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Oblique confinement at $\theta \neq 0$ in weakly coupled gauge theories with deformations

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The main focus of this work is to test the ideas related to the oblique confinement in a theoretically controllable manner using the "deformed QCD" as a toy model. We explicitly show that the oblique confinement in the weakly coupled gauge theories emerges as a result of condensation of N types of monopoles shifted by the phase $\exp(i\frac{\theta+2\pi m}{N})$ in Bloch type construction. It should be contrasted with the conventional and commonly accepted viewpoint that the confinement at $\theta \neq 0$ is due to the condensation of the electrically charged dyons which indeed normally emerge in the systems with $\theta \neq 0$ as a result of Witten's effect. We explain the basic reason why the "dyon" mechanism does not materialize—it is because the Witten's effect holds for a static magnetic monopole treated as an external source. It should be contrasted with our case when N-types of monopoles are not static, but rather the dynamical degrees of freedom which fluctuate and themselves determine the ground state of the system.

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I. INTRODUCTION AND MOTIVATION

A study of the QCD vacuum state in the strong coupling regime is the prerogative of numerical Monte Carlo lattice computations. However, a number of very deep and fundamental questions about the QCD vacuum structure can be addressed and, more importantly, answered using some simplified versions of QCD. In the present paper, we study a set of questions intimately connected to the ground state with $\theta \neq 0$. We use the so-called "deformed QCD" and similar toy models wherein we can work analytically.[1] These models belong to the class of the weakly coupled gauge theory, which, however, preserves many essential elements expected for true QCD, such as confinement, degenerate topological sectors, proper θ dependence, etc. This allows us to study difficult and nontrivial features, particularly related to vacuum structure at $\theta \neq 0$, in an analytically tractable manner.

The θ dependence in the system is intimately related to the presence of the metastable states which always accompany the gauge systems even at $\theta = 0$. The fact that some high energy metastable vacuum states must be present in a gauge theory system in the large N limit has been known for quite some time [1]. A similar conclusion also follows from the holographic description of QCD as originally discussed in [2]. Therefore, the understanding of the microscopical description of the ground state at $\theta \neq 0$ in terms of the monopoles (in the "deformed QCD" and other toy models as will be discussed in the present work) inevitably requires the microscopical understanding of these metastable states as both constructions, the $\theta \neq 0$ states and the metastable states at $\theta = 0$, must be described simultaneously in terms of the same degrees of freedom and in terms of the same fundamental gauge configurations.

A. $\theta \neq 0$: Phenomenological motivation

The questions being addressed in the present work, as highlighted above, are very deep and fundamental problems of the strongly coupled gauge theory. One could naively think that these problems with $\theta \neq 0$ are pure academic questions which have no physics applications, observable consequences or any phenomenological significance as it is known that $\theta = 0$ with extremely high accuracy in our Universe at present time. However we want to emphasize here that, in fact, the problems highlighted above were largely motivated by an attempt to understand the QCD transition in the early Universe when θ was not identically zero, but rather was slowly relaxing to zero field as a result of the axion dynamics, see original papers [3–9] and review articles [10–16] on the theory of axion and recent advances in the axion search experiments.

The recent lattice studies [17–20] addressing related questions on the axion dynamics during the QCD transition essentially are capable to compute the correlation functions, such as the topological susceptibility (1) at $\theta = 0$, while the gauge configurations at $\theta \neq 0$ are not accessible by conventional lattice methods. The study of the dynamics of the system at $\theta \neq 0$ represents a very challenging

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[1]We use one and the same term "deformed QCD" model for systems with and without quarks. We hope it does not confuse the readers as the specific description of the system should be obvious from the context of the discussions.

technical problem as a result of the so-called "sign problem". Therefore, at present time the lattice studies can provide limited information on microscopical dynamics of the strongly coupled gauge theories at finite $\theta \neq 0$ [21,22], especially the regions in the vicinity of $\theta \simeq \pi$ when the level-crossing phenomena are expected to occur, and metastable states become almost degenerate with the ground vacuum states.

At the same time, a precise understanding of the structure of the ground state at $\theta \neq 0$ and its microscopical description during this complicated time evolution plays a crucial role in computations of the axion production rate, possible formation of the axion domain walls,[2] possible role of the metastable states (which inevitably are present in the system as will be argued in this work), and many other related questions which essentially determine the dark sector of the Universe at present time.

The main claim which will be advocated in the present work is that the microscopical description of the oblique confinement at $\theta \neq 0$ is due to the condensation of the same fractionally charged monopoles in "deformed QCD" model which are responsible for the confinement at $\theta = 0$. The same microscopic description remains also valid for the metastable vacuum states which are always present in gauge theories. The only modification which occurs in the description for metastable states and $\theta \neq 0$ states is that the vacuum expectation value of the magnetization operator gets shifted by the phase $\exp(i\frac{\theta+2\pi m}{N})$ in Bloch type construction. We reiterate the same claim as follows: we do not see any room within our framework for the commonly accepted "dyon mechanism" for the oblique confinement, speculated long ago by 't' Hooft [25] when the electrically charged dyons condense.

We conjecture that this picture we have just described holds in strongly coupled regimes as well, not only in the weakly coupled "deformed QCD" model. We present few arguments supporting this conjecture in the next subsection.

B. Smooth transition between weakly coupled and strongly coupled regimes

When some deep questions are studied in a simplified version of a theory, there is always a risk that some effects which emerge in the simplified version of the theory could be just artifacts of the approximation, rather than genuine consequences of the original underlying theory. Our present studies in this work using the "deformed QCD" and other toy models are not free from this difficulty of possible misinterpretation of artifacts as inherent features of underlying QCD. Nevertheless, there are a few strong arguments suggesting that we indeed study some intrinsic features of the system rather than some artificial effects. The first argument has been presented in the original paper on "deformed QCD" [26] where it has been argued that this model describes a smooth interpolation between strongly coupled QCD and the weakly coupled "deformed QCD" without any phase transition. In addition, there are a few more arguments based on previous experience [26–39] with the "deformed QCD" and other toy models which also strongly suggest that we indeed study some intrinsic features of QCD rather than some artifacts of the deformations.

Most of the arguments, with very few exceptions, from the previous studies [26–39] of the system which are related to the θ dependent physics are purely analytical in nature as they cannot be independently verified or tested by using some other means, such as the numerical lattice simulations. Fortunately, some of the observables, such as the topological susceptibility χ defined as

$$\chi = \frac{\partial^2 E_{\rm vac}(\theta)}{\partial \theta^2}\bigg|_{\theta=0} = \lim_{k\to 0} \int e^{ikx} d^4x \langle q(x), q(0)\rangle \quad (1)$$

with $q(x)$ being the topological density operator, are highly sensitive to the θ behaviour even at $\theta = 0$ because χ measures the response of the system with respect to the insertion of the external parameter θ as one can see from the definition (1). What is more important is that the topological susceptibility χ can be also studied on the lattice at $\theta = 0$.

The topological susceptibility χ has been introduced into the theory long ago [40–42] in a course of studies related to the resolution of the $U(1)_A$ problem in QCD in the large N limit. As a result of its fundamental importance for the phenomenological particle physics the topological susceptibility χ has been extensively studied in lattice numerical simulations. The computations [28,34] of the topological susceptibility in the "deformed QCD" model is perfectly consistent with the lattice results, including some extremely nontrivial features related to the "wrong sign" of the contact term.[3] and exact cancellation (in the chiral limit) of the

[2]In particular, the so-called $N = 1$ domain walls corresponding to the interpolation of the axion $\theta(x)$ field between topologically distinct but physically identical states $\theta = 0$ and $\theta = 2\pi$ will inevitably form due to the 2π periodicity in θ and presence of the metastable states mentioned above. The formation of such kind of $N = 1$ domain walls happen irrespectively whether the inflation occurs before or after the PQ phase transition, see comments in [23,24].

[3]It is known that the contact term with a positive sign (in the Euclidean formulation) in χ is required for the resolution of the $U(1)_A$ problem [40–42]. At the same time, any physical propagating degrees of freedom must contribute with a negative sign, see [28] with details. In [40] this positive contact term has been simply postulated while in [41,42] an unphysical Veneziano ghost was introduced into the system to saturate this term with the "wrong" sign in the topological susceptibility. This entire, very nontrivial framework, has been successfully confirmed by a number of independent lattice computations and precisely reproduced in the "deformed QCD" model. In addition, one can explicitly see how the Veneziano ghost postulated in [41,42] is explicitly expressed in terms of auxiliary topological fields which saturate the contact term in this model [34]. One can also see that the η' becomes massive in this theory as a result of the mixture of a "would be" Goldstone field with auxiliary topological fields which saturate the contact term in (1).

contact term with the "wrong sign" with physical term in agreement with the Ward Identities, as described in the original papers [28,34].

Fortunately enough, there is still one more analytical study in the small circle limit that sheds light on the nature of the phase structure of gauge theories at $\theta \neq 0$. In [35] a conjectured continuity between mass deformed $\mathcal{N} = 1$ super Yang-Mills on a small circle and pure Yang-Mills at finite temperature was exploited to study the behavior of the thermal phase transition in the latter theories as a function of θ. According to this conjecture, quantum phase transitions in mass deformed $\mathcal{N} = 1$ on $\mathbb{R}^3 \times \mathbb{S}^1$ are analytically connected to thermal phase transitions in pure Yang-Mills [31,37]. Thus, one can perform all computations in the small circle limit, where the theory is under analytical control, and then extract conclusions about the strongly coupled theories. It was found in [35] that the deconfining temperature of any $SU(N)$ gauge theory decreases as θ increases and also the strength of the first order transition increases with θ. This is in accordance with the lattice simulations that were performed for small θ in strongly coupled theories [21,22] and arrived at the same conclusions of [35].

We conclude this subsection with the following generic comment. All the features related to the θ dependence which are known to be present in the strongly coupled regime also emerge in the weakly coupled "deformed QCD" and other toy models. Therefore, we interpret such nice agreement as a strong argument supporting our conjecture that these models properly describe, at least qualitatively, the microscopical features related to the θ dependent effects in the strongly coupled gauge theories.

C. The relation to $\mathcal{N} = 2$ Seiberg-Witten model and the structure of the paper

Our presentation is organized as follows. We start in Sec. II by reviewing a simplified ("deformed") version of QCD which, on one hand, is a weakly coupled gauge theory wherein computations can be performed in a theoretically controllable manner. On other hand, this deformation preserves all the elements relevant to our study such as confinement, degeneracy of topological sectors, nontrivial θ dependence, and other crucial aspects pertinent to the study of the oblique confinement for metastable states and $\theta \neq 0$ states. In Sec. III we explain the classification of the θ states while in Sec. IV we explicitly show that oblique confinement in this model is due to the *identically same* fractionally charged monopoles which are responsible for the confinement at $\theta = 0$.

This is obviously an expected result especially in view of the arguments presented above suggesting that this result holds in the strongly coupled regime as well due to the smooth transition between the weakly coupled "deformed QCD" and strongly coupled QCD realized in nature. At the same time the common lore in the community is that the oblique confinement at $\theta \neq 0$ is a result of condensation of the electrically charged dyons which emerge as a result of the Witten's effect [43]. This common lore is mostly based on analysis of the $\mathcal{N} = 2$ Seiberg-Witten model where the dyons are known to be part of spectrum. Therefore, it is indeed a quite natural assumption that these dyons will condense at $\theta \neq 0$, similar to the monopole's condensation in the original Seiberg-Witten model at $\theta = 0$.

Motivated by these arguments we turn to $\mathcal{N} = 2$ Seiberg-Witten model with the goal to understand the nature of the oblique confinement at $\theta \neq 0$ in supersymmetry (SUSY) gauge theories and its relation to studies in "deformed QCD" model presented in Sec. IV. We start, in Sec. V by reviewing the $\mathcal{N} = 2$ SUSY model defined on $\mathbb{R}^4$ with emphasis on the structure of the conventional static dyons and the monopoles in this model. As our goal is to understand the role of these particles in the confinement mechanism at $\theta \neq 0$ and the relation with oblique confinement in "deformed QCD" model, we formulate $\mathcal{N} = 2$ SUSY model on $\mathbb{R}^3 \times \mathbb{S}^1$ in Sec. VI and show that the nonperturbative spectrum of the theory on a sufficiently small circle consists of a tower of monopoles with higher winding numbers. In Sec. VII we explain Witten's effect [43] in the context of this work; i.e. we explain that the static magnetic monopoles, i.e. 't' Hooft lines, indeed become the dyons in the presence of $\theta \neq 0$. However, the magnetic monopoles which play the key role in the confinement mechanism are not static, but rather the dynamical degrees of freedom which fluctuate and themselves determine the ground state of the system. In the former case the monopoles become the dyons, while in the later case they remain pure monopoles with zero electric charges.

Throughout this work we use the word *dyon-particles* to mean *genuine particles (solitons) that carry both electric and magnetic charges*. They are genuine in the sense that they sweep timelike worldlines. We also use *dyon-instantons* to mean *pseudoparticles that carry both electric and magnetic charges*. They are pseudo since they are only *instantaneous events in the Euclidean space* and do not sweep worldlines. Dyons with zero electric charges are monopoles; these are either monopole-particles or monopoles-instantons. The words *dyons* or *monopoles* will be used to mean either particles or instantons when the distinction is either not important or understood from the context.

II. "DEFORMED QCD" MODEL

Here we overview the "center-stabilized" deformed Yang-Mills developed in [26]. In this section and in Sec. III and Sec. IV we use the words monopoles and dyons to mean monopole-instantons and dyon-instantons, respectively. In the deformed theory an extra "deformation" term is put into the Lagrangian in order to prevent the center symmetry breaking that characterizes the QCD phase transition between "confined" hadronic matter and "deconfined" quark-gluon plasma, thereby explicitly preventing that transition. Basically the extra term describes a potential for the order parameter. The basics of this model are reviewed in this section, while in Sec. III we classify the metastable states which are inherent elements of the system.

We start with pure Yang-Mills (gluodynamics) with gauge group $SU(N)$ on the manifold $\mathbb{R}^3 \times S^1$ with the standard action

$$S^{\mathrm{YM}} = \int_{\mathbb{R}^3 \times S^1} d^4x \frac{1}{2g^2} \mathrm{tr}[F_{\mu\nu}^2(x)], \tag{2}$$

and add to it a deformation action,

$$\Delta S \equiv \int_{\mathbb{R}^3} d^3x \frac{1}{L^3} P[\Omega(\mathbf{x})], \tag{3}$$

built out of the Wilson loop (Polyakov loop) wrapping the compact dimension

$$\Omega(\mathbf{x}) \equiv \mathcal{P}[e^{i\oint dx_4 A_4(\mathbf{x},x_4)}]. \tag{4}$$

The parameter L here is the length of the compactified dimension which is assumed to be small. The coefficients of the polynomial $P[\Omega(\mathbf{x})]$ can be suitably chosen such that the deformation potential (3) forces unbroken symmetry at any compactification scales. At small compactification L the gauge coupling is small so that the semiclassical computations are under complete theoretical control [26].

As described in [26], the proper infrared description of the theory is a dilute gas of N types of monopoles, characterized by their magnetic charges, which are proportional to the simple roots and affine root $\alpha_a \in \Delta_{\mathrm{aff}}$ of the Lie algebra of the gauge group $U(1)^N$. For a fundamental monopole with magnetic charge $\alpha_a \in \Delta_{\mathrm{aff}}$ (the affine root system), the topological charge is given by

$$Q = \int_{\mathbb{R}^3 \times S^1} d^4x \frac{1}{16\pi^2} \mathrm{tr}[F_{\mu\nu}\tilde{F}^{\mu\nu}] = \pm\frac{1}{N}, \tag{5}$$

and the Yang-Mills action is given by

$$S_{\mathrm{YM}} = \int_{\mathbb{R}^3 \times S^1} d^4x \frac{1}{2g^2} \mathrm{tr}[F_{\mu\nu}^2] = \frac{8\pi^2}{g^2}|Q|. \tag{6}$$

The θ-parameter in the Yang-Mills action can be included in the conventional way,

$$S_{\mathrm{YM}} \to S_{\mathrm{YM}} + i\theta \int_{\mathbb{R}^3 \times S^1} d^4x \frac{1}{16\pi^2} \mathrm{tr}[F_{\mu\nu}\tilde{F}^{\mu\nu}], \tag{7}$$

with $\tilde{F}^{\mu\nu} \equiv \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}/2$.

The system of interacting monopoles, including the θ parameter, can be represented in the dual sine-Gordon form as follows [26]

$$S_{\mathrm{dual}} = \int_{\mathbb{R}^3} d^3x \frac{1}{2L}\left(\frac{g}{2\pi}\right)^2 (\nabla\boldsymbol{\sigma})^2 - \zeta \int_{\mathbb{R}^3} d^3x \sum_{a=1}^{N} \cos\left(\alpha_a \cdot \boldsymbol{\sigma} + \frac{\theta}{N}\right), \tag{8}$$

where ζ is magnetic monopole fugacity which can be explicitly computed in this model using the conventional semiclassical approximation. The θ parameter enters the effective Lagrangian (8) as θ/N which is the direct consequence of the fractional topological charges of the monopoles (5). Nevertheless, the theory is still 2π periodic. This 2π periodicity of the theory is restored not due to the 2π periodicity of Lagrangian (8) as it was (incorrectly) claimed in the original Ref. [26]. Rather, it is restored as a result of summation over all branches of the theory when the levels cross at $\theta = \pi(\mathrm{mod}2\pi)$ and one branch replaces another and becomes the lowest energy state as presented in [28].

The dimensional parameter which governs the dynamics of the problem is the Debye correlation length of the monopole's gas,

$$m_\sigma^2 \equiv L\zeta\left(\frac{4\pi}{g}\right)^2. \tag{9}$$

The average number of monopoles in a "Debye volume" is given by

$$\mathcal{N} \equiv m_\sigma^{-3}\zeta = \left(\frac{g}{4\pi}\right)^3 \frac{1}{\sqrt{L^3\zeta}} \gg 1. \tag{10}$$

The last inequality holds since the monopole fugacity is exponentially suppressed, $\zeta \sim e^{-1/g^2}$, and in fact we can view (10) as a constraint on the region of validity where semiclassical approximation is justified. This parameter $\mathcal{N}$ measures the "semiclassicality" of the system.

It is convenient to express the action in terms of dimensionless variables as follows $x = x'/m_\sigma$ such that x' becomes a dimensionless coordinate. All distances now are measured in units of m_σ^{-1}. With this rescaling the action (8) assumes a very nice form:

$$S = \mathcal{N} \int_{\mathbb{R}^3} d^3x \sum_{n=1}^{N} \frac{1}{2}(\nabla\sigma_n)^2 - \mathcal{N} \int_{\mathbb{R}^3} d^3x \sum_{a=1}^{N} \cos\left(\sigma_n - \sigma_{n+1} + \frac{\theta}{N}\right), \tag{11}$$

with σ_{N+1} identified with σ_1. In formula (11) we used x as the dimensionless coordinate (rather than x') to simplify notations. The Lagrangian entering the action (11) is then dimensionless with a large semiclassical prefactor $\mathcal{N} \gg 1$ defined by (10).

III. CLASSIFICATION SCHEME OF THE VACUUM STATES

We start with a short overview of a well-known formal mathematical analogy between the construction of the $|\theta\rangle$ vacuum states in gauge theories and Bloch's construction of the allowed/forbidden bands in condensed matter (CM) physics (see e.g. [44]). The large gauge transformation operator $\mathcal{T}$ plays the role of the crystal translation operator in CM physics. $\mathcal{T}$ commutes with the Hamiltonian H and changes the topological sector of the system

$$\mathcal{T}|m\rangle = |m+1\rangle, \qquad [H,\mathcal{T}] = 0, \tag{12}$$

such that the $|\theta\rangle$-vacuum state is an eigenstate of the large gauge transformation operator $\mathcal{T}$:

$$|\theta\rangle = \sum_{m\in\mathbb{Z}} e^{im\theta}|m\rangle, \qquad \mathcal{T}|\theta\rangle = e^{-i\theta}|\theta\rangle.$$

The θ parameter in this construction plays the role of the "quasimomentum" $\theta \to qa$ of a quasiparticle propagating in the allowed energy band in a crystal lattice with unit cell length a.

An important element, which is typically skipped in presenting this analogy but which plays a key role in our studies is the presence of the Brillouin zones classified by integers k. Complete classification can be either presented in the so-called extended zone scheme where $-\infty < qa < +\infty$, or the reduced zone scheme where each state is classified by two numbers, the quasimomentum $-\pi \le qa \le +\pi$ and the Brillouin zone number k.

In the classification of the vacuum states, this corresponds to describing the system by two numbers $|\theta, k\rangle$, where θ is assumed to be varied in the conventional range $\theta \in [0, 2\pi)$, while the integer k describes the ground state (for $k = 0$) or the excited metastable vacuum states ($k \neq 0$). In most studies devoted to the analysis of the θ vacua, the questions related to the metastable vacuum states have not been addressed. Nevertheless, it has been known for some time that the metastable vacuum states must be present in non-Abelian gauge systems in the large N limit [1]. A similar conclusion also follows from the holographic description of QCD as originally discussed in [2].

In the present context the metastable vacuum states have been explicitly constructed in a weakly coupled "deformed QCD" model [39]. We follow this construction by keeping both: the metastable states as well as $\theta \neq 0$ states, such that our complete classification is $|\theta, m\rangle$ when the integer m describes the metastable states for a given $\theta \in [0, 2\pi)$. In terms of the CM physics we use the so-called reduced zone scheme, rather than the extended zone scheme as defined above.

The Euclidean potential density for the σ fields assumes the following form (11)

$$U(\boldsymbol{\sigma}, \theta) = \mathcal{N}\sum_{n=1}^{N}\left[1 - \cos\left(\sigma_n - \sigma_{n+1} + \frac{\theta}{N}\right)\right], \tag{13}$$

where we have added a constant term so that the potential is positive semidefinite. In Eq. (13) the field σ_{N+1} is identified with σ_1 as before.

The lowest energy state, is the state with all σ fields sitting at the same value ($\sigma_n = \sigma_{n+1}$) and has zero energy. This is clearly the true ground state of the system, but there are also potentially some higher energy metastable states even for $\theta = 0$. For an extremal state we must have

$$\frac{\partial U}{\partial \sigma_n} = 0, \tag{14}$$

for all n, which gives immediately

$$\sin\left(\sigma_n - \sigma_{n+1} + \frac{\theta}{N}\right) = \sin\left(\sigma_{n-1} - \sigma_n + \frac{\theta}{N}\right). \tag{15}$$

A necessary condition for a higher energy minimum of the potential is thus that the σ fields are evenly spaced around the unit circle or (up to a total rotation),

$$\sigma_n = m\frac{2\pi n}{N}, \tag{16}$$

where m is an integer which labels the metastable states in the extended classification scheme $|\theta, m\rangle$. This parameter plays the same role as the Brillouin zone number k in CM physics as discussed above. A sufficient condition is then

$$\frac{\partial^2 U}{\partial \sigma_n^2} > 0, \tag{17}$$

again for all n. This gives us

$$\cos\left(\sigma_n - \sigma_{n+1} + \frac{\theta}{N}\right) + \cos\left(\sigma_{n-1} - \sigma_n + \frac{\theta}{N}\right) > 0, \tag{18}$$

which using (16) gives

$$\cos\left(\frac{2\pi m}{N} - \frac{\theta}{N}\right) > 0. \tag{19}$$

This condition determines possible metastable states m for a given $\theta \in [0, 2\pi)$ and N. From (19) it is quite obvious that metastable states always exist for sufficiently large N even for $\theta = 0$, which is definitely consistent with old and very generic arguments [1]. In our simplified version of the theory one can explicitly see how these metastable states emerge in the system, and how they are classified in terms of the scalar magnetic potential fields $\boldsymbol{\sigma}(\mathbf{x})$ for arbitrary θ.

One should remark here that a nontrivial solution for $\theta = 0$ with $m \neq 0$ in (19) does not exist in the "deformed

QCD" model for the lowest $N = 2, 3, 4$ as it was originally discussed in [39]. However, for sufficiently large $\theta \neq 0$ the metastable states always emerge for $N \geq 3$, while $N = 2$, as usual, requires a special treatment [30]. What is more important is that Eq. (19) explicitly shows that at $\theta = \pi$ a metastable state with $m = 1$ becomes degenerate with the ground state with $m = 0$ and the level crossing phenomenon takes place precisely as it was originally described in [28] for this specific model. When θ further increases the metastable state becomes the lowest energy state of the system (13) for the given θ.

IV. OBLIQUE CONFINEMENT FOR $|\theta, m\rangle$ STATES

To understand the physical meaning of the solutions describing the nontrivial metastable vacuum states, one should compute the vacuum expectation value $\langle \mathcal{M}_a(\mathbf{x})\rangle$ of the magnetization for a given state $|\theta, m\rangle$ classified by two parameters m, θ as presented in previous Sec. III. The corresponding operator $\mathcal{M}_a(\mathbf{x})$ is defined as the creation operator of a single monopole of type α_a at point $\mathbf{x}$. It has been originally computed for the "deformed QCD" model in [28]. The corresponding computations can be easily generalized for arbitrary $\theta \neq 0$. The result of computations is

$$\mathcal{M}_a(\mathbf{x}) = e^{i(\alpha_a \cdot \boldsymbol{\sigma}(\mathbf{x}) + \frac{\theta}{N})}. \tag{20}$$

In the computation of (20) it has been assumed that the external magnetic source is infinitely heavy. If one identifies the corresponding magnetic source with the monopoles from the ensemble then the corresponding operator is accompanied by conventional classical contribution $8\pi^2/(g^2 N)$. Therefore, the resulting creation operator of a single monopole of type α_a at point $\mathbf{x}$ assumes the form

$$\mathcal{M}_a(\mathbf{x}) = e^{-\frac{8\pi^2}{g^2 N}} \cdot e^{i(\alpha_a \cdot \boldsymbol{\sigma}(\mathbf{x}) + \frac{\theta}{N})}. \tag{21}$$

This expression for the operator identically coincides for $N = 2$ with formula (63) derived in drastically different way by starting from $\mathcal{N} = 2$ supersymmetric model and breaking the supersymmetry.

Now we are in position to compute the vacuum expectation value $\langle \theta, m|\mathcal{M}_a(\mathbf{x})|\theta, m\rangle$ describing the magnetization of the system. It can be easily computed for each given state $|\theta, m\rangle$. Indeed, using the solutions (16), the magnetization assumes the form

$$\langle \theta, m|\mathcal{M}_a(\mathbf{x})|\theta, m\rangle \sim \exp\left[i\frac{\theta}{N} - i\frac{2\pi m}{N}\right], \tag{22}$$

where one should pick up a proper branch which satisfies condition (19) describing the lowest energy state.

A different, but equivalent way to describe all these $|\theta, m\rangle$ states is to compute the expectation values for the topological density operator for those states. By definition,

$$\begin{aligned}\langle \theta, m|\frac{1}{16\pi^2}\mathrm{tr}[F_{\mu\nu}\tilde{F}^{\mu\nu}]|\theta, m\rangle &\equiv -i\frac{\partial S_{\mathrm{dual}}(\theta)}{\partial\theta} \\ &= i\frac{\zeta}{L}\sin\left(\frac{2\pi m}{N} - \frac{\theta}{N}\right),\end{aligned} \tag{23}$$

where the dual action $S_{\mathrm{dual}}(\theta)$ is given by (8). The imaginary i in this expression should not confuse the readers as we work in the Euclidean space-time. In Minkowski space-time this expectation value is obviously a real number. A similar phenomenon is known to occur in the exactly solvable two dimensional Schwinger model wherein the expectation value for the electric field in the Euclidean space-time has an i. The expectation value (23) is the order parameter of a given $|\theta, m\rangle$ state.

As expected, the ground state with $m = 0$ at $\theta = 0$ the expectation value (23) vanishes, which of course, implies that the ground state respects $\mathcal{P}$ and $\mathcal{CP}$ symmetries. It is not the case for a generic states with $\theta \neq 0$. These symmetries are also broken for metastable states $m \neq 0$ even for $\theta = 0$ as emphasized in [39].

The fact that the confinement in this model is due to the condensation of fractionally charged monopoles has been known since the original paper [26]. Our original claim here is that the microscopical structure of the arbitrary $|\theta, m\rangle$ states can be also thought of as a condensate of the *same fractionally charged monopoles*. The only difference in comparison with the original construction [26] is that the corresponding magnetization receives a nontrivial phase (22) which depends on θ and integer number m which plays the same role as k-th Brillouin zone in the reduced classification scheme in CM physics.

Now we want to present a few additional arguments suggesting that the confinement in this system is indeed generated by the same magnetic monopoles with no trace for any dyons in this system which would provide a conventional "dyon mechanism". Indeed, the presence of the electrically charged dyons would imply that the interaction pattern between two Bogomol'ny-Prasad-Sommerfeld (BPS) dyons at distance r must have the following structure

$$\sim \frac{1}{4\pi r}\left[e^2 q \cdot q' + \left(\frac{2\pi}{e^2}\right) m \cdot m'\right]. \tag{24}$$

At the same time there is no trace for such kind of interaction in the original partition function which assumes the form [26]

$$e^{\frac{-2\pi^2 L}{g^2}\left[\sum_{a,b=1}^{N}\sum_{k=1}^{M^{(a)}}\sum_{l=0}^{M^{(b)}} \alpha_a \cdot \alpha_b Q_k^{(a)} Q_l^{(b)} G(\mathbf{x}_k^{(a)} - \mathbf{x}_l^{(b)})\right]}, \tag{25}$$

where $G(\mathbf{x}_k^{(a)} - \mathbf{x}_l^{(b)})$ is the corresponding Green's function. Precisely this interaction generates the dual action (8) which provides a proper low energy description of the system. One can explicitly see that there is no electric portion of the interaction in formula (25), in contrast with the anticipated structure expressed as (24) which is the conventional formula describing the interaction of two non-BPS dyons carrying simultaneously the magnetic and electric charges. In Sec. VI B we come back to this point re-emphasize it from a different perspective.

The argument presented above obviously dismisses the presence of the electric charge of the constituents. It also evidently raises the following question. How does the self-duality work in this case if the electric charges are not carried by the constituents? The answer is as follows: the BPS self-duality for the monopole's solutions is perfectly satisfied. However, the electric portion of the self-duality equation is due to the generation of the nontrivial holonomy rather than due to the electric charges of the dyons. Indeed, the self-duality equations for the monopoles assume the conventional form

$$D_i A_4^a = B_i^a, \qquad \langle A_4^{(a)} \rangle = \frac{2\pi}{NL}\mu^a, \qquad \mu^a \cdot \alpha_b = \delta_b^a, \tag{26}$$

which is precisely the key element in the original construction [26] when the holonomy $\langle A_4^{(a)} \rangle$ plays the role of the vacuum expectation value for the Higgs field.

Another related question can be formulated as follows. The topological charge operator is normally expressed as the product of the magnetic and electric fields, $q(\mathbf{x}) \sim \mathbf{E}^{(a)}(\mathbf{x}) \cdot \mathbf{B}^{(a)}(\mathbf{x})$. At the same time we claim that only magnetic monopoles are present in the system. These monopoles generate the oblique confinement, and saturate the vacuum expectation values (22) and (23). How does it work? The answer is as follows. The topological charge operator assumes the form

$$\int_{\mathbb{R}^3\times S^1} d^4x q(\mathbf{x}) = \int_{\mathbb{R}^3\times S^1} d^4x \frac{g}{4\pi^2}\sum_{a=1}^{N}\langle A_4^{(a)}\rangle[\nabla\cdot\mathbf{B}^{(a)}(\mathbf{x})]$$
$$= \int_{\mathbb{R}^3} d^3x \frac{1}{N}\sum_{a=1}^{N}\sum_{k=1}^{M^{(a)}} Q_k^{(a)}\delta(\mathbf{r}_k^{(a)}-\mathbf{x}), \tag{27}$$

where we integrated by parts and used formula (26) for the holonomy $\langle A_4^{(a)} \rangle$. One can explicitly see from (27) that the only constituents of the system are fractionally charged magnetic monopoles located at $\delta(\mathbf{r}_k^{(a)} - \mathbf{x})$ with zero electric charges as the corresponding sources are entirely determined by the divergence of the magnetic field $[\nabla \cdot \mathbf{B}^{(a)}(\mathbf{x})]$. In other words, there is no trace for the dyons to play any role in the system.[4]

[4]The electric field symbol, $\boldsymbol{E}^{(a)}$, that appears in the topological charge operator does not represent a genuine electric field irrespective of using the symbol $\boldsymbol{E}$. In fact, in the BPS limit (e.g. in $\mathcal{N} = 1$ supr Yang-Mills) this field mediates attractive force between similar charges; i.e., it plays the role of a dilaton or scalar field. As we break SUSY and go to the deformed pure Yang-Mills limit, the scalar field is gapped and we are left only with interactions due to the magnetic field, as given by Eq. (25); i.e., there are only objects that carry magnetic charges. Hence, there are no dyons. The same conclusion will be reached in Sec. VI.

Nevertheless, the gap is generated at $\theta \neq 0$, the confinement takes place in the conventional manner through the condensation of the monopoles (22), the θ parameter enters all the observables precisely as it should. This example explicitly shows that the conventional view that the confinement in gauge theories at $\theta \neq 0$ is a result of the condensation of the dyons cannot be correct, at least in this simplified "deformed QCD" model. Furthermore, as the transition between the "deformed QCD" model and strongly coupled gauge theories should be smooth, we expect that the picture presented above must hold in the strongly coupled regime as well. These results should be contrasted with the common lore which assumes that the oblique confinement at $\theta \neq 0$ is a result of condensation of the electrically charged dyons. In the next sections we consider supersymmetric models to understand the nature of this difference.

V. DYONS AND MONOPOLES IN $\mathcal{N} = 2$ SUPER YANG-MILLS

Dyons and monopoles are the main nonperturbative players in confinement in mass deformed $\mathcal{N} = 2$ super Yang-Mills on $\mathbb{R}^4$, as the monopole's condensation leads to the confinement of electric charge probes at $\theta = 0$ as originally discussed in [45]. It is commonly assumed that these monopoles at $\theta \neq 0$ become the dyons as a result of the Witten's effect [43]. The condensation of the dyons would lead to the oblique confinement speculated long ago by 't' Hooft [25]. On the other hand, it is the pure monopoles and not the dyons that lead to confinement in deformed Yang-Mills on $\mathbb{R}^3 \times \mathbb{S}^1$ as explained above. To elucidate this difference and track what really happens as we go from $\mathcal{N} = 2$ super Yang-Mills to deformed Yang-Mills on a circle, we start by reviewing the field contents of the former theory. We warn the reader that unlike in previous sections, now we care to distinguish between Minkowskian and Euclidean quantities. This is important to arrive at distinct conclusions.

$\mathcal{N} = 2$ super Yang-Mills theory has a massless $\mathcal{N} = 2$ hypermultiplet that contains four bosonic and four fermionic degrees of freedom; see e.g. the textbook [44]. Alternatively, one can decompose the $\mathcal{N} = 2$ multiplet into two $\mathcal{N} = 1$ mutiplets: vector and chiral mutiplets; both are in the adjoint representation of the gauge group. The bosonic part of the Lagrangian of $\mathcal{N} = 2$ super Yang-Mills is given by [in Minkowski space, where we work with the signature $\eta_{MN} = (+1, -1, -1, -1)$]

$$\mathcal{L} = \frac{1}{g^2}\mathrm{tr}\left[-\frac{1}{2}F_{MN}F^{MN} + D_M\phi^\dagger D^M\phi + \frac{1}{2}[\phi^\dagger, \phi]^2\right], \tag{28}$$

where $M, N = 0, 1, 2, 3$, the covariant derivative is $D_M = \partial_\mu + i[A_M,]$, the field strength is $F_{MN} = \partial_M A_N - \partial_N A_M + i[A_M, A_N]$, and the Lie algebra generators t^a are normalized as $\mathrm{tr}[t^a t^b] = \frac{\delta_{ab}}{2}$. For simplicity, we will mainly work with $SU(2)$ gauge group. Without loss of generality we can always choose the vev of ϕ to be along the Cartan generators, i.e. along the t^3 direction in the $SU(2)$ case. Therefore, we take $\phi = vt^3$ such that $SU(2)$ is broken down to $U(1)$, i.e., we are in the Coulomb branch, and the potential term $\mathrm{tr}[\phi^\dagger, \phi]^2$ vanishes, i.e., we are in the BPS limit. We can also take the gauge invariant field $u = \mathrm{tr}[\phi^2]$ to parametrize the moduli space of the gauge theory. The theory has a strong coupling scale Λ such that in the limit $u \gg \Lambda^2$ the theory is in the weakly coupled regime,[5] $g \ll 1$.

In the weakly coupled regime, both perturbative and nonperturbative spectra of the theory can be determined using semiclassical analysis. As we mentioned above, the theory is Higgsed down to $U(1)$, and therefore, the bosonic part of the perturbative spectrum consists of a massless photon and W-bosons of charges ± 1 with respect to the unbroken $U(1)$. Since we are in the BPS limit, the nonperturbative spectrum can be obtained via the Bogomol'nyi completion of the energy functional (see [46] for a review):

$$
\begin{aligned}
E &= \frac{1}{g^2}\int d^3x \mathrm{tr}[E_i^2 + B_i^2 + (D_0\phi)^2 + (D_i\phi)^2] \\
&= \frac{1}{g^2}\int d^3x \mathrm{tr}[(B_i \mp \cos\alpha D_i\phi)^2 + (E_i \mp \sin\alpha D_i\phi)^2 \\
&\quad + (D_0\phi)^2] \pm v\cos\alpha Q_M \pm v\sin\alpha Q_E \\
&\geq \pm v\cos\alpha Q_M \pm v\sin\alpha Q_E,
\end{aligned} \tag{29}
$$

where $E_i = F_{i0}$, $B_i = \frac{1}{2}\epsilon_{ijk}F^{jk}$, and we have used integration by parts and the Bianchi identity $D_iB_i = 0$. The magnetic and electric charges, Q_M and Q_E, are defined via

$$
\begin{aligned}
Q_M &= \frac{2}{g^2 v}\int d^2S_i \mathrm{tr}[\phi B_i], \\
Q_E &= \frac{2}{g^2 v}\int d^2S_i \mathrm{tr}[\phi E_i],
\end{aligned} \tag{30}
$$

where S_i is a two-sphere at spatial infinity. Similarly, one can define the scalar (dilaton) charge as

$$
Q_S = \frac{2}{g^2 v}\int d^2S_i \mathrm{tr}[\phi D_i\phi]. \tag{31}
$$

The most stringent inequality $E \geq v\sqrt{Q_E^2 + Q_M^2}$ is obtained by setting $\alpha = \tan^{-1}(\frac{Q_E}{Q_M})$. The equality is saturated by a configuration that satisfies the first order equations

$$
\begin{aligned}
B_i &= \pm\cos\alpha D_i\phi, \\
E_i &= \pm\sin\alpha D_i\phi, \\
D_0\phi &= 0.
\end{aligned} \tag{32}
$$

Equations (32), with the upper sign, are solved by the ansatz:

$$
\begin{aligned}
A_i^a &= \epsilon_{iam}\hat{r}^m\left[\frac{1-u(r)}{r}\right], \\
A_0^a &= \hat{r}^a j(r), \qquad \phi^a = \hat{r}^a h(r).
\end{aligned} \tag{33}
$$

Substituting (33) into (32) one finds the solution

$$
\begin{aligned}
u(r) &= \frac{\tilde{v}r}{\sinh(\tilde{v}r)}, \\
h(r) &= \frac{\sqrt{Q_M^2 + Q_E^2}}{Q_M}\left[\tilde{v}\coth(\tilde{v}r) - \frac{1}{r}\right], \\
j(r) &= -\frac{Q_E}{Q_M}\left[\tilde{v}\coth(\tilde{v}r) - \frac{1}{r}\right],
\end{aligned} \tag{34}
$$

where $\tilde{v} = v\frac{Q_M}{\sqrt{Q_E^2 + Q_M^2}}$. Equations (34) constitute Julia-Zee dyon-particle [47]. This configuration has a total energy (mass)

$$
E = M = v\sqrt{Q_M^2 + Q_E^2}. \tag{35}
$$

In the limit $r \to \infty$ one can use (33) and (34) to show that

$$
E_i \sim \frac{Q_E x_i}{r^3}, \qquad B_i \sim \frac{Q_M x_i}{r^3}, \qquad D_i\phi \sim -\frac{Q_S x_i}{r^3}, \tag{36}
$$

where $Q_S = \frac{Q_M}{\cos\alpha} = \sqrt{Q_E^2 + Q_M^2}$. Thus, the dyon mass satisfies the relation

$$
M = vQ_S. \tag{37}
$$

Using (34) in the energy functional (29) we obtain the interaction energy of two BPS dyon-particles with charges (Q_M, Q_E) and (Q'_M, Q'_E) and located at $\boldsymbol{r}$ and $\boldsymbol{r}'$:

$$
E_{\mathrm{int}} \sim g^2\frac{Q_E Q'_E + Q_M Q'_M - Q_S Q'_S}{|\boldsymbol{r} - \boldsymbol{r}'|}. \tag{38}
$$

We see that, as expected, the interaction force of the $U(1)$ field is repulsive, while the dilaton field is attractive [48].

Dyons are genuine particles[6] that carry both electric and magnetic charges. Classically, a dyon can have an arbitrary electric charge, while it can only have quantized magnetic charge $Q_M = \frac{4\pi n}{g^2}$, where n is a positive or negative integer,

[5] To one-loop order we have $\frac{4\pi}{g^2} = \frac{2}{\pi}\log\frac{v}{\Lambda}$.

[6] More precisely, they are solitons.

due to obvious topological reasons. However, quantum mechanical consistency demands that a pair of dyon-particles with charges (Q_M, Q_E) and (Q'_M, Q'_E) must satisfy the Dirac quantization condition $Q_E Q'_M - Q'_E Q_M = n\frac{4\pi}{g^2}$, where n is an integer. Therefore, both electric and magnetic charges must be quantized: $(Q_M, Q_E) = (\frac{4\pi n_M}{g^2}, n_E)$, where $n_M, n_E \in \mathbb{Z}$, and we find that the BPS spectrum is given by

$$M(n_M, n_E) = v\sqrt{n_M^2\left(\frac{4\pi}{g^2}\right)^2 + n_E^2}. \quad (39)$$

The BPS masses $M(n_M, n_E)$ do not receive quantum corrections, thanks to the high level of supersymmetry in $\mathcal{N}=2$ super Yang-Mills. In addition, one can take into account the effect of the θ-vacuum, $\frac{\theta}{32\pi^2}\tilde{F}_{MN}F^{MN}$, by making the substitution $n_E \to n_E + n_M\frac{\theta}{2\pi}$, which is Witten's effect [43]. One finally finds:

$$M(n_M, n_E, \theta) = v\sqrt{n_M^2\left(\frac{4\pi}{g^2}\right)^2 + \left(n_E + n_M\frac{\theta}{2\pi}\right)^2}. \quad (40)$$

We could also set $n_E = 0$, which is the limiting case of 't' Hooft Polyakov monopole particles. However, a single monopole has four collective coordinates: three translation coordinates and one coordinate corresponding to a $U(1)$ global transformation. The $U(1)$ collective coordinate is compact [remember that $U(1)$ is descendent from $SU(2)$, which is a compact group]. The Hamiltonian corresponding to the compact coordinate is $H_{U(1)} = \frac{p_\phi^2}{2I}$, with $I = \frac{4\pi}{g^2 v}$. Upon quantization, the magnetic monopole acquires an electric charge; this is one of the eigenvalues of $H_{U(1)}$. Thus, quantum fluctuations in the background of an 't' Hooft Polyakov monopole dresses it with a quantized electric charge and gives rise to a dyon, with its mass given by the BPS expression (40).

While the spectrum (40) is a well-established feature of $\mathcal{N}=2$ supersymmetry in weakly coupled regime at large v, the role of these dyons in confined strongly coupled regime is less understood. We review below some features of the system relevant for our studies by paying special attention to the dyon-particles. Precisely these degrees of freedom, according to the conventional wisdom, should condense at $\theta \neq 0$ and provide a precise realization for the oblique confinement as envisaged by 't' Hooft [25].

As we approach the strong coupling regime of the theory, $v \sim \Lambda$, most of the dyon-particles decay except the ones with lowest charges (1,0) or (1,1), which become massless. This theory is electrically strongly coupled and magnetically weakly coupled. Therefore, the theory can be described by a dual $\mathcal{N}=2$ supersymmetric electrodynamic of massless monopoles or dyons.

Now we insert a small mass term into the action with $m \ll \Lambda$. It breaks the symmetry from $\mathcal{N}=2$ down to $\mathcal{N}=1$. One could naively think that the oblique confinement might take place as a result of the dyon condensation. However, the oblique confinement does not occur, at least in weakly coupled regime [49,50]. The basic reason for that is that the "pure monopoles" rather than dyons condense at both points $u = \pm\Lambda^2$ as argued in [49,50]. This is in spite of the fact that near $u = -\Lambda^2$ the dyons (1,1) rather than monopoles (1,0) become massless particles. The absence of the oblique confinement in the system is obviously consistent with our analysis of the "deformed QCD" model in Sec. IV. However, one cannot make a definite conclusion with a large supersymmetry breaking in this construction when the question on oblique confinement remains open [49,50]. It should be contrasted with results of Sec. IV where the transition to strongly coupled regime of ordinary QCD is expected to be smooth as argued in Sec. I B.

One can also insert N_f flavours into the system [51]. It turns out that the oblique confinement occurs for $N_f = 3$ model, but does not occur for $N_f = 2$ nor for $N_f = 1$ models. All the arguments of Refs. [49–51] are crucially depend on the specific properties of supersymmetric theories. Therefore, it is not obvious if one can learn any lessons for ordinary QCD. With this motivation in mind we consider the Seiberg-Witten model being formulated on $\mathbb{R}^3 \times \mathbb{S}^1$ when one can approach the weakly coupled regime by varying the size of $\mathbb{S}^1$.

VI. DYON-INSTANTONS VS MONOPOLE-INSTANTONS ON $\mathbb{R}^3 \times \mathbb{S}^1$

In this section we show that the nonperturbative sector of $\mathcal{N}=2$ on $\mathbb{R}^3 \times \mathbb{S}^1$ consists of a variety of dyons, similar to our previous discussions. However, in this section we consider the Euclidean, rather than Minkowski formulation. Therefore we compute the Euclidean action generated by the pseudoparticles, rather than particles. To avoid confusion with terminology we coin the corresponding pseudoparticles the dyon-instantons to emphasize their Euclidean nature. We will show that for sufficiently small $\mathbb{S}^1$ circles in the weakly coupled regime where our computations are under control, the partition function is saturated by the tower of monopole-instantons rather than dyon-instantons. In our presentations we closely follow [27,52–55].

A. $\mathbb{R}^3 \times \mathbb{S}^1$: the large circle limit

We start our treatment by compactifying the x^3-direction over a circle and considering the Euclidean version of the theory. The Euclidean time direction will be denoted by x_4 such that $x_4 \equiv ix^0$, while the rest of coordinates are left intact. We also define the Euclidean fields $\hat{A}_i = -A^i$, and $\hat{A}_4 = -iA_0$. Again, we assume that $v \gg \Lambda$, and hence, the theory is in its semiclassical regime. The Euclidean action is given by

$$S_E = \frac{1}{g^2}\int_{\mathbb{R}^3\times\mathbb{S}^1} \mathrm{tr}\left[\frac{1}{2}\hat{F}_{MN}\hat{F}_{MN} + (\hat{D}_M\phi)^2\right] = \frac{1}{g^2}\int_{\mathbb{R}^3\times\mathbb{S}^1} \mathrm{tr}[(\tilde{E}_\mu)^2 + (\tilde{B}_\mu)^2 + (\tilde{D}_3\phi)^2 + (\hat{D}_\mu\phi)^2], \tag{41}$$

where M, $N = 1, 2, 3, 4$, $\hat{F}^a_{MN} = \partial_M\hat{A}^a_N - \partial_N\hat{A}^a_M + f^{abc}\hat{A}^b_M\hat{A}^c_N$, $\hat{D}_M\phi^a = \partial_M\phi^a + f^{abc}\hat{A}^b_M\phi^c$, where f^{abc} are the group structure constants. We also defined $\tilde{E}_\mu = \hat{F}_{\mu 3}$, $\tilde{B}_\mu = \frac{1}{2}\epsilon_{\mu\nu\alpha}\hat{F}_{\nu\alpha}$, where $\mu, \nu = 1, 2, 4$. Notice that here we distinguish between the electric and magnetic fields $\hat{E}_i$, $\hat{B}_i$, where $i = 1, 2, 3$, and $\tilde{E}_\mu$ and $\tilde{B}_\mu$. Although not mandatory, this distinction is convenient since it will enable us to keep track of various quantities. Comparing the Euclidean action (42) with the energy functional (29), we immediately reveal that a finite action solution can be obtained using the exact same procedure we followed to obtain Julia-Zee dyon-particles. The existence of a finite action solution demands that the fields profiles are independent of x_3 (exactly like the dyon-particle solution is independent of x^0). One then can think of this solution as wrapping around the x_3-direction, and hence, in the Euclidean setup we obtain *dyon-instantons*[7] to be contrasted with *dyon-particles* considered in Sec. V. Taking the length of the $\mathbb{S}^1$ circle to be L, we immediately find that the action of the BPS dyon-instanton is given by

$$S(n_M, n_E, \theta) = LM(n_M, n_E, \theta) = Lv\sqrt{n_M^2\left(\frac{4\pi}{g^2}\right)^2 + \left(n_E + n_M\frac{\theta}{2\pi}\right)^2}. \tag{42}$$

In addition, two BPS dyon-instantons carrying charges (Q_M, Q_E) and (Q'_M, Q'_E) and located at $\boldsymbol{r}$ and $\boldsymbol{r}'$ in the Euclidean space will interact as in (38):

$$S_{\mathrm{int}} \sim g^2\frac{Q_E Q'_E + Q_M Q'_M - Q_S Q'_S}{|\boldsymbol{r} - \boldsymbol{r}'|}, \tag{43}$$

where the Euclidean radial coordinate is defined as $r = \sqrt{x_1^2 + x_2^2 + x_4^2}$, such that the profile functions in (33) now depend on the newly defined r. Since dyon-instantons have a finite action, they will contribute to the Euclidean partition function. On $\mathbb{R}^3 \times \mathbb{S}^1$ the gauge potential $\hat{A}_3$ is a compact scalar with period L: $\hat{A}_3 \cong \hat{A}_3 + 1/L$. For convenience let us define ω as

$$\hat{A}_3 \equiv \omega/L. \tag{44}$$

In addition, we can go to a dual description such that

$$\hat{F}_{\mu\nu} = \frac{g^2}{2\pi L}\epsilon_{\mu\nu\alpha}\partial_\alpha\sigma. \tag{45}$$

Again, one can show that σ is a compact scalar with period 2π. Let us also define $\Phi = \phi/L$. Then, in terms of the scalars ω, σ, and Φ, the insertion of a dyon-instanton in the partition function can be represented by the vertex:

$$\mathcal{D} = e^{-S(n_M, n_E, \theta)} \times e^{i(n_E + \frac{\theta}{2\pi}n_M)\omega + in_M\sigma + \sqrt{n_M^2(\frac{4\pi}{g^2})^2 + (n_E + \frac{\theta}{2\pi}n_M)^2}\Phi} \times \text{fermion zero modes}. \tag{46}$$

In the absence of fermion zero modes one can easily show that this vertex will also reproduce the interaction (43) such that ω mediates the force between Q_E charged objects, σ mediates the force between Q_M charged objects, while ϕ mediates the force between Q_S charged objects.[8] This interaction is repulsive for both ω and σ fields [since both ω and σ are parts of the electromagnetic $U(1)$ field; notice the imaginary number i in front of these fields], while it is attractive for Φ (a scalar field; notice the absence of i in front of it).

A key point of this subsection is that the dyons are generic configurations of the system. The interaction pattern (43) obviously shows that they would be genuine static dyons if one treats the Euclidean x_3 coordinate as a time variable. Based on these configurations one could naively think that oblique confinement should occur as a result of the condensation of the dyons as the generic gauge configurations of the system. Nevertheless, as we demonstrate next in Sec. VI B this naive picture is incorrect: if one proceeds with computations in a theoretically controllable way, the confinement occurs as a result of the monopole's (not dyon's) condensation for arbitrary $\theta \neq 0$, similar to our analysis in Sec. IV in "deformed QCD" model.

B. $\mathbb{R}^3 \times \mathbb{S}^1$: the small circle limit

Our goal now is to consider the small circle limit where computations can be carried out in a theoretically controllable way. With this goal in mind we ignore the fermion

[7]One should not confuse these dyon-instantons with the gauge configurations considered in Refs. [56–58], which were (incorrectly) coined as the dyons or dyon-instantons. Those configurations representing the instanton constituents from Refs. [56–58] do not carry the electric charges and should be considered as monopoles-instantons in our classification scheme.

[8]This can be shown by writing the Abelian part of the kinetic term $F_{MN}F_{MN} + (D_M\phi)^2$ in terms of the three-dimensional fields ω, σ, and Φ:

$$\text{K.E.} = \frac{1}{2g^2}(\partial_\mu\omega)^2 + \frac{1}{2g^2}(\partial_\mu\Phi)^2 + \frac{g^2}{8\pi^2}(\partial_\mu\sigma)^2, \tag{47}$$

which is derived in Sec. (VII). Then we insert the vertex $\mathcal{D}(x_1)$ in the partition function and solve for the quadratic Lagrangian to obtain expression (43).

zero modes[9] and consider a tower of dyon-instantons with a unit magnetic charge and an arbitrary number of electric charges $(Q_M, Q_E) = (\frac{4\pi}{g^2}, n_E)$:

$$\mathcal{S} = \sum_{n_E \in \mathbb{Z}} e^{-S(1,n_E,\theta)} e^{i(n_E + \frac{\theta}{2\pi})\omega + i\sigma + \sqrt{(\frac{4\pi}{g^2})^2 + (n_E + \frac{\theta}{2\pi})^2}\Phi}. \quad (48)$$

Then, one can approximate the sum in (48) as

$$\mathcal{S} \cong e^{-\frac{4\pi(Lv-\Phi)}{g^2} + i\sigma} \sum_{n_E \in \mathbb{Z}} e^{i(n_E + \frac{\theta}{2\pi})\omega - \frac{g^2 Lv}{8\pi}(n_E + \frac{\theta}{2\pi})^2}. \quad (49)$$

A fast convergence of the series demands that $Lv \gg \frac{4\pi}{g^2} \gg 1$. Therefore, for a very large circle the sum will rapidly converge. For a small circle, however, the series is poorly convergent and a method of resummation is indispensable. To achieve this, we use the Poisson resummation formula defined as:

$$\sum_{n_E \in \mathbb{Z}} f(n_E) = \sum_{n_W \in \mathbb{Z}} \tilde{f}(n_W),$$
$$\tilde{f}(n_W) = \int dk f(k) e^{-2\pi n_W k}. \quad (50)$$

Applying this method to the series (48) we find [52], modulo pre-exponential factor,

$$S \cong \sum_{n_W \in \mathbb{Z}} e^{i\sigma - \frac{4\pi}{g^2}\sqrt{(Lv-\Phi)^2 + (\omega + 2\pi n_W)^2} + i n_W \theta}. \quad (51)$$

In the limit $vL \gg \omega$ we obtain the approximation

$$S \cong e^{-\frac{4\pi(Lv-\Phi)}{g^2} + i\sigma} \sum_{n_W \in \mathbb{Z}} e^{-\frac{2\pi}{g^2 Lv}(\omega + 2\pi n_W)^2 + i n_W \theta}. \quad (52)$$

The series (52) is rapidly convergent[10] in the small $\mathbb{S}^1$ limit $vL \ll 1$. A careful inspection of (51) reveals that the sum is over a tower of twisted monopole-instantons that carry magnetic charges ± 1 and winding numbers $n_W \in \mathbb{Z}$, as we show in details at the end of this section. This is a remarkable result since in the small circle limit we can think only about monopole-instantons instead of dyon-instantons. This claim holds for any $\theta \neq 0$, as is evident from (51).

Let us now make the shift $\omega \to \omega + \Omega$, where $0 < \Omega < 2\pi$ is a background holonomy (remember that ω is the scaled $\hat{A}_3$ component), in (51). It will suffice to consider only the two terms $n_W = 0$ and $n_W = -1$. For small fluctuations of Φ and ω the terms $n_W = 0$ and $n_W = -1$ are given by

$$\mathcal{M}_0 = e^{-S_0} e^{i\sigma + \frac{4\pi}{g^2}\frac{\Omega(\omega - \frac{Lv}{\Omega}\Phi)}{\sqrt{(Lv)^2 + \Omega^2}}},$$
$$\mathcal{M}_{-1} = e^{-S_{-1}} e^{i\sigma - i\theta + \frac{4\pi}{g^2}\frac{(\Omega - 2\pi)(\omega - \frac{Lv}{\Omega - 2\pi}\Phi)}{\sqrt{(Lv)^2 + (\Omega - 2\pi)^2}}}, \quad (53)$$

where $S_0 = \frac{4\pi\sqrt{(Lv)^2 + \Omega^2}}{g^2}$ and $S_{-1} = \frac{4\pi\sqrt{(Lv)^2 + (\Omega - 2\pi)^2}}{g^2}$ are, respectively, the actions of BPS and twisted (or Kaluza-Klein) monopole-instantons. Notice that both monopole-instantons have positive magnetic charges, $n_M = 1$, as is evident from the same sign in front of $i\sigma$ in (53). This should be expected since the series (52) originated from the sum over a tower of dyon-instantons all having the same magnetic charge $n_M = 1$. Also, the imaginary number in front of σ means that objects carrying the same magnetic charges will experience a repulsive force, which is also expected. The interesting thing, though, is the absence of i in front of ω and Φ, which means that the combination of the fields $\omega - \frac{Lv}{\Omega}\Phi$ or $\omega - \frac{Lv}{\Omega - 2\pi}\Phi$ mediates a scalar force rather than an electromagnetic one. This is a fascinating phenomenon since we start with a tower of dyon-instantons at large $\mathbb{S}^1$. The dyon-instantons experience a repulsive electromagnetic force (for both electric ω and magnetic σ parts) as in (43), in addition to a scalar force (mediated by the scalar field Φ). Then, we resum over the electric charges of this tower, using the Poisson resummation formula, to find that at a small $\mathbb{S}^1$ the electric force is incarnated as a scalar force. Using Q_b to denote the charge of the monopole-instanton under any of these combinations, we find that two monopoles carrying charges (Q_M, Q_b) and (Q'_M, Q'_b) and located at $\boldsymbol{r}$ and $\boldsymbol{r}'$ experience a force

$$S_{\rm int} \sim \frac{Q_M Q'_M - Q_b Q'_b}{|\boldsymbol{r} - \boldsymbol{r}'|}. \quad (54)$$

In particular, for the BPS and twisted monopole-instantons we have $|Q_M| = |Q_b| = \frac{4\pi}{g^2}$. This formula obviously shows that the only configurations which contribute to the partition function in the regime, where computations are under complete theoretical control, are the monopoles and twisted monopoles, but not the dyons carrying the electric charges.

The last element which completes our analysis of this subsection is the demonstration that the sum in (52) is indeed over a tower of twisted monopole-instantons. To this end, we set $\hat{D}_3\phi = 0$ in (42) [notice that this is exactly compatible with the third equation in (32)] and we also set $\partial_3 = 0$. This at least is enough to obtain

[9]Fermion zero modes in the duality we consider below is a subtle issue that has yet to be understood, see [27,52].

[10]Remember that we are still in the semiclassical regime $g \ll 1$. Thus, the series (52) is valid in the parameter range $\frac{4\pi}{g^2} \gg Lv \gg \omega$.

the zero-winding number (BPS) monopole-instanton. Monopoles with higher winding numbers (twisted monopoles) can be obtained by replacing $\hat{A}_3 \to \hat{A}_3 + \frac{2\pi n}{L}$. The lowest (zero-winding) monopole-instanton action reads

$$\begin{aligned} S_E &= \frac{2}{g^2}\int_{\mathbb{R}^3\times\mathbb{S}^1} \mathrm{tr}[(\hat{D}_\mu A_3)^2 + (\hat{B}_\mu)^2 + (\hat{D}_\mu\phi)^2] \\ &= \frac{2}{g^2}\int \mathrm{tr}[(\hat{D}_\mu\hat{A}_3 \mp \sin\beta\tilde{B}_\mu)^2 + (\hat{D}_\mu\phi \mp \sin\beta\tilde{B}_\mu)^2 \\ &\quad \pm 2\sin\beta\hat{D}_\mu\hat{A}_3\tilde{B}_\mu \pm 2\cos\beta\hat{D}_\mu\phi\tilde{B}_\mu] \\ &\geq LQ_M\left[\pm 2\frac{\Omega}{L}\sin\beta \pm 2v\cos\beta\right], \end{aligned} \tag{55}$$

where v and $\frac{\Omega}{L}$ are respectively the vevs of ϕ and $\hat{A}_3$ and the vevs are taken along the fourth direction in color space[11] We also defined the magnetic charge: $Q_M = \frac{1}{g^2}\int dS_\mu \hat{B}^4_\mu$, where the integral is over a two-sphere at infinity. The most stringent inequality $S_E \leq Q_M\sqrt{L^2v^2 + \Omega^2}$ is obtained by setting $\tan\beta = \frac{\Omega}{Lv}$, while the inequality is saturated by

$$\begin{aligned} \hat{D}_\mu\phi &= \pm\hat{B}_\mu\cos\beta, \\ \hat{D}_\mu A_3 &= \pm\hat{B}_\mu\sin\beta. \end{aligned} \tag{56}$$

A linear superposition of (56) can be written as

$$\begin{aligned} \hat{B}_\mu &= \pm\hat{D}_\mu\Psi_1, \\ 0 &= \hat{D}_\mu\Psi_2, \end{aligned} \tag{57}$$

where

$$\begin{aligned} \hat{\Psi}_1 &= \sin\beta\hat{A}_3 + \cos\beta\phi \\ \hat{\Psi}_2 &= \cos\beta\hat{A}_3 - \sin\beta\phi. \end{aligned} \tag{58}$$

Equation (57) is the (anti)self-dual BPS monopole-instanton equation. The solution of the self-dual equation is

$$\begin{aligned} \hat{A}^a_\mu &= \epsilon_{\mu a\nu}\hat{x}^\nu\left[\frac{1-u(r)}{r}\right], \\ \hat{\Psi}^a_1 &= \hat{x}h(r), \\ \hat{\Psi}_2 &= 0, \end{aligned} \tag{59}$$

where $r = \sqrt{x_1^2 + x_2^2 + x_4^2}$ and

$$\begin{aligned} u(r) &= \frac{\tilde{v}r}{\sinh(\tilde{v}r)}, \\ h(r) &= \tilde{v}\left[\cosh(\tilde{v}r) - \frac{1}{r}\right], \end{aligned} \tag{60}$$

and $\tilde{v} = \sqrt{\frac{\Omega^2}{L^2} + v^2}$.

Now, to obtain the twisted-monopole solutions we just need to make the substitution $\Omega \to \Omega + 2\pi n_W$ for all integers n_W. Thus, the action of the twisted monopole-instantons with magnetic charge $\frac{4\pi}{g^2}$ is given by

$$S_{n_W} = \frac{4\pi}{g^2}\sqrt{L^2v^2 + (\Omega + 2\pi n_W)^2}, \tag{61}$$

which is exactly the action in the sum (51).

The main lesson to be learnt from these computations is as follows. The generic gauge configurations of the system obviously include the dyons. However, if one tries to compute the partition function in a theoretically controllable region, the corresponding configurations can be described exclusively in terms of the monopoles, without any trace of the dyons. It is perfectly consistent with our analysis of the "deformed QCD" model in Sec. IV, where confinement is generated for any $\theta \neq 0$ as a result of condensation of the monopoles. In Sec. VI C we show that the picture also holds when supersymmetry is broken.

C. Supersymmetry breaking

In order to break supersymmetry in a controlled way we first add a suitable scalar mass term m_ϕ for the field ϕ and its super partner. In the limit $m_\phi \gg \Lambda$ the scalar decouples, which in turn breaks $\mathcal{N} = 2$ down to $\mathcal{N} = 1$. If this decoupling happens in the large $\mathbb{S}^1$ limit, then the theory flows to strong coupling regime, we loose theoretical control, and the dyon-instantons picture is no longer trusted. However, if the decoupling happens at a small $\mathbb{S}^1$, then the theory stays in its weakly coupled regime and preserves its center symmetry, i.e., $\Omega = \pi$. Setting $v = 0$ (since the scalar ϕ decouples), defining $b = \frac{4\pi}{g^2}\omega$, and shifting $\sigma \to \sigma + \frac{\theta}{2}$, we find that the monopole-instanton operators (53) are given by

$$\mathcal{M}_{0,1} = e^{-S_m}e^{i\sigma\pm(b+i\frac{\theta}{2})}, \tag{62}$$

and $S_m = \frac{4\pi^2}{g^2}$.

In order to further break $\mathcal{N} = 1$ we give the gaugino a mass larger than the strong scale. Again, we can guarantee that the theory is in the weakly coupled regime as long as the circle is kept sufficiently small. Preserving the center symmetry, however, requires that we add a double trace deformation. This theory is our "deformed QCD" model though in this case it represents pure gauge Yang-Mills

[11]Remember that we are in a Euclidean setup where our infinite dimensions are taken along x_1, x_2, x_4. Given our numbering convention, then we also take the color space index a to run over 1,2,4, where the diagonal Pauli matrix is taken along the 4-direction.

fields, see footnote in the Introduction regarding this terminological convention. In this case the scalar field b is gapped and we end with the monopole operators:

$$\mathcal{M}_{0,1} = e^{-S_m} e^{i\sigma \pm i\frac{\theta}{2}}. \tag{63}$$

This expression identically coincides with formula (21) for "deformed QCD" model derived in a drastically different way.

We conclude this section with the following generic comment. In all cases when the computations can be performed in a theoretically controllable way, the gauge configurations which saturate the partition function are the monopole-instantons for arbitrary $\theta \neq 0$. This claim holds for $\mathcal{N} = 2$, $\mathcal{N} = 1$, and the nonsupersymmetric "deformed QCD" model. This result should be contrasted with conventional wisdom that the oblique confinement in the system for $\theta \neq 0$ occurs as a result of the condensation of the electrically charged dyons.

VII. WITTEN'S EFFECT

Since there is no trace of dyons in the spectrum of theories on $\mathbb{R}^3 \times \mathbb{S}^1$ in the small circle limit, one may wonder how Witten's effect is realized in this case. The answer is that this effect can be seen for static (non dynamical) electric or magnetic charges, i.e. Wilson or 't' Hooft loops, which we use to probe the system [59]. In order to show this explicitly, we start from the Abelian action written in Minkowski space:

$$\begin{aligned} S &= \int_{\mathbb{R}^3 \times \mathbb{S}^1} \frac{1}{4g^2} F_{MN} F^{MN} + \frac{\theta}{32\pi^2} F_{MN} \tilde{F}_{MN} \\ &= \int_{\mathbb{R}^3 \times \mathbb{S}^1} \frac{1}{2g^2} (E_\mu E_\mu - B_\mu B_\mu) - \frac{\theta}{8\pi^2} E_\mu B_\mu, \end{aligned} \tag{64}$$

where $\tilde{F}_{MN} = \epsilon_{MNPQ} F^{PQ}/2$. Next, we dimensionally reduce the action (64) by neglecting all dependence on the x^3-direction and use the fields ω and σ defined via (44) and (45) (now in Minkowski space) to find[12]

$$F^{\nu\rho} = \frac{g^2}{2\pi L} \epsilon^{\mu\nu\rho} \left(\partial_\mu \sigma + \frac{\theta}{2\pi} \partial_\mu \omega \right), \tag{65}$$

and

$$S = -\frac{1}{2L} \int d^3x \frac{1}{g^2} (\partial_\mu \omega)^2 + \frac{g^2}{4\pi^2} \left(\partial_\mu \sigma + \frac{\theta}{2\pi} \partial_\mu \omega \right)^2. \tag{66}$$

From (65) we easily find (keeping in mind that the Greek letters run over 0,1,2, while the Latin letters M, N run over 0,1,2,3)

$$\begin{aligned} B_1 &= \frac{\partial_2 \omega}{L}, \qquad B_2 = -\frac{\partial_1 \omega}{L}, \\ B_3 &= \frac{g^2}{2\pi L} \left(\partial_t \sigma + \frac{\theta}{2\pi} \partial_t \omega \right), \\ E_1 &= -\frac{g^2}{2\pi L} \left(\partial_2 \sigma + \frac{\theta}{2\pi} \partial_2 \omega \right), \\ E_2 &= \frac{g^2}{2\pi L} \left(\partial_1 \sigma + \frac{\theta}{2\pi} \partial_1 \omega \right), \\ E_3 &= -\frac{\partial_t \omega}{L}. \end{aligned} \tag{67}$$

A Wilson loop operator that measures the magnetic flux in the $y - z$ plane and warps around the $\mathbb{S}^1$ circle is given by

$$\mathcal{W}(\mu_e) = e^{i\mu_e \oint \mathbf{A} \cdot d\ell} = e^{i\mu_e \int_{y_1}^{y_2} dy \int_0^L dz B_1} \to e^{i\mu_e \omega(x,y)}, \tag{68}$$

where μ_e is the electric charge of the Wilson line probe and we used (67). Also, the 't' Hooft loop operator that measures the electric flux penetrating the $y - z$ plane is given by

$$\mathcal{T}(\mu_m, \theta) = e^{-i\mu_m \frac{2\pi}{g^2} \int_{y-z} ds n^1 E_1} \to e^{i\mu_m (\sigma(x,y) + \frac{\theta}{2\pi} \omega(x,y))}, \tag{69}$$

and μ_m is the probe magnetic charge.

Now, starting with a pure 't' Hooft operator at $\theta = 0$, we find upon sending $\theta \to \theta + 2\pi$

$$\mathcal{T}(\mu_m, \theta \to \theta + 2\pi) = \mathcal{T}(\mu_m, \theta = 0) \mathcal{W}(\mu_m); \tag{70}$$

i.e. the magnetic probe acquires an electric charge μ_m. This is Witten's effect at work.

The main lesson to be learnt here is that the static monopole considered as the *external source* becomes the electrically charged dyon, in full agreement with the Witten's effect [43]. However, when the monopoles become the dynamical degrees of freedom and themselves determine the ground state of the ensemble they remain pure magnetic monopoles as demonstrated in Sec. VI.

At this point one may wonder how and why the θ parameter remains to be an observable parameter of the theory when exclusively Abelian gauge fields are present in the system. Indeed, normally we assume that the θ parameter in Maxwell Abelian QED is not physical because the θ term in Maxwell QED can be expressed as the total derivative which can be removed from the action due to the triviality of the topology. The key point relevant for our present discussions is that Witten's effect for the Abelian magnetic monopole is operational because the monopole itself determines the nontrivial topology and the θ parameter becomes the physical parameter in QED in the nontrivial topological (not vacuum) sector determined by the monopole's charge.

[12]The duality relation (45) can be incorporated into the action (64) using the auxiliary action $\frac{1}{4\pi} \int d^3x \epsilon_{\mu\nu\alpha} \partial_\mu \sigma F_{\nu\alpha}$.

A similar effect when θ becomes a physically observable parameter also holds for Maxwell QED when the external magnetic flux selects a nontrivial topological sector of the system, as argued in [60]. This effect, in fact, represents a novel idea on the axion search experiments when the system is sensitive to θ itself, rather than to $\partial_\mu \theta$ as in conventional axion search experiments.

In the context of the present work these arguments make it clear that the *external* magnetic monopoles become the electrically charged dyons in the presence of $\theta \neq 0$ in the given topological winding sector determined by the external magnetic charge itself. The *dynamical* magnetic monopoles remain pure monopoles as they cannot select the topological winding sector for the entire system. Precisely these *dynamical* monopoles condense and determine the ground state of the system. This interpretation is perfectly consistent with our conclusion at the end of Sec. VI that the confinement in supersymmetric and nonsupersymmetric theories at $\theta \neq 0$ is due to the condensation of the same magnetic monopoles, rather than dyons.

VIII. CONCLUSION

The main claim of the present work can be formulated as follows. We showed that the confinement in the gauge systems with $\theta \neq 0$ is a result of the condensation of the same monopole's configurations which generate the confinement at $\theta = 0$. It should be contrasted with a conventional lore that the confinement at $\theta \neq 0$ is a result of the condensation of the dyons.

The θ parameter is obviously a physical parameter of the system since all other observables, including the vacuum energy, are explicitly dependent on θ. Furthermore, $\mathcal{CP}$ invariance is explicitly broken for $\theta \neq 0$ as the computations of the vacuum expectation value of the topological density (23) suggest. However, the θ dependence emerges in the system not as a result of any modifications of any gauge configurations, in comparison with $\theta = 0$ case. Rather, the θ dependence emerges in the system as a result of selection of the specific superposition of the $|\theta, m\rangle$ states as discussed in Sec. III.

A simple way to interpret this result is to view the classification $|\theta, m\rangle$ in gauge theories in terms of the reduced Brillouin zone scheme as it is normally done in condensed matter physics, when θ parameter plays the role of the quasimomentum in the m-th Brillouin zone. In our classification the parameter m corresponds to the m−th metastable state. Using this analogy it is quite obvious that all the microscopical elements for any $|\theta, m\rangle$ states are the same. It is just a specific selection of the Bloch type superposition (constructed from the condensates of N different of monopole's species) which provides a complete description of the $|\theta, m\rangle$ state.

We conclude this work with the following short comments. It has been recent renewal interests in $\mathcal{CP}$ invariance of the gauge theories at $\theta = \pi$ [61,62]. While the questions addressed in [61,62] and in our work are somewhat different, nevertheless we observe a number of generic features discussed in [61,62] which have their counterparts in our simplified "deformed QCD" model. For example we obviously observe that there is a degeneracy at $\theta = \pi$ in our framework as one can see from classification scheme presented in Sec. III. Furthermore, one can explicitly see from (22), (23) that $\mathcal{CP}$ invariance is spontaneously broken at $\theta = \pi$, and the sign of $\mathcal{CP}$ violation is different depending on the direction this point is approached: $\theta = \pi \pm \epsilon$. These drastic changes correspond to complete reconstruction of the ground state when the system jumps to another Brillouin zone in the reduced classification scheme as described in Sec. III. Such a behavior obviously signals a phase transition at $\theta = \pi$. The superpositions of these two degenerate states at $\theta = \pi$ can make $\mathcal{CP}$ odd and $\mathcal{CP}$ even ground states.

One can trace the presence of the metastable states (which eventually become degenerate states at $\theta = \pi$) to the presence of nonlocal operator, the holonomy, in the system. Exactly this feature of nonlocality leads to a number of properties in "deformed QCD" model which are normally attributed to topologically ordered systems as argued in [34]. Precisely this sensitivity to arbitrary large distances in gapped theories might be the key element in understanding of the vacuum energy in cosmology because this type of the vacuum energy is generated by nonlocal physics and cannot be renormalized by any UV counter-terms, as recently advocated in [63].

There are many arguments, presented in Sec. I B, suggesting that this picture holds in the strongly coupled regime as well. Therefore, we strongly believe that in QCD we have precisely the same picture for the confinement at $\theta \neq 0$ including metastable states. If this is indeed the case, it may have profound observational effects on the axion production rate as mentioned in Sec. I A due to the nontrivial topological features of the system. It may be also important for understanding of the nature of the vacuum energy in cosmology as mentioned above. It may also affect the axion domain wall formation due to the 2π periodicity in θ and presence of the metastable states; see footnote for references. The very same metastable states, in general, violate $\mathcal{CP}$ invariance of the system as they effectively correspond to nonvanishing $\theta_{\mathrm{eff}} = 2\pi m/N$. One could speculate [39] that precisely these metastable states might be responsible for the $\mathcal{CP}$-odd correlations observed at RHIC and the LHC.

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[1] E. Witten, Ann. Phys. (Berlin) **128**, 363 (1980).
[2] E. Witten, Phys. Rev. Lett. **81**, 2862 (1998).
[3] R. D. Peccei and H. R. Quinn, Phys. Rev. D **16**, 1791 (1977).
[4] S. Weinberg, Phys. Rev. Lett. **40**, 223 (1978).
[5] F. Wilczek, Phys. Rev. Lett. **40**, 279 (1978).
[6] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B166**, 493 (1980).
[7] J. E. Kim, Phys. Rev. Lett. **43**, 103 (1979).
[8] M. Dine, W. Fischler, and M. Srednicki, Phys. Lett. **104B**, 199 (1981).
[9] A. R. Zhitnitsky, Yad. Fiz. **31**, 497 (1980) [Sov. J. Nucl. Phys. **31**, 260 (1980)].
[10] K. van Bibber and L. J. Rosenberg, Phys. Today **59**, No. 8, 30 (2006).
[11] S. J. Asztalos, L. J. Rosenberg, K. van Bibber, P. Sikivie, and K. Zioutas, Annu. Rev. Nucl. Part. Sci. **56**, 293 (2006).
[12] G. G. Raffelt, Lect. Notes Phys. **741**, 51 (2008).
[13] P. Sikivie, Int. J. Mod. Phys. A **25**, 554 (2010).
[14] L. J. Rosenberg, Proc. Natl. Acad. Sci. U.S.A. **112**, 12278 (2015).
[15] P. W. Graham, I. G. Irastorza, S. K. Lamoreaux, A. Lindner, and K. A. van Bibber, Annu. Rev. Nucl. Part. Sci. **65**, 485 (2015).
[16] A. Ringwald, *Proc. Sci.*, NOW2016 (2016) 081.
[17] R. Kitano and N. Yamada, J. High Energy Phys. 10 (2015) 136.
[18] C. Bonati, M. D'Elia, M. Mariti, G. Martinelli, M. Mesiti, F. Negro, F. Sanfilippo, and G. Villadoro, J. High Energy Phys. 03 (2016) 155.
[19] S. Borsanyi *et al.*, Nature (London) **539**, 69 (2016).
[20] P. Petreczky, H.-P. Schadler, and S. Sharma, Phys. Lett. B **762**, 498 (2016).
[21] M. D'Elia and F. Negro, Phys. Rev. Lett. **109**, 072001 (2012).
[22] M. D'Elia and F. Negro, Phys. Rev. D **88**, 034503 (2013).
[23] X. Liang and A. Zhitnitsky, Phys. Rev. D **94**, 083502 (2016).
[24] S. Ge, X. Liang, and A. Zhitnitsky, Phys. Rev. D **96**, 063514 (2017).
[25] G. 't Hooft, Nucl. Phys. **B190**, 455 (1981).
[26] M. Unsal and L. G. Yaffe, Phys. Rev. D **78**, 065035 (2008).
[27] E. Poppitz and M. Unsal, J. High Energy Phys. 07 (2011) 082.
[28] E. Thomas and A. R. Zhitnitsky, Phys. Rev. D **85**, 044039 (2012).
[29] M. M. Anber, E. Poppitz, and M. Unsal, J. High Energy Phys. 04 (2012) 040.
[30] M. Unsal, Phys. Rev. D **86**, 105012 (2012).
[31] E. Poppitz, T. Schäfer, and M. Unsal, J. High Energy Phys. 10 (2012) 115.
[32] E. Thomas and A. R. Zhitnitsky, Phys. Rev. D **87**, 085027 (2013).
[33] E. Poppitz, T. Schäfer, and M. Ünsal, J. High Energy Phys. 03 (2013) 087.
[34] A. R. Zhitnitsky, Ann. Phys. (Amsterdam) **336**, 462 (2013).
[35] M. M. Anber, Phys. Rev. D **88**, 085003 (2013).
[36] M. M. Anber, S. Collier, E. Poppitz, S. Strimas-Mackey, and B. Teeple, J. High Energy Phys. 11 (2013) 142.
[37] M. M. Anber, E. Poppitz, and B. Teeple, J. High Energy Phys. 09 (2014) 040.
[38] M. M. Anber and L. Vincent-Genod, arXiv:1704.08277.
[39] A. Bhoonah, E. Thomas, and A. R. Zhitnitsky, Nucl. Phys. **B890**, 30 (2014).
[40] E. Witten, Nucl. Phys. **B156**, 269 (1979).
[41] G. Veneziano, Nucl. Phys. **B159**, 213 (1979).
[42] P. Di Vecchia and G. Veneziano, Nucl. Phys. **B171**, 253 (1980).
[43] E. Witten, Phys. Lett. **86B**, 283 (1979).
[44] M. Shifman, *Advanced Topics in Quantum Field Theory* (Cambridge University Press, Cambridge, U.K., 2012).
[45] N. Seiberg and E. Witten, Nucl. Phys. **B426**, 19 (1994); **B430**, 485(E) (1994).
[46] E. J. Weinberg and P. Yi, Phys. Rep. **438**, 65 (2007).
[47] B. Julia and A. Zee, Phys. Rev. D **11**, 2227 (1975).
[48] D. Bak, C.-k. Lee, and K.-M. Lee, Phys. Rev. D **57**, 5239 (1998).
[49] M. Di Pierro and K. Konishi, Phys. Lett. B **388**, 90 (1996).
[50] K. Konishi, Phys. Lett. B **392**, 101 (1997).
[51] K. Konishi and H. Terao, Nucl. Phys. **B511**, 264 (1998).
[52] H.-Y. Chen, N. Dorey, and K. Petunin, J. High Energy Phys. 06 (2010) 024.
[53] H.-Y. Chen, N. Dorey, and K. Petunin, J. High Energy Phys. 11 (2011) 020.
[54] N. Dorey, J. High Energy Phys. 04 (2001) 008.
[55] N. Dorey and A. Parnachev, J. High Energy Phys. 08 (2001) 059.
[56] Y. Liu, E. Shuryak, and I. Zahed, Phys. Rev. D **92**, 085006 (2015).
[57] Y. Liu, E. Shuryak, and I. Zahed, Phys. Rev. D **92**, 085007 (2015).
[58] R. Larsen and E. Shuryak, Phys. Rev. D **92**, 094022 (2015).
[59] M. M. Anber and E. Poppitz, J. High Energy Phys. 10 (2015) 051.
[60] C. Cao and A. Zhitnitsky, Phys. Rev. D **96**, 015013 (2017).
[61] D. Gaiotto, A. Kapustin, Z. Komargodski, and N. Seiberg, J. High Energy Phys. 05 (**2017**) 091.
[62] D. Gaiotto, Z. Komargodski, and N. Seiberg, arXiv: 1708.06806.
[63] A. O. Barvinsky and A. R. Zhitnitsky, arXiv:1709.09671.