

$$
\mathrm{III}^{SO(2)-[2]^B_{++}}_{\mathcal{N},N,D,N+\Psi} = (q)_\infty \oint \frac{ds}{2\pi i s} \frac{1}{(q^{\frac{1-r}{2}}a^{-1}s^{\pm};q)^2_\infty}(q^{\frac{1}{2}}s^{\pm}u^{\pm};q)_\infty(q^{\frac{1}{2}}s^{\pm}u^{\mp};q)_\infty
$$
$$
\times (q^{1-r}a^{-2};q)^3_\infty \frac{1}{(q^{1-r}a^{-2};q)_\infty}, \tag{3.126}
$$

$$
\mathrm{III}^{SO(2)-[2]^B_{--}}_{\mathcal{N},N,D,N+\Psi} = (-q;q)_\infty \frac{1}{(q^{\frac{1-r}{2}}a^{-1};q)^2_\infty(-q^{\frac{1-r}{2}}a^{-1};q)^2_\infty}(q^{\frac{1}{2}}u^{\pm};q)_\infty(-q^{\frac{1}{2}}u^{\pm};q)_\infty
$$
$$
\times (q^{1-r}a^{-2};q)^3_\infty \frac{1}{(-q^{1-r}a^{-2};q)_\infty}, \tag{3.127}
$$

$$
\mathrm{III}^{SO(2)-[2]^B_{+-}}_{\mathcal{N},N,D,N+\Psi} = (-q;q)_\infty \frac{1}{(q^{\frac{1-r}{2}}a^{-1};q)^2_\infty(-q^{\frac{1-r}{2}}a^{-1};q)^2_\infty}(q^{\frac{1}{2}};q)^2_\infty(-q^{\frac{1}{2}};q)^2_\infty
$$
$$
\times (q^{1-r}a^{-2};q)^3_\infty \frac{1}{(q^{1-r}a^{-2};q)_\infty}, \tag{3.128}
$$

$$
\mathrm{III}^{SO(2)-[2]^B_{-+}}_{\mathcal{N},N,D,N+\Psi} = (q)_\infty \oint \frac{ds}{2\pi i s} \frac{1}{(q^{\frac{1-r}{2}}a^{-1}s^{\pm};q)^2_\infty}(q^{\frac{1}{2}}s^{\pm};q)^2_\infty(-q^{\frac{1}{2}}s^{\pm};q)^2_\infty
$$
$$
\times (q^{1-r}a^{-2};q)^3_\infty \frac{1}{(-q^{1-r}a^{-2};q)_\infty}. \tag{3.129}
$$

3.7 $N_c = 3, N_f = 3$ $(SO(3) - [3])$

We now consider examples with non-Abelian gauge groups. In the first case with $SO(3) - [3]^A$ the dual theory B is still Abelian with gauge group $SO(2)$. In the following section we will look at the case of $N_f = 4$ where theory B also has gauge group $SO(3)$.

The theory A indices are

$$I^{SO(3)-[3]^A_{++}} = \frac{1}{2}\sum_{m\in\mathbb{Z}}\oint\frac{ds}{2\pi is}(1-q^{\frac{|m|}{2}}s)(1-q^{\frac{|m|}{2}}s^{-1})$$

$$\times\frac{(q^{1-\frac{r+|m|}{2}}a^{-1}s^{-1};q)^3_\infty(q^{1-\frac{r+|m|}{2}}a^{-1}s;q)^3_\infty(q^{1-\frac{r}{2}}a^{-1};q)^3_\infty}{(q^{\frac{r+|m|}{2}}as;q)^3_\infty(q^{\frac{r+|m|}{2}}as^{-1};q)^3_\infty(q^{\frac{r}{2}}a;q)^3_\infty}q^{\frac{3(1-r)|m|}{2}-\frac{|m|}{2}}a^{-3|m|},$$

$$(3.130)$$

$$I^{SO(3)-[3]^A_{-+}} = \frac{1}{2}\sum_{m\in\mathbb{Z}}\oint\frac{ds}{2\pi is}(1-q^{\frac{|m|}{2}}s)(1-q^{\frac{|m|}{2}}s^{-1})$$

$$\times\frac{(q^{1-\frac{r+|m|}{2}}a^{-1}s^{-1};q)^3_\infty(q^{1-\frac{r+|m|}{2}}a^{-1}s;q)^3_\infty(q^{1-\frac{r}{2}}a^{-1};q)^3_\infty}{(q^{\frac{r+|m|}{2}}as;q)^3_\infty(q^{\frac{r+|m|}{2}}as^{-1};q)^3_\infty(q^{\frac{r}{2}}a;q)^3_\infty}(-1)^m q^{\frac{3(1-r)|m|}{2}-\frac{|m|}{2}}a^{-3|m|},$$

$$(3.131)$$

$$I^{SO(3)-[3]^A_{+-}} = \frac{1}{2}\sum_{m\in\mathbb{Z}}\oint\frac{ds}{2\pi is}(1+q^{\frac{|m|}{2}}s)(1+q^{\frac{|m|}{2}}s^{-1})$$

$$\times\frac{(q^{1-\frac{r+|m|}{2}}a^{-1}s^{-1};q)^3_\infty(q^{1-\frac{r+|m|}{2}}a^{-1}s;q)^3_\infty(-q^{1-\frac{r}{2}}a^{-1};q)^3_\infty}{(q^{\frac{r+|m|}{2}}as;q)^3_\infty(q^{\frac{r+|m|}{2}}as^{-1};q)^3_\infty(-q^{\frac{r}{2}}a;q)^3_\infty}q^{\frac{3(1-r)|m|}{2}-\frac{|m|}{2}}a^{-3|m|},$$

$$(3.132)$$

$$I^{SO(3)-[3]^A_{--}} = \frac{1}{2}\sum_{m\in\mathbb{Z}}\oint\frac{ds}{2\pi is}(1+q^{\frac{|m|}{2}}s)(1+q^{\frac{|m|}{2}}s^{-1})$$

$$\times\frac{(q^{1-\frac{r+|m|}{2}}a^{-1}s^{-1};q)^3_\infty(q^{1-\frac{r+|m|}{2}}a^{-1}s;q)^3_\infty(-q^{1-\frac{r}{2}}a^{-1};q)^3_\infty}{(q^{\frac{r+|m|}{2}}as;q)^3_\infty(q^{\frac{r+|m|}{2}}as^{-1};q)^3_\infty(-q^{\frac{r}{2}}a;q)^3_\infty}q^{\frac{3(1-r)|m|}{2}-\frac{|m|}{2}}a^{-3|m|}(-1)^m.$$

$$(3.133)$$

These indices coincide with theory B indices

$$I^{SO(2)-[3]^B_{++}} = \sum_{m\in\mathbb{Z}}\oint\frac{ds}{2\pi is}\frac{(q^{\frac{1+r+|m|}{2}}as^{-1};q)^3_\infty(q^{\frac{1+r+|m|}{2}}as;q)^3_\infty(q^{\frac{1+r}{2}}a;q)^3_\infty}{(q^{\frac{1-r+|m|}{2}}a^{-1}s;q)^3_\infty(q^{\frac{1-r+|m|}{2}}a^{-1}s^{-1};q)^3_\infty(q^{\frac{1-r}{2}}a^{-1};q)^3_\infty}q^{\frac{3r|m|}{2}}a^{3|m|}$$

$$\times\frac{(q^{1-r}a^{-2};q)^6_\infty}{(q^r a^2;q)^6_\infty}\frac{(q^{\frac{3r}{2}}a^3;q)_\infty}{(q^{1-\frac{3r}{2}}a^{-3};q)_\infty},\qquad(3.134)$$

$$I^{SO(2)-[3]^B_{--}} = \frac{(q^{\frac{1+r}{2}}a;q)^3_\infty(-q^{\frac{1+r}{2}}a;q)^3_\infty}{(q^{\frac{1-r}{2}}a^{-1};q)^3_\infty(-q^{\frac{1-r}{2}}a^{-1};q)^3_\infty}\frac{(q^{1-r}a^{-2};q)^6_\infty}{(q^r a^2;q)^6_\infty}\frac{(-q^{\frac{3r}{2}}a^3;q)_\infty}{(-q^{1-\frac{3r}{2}}a^{-3};q)_\infty},\qquad(3.135)$$

$$I^{SO(2)-[3]^B_{+-}} = \frac{(q^{\frac{1+r}{2}}a;q)^3_\infty(-q^{\frac{1+r}{2}}a;q)^3_\infty}{(q^{\frac{1-r}{2}}a^{-1};q)^3_\infty(-q^{\frac{1-r}{2}}a^{-1};q)^3_\infty}\frac{(q^{1-r}a^{-2};q)^6_\infty}{(q^r a^2;q)^6_\infty}\frac{(q^{\frac{3r}{2}}a^3;q)_\infty}{(q^{1-\frac{3r}{2}}a^{-3};q)_\infty},\qquad(3.136)$$

$$I^{SO(2)-[3]^B_{-+}} = \sum_{m\in\mathbb{Z}}\oint\frac{ds}{2\pi is}\frac{(q^{\frac{1+r+|m|}{2}}as^{-1};q)^3_\infty(q^{\frac{1+r+|m|}{2}}as;q)^3_\infty}{(q^{\frac{1-r+|m|}{2}}a^{-1}s;q)^3_\infty(q^{\frac{1-r+|m|}{2}}a^{-1}s^{-1};q)^3_\infty}q^{\frac{3r|m|}{2}}a^{3|m|}(-1)^m$$

$$\times\frac{(q^{1-r}a^{-2};q)^6_\infty}{(q^r a^2;q)^6_\infty}\frac{(-q^{\frac{3r}{2}}a^3;q)_\infty}{(-q^{1-\frac{3r}{2}}a^{-3};q)_\infty}.\qquad(3.137)$$

3.7.1 $SO(3) - [3]$ with $\mathcal{N}, (\mathrm{N}, \mathrm{N}, \mathrm{N}) + $ Fermis

The Neumann theory A half-indices are

$$
\mathrm{III}^{SO(3)-[3]^A_{++}}_{\mathcal{N},\mathrm{N}+\Psi} = \frac{1}{2}(q)_\infty \oint \frac{ds}{2\pi i s}(s^\pm; q)_\infty \frac{1}{(q^{\frac{r}{2}}as; q)^3_\infty (q^{\frac{r}{2}}as^{-1}; q)^3_\infty (q^{\frac{r}{2}}a; q)^3_\infty}
$$
$$
\times (q^{\frac{1}{2}}s^\pm u^\pm; q)_\infty (q^{\frac{1}{2}}s^\pm u^\mp; q)_\infty (q^{\frac{1}{2}}u^\pm; q)_\infty, \tag{3.138}
$$

$$
\mathrm{III}^{SO(3)-[3]^A_{-+}}_{\mathcal{N},\mathrm{N}+\Psi} = \frac{1}{2}(q)_\infty \oint \frac{ds}{2\pi i s}(s^\pm; q)_\infty \frac{1}{(q^{\frac{r}{2}}as; q)^3_\infty (q^{\frac{r}{2}}as^{-1}; q)^3_\infty (q^{\frac{r}{2}}a; q)^3_\infty}
$$
$$
\times (\pm q^{\frac{1}{2}}s^\pm; q)_\infty (\mp q^{\frac{1}{2}}s^\pm; q)_\infty (\pm q^{\frac{1}{2}}; q)_\infty, \tag{3.139}
$$

$$
\mathrm{III}^{SO(3)-[3]^A_{+-}}_{\mathcal{N},\mathrm{N}+\Psi} = \frac{1}{2}(q)_\infty \oint \frac{ds}{2\pi i s}(-s^\pm; q)_\infty \frac{1}{(q^{\frac{r}{2}}as; q)^3_\infty (q^{\frac{r}{2}}as^{-1}; q)^3_\infty (-q^{\frac{r}{2}}a; q)^3_\infty}
$$
$$
\times (\pm q^{\frac{1}{2}}s^\pm; q)_\infty (\mp q^{\frac{1}{2}}s^\pm; q)_\infty (\pm q^{\frac{1}{2}}; q)_\infty, \tag{3.140}
$$

$$
\mathrm{III}^{SO(3)-[3]^A_{--}}_{\mathcal{N},\mathrm{N}+\Psi} = \frac{1}{2}(q)_\infty \oint \frac{ds}{2\pi i s}(-s^\pm; q)_\infty \frac{1}{(q^{\frac{r}{2}}as; q)^3_\infty (q^{\frac{r}{2}}as^{-1}; q)^3_\infty (-q^{\frac{r}{2}}a; q)^3_\infty}
$$
$$
\times (q^{\frac{1}{2}}s^\pm u^\pm; q)_\infty (q^{\frac{1}{2}}s^\pm u^\mp; q)_\infty (-q^{\frac{1}{2}}u^\pm; q)_\infty \tag{3.141}
$$

with the matching theory B half-indices given by

$$
\mathrm{III}^{SO(2)-[3]^B_{++}}_{\mathcal{D},\mathrm{D},\mathrm{N},\mathrm{D}} = \frac{1}{(q)_\infty} \sum_{m\in\mathbb{Z}}(q^{\frac{1+r}{2}\pm m}au^\pm; q)^3_\infty (-1)^m q^{\frac{3}{2}m^2} u^{3m} \frac{1}{(q^r a^2; q)^6_\infty}(q^{\frac{3r}{2}}a^3; q)_\infty,
$$
$$
\tag{3.142}
$$

$$
\mathrm{III}^{SO(2)-[3]^B_{--}}_{\mathcal{D},\mathrm{D},\mathrm{N},\mathrm{D}} = \frac{1}{(-q; q)_\infty}(q^{\frac{1+r}{2}}a; q)^3_\infty (-q^{\frac{1-r}{2}}a; q)^3_\infty \frac{1}{(q^r a^2; q)^6_\infty}(-q^{\frac{3r}{2}}a^3; q)_\infty, \tag{3.143}
$$

$$
\mathrm{III}^{SO(2)-[3]^B_{+-}}_{\mathcal{D},\mathrm{D},\mathrm{N},\mathrm{D}} = \frac{1}{(-q; q)_\infty}(q^{\frac{1+r}{2}}a; q)^3_\infty (-q^{\frac{1-r}{2}}a; q)^3_\infty \frac{1}{(q^r a^2; q)^6_\infty}(q^{\frac{3r}{2}}a^3; q)_\infty, \tag{3.144}
$$

$$
\mathrm{III}^{SO(2)-[3]^B_{-+}}_{\mathcal{D},\mathrm{D},\mathrm{N},\mathrm{D}} = \frac{1}{(q)_\infty} \sum_{m\in\mathbb{Z}}(q^{\frac{1+r}{2}\pm m}au^\pm; q)^3_\infty q^{\frac{3}{2}m^2} u^{3m} \frac{1}{(q^r a^2; q)^6_\infty}(-q^{\frac{3r}{2}}a^3; q)_\infty . \tag{3.145}
$$

3.7.2 $SO(3) - [3]$ with $\mathcal{D}, (D, D, D)$

For Dirichlet conditions we have theory A half-indices

$$\mathbb{II}_{\mathcal{D},D}^{SO(3)-[3]_{++}^A} = \frac{1}{(q)_\infty} \sum_{m \in \mathbb{Z}} \frac{1}{(q^{1 \pm m} u^{\pm}; q)_\infty} (q^{1 - \frac{r}{2} \pm m} a^{-1} u^{\pm}; q)_\infty^3 (q^{1 - \frac{r}{2}} a^{-1}; q)_\infty^3 q^{m^2} u^{2m},$$

(3.146)

$$\mathbb{II}_{\mathcal{D},D}^{SO(3)-[3]_{-+}^A} = \frac{1}{(q)_\infty} \sum_{m \in \mathbb{Z}} \frac{1}{(q^{1 \pm m} u^{\pm}; q)_\infty} (q^{1 - \frac{r}{2} \pm m} a^{-1} u^{\pm}; q)_\infty^3 (q^{1 - \frac{r}{2}} a^{-1}; q)_\infty^3 (-1)^m q^{m^2} u^{2m},$$

(3.147)

$$\mathbb{II}_{\mathcal{D},D}^{SO(3)-[3]_{+-}^A} = \frac{1}{(-q;q)_\infty} \sum_{m \in \mathbb{Z}} \frac{1}{(-q^{1 \pm m} u^{\pm}; q)_\infty} (q^{1 - \frac{r}{2} \pm m} a^{-1} u^{\pm}; q)_\infty^3 (-q^{1 - \frac{r}{2}} a^{-1}; q)_\infty^3 (-1)^m q^{m^2} u^{2m},$$

(3.148)

$$\mathbb{II}_{\mathcal{D},D}^{SO(3)-[3]_{--}^A} = \frac{1}{(q)_\infty} \sum_{m \in \mathbb{Z}} \frac{1}{(-q^{1 \pm m} u^{\pm}; q)_\infty} (q^{1 - \frac{r}{2} \pm m} a^{-1} u^{\pm}; q)_\infty^3 (-q^{1 - \frac{r}{2}} a^{-1}; q)_\infty^3 q^{m^2} u^{2m} .$$

(3.149)

These match the theory B half-indices

$$\mathbb{II}_{\mathcal{N},N,D,N+\Psi}^{SO(2)-[3]_{++}^B} = (q)_\infty \oint \frac{ds}{2\pi i s} \frac{1}{(q^{\frac{1-r}{2}} a^{-1} s^{\pm}; q)_\infty^3} (q^{\frac{1}{2}} s^{\pm} u^{\pm}; q)_\infty (q^{\frac{1}{2}} s^{\pm} u^{\mp}; q)_\infty (q^{\frac{1}{2}} s^{\pm}; q)_\infty$$

$$\times (q^{1-r} a^{-2}; q)_\infty^6 \frac{1}{(q^{1 - \frac{3r}{2}} a^{-3}; q)_\infty},$$

(3.150)

$$\mathbb{II}_{\mathcal{N},N,D,N+\Psi}^{SO(2)-[3]_{--}^B} = (-q;q)_\infty \frac{1}{(q^{\frac{1-r}{2}} a^{-1}; q)_\infty^3 (-q^{\frac{1-r}{2}} a^{-1}; q)_\infty^3} (q^{\frac{1}{2}} u^{\pm}; q)_\infty (-q^{\frac{1}{2}} u^{\pm}; q)_\infty (\pm q^{\frac{1}{2}}; q)_\infty$$

$$\times (q^{1-r} a^{-2}; q)_\infty^6 \frac{1}{(-q^{1 - \frac{3r}{2}} a^{-3}; q)_\infty},$$

(3.151)

$$\mathbb{II}_{\mathcal{N},N,D,N+\Psi}^{SO(2)-[3]_{+-}^B} = (-q;q)_\infty \frac{1}{(q^{\frac{1-r}{2}} a^{-1}; q)_\infty^3 (-q^{\frac{1-r}{2}} a^{-1}; q)_\infty^3} (q^{\frac{1}{2}} u^{\pm}; q)_\infty (-q^{\frac{1}{2}} u^{\pm}; q)_\infty (\pm q^{\frac{1}{2}}; q)_\infty$$

$$\times (q^{1-r} a^{-2}; q)_\infty^6 \frac{1}{(q^{1 - \frac{3r}{2}} a^{-3}; q)_\infty},$$

(3.152)

$$\mathbb{II}_{\mathcal{N},N,D,N+\Psi}^{SO(2)-[3]_{-+}^B} = (q)_\infty \oint \frac{ds}{2\pi i s} \frac{1}{(q^{\frac{1-r}{2}} a^{-1} s^{\pm}; q)_\infty^3} (q^{\frac{1}{2}} s^{\pm} u^{\pm}; q)_\infty (q^{\frac{1}{2}} s^{\pm} u^{\mp}; q)_\infty (-q^{\frac{1}{2}} s^{\pm}; q)_\infty$$

$$\times (q^{1-r} a^{-2}; q)_\infty^6 \frac{1}{(-q^{1 - \frac{3r}{2}} a^{-3}; q)_\infty}.$$

(3.153)

3.8 $N_c = 3, N_f = 4 \ (SO(3) - [4])$

In the set of examples we have non-Abelian gauge group in both theories. The theory A full-indices are

$$
I^{SO(3)-[4]^A_{++}} = \frac{1}{2} \sum_{m \in \mathbb{Z}} \oint \frac{ds}{2\pi i s} (1 - q^{\frac{|m|}{2}} s)(1 - q^{\frac{|m|}{2}} s^{-1})
$$

$$
\times \frac{(q^{1 - \frac{r+|m|}{2}} a^{-1} s^{-1}; q)^4_\infty (q^{1 - \frac{r+|m|}{2}} a^{-1} s; q)^4_\infty (q^{1 - \frac{r}{2}} a^{-1}; q)^4_\infty}{(q^{\frac{r+|m|}{2}} as; q)^4_\infty (q^{\frac{r+|m|}{2}} as^{-1}; q)^4_\infty (q^{\frac{r}{2}} a; q)^4_\infty} q^{\frac{2(1-r)|m|}{2} - \frac{|m|}{2}} a^{-4|m|},
$$

(3.154)

$$
I^{SO(3)-[4]^A_{-+}} = \frac{1}{2} \sum_{m \in \mathbb{Z}} \oint \frac{ds}{2\pi i s} (1 - q^{\frac{|m|}{2}} s)(1 - q^{\frac{|m|}{2}} s^{-1})
$$

$$
\times \frac{(q^{1 - \frac{r+|m|}{2}} a^{-1} s^{-1}; q)^4_\infty (q^{1 - \frac{r+|m|}{2}} a^{-1} s; q)^4_\infty (q^{1 - \frac{r}{2}} a^{-1}; q)^4_\infty}{(q^{\frac{r+|m|}{2}} as; q)^4_\infty (q^{\frac{r+|m|}{2}} as^{-1}; q)^4_\infty (q^{\frac{r}{2}} a; q)^4_\infty} q^{\frac{2(1-r)|m|}{2} - \frac{|m|}{2}} a^{-4|m|}(-1)^m,
$$

(3.155)

$$
I^{SO(3)-[4]^A_{+-}} = \frac{1}{2} \sum_{m \in \mathbb{Z}} \oint \frac{ds}{2\pi i s} (1 + q^{\frac{|m|}{2}} s)(1 + q^{\frac{|m|}{2}} s^{-1})
$$

$$
\times \frac{(q^{1 - \frac{r+|m|}{2}} a^{-1} s^{-1}; q)^4_\infty (q^{1 - \frac{r+|m|}{2}} a^{-1} s; q)^4_\infty (-q^{1 - \frac{r}{2}} a^{-1}; q)^4_\infty}{(q^{\frac{r+|m|}{2}} as; q)^4_\infty (q^{\frac{r+|m|}{2}} as^{-1}; q)^4_\infty (-q^{\frac{r}{2}} a; q)^4_\infty} q^{\frac{2(1-r)|m|}{2} - \frac{|m|}{2}} a^{-4|m|},
$$

(3.156)

$$
I^{SO(3)-[4]^A_{--}} = \frac{1}{2} \sum_{m \in \mathbb{Z}} \oint \frac{ds}{2\pi i s} (1 + q^{\frac{|m|}{2}} s)(1 + q^{\frac{|m|}{2}} s^{-1})
$$

$$
\times \frac{(q^{1 - \frac{r+|m|}{2}} a^{-1} s^{-1}; q)^4_\infty (q^{1 - \frac{r+|m|}{2}} a^{-1} s; q)^4_\infty (-q^{1 - \frac{r}{2}} a^{-1}; q)^4_\infty}{(q^{\frac{r+|m|}{2}} as; q)^4_\infty (q^{\frac{r+|m|}{2}} as^{-1}; q)^4_\infty (-q^{\frac{r}{2}} a; q)^4_\infty} q^{\frac{2(1-r)|m|}{2} - \frac{|m|}{2}} a^{-4|m|}(-1)^m .
$$

(3.157)

The matching theory B indices are

$$I^{SO(3)-[4]_{++}^B} = \frac{1}{2}\sum_{m\in\mathbb{Z}}\oint\frac{ds}{2\pi i s}(1-q^{\frac{|m|}{2}}s)(1-q^{\frac{|m|}{2}}s^{-1})$$

$$\times\frac{(q^{\frac{1+r+|m|}{2}}as^{-1};q)_\infty^4(q^{\frac{1+r+|m|}{2}}as;q)_\infty^4(q^{\frac{1+r}{2}}a;q)_\infty^4}{(q^{\frac{1-r+|m|}{2}}a^{-1}s;q)_\infty^4(q^{\frac{1-r+|m|}{2}}a^{-1}s^{-1};q)_\infty^4(q^{\frac{1-r}{2}}a^{-1};q)_\infty^4}q^{2r|m|}a^{4|m|}$$

$$\times\frac{(q^{1-r}a^{-2};q)_\infty^{10}}{(q^ra^2;q)_\infty^{10}}\frac{(q^{\frac{-1+4r}{2}}a^4;q)_\infty}{(q^{\frac{3-4r}{2}}a^{-4};q)_\infty}, \tag{3.158}$$

$$I^{SO(3)-[4]_{--}^B} = \frac{1}{2}\sum_{m\in\mathbb{Z}}\oint\frac{ds}{2\pi i s}(1+q^{\frac{|m|}{2}}s)(1+q^{\frac{|m|}{2}}s^{-1})$$

$$\times\frac{(q^{\frac{1+r+|m|}{2}}as^{-1};q)_\infty^4(q^{\frac{1+r+|m|}{2}}as;q)_\infty^4(-q^{\frac{1+r}{2}}a;q)_\infty^4}{(q^{\frac{1-r+|m|}{2}}a^{-1}s;q)_\infty^4(q^{\frac{1-r+|m|}{2}}a^{-1}s^{-1};q)_\infty^4(-q^{\frac{1-r}{2}}a^{-1};q)_\infty^4}q^{2r|m|}a^{4|m|}(-1)^m$$

$$\times\frac{(q^{1-r}a^{-2};q)_\infty^{10}}{(q^ra^2;q)_\infty^{10}}\frac{(-q^{\frac{-1+4r}{2}}a^4;q)_\infty}{(-q^{\frac{3-4r}{2}}a^{-4};q)_\infty}, \tag{3.159}$$

$$I^{SO(3)-[4]_{+-}^B} = \frac{1}{2}\sum_{m\in\mathbb{Z}}\oint\frac{ds}{2\pi i s}(1+q^{\frac{|m|}{2}}s)(1+q^{\frac{|m|}{2}}s^{-1})$$

$$\times\frac{(q^{\frac{1+r+|m|}{2}}as^{-1};q)_\infty^4(q^{\frac{1+r+|m|}{2}}as;q)_\infty^4(-q^{\frac{1+r}{2}}a;q)_\infty^4}{(q^{\frac{1-r+|m|}{2}}a^{-1}s;q)_\infty^4(q^{\frac{1-r+|m|}{2}}a^{-1}s^{-1};q)_\infty^4(-q^{\frac{1-r}{2}}a^{-1};q)_\infty^4}q^{2r|m|}a^{4|m|}$$

$$\times\frac{(q^{1-r}a^{-2};q)_\infty^{10}}{(q^ra^2;q)_\infty^{10}}\frac{(q^{\frac{-1+4r}{2}}a^4;q)_\infty}{(q^{\frac{3-4r}{2}}a^{-4};q)_\infty}, \tag{3.160}$$

$$I^{SO(3)-[4]_{-+}^B} = \frac{1}{2}\sum_{m\in\mathbb{Z}}\oint\frac{ds}{2\pi i s}(1-q^{\frac{|m|}{2}}s)(1-q^{\frac{|m|}{2}}s^{-1})$$

$$\times\frac{(q^{\frac{1+r+|m|}{2}}as^{-1};q)_\infty^4(q^{\frac{1+r+|m|}{2}}as;q)_\infty^4(q^{\frac{1+r}{2}}a;q)_\infty^4}{(q^{\frac{1-r+|m|}{2}}a^{-1}s;q)_\infty^4(q^{\frac{1-r+|m|}{2}}a^{-1}s^{-1};q)_\infty^4(q^{\frac{1-r}{2}}a^{-1};q)_\infty^4}q^{2r|m|}a^{4|m|}(-1)^m$$

$$\times\frac{(q^{1-r}a^{-2};q)_\infty^{10}}{(q^ra^2;q)_\infty^{10}}\frac{(-q^{\frac{-1+4r}{2}}a^4;q)_\infty}{(-q^{\frac{3-4r}{2}}a^{-4};q)_\infty}. \tag{3.161}$$

3.8.1 $SO(3) - [4]$ **with** $\mathcal{N}, (\mathrm{N},\mathrm{N},\mathrm{N},\mathrm{N}) + \mathrm{Fermis}$

The theory A half-indices with Neumann boundary conditions are

$$
\mathbb{III}^{SO(3)-[4]^A_{++}}_{\mathcal{N},\mathrm{N}+\Psi} = \frac{1}{2}(q)_\infty \oint \frac{ds}{2\pi i s}(s^\pm;q)_\infty \frac{1}{(q^{\frac{r}{2}}as;q)^4_\infty(q^{\frac{r}{2}}as^{-1};q)^4_\infty(q^{\frac{r}{2}}a;q)^4_\infty}
$$
$$
\times (q^{\frac{1}{2}}s^\pm u^\pm;q)_\infty(q^{\frac{1}{2}}s^\pm u^\mp;q)_\infty(q^{\frac{1}{2}}s^\pm;q)_\infty(q^{\frac{1}{2}}u^\pm;q)_\infty(q^{\frac{1}{2}};q)_\infty, \qquad (3.162)
$$

$$
\mathbb{III}^{SO(3)-[4]^A_{-+}}_{\mathcal{N},\mathrm{N}+\Psi} = \frac{1}{2}(q)_\infty \oint \frac{ds}{2\pi i s}(s^\pm;q)_\infty \frac{1}{(q^{\frac{r}{2}}as;q)^4_\infty(q^{\frac{r}{2}}as^{-1};q)^4_\infty(q^{\frac{r}{2}}a;q)^4_\infty}
$$
$$
\times (q^{\frac{1}{2}}s^\pm u^\pm;q)_\infty(q^{\frac{1}{2}}s^\pm u^\mp;q)_\infty(-q^{\frac{1}{2}}s^\pm;q)_\infty(q^{\frac{1}{2}}u^\pm;q)_\infty(-q^{\frac{1}{2}};q)_\infty, \quad (3.163)
$$

$$
\mathbb{III}^{SO(3)-[4]^A_{+-}}_{\mathcal{N},\mathrm{N}+\Psi} = \frac{1}{2}(q)_\infty \oint \frac{ds}{2\pi i s}(-s^\pm;q)_\infty \frac{1}{(q^{\frac{r}{2}}as;q)^4_\infty(q^{\frac{r}{2}}as^{-1};q)^4_\infty(-q^{\frac{r}{2}}a;q)^4_\infty}
$$
$$
\times (q^{\frac{1}{2}}s^\pm u^\pm;q)_\infty(q^{\frac{1}{2}}s^\pm u^\mp;q)_\infty(-q^{\frac{1}{2}}s^\pm;q)_\infty(-q^{\frac{1}{2}}u^\pm;q)_\infty(q^{\frac{1}{2}};q)_\infty, \quad (3.164)
$$

$$
\mathbb{III}^{SO(3)-[4]^A_{--}}_{\mathcal{N},\mathrm{N}+\Psi} = \frac{1}{2}(q)_\infty \oint \frac{ds}{2\pi i s}(-s^\pm;q)_\infty \frac{1}{(q^{\frac{r}{2}}as;q)^4_\infty(q^{\frac{r}{2}}as^{-1};q)^4_\infty(-q^{\frac{r}{2}}a;q)^4_\infty}
$$
$$
\times (q^{\frac{1}{2}}s^\pm u^\pm;q)_\infty(q^{\frac{1}{2}}s^\pm u^\mp;q)_\infty(q^{\frac{1}{2}}s^\pm;q)_\infty(-q^{\frac{1}{2}}u^\pm;q)_\infty(-q^{\frac{1}{2}};q)_\infty. \quad (3.165)
$$

We find that these half-indices agree with those for theory B

$$
\mathbb{III}^{SO(3)-[4]^B_{++}}_{\mathcal{D},\mathrm{D},\mathrm{N},\mathrm{D}} = \frac{1}{(q)_\infty} \sum_{m\in\mathbb{Z}} \frac{1}{(q^{1\pm m}u^\pm;q)_\infty}(q^{\frac{1+r}{2}\pm m}au^\pm;q)^4_\infty(q^{\frac{1+r}{2}}a;q)^4_\infty
$$
$$
\times (-1)^m q^{\frac{3}{2}m^2}u^{3m}\frac{1}{(q^r a^2;q)^{10}_\infty}(q^{\frac{-1+4r}{2}}a^4;q)_\infty, \qquad (3.166)
$$

$$
\mathbb{III}^{SO(3)-[4]^B_{--}}_{\mathcal{D},\mathrm{D},\mathrm{N},\mathrm{D}} = \frac{1}{(q)_\infty} \sum_{m\in\mathbb{Z}} \frac{1}{(-q^{1\pm m}u^\pm;q)_\infty}(q^{\frac{1+r}{2}\pm m}au^\pm;q)^4_\infty(-q^{\frac{1+r}{2}}a;q)^4_\infty
$$
$$
\times (-1)^m q^{\frac{3}{2}m^2}u^{3m}\frac{1}{(q^r a^2;q)^{10}_\infty}(-q^{\frac{-1+4r}{2}}a^4;q)_\infty, \qquad (3.167)
$$

$$
\mathbb{III}^{SO(3)-[4]^B_{+-}}_{\mathcal{D},\mathrm{D},\mathrm{N},\mathrm{D}} = \frac{1}{(q)_\infty} \sum_{m\in\mathbb{Z}} \frac{1}{(-q^{1\pm m}u^\pm;q)_\infty}(q^{\frac{1+r}{2}\pm m}au^\pm;q)^4_\infty(-q^{\frac{1+r}{2}}a;q)^4_\infty
$$
$$
\times q^{\frac{3}{2}m^2}u^{3m}\frac{1}{(q^r a^2;q)^{10}_\infty}(q^{\frac{-1+4r}{2}}a^4;q)_\infty, \qquad (3.168)
$$

$$
\mathbb{III}^{SO(3)-[4]^B_{-+}}_{\mathcal{D},\mathrm{D},\mathrm{N},\mathrm{D}} = \frac{1}{(q)_\infty} \sum_{m\in\mathbb{Z}} \frac{1}{(q^{1\pm m}u^\pm;q)_\infty}(q^{\frac{1+r}{2}\pm m}au^\pm;q)^4_\infty(q^{\frac{1+r}{2}}a;q)^4_\infty
$$
$$
\times q^{\frac{3}{2}m^2}u^{3m}\frac{1}{(q^r a^2;q)^{10}_\infty}(-q^{\frac{-1+4r}{2}}a^4;q)_\infty \; . \qquad (3.169)
$$

3.8.2 $SO(3) - [4]$ with $\mathcal{D}, (D, D, D, D)$

With Dirichlet boundary conditions the theory A half-indices are

$$\mathbb{II}^{SO(3)-[4]^A_{++}}_{\mathcal{D},D} = \frac{1}{(q)_\infty} \sum_{m\in\mathbb{Z}} \frac{1}{(q^{1\pm m}u^\pm;q)_\infty} (q^{1-\frac{r}{2}\pm m}a^{-1}u^\pm;q)^4_\infty (q^{1-\frac{r}{2}}a^{-1};q)^4_\infty (-1)^m q^{\frac{3m^2}{2}} u^{3m},$$

$$(3.170)$$

$$\mathbb{II}^{SO(3)-[4]^A_{-+}}_{\mathcal{D},D} = \frac{1}{(q)_\infty} \sum_{m\in\mathbb{Z}} \frac{1}{(q^{1\pm m}u^\pm;q)_\infty} (q^{1-\frac{r}{2}\pm m}a^{-1}u^\pm;q)^4_\infty (q^{1-\frac{r}{2}}a^{-1};q)^4_\infty q^{\frac{3m^2}{2}} u^{3m},$$

$$(3.171)$$

$$\mathbb{II}^{SO(3)-[4]^A_{+-}}_{\mathcal{D},D} = \frac{1}{(q)_\infty} \sum_{m\in\mathbb{Z}} \frac{1}{(-q^{1\pm m}u^\pm;q)_\infty} (q^{1-\frac{r}{2}\pm m}a^{-1}u^\pm;q)^4_\infty (-q^{1-\frac{r}{2}}a^{-1};q)^4_\infty q^{\frac{3m^2}{2}} u^{3m},$$

$$(3.172)$$

$$\mathbb{II}^{SO(3)-[4]^A_{--}}_{\mathcal{D},D} = \frac{1}{(q)_\infty} \sum_{m\in\mathbb{Z}} \frac{1}{(-q^{1\pm m}u^\pm;q)_\infty} (q^{1-\frac{r}{2}\pm m}a^{-1}u^\pm;q)^4_\infty (-q^{1-\frac{r}{2}}a^{-1};q)^4_\infty (-1)^m q^{\frac{3m^2}{2}} u^{3m}.$$

$$(3.173)$$

We find agreement with the theory B half-indices

$$\mathbb{II}^{SO(3)-[4]^B_{++}}_{\mathcal{N},N,D,N+\Psi} = \frac{1}{2}(q)_\infty \oint \frac{ds}{2\pi i s} (s^\pm;q)_\infty \frac{1}{(q^{\frac{1-r}{2}}a^{-1}s^\pm;q)^4_\infty (q^{\frac{1-r}{2}}a^{-1};q)^4_\infty}$$
$$\times (q^{\frac{1}{2}}s^\pm u^\pm;q)_\infty (q^{\frac{1}{2}}s^\pm u^\mp;q)_\infty (q^{\frac{1}{2}}s^\pm;q)_\infty (q^{\frac{1}{2}}u^\pm;q)_\infty (q^{\frac{1}{2}};q)_\infty$$
$$\times (q^{1-r}a^{-2};q)^{10}_\infty \frac{1}{(q^{\frac{3-4r}{2}}a^{-4};q)_\infty}, \qquad (3.174)$$

$$\mathbb{II}^{SO(3)-[4]^B_{--}}_{\mathcal{N},N,D,N+\Psi} = \frac{1}{2}(q)_\infty \oint \frac{ds}{2\pi i s} (-s^\pm;q)_\infty \frac{1}{(q^{\frac{1-r}{2}}a^{-1}s^\pm;q)^4_\infty (-q^{\frac{1-r}{2}}a^{-1};q)^4_\infty}$$
$$\times (q^{\frac{1}{2}}s^\pm u^\pm;q)_\infty (q^{\frac{1}{2}}s^\pm u^\mp;q)_\infty (q^{\frac{1}{2}}s^\pm;q)_\infty (-q^{\frac{1}{2}}u^\pm;q)_\infty (-q^{\frac{1}{2}};q)_\infty$$
$$\times (q^{1-r}a^{-2};q)^{10}_\infty \frac{1}{(-q^{\frac{3-4r}{2}}a^{-4};q)_\infty}, \qquad (3.175)$$

$$\mathbb{II}^{SO(3)-[4]^B_{+-}}_{\mathcal{N},N,D,N+\Psi} = \frac{1}{2}(q)_\infty \oint \frac{ds}{2\pi i s} (-s^\pm;q)_\infty \frac{1}{(q^{\frac{1-r}{2}}a^{-1}s^\pm;q)^4_\infty (-q^{\frac{1-r}{2}}a^{-1};q)^4_\infty}$$
$$\times (q^{\frac{1}{2}}s^\pm u^\pm;q)_\infty (q^{\frac{1}{2}}s^\pm u^\mp;q)_\infty (-q^{\frac{1}{2}}s^\pm;q)_\infty (-q^{\frac{1}{2}}u^\pm;q)_\infty (q^{\frac{1}{2}};q)_\infty$$
$$\times (q^{1-r}a^{-2};q)^{10}_\infty \frac{1}{(q^{\frac{3-4r}{2}}a^{-4};q)_\infty}, \qquad (3.176)$$

$$\mathbb{II}^{SO(3)-[4]^B_{-+}}_{\mathcal{N},N,D,N+\Psi} = \frac{1}{2}(q)_\infty \oint \frac{ds}{2\pi i s} (s^\pm;q)_\infty \frac{1}{(q^{\frac{1-r}{2}}a^{-1}s^\pm;q)^4_\infty (q^{\frac{1-r}{2}}a^{-1};q)^4_\infty}$$
$$\times (q^{\frac{1}{2}}s^\pm u^\pm;q)_\infty (q^{\frac{1}{2}}s^\pm u^\mp;q)_\infty (-q^{\frac{1}{2}}s^\pm;q)_\infty (q^{\frac{1}{2}}u^\pm;q)_\infty (-q^{\frac{1}{2}};q)_\infty$$
$$\times (q^{1-r}a^{-2};q)^{10}_\infty \frac{1}{(-q^{\frac{3-4r}{2}}a^{-4};q)_\infty} \quad . \qquad (3.177)$$

3.9 Chern-Simons level $k \neq 0$

As shown in [21] we can induce a non-zero Chern-Simons coupling starting from the case of vanishing Chern-Simons level with $N_f + |k|$ fundamental chirals and giving masses to $|k|$ of them. The process is almost identical to that for symplectic gauge groups described in section 2.5. Taking the masses to $\pm\infty$ we integrate out these $|k|$ chirals and are left with N_f flavours and Chern-Simons coupling $k = \pm|k|$. Here the $\pm$ signs are the same for the masses and the sign of k. Note that for orthogonal gauge groups $k \in \mathbb{Z}$. In the dual theory B the corresponding chirals get masses of the opposite sign and consequently theory B gets a Chern-Simons level $-k$. This produces a duality $SO(N_c)_k \leftrightarrow SO(|k| + N_f - N_c + 2)_{-k}$ with N_f fundamental chirals in both theories. The matching with $\mathbb{Z}_2$ fugacities ζ and χ follows in the same way as for $k = 0$.

Again for simplicity to avoid chiral edge modes [43] we focus on the cases with negative masses for Neumann and positive masses for Dirichlet boundary conditions which don't require additional 2d multiplets other than those already present for $k = 0$. This results in the matching of half-indices for

- $(\mathcal{D}, \mathrm{D})$ b.c. in theory A with gauge group $SO(N_c)_k \leftrightarrow (\mathcal{N}, \mathrm{N}, \mathrm{D})$ b.c. in theory B with gauge group $SO(\tilde{N}_c)_{-k}$ with $k > 0$.

- $(\mathcal{N}, \mathrm{N})$ b.c. in theory A with gauge group $SO(N_c)_k \leftrightarrow (\mathcal{D}, \mathrm{D}, \mathrm{N})$ b.c. in theory B with gauge group $SO(\tilde{N}_c)_{-k}$ with $k < 0$.

where we note that again the singlet V is also integrated out for $k \neq 0$.

These half-indices are given by a simple prescription of removing the $\mathbb{III}_V$ contribution which is present in the half-indices for the $k = 0$ cases and noting that $\tilde{N}_c = |k| + N_f - N_c + 2$.

A derivation again follows from the limit taking masses to infinity as explained in section 2.5 following the procedure for the full index [21].

Again as we have not turned on background fluxes, our half-indices are not sensitive to other background Chern-Simons levels, but we note that these again arise from the process of integrating out the chirals. In particular, in addition to the $\pm k \operatorname{Tr}(\mathbf{s}^2)$ or $\mp k \operatorname{Tr}(\tilde{\mathbf{s}}^2)$ terms, we have the following contributions from background Chern-Simons

levels

$$\mathcal{A}_{CS\,\mathcal{N},N}^{SO(N_c)_{k<0}-[N_f]^A} = \mathcal{A}_{\mathcal{N},N}^{SO(N_c)-[N_f+|k|]^A} - \mathcal{A}_{\mathcal{N},N}^{SO(N_c)_{k<0}-[N_f]^A}$$

$$= \frac{1}{2}kN_c\mathbf{a}^2 + kN_c(r-1)\mathbf{ar} + \frac{1}{2}kN(r-1)^2\mathbf{r}^2, \qquad (3.178)$$

$$\mathcal{A}_{CS\,\mathcal{D},D,N}^{SO(\tilde{N}_c)_{k<0}-[N_f]^B} = \mathcal{A}_{\mathcal{D},D,N,D}^{SO(\tilde{N}_c)-[N_f+|k|]^B} - \mathcal{A}_{\mathcal{D},D,N}^{SO(\tilde{N}_c)_{k<0}-[N_f]^B}$$

$$= \frac{k}{2}\,\mathrm{Tr}(\mathbf{x}^2) + \frac{1}{2}\left(k(N_c+N_f+3)+N_f^2\right)\mathbf{a}^2$$

$$+ (k((r-1)N_c+r(N_f+3))-N_f(1-N_c+N_f(1-r)))\mathbf{ar}$$

$$+ \frac{1}{4}\left(k^2+k\left(2N\left(r^2-2r+2\right)+2N_f\left(r^2-1\right)+3(2r^2-1)\right)\right.$$

$$\left.+2(N+N_f(r-1)-1)^2\right)\mathbf{r}^2, \qquad (3.179)$$

$$\mathcal{A}_{CS\,\mathcal{D},D}^{SO(N_c)_{k>0}-[N_f]^A} = \mathcal{A}_{\mathcal{D},D}^{SO(N_c)-[N_f+|k|]^A} - \mathcal{A}_{\mathcal{D},D}^{SO(N_c)_{k>0}-[N_f]^A}$$

$$= \frac{1}{2}kN_c\mathbf{a}^2 + kN_c(r-1)\mathbf{ar} + \frac{1}{2}kN(r-1)^2\mathbf{r}^2, \qquad (3.180)$$

$$\mathcal{A}_{CS\,\mathcal{N},N,D}^{SO(\tilde{N}_c)_{k>0}-[N_f]^B} = \mathcal{A}_{\mathcal{N},N,D,N}^{SO(\tilde{N}_c)-[N_f+|k|]^B} - \mathcal{A}_{\mathcal{N},N,D}^{SO(\tilde{N}_c)_{k>0}-[N_f]^B}$$

$$= \frac{k}{2}\,\mathrm{Tr}(\mathbf{x}^2) + \frac{1}{2}\left(k(N_c+N_f+3)-N_f^2\right)\mathbf{a}^2$$

$$+ (k((r-1)N_c+r(N_f+3))+N_f(1-N_c+N_f(1-r)))\mathbf{ar}$$

$$+ \frac{1}{4}\left(-k^2+k\left(2N\left(r^2-2r+2\right)+2N_f\left(r^2-1\right)+3(2r^2-1)\right)\right.$$

$$\left.-2(N+N_f(r-1)-1)^2\right)\mathbf{r}^2. \qquad (3.181)$$

3.9.1 $SO(2)_{\pm k}-[2-k]$

Below we have $k \in \{1,2\}$.

For theory A we have

$$\mathbb{III}_{\mathcal{N},N+\Psi}^{SO(2)_{-k}-[2-k]_{++}^A} = (q)_\infty \oint \frac{ds}{2\pi i s}\frac{1}{(q^{\frac{r}{2}}as^\pm;q)_\infty^{2-k}}I_{2d}, \qquad (3.182)$$

$$I_{2d} = (q^{\frac{1}{2}}su;q)_\infty(q^{\frac{1}{2}}su^{-1};q)_\infty(q^{\frac{1}{2}}s^{-1}u;q)_\infty(q^{\frac{1}{2}}s^{-1}u^{-1};q)_\infty, \qquad (3.183)$$

$$\mathbb{III}_{\mathcal{N},N+\Psi}^{SO(2)_{-k}-[2-k]_{+-}^A} = (-q;q)_\infty \frac{1}{(\pm q^{\frac{r}{2}}a;q)_\infty^{2-k}}I_{2d}, \qquad (3.184)$$

$$I_{2d} = (q^{\frac{1}{2}};q)_\infty(-q^{\frac{1}{2}};q)_\infty(-q^{\frac{1}{2}};q)_\infty(q^{\frac{1}{2}};q)_\infty, \qquad (3.185)$$

$$\mathbb{III}_{\mathcal{N},N+\Psi}^{SO(2)_{-k}-[2-k]_{-+}^A} = (q)_\infty \oint \frac{ds}{2\pi i s}\frac{1}{(q^{\frac{r}{2}}as^\pm;q)_\infty^{2-k}}I_{2d}, \qquad (3.186)$$

$$I_{2d} = (q^{\frac{1}{2}}s;q)_\infty(-q^{\frac{1}{2}}s;q)_\infty(q^{\frac{1}{2}}s^{-1};q)_\infty(-q^{\frac{1}{2}}s^{-1};q)_\infty, \qquad (3.187)$$

$$\mathbb{III}_{\mathcal{N},N+\Psi}^{SO(2)_{-k}-[2-k]_{--}^A} = (-q;q)_\infty \frac{1}{(\pm q^{\frac{r}{2}}a;q)_\infty^{2-k}}I_{2d}, \qquad (3.188)$$

$$I_{2d} = (q^{\frac{1}{2}}u;q)_\infty(q^{\frac{1}{2}}u^{-1};q)_\infty(-q^{\frac{1}{2}}u;q)_\infty(-q^{\frac{1}{2}}u^{-1};q)_\infty, \qquad (3.189)$$

and

$$\mathbb{III}_{\mathcal{D},\mathrm{D}}^{SO(2)_k-[2-k]_{++}^A} = \frac{1}{(q)_\infty} \sum_{m\in\mathbb{Z}} q^{m^2} u^{2m} (u^\pm a q^{1-r/2\pm m}; q)_\infty^{2-k}, \tag{3.190}$$

$$\mathbb{III}_{\mathcal{D},\mathrm{D}}^{SO(2)_k-[2-k]_{+-}^A} = \frac{1}{(-q;q)_\infty} (\pm a q^{1-r/2}; q)_\infty^{2-k}, \tag{3.191}$$

$$\mathbb{III}_{\mathcal{D},\mathrm{D}}^{SO(2)_k-[2-k]_{-+}^A} = \frac{1}{(q)_\infty} \sum_{m\in\mathbb{Z}} (-1)^m q^{m^2} u^{2m} (u^\pm a q^{1-r/2\pm m}; q)_\infty^{2-k}, \tag{3.192}$$

$$\mathbb{III}_{\mathcal{D},\mathrm{D}}^{SO(2)_k-[2-k]_{--}^A} = \frac{1}{(-q;q)_\infty} (\pm a q^{1-r/2}; q)_\infty^{2-k}, \tag{3.193}$$

while for theory B we have

$$\mathbb{III}_{\mathcal{D},\mathrm{D},\mathrm{N},\mathrm{D}}^{SO(2)_{-k}-[2-k]_{++}^B} = \frac{1}{(q)_\infty} \sum_{m\in\mathbb{Z}} q^{m^2} u^{2m} (u^\pm a q^{(1+r)/2\pm m}; q)_\infty^{2-k} \frac{1}{(q^r a^2; q)_\infty^{(2-k)(3-k)/2}}, \tag{3.194}$$

$$\mathbb{III}_{\mathcal{D},\mathrm{D},\mathrm{N},\mathrm{D}}^{SO(2)_{-k}-[2-k]_{+-}^B} = \frac{1}{(-q;q)_\infty} (\pm a q^{(1+r)/2}; q)_\infty^{2-k} \frac{1}{(q^r a^2; q)_\infty^{(2-k)(3-k)/2}}, \tag{3.195}$$

$$\mathbb{III}_{\mathcal{D},\mathrm{D},\mathrm{N},\mathrm{D}}^{SO(2)_{-k}-[2-k]_{-+}^B} = \frac{1}{(q)_\infty} \sum_{m\in\mathbb{Z}} (-1)^m q^{m^2} u^{2m} (u^\pm a q^{(1+r)/2\pm m}; q)_\infty^{2-k} \frac{1}{(q^r a^2; q)_\infty^{(2-k)(3-k)/2}}, \tag{3.196}$$

$$\mathbb{III}_{\mathcal{D},\mathrm{D},\mathrm{N},\mathrm{D}}^{SO(2)_{-k}-[2-k]_{--}^B} = \frac{1}{(-q;q)_\infty} (\pm a q^{(1+r)/2}; q)_\infty^{2-k} \frac{1}{(q^r a^2; q)_\infty^{(2-k)(3-k)/2}}, \tag{3.197}$$

and

$$\mathbb{III}_{\mathcal{N},\mathrm{N},\mathrm{D},\mathrm{N}+\Psi}^{SO(2)_k-[2-k]_{++}^B} = (q)_\infty \oint \frac{ds}{2\pi i s} \frac{1}{(q^{\frac{1-r}{2}} a^{-1} s^\pm; q)_\infty^{2-k}} (q^{1-r} a^{-2}; q)_\infty^{(2-k)(3-k)/2} I_{2d}, \tag{3.198}$$

$$I_{2d} = (q^{\frac{1}{2}} s u; q)_\infty (q^{\frac{1}{2}} s^{-1} u; q)_\infty (q^{\frac{1}{2}} s u^{-1}; q)_\infty (q^{\frac{1}{2}} s^{-1} u^{-1}; q)_\infty, \tag{3.199}$$

$$\mathbb{III}_{\mathcal{N},\mathrm{N},\mathrm{D},\mathrm{N}+\Psi}^{SO(2)_k-[2-k]_{+-}^B} = (-q;q)_\infty \frac{1}{(\pm q^{\frac{1-r}{2}} a^{-1}; q)_\infty^{2-k}} (q^{1-r} a^{-2}; q)_\infty^{(2-k)(3-k)/2} I_{2d}, \tag{3.200}$$

$$I_{2d} = (q^{\frac{1}{2}}; q)_\infty (-q^{\frac{1}{2}}; q)_\infty (-q^{\frac{1}{2}}; q)_\infty (q^{\frac{1}{2}}; q)_\infty, \tag{3.201}$$

$$\mathbb{III}_{\mathcal{N},\mathrm{N},\mathrm{D},\mathrm{N}+\Psi}^{SO(2)_k-[2-k]_{-+}^B} = (q)_\infty \oint \frac{ds}{2\pi i s} \frac{1}{(q^{\frac{1-r}{2}} a^{-1} s^\pm; q)_\infty^{2-k}} (q^{1-r} a^{-2}; q)_\infty^{(2-k)(3-k)/2} I_{2d}, \tag{3.202}$$

$$I_{2d} = (q^{\frac{1}{2}} s; q)_\infty (q^{\frac{1}{2}} s^{-1}; q)_\infty (-q^{\frac{1}{2}} s; q)_\infty (-q^{\frac{1}{2}} s^{-1}; q)_\infty, \tag{3.203}$$

$$\mathbb{III}_{\mathcal{N},\mathrm{N},\mathrm{D},\mathrm{N}+\Psi}^{SO(2)_k-[2-k]_{--}^B} = (-q;q)_\infty \frac{1}{(\pm q^{\frac{1-r}{2}} a^{-1}; q)_\infty^{2-k}} (q^{1-r} a^{-2}; q)_\infty^{(2-k)(3-k)/2} I_{2d}, \tag{3.204}$$

$$I_{2d} = (q^{\frac{1}{2}} u; q)_\infty (-q^{\frac{1}{2}} u; q)_\infty (q^{\frac{1}{2}} u^{-1}; q)_\infty (-q^{\frac{1}{2}} u^{-1}; q)_\infty. \tag{3.205}$$

3.10   Checks on matching of indices

We have performed checks on the claimed matching half-indices using Mathematica to expand the q-series to at least order q^5 after first defining a fugacity $y = q^{r-1/2}$

to replace all terms with dependence on the parameter r in the R-charge assignment. Details of the expansion to this order are listed in appendix B.2.

Some of the identities can also be checked analytically. The cases where the indices do not include any integration or summation over monopole charges are particularly straightforward. This will occur in full-indices and half-indices with gauge group $SO(1)$ as well as in the case of gauge group $SO(2)$ with $\chi = -1$. The full-indices and half-indices in both theories will satisfy this property for precisely the following cases

$$SO(1) - [1]^A_{+-} \leftrightarrow SO(2) - [1]^B_{+-}, \tag{3.206}$$

$$SO(1) - [1]^A_{-+} \leftrightarrow SO(2) - [1]^B_{--}, \tag{3.207}$$

$$SO(2) - [2]^A_{+-} \leftrightarrow SO(2) - [2]^B_{+-}, \tag{3.208}$$

which we have presented and the cases

$$SO(2) - [1]^A_{+-} \leftrightarrow SO(2) - [1]^B_{+-}, \tag{3.209}$$

$$SO(2) - [1]^A_{--} \leftrightarrow SO(2) - [1]^B_{-+}, \tag{3.210}$$

which we have not presented. The variations with Chern-Simons levels can also be easily checked directly or by noting that they arise when taking the appropriate limits of these indices. For the case of $SO(2) - [2]_{+-}$ this requires first checking the indices including the flavour fugacities x_1 and x_2 but this is straightforward.

The analytic checks indicated above rely only on elementary Pochhammer identities

$$(\pm x; q)_\infty = (x^2; q^2)_\infty, \tag{3.211}$$

$$(x; q)_\infty = (x; q^2)_\infty (qx; q^2)_\infty . \tag{3.212}$$

When checking the cases with $SO(2)$ and $\chi = -1$ it is useful to note that setting $x = -q$ in the first of these identities and then using the second with $x = q$ gives

$$(-q; q)_\infty = \frac{(q^2; q^2)_\infty}{(q)_\infty} = \frac{1}{(q; q^2)_\infty}. \tag{3.213}$$

We give just one explicit example of $SO(2) - [2]_{+-}$ including the flavour fugacities.

$$
\mathbb{III}^{SO(2)-[2]^A_{+-}}_{\mathcal{N},\mathrm{N}+\Psi} = (-q; q)_\infty \frac{1}{\prod_{\alpha=1}^{2}(\pm q^{r/2} a x_\alpha; q)_\infty} (\pm q^{1/2}; q)_\infty^2
$$

$$
= (-q; q)_\infty \frac{(q; q^2)_\infty^2}{\prod_{\alpha=1}^{2}(q^r a^2 x_\alpha^2; q^2)_\infty} = \frac{1}{(-q; q)_\infty} \frac{1}{\prod_{\alpha=1}^{2}(q^r a^2 x_\alpha^2; q^2)_\infty}, \tag{3.214}
$$

$$
\mathbb{III}^{SO(2)-[2]^B_{+-}}_{\mathcal{D},\mathrm{D},\mathrm{N},\mathrm{D}} = \frac{1}{(-q; q)_\infty} \left(\prod_{\alpha=1}^{2} (\pm q^{(1+r)/2} a x_\alpha; q)_\infty \right) \frac{1}{\prod_{\alpha \le \beta}(\pm q^r a^2 x_\alpha x_\beta; q)_\infty} (\pm q^r a^2; q)_\infty
$$

$$
= \frac{1}{(-q; q)_\infty} \prod_{\alpha=1}^{2} \frac{(q^{1+r} a^2 x_\alpha^2; q^2)_\infty}{(q^r a^2 x_\alpha^2; q)_\infty} = \frac{1}{(-q; q)_\infty} \frac{1}{\prod_{\alpha=1}^{2}(q^r a^2 x_\alpha^2; q^2)_\infty} \tag{3.215}
$$

where for the theory B half-index we used $x_1 x_2 = 1$ for the $SU(2)$ flavour symmetry to see that the contribution for V cancels part of the contribution from M.

4 Other orthogonal gauge theories

For the Lie algebra $\mathfrak{so}(N_c)$, there exist other possibilities of the gauge groups, $O(N_c)_\pm$, $Spin(N_c)$ and $Pin(N_c)_\pm$, which lead to distinct gauge theories [24, 65]. Another possibility is $SO(2N)/\mathbb{Z}_2$ but we do not consider this case.

While the $SO(N_c)$ gauge theory has the monopole operators carrying weights as well as roots of the dual magnetic group, the $Spin(N_c)$ gauge theory only contains the monopole operators carrying roots. In the $Spin(N_c)$ gauge theory the minimal monopole operator which turns on one unit of magnetic flux and parametrises the Coulomb branch looks semi-classically $V_{Spin} \approx \exp(\frac{2\sigma_1}{\hat{g}_3^2} + 2i\gamma_1)$ where σ_i and γ_i are the adjoint scalar fields in the vector multiplet and the dual photons. It can be also described as

$$V_{Spin} = \begin{cases} V_1^2 V_2^2 \cdots V_{N-2}^2 V_{N-1} V_N & \text{for } \epsilon = 0 \\ V_1^2 V_2^2 \cdots V_{N-2}^2 V_{N-1}^2 V_N & \text{for } \epsilon = 1 \end{cases} \tag{4.1}$$

in terms of the semi-classical Coulomb branch coordinates

$$V_i \approx \exp\left[\frac{\sigma_i - \sigma_{i+1}}{\hat{g}_3^2} + i(\gamma_i - \gamma_{i+1})\right], \qquad \text{for } i = 1, \cdots, N-1, \tag{4.2}$$

$$V_N \approx \begin{cases} \exp\left[\frac{\sigma_{N-1} - \sigma_N}{\hat{g}_3^2} + i(\gamma_{N-1} - \gamma_N)\right] & \text{for } \epsilon = 0 \\ \exp\left[\frac{2\sigma_N}{\hat{g}_3^2} + i(2\gamma_N)\right] & \text{for } \epsilon = 1 \end{cases} \tag{4.3}$$

where $\hat{g}_3^2 = g_3^2/4\pi$ and g_3 is the 3d gauge coupling constant. In other words, it is the square of the minimal monopole operator $V \approx \exp(\frac{\sigma_1}{\hat{g}_3^2} + i\gamma_1)$ in the $SO(N_c)$ gauge theory

$$V_{Spin} = V^2. \tag{4.4}$$

One can also consider the gauge theories with disconnected orthogonal gauge groups $O(N_c)$ and $Pin(N_c)$ which include the reflections along the line bundle or determinant bundle. While the baryon $B = Q^{N_c}$ is an independent operator that is charged under $\mathbb{Z}_2^\mathcal{C}$ for $SO(N_c)$ and $Spin(N_c)$, it is not for $O(N_c)$ and $Pin(N_c)$ since $\mathbb{Z}_2^\mathcal{C}$ is gauged.

We can construct $O(N_c)_+$ theories from $SO(N_c)$ theories by gauging the charge conjugation symmetry $\mathbb{Z}_2^\mathcal{C}$ of the $SO(N_c)$. Alternatively, gauging $\mathbb{Z}_2^{\mathcal{MC}}$, the diagonal

subgroup of $\mathbb{Z}_2^{\mathcal{M}} \times \mathbb{Z}_2^{\mathcal{C}}$, leads to $O(N_c)_-$ theories [24, 65]. At the level of the Lagrangian description, they are distinguished by a discrete theta angle which is proportional to $w_1 \wedge w_2$ where $w_i \in H^i(X, \mathbb{Z})$, $i = 1, 2$ are $\mathbb{Z}_2$-valued Stiefel-Whitney characteristic classes of $O(N_c)$ bundle on a 3-manifold X. Both w_1 and w_2, i.e. theta angle are non-zero only for $O(N_c)$ but not for $SO(N_c)$, $Spin(N_c)$ or $Pin(N_c)_\pm$ [91, 92].

The $O(N_c)_+$ gauge theories and their dualities were discussed in [20, 21, 22, 23, 87]. In particular, the matching of the indices for the $O(N_c)_+$ gauge theories was tested in [22, 87]. Similarly to the $SO(N_c)$ gauge theory, the $O(N_c)_+$ gauge theory has the minimal monopole operator $V = \exp(\frac{\sigma_1}{\hat{g}_3^2} + i\gamma_1)$ which is charge-conjugation-even and gauge invariant.

On the other hand, in the $O(N_c)_-$ gauge theory the monopole operator V is charge-conjugation-odd and not gauge invariant. Instead, the product of V with the baryon B as well as the monopole operator V_{Spin} and the baryon-monopole operator $\beta = Q^{N_c-2}(v_+ - v_-)$ are gauge invariant where $v_+ \approx \exp(\frac{\sigma}{\hat{g}_3^2} + i\gamma)$ and $v_- \approx \exp(-\frac{\sigma_1}{\hat{g}_3^2} - i\gamma_1)$ are the $SO(2)$ monopoles which are even and odd under $\mathbb{Z}_2^{\mathcal{C}}$.

Furthermore, one can obtain the $Pin(N_c)$ gauge theories by gauging the global symmetry $\mathbb{Z}_2^{\mathcal{M}} \times \mathbb{Z}_2^{\mathcal{C}}$, which can also be viewed as gauging the $\mathbb{Z}_2^{\mathcal{C}}$ of $Spin(N_c)$ or the $\mathbb{Z}_2^{\mathcal{M}}$ of $O(N_c)_\pm$ [24, 65]. More precisely this gives $Pin(N_c)_+$ but a modification of the gauging process produces $Pin(N_c)_-$ [65].

The 4d duality between $Spin(N_c)$ and $O(\tilde{N}_c)_-$ gauge theories gives rise to the 3d IR dualities [24]:

- Theory A: $Spin(N_c)$ gauge theory with N_f chiral multiplets Q in the vector representation.

- Theory B: $O(\tilde{N}_c = N_f - N_c + 2)_-$ gauge theory with N_f chiral multiplets q in the vector representation, $N_f(N_f + 1)/2$ neutral chiral multiplets M in the rank-2 symmetric representation of $SU(N_f)$ and a chiral multiplet V which has a superpotential.

and similarly with $O(N_c)_-$ for theory A and $Spin(\tilde{N}_c)$ for theory B.

For the $Pin(N_c)_+$ gauge theories, the duality can be derived by gauging $\mathbb{Z}_2^{\mathcal{M}}$ of the $O(N_c)_+$ duality [20, 21, 22, 23, 87]. It gives rise to the duality between the $Pin(N_c)_+$ gauge theory and $Pin(\tilde{N}_c = N_f - N_c + 2)_+$ gauge theory.

The identities (3.17) imply the identities of the $O(N_c)_\pm$, $Spin(N_c)$ and $Pin(N_c)_+$

full-indices [24] as well as $Pin(N_c)_-$ full-indices:

$$I^{O(N_c)+-[N_f]^A_{\zeta,\chi'}} = \begin{cases} I^{O(\tilde{N}_c)+-[N_f]^B_{\zeta,\chi'}}, & (\zeta,\chi') \neq (-,-) \\ -I^{O(\tilde{N}_c)+-[N_f]^B_{-,-}}, & (\zeta,\chi') = (-,-) \end{cases}, \tag{4.5}$$

$$I^{O(N_c)--[N_f]^{A/B}_{\zeta,\chi'}} = \begin{cases} I^{Spin(\tilde{N}_c)-[N_f]^{B/A}_{\chi',\zeta}}, & (\zeta,\chi') \neq (-,-) \\ -I^{Spin(\tilde{N}_c)-[N_f]^{B/A}_{\chi',\zeta}}, & (\zeta,\chi') = (-,-) \end{cases}, \tag{4.6}$$

$$I^{Pin(N_c)\pm-[N_f]^A_{\zeta',\chi'}} = I^{Pin(\tilde{N}_c)\pm-[N_f]^B_{\pm\zeta'\chi',\chi'}}, \tag{4.7}$$

where [9] the indices can be constructed from the $SO(N_c)_{\zeta\chi}$ indices by summing over ζ to gauge $\mathbb{Z}_2^{\mathcal{M}}$, χ to gauge $\mathbb{Z}_2^{\mathcal{C}}$, or χ together with a change in sign of ζ to gauge $\mathbb{Z}_2^{\mathcal{MC}}$. The result is

$$I^{O(N_c)\pm-[N_f]_{\zeta,\chi'}} = \frac{1}{2}\left(I^{SO(N_c)-[N_f]_{\zeta,+}} + \chi' I^{SO(N_c)-[N_f]_{\pm\zeta,-}}\right), \tag{4.8}$$

$$I^{Spin(N_c)-[N_f]_{\zeta',\chi}} = \frac{1}{2}\left(I^{SO(N_c)-[N_f]_{+,\chi}} + \zeta' I^{SO(N_c)-[N_f]_{-,\chi}}\right), \tag{4.9}$$

$$I^{Pin(N_c)\pm-[N_f]_{\zeta',\chi'}} = \frac{1}{2}\left(I^{Spin(N_c)-[N_f]_{\zeta',+}} + \chi' I^{Spin(N_c)-[N_f]_{\pm\zeta',-}}\right), \tag{4.10}$$

where $\chi' = \pm 1$ corresponds to a projection onto even or odd states under, $\mathbb{Z}_2^{\mathcal{C}}$ for $O(N_c)_+$ and $Pin(N_c)_+$ or $\mathbb{Z}_2^{\mathcal{MC}}$ for $O(N_c)_-$, and $\zeta' = \pm 1$ determines a projection onto even or odd states under $\mathbb{Z}_2^{\mathcal{M}}$ for $Spin(N_c)$ and $Pin(N_c)_+$. Such an interpretation of ζ' and χ' is less straightforward for $Pin(N_c)_-$. These parameters can be viewed as discrete theta angles.

Similarly to the equalities (4.5)-(4.7) of the full-indices, we find the half-index version of the identities:

$$\mathbb{III}^{O(N_c)+-[N_f]^A_{\zeta,\chi'}}_{\mathcal{B}} = \begin{cases} \mathbb{III}^{O(\tilde{N}_c)+-[N_f]^B_{\zeta,\chi'}}_{\tilde{\mathcal{B}}}, & (\zeta,\chi') \neq (-,-) \\ -\mathbb{III}^{O(\tilde{N}_c)+-[N_f]^B_{-,-}}_{\tilde{\mathcal{B}}}, & (\zeta,\chi') = (-,-) \end{cases}, \tag{4.11}$$

$$\mathbb{III}^{O(N_c)--[N_f]^A_{\zeta,\chi'}}_{\mathcal{B}} = \begin{cases} \mathbb{III}^{Spin(\tilde{N}_c)-[N_f]^B_{\chi',\zeta}}_{\tilde{\mathcal{B}}}, & (\zeta,\chi') \neq (-,-) \\ -\mathbb{III}^{Spin(\tilde{N}_c)-[N_f]^B_{\chi',\zeta}}_{\tilde{\mathcal{B}}}, & (\zeta,\chi') = (-,-) \end{cases}, \tag{4.12}$$

$$\mathbb{III}^{Spin(N_c)-[N_f]^A_{\zeta,\chi'}}_{\mathcal{B}} = \begin{cases} \mathbb{III}^{O(\tilde{N}_c)--[N_f]^B_{\chi',\zeta}}_{\tilde{\mathcal{B}}}, & (\zeta,\chi') \neq (-,-) \\ -\mathbb{III}^{O(\tilde{N}_c)--[N_f]^B_{\chi',\zeta}}_{\tilde{\mathcal{B}}}, & (\zeta,\chi') = (-,-) \end{cases}, \tag{4.13}$$

$$\mathbb{III}^{Pin(N_c)\pm-[N_f]^A_{\zeta',\chi'}}_{\mathcal{B}} = \mathbb{III}^{Pin(\tilde{N}_c)\pm-[N_f]^B_{\pm\zeta'\chi',\chi'}}_{\tilde{\mathcal{B}}}, \tag{4.14}$$

where we have boundary conditions $(\mathcal{B};\tilde{\mathcal{B}}) = (\mathcal{N}, N + \Psi; \mathcal{D}, D, N, D)$ or $(\mathcal{B};\tilde{\mathcal{B}}) = (\mathcal{D}, D; \mathcal{N}, N, D, N + \Psi)$. In both theory A and theory B these half-indices are defined, with boundary conditions $\mathcal{B}$, by gauging the same symmetries as described for

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[9]See [24] for more general expressions with dynamical and background CS couplings.

the full-indices

$$\mathbb{II}_{\mathcal{B}}^{O(N_c)_\pm - [N_f]_{\zeta, \chi'}} = \frac{1}{2} \left(\mathbb{II}_{\mathcal{B}}^{SO(N_c) - [N_f]_{\zeta, +}} + \chi' \mathbb{II}_{\mathcal{B}}^{SO(N_c) - [N_f]_{\pm \zeta, -}} \right), \tag{4.15}$$

$$\mathbb{II}_{\mathcal{B}}^{Spin(N_c) - [N_f]_{\zeta', \chi}} = \frac{1}{2} \left(\mathbb{II}_{\mathcal{B}}^{SO(N_c) - [N_f]_{+, \chi}} + \zeta' \mathbb{II}_{\mathcal{B}}^{SO(N_c) - [N_f]_{-, \chi}} \right), \tag{4.16}$$

$$\mathbb{II}_{\mathcal{B}}^{Pin(N_c)_\pm - [N_f]_{\zeta', \chi'}} = \frac{1}{2} \left(\mathbb{II}_{\mathcal{B}}^{Spin(N_c) - [N_f]_{\zeta', +}} + \chi' \mathbb{II}_{\mathcal{B}}^{Spin(N_c) - [N_f]_{\pm \zeta', -}} \right). \tag{4.17}$$

The equalities (4.11)-(4.14) indicate the dualities of the basic $\mathcal{N} = (0, 2)$ half-BPS boundary conditions for 3d $\mathcal{N} = 2$ theories with orthogonal gauge groups; $O(N_c)_\pm$, $Spin(N_c)$ and $Pin(N_c)_\pm$. These can easily be extended to include Chern-Simons levels for the gauge groups by integrating out some of the fundamental chirals as explained for the $SO(N_c)$ half-indices in section 3.9.

Acknowledgements

A   General results for 2d anomalies and half-indices

A.1   Notations

We use the standard notation by defining q-shifted factorial

$$(a; q)_0 := 1, \qquad (a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k), \qquad (q)_n := \prod_{k=1}^{n} (1 - q^k), \qquad n \geq 1,$$

$$(a; q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k), \qquad (a^\pm; q)_\infty := (a; q)_\infty (a^{-1}; q)_\infty,$$

$$(a^\pm b^\mp; q)_\infty := (ab^{-1}; q)_\infty (a^{-1}b; q)_\infty, \qquad (q)_\infty := \prod_{k=1}^{\infty} (1 - q^k) \tag{A.1}$$

where a and q are complex numbers with $|q| < 1$.

A.2   General contributions to 2d anomalies

Here we summarise the general results presented in [43] to calculate the 2d boundary anomaly polynomial.

The conventions are chosen so that a left-handed 2d complex fermion with charge c under a $U(1)$ symmetry with field strength $\mathbf{f}$ contributes $+(c\mathbf{f})^2$ to the anomaly

polynomial. This is therefore the contribution from Fermi multiplet charged under the $U(1)$. A right-handed fermion, and hence a chiral multiplet, contribute with the opposite sign, i.e. $-(c\mathbf{f})^2$.

A key result is that 3d chiral multiplets with Dirichlet (Neumann) boundary conditions will project out the right-handed (left-handed) fermion, leaving a boundary left-handed (right-handed) fermion, but that the anomaly contribution will be half that of a 2d fermion. Therefore a 3d chiral multiplet with charge c under the $U(1)$ symmetry will contribute $\frac{1}{2}(c\mathbf{f})^2$ for Dirichlet and $-\frac{1}{2}(c\mathbf{f})^2$ for Neumann boundary conditions. One slight subtlety is that for the $U(1)$ R-symmetry the fermions have a shifted R-charge so if a chiral multiplet has R-charge ρ there will be anomaly contributions $\pm\frac{1}{2}((\rho-1)\mathbf{r})^2$.

If we have several $U(1)$ factors we add up the contributions of the form $c\mathbf{f}$ before squaring.

These results generalise to irreducible representations $\mathbf{R}$ of non-Abelian simple compact groups G by setting $c = 1$ and replacing $\mathbf{f}^2$ with $\mathrm{Tr}_{\mathbf{R}}(\mathbf{f}^2)$. The trace in representation $\mathbf{R}$ is given by $\mathrm{Tr}_{\mathbf{R}}(\mathbf{f}^2) = T_{\mathbf{R}}\,\mathrm{Tr}(\mathbf{f}^2) = \frac{T_{\mathbf{R}}}{2h}\,\mathrm{Tr}_{adjoint}(\mathbf{f}^2)$ where $T_{\mathbf{R}}$ is the quadratic index of $\mathbf{R}$ (proportional to the quadratic Casimir multiplied by the dimension of $\mathbf{R}$) normalised so that $T_{adjoint} = 2h$ where h is the dual Coxeter number. We have the following values for $SU(N)$

$$T_{Fundamental} = 1, \tag{A.2}$$

$$T_{Adjoint} = 2h = 2N, \tag{A.3}$$

$$T_{Rank\ 2\ symmetric} = N + 2, \tag{A.4}$$

$$T_{Rank\ 2\ antisymmetric} = N - 2\ . \tag{A.5}$$

For $SO(N)$ we have

$$T_{Fundamental} = 2, \tag{A.6}$$

$$T_{Adjoint} = 2h = 2(N - 2) \tag{A.7}$$

while for $USp(2N)$

$$T_{Fundamental} = 1, \tag{A.8}$$

$$T_{Adjoint} = 2h = 2(N + 1)\ . \tag{A.9}$$

More precisely the results for $SO(N)$ apply to $N \geq 4$ but we choose the same normalisation for $SO(2)$ and $SO(3)$.

In addition, a 3d Chern-Simons coupling at level k gives a contribution $k\mathbf{f}^2$.

A.3   General expressions for half-indices

Here we summarise the general results presented in [39, 41, 42, 43] (also see [93, 94, 95] for 2d indices) to calculate the half-indices. For a 2d chiral multiplet with R-charge ρ and other $U(1)$ charges giving fugacity x the contribution to the half-index is

$$C(q^{\rho/2}x; q) = \frac{1}{(q^{\rho/2}x; q)_\infty (q^{1-\rho/2}x^{-1}; q)_\infty} \tag{A.10}$$

while for a Fermi multiplet we have

$$F(q^{(1+\rho)/2}x; q) = (q^{(1+\rho)/2}x; q)_\infty (q^{(1-\rho)/2}x^{-1}; q)_\infty. \tag{A.11}$$

A 3d chiral with these charges will give contributions

$$\mathbb{II}_{\mathrm{N}} = \frac{1}{(q^{\rho/2}x; q)_\infty}, \tag{A.12}$$

$$\mathbb{II}_{\mathrm{D}} = (q^{1-\rho/2}x^{-1}; q)_\infty \tag{A.13}$$

for Neumann or Dirichlet boundary conditions.

   If we have a representation $\mathbf{R}$ of a non-Abelian simple group G we take the product of similar expressions over the weights of $\mathbf{R}$. So for a 3d chiral with Neumann boundary conditions we have

$$\mathbb{II}_{\mathrm{N}} = \prod_{\alpha \in weights(\mathbf{R})} \frac{1}{(q^{\rho/2}xs^\alpha; q)_\infty}. \tag{A.14}$$

For example, the s^α could be labelled $s_1, s_2, \ldots, s_N$ with the constraint $s_1 s_2 \cdots s_N = 1$ for the fundamental representation of $SU(N)$ – for $U(N)$ we simply drop the constraint $s_1 s_2 \cdots s_N = 1$. These fugacities s_i correspond to the fundamental weights μ_i, with the constraint $\sum_i \mu_i = 0$ for $SU(N)$.

   One point to note is that the weights of the adjoint representation include zero weights and the vector multiplet fermions have opposite chirality to those from chiral multiplets for the same boundary condition. So, for $U(N)$, the contribution of the vector multiplet with Neumann boundary conditions is given by the gauginos (with R-charge 1) in the surviving 2d vector multiplet

$$\mathbb{II}_{\mathcal{N}}^{\mathrm{VM}} = \prod_{\alpha \in weights(\mathbf{R})} (qs^\alpha; q)_\infty = \prod_{i,j}(qs_i s_j^{-1}; q)_\infty = (q)_\infty^N \prod_{i \neq j}(qs_i s_j^{-1}; q)_\infty = (q)_\infty^{rank(G)} \prod_{\alpha \in roots(G)} (qs^\alpha; q)_\infty. \tag{A.15}$$

In the half-index the projection onto gauge invariant state is imposed by contour integration over the gauge fugacities s_i and this includes a Vandermonde determinant which introduces factors $(1 - s^\alpha)$ in the product over roots, turning

$$\prod_{\alpha \in roots(G)} (qs^\alpha; q)_\infty \to \prod_{\alpha \in roots(G)} (1 - s^\alpha)(qs^\alpha; q)_\infty = \prod_{\alpha \in roots(G)} (s^\alpha; q)_\infty. \tag{A.16}$$

Instead, for Dirichlet boundary conditions the contribution is from the surviving 2d chiral multiplet (with R-charge 0) giving contribution

$$\mathbb{III}_{\mathcal{D}}^{\text{VM}} = \prod_{\alpha \in weights(\mathbf{R})} \frac{1}{(qs^\alpha; q)_\infty} = \prod_{i,j} \frac{1}{(qs_i s_j^{-1}; q)_\infty} = \frac{1}{(q)_\infty^N} \prod_{i \neq j} \frac{1}{(qs_i s_j^{-1}; q)_\infty} = \frac{1}{(q)_\infty^{rank(G)}} \prod_{\alpha \in roots(G)} \frac{1}{(qs^\alpha; q)_\infty} \,.$$
(A.17)

and now the gauge symmetry is broken so G is a global symmetry, and there is no projection onto gauge invariant states.

However, as explained in [43] the complete non-perturbative half-index with Dirichlet boundary conditions for the vector multiplet is given after including a sum over monopole charges and including factors for the effective Chern-Simons levels. Specifically, there is a sum over all magnetic charges $m \in cochar(G)$. Each term in this sum is given by the same half-index but with the fugacities s_i shifted by a factor q^{m_i} reflecting the spin induced by magnetic charge m_i and a factor for the monopole contribution. This monopole factor is [43]

$$(-q^{1/2})^{k_{eff}[m,m]} s^{k_{eff}[m,-]}$$
(A.18)

where k_{eff} is a bilinear form defined by the anomaly polynomial. The interpretation is that for $U(1)$ at Chern-Simons level k we would have contribution $(-q^{1/2})^{km^2} s^{km}$ reflecting the induced electric charge km of the monopole due to the Chern-Simons level and consequently the induced spin $km^2/2$. Note that our convention is that the indices involve tracing $(-1)^F$, whereas in [43] the convention was $(-1)^R$. These conventions are simply related by $q^{1/2} \leftrightarrow (-q)^{1/2}$ and can lead to some sign differences when comparing explicit expressions for indices and half-indices. In particular, for the Dirichlet half-indices in our convention we may get sign factors depending on the magnetic charges in the sum, of the form $(-1)^{km^2}$ as objects with spin $km^2/2$ are fermions for odd km^2. Such terms indeed appear for the orthogonal half-indices, but not for the symplectic case where in all cases $(-1)^{k_{eff}[m,m]} = 1$.

The results above in terms of $rank(G)$ and product over roots hold also for $SU(N)$, $USp(2N)$ and $SO(N_c)$.

The details for the symplectic case are very similar to the unitary case. The fundamental weights for $USp(2N)$ are $\pm\mu_1, \pm\mu_2, \ldots, \pm\mu_N$ giving fugacities $s_i^{\pm 1}$, while the roots are $\mu_i - \mu_j \, \forall i \neq j$ and $\pm(\mu_i + \mu_j) \, \forall i \leq j$.

For $SO(N_c)$ we need to consider separately the cases of N_c even or odd. For the even case we write $N_c = 2N$ and we have fundamental weights $\pm\mu_1, \pm\mu_2, \ldots, \pm\mu_N$ and the roots are $\pm\mu_i \pm \mu_j \, \forall i < j$ where all four sign combinations are taken.

For the case of odd N_c we write $N_c = 2N + 1$ and we have one additional fundamental weight $\mu_0 = 0$ giving additional roots $\pm\mu_i + \mu_0 = \pm\mu_i \, \forall i$.

The above results, along with details of the monopole sums and effective Chern-Simons levels define the half-indices, although we found that the effective Chern-Simons levels need a different normalisation factor for symplectic gauge groups. However, there is a further complication for the $SO(N_c)$ indices when the discrete fugacity χ for the $\mathbb{Z}_2^{\mathcal{C}}$ global symmetries is included.

For the case of odd N_c if we have $\chi = +1$ we can consider that we have a gauge fugacity $s_0 = 1$ arising from the fundamental weight $\mu_0 = 0$. If instead we have $\chi = -1$ we set $s_0 = -1$. So, in general we have $s_0 = \chi$ meaning that the fugacities associated to the roots $\pm\mu_i + \mu_0$ are $\chi s_i^{\pm 1}$.

Note that these four different sectors for $SO(N_c)$, labelled by ζ and χ arise in 3d and so are inherited on the boundary since we have bulk gauge fields with boundary conditions. In 2d theories on T^2 there are additional sectors as there is a different classification of flat connections on T^2 as clearly explained in [88]. The point to note here is that this means the half-index for orthogonal groups is quite different from the 2d elliptic genus.

For even $N_c = 2N$ if we have $\chi = -1$ we must replace the gauge fugacities corresponding to fundamental weights μ_N and $-\mu_N$ with $s_N \to 1$ and $s_N^{-1} \to -1$. In cases of ambiguity, it is necessary to check whether the factor of s_N originated from fundamental weight μ_N or $-\mu_N$. One point to note in this case is that in the derivation of the vector multiplet contribution to the index one of the factors of $(q)_\infty$ (for Neumann or Dirichlet boundary conditions) arose from the adjoint weight $\mu_N + (-\mu_N) = 0$ giving factor $(qs_N s_N^{-1}; q)_\infty = (q)_\infty$. In the case of $\chi = -1$ this is replaced, since $s_N s_N^{-1} \to 1 \times (-1) = -1$ by the factor $(-q; q)_\infty$. I.e. in the half-index we replace $(q)_\infty^{rank(G)}$ with $(q)_\infty^{rank(G)-1}(-q; q)_\infty$ for $G = SO(2N)$ with $\chi = -1$.

Finally, our notation for fugacities is to label gauge fugacities s_i as above for half-indices with Neumann boundary conditions for the vector multiplet and instead u_i if we have Dirichlet boundary conditions. This applies to both theories A and B in the proposed dualities and is unambiguous as all these dualities involve Neumann for one theory and Dirichlet for the other. Of course, gauge invariance is imposed by integrating over the s_i so the results will depend on the u_i only.

Our notation for other fugacities is a for $U(1)_a$ and x_α for global $SU(N_F)$ flavour symmetries. In numerical checks we often set $x_\alpha = 1$ for simplicity. Note that $\prod_\alpha x_\alpha = 1$ but we could also use fugacities $\hat{x}_\alpha$ which don't obey such a constraint for the global $U(N_f)$ flavour symmetry by combining $SU(N_F)$ and $U(1)_a$. We comment on this when discussing dualities for non-zero Chern-Simons levels.

B   Series expansions of indices

We show several terms in the expansions of indices obtained by using Mathematica.

B.1 $USp(2N_c) - [2N_f]$

We show the expansions of full-indices and the half-indices in powers of q by defining a fugacity $y := q^{r-\frac{1}{2}}$. We find the matching at least up to q^5.

B.1.1 $USp(2N_c) - [2N_f]$

N_c	N_f	$\tilde{N}_c$	$I^{USp(2N_c)-[2N_f]^A} = I^{USp(2\tilde{N}_c)-[2N_f]^B}$
1	3	1	$1+\sqrt{q}\left(\frac{1}{a^6 y^3}+15a^2 y\right)+q\left(105a^4 y^2+\frac{1}{a^{12}y^6}-36\right)$ $+q^{3/2}\left(490a^6 y^3+\frac{1}{a^{18}y^9}-384a^2 y+\frac{21}{a^2 y}\right)$ $+q^2\left(1764a^8 y^4-2100a^4 y^2+\frac{36}{a^4 y^2}+\frac{1}{a^{24}y^{12}}+558\right)$ $+q^{5/2}\left(5292a^{10}y^5-8064a^6 y^3-\frac{71}{a^6 y^3}+\frac{1}{a^{30}y^{15}}+3690a^2 y-\frac{384}{a^2 y}\right)$ $+q^3\left(13860a^{12}y^6-24696a^8 y^4+14526a^4 y^2+\frac{135}{a^4 y^2}+\frac{36}{a^8 y^4}+\frac{1}{a^{36}y^{18}}-3536\right)+\cdots$
2	5	2	$1+\sqrt{q}\left(\frac{1}{a^{10}y^5}+\left(45a^2\right)y\right)$ $+q\left(\left(1035a^4\right)y^2+\frac{45}{a^8 y^4}+\frac{1}{a^{20}y^{10}}-100\right)$ $+q^{3/2}\left(\left(16005a^6\right)y^3+\frac{825}{a^6 y^3}-\frac{99}{a^{10}y^5}+\frac{45}{a^{18}y^9}+\frac{1}{a^{30}y^{15}}-\left(4400a^2\right)y+\frac{55}{a^2 y}\right)$ $+q^2\left(\left(186285a^8\right)y^4-\left(96480a^4\right)y^2+\frac{9075}{a^4 y^2}-\frac{3200}{a^8 y^4}+\frac{55}{a^{12}y^6}+\frac{825}{a^{16}y^8}-\frac{99}{a^{20}y^{10}}+\frac{45}{a^{28}y^{14}}+\frac{1}{a^{40}y^{20}}+7225\right)+\cdots$

$$\text{(B.1)}$$

B.1.2 $USp(2N_c) - [2N_f]$ with $\mathcal{N}, (\text{N}, \text{N}, \text{N}; \text{N}, \text{N}, \text{N}) + \text{Fermis}$

N_c	N_f	$\tilde{N}_c$	$\mathrm{III}^{USp(2N_c)-[2N_f]^A}_{\mathcal{N},\text{N}+\{\Psi\}} = \mathrm{III}^{USp(2\tilde{N}_c)-[2N_f]^B}_{\mathcal{D},\text{D},\text{N},\text{D}}$
1	3	1	$1+15a^2\sqrt{q}\,y-\frac{6q^{3/4}\left(a\left(u^2+1\right)\sqrt{y}\right)}{u}+q\left(105a^4 y^2+u^2+\frac{1}{u^2}+1\right)-\frac{70q^{5/4}\left(a^3\left(u^2+1\right)y^{3/2}\right)}{u}$ $+\frac{a^2 q^{3/2}y\left(u^2\left(490a^4 y^2+51\right)+15u^4+15\right)}{u^2}-\frac{12q^{7/4}\left(a\left(u^2+1\right)\sqrt{y}\left(35a^4 y^2+1\right)\right)}{u}$ $+q^2\left(u^2\left(105a^4 y^2+1\right)+\frac{105a^4 y^2+1}{u^2}+1764a^8 y^4+525a^4 y^2+2\right)-\frac{2q^{9/4}\left(a^3\left(u^2+1\right)y^{3/2}\left(882a^4 y^2+115\right)\right)}{u}$ $+\frac{a^2 q^{5/2}y\left(u^4\left(490a^4 y^2+51\right)+u^2\left(5292a^8 y^4+3010a^4 y^2+117\right)+490a^4 y^2+51\right)}{u^2}+\cdots$
2	5	2	$1+45a^2\sqrt{q}\,y-\frac{10q^{3/4}\left(a\sqrt{y}\left(u1^2 u2+u1 u2^2+u1+u2\right)\right)}{u1 u2}$ $+q\left(1035a^4 y^2+u1^2+\frac{1}{u1^2}+u1\left(u2+\frac{1}{u2}\right)+\frac{u2+\frac{1}{u2}}{u1}+u2^2+\frac{1}{u2^2}+2\right)$ $-\frac{450q^{5/4}\left(a^3 y^{3/2}\left(u1^2 u2+u1 u2^2+u1+u2\right)\right)}{u1 u2}$ $+\frac{5a^2 q^{3/2}y\left(u1^2\left(u2^2\left(3201a^4 y^2+67\right)+18u2^4+18\right)+18u1^4 u2^2+29u1^3\left(u2^3+u2\right)+29u1\left(u2^3+u2\right)+18u2^2\right)}{u1^2 u2^2}+\cdots$

$$\text{(B.2)}$$

B.1.3 $USp(2N_c) - [2N_f]$ with $\mathcal{D}, (D, D, D; D, D, D)$

N_c	N_f	$\tilde{N}_c$	$\mathrm{III}_{\mathcal{D},D}^{USp(2N_c)-[2N_f]^A} = \mathrm{III}_{\mathcal{N},N,D,N}^{USp(2\tilde{N}_c)-[2N_f]^B}$
1	3	1	$1 + \left(10a^2\right)\sqrt{q}\,y - \dfrac{q^{3/4}\sqrt{y}\left((5a)\left(u^2+1\right)\right)}{u} + q\left(\left(50a^4\right)y^2 + u^2 + \frac{1}{u^2} + 1\right) - \dfrac{q^{5/4}y^{3/2}\left(\left(40a^3\right)\left(u^2+1\right)\right)}{u}$ $+ \dfrac{q^{3/2}\left(\left(5a^2\right)y\right)\left(\left(7u^2\right)\left(\left(5a^4\right)y^2+1\right)+2u^4+2\right)}{u^2} - \dfrac{q^{7/4}\left(\left(35a^4\right)y^2+2\right)\left(\sqrt{y}\left((5a)\left(u^2+1\right)\right)\right)}{u}$ $+ q^2\left(u^2\left(\left(50a^4\right)y^2+1\right) + \dfrac{\left(50a^4\right)y^2+1}{u^2} + \left(490a^8\right)y^4 + \left(250a^4\right)y^2 + 2\right) - \dfrac{q^{9/4}\left(\left(56a^4\right)y^2+13\right)\left(y^{3/2}\left(\left(10a^3\right)\left(u^2+1\right)\right)\right)}{u}$ $+ \dfrac{q^{5/2}\left(a^2y\right)\left(\left(35u^4\right)\left(\left(5a^4\right)y^2+1\right) + \left(2u^2\right)\left(\left(588a^8\right)y^4 + \left(525a^4\right)y^2 + 40\right) + 35\left(\left(5a^4\right)y^2+1\right)\right)}{u^2} + \cdots$
2	5	2	$1 + \dfrac{\sqrt{q}}{a^{10}y^5} - \dfrac{10q^{3/4}\left(u1^2u2 + u1u2^2 + u1 + u2\right)}{au1u2\sqrt{y}}$ $+ q\left(\dfrac{1}{a^{20}y^{10}} + u1^2 + \dfrac{1}{u1^2} + u1\left(u2 + \frac{1}{u2}\right) + \dfrac{u2 + \frac{1}{u2}}{u1} + u2^2 + \dfrac{1}{u2^2} + 2\right)$ $- \dfrac{10q^{5/4}\left(u1^2u2 + u1u2^2 + u1 + u2\right)}{a^{11}u1u2y^{11/2}} + \cdots$

$$(B.3)$$

B.2 $SO(N_c) - [N_f]_{\zeta\chi}$

We show several terms in the expansions of full-indices and half-indices in powers of q by defining a fugacity $y := q^{r-\frac{1}{2}}$ or $t := q^{r-\frac{3}{4}}$. We find the agreement at least up to q^5.

B.2.1 $\quad SO(N_c) - [N_f]_{++}$

N_c	N_f	$\tilde{N}_c$	$I^{SO(N_c)-[N_f]^A_{++}} = I^{SO(\tilde{N}_c)-[N_f]^B_{++}}$
1	1	2	$1+a\sqrt[4]{q}\sqrt{y}+a^2\sqrt{q}y+\dfrac{q^{3/4}\left(a^4y^2-1\right)}{a\sqrt{y}}$ $+q\left(a^4y^2-1\right)+a^5q^{5/4}y^{5/2}+a^6q^{3/2}y^3+\dfrac{q^{7/4}\left(a^8y^4-1\right)}{a\sqrt{y}}$ $+q^2\left(a^8y^4-2\right)+q^{9/4}\left(a\sqrt{y}\right)\left(a^8y^4-1\right)+q^{5/2}\left(a^{10}y^5+\dfrac{1}{a^2y}\right)$ $+a^{11}q^{11/4}y^{11/2}+q^3\left(a^{12}y^6-2\right)+q^{13/4}\left(a\sqrt{y}\right)\left(a^{12}y^6-2\right)$ $+q^{7/2}\left(a^{14}y^7-a^2y+\dfrac{1}{a^2y}\right)+\dfrac{q^{15/4}\left(a^{16}y^8+1\right)}{a\sqrt{y}}+q^4\left(a^{16}y^8-2\right)$ $+q^{17/4}\left(a\sqrt{y}\right)\left(a^{16}y^8-3\right)+q^{9/2}\left(a^{18}y^9-\left(2a^2\right)y+\dfrac{2}{a^2y}\right)+\dfrac{q^{19/4}\left(a^{20}y^{10}-a^4y^2+3\right)}{a\sqrt{y}}+\cdots$
1	2	3	$1+(2a)q^{3/8}\sqrt{t}-\dfrac{2q^{5/8}}{a\sqrt{t}}+\left(3a^2\right)q^{3/4}t-4q+\left(4a^3\right)q^{9/8}t^{3/2}$ $+\dfrac{q^{5/4}}{a^2t}+(-4a)q^{11/8}\sqrt{t}+\left(5a^4\right)q^{3/2}t^2+\left(-4a^2\right)q^{7/4}t$ $+\left(6a^5\right)q^{15/8}t^{5/2}-5q^2+\left(-4a^3\right)q^{17/8}t^{3/2}+\dfrac{q^{9/4}\left(\left(7a^8\right)t^4+4\right)}{a^2t}$ $+(-8a)q^{19/8}\sqrt{t}+\left(-4a^4\right)q^{5/2}t^2+\dfrac{8q^{21/8}\left(a^8t^4+1\right)}{a\sqrt{t}}+\left(-8a^2\right)q^{11/4}t-\dfrac{2q^{23/8}\left(\left(2a^8\right)t^4+1\right)}{a^3t^{3/2}}+\cdots$
1	3	4	$1+(3a)q^{3/8}\sqrt{t}-\dfrac{3q^{5/8}}{a\sqrt{t}}+\left(6a^2\right)q^{3/4}t-9q+\left(10a^3\right)q^{9/8}t^{3/2}$ $+\dfrac{3q^{5/4}}{a^2t}+(-15a)q^{11/8}\sqrt{t}+\left(15a^4\right)q^{3/2}t^2+\dfrac{6q^{13/8}}{a\sqrt{t}}+\left(-21a^2\right)q^{7/4}t$ $+\dfrac{q^{15/8}\left(\left(21a^8\right)t^4-1\right)}{a^3t^{3/2}}+\left(-27a^3\right)q^{17/8}t^{3/2}+\dfrac{q^{9/4}\left(\left(28a^8\right)t^4+6\right)}{a^2t}$ $+(-12a)q^{19/8}\sqrt{t}+\left(-33a^4\right)q^{5/2}t^2+\dfrac{9q^{21/8}\left(\left(4a^8\right)t^4+3\right)}{a\sqrt{t}}+\left(-24a^2\right)q^{11/4}t-\dfrac{3q^{23/8}\left(\left(13a^8\right)t^4+3\right)}{a^3t^{3/2}}+\cdots$
2	2	2	$1+\sqrt{q}\left(\left(4a^2\right)y+\dfrac{2}{a^2y}\right)+q\left(\left(9a^4\right)y^2+\dfrac{2}{a^4y^2}-8\right)$ $+\dfrac{2q^{3/2}\left(\left(8a^{12}\right)y^6-\left(8a^8\right)y^4+\left(2a^4\right)y^2+1\right)}{a^6y^3}$ $+q^2\left(\left(25a^8\right)y^4-\left(24a^4\right)y^2+\dfrac{2}{a^8y^4}+14\right)+q^{5/2}\left(\left(36a^{10}\right)y^5-\left(32a^6\right)y^3+\dfrac{2}{a^{10}y^5}-\dfrac{16}{a^2y}-\left(16a^2\right)y\right)$ $+q^3\left(\left(49a^{12}\right)y^6-\left(40a^8\right)y^4-\left(32a^4\right)y^2+\dfrac{9}{a^4y^2}+\dfrac{2}{a^{12}y^6}+64\right)+\cdots$
3	3	2	$1+\dfrac{\sqrt[4]{q}}{a^3y^{3/2}}+\sqrt{q}\left(\dfrac{1}{a^6y^3}+\left(6a^2\right)y+\dfrac{3}{a^2y}\right)$ $+\dfrac{q^{3/4}\left(a^{12}y^6+\left(6a^8\right)y^4+\left(3a^4\right)y^2+1\right)}{a^9y^{9/2}}$ $+q\left(\left(21a^4\right)y^2+\dfrac{3}{a^4y^2}+\dfrac{3}{a^8y^4}+\dfrac{1}{a^{12}y^6}+1\right)$ $+\dfrac{q^{5/4}\left(\left(6a^{20}\right)y^{10}+\left(6a^{16}\right)y^8+\left(2a^{12}\right)y^6+\left(3a^8\right)y^4+\left(3a^4\right)y^2+1\right)}{a^{15}y^{15/2}}$ $+\dfrac{q^{3/2}\left(a^{12}y^6+\left(56a^{24}\right)y^{12}-\left(24a^{20}\right)y^{10}+\left(3a^8\right)y^4+\left(3a^4\right)y^2+1\right)}{a^{18}y^9}+\cdots$

$$(\text{B.4})$$

B.2.2 $SO(N_c) - [N_f]_{-+}$

N_c	N_f	$\tilde{N}_c$	$I^{SO(N_c)-[N_f]^A_{-+}} = I^{SO(\tilde{N}_c)-[N_f]^B_{--}}$
1	1	2	$1 + a\sqrt[4]{q}\sqrt{y} + a^2\sqrt{q}y + \frac{q^{3/4}\left(a^4y^2-1\right)}{a\sqrt{y}} + q\left(a^4y^2-1\right)$ $+ a^5q^{5/4}y^{5/2} + a^6q^{3/2}y^3 + \frac{q^{7/4}\left(a^8y^4-1\right)}{a\sqrt{y}}$ $+ q^2\left(a^8y^4-2\right) + q^{9/4}\left(a\sqrt{y}\right)\left(a^8y^4-1\right) + q^{5/2}\left(a^{10}y^5+\frac{1}{a^2y}\right) + a^{11}q^{11/4}y^{11/2} + \cdots$
1	2	3	$1 + (2a)q^{3/8}\sqrt{t} - \frac{2q^{5/8}}{a\sqrt{t}} + \left(3a^2\right)q^{3/4}t - 4q + \left(4a^3\right)q^{9/8}t^{3/2}$ $+ \frac{q^{5/4}}{a^2t} + (-4a)q^{11/8}\sqrt{t} + \left(5a^4\right)q^{3/2}t^2 + \left(-4a^2\right)q^{7/4}t$ $+ \left(6a^5\right)q^{15/8}t^{5/2} - 5q^2 + \left(-4a^3\right)q^{17/8}t^{3/2} + \frac{q^{9/4}\left(\left(7a^8\right)t^4+4\right)}{a^2t}$ $+ (-8a)q^{19/8}\sqrt{t} + \left(-4a^4\right)q^{5/2}t^2 + \frac{8q^{21/8}\left(a^8t^4+1\right)}{a\sqrt{t}} + \left(-8a^2\right)q^{11/4}t - \frac{2q^{23/8}\left(\left(2a^8\right)t^4+1\right)}{a^3t^{3/2}} + \cdots$
1	3	4	$1 + (3a)q^{3/8}\sqrt{t} - \frac{3q^{5/8}}{a\sqrt{t}} + \left(6a^2\right)q^{3/4}t - 9q + \left(10a^3\right)q^{9/8}t^{3/2}$ $+ \frac{3q^{5/4}}{a^2t} + (-15a)q^{11/8}\sqrt{t} + \left(15a^4\right)q^{3/2}t^2 + \frac{6q^{13/8}}{a\sqrt{t}} + \left(-21a^2\right)q^{7/4}t$ $+ \frac{q^{15/8}\left(\left(21a^8\right)t^4-1\right)}{a^3t^{3/2}} + \left(-27a^3\right)q^{17/8}t^{3/2} + \frac{q^{9/4}\left(\left(28a^8\right)t^4+6\right)}{a^2t}$ $+ (-12a)q^{19/8}\sqrt{t} + \left(-33a^4\right)q^{5/2}t^2 + \frac{9q^{21/8}\left(\left(4a^8\right)t^4+3\right)}{a\sqrt{t}} + \left(-24a^2\right)q^{11/4}t - \frac{3q^{23/8}\left(\left(13a^8\right)t^4+3\right)}{a^3t^{3/2}} + \cdots$
2	2	2	$1 + \sqrt{q}\left(\left(4a^2\right)y - \frac{2}{a^2y}\right) + q\left(\left(9a^4\right)y^2 + \frac{2}{a^4y^2} - 8\right)$ $+ \frac{2q^{3/2}\left(\left(8a^{12}\right)y^6 - \left(8a^8\right)y^4 + \left(2a^4\right)y^2 - 1\right)}{a^6y^3}$ $+ q^2\left(\left(25a^8\right)y^4 - \left(24a^4\right)y^2 + \frac{2}{a^8y^4} - 2\right) + q^{5/2}\left(\left(36a^{10}\right)y^5 - \left(32a^6\right)y^3 - \frac{2}{a^{10}y^5} - \left(16a^2\right)y + \frac{16}{a^2y}\right)$ $+ q^3\left(\left(49a^{12}\right)y^6 - \left(40a^8\right)y^4 - \left(32a^4\right)y^2 - \frac{7}{a^4y^2} + \frac{2}{a^{12}y^6} + 32\right) + \cdots$
3	3	2	$1 - \frac{\sqrt[4]{q}}{a^3y^{3/2}} + \sqrt{q}\left(\frac{1}{a^6y^3} - \frac{3}{a^2y} + \left(6a^2\right)y\right)$ $+ \frac{q^{3/4}\left(a^{12}y^6 - \left(6a^8\right)y^4 + \left(3a^4\right)y^2 - 1\right)}{a^9y^{9/2}}$ $+ q\left(\left(21a^4\right)y^2 + \frac{9}{a^4y^2} - \frac{3}{a^8y^4} + \frac{1}{a^{12}y^6} - 19\right)$ $+ \frac{q^{5/4}\left(\left(6a^{20}\right)y^{10} - \left(24a^{16}\right)y^8 + \left(18a^{12}\right)y^6 - \left(9a^8\right)y^4 + \left(3a^4\right)y^2 - 1\right)}{a^{15}y^{15/2}}$ $+ \frac{q^{3/2}\left(\left(56a^{24}\right)y^{12} - \left(66a^{20}\right)y^{10} + \left(36a^{16}\right)y^8 - \left(19a^{12}\right)y^6 + \left(9a^8\right)y^4 - \left(3a^4\right)y^2 + 1\right)}{a^{18}y^9} + \cdots$

$$(B.5)$$

B.2.3 $SO(N_c) - [N_f]_{+-}$

N_c	N_f	$\tilde{N}_c$	$I^{SO(N_c)-[N_f]^A_{+-}} = I^{SO(\tilde{N}_c)-[N_f]^B_{+-}}$
1	1	2	$1-a\sqrt[4]{q}\sqrt{y}+a^2\sqrt{q}y+\frac{q^{3/4}\left(1-a^4y^2\right)}{a\sqrt{y}}+q\left(a^4y^2-1\right)$ $-a^5q^{5/4}y^{5/2}+a^6q^{3/2}y^3+\frac{q^{7/4}\left(1-a^8y^4\right)}{a\sqrt{y}}+q^2\left(a^8y^4-2\right)$ $+q^{9/4}\left(a\sqrt{y}\right)\left(1-a^8y^4\right)+q^{5/2}\left(a^{10}y^5+\frac{1}{a^2y}\right)-a^{11}q^{11/4}y^{11/2}+\cdots$
1	2	3	$1+(-2a)q^{3/8}\sqrt{t}+\frac{2q^{5/8}}{a\sqrt{t}}+\left(3a^2\right)q^{3/4}t-4q+\left(-4a^3\right)q^{9/8}t^{3/2}$ $+\frac{q^{5/4}}{a^2t}+(4a)q^{11/8}\sqrt{t}+\left(5a^4\right)q^{3/2}t^2+\left(-4a^2\right)q^{7/4}t$ $+\left(-6a^5\right)q^{15/8}t^{5/2}-5q^2+\left(4a^3\right)q^{17/8}t^{3/2}+\frac{q^{9/4}\left(\left(7a^8\right)t^4+4\right)}{a^2t}$ $+(8a)q^{19/8}\sqrt{t}+\left(-4a^4\right)q^{5/2}t^2-\frac{8q^{21/8}\left(a^8t^4+1\right)}{a\sqrt{t}}+\left(-8a^2\right)q^{11/4}t+\frac{q^{23/8}\left(\left(4a^8\right)t^4+2\right)}{a^3t^{3/2}}+\cdots$
1	3	4	$1+(-3a)q^{3/8}\sqrt{t}+\frac{3q^{5/8}}{a\sqrt{t}}+\left(6a^2\right)q^{3/4}t-9q+\left(-10a^3\right)q^{9/8}t^{3/2}$ $+\frac{3q^{5/4}}{a^2t}+(15a)q^{11/8}\sqrt{t}+\left(15a^4\right)q^{3/2}t^2-\frac{6q^{13/8}}{a\sqrt{t}}+\left(-21a^2\right)q^{7/4}t$ $+\frac{q^{15/8}\left(1-\left(21a^8\right)t^4\right)}{a^3t^{3/2}}+\left(27a^3\right)q^{17/8}t^{3/2}+\frac{q^{9/4}\left(\left(28a^8\right)t^4+6\right)}{a^2t}$ $+(12a)q^{19/8}\sqrt{t}+\left(-33a^4\right)q^{5/2}t^2-\frac{9q^{21/8}\left(\left(4a^8\right)t^4+3\right)}{a\sqrt{t}}+\left(-24a^2\right)q^{11/4}t+\frac{q^{23/8}\left(\left(39a^8\right)t^4+9\right)}{a^3t^{3/2}}+\cdots$
2	2	2	$1+\left(2a^2\right)\sqrt{q}y+\left(3a^4\right)qy^2+\frac{q^{3/2}\left(\left(4a^8\right)y^4-2\right)}{a^2y}$ $+q^2\left(\left(5a^8\right)y^4-4\right)+q^{5/2}\left(\left(2a^2\right)y\right)\left(\left(3a^8\right)y^4-2\right)$ $+q^3\left(\left(7a^{12}\right)y^6-\left(4a^4\right)y^2+\frac{1}{a^4y^2}\right)+q^{7/2}\left(\left(4a^6\right)y^3\right)\left(\left(2a^8\right)y^4-1\right)+q^4\left(\left(9a^{16}\right)y^8-\left(4a^8\right)y^4-5\right)$ $+q^{9/2}\left(\left(2a^2\right)y\right)\left(\left(5a^{16}\right)y^8-\left(2a^8\right)y^4-4\right)+\cdots$
3	3	2	$1+\frac{\sqrt[4]{q}}{a^3y^{3/2}}+\sqrt{q}\left(\frac{1}{a^6y^3}-\frac{3}{a^2y}+\left(6a^2\right)y\right)$ $+\frac{q^{3/4}\left(a^{12}\left(-y^6\right)+\left(6a^8\right)y^4-\left(3a^4\right)y^2+1\right)}{a^9y^{9/2}}$ $+q\left(\left(21a^4\right)y^2+\frac{9}{a^4y^2}-\frac{3}{a^8y^4}+\frac{1}{a^{12}y^6}-19\right)$ $+\frac{q^{5/4}\left(-\left(6a^{20}\right)y^{10}+\left(24a^{16}\right)y^8-\left(18a^{12}\right)y^6+\left(9a^8\right)y^4-\left(3a^4\right)y^2+1\right)}{a^{15}y^{15/2}}$ $+\frac{q^{3/2}\left(\left(56a^{24}\right)y^{12}-\left(66a^{20}\right)y^{10}+\left(36a^{16}\right)y^8-\left(19a^{12}\right)y^6+\left(9a^8\right)y^4-\left(3a^4\right)y^2+1\right)}{a^{18}y^9}+\cdots$

$$(\text{B.6})$$

B.2.4 $SO(N_c) - [N_f]_{--}$

N_c	N_f	$\tilde{N}_c$	$I^{SO(N_c)-[N_f]^A_{--}} = I^{SO(\tilde{N}_c)-[N_f]^B_{-+}}$
1	1	2	$1-a\sqrt[4]{q}\sqrt{y}+a^2\sqrt{q}y+\frac{q^{3/4}\left(1-a^4y^2\right)}{a\sqrt{y}}+q\left(a^4y^2-1\right)$ $-a^5q^{5/4}y^{5/2}+a^6q^{3/2}y^3+\frac{q^{7/4}\left(1-a^8y^4\right)}{a\sqrt{y}}+q^2\left(a^8y^4-2\right)$ $+q^{9/4}\left(a\sqrt{y}\right)\left(1-a^8y^4\right)+q^{5/2}\left(a^{10}y^5+\frac{1}{a^2y}\right)-a^{11}q^{11/4}y^{11/2}+\cdots$
1	2	3	$1+(-2a)q^{3/8}\sqrt{t}+\frac{2q^{5/8}}{a\sqrt{t}}+\left(3a^2\right)q^{3/4}t-4q+\left(-4a^3\right)q^{9/8}t^{3/2}$ $+\frac{q^{5/4}}{a^2t}+(4a)q^{11/8}\sqrt{t}+\left(5a^4\right)q^{3/2}t^2+\left(-4a^2\right)q^{7/4}t$ $+\left(-6a^5\right)q^{15/8}t^{5/2}-5q^2+\left(4a^3\right)q^{17/8}t^{3/2}+\frac{q^{9/4}\left(\left(7a^8\right)t^4+4\right)}{a^2t}$ $+(8a)q^{19/8}\sqrt{t}+\left(-4a^4\right)q^{5/2}t^2-\frac{8q^{21/8}\left(a^8t^4+1\right)}{a\sqrt{t}}+\left(-8a^2\right)q^{11/4}t+\frac{q^{23/8}\left(\left(4a^8\right)t^4+2\right)}{a^3t^{3/2}}+\cdots$
1	3	4	$1+(-3a)q^{3/8}\sqrt{t}+\frac{3q^{5/8}}{a\sqrt{t}}+\left(6a^2\right)q^{3/4}t-9q+\left(-10a^3\right)q^{9/8}t^{3/2}$ $+\frac{3q^{5/4}}{a^2t}+(15a)q^{11/8}\sqrt{t}+\left(15a^4\right)q^{3/2}t^2-\frac{6q^{13/8}}{a\sqrt{t}}+\left(-21a^2\right)q^{7/4}t$ $+\frac{q^{15/8}\left(1-\left(21a^8\right)t^4\right)}{a^3t^{3/2}}+\left(27a^3\right)q^{17/8}t^{3/2}+\frac{q^{9/4}\left(\left(28a^8\right)t^4+6\right)}{a^2t}$ $+(12a)q^{19/8}\sqrt{t}+\left(-33a^4\right)q^{5/2}t^2-\frac{9q^{21/8}\left(\left(4a^8\right)t^4+3\right)}{a\sqrt{t}}+\left(-24a^2\right)q^{11/4}t+\frac{q^{23/8}\left(\left(39a^8\right)t^4+9\right)}{a^3t^{3/2}}+\cdots$
2	2	2	$1+\left(2a^2\right)\sqrt{q}y+\left(3a^4\right)qy^2+\frac{q^{3/2}\left(\left(4a^8\right)y^4-2\right)}{a^2y}$ $+q^2\left(\left(5a^8\right)y^4-4\right)+q^{5/2}\left(\left(2a^2\right)y\right)\left(\left(3a^8\right)y^4-2\right)$ $+q^3\left(\left(7a^{12}\right)y^6-\left(4a^4\right)y^2+\frac{1}{a^4y^2}\right)+q^{7/2}\left(\left(4a^6\right)y^3\right)\left(\left(2a^8\right)y^4-1\right)+q^4\left(\left(9a^{16}\right)y^8-\left(4a^8\right)y^4-5\right)$ $+q^{9/2}\left(\left(2a^2\right)y\right)\left(\left(5a^{16}\right)y^8-\left(2a^8\right)y^4-4\right)+\cdots$
3	3	2	$1-\frac{\sqrt[4]{q}}{a^3y^{3/2}}+\sqrt{q}\left(\frac{1}{a^6y^3}+6a^2y+\frac{3}{a^2y}\right)$ $+\frac{q^{3/4}\left(a^{12}\left(-y^6\right)-6a^8y^4-3a^4y^2-1\right)}{a^9y^{9/2}}$ $+q\left(21a^4y^2+\frac{3}{a^4y^2}+\frac{3}{a^8y^4}+\frac{1}{a^{12}y^6}+1\right)$ $+\frac{q^{5/4}\left(-6a^{20}y^{10}-6a^{16}y^8-2a^{12}y^6-3a^8y^4-3a^4y^2-1\right)}{a^{15}y^{15/2}}$ $+\frac{q^{3/2}\left(56a^{24}y^{12}-24a^{20}y^{10}+a^{12}y^6+3a^8y^4+3a^4y^2+1\right)}{a^{18}y^9}+\cdots$

$$(\text{B.7})$$

N_c	N_f	$\tilde{N}_c$	$\mathbb{III}^{SO(N_c)-[N_f]^A_{++}}_{\mathcal{N},N+\{\Psi\}} = \mathbb{III}^{SO(\tilde{N}_c)-[N_f]^B_{++}}_{\mathcal{D},D,N,D}$
1	1	2	$1+a\sqrt[4]{q}\sqrt{y}+\sqrt{q}\left(a^2y-u-\frac{1}{u}\right)+\frac{q^{3/4}(a\sqrt{y})\left(y\left(a^2u\right)-u^2-1\right)}{u}$ $+q\left(-\frac{y\left(a^2\left(u^2+1\right)\right)}{u}+a^4y^2+1\right)+q^{5/4}(a\sqrt{y})\left(-\frac{y\left(a^2\left(u^2+1\right)\right)}{u}+a^4y^2+2\right)$ $-\frac{q^{3/2}\left(y^2\left(a^4\left(u^2+1\right)\right)+y^3\left(-\left(a^6u\right)\right)-y\left(\left(2a^2\right)u\right)+u^2+1\right)}{u}+\cdots$
1	2	3	$1+(2a)q^{3/8}\sqrt{t}-\frac{\sqrt{q}\left(u^2+u+1\right)}{u}+\left(3a^2\right)q^{3/4}t-\frac{q^{7/8}\left(u^2+u+1\right)\left((2a)\sqrt{t}\right)}{u}$ $+q\left(u+\frac{1}{u}+1\right)+\left(4a^3\right)q^{9/8}t^{3/2}-\frac{q^{5/4}\left(u^2+u+1\right)\left(\left(3a^2\right)t\right)}{u}$ $+\frac{q^{11/8}(u+1)^2\left((2a)\sqrt{t}\right)}{u}+q^{3/2}\left(\left(5a^4\right)t^2-\frac{(u+1)^2}{u}\right)+\cdots$
1	3	4	$1+(3a)q^{3/8}\sqrt{t}+\sqrt{q}\left(-u1-\frac{1}{u1}-u2-\frac{1}{u2}\right)$ $+\left(6a^2\right)q^{3/4}t-\frac{q^{7/8}\left((3a)\sqrt{t}\right)\left(u1^2u2+u1u2^2+u1+u2\right)}{u1u2}$ $+q\left(u1\left(u2+\frac{1}{u2}\right)+\frac{u2+\frac{1}{u2}}{u1}+2\right)+\left(10a^3\right)q^{9/8}t^{3/2}-\frac{q^{5/4}\left(\left(6a^2\right)t\right)\left(u1^2u2+u1u2^2+u1+u2\right)}{u1u2}+\cdots$
2	2	2	$1+\left(4a^2\right)\sqrt{q}y-\frac{q^{3/4}\sqrt{y}\left((4a)\left(u^2+1\right)\right)}{u}$ $+q\left(\left(9a^4\right)y^2+u^2+\frac{1}{u^2}+1\right)-\frac{q^{5/4}y^{3/2}\left(\left(12a^3\right)\left(u^2+1\right)\right)}{u}$ $+\frac{q^{3/2}\left(\left(2a^2\right)y\right)\left(u^2\left(\left(8a^4\right)y^2+9\right)+2u^4+2\right)}{u^2}$ $-\frac{q^{7/4}\left(\left(3a^4\right)y^2+1\right)\left(\sqrt{y}\left((8a)\left(u^2+1\right)\right)\right)}{u}+\cdots$
3	3	2	$1+\left(6a^2\right)\sqrt{q}y+\frac{q^{3/4}(a\sqrt{y})\left(y\left(a^2u\right)-3u^2-3\right)}{u}$ $+q\left(-\frac{y\left(\left(3a^2\right)\left(u^2+1\right)\right)}{u}+\left(21a^4\right)y^2+1\right)$ $\frac{q^{5/4}((3a)\sqrt{y})\left(y^2\left(\left(2a^4\right)u^2\right)-y\left(\left(6a^2\right)u^3\right)-y\left(\left(6a^2\right)u\right)+u^4+1\right)}{u^2}$ $+q^{3/2}\left(\frac{\left(3a^2\right)y}{u^2}+y\left(\left(3a^2\right)u^2\right)-\frac{\left(18a^4\right)y^2}{u}-y^2\left(\left(18a^4\right)u\right)+\left(\left(7a^2\right)y\right)\left(\left(8a^4\right)y^2+3\right)-u^3-\frac{1}{u^3}\right)+\cdots$

$$(B.8)$$

B.2.6 $SO(N_c) - [N_f]_{-+}$ with $\mathcal{N}, (\mathrm{N}, \mathrm{N}, \mathrm{N}; \mathrm{N}, \mathrm{N}, \mathrm{N}) + \mathrm{Fermis}$

N_c	N_f	$\tilde{N}_c$	$\mathbb{III}^{SO(N_c)-[N_f]^A_{-+}}_{\mathcal{N},\mathrm{N}+\{\Psi\}} = \mathbb{III}^{SO(\tilde{N}_c)-[N_f]^B_{--}}_{\mathcal{D},\mathrm{D},\mathrm{N},\mathrm{D}}$
1	1	2	$1+a\sqrt[4]{q}\sqrt{y}+a^2\sqrt{q}y+a^3q^{3/4}y^{3/2}+q\left(a^4y^2-1\right)+a^5q^{5/4}y^{5/2}$ $+a^6q^{3/2}y^3+a^7q^{7/4}y^{7/2}+a^8q^2y^4+a^9q^{9/4}y^{9/2}+q^{5/2}\left(a^2y\right)\left(a^8y^4+1\right)+\cdots$
1	2	3	$1+(2a)q^{3/8}\sqrt{t}+\sqrt{q}\left(-u-\frac{1}{u}+1\right)+\left(3a^2\right)q^{3/4}t$ $+q^{7/8}\left(-u-\frac{1}{u}+1\right)\left((2a)\sqrt{t}\right)+q\left(-u-\frac{1}{u}+1\right)+\left(4a^3\right)q^{9/8}t^{3/2}$ $+q^{5/4}\left(-u-\frac{1}{u}+1\right)\left(\left(3a^2\right)t\right)-\frac{q^{11/8}(u-1)^2\left((2a)\sqrt{t}\right)}{u}+q^{3/2}\left(\left(5a^4\right)t^2-u-\frac{1}{u}+2\right)+\cdots$
1	3	4	$1+(3a)q^{3/8}\sqrt{t}-\frac{\sqrt{q}\left(u^2+1\right)}{u}+\left(6a^2\right)q^{3/4}t-\frac{q^{7/8}\left(u^2+1\right)\left((3a)\sqrt{t}\right)}{u}$ $+\left(10a^3\right)q^{9/8}t^{3/2}-\frac{q^{5/4}\left(u^2+1\right)\left(\left(6a^2\right)t\right)}{u}+(3a)q^{11/8}\sqrt{t}$ $+\left(15a^4\right)q^{3/2}t^2-\frac{q^{13/8}\left(u^2+1\right)\left(\left(10a^3\right)t^{3/2}\right)}{u}+\left(9a^2\right)q^{7/4}t+\cdots$
2	2	2	$1+\left(4a^2\right)\sqrt{q}y+q\left(\left(9a^4\right)y^2-1\right)+q^{3/2}\left(\left(2a^2\right)y\right)\left(\left(8a^4\right)y^2-1\right)$ $+q^2\left(a^4y^2\right)\left(\left(25a^4\right)y^2-1\right)+q^{5/2}\left(\left(2a^2\right)y\right)\left(a^4y^2+\left(18a^8\right)y^4+1\right)+\cdots$
3	3	2	$1+6a^2\sqrt{q}y+a^3q^{3/4}y^{3/2}+q\left(21a^4y^2-1\right)+6a^5q^{5/4}y^{5/2}+a^2q^{3/2}y\left(56a^4y^2-3\right)$ $+21a^7q^{7/4}y^{7/2}+3a^4q^2y^2\left(42a^4y^2-1\right)+a^5q^{9/4}y^{5/2}\left(56a^4y^2+3\right)+a^2q^{5/2}y\left(252a^8y^4+8a^4y^2+3\right)+\cdots$

$$\text{(B.9)}$$

B.2.7 $SO(N_c) - [N_f]_{+-}$ with $\mathcal{N}, (\mathrm{N}, \mathrm{N}, \mathrm{N}; \mathrm{N}, \mathrm{N}, \mathrm{N}) + \mathrm{Fermis}$

N_c	N_f	$\tilde{N}_c$	$\mathbb{III}^{SO(N_c)-[N_f]^A_{+-}}_{\mathcal{N},\mathrm{N}+\{\Psi\}} = \mathbb{III}^{SO(\tilde{N}_c)-[N_f]^B_{+-}}_{\mathcal{D},\mathrm{D},\mathrm{N},\mathrm{D}}$
1	1	2	$1-a\sqrt[4]{q}\sqrt{y}+a^2\sqrt{q}y-a^3q^{3/4}y^{3/2}+q\left(a^4y^2-1\right)-a^5q^{5/4}y^{5/2}$ $+a^6q^{3/2}y^3-a^7q^{7/4}y^{7/2}+a^8q^2y^4-a^9q^{9/4}y^{9/2}+q^{5/2}\left(a^2y\right)\left(a^8y^4+1\right)+\cdots$
1	2	3	$1+(-2a)q^{3/8}\sqrt{t}+\sqrt{q}\left(u+\frac{1}{u}-1\right)+\left(3a^2\right)q^{3/4}t$ $+q^{7/8}\left(u+\frac{1}{u}-1\right)\left((-2a)\sqrt{t}\right)+q\left(-u-\frac{1}{u}+1\right)+\left(-4a^3\right)q^{9/8}t^{3/2}$ $+q^{5/4}\left(u+\frac{1}{u}-1\right)\left(\left(3a^2\right)t\right)+\frac{q^{11/8}(u-1)^2\left((2a)\sqrt{t}\right)}{u}+q^{3/2}\left(\left(5a^4\right)t^2+u+\frac{1}{u}-2\right)+\cdots$
1	3	4	$1+(-3a)q^{3/8}\sqrt{t}+\sqrt{q}\left(u+\frac{1}{u}\right)+\left(6a^2\right)q^{3/4}t+q^{7/8}\left(u+\frac{1}{u}\right)\left((-3a)\sqrt{t}\right)$ $+\left(-10a^3\right)q^{9/8}t^{3/2}+q^{5/4}\left(u+\frac{1}{u}\right)\left(\left(6a^2\right)t\right)+(-3a)q^{11/8}\sqrt{t}$ $+\left(15a^4\right)q^{3/2}t^2+q^{13/8}\left(u+\frac{1}{u}\right)\left(\left(-10a^3\right)t^{3/2}\right)+\left(9a^2\right)q^{7/4}t+\cdots$
2	2	2	$1+\left(2a^2\right)\sqrt{q}y+q\left(\left(3a^4\right)y^2-1\right)+q^{3/2}\left(\left(4a^6\right)y^3-\left(2a^2\right)y\right)$ $+q^2\left(a^4y^2\right)\left(\left(5a^4\right)y^2-3\right)+q^{5/2}\left(\left(2a^2\right)y\right)\left(\left(3a^8\right)y^4-\left(2a^4\right)y^2+1\right)+\cdots$
3	3	2	$1+\left(6a^2\right)\sqrt{q}y-a^3q^{3/4}y^{3/2}+q\left(\left(21a^4\right)y^2-1\right)+\left(-6a^5\right)q^{5/4}y^{5/2}+q^{3/2}\left(a^2y\right)\left(\left(56a^4\right)y^2-3\right)$ $+\left(-21a^7\right)q^{7/4}y^{7/2}+q^2\left(\left(3a^4\right)y^2\right)\left(\left(42a^4\right)y^2-1\right)$ $+q^{9/4}\left(-a^5y^{5/2}\right)\left(\left(56a^4\right)y^2+3\right)+q^{5/2}\left(a^2y\right)\left(\left(252a^8\right)y^4+\left(8a^4\right)y^2+3\right)+\cdots$

$$\text{(B.10)}$$

B.2.8 $SO(N_c) - [N_f]_{--}$ with $\mathcal{N}, (\text{N}, \text{N}, \text{N}; \text{N}, \text{N}, \text{N}) + \text{Fermis}$

N_c	N_f	$\tilde{N}_c$	$\mathbb{III}^{SO(N_c)-[N_f]^A_{--}}_{\mathcal{N},\text{N}+\{\Psi\}} = \mathbb{III}^{SO(\tilde{N}_c)-[N_f]^B_{-+}}_{\mathcal{D},\text{D},\text{N},\text{D}}$
1	1	2	$1 - a\sqrt[4]{q}\sqrt{y} + \sqrt{q}\left(a^2 y + u + \frac{1}{u}\right) - \frac{q^{3/4}(a\sqrt{y})\left(y\left(a^2 u\right) + u^2 + 1\right)}{u}$ $+ q\left(y\left(a^2\left(u + \frac{1}{u}\right)\right) + a^4 y^2 + 1\right) + q^{5/4}(a\sqrt{y})\left(-y\left(a^2\left(u + \frac{1}{u}\right)\right) + a^4\left(-y^2\right) - 2\right)$ $+ q^{3/2}\left(y^2\left(a^4\left(u + \frac{1}{u}\right)\right) + a^6 y^3 + \left(2a^2\right)y + u + \frac{1}{u}\right) + \cdots$
1	2	3	$1 + (-2a)q^{3/8}\sqrt{t} + \sqrt{q}\left(u + \frac{1}{u} + 1\right) + \left(3a^2\right)q^{3/4}t$ $+ q^{7/8}\left(u + \frac{1}{u} + 1\right)\left((-2a)\sqrt{t}\right) + q\left(u + \frac{1}{u} + 1\right) + \left(-4a^3\right)q^{9/8}t^{3/2}$ $+ q^{5/4}\left(u + \frac{1}{u} + 1\right)\left(\left(3a^2\right)t\right) - \frac{q^{11/8}(u+1)^2\left((2a)\sqrt{t}\right)}{u} + q^{3/2}\left(\left(5a^4\right)t^2 + u + \frac{1}{u} + 2\right) + \cdots$
1	3	4	$1 + (-3a)q^{3/8}\sqrt{t} + \sqrt{q}\left(\text{u1} + \frac{1}{\text{u1}} + \text{u2} + \frac{1}{\text{u2}}\right) + \left(6a^2\right)q^{3/4}t$ $+ q^{7/8}\left((-3a)\sqrt{t}\right)\left(\text{u1} + \frac{1}{\text{u1}} + \text{u2} + \frac{1}{\text{u2}}\right) + q\left(\text{u1}\left(\text{u2} + \frac{1}{\text{u2}}\right) + \frac{\text{u2} + \frac{1}{\text{u2}}}{\text{u1}} + 2\right)$ $+ \left(-10a^3\right)q^{9/8}t^{3/2} + q^{5/4}\left(\left(6a^2\right)t\right)\left(\text{u1} + \frac{1}{\text{u1}} + \text{u2} + \frac{1}{\text{u2}}\right) + \cdots$
2	2	2	$1 + \left(2a^2\right)\sqrt{q}y + q\left(\left(3a^4\right)y^2 - u^2 - \frac{1}{u^2} + 1\right) + \frac{q^{3/2}\left(\left(2a^2\right)y\right)\left(u^2\left(\left(2a^4\right)y^2 + 1\right) - u^4 - 1\right)}{u^2}$ $+ q^2\left(-u^2\left(\left(3a^4\right)y^2 + 1\right) - \frac{\left(3a^4\right)y^2 + 1}{u^2} + \left(5a^8\right)y^4 + \left(3a^4\right)y^2 + 2\right) + \cdots$
3	3	2	$1 + \left(6a^2\right)\sqrt{q}y - \frac{q^{3/4}(a\sqrt{y})\left(y\left(a^2 u\right) + 3u^2 + 3\right)}{u} + q\left(\frac{y\left(\left(3a^2\right)\left(u^2 + 1\right)\right)}{u} + \left(21a^4\right)y^2 + 1\right)$ $- \frac{q^{5/4}((3a)\sqrt{y})\left(y^2\left(\left(2a^4\right)u^2\right) + y\left(\left(6a^2\right)u^3\right) + y\left(\left(6a^2\right)u\right) + u^4 + 1\right)}{u^2}$ $+ q^{3/2}\left(\frac{\left(3a^2\right)y}{u^2} + y\left(\left(3a^2\right)u^2\right) + \frac{\left(18a^4\right)y^2}{u} + y^2\left(\left(18a^4\right)u\right) + \left(\left(7a^2\right)y\right)\left(\left(8a^4\right)y^2 + 3\right) + u^3 + \frac{1}{u^3}\right) + \cdots$

$$(\text{B.11})$$

B.2.9 $SO(N_c) - [N_f]_{++}$ with $\mathcal{D}, (\text{D}, \text{D}, \text{D}; \text{D}, \text{D}, \text{D})$

N_c	N_f	$\tilde{N}_c$	$\mathbb{III}^{SO(N_c)-[N_f]^A_{++}}_{\mathcal{D},\text{D}} = \mathbb{III}^{SO(\tilde{N}_c)-[N_f]^B_{++}}_{\mathcal{N},\text{N},\text{D},\text{N}}$
1	1	2	$1 - \frac{q^{3/4}}{a\sqrt{y}} - \frac{q^{7/4}}{a\sqrt{y}} + \frac{q^{5/2}}{a^2 y} - \frac{q^{11/4}}{a\sqrt{y}} + \frac{q^{7/2}}{a^2 y} - \frac{q^{15/4}}{a\sqrt{y}} + \frac{2q^{9/2}}{a^2 y} - \frac{q^{19/4}}{a\sqrt{y}} - \frac{q^{21/4}}{a^3 y^{3/2}} + \frac{2q^{11/2}}{a^2 y} - \frac{q^{23/4}}{a\sqrt{y}} - \frac{q^{25/4}}{a^3 y^{3/2}} + \cdots$
1	2	3	$1 - \frac{2q^{5/8}}{a\sqrt{t}} + \frac{q^{5/4}}{a^2 t} - \frac{2q^{13/8}}{a\sqrt{t}} + \frac{4q^{9/4}}{a^2 t} - \frac{2q^{21/8}}{a\sqrt{t}} - \frac{2q^{23/8}}{a^3 t^{3/2}} + \frac{5q^{13/4}}{a^2 t} - \frac{2q^{29/8}}{a\sqrt{t}} - \frac{4q^{31/8}}{a^3 t^{3/2}} + \cdots$
1	3	4	$1 - \frac{3q^{5/8}}{a\sqrt{t}} + \frac{3q^{5/4}}{a^2 t} - \frac{3q^{13/8}}{a\sqrt{t}} - \frac{q^{15/8}}{a^3 t^{3/2}} + \frac{9q^{9/4}}{a^2 t} - \frac{3q^{21/8}}{a\sqrt{t}} - \frac{9q^{23/8}}{a^3 t^{3/2}} + \cdots$
2	2	2	$1 + \frac{2\sqrt{q}}{a^2 y} - \frac{q^{3/4}\left(4\left(u^2 + 1\right)\right)}{\sqrt{y}(au)} + q\left(\frac{2}{a^4 y^2} + u^2 + \frac{1}{u^2} + 1\right)$ $- \frac{q^{5/4}\left(4\left(u^2 + 1\right)\right)}{y^{3/2}\left(a^3 u\right)} + \frac{2q^{3/2}\left(y^2\left(a^4 u^4\right) + u^2\left(\left(7a^4\right)y^2 + 1\right) + a^4 y^2\right)}{y^3\left(a^6 u^2\right)}$ $- \frac{q^{7/4}\left(4\left(u^2 + 1\right)\right)\left(\left(2a^4\right)y^2 + 1\right)}{y^{5/2}\left(a^5 u\right)} + q^2\left(u^2\left(\frac{2}{a^4 y^2} + 1\right) + \frac{\frac{2}{a^4 y^2} + 1}{u^2} + \frac{10}{a^4 y^2} + \frac{2}{a^8 y^4} + 2\right) + \cdots$
3	3	2	$1 + \frac{\sqrt[4]{q}}{a^3 y^{3/2}} + \frac{\sqrt{q}\left(\left(3a^4\right)y^2 + 1\right)}{a^6 y^3} + \frac{q^{3/4}\left(-y^4\left(\left(6a^8\right)u^2\right) - y^4\left(\left(6a^8\right)u\right) + y^2\left(\left(3a^4\right)u\right) - \left(6a^8\right)y^4 + u\right)}{y^{9/2}\left(a^9 u\right)}$ $+ q\left(u\left(2 - \frac{6}{a^4 y^2}\right) + \frac{2 - \frac{6}{a^4 y^2}}{u} - \frac{9}{a^4 y^2} + \frac{3}{a^8 y^4} + \frac{1}{a^{12} y^6} + u^2 + \frac{1}{u^2} + 2\right)$ $+ \frac{q^{5/4}\left(y^6\left(a^{12}u^4\right) + \left(a^4 y^2 - 3\right)\left(y^4\left(\left(2a^8\right)u^3\right)\right) + u^2\left(\left(3a^{12}\right)y^6 - \left(9a^8\right)y^4 + \left(3a^4\right)y^2 + 1\right) + \left(a^4 y^2 - 3\right)\left(y^4\left(\left(2a^8\right)u\right)\right) + a^{12}y^6\right)}{y^{15/2}\left(a^{15}u^2\right)} + \cdots$

$$(\text{B.12})$$

B.2.10 $SO(N_c) - [N_f]_{-+}$ **with** $\mathcal{D}, (\mathrm{D},\mathrm{D},\mathrm{D};\mathrm{D},\mathrm{D},\mathrm{D})$

N_c	N_f	$\tilde{N}_c$	$\mathbb{III}_{\mathcal{D},\mathrm{D}}^{SO(N_c)-[N_f]_{-+}^A} = \mathbb{III}_{\mathcal{N},\mathrm{N},\mathrm{D},\mathrm{N}}^{SO(\tilde{N}_c)-[N_f]_{--}^B}$
1	1	2	$1 - \frac{q^{3/4}}{a\sqrt{y}} - \frac{q^{7/4}}{a\sqrt{y}} + \frac{q^{5/2}}{a^2 y} - \frac{q^{11/4}}{a\sqrt{y}} + \frac{q^{7/2}}{a^2 y} - \frac{q^{15/4}}{a\sqrt{y}} + \frac{2q^{9/2}}{a^2 y} - \frac{q^{19/4}}{a\sqrt{y}} - \frac{q^{21/4}}{a^3 y^{3/2}} + \frac{2q^{11/2}}{a^2 y} - \frac{q^{23/4}}{a\sqrt{y}} - \frac{q^{25/4}}{a^3 y^{3/2}} + \frac{3q^{13/2}}{a^2 y} + \cdots$
1	2	3	$1 - \frac{2q^{5/8}}{a\sqrt{t}} + \frac{q^{5/4}}{a^2 t} - \frac{2q^{13/8}}{a\sqrt{t}} + \frac{4q^{9/4}}{a^2 t} - \frac{2q^{21/8}}{a\sqrt{t}} - \frac{2q^{23/8}}{a^3 t^{3/2}} + \frac{5q^{13/4}}{a^2 t} - \frac{2q^{29/8}}{a\sqrt{t}} - \frac{4q^{31/8}}{a^3 t^{3/2}} + \cdots$
1	3	4	$1 - \frac{3q^{5/8}}{a\sqrt{t}} + \frac{3q^{5/4}}{a^2 t} - \frac{3q^{13/8}}{a\sqrt{t}} - \frac{q^{15/8}}{a^3 t^{3/2}} + \frac{9q^{9/4}}{a^2 t} - \frac{3q^{21/8}}{a\sqrt{t}} - \frac{9q^{23/8}}{a^3 t^{3/2}} + \cdots$
2	2	2	$1 - \frac{2\sqrt{q}}{a^2 y} + q\left(\frac{2}{a^4 y^2} - u^2 - \frac{1}{u^2} + 1\right) + \frac{2q^{3/2}\left(y^2\left(a^4 u^4\right) - u^2\left(\left(3a^4\right)y^2 + 1\right) + a^4 y^2\right)}{y^3\left(a^6 u^2\right)}$ $+ q^2\left(u^2\left(-\frac{2}{a^4 y^2} - 1\right) + \frac{-\frac{2}{a^4 y^2} - 1}{u^2} + \frac{10}{a^4 y^2} + \frac{2}{a^8 y^4} + 2\right)$ $+ \frac{2q^{5/2}\left(u^4\left(a^4 y^2 + \left(3a^8\right)y^4\right) - u^2\left(\left(5a^8\right)y^4 + \left(5a^4\right)y^2 + 1\right) + a^4 y^2 + \left(3a^8\right)y^4\right)}{y^5\left(a^{10} u^2\right)} + \cdots$
3	3	2	$1 - \frac{\sqrt[4]{q}}{a^3 y^{3/2}} + \frac{\sqrt{q}\left(1 - \left(3a^4\right)y^2\right)}{a^6 y^3} + \frac{q^{3/4}\left(\left(3a^4\right)y^2 - 1\right)}{a^9 y^{9/2}}$ $+ q\left(\frac{\left(3a^8\right)y^4 - \left(3a^4\right)y^2 + 1}{a^{12} y^6} - u^2 - \frac{1}{u^2}\right)$ $+ \frac{q^{5/4}\left(y^6\left(a^{12} u^4\right) - u^2\left(a^{12} y^6 + \left(3a^8\right)y^4 - \left(3a^4\right)y^2 + 1\right) + a^{12} y^6\right)}{y^{15/2}\left(a^{15} u^2\right)} + \cdots$

$$(\mathrm{B}.13)$$

B.2.11 $SO(N_c) - [N_f]_{+-}$ **with** $\mathcal{D}, (\mathrm{D},\mathrm{D},\mathrm{D};\mathrm{D},\mathrm{D},\mathrm{D})$

N_c	N_f	$\tilde{N}_c$	$\mathbb{III}_{\mathcal{D},\mathrm{D}}^{SO(N_c)-[N_f]_{+-}^A} = \mathbb{III}_{\mathcal{N},\mathrm{N},\mathrm{D},\mathrm{N}}^{SO(\tilde{N}_c)-[N_f]_{+-}^B}$
1	1	2	$1 + \frac{q^{3/4}}{a\sqrt{y}} + \frac{q^{7/4}}{a\sqrt{y}} + \frac{q^{5/2}}{a^2 y} + \frac{q^{11/4}}{a\sqrt{y}} + \frac{q^{7/2}}{a^2 y} + \frac{q^{15/4}}{a\sqrt{y}} + \frac{2q^{9/2}}{a^2 y} + \frac{q^{19/4}}{a\sqrt{y}} + \frac{q^{21/4}}{a^3 y^{3/2}} + \frac{2q^{11/2}}{a^2 y} + \frac{q^{23/4}}{a\sqrt{y}} + \frac{q^{25/4}}{a^3 y^{3/2}} + \frac{3q^{13/2}}{a^2 y} + \cdots$
1	2	3	$1 + \frac{2q^{5/8}}{a\sqrt{t}} + \frac{q^{5/4}}{a^2 t} + \frac{2q^{13/8}}{a\sqrt{t}} + \frac{4q^{9/4}}{a^2 t} + \frac{2q^{21/8}}{a\sqrt{t}} + \frac{2q^{23/8}}{a^3 t^{3/2}} + \frac{5q^{13/4}}{a^2 t} + \frac{2q^{29/8}}{a\sqrt{t}} + \frac{4q^{31/8}}{a^3 t^{3/2}} + \cdots$
1	3	4	$1 + \frac{3q^{5/8}}{a\sqrt{t}} + \frac{3q^{5/4}}{a^2 t} + \frac{3q^{13/8}}{a\sqrt{t}} + \frac{q^{15/8}}{a^3 t^{3/2}} + \frac{9q^{9/4}}{a^2 t} + \frac{3q^{21/8}}{a\sqrt{t}} + \frac{9q^{23/8}}{a^3 t^{3/2}} + \cdots$
2	2	2	$1 - q - \frac{2q^{3/2}}{a^2 y} + \frac{2q^{5/2}}{a^2 y} + q^3\left(\frac{1}{a^4 y^2} - 1\right) - \frac{2q^{7/2}}{a^2 y} + q^4\left(1 - \frac{1}{a^4 y^2}\right) + \frac{4q^{9/2}}{a^2 y} + \cdots$
3	3	2	$1 + \frac{\sqrt[4]{q}}{a^3 y^{3/2}} + \frac{\sqrt{q}\left(1 - \left(3a^4\right)y^2\right)}{a^6 y^3} + \frac{q^{3/4}\left(1 - \left(3a^4\right)y^2\right)}{a^9 y^{9/2}}$ $+ q\left(\frac{3}{a^4 y^2} - \frac{3}{a^8 y^4} + \frac{1}{a^{12} y^6} - u^2 - \frac{1}{u^2} - 2\right) + \frac{q^{5/4}\left(-y^6\left(a^{12} u^4\right) - u^2\left(a^4 y^2 - 1\right)^3 - a^{12} y^6\right)}{y^{15/2}\left(a^{15} u^2\right)}$ $+ \frac{q^{3/2}\left(\frac{y^8\left(\left(3a^{16}\right)\left(u^4 + 1\right)\right)}{u^2} - \frac{y^6\left(a^{12}\left(u^2 + 1\right)^2\right)}{u^2} + \left(3a^8\right)y^4 - \left(3a^4\right)y^2 + 1\right)}{a^{18} y^9} + \cdots$

$$(\mathrm{B}.14)$$

B.2.12 $SO(N_c)-[N_f]_{--}$ **with** $\mathcal{D},(\mathrm{D},\mathrm{D},\mathrm{D};\mathrm{D},\mathrm{D},\mathrm{D})$

N_c	N_f	$\tilde{N}_c$	$\mathbb{III}_{\mathcal{D},\mathrm{D}}^{SO(N_c)-[N_f]^A_{--}} = \mathbb{III}_{\mathcal{N},\mathrm{N},\mathrm{D},\mathrm{N}}^{SO(\tilde{N}_c)-[N_f]^B_{-+}}$
1	1	2	$1+\frac{q^{3/4}}{a\sqrt{y}}+\frac{q^{7/4}}{a\sqrt{y}}+\frac{q^{5/2}}{a^2y}+\frac{q^{11/4}}{a\sqrt{y}}+\frac{q^{7/2}}{a^2y}+\frac{q^{15/4}}{a\sqrt{y}}+\frac{2q^{9/2}}{a^2y}+\frac{q^{19/4}}{a\sqrt{y}}+\frac{q^{21/4}}{a^3y^{3/2}}+\frac{2q^{11/2}}{a^2y}+\frac{q^{23/4}}{a\sqrt{y}}+\frac{q^{25/4}}{a^3y^{3/2}}+\frac{3q^{13/2}}{a^2y}+\cdots$
1	2	3	$1+\frac{2q^{5/8}}{a\sqrt{t}}+\frac{q^{5/4}}{a^2t}+\frac{2q^{13/8}}{a\sqrt{t}}+\frac{4q^{9/4}}{a^2t}+\frac{2q^{21/8}}{a\sqrt{t}}+\frac{2q^{23/8}}{a^3t^{3/2}}+\frac{5q^{13/4}}{a^2t}+\frac{2q^{29/8}}{a\sqrt{t}}+\frac{4q^{31/8}}{a^3t^{3/2}}+\cdots$
1	3	4	$1+\frac{3q^{5/8}}{a\sqrt{t}}+\frac{3q^{5/4}}{a^2t}+\frac{3q^{13/8}}{a\sqrt{t}}+\frac{q^{15/8}}{a^3t^{3/2}}+\frac{9q^{9/4}}{a^2t}+\frac{3q^{21/8}}{a\sqrt{t}}+\frac{9q^{23/8}}{a^3t^{3/2}}+\cdots$
2	2	2	$1-q-\frac{2q^{3/2}}{a^2y}+\frac{2q^{5/2}}{a^2y}+q^3\left(\frac{1}{a^4y^2}-1\right)-\frac{2q^{7/2}}{a^2y}+q^4\left(1-\frac{1}{a^4y^2}\right)+\frac{4q^{9/2}}{a^2y}+\cdots$
3	3	2	$1-\frac{\sqrt[4]{q}}{a^3y^{3/2}}+\frac{\sqrt{q}\left(\left(3a^4\right)y^2+1\right)}{a^6y^3}+\frac{q^{3/4}\left(-y^4\left(\left(6a^8\right)u^2\right)+u\left(\left(6a^8\right)y^4-\left(3a^4\right)y^2-1\right)+\left(-6a^8\right)y^4\right)}{y^{9/2}\left(a^9u\right)}$ $+q\left(u\left(\frac{6}{a^4y^2}-2\right)+\frac{\frac{6}{a^4y^2}-2}{u}-\frac{9}{a^4y^2}+\frac{3}{a^8y^4}+\frac{1}{a^{12}y^6}+u^2+\frac{1}{u^2}+2\right)$ $+\frac{q^{5/4}\left(-y^6\left(a^{12}u^4\right)+\left(a^4y^2-3\right)\left(y^4\left(\left(2a^8\right)u^3\right)\right)-u^2\left(\left(3a^{12}\right)y^6-\left(9a^8\right)y^4+\left(3a^4\right)y^2+1\right)+\left(a^4y^2-3\right)\left(y^4\left(\left(2a^8\right)u\right)\right)-a^{12}y^6\right)}{y^{15/2}\left(a^{15}u^2\right)}+\cdots$

$$(\mathrm{B}.15)$$

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