ON FUNCTIONAL EQUATIONS FOR NIELSEN POLYLOGARITHMS

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Abstract. We derive new functional equations for Nielsen polylogarithms. We show that, when viewed modulo Li_5 and products of lower weight functions, the weight 5 Nielsen polylogarithm $S_{3,2}$ satisfies the dilogarithm five-term relation. We also give some functional equations and evaluations for Nielsen polylogarithms in weights up to 8, and general families of identities in higher weight.

1. Introduction

The classical m-th polylogarithm function Li_m is an analytic function defined by the Taylor series

$$\mathrm{Li}_m(z) \;=\; \sum_{n=1}^{\infty} \frac{z^n}{n^m}\,,$$

convergent for $|z| < 1$. For $m \geq 1$ the function Li_m extends to a multivalued analytic function on $\mathbb{P}^1(\mathbb{C}) \smallsetminus \{0, 1, \infty\}$, which can be seen, for example, from the recursive formula

$$\mathrm{Li}_m(z) \;=\; \int_0^z \mathrm{Li}_{m-1}(t)\frac{dt}{t}\,,$$

together with the initial condition $\mathrm{Li}_1(z) \;=\; -\log(1-z)$. Polylogarithms appear in several areas of mathematics: for example, the Euler dilogarithm Li_2 (or, more precisely, a single-valued version) can be used to compute volumes of hyperbolic 3-folds, special values of Dedekind zeta functions at $s = 2$, and it is intimately related to algebraic K-theory (more precisely to K_3 and K_2) of number fields, see [53], [55], [56]. One of the most curious features of polylogarithms is that they satisfy a plethora of identities and functional equations, the most famous of which is undoubtedly the five-term relation,

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(y) - \mathrm{Li}_2\Big(\frac{x}{1-y}\Big) - \mathrm{Li}_2\Big(\frac{y}{1-x}\Big) + \mathrm{Li}_2\Big(\frac{xy}{(1-x)(1-y)}\Big) \;=\; -\log(1-x)\log(1-y)\,, \quad (1)$$

(for $|x|+|y| < 1$) in this or any of its equivalent forms (see Section 1.5 in [44]). Numerous other identities of this kind are known, both for Li_2 and for Li_m for $m > 2$, but as soon as m becomes greater than 7, the only relations that are known in general are the inversion identity that relates $\mathrm{Li}_m(z)$ and $\mathrm{Li}_m(z^{-1})$, and the distribution relations $n^{1-m}\,\mathrm{Li}_m(z^n) \;=\; \sum_{\lambda^n=1} \mathrm{Li}_m(\lambda z)$ for $n \geq 1$. For an introduction to functional equations for polylogarithms we refer the reader to [54]. Many examples of functional equations for Li_m up to $m = 5$ can already be found in Lewin's classical book [44], while newer results are e.g. given in [39] for $m = 2$ and in [50], [32] for $m \leq 3$. Inaugural results in weight 6 and 7 are treated in [27],[28]. For further examples, and background, we refer also to these theses [15, 26, 47].

In [46] Nielsen defined and studied the functions $S_{n,p}$ given by the following integral

$$S_{n,p}(z) \;\coloneqq\; \frac{(-1)^{n+p-1}}{(n-1)!\,p!} \int_0^1 \log^{n-1}(t)\log^p(1-zt)\frac{dt}{t}\,. \quad (2)$$

It is easy to show that $S_{m-1,1} \;=\; \mathrm{Li}_m$, so that classical polylogarithms are a special case of Nielsen's generalised polylogarithms. On the other hand, Nielsen polylogarithms themselves are special cases of multiple polylogarithms and iterated integrals:

$$\begin{aligned} S_{n,p}(z) \;&=\; \mathrm{Li}_{\{1\}^{p-1},n+1}(1,\ldots,1,z) \\ &=\; (-1)^p I(0;\{1\}^p,\{0\}^n;z)\,, \end{aligned} \quad (3)$$

where $\{a\}^k$ denotes the string a repeated k times. Here $\mathrm{Li}_{n_1,\ldots,n_d}$ is the multiple polylogarithm function

$$\mathrm{Li}_{n_1,\ldots,n_d}(z_1,\ldots,z_d) \;\coloneqq\; \sum_{0<k_1<\cdots<k_d} \frac{z_1^{k_1}\cdots z_d^{k_d}}{k_1^{n_1}\cdots k_d^{n_d}}\,,$$

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convergent for $|z_i| < 1$, $1 \leq i \leq d$, and I is the iterated integral

$$I(x_0; x_1, \ldots, x_N; x_{N+1}) \;:=\; \int_{x_0 < t_1 < \cdots < t_N < x_{N+1}} \frac{dt_1}{t_1 - x_1} \wedge \frac{dt_2}{t_2 - x_2} \wedge \cdots \wedge \frac{dt_N}{t_N - x_N}\,.$$

(The integral implicitly depends on the choice of a path going from x_0 to x_{N+1}, where the integration variables t_i are considered to be ordered on the path.) Note that, as a multiple polylogarithm, $S_{n,p}$ has weight $n + p$ and depth $\leq p$. Moreover, the iterated integral identity extends the range of definition of $S_{n,p}$ to the case $n = 0$ and $p = 0$, giving

$$\begin{aligned} S_{0,p}(z) &= \tfrac{(-1)^p}{p!} \log^p(1 - z) = \tfrac{1}{p!} \operatorname{Li}_1^p(z) \\ S_{n,0}(z) &= \tfrac{1}{n!} \log^n(z)\,. \end{aligned}$$

While they are certainly not as well-studied as their classical counterparts, Nielsen polylogarithms do naturally appear in some calculations in quantum electrodynamics (for some references see [40]), and they also provide the simplest examples (aside from Li_m) of harmonic polylogarithms that appear, for example, in computations of planar scattering amplitudes [20]. For the original paper on harmonic polylogarithms, see [48]. There is also some interest in computing special values of $S_{n,p}(z)$, at least when z is a root of unity, since they arise in ε-expansions of some Feynman diagrams [18], and also in connection with Mahler measures and planar random walks [4] (see also [3]). For approaches to the numerical computation of values of Nielsen polylogarithms and harmonic polylogarithms, see respectively [41] and [30].

Despite this there appear to be very few results concerning functional equations for $S_{n,p}$. In fact, excluding the case of classical polylogarithms $\operatorname{Li}_m = S_{m-1,1}$, the most general functional equations that we have found in the literature are the relations that express $S_{n,p}(\gamma(z))$ in terms of $S_{n',p'}(z)$, where $n' + p' = n + p$ and $\gamma(z)$ is one of the Möbius transformations $\{z, 1 - z, \frac{1}{z}, \frac{z-1}{z}, \frac{z}{z-1}, \frac{1}{1-z}\}$ (see [40], or more recent [49]). There is no known analogue of distribution relations for $S_{n,p}$ when $p \geq 2$.

Part of our initial motivation comes from the conjectures of Goncharov regarding the structure of the so-called motivic Lie coalgebra, the existence of which is known in the case of number fields, which predicts the reduction in depth of certain linear combinations of iterated integrals. One of these predictions is that $S_{3,2}$ of the five-term relation reduces to Li_5, i.e.

$$S_{3,2}(x) + S_{3,2}(y) - S_{3,2}\Big(\frac{x}{1-y}\Big) - S_{3,2}\Big(\frac{y}{1-x}\Big) + S_{3,2}\Big(\frac{xy}{(1-x)(1-y)}\Big) \;=\; 0 \ (\mathrm{mod}\ \operatorname{Li}_5,\ \text{products})\,.$$

Indeed we establish this in Theorem 16 below, together with the explicit form of the Li_5 terms, thus corroborating part of these conjectures in weight 5. We also establish other examples of depth reduction for Nielsen polylogarithms, by way of giving functional equations for various $S_{n,p}$ up to and including weight 8. Extending the results of Nielsen and of Kölbig we also consider various evaluations and ladders (i.e. relations modulo products among $S_{n,p}$ at $\pm\theta^k$ for some algebraic number θ and $k \in \mathbb{Z}$) of Nielsen polylogarithms, most of them apparently new. In particular, in (33), we give another example of the Broadhurst-Kreimer 'push down' phenomenon (in the terminology of [1], for MZV's), whereby we conjecturally reduce $S_{4,2}(-1)$, an irreducible algebra generator of Deligne's [21] fundamental group of $\mathbb{P}^1 \smallsetminus \{-1, 0, 1, \infty\}$, to Li_6 and products by allowing more general arguments and viewing it as an element of a larger set of periods.

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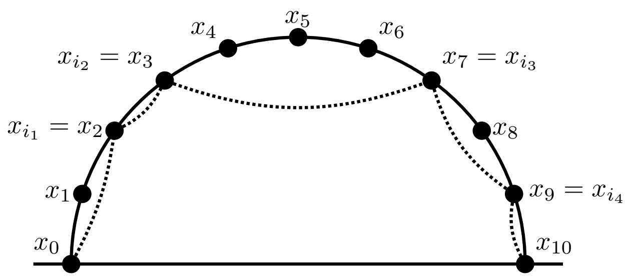
2. Motivic framework, and symbols

We first briefly recall some of the motivic framework for multiple polylogarithms from the works of Goncharov [34], [33] and Brown [11, 10, 13]. In particular, we recall Goncharov's Hopf algebra $\mathcal{A}_\bullet$ of motivic iterated integrals $I^{\mathfrak{u}}$ and the 'symbols' thereof, and Brown's $\mathcal{A}_\bullet$-comodule $\mathcal{H}_\bullet$ of motivic iterated integrals $I^{\mathfrak{m}}$ which 'refines' Goncharov's Hopf algebra.

2.1. **Goncharov's Hopf algebra of motivic iterated integrals.** In [34], Goncharov upgraded the iterated integrals $I(x_0; x_1, \ldots, x_N; x_{N+1})$, $x_i \in \overline{\mathbb{Q}}$, to framed mixed Tate motives to define motivic iterated integrals $I^{\mathfrak{u}}(x_0; x_1, \ldots, x_N; x_{N+1})$, living in a graded (by the weight N) connected Hopf algebra $\mathcal{A}_\bullet = \mathcal{A}_\bullet(\overline{\mathbb{Q}})$. The Hopf algebra $\mathcal{A}_\bullet$ is the ring of regular functions on the unipotent part of the motivic Galois group. In [34], they are denoted by $I^{\mathscr{M}}$, but when incorporated into Brown's motivic framework below, they are commonly denoted $I^{\mathfrak{u}}$, for the unipotent part, a convention we adopt. The coproduct Δ on this Hopf algebra is computed via Theorem 1.2 in [34] as

$$\Delta I^{\mathfrak{u}}(x_0; x_1, \ldots, x_N; x_{N+1}) = \sum_{\substack{0=i_0<i_1<\cdots \\ <i_k<i_{k+1}=N+1}} I^{\mathfrak{u}}(x_0; x_{i_1}, \ldots, x_{i_k}; x_{N+1}) \otimes \prod_{p=0}^{k} I^{\mathfrak{u}}(x_{i_p}; x_{i_p+1}, \ldots, x_{i_{p+1}-1}; x_{i_{p+1}}).$$

This is often stated mnemonically as a sum over all semicircular polygons, with the left hand factor corresponding to the main polygon and the right hand factor corresponding to the product over all small cut-off polygons. A typical term is given by the following picture (for easier visibility we draw curved lines rather than straight ones)



This term has $i_1 = 2, i_2 = 3, i_3 = 7$ and $i_4 = 9$. We shall refer to all the terms in Δ which arise from a fixed value of the above k as the $(k, N-k)$-part of the coproduct, and denote them by $\Delta^{(k,N-k)}$. It is often convenient to invoke the reduced coproduct $\Delta' = \Delta - 1 \otimes \mathrm{id} - \mathrm{id} \otimes 1$.

Conjecturally one can think of Goncharov's motivic iterated integrals as polylogarithms modulo the ideal generated by πi. In particular $\zeta^{\mathfrak{u}}(2) = 0$, so Goncharov's framework does not see the terms involving even zeta values.

2.2. **The mod-products symbol.** Recall from [34, Section 4.4] the '$\otimes^N$-invariant', or *symbol*, of a motivic iterated integral $I^{\mathfrak{u}}$ of weight N. The symbol $\mathrm{Symb}(I^{\mathfrak{u}}) \in \mathcal{A}_1^{\otimes N}$ is an algebraic invariant of $I^{\mathfrak{u}}$, which respects functional equations among iterated integrals. More precisely, $\mathrm{Symb}\colon (\mathcal{A}_\bullet, \sqcup\!\sqcup, \Delta) \to (T(\mathcal{A}_1), \sqcup\!\sqcup, \Delta_{\mathrm{dec}})$ is a map of graded Hopf algebras, where $T(\mathcal{A}_1)$ is the tensor algebra of $\mathcal{A}_1$, and Δ_{dec} is the deconcatenation coproduct on tensors. The map Symb can be obtained by iterating the reduced coproduct Δ' precisely $N-1$ times.

Recall also the projectors $\Pi_\bullet$ from [22, Section 5.5] which annihilate the symbols of products. The projector $D_N = N\Pi_N$ acts on length N tensors as follows. (We prefer D_N to Π_N in order to avoid unnecessary scaling factors, at the expense of that operator no longer being idempotent.)

$$D_N(x_1 \otimes \cdots \otimes x_N) = D_{N-1}(x_1 \otimes \cdots \otimes x_{N-1}) \otimes x_N - D_{N-1}(x_2 \otimes \cdots \otimes x_N) \otimes x_1 \,.$$

This is actually the classical Dynkin operator in the theory of Hopf algebras (rediscovered many times since, and we thank the referee for alerting us to this construction, and to the reference below), which can be written as $D = S \star Y \coloneqq \mu(S \otimes Y)\Delta$, in terms of the grading $Y(x_1 \otimes \cdots \otimes x_N) = N(x_1 \otimes \cdots \otimes x_N)$, the antipode $S(x_1 \otimes \cdots \otimes x_N) = (-1)^N x_N \otimes \cdots \otimes x_1$, and the multiplication μ in the Hopf algebra. (See [24, Section 4] for more details in the current setup of a commutative, non-cocommutative Hopf algebra.)

The composition $N\Pi_N \circ \mathrm{Symb} \;=:\; \mathrm{Symb}^{\sqcup\!\sqcup}\colon \mathcal{A}_\bullet \to T(\mathcal{A}_1)$ is the so-called *mod-products symbol.* Since $\mathrm{Symb}^{\sqcup\!\sqcup}$ annihilates products it is also well-defined as a map from the corresponding quotient, see the parenthetical remark in Section 2.5. By considering how the projector $N\Pi_N$ acts on Symb when written via the iterated coproduct in both ways (iterating the $(N-1,1)$-part and the $(1,N-1)$-part respectively), we derive the following recursion

$$\begin{aligned}\operatorname{Symb}^{ш} I^{\mathfrak{u}}(x_0;\dots;x_{N+1}) \;=\; \sum_{j=1}^{N} & \operatorname{Symb}^{ш} I^{\mathfrak{u}}(x_0;x_1,\dots,\widehat{x_j},\dots,x_N;x_{N+1}) \otimes I^{\mathfrak{u}}(x_{j-1};x_j;x_{j+1}) \\ & - \operatorname{Symb}^{ш} I^{\mathfrak{u}}(x_1;x_2,\dots,x_N;x_{N+1}) \otimes I^{\mathfrak{u}}(x_0;x_1;x_{N+1}) \\ & - \operatorname{Symb}^{ш} I^{\mathfrak{u}}(x_0;x_1,\dots,x_{N-1};x_N) \otimes I^{\mathfrak{u}}(x_0;x_N;x_{N+1})\,.\end{aligned} \tag{4}$$

Here $I^{\mathfrak{u}}(a;b;c)$ is regularised (cf. (6) in [34]) as

$$I^{\mathfrak{u}}(a;b;c) = \begin{cases} \log^{\mathfrak{u}}(1) = 0 & \text{if } a = b \text{ and } b = c\,, \\ \log^{\mathfrak{u}}(\frac{1}{b-a}) & \text{if } a \neq b \text{ and } b = c\,, \\ \log^{\mathfrak{u}}(b-c) & \text{if } a = b \text{ and } b \neq c\,, \\ \log^{\mathfrak{u}}(\frac{b-c}{b-a}) & \text{otherwise}\,. \end{cases}$$

Note that $I^{\mathfrak{u}}(a;b;c) = 0$, whenever $a,b,c \in \{0,1\}$, so the symbol of every multiple zeta value (MZV) is zero in this setup.

As usual with symbols, we will drop the $\log^{\mathfrak{u}}$ from the notation and write tensors multiplicatively. The notation would then suggest that the symbol entries are elements $x \in \overline{\mathbb{Q}}^{\times}$, whereas they are really elements of $\overline{\mathbb{Q}}^{\times} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathcal{A}_1$. In particular, on the level of the symbol 2-torsion (indeed torsion generally) vanishes, because in the Hopf algebra $\mathcal{A}_\bullet$ (as $(2\pi i)^{\mathfrak{u}}$ is zero) one has the exact equality of motivic logarithms $\log^{\mathfrak{u}}(x) = \log^{\mathfrak{u}}(-x)$. We can therefore ignore signs in the tensor entries, and freely interchange between $\otimes(-x)$ and $\otimes x$. To emphasise that certain identities hold only on the level of the mod-products symbol, we shall write $f \overset{ш}{=} g$ to mean $\operatorname{Symb}^{ш} f = \operatorname{Symb}^{ш} g$.

2.3. **Lie coalgebra.** The coproduct induces a cobracket $\delta = \Delta - \Delta^{\mathrm{op}}$, with Δ^{op} the opposite coproduct, on the Lie coalgebra of irreducibles

$$\mathcal{L}_\bullet \;\coloneqq\; \mathcal{A}_{>0}/\mathcal{A}_{>0}^2\,.$$

The image in $\mathcal{L}_\bullet$ of some motivic iterated integral $I^{\mathfrak{u}}$ shall be denoted by $I^{\mathfrak{L}}$. We use the notation $\{z\}_m$ for elements in the weight m pre-Bloch group $B_m(F)$ (also called 'polylogarithmic group' in the literature), where F is any field. For a rigorous definition of $B_m(F)$ see [32] §1.9, but roughly one can think about it as the quotient space of formal linear combinations of elements of F modulo the subspace given by specialisations of all functional equations for Li_m. The conjectural structure of $\mathcal{L}_\bullet$ alluded to above implies that $B_m(F)$ is isomorphic to a subgroup of $\mathcal{L}_m(F)$, and hence one can think about $\{z\}_m$ as the image of $\mathrm{Li}_m(z)$ modulo products, i.e. $\{z\}_m = \mathrm{Li}_m^{\mathfrak{L}}(z)$, where $\mathrm{Li}_m^{\mathfrak{m}}$ is the motivic version of Li_m, viewed as an iterated integral, and $\mathrm{Li}_m^{\mathfrak{L}}$ is the image of $\mathrm{Li}_m^{\mathfrak{m}}$ in the Lie coalgebra. In Section 8.3 we will also work with higher Bloch groups $\mathcal{B}_m(\mathbb{Q})$. These were originally defined (for number fields) in [53]. One can define $\mathcal{B}_m(F)$ as the kernel of δ restricted to the pre-Bloch group $B_m(F)$.

We define $\delta^{\geq 2}$, the 2-part of the cobracket in weight N, as the projection to $\bigoplus_{k=2}^{N-2} \mathcal{L}_k \wedge \mathcal{L}_{N-k}$. Then $\delta^{\geq 2}$ is seen to annihilate all classical polylogarithms. Conjecture 1.20 and Section 1.6 in [31] on the structure of the motivic Lie coalgebra would imply that the kernel of the motivic cobracket should coincide with classical polylogarithms.

For example, up to weight 3 the 2-part vanishes identically for trivial reasons. This corresponds to the fact that in weight 3 every iterated integral can be expressed in terms of the classical trilogarithm Li_3 (and products of lower weight), which was already proven by Kummer [42, p. 328]. This reduction is also given in Equation A.3.5 in (the appendix of) [44], and in a somewhat different form in [37]. The next case is weight 4, in which Goncharov predicted that

$$I_{3,1}(V(x,y),z) = 0 \text{ (mod } \mathrm{Li}_4\text{, products)}\,.$$

Here $I_{3,1}(x,y) = I(0;x,0,0,y;1)$ and $V(x,y)$ is any version of the five-term relation, such as in (1) above. This was established by the second author in [29], and was subsequently also shown by Goncharov and Rudenko in [35] where it played a key role in the proof of Zagier's Polylogarithm Conjecture for weight 4.

In weight 5, one of the predictions is that

$$I_{4,1}(V(x,y),z) + I_{4,1}(V(x,y),z^{-1}) = 0 \text{ (mod } \mathrm{Li}_5\text{, products)}\,, \tag{5}$$

where $I_{4,1}(x,y) = I(0;x,0,0,0,y;1)$, and this reduction is expected to play a similarly important role in any proof of Zagier's Polylogarithm Conjecture for weight 5. It follows from the special case of (5) at $z = 1$, that one also expects

$$S_{3,2}(V(x,y)) = 0 \text{ (mod } \mathrm{Li}_5\text{, products)}\,,$$

which we prove in Theorem 16 below.

2.4. **Brown's $\mathcal{A}_\bullet$-comodule of motivic iterated integrals.** Motivic iterated integrals $I^{\mathfrak{m}}(x_0; x_1, \ldots, x_N; x_{N+1})$ in the sense of Brown [11, 13] are elements of the $\mathcal{A}_\bullet$-comodule $\mathcal{H}_\bullet$ of regular functions on the torsor of tensor isomorphisms between Betti and de Rham realisations. The superscript $(\cdot)^{\mathfrak{m}}$ for Li_m or ζ will refer to the respective elements in this setting.

This comodule is endowed with a coaction $\Delta\colon \mathcal{H}_\bullet \to \mathcal{H}_\bullet \otimes \mathcal{A}_\bullet$ which, as noted in [13], is given by the same formula as Goncharov's coproduct, transposed to this setting, i.e.

$$\Delta I^{\mathfrak{m}}(x_0; x_1, \ldots, x_N; x_{N+1}) \; = \\ \sum_{\substack{0=i_0<i_1<\cdots \\ <i_k<i_{k+1}=N+1}} I^{\mathfrak{m}}(x_0; x_{i_1}, \ldots, x_{i_k}; x_{N+1}) \otimes \prod_{p=0}^{k} I^{\mathfrak{u}}(x_{i_p}; x_{i_p+1}, \ldots, x_{i_{p+1}-1}; x_{i_{p+1}}).$$

As above, we denote by $\Delta' = \Delta - 1 \otimes \mathrm{id} - \mathrm{id} \otimes 1$ the reduced coaction, and we speak about the $(k, N-k)$-part $\Delta^{(k,N-k)}$ of the coaction to mean the terms arising from a fixed value of the above k.

In Brown's setting $\zeta^{\mathfrak{m}}(2) \neq 0$, and therefore much more information about motivic iterated integrals is retained. In particular, the coaction can be used to also fix the coefficient of product terms involving $\zeta(2n)$, in contrast to the coproduct above. By computing the coaction, we can establish relations modulo certain primitive elements (those having trivial coaction), namely polylogarithms at roots of unity $\mathrm{Li}_m^{\mathfrak{m}}(e^{2\pi i n/N})$. We will return to this point in Sections 7.3 and 9.3.

2.5. **Summary of the versions of motivic iterated integrals.** We briefly summarise how the various notions of motivic iterated integral relate, and what information is lost at each step.

Brown's comodule		Goncharov's Hopf algebra		Lie coalgebra		Tensor algebra
$\mathcal{H}_\bullet$	$\to$	$\mathcal{A}_\bullet$	$\to$	$\mathcal{L}_\bullet$	$\to$	$T(\mathcal{A}_1)$
$I^{\mathfrak{m}}$	$\mapsto$	$I^{\mathfrak{u}}$	$\mapsto$	$I^{\mathfrak{L}}$	$\mapsto$	$\mathrm{Symb}^{\sqcup\!\sqcup}(I^{\mathfrak{L}})$

At the first step $I^{\mathfrak{m}} \mapsto I^{\mathfrak{u}}$ we lose $(2\pi i)^{\mathfrak{m}}$. Passing to $I^{\mathfrak{L}}$ loses products. (Note that we only defined $\mathrm{Symb}^{\sqcup\!\sqcup}(I^{\mathfrak{u}})$ above, but computing $\mathrm{Symb}^{\sqcup\!\sqcup}(I^{\mathfrak{L}})$ makes sense since $\mathrm{Symb}^{\sqcup\!\sqcup}(I^{\mathfrak{u}})$ depends only on the class modulo products of $I^{\mathfrak{u}}$.) Finally passing to $\mathrm{Symb}^{\sqcup\!\sqcup}(I^{\mathfrak{L}})$ loses everything which is built out of primitives of weight > 1.

3. General properties of Nielsen polylogarithms

In this section we recall the basic two-term relations found by Nielsen himself [46] (and related to us by Kölbig [40]) for Nielsen polylogarithms of arbitrary weight (Propositions 2 and 4). We then reduce $S_{2,2}(z)$ to Li_4 (Proposition 5) and determine a basis of the space of mod-products symbols for Nielsen polylogarithms of anharmonic ratios in a given weight (Theorem 7).

3.1. **General relations.** We first recall the differential behaviour of Nielsen polylogarithms, which can be used to verify some of the identities we give later. The behaviour follows by differentiating the integral (2) defining $S_{n,p}$.

Proposition 1 (Derivative, Equation 2.11 in [40])**.** *For $n, p \in \mathbb{Z}_{>0}$, the Nielsen polylogarithm $S_{n,p}(z)$ satisfies the differential equation*

$$\Big(z \frac{d}{dz}\Big) S_{n,p}(z) \; = \; S_{n-1,p}(z)\,.$$

We use the convention $S_{0,p}(z) = \frac{(-1)^p}{p!} \log^p(1-z) = \frac{1}{p!} \mathrm{Li}_1^p(z)$, via the iterated integral definition (3).

Nielsen already established a general inversion and reflection relation for the Nielsen polylogarithms. Henceforth we fix the principal branch of the logarithm in all results.

Proposition 2 (Reflection, Section 5.1 in [40])**.** *For all $z \in \mathbb{C} \smallsetminus \big((-\infty, 0] \cup [1, \infty)\big)$, and all $n, p \in \mathbb{Z}_{>0}$, we have*

$$S_{n,p}(1-z) \; = \; \frac{(-1)^p}{n!\,p!} \log^n(1-z) \log^p(z) \\ + \sum_{j=0}^{n-1} \frac{\log^j(1-z)}{j!} \left(S_{n-j,p}(1) - \sum_{k=0}^{p-1} \frac{(-1)^k \log^k(z)}{k!} S_{p-k,n-j}(z) \right).$$

In particular, after neglecting products, one has

$$S_{p,n}(z) = -S_{n,p}(1-z) + S_{n,p}(1) \text{ (mod products)}\,.$$

This also follows directly from the functoriality, shuffle product and path (de)composition properties of iterated integrals [17].

Remark 3. From (3), we have $S_{n,p}(1) = (-1)^p I(0;\{1\}^p,\{0\}^n;1) = \zeta(\{1\}^{p-1}, n+1)$, where $\zeta(k_1,\ldots,k_r) := \mathrm{Li}_{k_1,\ldots,k_r}(1,\ldots,1)$, with $k_i \in \mathbb{Z}_{>0}$ and $k_r > 1$, is a multiple zeta value (MZV). From [2, Equation 10], the following generating function expansion

$$\sum_{m,n\geq 0} x^{m+1}y^{n+1}\zeta(\{1\}^n, m+2) \;=\; 1 - \exp\bigg(\sum_{k\geq 2}\frac{1}{k}\big(x^k + y^k - (x+y)^k\big)\zeta(k)\bigg)$$

shows that this class of MZV's consists of polynomials in the Riemann zeta values $\zeta(q)$.

Proposition 4 (Inversion, Section 5.3 in [40])**.** *For all $z \in \mathbb{C} \smallsetminus [0,\infty)$, and all $n,p \in \mathbb{Z}_{>0}$, we have*

$$\begin{aligned} S_{n,p}\Big(\frac{1}{z}\Big) \;=\; & (-1)^n \sum_{k=0}^{p-1}(-1)^k \sum_{m=0}^{k} \frac{\log^m(-z^{-1})}{m!}\binom{n+k-m-1}{k-m} S_{n+k-m,p-k}(z) \\ & + (-1)^p\bigg(\frac{\log^{n+p}(-z^{-1})}{(n+p)!} + \sum_{j=0}^{n-1}\frac{\log^j(-z^{-1})}{j!}C_{n-j,p}\bigg)\,, \end{aligned}$$

where $C_{n,p}$ is some explicit polynomial in $S_{a,b}(1) = \zeta(\{1\}^{a-1}, b+1)$, of homogeneous weight $n+p$.

The explicit form of $C_{n,p}$ is given in [40, Equation 7.2, Theorem 1]. By the remark above $C_{n,p}$ is in fact a polynomial in Riemann zeta values $\zeta(q)$, so consists of products and a single depth 1 term $\zeta(n+p)$ (which itself reduces to products in even weight). In particular, for $p > 1$ the following depth p combination

$$S_{n,p}\Big(\frac{1}{z}\Big) - (-1)^n S_{n,p}(z)$$

reduces to lower depth and products.

3.2. **Nielsen polylogarithms in weight ≤ 4.** In weights up to 4, Nielsen polylogarithms give the same class of functions as the classical polylogarithms Li_m. More precisely, in weight 2, we only have $S_{1,1} = \mathrm{Li}_2$ identically. In weight 3, $S_{2,1} = \mathrm{Li}_3$ identically and then $S_{1,2}(z)$ is expressible in terms of Li_3 by the reflection in Proposition 2.

In weight 4, $S_{3,1} = \mathrm{Li}_4$ and $S_{1,3}(z)$ is also expressible in terms of Li_4 by reflection. The function $S_{2,2}$ is potentially new, but it too can be reduced to Li_4's. In [40, Section 6], Kölbig notes that one can in principle find a 'complicated' expression for $S_{2,2}(z)$ in terms of polylogarithms, by studying the formulae for $S_{n,p}$ under the 6 anharmonic transformations. He references an equation in [44, p. 204], from which such a formula could also be derived.

Kölbig perhaps overstates the complexity of this formula, and of the manner in which it should be derived. Note that from Proposition 4 we have the following identity, since the constant $C_{1,3} = \zeta(4)$ [40, Table 1] is already a product,

$$S_{1,3}(z^{-1}) \;=\; -S_{1,3}(z) + S_{2,2}(z) - S_{3,1}(z) \text{ (mod products)}\,.$$

So immediately $S_{2,2}(z)$ can be expressed in terms of the other weight 4 Nielsen polylogarithms and products. Applying reflection to write $S_{1,3}(z) = -S_{3,1}(1-z) = -\mathrm{Li}_4(1-z)$ (mod products) gives a reduction in depth to Li_4. (The constant is a weight 4 MZV, so it is necessarily a product.) Wojtkowiak already gives a version of this reduction in [52, Equation 8.3.7] for some single-valued analogue of $S_{2,2}$.

Proposition 5 (Reduction of $S_{2,2}$)**.** *The function $S_{2,2}(z)$ can be reduced to the classical* Li_4*, and products of lower weight classical polylogarithms, as follows. For all $z \in \mathbb{C} \smallsetminus \big((-\infty,0] \cup [1,\infty)\big)$, we have*

$$\begin{aligned} S_{2,2}(z) \;=\; & -\mathrm{Li}_4(1-z) + \mathrm{Li}_4(z) + \mathrm{Li}_4\Big(\frac{z}{z-1}\Big) - \mathrm{Li}_3(z)\log(1-z) \\ & + \frac{1}{4!}\log^4(1-z) - \frac{1}{3!}\log(z)\log^3(1-z) \\ & + \frac{1}{2!}\zeta(2)\log^2(1-z) + \zeta(3)\log(1-z) + \zeta(4)\,. \end{aligned}$$

Proof. This identity can be verified by differentiation, and checking the resulting weight 3 combination is identically 0. Since $S_{2,2}(0) = 0$, the constant of integration is fixed to $\zeta(4)$ to ensure the right hand side also vanishes at $z = 0$. □

The same strategy also reduces $S_{n,n}(z)$, $n \geq 2$, to lower depth Nielsen polylogarithms. Moreover, we can determine a spanning set for weight N Nielsen polylogarithms, as follows.

3.3. **Generators for Nielsen polylogarithms.** We first state a lemma about the mod-products symbols of Nielsen polylogarithms.

Lemma 6. *The mod-products symbol of $S^{\mathfrak{u}}_{n,p}(z)$, $n,p > 0$, lies in the subspace $\bigwedge^2(\mathbb{Q}(z)^\times) \otimes \mathrm{Sym}^{p+n-2}(\mathbb{Q}(z)^\times) \otimes_\mathbb{Z} \mathbb{Q}$ of the symbol tensor algebra $\bigotimes_{i=1}^{p+n} \mathbb{Q}(z)^\times \otimes_\mathbb{Z} \mathbb{Q}$, and is given by*

$$\mathrm{Symb}^{ш}(S^{\mathfrak{u}}_{n,p}(z)) = -(1-z) \wedge z \otimes \Big((1-z)^{\otimes p-1} ш z^{\otimes n-1}\Big)\,. \tag{6}$$

Here $a \wedge b = a \otimes b - b \otimes a$, and ш is the shuffle product of tensors, recursively defined on words via

$$\big(a \otimes w_1\big) ш \big(b \otimes w_2\big) = a \otimes \big(w_1 ш \big(b \otimes w_2\big)\big) + b \otimes \big(\big(a \otimes w_1\big) ш w_2\big)\,,$$

with the empty word $\mathbb{1}$ satisfying $w ш \mathbb{1} = \mathbb{1} ш w = w$.

Proof. For $S^{\mathfrak{u}}_{n,p}(z)$, only the terms $\widehat{x_{n+p}}$ and $\widehat{x_0}$ contribute in the recursion (4), so

$$\begin{aligned}\mathrm{Symb}^{ш}\, S^{\mathfrak{u}}_{n,p}(z) &= (-1)^p\, \mathrm{Symb}^{ш}\, I^{\mathfrak{u}}(0;\{1\}^p,\{0\}^n;z) \\ &= (-1)^p \Big(\mathrm{Symb}^{ш}\, I^{\mathfrak{u}}(0;\{1\}^p,\{0\}^{n-1};z) \otimes z \\ &\qquad - \mathrm{Symb}^{ш}\, I^{\mathfrak{u}}(1;\{1\}^{p-1},\{0\}^n;z) \otimes (1-z)\Big)\,.\end{aligned}$$

We can deduce $I^{\mathfrak{u}}(1;\{1\}^{p-1},\{0\}^n;z) = I^{\mathfrak{u}}(0;\{1\}^{p-1},\{0\}^n;z)$ (mod products, constants), by splitting the integration path at 0. Since the constants are MZV's, they do not contribute to the symbol, so

$$\mathrm{Symb}^{ш}\, S^{\mathfrak{u}}_{n,p}(z) = \mathrm{Symb}^{ш}\, S^{\mathfrak{u}}_{n-1,p}(z) \otimes z + \mathrm{Symb}^{ш}\, S^{\mathfrak{u}}_{n,p-1}(z) \otimes (1-z)\,. \tag{7}$$

For the base case $n = p = 1$, we read off $\mathrm{Symb}^{ш}\, S^{\mathfrak{u}}_{1,1}(x) = -(1-z) \otimes z + z \otimes (1-z) = -(1-z) \wedge z$, since $S^{\mathfrak{u}}_{0,1}(z) = -\log^{\mathfrak{u}}(1-z)$ and $S^{\mathfrak{u}}_{1,0}(z) = \log^{\mathfrak{u}}(z)$. If $n > 1, p = 1$ then (7) collapses to the single summand

$$\mathrm{Symb}^{ш}\, S^{\mathfrak{u}}_{n,1}(z) = \mathrm{Symb}^{ш}\, S^{\mathfrak{u}}_{n-1,1}(z) \otimes z\,.$$

Now apply (6) as the induction hypothesis; in this case one factor of the shuffle product is the empty word $\mathbb{1}$, and we obtain

$$\mathrm{Symb}^{ш}\, S^{\mathfrak{u}}_{n,1}(z) = -(1-z) \wedge z \otimes \big(z^{\otimes n-2}\big) \otimes z = -(1-z) \wedge z \otimes \big(z^{\otimes n-1}\big)\,,$$

so the claim holds for $p = 1$. Similarly, if $n = 1, p > 1$, then (7) collapses to the single summand

$$\mathrm{Symb}^{ш}\, S^{\mathfrak{u}}_{1,p}(z) = \mathrm{Symb}^{ш}\, S^{\mathfrak{u}}_{1,p-1}(z) \otimes (1-z)\,,$$

and the claim follows.

Otherwise if $n, p \geq 2$, both terms of (7) contribute, and we have

$$\begin{aligned}\mathrm{Symb}^{ш}\, S^{\mathfrak{u}}_{n,p}(z) &= -(1-z) \wedge z \otimes \Big(((1-z)^{\otimes p-1} ш z^{\otimes n-2}) \otimes z \\ &\qquad + ((1-z)^{\otimes p-2} ш z^{\otimes n-1}) \otimes (1-z)\Big)\,.\end{aligned} \tag{8}$$

By the recursive definition of the shuffle product ш, we obtain the result. □

The above identities and mod-products symbol expressions show that $\mathfrak{S}_3$ acts on the set of Nielsen polylogarithms of anharmonic ratios (we identify $\mathfrak{S}_3$ with the anharmonic group of order 6 generated by $z \mapsto 1-z$ and $z \mapsto 1/z$). The symbol of the Nielsen polylogarithm $S^{\mathfrak{u}}_{n,p}$ of weight $N = n+p$ is of the form $\mathrm{Symb}^{ш}(\mathrm{Li}^{\mathfrak{u}}_2(z)) \otimes ((1-z)^{\otimes p-1} ш z^{\otimes n-1})$, i.e. is symmetric in the last $N-2$ factors and antisymmetric in the first 2. The representation of $\mathfrak{S}_3$ on Nielsen polylogarithms is then isomorphic to the representation of $\mathfrak{S}_3$ on 2-variable homogeneous polynomials of degree $N-2$, under $z \leftrightarrow X$ and $1-z \leftrightarrow Y$, and then tensored with the sign representation for the $\mathrm{Li}^{\mathfrak{u}}_2(z)$ factor. More precisely, under the following identification, relations among $S^{\mathfrak{u}}_{n,p}$ and their associated polynomials correspond to each other in a bijective manner

$$\begin{aligned} S^{\mathfrak{u}}_{n,p}(z) &\mapsto \binom{N-2}{p-1} X^{n-1}Y^{p-1}\,, \\ S^{\mathfrak{u}}_{n,p}(1-z) &\mapsto -\binom{N-2}{p-1} Y^{n-1}X^{p-1}\,, \\ S^{\mathfrak{u}}_{n,p}\Big(1-\frac{1}{z}\Big) &\mapsto \binom{N-2}{p-1} (Y-X)^{n-1}(-X)^{p-1}\,. \end{aligned}$$

Theorem 7. *Writing $d = \lfloor (N+1)/3 \rfloor$, for $N \geq 2$, then the following set forms a basis for the symbols of Nielsen polylogarithms of weight N modulo products, under the anharmonic ratios*

$$\mathcal{B} \;=\; \{S^{\mathfrak{u}}_{N-i,i}(z), S^{\mathfrak{u}}_{N-i,i}(1-z), S^{\mathfrak{u}}_{N-i,i}(1-z^{-1})\}_{i=1}^{d-1} \,\cup\, \mathcal{X}_N^d \,,$$

where

$$\mathcal{X}_N^i \;=\; \begin{cases} \{S^{\mathfrak{u}}_{N-i,i}(z)\} & \text{if } N \equiv -1 \pmod 3\,, \\ \{S^{\mathfrak{u}}_{N-i,i}(z), S^{\mathfrak{u}}_{N-i,i}(1-z)\} & \text{if } N \equiv 0 \pmod 3\,, \\ \{S^{\mathfrak{u}}_{N-i,i}(z), S^{\mathfrak{u}}_{N-i,i}(1-z), S^{\mathfrak{u}}_{N-i,i}(1-z^{-1})\} & \text{if } N \equiv 1 \pmod 3\,. \end{cases}$$

In particular, depth $d = \lfloor (N+1)/3 \rfloor$ suffices to generate all the Nielsen polylogarithms of weight N modulo products.

Remark 8. Depth $d = \lfloor (N+1)/3 \rfloor$ is the expected depth necessary. Since the cobracket of depth p involves only terms of depth $< p$, one can iterate the 2-cobracket on the wedge factors in each term of $\delta^{\geq 2} S^{\mathfrak{L}}_{n,p}(z)$ to determine a lower bound on the depth. The cobracket $\delta^{\geq 2} S^{\mathfrak{L}}_{n,p}(z)$ involves both $S^{\mathfrak{L}}_{n-2,p-1}(z) \wedge \{1\}_3$ and $S^{\mathfrak{L}}_{n-1,p-2}(z) \wedge \{1\}_3$. Note that the highest depth contributions at most come from $S^{\mathfrak{L}}_{n-2k,p-1}(z) \wedge \{1\}_{2k+1}$ for $0 < k < \frac{n}{2}$, so we can very informally say that, in the cases $(n,p) = (2m-\varepsilon, m)$ for $\varepsilon \in \{0,1,2\}$, that modulo lower depth $S_{n,p}(z)$ "behaves like $S_{n-2,p-1}(z)$". (We will see instances of such a behaviour below, e.g., for $S_{3,2}$ and $S_{5,3}$, and, in a weaker form, for the cases $(n,p) = (2m-\varepsilon, m)$ as evidenced in Theorem 47 below.) So in weight $3M+k$, $k = 2,3,4$, with $n = 2M+k-1, p = M+1$, we can iterate down M times until we reach $((S^{\mathfrak{L}}_{n',p'}(z) \wedge \{1\}_3) \wedge \cdots) \wedge \{1\}_3$, $n'+p' = k$.

Since there are no (motivic) identities between the single term $\mathrm{Li}^{\mathfrak{L}}_2(z)$, between the terms $\mathrm{Li}^{\mathfrak{L}}_3(z)$ and $S^{\mathfrak{L}}_{1,2}(z)$, or between the terms $\mathrm{Li}^{\mathfrak{L}}_4(z), S^{\mathfrak{L}}_{2,2}(z), S^{\mathfrak{L}}_{1,3}(z)$, the left hand factor $S^{\mathfrak{L}}_{n',p'}(z)$, $n'+p' = 2,3,4$, cannot simplify to 0. This shows that depth $M+1$ is necessary for weight $3M+k$.

Proof of Theorem 7. For simplicity, we focus mainly on the case $N \equiv 1 \pmod 3$, say $N = 3M+1$. In this case, $\mathcal{X}_N^M$ consists of 3 elements, and we claim the full basis is

$$\mathcal{B} \;=\; \{S^{\mathfrak{u}}_{3M+1-i,i}(z), S^{\mathfrak{u}}_{3M+1-i,i}(1-z), S^{\mathfrak{u}}_{3M+1-i,i}(1-z^{-1})\}_{i=1}^{M} \,.$$

We need to check the image of $\mathcal{B}$ has full rank, in terms of the basis $\{X^i Y^{3M-1-i}\}_{i=0}^{3M-1}$ of 2-variable homogeneous polynomials of degree $3M-1$. Up to scalars, $S^{\mathfrak{u}}_{3M+1-i,i}(z) \mapsto X^{3M-i}Y^{i-1}$ and $S^{\mathfrak{u}}_{3M+1-i,i}(1-z) \mapsto X^{i-1}Y^{3M-i}$. So we can project the vector space of 2-variable degree $3M-1$ homogeneous polynomials down to the quotient by the subspace

$$\langle X^{3M-i}Y^{i-1}, X^{i-1}Y^{3M-i} \mid 1 \leq i \leq M \rangle \,.$$

This leaves only basis monomials $\{X^i Y^{3M-1-i}\}_{i=M}^{2M-1}$, and we have to consider whether the projection of the image of

$$\{S^{\mathfrak{u}}_{3M+1-i,i}(1-z^{-1})\}_{i=1}^{M} \,,$$

has full rank in this quotient space.

Up to scalars, we have

$$\begin{aligned} S^{\mathfrak{u}}_{3M+1-i,i}(1-z^{-1}) &\mapsto (Y-X)^{3M-i} X^{i-1} \\ &= \sum_{j=0}^{3M-i} (-1)^{3M-i-j} \binom{3M-i}{j} X^{3M-1-j} Y^j \,. \end{aligned}$$

In the quotient space only terms $j = M, \ldots, 2M-1$ survive, so the matrix of the map is given by

$$A_M = \left((-1)^{3M-i-j} \binom{3M-i}{j} \right)_{1 \leq i \leq M,\, M \leq j \leq 2M-1} \,.$$

After factoring out scalars from each row and column, reversing the order of the columns, and reindexing, we obtain the matrix

$$A'_M = \left(\binom{3M-i}{2M-j} \right)_{i,j=1}^{M} \,.$$

Standard evaluations show that

$$\det(A'_M) = \prod_{i=0}^{M-1} \frac{i!\,(i+2M)!}{(i+M)!^2} > 0 \,,$$

which proves that $\mathcal{B}$ is a basis in weight $3M+1$.

For the case of weight $N \equiv 0 \pmod 3$, say $N = 3M$, the quotient matrix up to scalars is

$$B'_M = \left(\binom{3M-1-i}{2M-1-j} \right)_{i,j=1}^{M-1},$$

with

$$\det(B'_M) = \frac{(2M-1)!}{(M-1)!} \prod_{i=0}^{M-1} \frac{i!\,(i+2M-1)!}{(M+i)!^2} > 0\,.$$

Finally, for the case of weight $N \equiv -1 \pmod 3$, say $N = 3M+2$, the quotient matrix up to scalars is

$$C'_M = \left(\binom{3M+1-i}{2M+1-j} \right)_{i,j=1}^{M},$$

with

$$\det(C'_M) = \prod_{i=0}^{M} \frac{i!\,(i+2M)!}{(M+i)!^2} > 0\,.$$

So the set $\mathcal{B}$ always forms a basis, as claimed. □

4. Clean single-valued Nielsen polylogarithms

For the purposes of numerical experimentation with Nielsen and classical polylogarithm identities, we can apply the 'clean single-valued' procedure from [16], to obtain functions which automatically lift mod-products symbol level identities to analytic identities, up to a constant of integration. We are grateful to the referee for providing us with the following very concise formulation of the procedure.

4.1. **Cleaning procedure.** We give a brief self-contained account of the procedure to obtain clean single-valued functions, by combining the single-valued map sv defined in [8], with a 'cleaning' map $R_\bullet$.

Cleaning map $R_\bullet$: Let K be a graded connected Hopf algebra $K = \bigoplus_N K_N$, with (reduced) coproduct Δ' and multiplication μ. Define the linear map $R_N \colon K_N \to K_N$ in grading N by

$$R_N = N \operatorname{id} - \mu(\operatorname{id} \otimes R_\bullet)\Delta'\,.$$

In terms of the grading $Y(x) = N \cdot x$, $x \in K_N$, this can be written

$$R_\bullet = Y - (\operatorname{id} \star R_\bullet - R_\bullet)\,,$$

so that $Y = \operatorname{id} \star R_\bullet$. Convoluting with the antipode S gives $S \star Y = R_\bullet$, so that $R_\bullet$ is again the Dynkin map (see $D_N = N\Pi_N$ in Section 2.2 above). An important property of the Dynkin map is that it annihilates products. Moreover since $\operatorname{Symb}\colon (\mathcal{A}_\bullet, ⧢, \Delta) \to (T(\mathcal{A}_1), ⧢, \Delta_{\mathrm{dec}})$ is a map of Hopf algebras, it intertwines the respective Dynkin operators, giving

$$\operatorname{Symb}^{⧢} = Y\Pi_\bullet \circ \operatorname{Symb} = \operatorname{Symb} \circ R_\bullet\,, \tag{9}$$

where $R_\bullet$ here is defined on the Hopf algebra of Goncharov's motivic iterated integrals.

Single valued map sv*:* The single valued map sv is a well-defined algebra homomorphism from $\mathcal{A}_\bullet$ to single-valued, real-analytic functions (see [8, 14], [13, Section 8.3], also for more general de Rham periods [12]). In terms of the period matrix P of an iterated integral, where a basis of the Betti cohomology is paired with a basis of the de Rham cohomology, the single-valued map sv may be computed via $\overline{P}^{-1}P$, where $\overline{P}$ denotes complex conjugation (corresponding to the real Frobenius involution F_∞ on Betti cohomology).

Main result on clean single-valued functions: For an iterated integral $I^{\mathfrak{u}} \in \mathcal{A}_N$, define the associated clean single-valued function $\mathcal{I}$ by

$$\mathcal{I} \coloneqq \frac{1}{N}(\operatorname{sv} \circ R_N)(I^{\mathfrak{u}})\,.$$

(Note that $\mathcal{I}$ has $\operatorname{sv} I^{\mathfrak{u}}$ as its main term.) The main result in [16] is that the clean single-valued functions $\mathcal{I}_i$ (associated to $I_i^{\mathfrak{u}}$ iterated integrals) automatically lift a mod-products symbol identity $\operatorname{Symb}^{⧢}\left(\sum_i \lambda_i I_i^{\mathfrak{u}}\right) = 0$, $\lambda_i \in \mathbb{Q}$, to an analytic identity

$$\sum\nolimits_i \lambda_i \mathcal{I}_i = \text{constant}\,.$$

Proof sketch. Suppose $\Lambda = \sum_i \lambda_i I_i^{\mathfrak{u}}$ satisfies $\mathrm{Symb}^{\sqcup\!\sqcup}(\Lambda) = 0$. Then (9) implies $R_\bullet(\Lambda) = 0$ in $\mathcal{A}_\bullet$, modulo iterated integrals with vanishing symbol (i.e. products of lower weight integrals a_i with constants $b_i \in \mathbb{C}$ whose symbols satisfy $\mathrm{Symb}(b_i) = 0$). So

$$R_\bullet(\Lambda) = \sum_i a_i b_i + c\,,$$

for some constant c. Idempotency (up to scaling) of $R_\bullet$ and the fact that $R_\bullet$ annihilates products (since it is the Dynkin map) implies $R_\bullet(\Lambda) = \frac{1}{N} R_\bullet(c)$, a constant. Now applying the single valued map shows

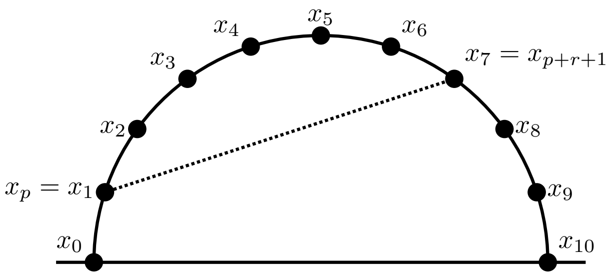
$$\frac{1}{N} \operatorname{sv} R_\bullet(\Lambda) = \text{constant}\,,$$

as claimed. □

Remark 9. The map $R_\bullet$ kills products, so when applying it to Δ' one only needs to keep the terms where the right-hand factor is not (trivially) a product. For Goncharov's motivic iterated integrals, such terms are encoded by an analogue of the *infinitesimal coproduct* D (cf. [9, Definition 4.4]) given by

$$DI^{\mathfrak{u}}(x_0; x_1, \ldots, x_N; x_{N+1}) = \\ \sum_{r=1}^{N-1} \sum_{p=0}^{N-r} I^{\mathfrak{u}}(x_0; x_1, \ldots, x_p, x_{p+r+1}, \ldots, x_N; x_{N+1}) \otimes I^{\mathfrak{u}}(x_p; x_{p+1}, \ldots x_{p+r}; x_{p+r+1})\,.$$

The terms in this can be mnemonically represented as the following type of segments cut out of a semicircular polygon



Here the main part containing the integration end points x_0 and x_{N+1} gives the left hand factor, and the cut-off segment gives the right hand factor in the tensor.

One can quickly check how $R_\bullet$ acts in the following cases.

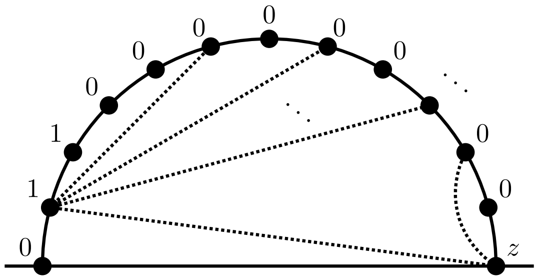
$$\begin{aligned} R_\bullet \zeta^{\mathfrak{u}}(2k+1) &= (2k+1)\zeta^{\mathfrak{u}}(2k+1)\,, \\ R_\bullet \log^{\mathfrak{u}}(z) &= \log^{\mathfrak{u}}(z)\,, \text{ and} \\ R_\bullet \operatorname{Li}_n^{\mathfrak{u}}(z) &= n \operatorname{Li}_n^{\mathfrak{u}}(z) - \log^{\mathfrak{u}}(z) \operatorname{Li}_{n-1}^{\mathfrak{u}}(z)\,. \end{aligned} \tag{10}$$

4.2. **Clean $S_{n,2}$ Nielsen polylogarithms.** One can easily derive a 'clean' version of $S_{n,p}^{\mathfrak{u}}$ for the symbol level, for all $n, p \in \mathbb{Z}_{>0}$, because MZV constants go to 0 under the symbol map. For a 'clean' analytic version of $S_{n,p}^{\mathfrak{u}}$ it is important to retain the constants, but this makes a general formula more difficult to obtain. We focus only on the clean version of $S_{n,2}^{\mathfrak{u}}(z)$ for the purposes of this paper.

We find that only the following terms contribute to the infinitesimal coproduct $DS_{n,2}^{\mathfrak{u}}(z)$,

$$\begin{aligned} &I^{\mathfrak{u}}(0; 1, \{0\}^{n-2j}; z) \otimes I^{\mathfrak{u}}(1; 1, \{0\}^{2j}; 0)\,, \quad j \geq 1\,, \\ &I^{\mathfrak{u}}(0; 1, z) \otimes I^{\mathfrak{u}}(1; 1, \{0\}^n; z)\,, \\ &I^{\mathfrak{u}}(0; 1, 1, \{0\}^{n-1}; z) \otimes I^{\mathfrak{u}}(0; 0; z)\,. \end{aligned}$$

Illustrated diagrammatically, they are the following segments (respectively the family in the upper left, connecting the first vertex '1' with any of the subsequent '0's, the long segment from '1' to 'z' at the bottom, and the short segment at the bottom right ending in 'z')



Moreover, after rewriting

$$I^{\mathfrak{u}}(1;1,\{0\}^n;z) \;=\; \underbrace{I^{\mathfrak{u}}(0;1,\{0\}^n;z)}_{=-\mathrm{Li}^{\mathfrak{u}}_{n+1}(z)} + \underbrace{I^{\mathfrak{u}}(1;1,\{0\}^n;0)}_{=\zeta^{\mathfrak{u}}(n+1)} \; (\text{mod products})\,,$$

we obtain, modulo products in the right hand tensor factor,

$$\begin{aligned} DS^{\mathfrak{u}}_{n,2}(z) \;=\; & S^{\mathfrak{u}}_{n-1,2}(z) \otimes \log^{\mathfrak{u}}(z) - \log^{\mathfrak{u}}(1-z) \otimes \mathrm{Li}^{\mathfrak{u}}_{n+1}(z) \\ & - \sum_{j=1}^{\lfloor n/2 \rfloor} \mathrm{Li}^{\mathfrak{u}}_{n+1-2j}(z) \otimes \zeta^{\mathfrak{u}}(2j+1) \; (\text{mod right-}\otimes\text{-factor products})\,. \end{aligned} \tag{11}$$

The clean version of $S^{\mathfrak{u}}_{n,2}(z)$ is therefore given by

$$\begin{aligned} S^{\sqcup\!\sqcup}_{n,2}(z) \;\coloneqq\; & \Big(\mathrm{id} - \frac{1}{n+2}\mu(\mathrm{id} \otimes R_\bullet)D\Big) S^{\mathfrak{u}}_{n,2}(z) \\ =\; & S^{\mathfrak{u}}_{n,2}(z) - \frac{1}{n+2} S^{\mathfrak{u}}_{n-1,2}(z) \log^{\mathfrak{u}}(z) + \frac{n+1}{n+2} \log^{\mathfrak{u}}(1-z)\,\mathrm{Li}^{\mathfrak{u}}_{n+1}(z) \\ & - \frac{1}{n+2} \log^{\mathfrak{u}}(1-z) \log^{\mathfrak{u}}(z)\,\mathrm{Li}^{\mathfrak{u}}_{n}(z) + \sum_{j=1}^{\lfloor n/2 \rfloor} \frac{2j+1}{n+2} \zeta^{\mathfrak{u}}(2j+1)\,\mathrm{Li}^{\mathfrak{u}}_{n+1-2j}(z)\,. \end{aligned} \tag{12}$$

Using (10), the clean version of Li_n is given by

$$\mathrm{Li}^{\sqcup\!\sqcup}_n(z) \;\coloneqq\; \mathrm{Li}^{\mathfrak{u}}_n(z) - \frac{1}{n} \log^{\mathfrak{u}}(z)\,\mathrm{Li}^{\mathfrak{u}}_{n-1}(z)\,.$$

4.3. **Single-valued $S_{n,2}$ Nielsen polylogarithms.** To obtain a single-valued version of the Nielsen polylogarithm $S_{n,2}$, we can apply Brown's single-valued map sv to $S^{\mathfrak{u}}_{n,2}$. We do not go through the whole calculation via period matrices $\overline{P}^{-1}P$ here, we note the resulting combination satisfies four key defining properties, explained below. Using, as in the introduction, the convention $S_{0,p}(z) = \frac{(-1)^p}{p!} \log^p(1-z)$, via (3), we obtain the following

$$\begin{aligned} \mathrm{sv}\, S^{\mathfrak{u}}_{n,2}(z) \;=\; & \Big(S_{n,2}(z) + (-1)^{n+1} S_{n,2}(\bar z)\Big) - \log(1-\bar z)\big(\mathrm{Li}_{n+1}(z) + (-1)^n \mathrm{Li}_{n+1}(\bar z)\big) \\ & - \sum_{j=0}^{n-1} \frac{(-1)^j}{(n-j)!} \log^{n-j}\big(|z|^2\big) \Big(S_{j,2}(\bar z) + \log(1-\bar z)\,\mathrm{Li}_{j+1}(\bar z)\Big) \\ & + \sum_{\substack{k=1 \\ k \text{ odd}}}^{n-1} \sum_{j=1}^{n-k} \frac{2(-1)^j \zeta(k+2)}{(n-j-k)!} \mathrm{Li}_j(\bar z) \log^{n-j-k}\big(|z|^2\big)\,. \end{aligned}$$

We outline the defining properties, through the single-valued map, that sv $S^{\mathfrak{u}}_{n,2}$ must satisfy.

i) $\frac{\partial}{\partial z} \mathrm{sv}\, S^{\mathfrak{u}}_{n,2}(z) = \frac{1}{z} \mathrm{sv}\, S^{\mathfrak{u}}_{n-1,2}(z)$,
ii) No monodromy at $z = 0$,
iii) No monodromy at $z = 1$,
iv) Vanishing integration constant: $\lim_{z \to 0} \mathrm{sv}\, S^{\mathfrak{u}}_{n,2}(z) = 0$.

Properties i), ii) and iv) are straightforward to check. For i), one directly checks sv $S^{\mathfrak{u}}_{n,2}$ satisfies the given recursion. For ii), use that $\log|z|$ is single-valued at 0, and all polylogarithms with argument z or $\bar z$ are holomorphic or anti-holomorphic at 0, respectively.

For iii), one uses that the monodromy of $\mathrm{Li}_j(z)$ clockwise around 1 is

$$\frac{2\pi i}{(j-1)!} \log^{j-1}(z)\,.$$

We can then compute the monodromy of $S_{n,2}(z)$ clockwise around 1, using path (de)composition and that $I(0;\{0\}^j,1;1) = (-1)^{j+1}\zeta(j+1)$ from path reversal and shuffle regularisation. One finds the monodromy is

$$\frac{1}{2\cdot n!}(2\pi i)^2 \log^n(z) + 2\pi i \operatorname{Li}_{n+1}(z) - 2\pi i \sum_{j=1}^{n} \frac{\zeta(j+1)}{(n-j)!} \log^{n-j}(z)\,.$$

With these two results, we can check that $\operatorname{sv} S^{\mathfrak{u}}_{n,2}(z)$ has no monodromy at $z=1$.

Computed already in [8] is the following single-valued version of $\operatorname{Li}^{\mathfrak{u}}_n$, obtained from the single-valued map sv:

$$\operatorname{sv}\operatorname{Li}^{\mathfrak{u}}_n(z) \;=\; \Big(\operatorname{Li}_n(z) - (-1)^n \operatorname{Li}_n(\bar{z})\Big) - \sum_{j=1}^{n-1} \frac{(-1)^j}{(n-j)!} \operatorname{Li}_j(\bar{z}) \log^{n-j}\big(|z|^2\big)\,,$$

although this does not yet satisfy clean functional equations. The single-valued version of $\operatorname{Li}^{ш}_n(z)$, namely

$$\begin{aligned}\mathcal{L}^{ш}_n(z) \;\coloneqq\; & \Big(\operatorname{Li}_n(z) - (-1)^n \operatorname{Li}_n(\bar{z})\Big) - \frac{1}{n}\operatorname{Li}_{n-1}(z)\log\big(|z|^2\big) \\ & - \sum_{j=1}^{n-1} \frac{j(-1)^j}{n(n-j)!} \operatorname{Li}_j(\bar{z}) \log^{n-j}\big(|z|^2\big)\,,\end{aligned} \tag{13}$$

does have this property.

Remark 10. This single-valued polylogarithm differs from Zagier's single-valued version (denoted $P_n(z)$ in [53])

$$\mathscr{L}_n(z) \;\coloneqq\; \operatorname{Re}_n\left(\sum_{j=0}^{n-1} \frac{2^j B_j}{j!} \log^j|z| \operatorname{Li}_{n-j}(z)\right),$$

where $\operatorname{Re}_n = \operatorname{Re}$ for n odd, $\operatorname{Re}_n = \operatorname{Im}$ for n even and B_j is the j-th Bernoulli number. It is shown in [16] how Zagier's single-valued version $\mathscr{L}_n$ and the clean single-valued version $\mathcal{L}^{ш}_n$ are related.

Applying the single-valued map to the expression for the clean Nielsen polylogarithm $S^{ш}_{n,2}(z)$ in (12) gives the following clean single-valued Nielsen polylogarithm

$$\begin{aligned}\mathcal{S}^{ш}_{n,2}(z) \;\coloneqq\; & \Big(S_{n,2}(z) - (-1)^{n+2} S_{n,2}(\bar{z})\Big) - \frac{1}{n+2}\log\big(|z|^2\big)\Big(S_{n-1,2}(z) + \log(1-z)\operatorname{Li}_n(z)\Big) \\ & + \frac{1}{n+2}\Big((n+1)\log(1-z) - \log(1-\bar{z})\Big)\Big(\operatorname{Li}_{n+1}(z) - (-1)^{n+1}\operatorname{Li}_{n+1}(\bar{z})\Big) \\ & + \sum_{j=1}^{n} \frac{(-1)^j}{n+2}\Big\{(j+1)S_{j-1,2}(\bar{z}) - \Big(j\log(1-z) - \log(1-\bar{z})\Big)\operatorname{Li}_j(\bar{z})\Big\}\frac{\log^{n-j+1}\big(|z|^2\big)}{(n-j+1)!} \\ & + \sum_{\substack{k=1\\ k \text{ odd}}}^{n-1} \frac{2\zeta(k+2)}{n+2}\Big\{(k+2)\operatorname{Li}_{n-k}(z) + \sum_{j=1}^{n-k} j(-1)^j \frac{\log^{n-j-k}\big(|z|^2\big)}{(n-j-k)!}\operatorname{Li}_j(\bar{z})\Big\}\,.\end{aligned}$$

In particular, the main term is $\operatorname{Re}_{n+2} S_{n,2}(z)$, just as $\operatorname{Re}_n \operatorname{Li}_n(z)$ is the main term for $\mathcal{L}^{ш}_n$ and $\mathscr{L}_n$.

Combined with Lemma 6, the main result on clean single-valued functions shows that $\mathcal{S}^{ш}_{n,p}$ satisfies mod-products symbol level identities up to an integration constant, i.e. if

$$\operatorname{Symb}^{ш}\Big(\sum\nolimits_i \alpha_i S_{n_i,p_i}(x_i)\Big) = 0\,,$$

then

$$\sum\nolimits_i \alpha_i \mathcal{S}^{ш}_{n_i,p_i}(x_i) = \text{constant}\,.$$

In particular, we obtain (with a small computation to determine the constants, given below) the following clean-single-valued versions of the inversion and reflection results in Propositions 2 and 4.

$$\begin{aligned}\mathcal{S}^{ш}_{n,p}(1-z) \;=\; & -\mathcal{S}^{ш}_{p,n}(z) + \frac{1}{n+p}\binom{n+p}{n}\zeta^{\mathrm{sv}}(n+p)\,, \\ \mathcal{S}^{ш}_{n,p}\Big(\frac{1}{z}\Big) \;=\; & (-1)^n \sum_{k=0}^{p-1}(-1)^k\binom{n+k-1}{k}\mathcal{S}^{ш}_{n+k,p-k}(z) \\ & - (-1)^n \frac{\zeta^{\mathrm{sv}}(n+p)}{n+p}\left(1 + (-1)^p\binom{n+p-1}{n}\right),\end{aligned} \tag{14}$$

where $\zeta^{\mathrm{sv}}(n)$ is the single-valued MZV given by

$$\zeta^{\mathrm{sv}}(n) \;=\; \begin{cases} 2\zeta(n) & n \text{ odd}\,, \\ 0 & n \text{ even}\,. \end{cases} \tag{15}$$

In both cases, a small computation must be done to determine the constant in terms of (single-valued) MZV's. For the first identity, we would directly obtain the constant $\mathcal{S}^{ш}_{n,p}(1)$ from Proposition 2. By the definition of $\mathcal{S}^{ш}_{n,p}$ we compute

$$\mathcal{S}^{ш}_{n,p}(1) = \frac{1}{n+p} R_{n+p} S_{n,p}(1) = \frac{1}{n+p} R_{n+p} \zeta(\{1\}^{p-1}, n+1)$$

By Remark 3, $\zeta(\{1\}^{p-1}, n+1)$ is a polynomial in Riemann zeta values and since $R_\bullet$ kills products, it suffices to extract the coefficient of $\zeta(n+p)$ from the generating series formula therein. This leads to the claimed constant. In the second case, the constant should directly be

$$\frac{1}{n+p} \operatorname{sv} R_{n+p}\big((-1)^p C_{n,p}\big)\,.$$

We can look up the following expression for $C_{n,p}$ in terms of $S_{n',p'}(1)$, from [40, Theorem 1, Equation 7.2],

$$C_{n,p} = (-1)^{n+p-1} \sum_{k=0}^{p-1} (-1)^k (1-(-1)^n \delta_{k,0}) \binom{n+k-1}{k} S_{n+k,p-k}(1) \text{ (mod products)}\,,$$

and apply the previous computation of $\mathcal{S}^{ш}_{n',p'}(1)$ to it. Alternatively, we can set $z=1$ in the identity to fix the constant; this essentially leads to the clean single-valued version of the above combination directly. After computing the binomial sum, we obtain the claimed constant.

5. The algebraic Li_2, Li_3 and Li_4 functional equations

We recall the following infinite family of functional equations given in [26], for Li_2, Li_3 and Li_4. We will use them in later sections, particularly Sections 6.2, 7.2, 8.1 and 9.4, to provide some additional evidence for the behaviour we expect of Nielsen polylogarithms modulo the classical polylogarithms Li_n.

Let $a, b, c \in \mathbb{Z} \smallsetminus \{0\}$ be such that $a+b+c=0$, and let $\{p_i(t)\}_{i=1}^r$ be the roots (counted with multiplicity) of $x^a(1-x)^b = t$, or strictly speaking the roots of the associated polynomial function $x^{\max(0,a)}(1-x)^{\max(0,b)} - t x^{\max(0,-a)}(1-x)^{\max(0,-b)}$. Furthermore, assume $a>0$ for convenience. (If $a<0$, we can simply invert both sides to obtain $x^{-a}(1-x)^{-b} = t^{-1}$, to reduce to that case.) Then with the earlier notation that $\{z\}_n$ means the image of the motivic $\mathrm{Li}_n(z)$ modulo products, we have

$$\sum_{i=1}^r \{p_i(t)\}_2 \;=\; 0\,, \tag{16}$$

$$\sum_{i=1}^r -\frac{1}{a}\{1-p_i(t)\}_3 + \frac{1}{b}\{p_i(t)\}_3 \;=\; \begin{cases} 0 & \text{if } 0<a,b\,, \\ -\{1\}_3 & \text{if } b<0<a\,, \end{cases} \tag{17}$$

$$\sum_{i=1}^r -\frac{1}{a}\{1-p_i(t)\}_4 + \frac{1}{b}\{p_i(t)\}_4 + \frac{1}{c}\{1-p_i(t)^{-1}\}_4 \;=\; 0\,. \tag{18}$$

As in [26], we observe the following facts about p_i, where again we assume $a>0$ for convenience:

$$\prod_{i=1}^r p_i(t) \;=\; \begin{cases} \pm t & \text{if } a+b>0\,, \\ \pm 1 & \text{if } a+b<0\,, \end{cases}$$

$$1-p_i(t) \;=\; \frac{t^{1/b}}{p_i^{a/b}(t)}\,, \text{ up to a } b\text{-th root of unity}\,.$$

Given the formulae for $\prod_i p_i(t)$ and $1-p_i(t)$ above, it is straightforward to check the mod-products symbols of equations (16), (17), and (18) vanish. To fix the constants, we consider the limit $t\to 0$, and use that $\{0\}_n = \{\infty\}_n = 0$, for $n=2,3,4$ and $\{1\}_2 = \{1\}_4 = 0$ (i.e. $\zeta^{\mathfrak{L}}(2)=0$ and $\zeta^{\mathfrak{L}}(4)=0$ taken modulo products, whereas $\{1\}_3 \neq 0$ or equivalently $\zeta^{\mathfrak{L}}(3)\neq 0$). If $a>b>0$ we obtain roots $p_i(t)=0$ with multiplicity a, and roots $p_i(t)=1$ with multiplicity b; in (17) these contributions cancel and in the other identities each term already vanishes. Whereas if $-a<b<0$ we obtain roots $p_i(t)=0$ with multiplicity a, giving the above constants. Likewise if $b<-a<0$ we obtain roots $p_i(t)=0$ with multiplicity a, and roots $p_i(t)=\infty$ with multiplicity $-b-a$, again giving the above constants.

Note that the case $(a,b,c) = (1,2,-3)$, or any permutation thereof, can be rationally parametrised over $\mathbb{Q}$. Namely the solutions to

$$x(1-x)^2 \;=\; \frac{(1-t)^2t^2}{(1-t+t^2)^3}$$

are given by

$$p_1(t) \;=\; \frac{1}{1-t+t^2}\,, \quad p_2(t) \;=\; \frac{t^2}{1-t+t^2}\,, \quad p_3(t) \;=\; \frac{(1-t)^2}{1-t+t^2}\,.$$

The case $(a,b,c) = (1,3,-4)$, or any permutation thereof, can also be rationally parametrised but this time only over $\mathbb{Q}(i)$. In fact, for some variable t let

$$\begin{aligned} U_1 &= -1-(1-2i)t+it^2\,, \quad U_2 \;=\; 1+t+\;\;t^2\,, \\ U_3 &= -i-(1+2i)t-\;\;t^2\,, \quad U_4 \;=\; i+t-it^2\,, \\ V &= -(U_1+U_2)(U_1+U_3)(U_2+U_3)\,. \end{aligned}$$

Then the roots of

$$x(1-x)^3 \;=\; \prod_{j=1}^{4}(U_j^3/V)\,,$$

are given by

$$p_j(t) \;=\; U_j^3/V\,,$$

for $j = 1, \ldots, 4$.

6. Nielsen polylogarithms in weight 5

In this section we prove one of our main results, stating that "$S_{3,2}$ evaluated on functional equations of Li_2 is expressible in terms of Li_5". We first corroborate this for the simpler two-term relations in Propositions 11 and 12 as well as for a family of algebraic functional equations (which are not known to be consequences of the five-term relation) in Proposition 14, all of which have already been proved in [15], before turning to the basic five-term relation itself (Theorem 16) and subsequent specialisations like distribution relations as well as ladders and special values. As a further corollary we recover a functional equation for Li_5 recently obtained in [47].

Preconsideration: Following Section 2, and the result in Equation (11), the 2-part of the motivic cobracket of $S_{3,2}(z)$ is computed to be

$$\delta^{\geq 2} S^{\mathfrak{L}}_{3,2}(z) \;=\; -\,\{z\}_2 \wedge \{1\}_3\,.$$

Since $\{1\}_3 \neq 0$, this does not vanish in general, and we cannot reduce $S_{3,2}(z)$ to Li_5 on the motivic level, hence we should not expect this on a function level, either.

On the other hand, combinations $\sum_i \alpha_i[x_i]$ such that $\sum_i \alpha_i\{x_i\}_2 = 0$, i.e. functional equations for Li_2, will automatically annihilate $\delta^{\geq 2}\sum_i \alpha_i S^{\mathfrak{L}}_{3,2}(x_i)$. On this basis we expect the Nielsen polylogarithm $S_{3,2}$ to behave like Li_2, when viewed modulo Li_5 and products.

6.1. Two-term identities.

We can give relatively simple analytic identities for $S_{3,2}$ under the basic two-term identities $\{z\}_2 + \{z^{-1}\}_2 = 0$ and $\{z\}_2 + \{1-z\}_2 = 0$ for Li_2. These identities are already contained within the reflection and inversion results, and so can be shown without the need to invoke the clean single-valued functions.

Proposition 11. *For all $z \in \mathbb{C} \smallsetminus [0,\infty)$, the following identity holds*

$$\begin{aligned} S_{3,2}(z) + S_{3,2}\Big(\frac{1}{z}\Big) \;=\;& 3\,\mathrm{Li}_5(z) - \mathrm{Li}_4(z)\log(-z) - \frac{1}{5!}\log^5(-z) \\ &+ \frac{1}{2!}\zeta(3)\log^2(-z) + \frac{7}{4}\zeta(4)\log(-z) + \Big(\zeta(5) + \zeta(2)\zeta(3)\Big)\,. \end{aligned}$$

Proof. This is just the case $S_{3,2}$ of Proposition 4. ∎

Proposition 12. *For all $z \in \mathbb{C} \smallsetminus \big((-\infty,0] \cup [1,\infty)\big)$, the following identity holds*

$$\begin{aligned} S_{3,2}(1-z) + S_{3,2}(z) \;=\; & \operatorname{Li}_5(1-z) + \operatorname{Li}_5(1-z^{-1}) + \operatorname{Li}_5(z) - \operatorname{Li}_4(1-z)\log(z) - \operatorname{Li}_4(z)\log(1-z) \\ & - \frac{1}{5!}\log^5(z) + \frac{1}{4!}\log^4(z)\log(1-z) - \frac{1}{3!\,2!}\log^3(z)\log^2(1-z) \\ & - \frac{1}{3!}\zeta(2)\log^3(z) + \frac{1}{2!}\zeta(2)\log^2(z)\log(1-z) + \zeta(3)\log(z)\log(1-z) \\ & + \zeta(4)\log(1-z) - \frac{3}{4}\zeta(4)\log(z) + \Big(\zeta(5) - \zeta(2)\zeta(3)\Big)\,. \end{aligned}$$

Proof. Use differentiation to reduce to a weight 4 identity, which can also be verified. The constant of integration is fixed by evaluating as $z \to 1$, to obtain $S_{3,2}(1) - \operatorname{Li}_5(1) = -\zeta(2)\zeta(3) + \zeta(5)$.

Because of the argument in Section 3.3, one can verify that such an identity exists, and that it follows from the reflection and inversion just by computing the mod-products symbol. From the recursion (7) for $\operatorname{Symb}^{\sqcup\!\sqcup} S_{n,p}(z)$, and the reduction of weight 4 Nielsen polylogarithms to Li_4, we have

$$\begin{aligned} \operatorname{Symb}^{\sqcup\!\sqcup} S^{\mathfrak{u}}_{3,2}(z) \;&=\; \operatorname{Symb}^{\sqcup\!\sqcup} S^{\mathfrak{u}}_{2,2}(z) \otimes z \;+\; \operatorname{Symb}^{\sqcup\!\sqcup} S^{\mathfrak{u}}_{3,1}(z) \otimes (1-z) \\ &=\; \operatorname{Symb}^{\sqcup\!\sqcup} \Big(-\operatorname{Li}^{\mathfrak{u}}_4(1-z) + \operatorname{Li}^{\mathfrak{u}}_4(z) - \operatorname{Li}^{\mathfrak{u}}_4(1-z^{-1})\Big) \otimes z \\ &\quad + \operatorname{Symb}^{\sqcup\!\sqcup} \operatorname{Li}^{\mathfrak{u}}_4(z) \otimes (1-z)\,. \end{aligned}$$

We also have $\operatorname{Symb}^{\sqcup\!\sqcup} \operatorname{Li}^{\mathfrak{u}}_5(z) = \operatorname{Symb}^{\sqcup\!\sqcup} \operatorname{Li}^{\mathfrak{u}}_4(z) \otimes z$. Recall too that $\operatorname{Symb}^{\sqcup\!\sqcup} \operatorname{Li}^{\mathfrak{u}}_4(z) = -\operatorname{Symb}^{\sqcup\!\sqcup} \operatorname{Li}^{\mathfrak{u}}_4(z^{-1})$, and that our tensor symbols are written multiplicatively in each slot. We compute directly that the mod-products symbol of the left hand side is

$$\begin{aligned} & \operatorname{Symb}^{\sqcup\!\sqcup} \Big(S^{\mathfrak{u}}_{3,2}(1-z) + S^{\mathfrak{u}}_{3,2}(z) \Big) \\ &=\; \operatorname{Symb}^{\sqcup\!\sqcup} \Big(-\operatorname{Li}^{\mathfrak{u}}_4(1-z) + \operatorname{Li}^{\mathfrak{u}}_4(z) - \operatorname{Li}^{\mathfrak{u}}_4(1-z^{-1}) \Big) \otimes z + \operatorname{Li}^{\mathfrak{u}}_4(z) \otimes (1-z) \\ &\quad + \operatorname{Symb}^{\sqcup\!\sqcup} \Big(-\operatorname{Li}^{\mathfrak{u}}_4(z) + \operatorname{Li}^{\mathfrak{u}}_4(1-z) - \operatorname{Li}^{\mathfrak{u}}_4\Big(\frac{z}{z-1}\Big) \Big) \otimes (1-z) + \operatorname{Li}^{\mathfrak{u}}_4(1-z) \otimes z \\ &=\; \operatorname{Symb}^{\sqcup\!\sqcup} \Big(\operatorname{Li}^{\mathfrak{u}}_4(z) - \operatorname{Li}^{\mathfrak{u}}_4(1-z^{-1}) \Big) \otimes z + \operatorname{Symb}^{\sqcup\!\sqcup} \Big(\operatorname{Li}^{\mathfrak{u}}_4(1-z) - \operatorname{Li}^{\mathfrak{u}}_4\Big(\frac{z}{z-1}\Big) \Big) \otimes (1-z) \\ &=\; \operatorname{Symb}^{\sqcup\!\sqcup} \operatorname{Li}^{\mathfrak{u}}_4(1-z) \otimes (1-z) + \operatorname{Symb}^{\sqcup\!\sqcup} \operatorname{Li}^{\mathfrak{u}}_4\Big(\frac{z-1}{z}\Big) \otimes \frac{z-1}{z} + \operatorname{Symb}^{\sqcup\!\sqcup} \operatorname{Li}^{\mathfrak{u}}_4(z) \otimes z\,. \end{aligned}$$

The last expression is already the mod-products symbol of the right hand side, i.e. it equals:

$$\operatorname{Symb}^{\sqcup\!\sqcup} \Big(\operatorname{Li}^{\mathfrak{u}}_5(1-z) + \operatorname{Li}^{\mathfrak{u}}_5(1-z^{-1}) + \operatorname{Li}^{\mathfrak{u}}_5(z) \Big)\,.$$

The remaining terms on the right hand side do not contribute, as they are already non-trivial products.

Alternatively, the above symbol calculation translates to the following straightforward to check equality of polynomial invariants (see Section 3.3):

$$\begin{aligned} S^{\mathfrak{u}}_{3,2}(z) + S^{\mathfrak{u}}_{3,2}(1-z) \;&\mapsto\; 3YX^2 - 3XY^2\,, \\ \operatorname{Li}^{\mathfrak{u}}_5(z) + \operatorname{Li}^{\mathfrak{u}}_5(1-z) + \operatorname{Li}^{\mathfrak{u}}_5(1-z^{-1}) \;&\mapsto\; X^3 - Y^3 + (Y-X)^3\,. \end{aligned}$$ ☐

By setting $z = -1$ in the first identity, and $z = \frac{1}{2}$ in the second, we recover the following evaluations, contained in Table 2 and Equation 9.9 [40].

$$S_{3,2}(-1) \;=\; -\frac{29}{32}\zeta(5) + \frac{1}{2}\zeta(2)\zeta(3)\,, \tag{19}$$

$$\begin{aligned} S_{3,2}\Big(\frac{1}{2}\Big) \;=\; & \operatorname{Li}_5\Big(\frac{1}{2}\Big) + \operatorname{Li}_4\Big(\frac{1}{2}\Big)\log(2) + \frac{1}{2}\Big(\frac{1}{16}\zeta(5) - \zeta(2)\zeta(3)\Big) - \frac{1}{8}\zeta(4)\log(2) \\ & + \frac{1}{2!}\zeta(3)\log^2(2) - \frac{1}{3!}\zeta(2)\log^3(2) + \frac{3}{5!}\log^5(2)\,. \end{aligned} \tag{20}$$

These reductions correspond to the fact that $\{\frac{1}{2}\}_2 = \{-1\}_2 = 0$, so that the 2-parts of the motivic cobrackets of $S^{\mathfrak{L}}_{3,2}(\frac{1}{2})$ and of $S^{\mathfrak{L}}_{3,2}(-1)$ vanish.

Remark 13 (MZV Data Mine for $S_{n,p}(\frac{1}{2})$)**.** Note that any Nielsen polylogarithm $S_{n,p}(\frac{1}{2})$, including the classical polylogarithms, can be rewritten as an alternating MZV via the Möbius transformation

$$\begin{gathered}\mathbb{P}^1 \smallsetminus \{\infty, 0, 1, \tfrac{1}{2}\} \to \mathbb{P}^1 \smallsetminus \{\infty, 1, -1, 0\} \\ z \mapsto 1 - 2z\end{gathered}$$

which identifies the indicated punctured curves. For example,

$$\begin{aligned} S_{3,2}(\tfrac{1}{2}) &= I(0; 1, 1, 0, 0, 0; \tfrac{1}{2}) \\ &= I(1; -1, -1, 1, 1, 1; 0) \\ &= -I(0; 1, 1, 1, -1, -1; 1) \\ &= -\operatorname{Li}_{1,1,1,1,1}(1, 1, -1, 1, -1) \\ &= -\zeta(1, 1, \overline{1}, 1, \overline{1}) \,, \end{aligned}$$

via the reversal of paths property of iterated integrals.

In particular, any identity between special values of Nielsen polylogarithms at $z = \frac{1}{2}, \pm 1$ can be verified by appealing to the MZV Data Mine [1].

6.2. **Algebraic Li_2 functional equation.** Before dealing with the full five-term identity, we consider the simplest case of the algebraic Li_2 functional equation from Section 5. In more general cases, we have greater success reducing these algebraic functional equations, and so this is a good place to introduce them. This identity was already observed in [15], where it was used to obtain a new functional equation for Li_5. Note that the special case $a = b = 1$ is essentially Proposition 12.

Proposition 14 (Proposition 7.4.19 in [15])**.** *Let $a, b, c \in \mathbb{Z} \smallsetminus \{0\}$, with $a + b + c = 0$, and let $\{p_i(t)\}_{i=1}^r$ be the roots of $x^a(1-x)^b = t$. Then the following reduction holds on the level of the mod-products symbol*

$$\sum_{i=1}^r S^{\mathfrak{u}}_{3,2}(p_i(t)) \overset{\sqcup\!\sqcup}{=} \sum_{i=1}^r \bigg\{ \frac{b-a}{b} \operatorname{Li}^{\mathfrak{u}}_5(p_i(t)) + \frac{b}{a} \operatorname{Li}^{\mathfrak{u}}_5(1 - p_i(t)) + \frac{b}{a+b} \operatorname{Li}^{\mathfrak{u}}_5(1 - p_i(t)^{-1}) \bigg\} \,.$$

Corollary 15. *Assuming $a > 0$ for convenience, we have the clean single-valued identity*

$$\sum_{i=1}^r \mathcal{S}^{\sqcup\!\sqcup}_{3,2}(p_i(t)) - \sum_{i=1}^r \bigg\{ \frac{b-a}{b} \mathcal{L}^{\sqcup\!\sqcup}_5(p_i(t)) + \frac{b}{a} \mathcal{L}^{\sqcup\!\sqcup}_5(1 - p_i(t)) + \frac{b}{a+b} \mathcal{L}^{\sqcup\!\sqcup}_5(1 - p_i(t)^{-1}) \bigg\}$$

$$= \begin{cases} 2a\zeta(5) & \text{if } b > 0 \,, \\ -2b\zeta(5) & \text{if } -a < b < 0 \,, \\ -2(a+b)\zeta(5) & \text{if } b < -a \,. \end{cases}$$

Proof. Consider the limit $t \to 0$ and use $\mathcal{L}^{\sqcup\!\sqcup}_5(0) = \mathcal{L}^{\sqcup\!\sqcup}_5(\infty) = 0$, $\mathcal{L}^{\sqcup\!\sqcup}_5(1) = 2\zeta(5)$, $\mathcal{S}^{\sqcup\!\sqcup}_{3,2}(0) = 0$, $\mathcal{S}^{\sqcup\!\sqcup}_{3,2}(1) = 4\zeta(5)$, and $\mathcal{S}^{\sqcup\!\sqcup}_{3,2}(\infty) = 2\zeta(5)$.

If $b > 0$, we obtain roots $p_i(t) = 0$ with multiplicity a and $p_i(t) = 1$ with multiplicity b, giving the constant $2a\zeta(5)$. If $-a < b < 0$, we obtain roots $p_i(t) = 0$ with multiplicity a, giving the constant $-2b\zeta(5)$. Finally if $b < -a$, we obtain roots $p_i(t) = 0$ with multiplicity a and roots $p_i(t) = \infty$ with multiplicity $-b - a$, giving the constant $-2(a + b)\zeta(5)$. □

Proof of Proposition 14. For simplicity, we shall henceforth write $p_i = p_i(t)$. Then $1 - p_i = t^{1/b} p_i^{-a/b}$ by the observations in Section 5. Using the recursion from Lemma 6, and the reduction of $S_{2,2}$ to Li_4 in Proposition 5, the mod-products symbol of the left hand side is

$$\begin{aligned} \sum_{i=1}^r \bigg\{ &\operatorname{Symb}^{\sqcup\!\sqcup} \Big(-\operatorname{Li}^{\mathfrak{u}}_4(1 - p_i) + \operatorname{Li}^{\mathfrak{u}}_4(p_i) - \operatorname{Li}^{\mathfrak{u}}_4(1 - p_i^{-1}) \Big) \otimes p_i \\ &+ \frac{1}{b} \operatorname{Symb}^{\sqcup\!\sqcup} \operatorname{Li}^{\mathfrak{u}}_4(p_i) \otimes t - \frac{a}{b} \operatorname{Symb}^{\sqcup\!\sqcup} \operatorname{Li}^{\mathfrak{u}}_4(p_i) \otimes p_i \bigg\} \,. \end{aligned}$$

The mod-products symbol of the right hand side is

$$\begin{aligned} \sum_{i=1}^r \bigg\{ &\frac{b-a}{b} \operatorname{Symb}^{\sqcup\!\sqcup} \operatorname{Li}^{\mathfrak{u}}_4(p_i) \otimes p_i + \frac{b}{a} \Big(\frac{1}{b} \operatorname{Symb}^{\sqcup\!\sqcup} \operatorname{Li}^{\mathfrak{u}}_4(1 - p_i) \otimes t - \frac{a}{b} \operatorname{Symb}^{\sqcup\!\sqcup} \operatorname{Li}^{\mathfrak{u}}_4(1 - p_i) \otimes p_i \Big) \\ &+ \frac{b}{a+b} \Big(\frac{1}{b} \operatorname{Symb}^{\sqcup\!\sqcup} \operatorname{Li}^{\mathfrak{u}}_4(1 - p_i^{-1}) \otimes t - \frac{a}{b} \operatorname{Symb}^{\sqcup\!\sqcup} \operatorname{Li}^{\mathfrak{u}}_4(1 - p_i^{-1}) \otimes p_i - \operatorname{Symb}^{\sqcup\!\sqcup} \operatorname{Li}^{\mathfrak{u}}_4(1 - p_i^{-1}) \otimes p_i \Big) \bigg\} \,. \end{aligned}$$

In the difference of the left hand side and the right hand side, all terms ending in $\otimes p_i$ cancel. We are left with

$$\Big\{ \sum_{i=1}^{r} \mathrm{Symb}^{\sqcup\!\sqcup} \Big(\frac{1}{b} \operatorname{Li}_4^{\mathfrak{u}}(p_i) - \frac{1}{a} \operatorname{Li}_4^{\mathfrak{u}}(1-p_i) - \frac{1}{a+b} \operatorname{Li}_4^{\mathfrak{u}}(1-p_i^{-1}) \Big) \Big\} \otimes t \,,$$

which vanishes since the expression in brackets is an algebraic functional equation for Li_4. □

6.3. **Five-term identity.** Our main result is that $S_{3,2}$, evaluated on the five-term relation, can be reduced to explicit Li_5 terms. On account of the known two-term inversion and reflection identities for $S_{3,2}$ in Propositions 11 and 12 above, we can without loss of generality fully antisymmetrise the five-term relation over $\mathfrak{S}_5$. Here and below we use the notation

$$f\big(\sum\nolimits_j \nu_j [x_j]\big) \coloneqq \sum\nolimits_j \nu_j f(x_j) \,,$$

i.e. we extend functions to formal linear combinations $\sum_j \nu_j [x_j]$ by linearity.

Theorem 16 ($S_{3,2}$ of the five-term relation)**.** *For indeterminates $x_1, \dots, x_5$, we have the following identity between the mod-products symbols of $S_{3,2}^{\mathfrak{u}}$ and $\operatorname{Li}_5^{\mathfrak{u}}$ in $\bigotimes_{i=1}^{5} \mathbb{Q}(x_1,\dots,x_5)^\times \otimes_{\mathbb{Z}} \mathbb{Q}$*

$$\mathrm{Alt}_5 \bigg(11 S_{3,2}^{\mathfrak{u}}(\mathrm{cr}(x_1,x_2,x_3,x_4)) + \operatorname{Li}_5^{\mathfrak{u}} \Big(15[\mathrm{r}_1(x_1,\dots,x_5)] - 9[\mathrm{r}_2(x_1,\dots,x_5)] + [\mathrm{r}_3(x_1,\dots,x_5)] \Big) \bigg) \overset{\sqcup\!\sqcup}{\equiv} 0 \,. \quad (21)$$

Here

$$\mathrm{cr}(x_1,x_2,x_3,x_4) \coloneqq \frac{(x_1-x_3)(x_2-x_4)}{(x_1-x_4)(x_2-x_3)}$$

is the classical cross-ratio, and $\mathrm{r}_1, \mathrm{r}_2, \mathrm{r}_3$ are the following 'higher ratios'

$$\begin{aligned} \mathrm{r}_1(x_1,\dots,x_5) &\coloneqq -\frac{(x_1-x_2)(x_1-x_4)(x_3-x_5)}{(x_1-x_3)(x_1-x_5)(x_2-x_4)} \,, \\ \mathrm{r}_2(x_1,\dots,x_5) &\coloneqq -\frac{(x_1-x_2)^2(x_3-x_4)(x_3-x_5)}{(x_1-x_3)(x_1-x_4)(x_2-x_3)(x_2-x_5)} \,, \\ \mathrm{r}_3(x_1,\dots,x_5) &\coloneqq -\frac{(x_1-x_2)^3(x_1-x_5)(x_3-x_4)^2(x_3-x_5)}{(x_1-x_3)^3(x_1-x_4)(x_2-x_4)(x_2-x_5)^2} \,. \end{aligned}$$

Corollary 17. *For $x_1,\dots,x_5 \in \mathbb{P}^1(\mathbb{C})$ we have the following identity for the clean single-valued functions*

$$\mathrm{Alt}_5 \bigg(11 \mathcal{S}_{3,2}^{\sqcup\!\sqcup}(\mathrm{cr}(x_1,x_2,x_3,x_4)) + \mathcal{L}_5^{\sqcup\!\sqcup} \Big(15[\mathrm{r}_1(x_1,\dots,x_5)] - 9[\mathrm{r}_2(x_1,\dots,x_5)] + [\mathrm{r}_3(x_1,\dots,x_5)] \Big) \bigg) = 0 \,.$$

Proof. By the antisymmetry, the constant in the clean single-valued identity must be 0. □

Proof of Theorem 16. Let us define two polynomials π_1 and π_2 by $\pi_j = \mathrm{numerator}(1-\mathrm{r}_j)$. All computations will be done modulo 2-torsion, and therefore we ignore the ambiguity of the choice of sign of π_j. One can easily check that the subgroup $G_j \subset \mathfrak{S}_5$ fixes π_j (up to sign), where

$$G_1 = \langle (23), (2435) \rangle \,, \qquad G_2 = \langle (123) \rangle \,.$$

We have $|G_1| = 8$ and $|G_2| = 3$. It is also easy to check that $\mathrm{numerator}(1-\mathrm{r}_3) = \pm \pi_1 \pi_2$. We claim that

$$\begin{aligned} \mathrm{Alt}_{G_1} \Big(\;\; 15\, \mathrm{r}_1^{\otimes 4} + \mathrm{r}_3^{\otimes 4} \Big) &= 0 \,, \\ \mathrm{Alt}_{G_2} \Big(-9\, \mathrm{r}_2^{\otimes 4} + \mathrm{r}_3^{\otimes 4} \Big) &= 0 \,. \end{aligned} \quad (22)$$

To see this, let us denote $\mathrm{s}_1 = \sigma_{(45)}(\mathrm{r}_1)$, $\mathrm{s}_2 = \sigma_{(123)}(\mathrm{r}_2)$, where σ_g denotes the action of $g \in \mathfrak{S}_5$ on $\mathbb{Q}(x_1,\dots,x_5)$. Note the identities

$$\mathrm{r}_3 = \mathrm{r}_1 \cdot \mathrm{s}_1^2 = \mathrm{r}_2^2 \cdot \mathrm{s}_2 \,.$$

The group G_1 acts in the following way on r_1 and s_1:

$$\begin{array}{llll} \sigma_e(\mathrm{r}_1) = \mathrm{r}_1 \,, & \sigma_{(23)(45)}(\mathrm{r}_1) = \mathrm{r}_1^{-1} \,, & \sigma_{(24)(35)}(\mathrm{r}_1) = \mathrm{r}_1 \,, & \sigma_{(25)(34)}(\mathrm{r}_1) = \mathrm{r}_1^{-1} \,, \\ \sigma_e(\mathrm{s}_1) = \mathrm{s}_1 \,, & \sigma_{(23)(45)}(\mathrm{s}_1) = \mathrm{s}_1^{-1} \,, & \sigma_{(24)(35)}(\mathrm{s}_1) = \mathrm{s}_1^{-1} \,, & \sigma_{(25)(34)}(\mathrm{s}_1) = \mathrm{s}_1 \,, \\ \sigma_{(23)}(\mathrm{r}_1) = \mathrm{s}_1^{-1} \,, & \sigma_{(45)}(\mathrm{r}_1) = \mathrm{s}_1 \,, & \sigma_{(2435)}(\mathrm{r}_1) = \mathrm{s}_1^{-1} \,, & \sigma_{(5342)}(\mathrm{r}_1) = \mathrm{s}_1 \,, \\ \sigma_{(23)}(\mathrm{s}_1) = \mathrm{r}_1^{-1} \,, & \sigma_{(45)}(\mathrm{s}_1) = \mathrm{r}_1 \,, & \sigma_{(2435)}(\mathrm{s}_1) = \mathrm{r}_1 \,, & \sigma_{(5342)}(\mathrm{s}_1) = \mathrm{r}_1^{-1} \,. \end{array}$$

(That is, the representation of G_1 on the multiplicative group generated by $\mathrm{r}_1, \mathrm{s}_1$ is isomorphic to the standard 2-dimensional representation of the dihedral group D_4.) Thus the first identity in (22) holds by

$$\mathrm{Alt}_{G_1}\Big(15\,\mathrm{r}_1^{\otimes 4} + \mathrm{r}_3^{\otimes 4}\Big) \;=\; 60\,\mathrm{r}_1^{\otimes 4} - 60\,\mathrm{s}_1^{\otimes 4} + 2(\mathrm{r}_1\,\mathrm{s}_1^2)^{\otimes 4} + 2\Big(\frac{\mathrm{r}_1}{\mathrm{s}_1^2}\Big)^{\otimes 4} - 2(\mathrm{s}_1\,\mathrm{r}_1^2)^{\otimes 4} - 2\Big(\frac{\mathrm{s}_1}{\mathrm{r}_1^2}\Big)^{\otimes 4} \;=\; 0\,,$$

where the vanishing is equivalent to the following easily checked polynomial identity

$$2(X+2Y)^4 + 2(X-2Y)^4 - 2(2X+Y)^4 - 2(2X-Y)^4 \;=\; 60(Y^4 - X^4)\,.$$

Similarly, for G_2 we have

$$\begin{aligned} &\sigma_e(\mathrm{r}_2) = \mathrm{r}_2\,, \quad \sigma_{(123)}(\mathrm{r}_2) = \mathrm{s}_2\,, \qquad \sigma_{(321)}(\mathrm{r}_2) = (\mathrm{r}_2\,\mathrm{s}_2)^{-1}\,, \\ &\sigma_e(\mathrm{s}_2) = \mathrm{s}_2\,, \quad \sigma_{(123)}(\mathrm{s}_2) = (\mathrm{r}_2\,\mathrm{s}_2)^{-1}\,, \quad \sigma_{(321)}(\mathrm{s}_2) = \mathrm{r}_2\,. \end{aligned}$$

Therefore

$$\mathrm{Alt}_{G_2}\Big(-9\,\mathrm{r}_2^{\otimes 4} + \mathrm{r}_3^{\otimes 4}\Big) \;=\; -9\,\mathrm{r}_2^{\otimes 4} - 9\,\mathrm{s}_2^{\otimes 4} - 9(\mathrm{r}_2^{-1}\,\mathrm{s}_2^{-1})^{\otimes 4} + (\mathrm{r}_2\,\mathrm{s}_2^2)^{\otimes 4} + (\mathrm{r}_2^2\,\mathrm{s}_2)^{\otimes 4} + \Big(\frac{\mathrm{r}_2}{\mathrm{s}_2}\Big)^{\otimes 4} \;=\; 0\,,$$

where we used the polynomial identity

$$(X-Y)^4 + (2X+Y)^4 + (X+2Y)^4 \;=\; 9(X^4 + Y^4 + (X+Y)^4)\,.$$

Since $G_j \subset \mathrm{Aut}(\pi_j)$ for $j = 1, 2$, and also $G_1 \subset \mathrm{Aut}(x_{12}x_{13}x_{14}x_{15}x_{23}x_{45})$ and $G_2 \subset \mathrm{Aut}(x_{12}x_{13}x_{23}x_{45})$, where we denote $x_{ij} \coloneqq x_i - x_j$ (here Aut is considered modulo ± 1, i.e. G_1 fixes $x_{12}x_{13}x_{14}x_{15}x_{23}x_{45}$ up to sign, and similarly for G_2 and $x_{12}x_{13}x_{23}x_{45}$), we obtain that

$$\begin{aligned} \mathrm{Alt}_{G_1}\bigg(\frac{\pi_1^2}{x_{12}x_{13}x_{14}x_{15}x_{23}x_{45}} \otimes \Big(\;15\;\mathrm{r}_1^{\otimes 4} + \mathrm{r}_3^{\otimes 4}\Big)\bigg) \;&=\; 0\,, \\ \mathrm{Alt}_{G_2}\bigg(\frac{\pi_2}{x_{12}x_{13}x_{23}x_{45}} \otimes \Big(-9\;\mathrm{r}_2^{\otimes 4} + \mathrm{r}_3^{\otimes 4}\Big)\bigg) \;&=\; 0\,. \end{aligned}$$

Since $\mathrm{Symb}^{\sqcup\!\sqcup}\big(\mathrm{Li}_5^{\mathfrak{u}}(z)\big) \;=\; -(1-z)\wedge z \otimes z^{\otimes 3}$, where again $a \wedge b \;=\; a\otimes b - b \otimes a$, we see from these two identities that the mod-products symbol of the Li_5 part of (21) is equal to

$$\begin{aligned} \mathrm{Alt}_5\bigg(-\frac{15}{2}\,\frac{x_{12}x_{13}x_{14}x_{15}x_{23}x_{45}}{x_{13}^2x_{15}^2x_{24}} \wedge \mathrm{r}_1 \otimes \mathrm{r}_1^{\otimes 3} \;+ 9\,\frac{x_{12}x_{13}x_{23}x_{45}}{x_{13}x_{14}x_{23}x_{25}} \wedge \mathrm{r}_2 \otimes \mathrm{r}_2^{\otimes 3} \\ -\frac{1}{2}\,\frac{x_{12}^3x_{13}^3x_{23}^3x_{45}^3x_{14}x_{15}}{x_{13}^6x_{14}^2x_{24}^2x_{25}^4} \wedge \mathrm{r}_3 \otimes \mathrm{r}_3^{\otimes 3}\bigg)\,. \end{aligned}$$

For the r_1 term we compute

$$\frac{x_{12}x_{13}x_{14}x_{15}x_{23}x_{45}}{x_{13}^2x_{15}^2x_{24}} \wedge \mathrm{r}_1 \otimes \mathrm{r}_1^{\otimes 3} \;=\; \frac{x_{23}x_{45}}{x_{24}x_{35}} \wedge \mathrm{r}_1 \otimes \mathrm{r}_1^{\otimes 3} \;=\; [2534] \wedge \mathrm{r}_1 \otimes \mathrm{r}_1^{\otimes 3}\,,$$

where we denote by $[ijkl]$ the cross-ratio $\frac{x_{ik}x_{jl}}{x_{il}x_{jk}}$. By factoring out r_3 from the left-hand factor of the r_3 term we compute

$$\frac{x_{12}^3x_{23}^3x_{45}^3x_{15}}{x_{13}^3x_{14}x_{24}^2x_{25}^4} \wedge \mathrm{r}_3 \;=\; \frac{x_{23}^3x_{45}^3}{x_{24}x_{35}x_{25}^2x_{34}^2} \wedge \mathrm{r}_3 \;=\; [2435]^2[2534] \wedge \mathrm{r}_3\,.$$

From these observations we see that the mod-products symbol of the Li_5 part of (21) is equal to

$$\mathrm{Alt}_5\Big(-\frac{15}{2}\,[2534] \wedge \mathrm{r}_1 \otimes \mathrm{r}_1^{\otimes 3} \;+ 9\,[1524] \wedge \mathrm{r}_2 \otimes \mathrm{r}_2^{\otimes 3} \;- \frac{1}{2}\,[2435]^2[2534] \wedge \mathrm{r}_3 \otimes \mathrm{r}_3^{\otimes 3}\Big)\,. \tag{23}$$

Next, we introduce the variables $u_j \coloneqq [j, j+1, j+2, j+3]$, where as before $[ijkl] \coloneqq \mathrm{cr}(x_i, x_j, x_k, x_l)$ and all indices are written modulo 5. The action of $\mathfrak{S}_5$ on u_j gives rise to an irreducible 5-dimensional representation V (written multiplicatively), in which $\sigma_{(12345)}(u_j) = u_{j+1}$ and $\sigma_{(12)}$ acts by

$$(u_1, u_2, u_3, u_4, u_5) \;\mapsto\; \Big(u_1^{-1}, u_2u_4, u_1u_3, u_4^{-1}, -\frac{u_5}{u_1u_4}\Big)\,.$$

Since $\mathrm{Symb}^{\sqcup\!\sqcup}\big(S_{3,2}^{\mathfrak{u}}(z)\big) = -\big((1-z)\wedge z\big) \otimes \big((1-z) \sqcup\!\sqcup (z \otimes z)\big)$, we have under the mod-products symbol

$$\mathrm{Alt}_5\, S_{3,2}([1234]) \;=\; \mathrm{Alt}_5\Big(-(u_1 \wedge u_2u_5) \otimes (\frac{u_1}{u_2u_5} \sqcup\!\sqcup (u_1 \otimes u_1))\Big) \quad \in \bigwedge\nolimits^2 V \otimes \mathrm{Sym}^3(V)\,. \tag{24}$$

Therefore, (21) is an identity in the skew-symmetric part of an $\mathfrak{S}_5$-module $\bigwedge^2 V \otimes \mathrm{Sym}^3(V)$. The 10-dimensional representation $\bigwedge^2 V$ decomposes into a direct sum $V_4 \oplus V_6$ of a 4-dimensional and a

6-dimensional irreducible representation. We can take a basis for V_4 to be $u_j \wedge u_{j-1}u_{j+1}$, $j = 1, \dots, 4$, and a basis for V_6 to be given by w and $\sigma_{(j,j+1)}(w)$, $j = 1, \dots, 5$, where

$$w = \frac{1}{5}\Big(u_1 \wedge u_2 + u_2 \wedge u_3 + u_3 \wedge u_4 + u_4 \wedge u_5 + u_5 \wedge u_1\Big)\,.$$

First, we want to show that (23) projects trivially onto $V_6 \otimes \mathrm{Sym}^3(V)$. We compute

$$\begin{aligned}
\mathrm{pr}_{V_6}([2534] \wedge \mathrm{r}_1) &= \sigma_{(12)}(w) + \sigma_{(51)}(w)\,,\\
\mathrm{pr}_{V_6}([1524] \wedge \mathrm{r}_2) &= 2w + \sigma_{(23)}(w) + \sigma_{(34)}(w) + 2\sigma_{(51)}(w)\,,\\
\mathrm{pr}_{V_6}([2435]^2[2534] \wedge \mathrm{r}_3) &= 4w - 3\sigma_{(12)}(w) + 6\sigma_{(23)}(w) - 2\sigma_{(45)}(w) + 9\sigma_{(51)}(w)\,.
\end{aligned}$$

From this we see that the projection of (23) onto $V_6 \otimes \mathrm{Sym}^3(V)$ is equal to

$$\begin{aligned}
\mathrm{Alt}_5 \bigg(w \otimes \Big(&\frac{15}{2}\sigma_{(12)}(\mathrm{r}_1)^{\otimes 3} + \frac{15}{2}\sigma_{(51)}(\mathrm{r}_1)^{\otimes 3} + 18\,\mathrm{r}_2^{\otimes 3} - 9\sigma_{(23)}(\mathrm{r}_2)^{\otimes 3} - 9\sigma_{(34)}(\mathrm{r}_2)^{\otimes 3}\\
&- 18\sigma_{(51)}(\mathrm{r}_2)^{\otimes 3} - 2\,\mathrm{r}_3^{\otimes 3} - \frac{3}{2}\sigma_{(12)}(\mathrm{r}_3)^{\otimes 3} + 3\sigma_{(23)}(\mathrm{r}_3)^{\otimes 3} - \sigma_{(45)}(\mathrm{r}_3)^{\otimes 3} + \frac{9}{2}\sigma_{(51)}(\mathrm{r}_3)^{\otimes 3}\Big)\bigg)\,.
\end{aligned}$$

Factorising in terms of u_j and switching to additive notation with indeterminate U_j corresponding to u_j, we can rewrite the last expression as

$$\begin{aligned}
\mathrm{Alt}_5 \bigg(w \otimes \Big(&\frac{15}{2}(U_1 - U_5)^3 + \frac{15}{2}(U_4 - U_3)^3 + 18(U_1 - U_2 - 2U_5)^3 - 9(U_1 + U_5)^3 - 9(-U_2 - U_3 - 2U_5)^3\\
&- 18(U_1 - U_2 + U_3 + U_5)^3 - 2(-2U_2 - U_4 - 3U_5)^3 - \frac{3}{2}(3U_1 - 2U_2 + 2U_4 - 3U_5)^3\\
&+ 3(U_2 - U_4 + 3U_5)^3 - (-U_2 + U_4 - 3U_5)^3 + \frac{9}{2}(-2U_2 + U_3 - U_4 + 2U_5)^3\Big)\bigg)\,.
\end{aligned}$$

Note that for any dihedral permutation $g \in D_5 = \langle (12345), (12)(35)\rangle \subset \mathfrak{S}_5$ we have $\sigma_g(w) = \chi(g)w$, where $\chi\colon D_5 \to \{\pm 1\}$ takes value 1 on rotations and -1 on reflections. From this we see that $\mathrm{Alt}_5\big(w \otimes v\big) = \frac{1}{10}\mathrm{Alt}_5\big(w \otimes (\mathrm{Alt}_{D_5} v)\big)$, where $\mathrm{Alt}_{D_5}(v) = \sum_{g \in D_5}\chi(g)\sigma_g(v)$. The dihedral group D_5 acts on U_j as on the vertices of a regular pentagon, and it is not hard to see that $\mathrm{Alt}_{D_5} U_j^3 = \mathrm{Alt}_{D_5} U_iU_jU_k = 0$, hence the image of Alt_{D_5} on cubic polynomials is two-dimensional and it is spanned by $\mathrm{Alt}_{D_5} U_1U_j^2$ for $j = 2, 3$. From this we get that the projection of (23) onto $V_6 \otimes \mathrm{Sym}^3(V)$ is equal to

$$192\,\mathrm{Alt}_5\Big(w \otimes \Big(-3U_1U_2^2 + U_1U_3^2\Big)\Big)\,.$$

On the other hand, one can easily check that

$$w + \sigma_{(23)}(w) - \sigma_{(24)}(w) - \sigma_{(243)}(w) = 0\,,$$

and therefore

$$\begin{aligned}
0 &= \mathrm{Alt}_5\Big((-w - \sigma_{(23)}(w) + \sigma_{(24)}(w) + \sigma_{(243)}(w)) \otimes (U_1 - U_2)^3\Big)\\
&= \mathrm{Alt}_5\Big(w \otimes \Big(-(U_1 - U_2)^3 + (U_1 - U_5)^3 - (-U_1 + U_2 - U_3 + U_5)^3 + (-U_1 - U_3 - U_5)^3\Big)\Big)\\
&= 6\,\mathrm{Alt}_5\Big(w \otimes \big(-3U_1U_2^2 + U_1U_3^2\big)\Big)\,.
\end{aligned}$$

This shows that (23) is an element of $V_4 \otimes \mathrm{Sym}^3(V)$. Next, we compute the projections onto V_4:

$$\begin{aligned}
\mathrm{pr}_{V_4}([2534] \wedge \mathrm{r}_1) &= \frac{2}{5}u_1 \wedge u_5u_2 + \frac{1}{5}u_2 \wedge u_1u_3 + \frac{2}{5}u_4 \wedge u_3u_5\,,\\
\mathrm{pr}_{V_4}([1524] \wedge \mathrm{r}_2) &= \frac{2}{5}u_2 \wedge u_1u_3 + \frac{2}{5}u_3 \wedge u_2u_4 + \frac{1}{5}u_4 \wedge u_3u_5\,,\\
\mathrm{pr}_{V_4}([2435]^2[2534] \wedge \mathrm{r}_3) &= \frac{26}{5}u_1 \wedge u_5u_2 + \frac{13}{5}u_2 \wedge u_1u_3 + \frac{16}{5}u_3 \wedge u_2u_4 + 2u_4 \wedge u_3u_5\,.
\end{aligned}$$

Then (23) is equal to

$$\begin{aligned}
\frac{1}{5}\mathrm{Alt}_5\bigg((u_1 \wedge u_2u_5) \otimes \Big(&-15(U_4 - U_5)^3 - \frac{15}{2}(U_3 - U_4)^3 - 15(U_1 - U_2)^3 + 18(U_5 - U_1 - 2U_4)^3\\
&+ 18(U_4 - U_5 - 2U_3)^3 + 9(U_3 - U_4 - 2U_2)^3 - 13(-2U_2 - U_4 - 3U_5)^3\\
&- \frac{13}{2}(-2U_1 - U_3 - 3U_4)^3 - 8(-2U_5 - U_2 - 3U_3)^3 - 5(-2U_4 - U_1 - 3U_2)^3\Big)\bigg)\,.
\end{aligned}$$

Let us denote the parenthesised polynomial by $P(U_1, \ldots, U_5)$. Then, combining this identity with (24) we get that the mod-products symbol of the left-hand-side of (21) is equal to

$$\operatorname{Alt}_5 \Big((u_1 \wedge u_2 u_5) \otimes \Big(\frac{1}{5} P(U_1, U_2, U_3, U_4, U_5) - 33(U_1 - U_2 - U_5) U_1^2 \Big) \Big) . \tag{25}$$

The term $(u_1 \wedge u_2 u_5)$ is skew-symmetric under the subgroup $\mathfrak{S}_4 \subset \mathfrak{S}_5$ that permutes $x_1, \ldots, x_4$, therefore

$$\operatorname{Alt}_5 \Big((u_1 \wedge u_2 u_5) \otimes v \Big) \ = \ \operatorname{Alt}_5 \Big((u_1 \wedge u_2 u_5) \otimes \frac{1}{24} (\operatorname{Sym}_{\mathfrak{S}_4} v) \Big) .$$

In view of this we compute

$$\begin{aligned} &\frac{1}{24} \operatorname{Sym}_{\mathfrak{S}_4} \Big(\frac{1}{5} P(U_1, U_2, U_3, U_4, U_5) - 33(U_1 - U_2 - U_5) U_1^2 \Big) \\ &= \ 12 \sum_{j \pmod 5} \big(- U_j^3 + U_j U_{j+1} (U_j + U_{j+1} + U_{j+2}) - U_j U_{j+2} (U_j + U_{j+2}) \big) . \end{aligned}$$

Finally, combining this with

$$\sum_{j \pmod 5} (u_j \wedge u_{j-1} u_{j+1}) \ = \ \sum_{j \pmod 5} (u_j \wedge u_{j+1} - u_{j-1} \wedge u_j) \ = \ 0$$

we get that (25) is equal to

$$\begin{aligned} &12 \operatorname{Alt}_5 \Big((u_1 \wedge u_2 u_5) \otimes \sum_{j \pmod 5} \big(- U_j^3 + U_j U_{j+1} (U_j + U_{j+1} + U_{j+2}) - U_j U_{j+2} (U_j + U_{j+2}) \big) \Big) \\ &= \ 12 \operatorname{Alt}_5 \Big(\sum_{j \pmod 5} (u_j \wedge u_{j-1} u_{j+1}) \otimes \big(- U_1^3 + U_1 U_2 (U_1 + U_2 + U_3) - U_1 U_3 (U_1 + U_3) \big) \Big) = 0 , \end{aligned}$$

concluding the proof of (21). □

Remark 18. If we utilise the results of Brown from [7], one can potentially obtain a simpler proof of the five-term identity for $S_{3,2}$ given in (21). Indeed, from (22), we see that the mod-products symbol of

$$\operatorname{Alt}_5 \Big(\operatorname{Li}_5^{\mathfrak{u}} \big(15[\mathrm{r}_1(x_1, \ldots, x_5)] - 9[\mathrm{r}_2(x_1, \ldots, x_5)] + [\mathrm{r}_3(x_1, \ldots, x_5)] \big) \Big) \tag{26}$$

lands in the space of (integrable) tensors of iterated integrals on $\mathfrak{M}_{0,5}$. This happens because the tensors involving each of the irreducibles π_1 and π_2 cancel out, as was shown in the first part of the proof above.

Since the 2-part of the deconcatenation cobracket (or functional cobracket, rather than motivic cobracket) of (26) also vanishes (it is a combination of depth 1 polylogarithms), Theorem 56 in [7] implies that it must be expressible in terms of Nielsen polylogarithms of weight 5, with cross-ratio arguments. Using the Alt_5-symmetry and the fact that, by Theorem 7, $S_{3,2}$ and Li_5 suffice to express all Nielsen polylogarithms in weight 5 we see that the above combination must be equal (on the symbol level) to

$$\operatorname{Alt}_5 \Big(c_1 S_{3,2}^{\mathfrak{u}} (\operatorname{cr}(x_1, \ldots, x_4)) + c_2 \operatorname{Li}_5^{\mathfrak{u}} (\operatorname{cr}(x_1, \ldots, x_4)) \Big)$$

for some constants c_1 and c_2. Moreover, the term $\operatorname{Alt}_5 \operatorname{Li}_5^{\mathfrak{u}}(\operatorname{cr}(x_1, \ldots, x_4))$ vanishes, as can be seen from the inversion identity for Li_5. To fix the value of c_1, one can compare the coefficients of the tensor $(x_1 - x_2) \wedge (x_2 - x_3) \otimes (x_1 - x_2)^{\otimes 3}$.

If we alternate (21) over a sixth point x_6, the Nielsen term vanishes as it depends on only four points. As a corollary, we obtain the following non-trivial functional equation for Li_5 that was previously found by the third author as a result of an extensive computer search.

Corollary 19 ([47, Theorem 5.13])**.** *For any $x_1, \ldots, x_6 \in \mathbb{P}^1(\mathbb{C})$ we have*

$$\operatorname{Alt}_6 \Big(L_5 \big(15[\mathrm{r}_1(x_1, \ldots, x_5)] - 9[\mathrm{r}_2(x_1, \ldots, x_5)] + [\mathrm{r}_3(x_1, \ldots, x_5)] \big) \Big) \ = \ 0 .$$

Here we can choose either $L_5 = \mathcal{L}_5^{\sqcup\!\sqcup}$, the clean single-valued polylogarithm from (13)*, or $L_5 = \mathscr{L}_5$, Zagier's single-valued polylogarithm defined in Remark 10.*

For completeness and application to numerical identities below, we give a version of Theorem 16 which produces a single five-term relation for $S_{3,2}$ on the nose.

Corollary 20 ($S_{3,2}$ five-term relation)**.** *For indeterminates $x_1, \ldots, x_5$, we have the following identity between the mod-products symbols of $S_{3,2}$ and* Li_5 *in $\bigotimes_{i=1}^5 \mathbb{Q}(x_1, \ldots, x_5)^\times \otimes_{\mathbb{Z}} \mathbb{Q}$:*

$$\sum_{i=1}^5 (-1)^i S^{\mathrm{u}}_{3,2}(\mathrm{cr}(x_1, \ldots, \widehat{x_i}, \ldots, x_5)) \overset{\sqcup\!\sqcup}{=}$$
$$\frac{1}{4! \cdot 11} \mathrm{Alt}_5 \Big(\mathrm{Li}^{\mathrm{u}}_5 \left(15\big[\,\mathrm{r}_1(x_1, \ldots, x_5)\big] - 9\big[\,\mathrm{r}_2(x_1, \ldots, x_5)\big] + \big[\,\mathrm{r}_3(x_1, \ldots, x_5)\big]\right)\Big)$$
$$+ \frac{1}{2} \sum_{i=1}^5 (-1)^i \mathrm{Li}^{\mathrm{u}}_5 \left(3\big[c_i\big] - \big[1 - c_i\big] + \big[1 - c_i^{-1}\big]\right),$$

where $c_i := \mathrm{cr}(x_1, \ldots, \widehat{x_i}, \ldots, x_5)$. Moreover, in the clean-single valued version of the identity, the constant on the right hand side is $-\zeta(5)$.

Proof. Using the two-term relations $S_{3,2}(x) + S_{3,2}(1-x) = 0$ (mod Li_5, products) and $S_{3,2}(x) + S_{3,2}(x^{-1}) = 0$ (mod Li_5, products) from Propositions 11 and 12, we can relate every cross-ratio term $S_{3,2}(\mathrm{cr}(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}))$, $\sigma \in \mathfrak{S}_4$, back to $S_{3,2}(\mathrm{cr}(x_1, \ldots, x_4))$, along with some Li_5 terms. Applying this to Theorem 16 produces the above identity. Keeping track of the additional constants gives $-\zeta(5) = -\frac{1}{2}\zeta^{\mathrm{sv}}(5)$. □

Remark 21. Specialising the above to $(x_1, \ldots, x_5) = (\infty, 0, 1, 1-x, y)$ produces

$$S_{3,2}\Big(- [1-x] + [y] - \Big[\frac{y}{1-x}\Big] + \Big[\frac{1-y}{x}\Big] - \Big[\frac{(1-x)(1-y)}{xy}\Big]\Big) = 0 \text{ (mod } \mathrm{Li}_5\text{, products)}\,.$$

Up to the two-term relations, this is the version of the five-term we claimed in the introduction.

Since every rational functional equation for Li_2, by which we understand any element $\sum_i n_i[F_i(x_1, \ldots, x_k)]$ in $\ker(\mathrm{Symb}^{\sqcup\!\sqcup})$ with $F_i \in \mathbb{Q}(x_1, \ldots, x_k)$, is essentially a linear combination of five-term relations (for a more precise statement with detailed proof see [19]) we see that $S_{3,2}$ satisfies all those dilogarithm functional equations modulo Li_5 terms, as well as certain algebraic ones as in Section 6.2.

Corollary 22 (Distribution relations for $S_{3,2}$)**.** *The Nielsen polylogarithm $S_{3,2}$ satisfies the distribution relations*

$$\frac{1}{n} S_{3,2}(z^n) - \sum_{\lambda^n = 1} S_{3,2}(\lambda z) \;=\; 0 \text{ (mod } \mathrm{Li}_5\text{, products)}\,,$$

with algorithmically determinable Li_5 *terms.*

Proof. Wojtkowiak [51] gives an algorithm which reduces any functional equation for Li_2 in a single variable z, with arguments in $\mathbb{C}(z)$, to a combination of five-term relations (for a condensed version of the proof see also [55], Proposition 4). From this we can write any Li_2 distribution relation as a sum of five-term relations, and obtain the corresponding statement for $S_{3,2}$. □

Corollary 23 (Duplication for $S_{3,2}$)**.** *The Nielsen polylogarithm $S_{3,2}$ satisfies the duplication relation*

$$132\, S_{3,2}\Big(\frac{1}{2}\big[x^2\big] - \big[x\big] - \big[-x\big]\Big) = \mathrm{Li}_5\left(f(x) + f(-x)\right) - 198\zeta(5) \text{ (mod products)}$$

where

$$\begin{aligned} f(x) \;=\; & 7 \cdot 198\,\big[x\big] + 12\Big[\tfrac{-x^2}{(1-x)(1+x)}\Big] - 12\big[(1-x)(1+x)\big] + 8\Big[-\tfrac{1+x}{1-x}\Big] - 8\Big[\tfrac{1+x}{1-x}\Big] \\ & + 75\Big[\tfrac{1}{1+x}\Big] - 75\Big[\tfrac{x}{1+x}\Big] + 30\Big(\big[x(1+x)\big] - \big[-x(1+x)\big] + \Big[-\tfrac{x^2}{1+x}\Big] - \Big[\tfrac{x^2}{1+x}\Big]\Big) \\ & + 9\Big(\Big[-\tfrac{x}{1+x}\Big] + \Big[x^2(1+x)\Big] + \Big[\tfrac{-x^3}{(1-x)(1+x)}\Big] + \Big[\tfrac{x}{(1-x)(1+x)^2}\Big] + \tfrac{1}{2}\Big[\tfrac{x^2}{(1+x)^2}\Big] \\ & \qquad - \Big[\tfrac{x^3}{1+x}\Big] - \big[-(1-x)\big] - \big[-(1-x)x(1+x)\big] - \Big[\tfrac{-x^2}{(1-x)(1+x)^2}\Big] - \tfrac{1}{2}\Big[(1+x)^2\Big]\Big) \\ & + \Big(\Big[-x^4(1+x)\Big] + \Big[-(1+x)^3\Big] + \Big[\tfrac{-x^5}{(1-x)(1+x)^3}\Big] + \Big[\tfrac{-x^3}{(1-x)(1+x)^2}\Big] + \Big[\tfrac{x}{(1-x)^2(1+x)^3}\Big] \\ & \qquad - \Big[-\tfrac{x^5}{1+x}\Big] - \Big[\tfrac{-x^3}{(1+x)^3}\Big] - \Big[(1-x)x(1+x)^3\Big] - \Big[(1-x)(1+x)^2\Big] - \Big[\tfrac{x^4}{(1-x)^2(1+x)^3}\Big]\Big)\,. \end{aligned}$$

Proof. Specialise the five-term relation Corollary 20 to $(x_1, \ldots, x_5) = (\infty, 0, 1, 1/x, x)$, use inversion on the resulting $S_{3,2}(\frac{1}{x^2})$, and apply the duplication relation $\mathrm{Li}_5(x^2) = 16\,\mathrm{Li}_5([x] + [-x])$ to obtain the above terms. □

Remark 24. Despite a brute force search on the level of the symbol, using arguments inspired by the above, no significantly simpler reduction was found. Moreover, a uniform description of the Li_5 terms for $S_{3,2}$ of higher distribution relations (to match the uniform nature of the distribution relations themselves) is also lacking, and one must instead resort to applying Wojtkowiak's algorithm.

Corollary 25. *Any* Li_2 *evaluation which is accessible via the five-term relation (i.e. following explicitly from the five-term relation, see* [39]*) can be upgraded to an* $S_{3,2}$ *evaluation, with explicit* Li_5 *terms. In particular, the Nielsen polylogarithm* $S_{3,2}(z)$ *can be evaluated in terms of* Li_5 *whenever* $\mathrm{Li}_2(z)$ *can be evaluated in terms of products of logarithms.*

6.4. **Special values and ladders.** Corollary 25 gives us the previous formulae for $S_{3,2}(1) = \zeta(1,4) = -\zeta(2)\zeta(3) + 2\zeta(5)$, and for $S_{3,2}(-1)$, $S_{3,2}(\frac{1}{2})$ given in (19), and (20) above. It also gives the following new identities involving the golden ratio, and ladders involving $\frac{1}{3}$ or $\sqrt{2}-1$.

Values involving the golden ratio: Recall the following evaluation involving the golden ratio $\phi = \frac{1}{2}(1+\sqrt{5})$ for Li_2 (see [44, Equations 1.20 and 1.21], or [55, Section 1.1]):

$$\mathrm{Li}_2(\phi^{-2}) \;=\; \frac{2}{5}\zeta(2) - \log^2(\phi)\,.$$

The same specialisations of the Li_2 five-term identity which produce this will give an evaluation for $S_{3,2}(\phi^{-2})$. Specialise Corollary 20 to $(x_1,\ldots,x_5) = (\infty, 0, 1, \phi, \phi^{-1})$. Every Li_5 argument can be written as $\pm\phi^{-n}$ using inversion, and after doing so, we obtain

$$\begin{aligned}&\mathcal{S}^{\sqcup\!\sqcup}_{3,2}\big([-\phi^{-1}] + [\phi^{-1}] - [\phi] - [-\phi] - [\phi^{-2}]\big) \;=\\ &\frac{1}{66}\mathcal{L}^{\sqcup\!\sqcup}_5\Big(-80\big[-\phi^{-3}\big] + 80\big[\phi^{-3}\big] - 99\big[\phi^{-2}\big] + 78\big[-\phi^{-1}\big] - 78\big[\phi^{-1}\big]\Big) - \zeta(5)\,.\end{aligned} \tag{27}$$

By the inversion relation for $\mathcal{S}^{\sqcup\!\sqcup}_{3,2}$, we have that $\mathcal{S}^{\sqcup\!\sqcup}_{3,2}(-\phi) = -\mathcal{S}^{\sqcup\!\sqcup}_{3,2}(-\phi^{-1}) + 3\mathcal{L}^{\sqcup\!\sqcup}_5(-\phi^{-1}) + 2\zeta(5)$ and $\mathcal{S}^{\sqcup\!\sqcup}_{3,2}(\phi) = -\mathcal{S}^{\sqcup\!\sqcup}_{3,2}(\phi^{-1}) + 3\mathcal{L}^{\sqcup\!\sqcup}_5(\phi^{-1}) + 2\zeta(5)$. Note that

$$1 - \frac{1}{\phi} = \frac{1}{\phi^2}\,,$$

so applying the clean single-valued version of $S_{3,2}(x) + S_{3,2}(1-x) = 0$ (mod Li_5, products) from Proposition 12, with $x = \phi^{-1}$ gives

$$\mathcal{S}^{\sqcup\!\sqcup}_{3,2}(\phi^{-1}) = -\mathcal{S}^{\sqcup\!\sqcup}_{3,2}(\phi^{-2}) + \mathcal{L}^{\sqcup\!\sqcup}_5(\phi^{-2}) + \mathcal{L}^{\sqcup\!\sqcup}_5(\phi^{-1}) + \mathcal{L}^{\sqcup\!\sqcup}_5(-\phi^{-1}) + 2\zeta(5)\,.$$

Similarly, since

$$1 + \frac{1}{\phi} = \phi\,,$$

we obtain

$$\mathcal{S}^{\sqcup\!\sqcup}_{3,2}(-\phi^{-1}) = -\mathcal{S}^{\sqcup\!\sqcup}_{3,2}(\phi) + \mathcal{L}^{\sqcup\!\sqcup}_5(\phi^{-2}) + \mathcal{L}^{\sqcup\!\sqcup}_5(-\phi^{-1}) + \mathcal{L}^{\sqcup\!\sqcup}_5(\phi^{-1}) + 2\zeta(5)\,.$$

In particular, the $\mathcal{S}^{\sqcup\!\sqcup}_{3,2}$ combination in (27) can be re-written as $5\mathcal{S}^{\sqcup\!\sqcup}_{3,2}(\phi^{-2})$, modulo Li_5's. After doing this, and using inversion and the duplication relation to write all $\mathcal{L}^{\sqcup\!\sqcup}_5$ arguments as ϕ^{-k}, we obtain

$$\mathcal{S}^{\sqcup\!\sqcup}_{3,2}(\phi^{-2}) = \frac{1}{66}\mathcal{L}^{\sqcup\!\sqcup}_5\Big(\big[\phi^{-6}\big] - 32\big[\phi^{-3}\big] + \frac{201}{2}\big[\phi^{-2}\big] - 48\big[\phi^{-1}\big]\Big) + \zeta(5)\,.$$

From the clean single-valued identity, we can extract an analytic identity for $S_{3,2}(\phi^{-2})$ since the main term in $\mathcal{S}^{\sqcup\!\sqcup}_{3,2}(x)$ is $2\,\mathrm{Re}\,S_{3,2}(x)$. To simplify the resulting analytic identity we can use the reduction

$$\begin{aligned}S_{2,2}\big(\phi^{-2}\big) \;=\;& \mathrm{Li}_4\left(\phi^{-2}\right) + \mathrm{Li}_4\left(-\phi^{-1}\right) - \mathrm{Li}_4\left(\phi^{-1}\right) + \zeta(4) + \mathrm{Li}_3\left(\phi^{-2}\right)\log(\phi)\\ &- \zeta(3)\log(\phi) + \frac{1}{2}\zeta(2)\log^2(\phi) - \frac{7}{24}\log^4(\phi)\end{aligned}$$

obtained from Proposition 5, along with some polylogarithm ladders for ϕ, in weight 2, 3 and 4, such as

$$\mathrm{Li}_4(\phi^{-6}) \;=\; \mathrm{Li}_4\Big(16\big[\phi^{-3}\big] + \frac{9}{4}\big[\phi^{-2}\big] - 36\big[\phi^{-1}\big]\Big) + 20\zeta(4) - 9\zeta(2)\log^2(\phi) + \frac{27}{4}\log^4(\phi)\,.$$

See Equations 3.71 and 3.105 in [45], wherein $\rho = \phi^{-1}$. One obtains the analytic identity

$$\begin{aligned}S_{3,2}\big(\phi^{-2}\big) \;=\;& \frac{1}{66}\,\mathrm{Li}_5\Big([\phi^{-6}] - 32[\phi^{-3}] + \frac{201}{2}[\phi^{-2}] - 48[\phi^{-1}]\Big) + \mathrm{Li}_4\left(\phi^{-2}\right)\log(\phi)\\ &+ \frac{1}{2}\zeta(5) - \frac{2}{11}\zeta(4)\log(\phi) - \zeta(3)\,\mathrm{Li}_2\left(\phi^{-2}\right) - \frac{20}{33}\zeta(2)\log^3(\phi) + \frac{79}{330}\log^5(\phi)\,.\end{aligned}$$

Note that the coefficient of $\zeta(5)$ in the analytic identity is $\frac{1}{2}$ of the coefficient in the single-valued identity. The $\zeta(5)$ appearing in the single-valued identity is really $\frac{1}{2}\zeta^{\mathrm{sv}}(5) = \frac{1}{2}\mathcal{L}^{\sqcup\!\sqcup}_5(1)$ (where ζ^{sv} and $\mathcal{L}^{\sqcup\!\sqcup}_5$

are introduced in (15) and (13), respectively), and this coefficient becomes manifest when passing to the analytic identity.

There are three related evaluations for $-\phi$, ϕ^{-1} and $-\phi^{-1}$ (obtained using the inversion and two-term relation), which we reproduce for the sake of completeness in Appendix A.

Remark 26. We note that a reduction of this type has already been conjectured in [6, Equation (35)], for the multiple polylogarithm $\mathrm{Li}_{4,1}(1,z)$, rather than for $S_{3,2}(z) = \mathrm{Li}_{1,4}(1,z)$. The lower order product terms are omitted from [6], but we give them here for completeness. (It appears that there is a sign error in the expression therein, and the coefficient of $\mathrm{Li}_5(\phi^{-2})$ seems to be $\frac{357}{4}$ rather than $\frac{27}{4}$.)

$$\begin{aligned}\frac{165}{2}\,\mathrm{Li}_{4,1}\left(1,\phi^{-2}\right) \overset{?}{=} {} & \mathrm{Li}_5\left(\left[\phi^{-6}\right] - 32\left[\phi^{-3}\right] - \frac{357}{4}\left[\phi^{-2}\right] + 216\left[\phi^{-1}\right]\right) - \frac{165}{2}\log(\phi)\,\mathrm{Li}_4\left(\phi^{-2}\right) \\ & - 165\zeta(5) + 120\zeta(4)\log(\phi) + 66\zeta(3)\,\mathrm{Li}_2\left(\phi^{-2}\right) + 26\zeta(2)\log^3(\phi) - \frac{139}{10}\log^5(\phi)\,.\end{aligned}$$

However, it is not immediately clear how to go from our evaluation to this evaluation. The function $\mathrm{Li}^{\mathfrak{L}}_{4,1}(1,y)$ has 2-coboundary $2\{y\}_2 \wedge \{y\}_3$, so it cannot be expressed in terms of Nielsen polylogarithms alone as $\delta^{\geq 2} S^{\mathfrak{L}}_{3,2}(x) = \{x\}_2 \wedge \{1\}_3$ has constant weight 3 component. However, one has

$$\mathrm{Li}^{\mathfrak{u}}_{4,1}(1,y) - \mathrm{Li}^{\mathfrak{u}}_{4,1}(1,1-y) + \mathrm{Li}^{\mathfrak{u}}_{4,1}(1,1-y^{-1}) \overset{\shuffle}{=} -2S^{\mathfrak{u}}_{3,2}(y) + 2\,\mathrm{Li}^{\mathfrak{u}}_5(y)\,,$$

so by specialising in various ways, one could potentially extract the evaluation at $y = \phi^{-2}$. On the other hand

$$\delta^{\geq 2}\,\mathrm{Li}^{\mathfrak{L}}_{4,1}(x,y) = \{xy\}_2 \wedge \{y\}_3 + \{y\}_2 \wedge \{xy\}_3\,.$$

Since $\{\phi^{-2}\}_3 = \frac{4}{5}\{1\}_3$ and $\{\phi^{-2}\}_2 = 0$, one has

$$\delta^{\geq 2}\,\mathrm{Li}^{\mathfrak{L}}_{4,1}(x,\phi^{-2}) = \frac{4}{5}\{x\phi^{-2}\}_2 \wedge \{1\}_3 = \frac{4}{5}\,\delta^{\geq 2} S_{3,2}(x\phi^{-2})\,.$$

It should therefore be possible to write $\mathrm{Li}_{4,1}(x,\phi^{-2})$ in terms of $S_{3,2}$ and Li_5 modulo products, as a route to evaluating $\mathrm{Li}_{4,1}(1,\phi^{-2})$.

We point out that similar conjectural evaluations exist for $y = \pm\phi^{\pm 1}$, but they are slightly more complicated, involving also higher powers ϕ^{-12} and ϕ^{-4}, for example

$$\begin{aligned}\mathrm{Li}_{4,1}\left(1,\phi^{-1}\right) \overset{?}{=} {} & \frac{1}{660}\,\mathrm{Li}_5\left(-\frac{1}{2}\left[\phi^{-12}\right] + 26\left[\phi^{-6}\right] + \frac{81}{2}\left[\phi^{-4}\right] - 448\left[\phi^{-3}\right] - 243\left[\phi^{-2}\right] + 384\left[\phi^{-1}\right]\right) \\ & - \frac{101}{88}\zeta(5) + \frac{36}{11}\zeta(4)\log(\phi) - 2\,\mathrm{Li}_4\left(\phi^{-1}\right)\log(\phi) \\ & + \mathrm{Li}_3\left(\phi^{-1}\right)\mathrm{Li}_2\left(\phi^{-1}\right) - \frac{8}{11}\zeta(2)\log^3(\phi) + \frac{37}{110}\log^5(\phi)\,.\end{aligned}$$

Ladder with $\frac{1}{3}$: Corresponding to the evaluation

$$\mathrm{Li}_2\left(\left[\frac{1}{9}\right] - 6\left[\frac{1}{3}\right]\right) = -2\zeta(2) + \log^2(3)\,,$$

we can derive an evaluation for the clean single-valued $\mathcal{S}^{\shuffle}_{3,2}$. Write

$$f(a,b) = \sum_{i=1}^{5} (-1)^i [\mathrm{cr}(x_1,\ldots,\widehat{x_i},\ldots,x_5)]\,,$$

with $(x_1,\ldots,x_5) = (\infty,0,1,a,b)$, as our usual presentation of the five-term relation. Then

$$\begin{aligned} f\left(\frac{1}{3},3\right) - 2f\left(\frac{1}{2},2\right) = {} & [-3] - 2[-2] + 2\left[-\frac{1}{2}\right] - \left[-\frac{1}{3}\right] - \left[\frac{1}{3}\right] \\ & + 2\left[\frac{1}{2}\right] - 2[2] + [3] + 2[4] - [9] \end{aligned} \tag{28}$$

gives the main contribution to this ladder. Under the two-term relations $[x] + [1-x] \sim 0$ and $[x] + [x^{-1}] \sim 0$, we see $-[\frac{1}{2}] \sim [\frac{1}{2}] \sim -[2] \sim 0$, $[9] \sim -[\frac{1}{9}]$, $[4] \sim -[-3] \sim [-\frac{1}{3}]$ and $[\frac{1}{3}] \sim -[3] \sim [-2] \sim -[-\frac{1}{2}]$. With these (28) simplifies to the desired combination

$$\left[\frac{1}{9}\right] - 6\left[\frac{1}{3}\right].$$

Applying Corollary 20 to the five-term combinations from (28) and simplifying via the two-term relations for $\mathcal{S}^{ш}_{3,2}$ from Propositions 11 and 12 gives us the desired $\mathcal{S}^{ш}_{3,2}$ evaluation. Because of its length, this initial reduction is only reproduced in Appendix B.

Since the original reduction is rather long, we have applied a well-known lattice reduction algorithm ('LLL') to obtain the following shorter, but only numerically checked, identity instead

$$\mathcal{S}^{ш}_{3,2}\left(\left[\frac{1}{9}\right] - 6\left[\frac{1}{3}\right]\right) \overset{?}{=} \mathcal{L}^{ш}_5\left(\frac{1}{16}\left[\frac{1}{9}\right] + \frac{21}{2}\left[\frac{1}{4}\right] + 36\left[\frac{1}{3}\right] - 100\left[\frac{1}{2}\right] - 60\left[\frac{2}{3}\right] + \frac{69}{2}\left[\frac{3}{4}\right] - 2\left[\frac{8}{9}\right]\right) + \frac{1855}{12}\zeta(5)\,.$$

From this identity we extract a corresponding analytic identity for $S_{3,2}\big(\big[\frac{1}{9}\big] - 6\big[\frac{1}{3}\big]\big)$ which we reproduce in Appendix B.

Lewin's ladder with $\alpha = \sqrt{2}-1$: Corresponding to one of Lewin's ladders (Equation 94(a) in [43])

$$\mathrm{Li}_2\left(\left[\alpha^2\right] - 4[\alpha]\right) \;=\; \log^2(\alpha) - \frac{3}{2}\zeta(2)\,,$$

where $\alpha = \sqrt{2}-1$, we have a corresponding $\mathcal{S}^{ш}_{3,2}$ identity. The ladder

$$\left[\alpha^2\right] - 4[\alpha]$$

is obtained from the following five-term combination

$$f(\alpha, \alpha^{-1}) + f(\alpha, -\alpha)$$

after simplification using the two-term inversion relation for α^2 and for $-\alpha$. Applying Corollary 20, and simplifying using $\mathcal{L}^{ш}_5$ inversion produces a reduction involving around 52 terms, whose arguments are of the form $\frac{1}{8}(a+b\sqrt{2})$, $a, b \in \mathbb{Z}$. We have applied the LLL lattice reduction algorithm to find the following shorter, but only numerically checked identity

$$\begin{aligned}\mathcal{S}^{ш}_{3,2}\big(\left[\alpha^2\right] - 4[\alpha]\big) \overset{?}{=} \frac{1}{117}\mathcal{L}^{ш}_5\Bigg(& 14\left[-\frac{\beta^5}{\alpha}\right] + 28\left[\alpha\beta^4\right] + 62\left[\alpha\beta^3\right] - 252\left[-\beta^3\right] \\ & + 44\left[\frac{\beta^3}{\alpha}\right] - 574\left[\alpha\beta^2\right] - 252\left[\frac{\beta^2}{\alpha}\right] - 22\left[-\frac{\beta^2}{\alpha}\right] \\ & + 354\left[\alpha\beta\right] - 252\left[-\alpha\beta\right] - 2488\left[\beta\right] - 2896\left[-\beta\right] \\ & + 70\left[\frac{\beta}{\alpha}\right] + 28\left[-\frac{\beta}{\alpha^2}\right] + 1260\left[\alpha\right] + 1824\left[-\alpha\right]\Bigg) - \frac{659}{117}\zeta(5)\,,\end{aligned}$$

where we write $\beta = \sqrt{2}$ for convenience. From this, an analytic identity can again be extracted.

6.5. **Evaluation of $S_{3,2}([\omega^2] + 2[\omega])$ for ω a root of the polynomial $u^3 + u^2 - 1$.** By combining different functional equations of $S_{3,2}$ we can give another ladder evaluation. Let ω be a root of the polynomial $u^3 + u^2 - 1$. Then we use the depth reduction of $S_{3,2}$ applied to the following algebraic Li_2 functional equation $\big[t(1-t)\big] + \big[-\frac{t}{(1-t)^2}\big] + \big[-\frac{1-t}{t^2}\big]$ from the three roots of $x^2(1-x)^{-3} = \frac{t^2(1-t)^2}{(1-t+t^2)^3}$ (case $a = 2, b = -3$ in Proposition 14 above) and specialise to $t = -\omega$. The three arguments turn actually out to be equal to $-\omega^{-1}, \omega^5$ and $-\omega^{-4}$, respectively. Now using further algebraic relations for ω like $1 + \omega^4 = \omega^{-1}$ and $1 - \omega^5 = \omega$ together with inversion and reflection relations as well as the duplication relation, we can rewrite the given combination as $-\frac{1}{2}\mathcal{S}^{ш}_{3,2}([\omega^2] + 2[\omega])$ modulo explicit $\mathcal{L}^{ш}_5$ terms. Moreover, if we consider the *real* embedding of ω the same ladder holds even for $S_{3,2}$ modulo Li_5.

7. Identities in weight 6

In this section we first show that the depth 3 integral $S_{3,3}(z)$ can be reduced to $S_{4,2}$ and $S_{5,1} = \mathrm{Li}_6$ (Proposition 27). Moreover, in analogy to the situation for $S_{3,2}$ and functional equations of Li_2 above, we expect that $S_{4,2}$, evaluated on any functional equation of Li_3, can itself be depth-reduced to Li_6, at least modulo products. As evidence we show the corresponding statement for the three-term relation (Proposition 29) and for an algebraic family of functional equations (Proposition 31). As a consequence, we evaluate $S_{3,3}$ at certain roots of unity (Corollary 30), and using polylogarithms we match the coaction for $S^{\mathfrak{m}}_{3,3}(-1)$ and for $S^{\mathfrak{m}}_{4,2}$ evaluated at -1, $\frac{1}{2}$ and $-\phi^{-2}$, where ϕ again denotes the golden ratio.

Preconsideration: The 2-part of the motivic coboundary of $S^{\mathfrak{L}}_{4,2}(z)$ and $S^{\mathfrak{L}}_{3,3}(z)$ is computed to be

$$\begin{aligned}\delta^{\geq 2} S^{\mathfrak{L}}_{4,2}(z) &= -\{z\}_3 \wedge \{1\}_3\,,\\ \delta^{\geq 2} S^{\mathfrak{L}}_{3,3}(z) &= -\{z\}_3 \wedge \{1\}_3 + \{1-z\}_3 \wedge \{1\}_3\,.\end{aligned}$$

This suggests that $S_{4,2}(z)$ should behave like Li_3 modulo Li_6, and gives a candidate for reducing $S_{3,3}$ to $S_{4,2}$ by matching their cobrackets.

7.1. **Depth reduction of $S_{3,3}$.** From Theorem 7 we know that $S_{3,3}(z)$ can be reduced to $S_{4,2}$ and Li_6, but from the motivic cobracket evaluated above we expect the combination

$$S_{3,3}(z) + \big(S_{4,2}(1-z) - S_{4,2}(z)\big)$$

in particular to reduce modulo products to Li_6's. Indeed, we find the following reduction of $S_{3,3}(z)$ to Nielsen polylogarithms of depth ≤ 2.

Proposition 27. *The following identity holds for all* $z \in \mathbb{C} \smallsetminus \big((-\infty,0] \cup [1,\infty)\big)$

$$\begin{aligned}S_{3,3}(z) \;=\;& S_{4,2}(z) - S_{4,2}(1-z) + \mathrm{Li}_6(1-z) - \mathrm{Li}_6(z) - \mathrm{Li}_6\Big(\frac{z}{z-1}\Big) - S_{3,2}(z)\log(1-z)\\ &+ \mathrm{Li}_5(z)\log(1-z) - \mathrm{Li}_5(1-z)\log(z) - \frac{1}{2!}\,\mathrm{Li}_4(z)\log^2(1-z)\\ &- \frac{1}{6!}\log^6(1-z) + \frac{1}{5!}\log(z)\log^5(1-z) - \frac{1}{2!\,4!}\log^2(z)\log^4(1-z)\\ &- \frac{1}{4!}\zeta(2)\log^4(1-z) + \frac{1}{3!}\zeta(2)\log(z)\log^3(1-z) + \frac{1}{2!}\zeta(3)\log(z)\log^2(1-z)\\ &+ \zeta(4)\log(z)\log(1-z) - \frac{3}{4\cdot 2!}\zeta(4)\log^2(1-z) + \zeta(5)\log(z)\\ &+ \Big(\zeta(5) - \zeta(2)\zeta(3)\Big)\log(1-z) - \Big(\frac{1}{4}\zeta(6) + \frac{1}{2}\zeta(3)^2\Big)\,.\end{aligned}$$

Proof. Differentiate, and use weight 5 identities to see the result is constant. By taking $z \to 0$ we can fix the constant as $-S_{4,2}(1) + \mathrm{Li}_6(1) = \frac{1}{4}\zeta(6) + \frac{1}{2}\zeta(3)^2$.

One can also check this identity using the polynomial invariant from Section 3.3. Moreover, since any identity between these invariants can be derived from inversion (Proposition 4) and reflection (Proposition 2) one can also get the exact form of the product terms and constants. We have

$$\begin{aligned}& S^{\mathfrak{u}}_{3,3}(z) - \Big(S^{\mathfrak{u}}_{4,2}(z) - S^{\mathfrak{u}}_{4,2}(1-z) + \mathrm{Li}^{\mathfrak{u}}_6(1-z) - \mathrm{Li}^{\mathfrak{u}}_6(z) - \mathrm{Li}^{\mathfrak{u}}_6\Big(\frac{z}{z-1}\Big)\Big)\\ \mapsto\;& 6X^2Y^2 - \big(4X^3Y + 4XY^3 - Y^4 - X^4 + (X-Y)^4\big) = 0\,.\end{aligned}$$

□

By specialising the above proposition to $z = \frac{1}{2}$ and to $z = -1$ via analytic continuation, respectively, we get the following identities. Each may be confirmed using the MZV Data Mine [1] via Remark 13.

Corollary 28. (i) *One has the reduction*

$$\begin{aligned}S_{3,3}\Big(\frac{1}{2}\Big) \;=\;& \mathrm{Li}_5\Big(\frac{1}{2}\Big)\log(2) + \frac{1}{2}\,\mathrm{Li}_4\Big(\frac{1}{2}\Big)\log^2(2) + \frac{23}{32}\zeta(6) - \frac{1}{2}\zeta(3)^2 - \frac{63}{32}\zeta(5)\log(2)\\ &+ \frac{1}{2}\zeta(2)\zeta(3)\log(2) + \frac{1}{2!}\zeta(4)\log^2(2) - \frac{1}{4!}\zeta(2)\log^4(2) + \frac{8}{6!}\log^6(2)\,.\end{aligned}$$

(ii) *We can reduce* $S_{3,3}(-1)$ *to* $S_{4,2}(-1) - S_{4,2}(\frac{1}{2})$ *modulo* Li_6 *and products as*

$$\begin{aligned}S_{3,3}(-1) \;=\;& S_{4,2}(-1) - S_{4,2}\Big(\frac{1}{2}\Big) + 2\,\mathrm{Li}_6\Big(\frac{1}{2}\Big) + \mathrm{Li}_5\Big(\frac{1}{2}\Big)\log(2) - \frac{41}{32}\zeta(6) - \frac{1}{2}\zeta(3)^2\\ &- \frac{1}{2}\Big(\frac{1}{16}\zeta(5) - \zeta(2)\zeta(3)\Big)\log(2) + \frac{1}{8\cdot 2!}\zeta(4)\log^2(2)\\ &- \frac{1}{3!}\zeta(3)\log^3(2) + \frac{1}{4!}\zeta(2)\log^4(2) - \frac{6}{2\cdot 6!}\log^6(2)\,.\end{aligned} \tag{29}$$

Note that an evaluation of $S_{3,3}(\frac{1}{2})$ is already known, but the general result in [40, Theorem 4] would only express it in terms of $S_{2,4}(-1)$ (equivalently of $S_{4,2}(\frac{1}{2})$, by reflection and inversion) and $S_{3,3}(-1)$. The above reduction corresponds to the fact that $\delta^{\geq 2} S^{\mathfrak{L}}_{3,3}(\frac{1}{2}) = 0$.

We also stress that this reduction still contains weight 6 Nielsen polylogarithms. However, we expect that both $S_{4,2}(-1)$ and $S_{4,2}(\frac{1}{2})$ reduce further, since the motivic 2-coboundaries vanish of their motivic analogues. From the three-term and duplication relation for Li_3, we obtain that both $\{\frac{1}{2}\}_3$ and $\{-1\}_3$ are rational multiples of $\{1\}_3$, so each coboundary reduces to 0 via the antisymmetry of the wedge product $\{1\}_3 \wedge \{1\}_3 = 0$. These reductions would imply also that $S_{3,3}(-1)$ reduces to depth 1, as opposed to depth 2 above. We return to these questions in Section 7.3 below.

7.2. **Functional equations for $S_{4,2}$.** As mentioned above, in analogy with the case of $S_{3,2}$ we expect that $S_{4,2}$ of any Li_3 functional equation can be reduced to Li_6 terms. As evidence for this, we show this for the three term relation and the algebraic family of functional equations from Section 5.

Corresponding to the three-term relation for Li_3, namely

$$\mathrm{Li}_3(1-z) + \mathrm{Li}_3(z) + \mathrm{Li}_3\Big(\frac{z}{z-1}\Big) = \zeta(3) \text{ (mod products)}\,, \tag{30}$$

we have the following functional equation for $S_{4,2}$.

Proposition 29 (Three-term relation)**.** *For all $z \in \mathbb{C} \smallsetminus \big((-\infty,0] \cup [1,\infty)\big)$, the following three-term identity for $S_{4,2}$ holds*

$$\begin{aligned}
&S_{4,2}(1-z) + S_{4,2}(z) + S_{4,2}\Big(\frac{z}{z-1}\Big) = \\
&2\,\mathrm{Li}_6(1-z) + 2\,\mathrm{Li}_6(z) + 2\,\mathrm{Li}_6\Big(\frac{z}{z-1}\Big) - \Big(\mathrm{Li}_5(z) - \mathrm{Li}_5\Big(\frac{z}{z-1}\Big)\Big)\log(1-z) - \mathrm{Li}_5(1-z)\log(z) \\
&- \frac{3}{6!}\log^6(1-z) + \frac{2}{5!}\log(z)\log^5(1-z) - \frac{1}{2!\,4!}\log^2(z)\log^4(1-z) - \frac{2}{4!}\zeta(2)\log^4(1-z) \\
&+ \frac{1}{3!}\zeta(2)\log(z)\log^3(1-z) - \frac{1}{3!}\zeta(3)\log^3(1-z) + \frac{1}{2!}\zeta(3)\log(z)\log^2(1-z) - \frac{7}{4\cdot 2!}\zeta(4)\log^2(1-z) \\
&+ \zeta(4)\log(z)\log(1-z) + \zeta(5)\log(z) - \zeta(2)\zeta(3)\log(1-z) - \Big(\frac{1}{2}\zeta(3)^2 + \frac{5}{4}\zeta(6)\Big)\,.
\end{aligned}$$

Proof. Differentiate, and take $z \to 0$ to fix the constant as $S_{4,2}(1) - 2\,\mathrm{Li}_6(1) = -\frac{1}{2}\zeta(3)^2 - \frac{5}{4}\zeta(6)$.

Alternatively, we can also verify that this follows from reflection and inversion, by checking the polynomial invariant from Section 3.3:

$$\begin{aligned}
&S^{\mathrm{u}}_{4,2}(1-z) + S^{\mathrm{u}}_{4,2}(z) + S^{\mathrm{u}}_{4,2}\Big(\frac{z}{z-1}\Big) - 2\Big(\mathrm{Li}^{\mathrm{u}}_6(1-z) + \mathrm{Li}^{\mathrm{u}}_6(z) + \mathrm{Li}^{\mathrm{u}}_6\Big(\frac{z}{z-1}\Big)\Big) \\
\mapsto\quad & -4XY^3 + 4X^3Y + 4Y(X-Y)^3 - 2\big(-Y^4 + X^4 - (X-Y)^4\big) = 0\,. \qquad \square
\end{aligned}$$

We note the following reductions of $S_{3,3}$ at roots of unity (the latter of which is confirmed via the evaluation $S_{3,3}(e^{2\pi i/6}) = Z(AAADDD)$ in Broadhurst's multiple Deligne Value Data Mine [5], and the former of which follows from the MZV Data Mine [1] as the Nielsen polylogarithms at -1 are alternating MZV's).

Corollary 30. *We have the following specialisations.*

(i)
$$S_{3,3}(-1) = \frac{3}{2}S_{4,2}(-1) + \frac{5}{16}\zeta(6) - \frac{1}{4}\zeta(3)^2\,. \tag{31}$$

(ii)
$$\begin{aligned}
S_{3,3}(e^{2\pi i/6}) &= 3\,\mathrm{Li}_6(e^{2\pi i/6}) - \frac{1}{2}\zeta(3)^2 - \frac{1829}{1944}\zeta(6) + \frac{1}{3}i\pi\Big(S_{3,2}(e^{2\pi i/6}) - 2\zeta(5)\Big) \\
&\quad + \frac{1}{3}\zeta(2)\,\mathrm{Li}_4(e^{2\pi i/6}) + \frac{1}{324}(2\pi i)^3\zeta(3)\,.
\end{aligned}$$

Proof. (i) Setting $z = \frac{1}{2}$ in Proposition 29 leads to the following two-term identity

$$\begin{aligned}
S_{4,2}\Big([-1] + 2\Big[\frac{1}{2}\Big]\Big) &= 4\,\mathrm{Li}_6\Big(\frac{1}{2}\Big) + 2\,\mathrm{Li}_5\Big(\frac{1}{2}\Big)\log(2) - \frac{51}{16}\zeta(6) - \frac{1}{2}\zeta(3)^2 \\
&\quad - \Big(\frac{1}{16}\zeta(5) - \zeta(2)\zeta(3)\Big)\log(2) + \frac{1}{4\cdot 2!}\zeta(4)\log^2(2) \\
&\quad - \frac{2}{3!}\zeta(3)\log^3(2) + \frac{2}{4!}\zeta(2)\log^4(2) - \frac{6}{6!}\log^6(2)\,.
\end{aligned} \tag{32}$$

Now note that (29) and (32) together imply (31).

(ii) This follows from the $S_{3,3}$ to $S_{4,2}$ reduction (Proposition 27) and $S_{4,2}$ inversion since

$$\begin{aligned}
S_{3,3}(e^{2\pi i/6}) &= S_{4,2}(e^{2\pi i/6}) - S_{4,2}(1 - e^{2\pi i/6}) \text{ (mod products)} \\
&= S_{4,2}(e^{2\pi i/6}) - S_{4,2}(e^{-2\pi i/6})\,. \qquad \square
\end{aligned}$$

For the algebraic Li_3 functional equation from Section 5, we can reduce $S_{4,2}$ to Li_6, as expected. This was also used in [15] to obtain pure Li_6 functional equations, from certain depth reductions of the depth 2 integral $I_{5,1}(x,y) = I(0; x, 0, 0, 0, 0, y; 1)$ under trilogarithm functional equations.

Proposition 31 (Proposition 7.6.12 in [15])**.** *Let $a, b, c \in \mathbb{Z} \setminus \{0\}$, with $a+b+c=0$, and let $\{p_i(t)\}_{i=1}^r$ be the roots of $x^a(1-x)^b = t$. Then the following reduction holds on the level of the mod-products symbol*

$$\sum_{i=1}^r -\frac{1}{a} S^{\mathfrak{u}}_{4,2}(1-p_i(t)) + \frac{1}{b} S^{\mathfrak{u}}_{4,2}(p_i(t)) \overset{\sqcup\!\sqcup}{=}$$
$$\sum_{i=1}^r \frac{b-a}{a^2} \mathrm{Li}^{\mathfrak{u}}_6(1-p_i(t)) - \frac{a-b}{b^2} \mathrm{Li}^{\mathfrak{u}}_6(p_i(t)) - \frac{1}{a+b} \mathrm{Li}^{\mathfrak{u}}_6(1-p_i(t)^{-1}) \,.$$

Corollary 32. *We have the clean single-valued identity*

$$\sum_{i=1}^r -\frac{1}{a} \mathcal{S}^{\sqcup\!\sqcup}_{4,2}(1-p_i(t)) + \frac{1}{b} \mathcal{S}^{\sqcup\!\sqcup}_{4,2}(p_i(t)) \;=$$
$$\sum_{i=1}^r \frac{b-a}{a^2} \mathcal{L}^{\sqcup\!\sqcup}_6(1-p_i(t)) - \frac{a-b}{b^2} \mathcal{L}^{\sqcup\!\sqcup}_6(p_i(t)) - \frac{1}{a+b} \mathcal{L}^{\sqcup\!\sqcup}_6(1-p_i(t)^{-1}) \,.$$

Proof. Consider the limit $t \to 0$, and use $\mathcal{L}^{\sqcup\!\sqcup}_6(0) = \mathcal{L}^{\sqcup\!\sqcup}_6(1) = \mathcal{L}^{\sqcup\!\sqcup}_6(\infty) = 0$ and $\mathcal{S}^{\sqcup\!\sqcup}_{4,2}(0) = \mathcal{S}^{\sqcup\!\sqcup}_{4,2}(1) = \mathcal{S}^{\sqcup\!\sqcup}_{4,2}(\infty) = 0$. Since both functions $\mathcal{S}^{\sqcup\!\sqcup}_{4,2}$ and $\mathcal{L}^{\sqcup\!\sqcup}_6$ vanish at all three of the points in $\{0, 1, \infty\}$, the constant in this clean single-valued identity is always identically 0. ☐

Proof of Proposition 31. Expand out as in the proof of Proposition 14, using the recursive definition of the mod-products symbol of $S^{\mathfrak{u}}_{n,p}(z)$, and replace $1-p_i$ in the last tensor factor by $t^{1/b} p_i^{-a/b}$. The difference becomes

$$\sum_{i=1}^r \mathrm{Symb}^{\sqcup\!\sqcup} \bigg(\frac{1}{b} \big\{ S^{\mathfrak{u}}_{3,2}(1-p_i) + S^{\mathfrak{u}}_{3,2}(p_i) - \mathrm{Li}^{\mathfrak{u}}_5(1-p_i) - \mathrm{Li}^{\mathfrak{u}}_5(1-p_i^{-1}) - \mathrm{Li}^{\mathfrak{u}}_5(p_i) \big\} \otimes p_i$$
$$- \frac{1}{ab} \mathrm{Symb}^{\sqcup\!\sqcup} \Big\{ S^{\mathfrak{u}}_{3,2}(1-p_i) - \frac{a-b}{a} \mathrm{Li}^{\mathfrak{u}}_5(1-p_i) - \frac{a}{a+b} \mathrm{Li}^{\mathfrak{u}}_5(1-p_i^{-1}) - \frac{a}{b} \mathrm{Li}^{\mathfrak{u}}_5(p_i) \Big\} \otimes t \bigg) \,.$$

The first bracket cancels using the two-term identity $S_{3,2}(z) + S_{3,2}(1-z)$ from Proposition 12. The second factor cancels using this, and the reduction for $S_{3,2}$ of the algebraic Li_2 equation from Proposition 14. ☐

In particular, we expect a reduction of $S_{4,2}$ to Li_6, when applied to the 840-term trilogarithm relation found by Zagier via antisymmetrising Goncharov's 22-term relation (see Lemma 3.9, in [31]). By analogy with the weight 5 case (Corollary 19), we anticipate an interesting many variable functional equation for Li_6 to arise from applying the obvious 8-fold antisymmetrisation to such a reduction.

7.3. Depth reductions of $S_{4,2}(-1)$, $S_{4,2}(\frac{1}{2})$ and $S_{4,2}(\phi^{-2})$.

The motivic coproduct yoga suggests that one can reduce $S_{4,2}(-1)$ alone to classical polylogarithms and lower weight products, and that one can similarly reduce $S_{4,2}(\phi^{-2})$, where $\phi = \frac{1}{2}(1+\sqrt{5})$. We expect these claims to hold since the following trilogarithm identities

$$\{-1\}_3 \;=\; -\frac{3}{4}\{1\}_3 \,, \quad \{\phi^{-2}\}_3 \;=\; \frac{4}{5}\{1\}_3 \,,$$

lead to the vanishing 2-cobrackets for $S_{4,2}$:

$$\delta^{\geq 2} S^{\mathfrak{L}}_{4,2}(-1) \;=\; -\{-1\}_3 \wedge \{1\}_3 \;=\; \frac{3}{4}\{1\}_3 \wedge \{1\}_3 \;=\; 0 \,,$$
$$\delta^{\geq 2} S^{\mathfrak{L}}_{4,2}(\phi^{-2}) \;=\; -\{\phi^{-2}\}_3 \wedge \{1\}_3 \;=\; -\frac{4}{5}\{1\}_3 \wedge \{1\}_3 \;=\; 0 \,.$$

The first trilogarithm identity above is just the specialization to $z = -1$ of the duplication relation

$$\mathrm{Li}_3(z) + \mathrm{Li}_3(-z) \;=\; \frac{1}{4} \mathrm{Li}_3(z^2) \,.$$

The second identity, known to Landen, follows from duplication and the three-term relation (30). More precisely, we have

$$\mathrm{Li}_3(-1) \;=\; -\frac{3}{4}\zeta(3) \,,$$
$$\mathrm{Li}_3(\phi^{-2}) \;=\; \frac{4}{5}\zeta(3) - \frac{4}{5}\zeta(2)\log(\phi) + \frac{2}{3}\log^3(\phi) \,.$$

The corresponding depth reductions for $S_{4,2}(-1)$ and $S_{4,2}(\phi^{-2})$ would immediately follow from the conjectured identity for $S_{4,2}$ corresponding to the trilogarithm duplication relation $\mathrm{Li}_3(z) + \mathrm{Li}_3(-z) - \frac{1}{2^2}\mathrm{Li}_3(z^2) = 0$. Unfortunately, we do not have such an identity. Nevertheless, we can still investigate these reductions numerically and via other functional identities.

Strategy for finding an $S_{4,2}(-1)$ evaluation via functional identities. We are able to find a certain mod-products symbol level identity relating $\mathrm{Li}^{\mathfrak{L}}_{5,1}(-x,-1) = I^{\mathfrak{L}}(0; x^{-1}, 0, 0, 0, 0, -1; 1) = I^{\mathfrak{L}}_{5,1}(x^{-1}, -1)$ to a combination of $S^{\mathfrak{L}}_{4,2}$ and $\mathrm{Li}^{\mathfrak{L}}_6$ terms.

$$\begin{aligned}&\mathrm{Li}^{\mathfrak{L}}_{5,1}(-x,-1) \overset{\sqcup\!\sqcup}{=} \\ &S^{\mathfrak{L}}_{4,2}\bigg(-\frac{1}{32}\big[x^2\big] + \frac{17}{4}\big[-x\big] - \frac{13}{4}\big[x\big] - \frac{33}{8}\bigg[\frac{1-x}{2}\bigg] + \frac{33}{8}\bigg[\frac{1-x}{1+x}\bigg] + \frac{33}{4}\bigg[\frac{2x}{1+x}\bigg] \\ &\qquad + \frac{33}{8}\bigg[\frac{1+x}{2}\bigg] + \frac{33}{16}\bigg[-\frac{4x}{(1-x)^2}\bigg] + \frac{33}{32}\bigg[\frac{(1-x)^2}{(1+x)^2}\bigg]\bigg) \quad (\text{mod explicit } \mathrm{Li}^{\mathfrak{L}}_6\text{'s})\,.\end{aligned}$$

The full mod-products symbol identity involves 117 $\mathrm{Li}^{\mathfrak{L}}_6$ terms, a typical term of which is

$$\mathrm{Li}^{\mathfrak{L}}_6\Big(\frac{-2(1-x)x}{(1+x)^2}\Big)\,,$$

and the full expression is reproduced in Appendix C.

In a small interval to the right of 0, $(0, \frac{1}{10})$ say, where the values of all arguments lie along $(-\infty, 1]$, we have lifted this to an analytic identity, verifiable by differentiation, the constant of which is fixed by evaluation in the limit $x \to 0$. Unfortunately, the resulting identity contains many Li_6 terms which are ill-defined in the limit $x \to 1$, so that one would have to perform a *cumbersome* analytic continuation and to separate the imaginary parts. Nevertheless, except for $S_{4,2}(-\frac{4x}{(1-x)^2})$, the remaining $S_{4,2}$ terms are well-defined as $x \to 1$, and that term can be replaced by its inverse modulo Li_6 and products, using the $S_{4,2}$ inversion relation. This gives

$$\mathrm{Li}_{5,1}(-1,-1) \;=\; S_{4,2}\bigg(\frac{17}{4}\big[-1\big] + \frac{99}{32}\big[0\big] + \frac{291}{32}\big[1\big]\bigg) \;(\text{mod products, } \mathrm{Li}_6)\,.$$

On the other hand, $\mathrm{Li}_{5,1}(-1,-1) = \zeta(\overline{5},\overline{1})$, and $S_{4,2}(-1) = \zeta(1,\overline{5})$ as alternating MZV's. From the MZV Data Mine [1], we directly have

$$\mathrm{Li}_{5,1}(-1,-1) = S_{4,2}(-1) + \frac{23}{16}\zeta(6) - \frac{3}{4}\zeta(3)^2 - \zeta(5)\log(2)\,.$$

We can therefore obtain an expression for $S_{4,2}(-1)$ in terms of known quantities $S_{4,2}(0) = 0, S_{4,2}(1) = \frac{3}{4}\zeta(6) - \frac{1}{2}\zeta(3)^2$, Li_6's of more complicated arguments, and products of lower weight terms. At $z \to 1$, the Li_6 arguments in fact specialise to $\pm 1, \pm\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}$ (up to inverses), and after performing the analytic continuation (with computer assistance) via the $S_{4,2}, S_{3,2}$ and polylogarithm inversion relations we obtain such an expression. After some simplification with the Li_5 and Li_6 duplication relations, and a certain weight 5 evaluation (actually, (36) below), we obtain the following identity. (The veracity of the following identity does rely on the long aforementioned analytic weight 6 identity, lifting the identity in Appendix C, and on computer assistance for analytic continuation.)

$$\begin{aligned}S_{4,2}(-1) \;=\; &\frac{1}{13}\bigg(\frac{1}{3}\mathrm{Li}_6\Big(-\frac{1}{8}\Big) - 162\,\mathrm{Li}_6\Big(-\frac{1}{2}\Big) - 126\,\mathrm{Li}_6\Big(\frac{1}{2}\Big)\bigg) - \frac{1787}{624}\zeta(6) + \frac{3}{8}\zeta(3)^2 \\ &+ \frac{31}{16}\zeta(5)\log(2) - \frac{15}{26}\zeta(4)\log^2(2) + \frac{3}{104}\zeta(2)\log^4(2) - \frac{1}{208}\log^6(2)\,.\end{aligned} \tag{33}$$

Note that the coefficient of $\mathrm{Li}_6(-\frac{1}{8})$ is written deliberately as $\frac{1}{3}$ inside the parentheses, for structural reasons as explained in the coaction analysis below.

This provides another example of the so-called 'push-down' phenomenon (in the terminology of [1] for MZV's), whereby one can reduce the depth of a period by viewing it within a larger set of periods. In the case of MZV's, the simplest examples occur in weight 12 (see Equation (10.1) in [1]), where the apparently irreducible depth 4 MZV $\zeta(1,1,4,6)$ becomes reducible to depth 2 when viewed as an alternating MZV (Euler sum).

Strategy for finding an $S_{4,2}(-1)$ evaluation by matching the coaction. Using the motivic coaction (recall in Brown's comodule $\mathcal{H}_\bullet$ from Section 2.4 we have $\zeta^{\mathfrak{m}}(2) \neq 0$, so more information is retained) we can better understand the nature and structure of this reduction and attempt to generalise it to higher cases. This equality on the motivic level means that the reduced coactions of both sides must agree. We compute

$$\Delta' S^{\mathfrak{m}}_{4,2}(-1) \;=\; \frac{3}{4}\zeta^{\mathfrak{m}}(3) \otimes \zeta^{\mathfrak{u}}(3) + \frac{31}{16}\log^{\mathfrak{m}}(2) \otimes \zeta^{\mathfrak{u}}(5)\,. \tag{34}$$

Firstly, it is straightforward to see that $\Delta' \zeta^{\mathfrak{m}}(3)^2 \;=\; 2\zeta^{\mathfrak{m}}(3) \otimes \zeta^{\mathfrak{u}}(3)$. So the $\zeta^{\mathfrak{m}}(3) \otimes \zeta^{\mathfrak{u}}(3)$ component of $\Delta' S^{\mathfrak{m}}_{4,2}(-1)$ can be matched using

$$\frac{3}{8}\zeta^{\mathfrak{m}}(3)^2\,,$$

exactly as appears in the reduction (33). To match the rest of the coaction recall that

$$\Delta' \operatorname{Li}^{\mathfrak{m}}_n(x) \;=\; \sum_{k=1}^{n-1} \frac{1}{(n-k)!} \operatorname{Li}^{\mathfrak{m}}_k(x) \otimes (\log^{\mathfrak{u}}(x))^{n-k}\,.$$

Let

$$A = \frac{1}{3}\operatorname{Li}^{\mathfrak{m}}_6\left(-\frac{1}{8}\right) - 162\operatorname{Li}^{\mathfrak{m}}_6\left(-\frac{1}{2}\right) - 126\operatorname{Li}^{\mathfrak{m}}_6\left(\frac{1}{2}\right),$$

so the associated weight $(5,1)$-part of the coaction becomes

$$\Delta^{(5,1)} A \;=\; -\left(\operatorname{Li}^{\mathfrak{m}}_5\left(-\frac{1}{8}\right) - 162\operatorname{Li}^{\mathfrak{m}}_5\left(-\frac{1}{2}\right) - 126\operatorname{Li}^{\mathfrak{m}}_5\left(\frac{1}{2}\right)\right) \otimes \log^{\mathfrak{u}}(2)\,, \tag{35}$$

the factor $\frac{1}{3}$ annihilating with the $\log^{\mathfrak{u}}(-\frac{1}{8}) \;=\; \log^{\mathfrak{u}}(\frac{1}{8}) \;=\; -3\log^{\mathfrak{u}}(2)$. This is essentially an avatar of Lewin's pseudo-integration process [45, Section 1.4].

Recall now the following identity [53, p. 419] for the single-valued polylog

$$\mathscr{L}_5\Big(-\frac{1}{8}\Big) - 162\mathscr{L}_5\Big(-\frac{1}{2}\Big) - 126\mathscr{L}_5\Big(\frac{1}{2}\Big) \;=\; \frac{403}{16}\zeta(5)\,.$$

A version of this identity is already given for the analytic function Li_5 including explicit lower order terms as follows in [44, Equation 7.100] (note however, that the coefficient of $\pi^2\log(2)^3$ therein appears to be incorrect, and it should be $\frac{1}{4}$)

$$\begin{aligned} &\operatorname{Li}_5\left(-\frac{1}{8}\right) - 162\operatorname{Li}_5\left(-\frac{1}{2}\right) - 126\operatorname{Li}_5\left(\frac{1}{2}\right) \;= \\ &\frac{403}{16}\zeta(5) - 15\zeta(4)\log(2) + \frac{3}{2}\zeta(2)\log^3(2) - \frac{3}{8}\log^5(2)\,. \end{aligned} \tag{36}$$

In particular, from a motivic version of this, we obtain the following term in the coaction

$$-\frac{403}{16}\zeta^{\mathfrak{m}}(5) \otimes \log^{\mathfrak{u}}(2)\,.$$

So to match the actual term $\frac{31}{16}\log^{\mathfrak{m}}(2) \otimes \zeta^{\mathfrak{u}}(5)$ appearing in $\Delta' S_{4,2}(-1)$ we can take the combination

$$\frac{1}{13}\left(\frac{1}{3}\operatorname{Li}^{\mathfrak{m}}_6\left(-\frac{1}{8}\right) - 162\operatorname{Li}^{\mathfrak{m}}_6\left(-\frac{1}{2}\right) - 126\operatorname{Li}^{\mathfrak{m}}_6\left(\frac{1}{2}\right)\right) + \frac{31}{16}\zeta^{\mathfrak{m}}(5)\log^{\mathfrak{m}}(2)\,,$$

as is manifest in the reduction in (33). This explains the main term of the reduction. We have, for simplicity, ignored much of the coaction, not just the lower order product terms in the weight $(5,1)$-part, but also the weight $(k, 6-k)$-parts, for $k = 1, \ldots, 4$. This is not a cause for concern, since these parts are strictly simpler and so easier to deal with; they involve only products in the left hand factor, or higher powers of $\log^{\mathfrak{u}}(2)$ in the right hand factor.

In fact, it is clear that for (non-multiple) polylogarithm, one can recover the rest of the reduced coaction Δ from the $(n-1,1)$-part of the coaction. We thank the referee for pointing this out. Specifically, we have

$$\Delta' \operatorname{Li}^{\mathfrak{m}}_n(x) = \sum_{k=0}^{n-2} \mu_k \circ (\Delta^{(n-1-k,k)} \otimes \mathrm{id}) \circ \Delta^{(n-1,1)} \operatorname{Li}^{\mathfrak{m}}_n(x)\,, \tag{37}$$

where $\mu_k(a \otimes b \otimes c) = a \otimes \frac{bc}{k+1}$. With this we can recover the full coaction of the $\operatorname{Li}^{\mathfrak{m}}_6$ combination A from the $(5,1)$-part in (35), using the evaluation in (36). Denote the right hand side of the motivic version of (36) as

$$B = \frac{403}{16}\zeta^{\mathfrak{m}}(5) - 15\zeta^{\mathfrak{m}}(4)\log^{\mathfrak{m}}(2) + \frac{3}{2}\zeta^{\mathfrak{m}}(2)\log^{\mathfrak{m}}(2)^3 - \frac{3}{8}\log^{\mathfrak{m}}(2)^5\,,$$

so that $\Delta^{(5,1)}A = -B \otimes \log(2)$. We find that $\Delta\zeta^{\mathfrak{m}}(4)\log^{\mathfrak{m}}(2) = \zeta^{\mathfrak{m}}(4)\otimes\log^{\mathfrak{u}}(2) + (\zeta^{\mathfrak{m}}(4)\log^{\mathfrak{m}}(2))\otimes 1$, so that only the $k=0$ and $k=1$ terms contribute for this summand. So the term $15\zeta^{\mathfrak{m}}(4)\log^{\mathfrak{m}}(2)$ in $-B$ contributes (note the sign)

$$\begin{aligned}&15\Big(\mu_0(\zeta^{\mathfrak{m}}(4)\log^{\mathfrak{m}}(2)\otimes 1\otimes\log^{\mathfrak{u}}(2)) + \mu_1(\zeta^{\mathfrak{m}}(4)\otimes\log^{\mathfrak{u}}(2),\otimes\log^{\mathfrak{u}}(2))\Big)\\ &\quad = \frac{15}{2}\Big(\zeta^{\mathfrak{m}}(4)\otimes\log^{\mathfrak{u}}(2)^2 + 2(\zeta^{\mathfrak{m}}(4)\log^{\mathfrak{m}}(2))\otimes\log^{\mathfrak{u}}(2)\Big) \;=\; \Delta'\Big(\frac{15}{2}\zeta^{\mathfrak{m}}(4)\log^{\mathfrak{m}}(2)^2\Big)\end{aligned}$$

to the full coaction of A. Similarly the terms

$$-\frac{3}{2}\zeta^{\mathfrak{m}}(2)\log^{\mathfrak{m}}(2)^3 \quad\text{and}\quad \frac{3}{8}\log^{\mathfrak{m}}(2)^5$$

contribute

$$\Delta'\Big(-\frac{3}{8}\zeta^{\mathfrak{m}}(2)\log^{\mathfrak{m}}(2)^4\Big) \quad\text{and}\quad \Delta'\Big(\frac{1}{16}\log^{\mathfrak{m}}(2)^6\Big)\,,$$

respectively. (We effectively integrate B with respect to $\log^{\mathfrak{m}}(2)$, in this case.) Since $\zeta^{\mathfrak{m}}(5)$ is primitive, the term $-\frac{403}{16}\zeta^{\mathfrak{m}}(5)$ contributes only $-\frac{403}{16}\zeta^{\mathfrak{m}}(5)\otimes\log^{\mathfrak{u}}(2)$ from $k=0$, and this cannot be recognised as the coaction of a product. Overall, this says

$$\Delta' A = -\frac{403}{16}\zeta^{\mathfrak{m}}(5)\otimes\log^{\mathfrak{m}}(2) + \Delta'\Big(\frac{15}{2}\zeta^{\mathfrak{m}}(4)\log^{\mathfrak{m}}(2)^2 - \frac{3}{8}\zeta^{\mathfrak{m}}(2)\log^{\mathfrak{m}}(2)^4 + \frac{1}{16}\log^{\mathfrak{m}}(2)^6\Big)\,.$$

Since $\Delta'\zeta^{\mathfrak{m}}(5)\log^{\mathfrak{m}}(2) = \zeta^{\mathfrak{m}}(5)\otimes\log^{\mathfrak{u}}(2) + \log^{\mathfrak{m}}(2)\otimes\zeta^{\mathfrak{u}}(5)$, we conclude that

$$\frac{1}{13}\left(A + \frac{403}{16}\zeta^{\mathfrak{m}}(5)\log^{\mathfrak{m}}(2) - \frac{15}{2}\zeta(4)\log^{\mathfrak{m}}(2)^2 + \frac{3}{8}\zeta^{\mathfrak{m}}(2)\log^{\mathfrak{m}}(2)^4 - \frac{1}{16}\log^{\mathfrak{m}}(2)^6\right)$$

has reduced coaction exactly $\frac{31}{16}\log^{\mathfrak{m}}(2)\otimes\zeta^{\mathfrak{u}}(5)$, matching the second term of $\Delta' S^{\mathfrak{m}}_{4,2}(-1)$ in (34).

At this point, we have only shown that the equality (33) holds up to primitives, $\mathrm{Li}^{\mathfrak{m}}_6(e^{2\pi i k/N})$, where k, N are integers. However it follows from Jonquière's inversion formula [38]

$$\mathrm{Li}_n(e^{2\pi i x}) + (-1)^n\,\mathrm{Li}_n(e^{-2\pi i x}) = -\frac{(2\pi i)^n}{n!}\mathcal{B}_n(x)\,,$$

where $\mathcal{B}_n(x)$ is the n-th periodic Bernoulli function, that the real part of any such weight 6 primitive is proportional to $\zeta(6)$. Since the rest of the terms in (33) are real, we only care about the real part of the primitives, so the identity indeed holds up to some rational multiple of $\zeta(6)$.

Overall we have shown that, for some rational $q\in\mathbb{Q}$, the following motivic identity holds

$$\begin{aligned}S^{\mathfrak{m}}_{4,2}(-1) \;=\; & q\cdot\zeta^{\mathfrak{m}}(6) + \frac{1}{13}\Big(\frac{1}{3}\mathrm{Li}^{\mathfrak{m}}_6\Big(-\frac{1}{8}\Big) - 162\,\mathrm{Li}^{\mathfrak{m}}_6\Big(-\frac{1}{2}\Big) - 126\,\mathrm{Li}^{\mathfrak{m}}_6\Big(\frac{1}{2}\Big)\Big) + \frac{3}{8}\zeta^{\mathfrak{m}}(3)^2\\ & + \frac{31}{16}\zeta^{\mathfrak{m}}(5)\log^{\mathfrak{m}}(2) - \frac{15}{26}\zeta^{\mathfrak{m}}(4)\log^{\mathfrak{m}}(2)^2 + \frac{3}{104}\zeta^{\mathfrak{m}}(2)\log^{\mathfrak{m}}(2)^4 - \frac{1}{208}\log^{\mathfrak{m}}(2)^6\,.\end{aligned} \tag{38}$$

From numerical evaluation we find

$$q \overset{?}{=} -\frac{1787}{624}\,.$$

Moreover, the question mark can be removed if one accepts the computer-aided proof of (33) above. We can then use (32) and (31) to obtain the following.

Remark 33. One has the following reductions of $S_{4,2}(\frac{1}{2})$ and $S_{3,3}(-1)$ to polynomials in classical polylogarithms, for certain rational numbers $p, r\in\mathbb{Q}$.

$$\begin{aligned}S_{4,2}\Big(\frac{1}{2}\Big) \;\overset{?}{=}\; & r\cdot\zeta(6) - \frac{1}{26}\Big(\frac{1}{3}\mathrm{Li}_6\Big(-\frac{1}{8}\Big) - 162\,\mathrm{Li}_6\Big(-\frac{1}{2}\Big) - 178\,\mathrm{Li}_6\Big(\frac{1}{2}\Big)\Big) - \frac{7}{16}\zeta(3)^2\\ & + \mathrm{Li}_5\Big(\frac{1}{2}\Big)\log(2) - \Big(\zeta(5) - \frac{1}{2}\zeta(2)\zeta(3)\Big)\log(2) + \frac{73}{208}\zeta(4)\log^2(2) - \frac{1}{6}\zeta(3)\log^3(2)\\ & + \frac{17}{624}\zeta(2)\log^4(2) - \frac{11}{6240}\log^6(2)\,,\end{aligned}$$

$$\begin{aligned}S_{3,3}(-1) \;\overset{?}{=}\; & s\cdot\zeta(6) + \frac{3}{26}\Big(\frac{1}{3}\mathrm{Li}_6\Big(-\frac{1}{8}\Big) - 162\,\mathrm{Li}_6\Big(-\frac{1}{2}\Big) - 126\,\mathrm{Li}_6\Big(\frac{1}{2}\Big)\Big) + \frac{5}{16}\zeta(3)^2\\ & + \frac{93}{32}\zeta(5)\log(2) - \frac{45}{52}\zeta(4)\log^2(2) + \frac{9}{208}\zeta(2)\log^4(2) - \frac{3}{416}\log^6(2)\,,\end{aligned}$$

where

$$r = \frac{1}{2}\Big(-\frac{51}{16} - q\Big) \overset{?}{=} -\frac{101}{624} \text{ and } s = \frac{5}{16} + \frac{3}{2}q \overset{?}{=} -\frac{1657}{416}$$

are obtained from the numerically fixed value of q in (38) above.

We emphasise yet again that the only uncertainty in these equations lies in the respective coefficient of $\zeta(6)$, as the coaction expressions of both sides agree in each case.

Aside: application to alternating MZV's. The reduction from (33) allows us to give an apparently new evaluation for some weight 6 alternating MZV's (Euler sums), and thence reduce all weight ≤ 6 alternating MZV's to polynomials in classical polylogarithms.

More explicitly, one has the following equalities, as verified by the MZV Data Mine via Remark 13.

$$\begin{aligned} S_{4,2}(-1) &= \zeta(1,\overline{5})\,, \\ \zeta(1,1,1,\overline{3}) - \frac{1}{2}\zeta(1,\overline{5}) &= 2\operatorname{Li}_6\Big(\frac{1}{2}\Big) + 2\operatorname{Li}_5\Big(\frac{1}{2}\Big)\log(2) + \operatorname{Li}_4\Big(\frac{1}{2}\Big)\log^2(2) - \frac{1}{4}\zeta(3)^2 \\ &\quad + \frac{7}{24}\zeta(3)\log^3(2) - \frac{53}{32}\zeta(6) + \frac{1}{36}\log^6(2) - \frac{1}{8}\zeta(2)\log^4(2)\,, \end{aligned}$$

where

$$\begin{aligned} \zeta(1,\overline{5}) &\coloneqq \sum_{0<n_1<n_2} \frac{(-1)^{n_2}}{n_1 n_2^5} = \operatorname{Li}_{1,5}(1,-1)\,, \\ \zeta(1,1,1,\overline{3}) &\coloneqq \sum_{0<n_1<n_2<n_3<n_4} \frac{(-1)^{n_4}}{n_1 n_2 n_3 n_4^3} = \operatorname{Li}_{1,1,1,3}(1,1,1,-1) \end{aligned}$$

are alternating MZV's of weight 6.

Using the MZV Data Mine [1], a set of algebra generators of alternating MZV's is given up to weight 6 by

$$\big\{\log(2), \zeta(2), \zeta(3), \zeta(5), \zeta(1,\overline{3}), \zeta(1,1,\overline{3}), \zeta(1,1,1,\overline{3}), \zeta(1,\overline{5})\big\}\,.$$

The strictly alternating MZV's $\zeta(1,\overline{3})$ and $\zeta(1,1,\overline{3})$ are already known to be polynomials in classical polylogarithms, as verified by the MZV Data Mine via Remark 13. Namely

$$\begin{aligned} \zeta(1,\overline{3}) &= 2\operatorname{Li}_4\Big(\frac{1}{2}\Big) - \frac{15}{8}\zeta(4) + \frac{7}{4}\zeta(3)\log(2) - \frac{1}{2!}\zeta(2)\log^2(2) + \frac{2}{4!}\log^4(2), \\ \zeta(1,1,\overline{3}) &= -2\operatorname{Li}_5\Big(\frac{1}{2}\Big) - 2\operatorname{Li}_4\Big(\frac{1}{2}\Big)\log(2) + \frac{33}{32}\zeta(5) + \frac{1}{2}\zeta(2)\zeta(3) \\ &\quad - \frac{7}{8}\zeta(3)\log^2(2) + \frac{1}{3}\zeta(2)\log^3(2) - \frac{1}{15}\log^5(2)\,. \end{aligned}$$

Together with the above reduction for $\zeta(1,\overline{5})$, and consequently $\zeta(1,1,1,\overline{3})$, one obtains a reduction, albeit complicated, of all alternating MZV's of weight ≤ 6 to polynomials in classical polylogarithm values.

Reduction of $S_{4,2}(\phi^{-2})$ obtained using the coaction. As explained in the paragraph on the strategy for finding $S_{4,2}(-1)$ after (33), a great deal of structure in the $S_{4,2}(-1)$ reduction above becomes manifest in the coaction. By combining this understanding with the $S_{3,2}(\phi^{-2})$ reduction found earlier, we can produce a very short list of potentially relevant polylogarithm arguments for a candidate $S_{4,2}(\phi^{-2})$ reduction. With the LLL lattice reduction algorithm we quickly found the following to high precision, which was then subsequently verified to 10,000 decimal places in PARI/GP [36]. A complete analysis of the coaction, similar to the case $S_{4,2}(-1)$ above, explains all of the coefficients and terms, except for the $\zeta(6)$ coefficient which must be numerically fixed. Specifically, for some rational $t \in \mathbb{Q}$, we have

$$\begin{aligned} S_{4,2}(\phi^{-2}) &= t\cdot\zeta(6) + \frac{1}{396}\operatorname{Li}_6\Big(2\big[\phi^{-6}\big] - 128\big[\phi^{-3}\big] + 801\big[\phi^{-2}\big] - 576\big[\phi^{-1}\big]\Big) + \frac{2}{5}\zeta(3)^2 \\ &\quad + \operatorname{Li}_5\big(\phi^{-2}\big)\log(\phi) - \zeta(5)\log(\phi) + \frac{2}{11}\zeta(4)\log^2(\phi) - \zeta(3)\operatorname{Li}_3\big(\phi^{-2}\big) \\ &\quad + \frac{10}{33}\zeta(2)\log^4(\phi) - \frac{79}{990}\log^6(\phi)\,. \end{aligned}$$

From numerical evaluation, we find

$$t \overset{?}{=} \frac{35}{99}\,.$$

8. Identities in weight 7

In this section we 'depth reduce' $S_{4,3}(z)$, and give evaluations of it at -1 and $\frac{1}{2}$. Furthermore, in order to guarantee the vanishing of the cobracket terms and hence to have a chance to depth reduce $S_{5,2}$ we need to invoke functional equations which hold simultaneously for Li_2 and Li_4. This is the smallest weight where such a requirement is needed, and in higher weights one would need to understand simultaneous functional equations for different Li_a. An approach for finding equations of that type, at least with algebraic arguments, is outlined in Section 8.2. Finally, in Section 8.3, we corroborate our expectations on linear combinations which simultaneously represent an element of both (higher) Bloch groups $\mathcal{B}_2(\mathbb{Q})$ and $\mathcal{B}_4(\mathbb{Q})$.

Preconsiderations. The 2-part of the motivic cobracket of $S^{\mathfrak{L}}_{5,2}(z)$ and $S^{\mathfrak{L}}_{4,3}(z)$ are computed to be

$$\begin{aligned}\delta^{\geq 2} S^{\mathfrak{L}}_{5,2}(z) &= -\{z\}_2 \wedge \{1\}_5 - \{z\}_4 \wedge \{1\}_3\,,\\ \delta^{\geq 2} S^{\mathfrak{L}}_{4,3}(z) &= -2\{z\}_2 \wedge \{1\}_5 - \big(\{z\}_4 + S^{\mathfrak{L}}_{2,2}(z)\big) \wedge \{1\}_3\\ &= -2\{z\}_2 \wedge \{1\}_5 - \Big(2\{z\}_4 - \{1-z\}_4 + \Big\{\frac{z}{z-1}\Big\}_4\Big) \wedge \{1\}_3\,.\end{aligned}$$

(The latter requires the reduction of $S_{2,2}(z)$ to Li_4's from Proposition 5, and the evaluation $S_{3,2}(1) = 2\zeta(5)$ (mod products).)

Hence we expect a reduction of $S_{5,2}$ to Li_7 only when $\sum \alpha_i[x_i]$ simultaneously satisfies a Li_2 and a Li_4 identity. On the other hand, we showed in Theorem 7 that $S_{4,3}(z)$ can be reduced to lower depth. Since its cobracket is matched by

$$-S_{5,2}(1-z) + 2S_{5,2}(z) + S_{5,2}\Big(\frac{z}{z-1}\Big)\,,$$

the difference should be expressible in terms of Li_7's.

Proposition 34. *The following identity follows from inversion and reflection, and it reduces $S_{4,3}(z)$ to lower depth*

$$\begin{aligned}S_{4,3}(z) &= -S_{5,2}(1-z) + 2S_{5,2}(z) + S_{5,2}\Big(\frac{z}{z-1}\Big)\\ &\quad + 2\,\mathrm{Li}_7(1-z) - 3\,\mathrm{Li}_7(z) - 3\,\mathrm{Li}_7\Big(\frac{z}{z-1}\Big) + \zeta(7) \text{ (mod products)}\,.\end{aligned}$$

The full version of this reduction, including product terms, can be found in Appendix D.

Proof. The polynomial invariant of the difference of the left hand side and the right hand side is

$$10X^3Y^2 - \Big(5XY^4 + 10X^4Y + 5(X-Y)^4Y - 2Y^5 - 3X^5 + 3(X-Y)^5\Big) = 0\,.$$

The irreducible part of the constant is fixed by sending $z \to 0$. □

At the value $z = -1$, it follows from the inversion identity of $S_{n,2}$ in Proposition 4 that $S_{5,2}(-1)$ is reducible, and

$$S_{5,2}(-1) = -\frac{251}{128}\zeta(7) + \frac{1}{2}\zeta(2)\zeta(5) + \frac{7}{8}\zeta(3)\zeta(4)\,.$$

However, since $\delta^{\geq 2} S^{\mathfrak{L}}_{5,2}(\frac{1}{2}) = -\{\frac{1}{2}\}_4 \wedge \{1\}_3 \neq 0$, we do not expect a reduction of this to lower depth. Similarly $S_{4,3}(-1)$ and $S_{4,3}(\frac{1}{2})$ both have non-vanishing cobracket involving $\{\frac{1}{2}\}_4 \wedge \{1\}_3$. But there *are* the following reductions of each to $S_{5,2}(\frac{1}{2})$ and simpler objects

$$\begin{aligned}S_{4,3}\Big(\frac{1}{2}\Big) &= S_{5,2}\Big(\frac{1}{2}\Big) - \mathrm{Li}_7\Big(\frac{1}{2}\Big) + \Big(S_{4,2}\Big(\frac{1}{2}\Big) - \mathrm{Li}_6\Big(\frac{1}{2}\Big)\Big)\log(2) - \frac{1}{2}\mathrm{Li}_5\Big(\frac{1}{2}\Big)\log^2(2) + \frac{255}{128}\zeta(7)\\ &\quad - \frac{1}{8}\zeta(3)\zeta(4) - \frac{1}{2}\zeta(2)\zeta(5) - \Big(\frac{23}{32}\zeta(6) - \frac{1}{2}\zeta(3)^2\Big)\log(2) + \Big(\zeta(5) - \frac{1}{2}\zeta(2)\zeta(3)\Big)\log^2(2)\\ &\quad - \frac{5}{4!}\zeta(4)\log^3(2) + \frac{3}{4!}\zeta(3)\log^4(2) - \frac{3}{5!}\zeta(2)\log^5(2) + \frac{10}{7!}\log^7(2)\,,\end{aligned}$$

$$\begin{aligned}S_{4,3}(-1) &= 2S_{5,2}\Big(\frac{1}{2}\Big) - 6\,\mathrm{Li}_7\Big(\frac{1}{2}\Big) - \Big(S_{4,2}(-1) + 2\,\mathrm{Li}_6\Big(\frac{1}{2}\Big)\Big)\log(2) - \frac{31}{32}\zeta(7) + \frac{11}{4}\zeta(4)\zeta(3)\\ &\quad + 2\zeta(2)\zeta(5) - \Big(\frac{51}{16}\zeta(6) + \frac{1}{2}\zeta(3)^2\Big)\log(2) + \Big(\frac{1}{2}\zeta(2)\zeta(3) - \frac{1}{32}\zeta(5)\Big)\log^2(2)\\ &\quad + \frac{1}{4!}\zeta(4)\log^3(2) - \frac{2}{4!}\zeta(3)\log^4(2) + \frac{2}{5!}\zeta(2)\log^5(2) - \frac{6}{7!}\log^7(2)\,.\end{aligned}$$

Both evaluations are obtained by specialisation of the full version of Proposition 34 from Appendix D (with analytic continuation where necessary), and are confirmed by the MZV Data Mine via Remark 13. Moreover $S_{5,2}(\frac{1}{2})$ should be the only new irreducible object needed, by combining with the earlier reductions of $S_{4,2}(-1)$ and $S_{4,2}(\frac{1}{2})$ in (33) and Remark 33.

8.1. $\mathrm{Li}_2 + \mathrm{Li}_4$ **functional equations.** Recall from Section 5 that we have the following Li_4 functional equation

$$\sum_{i=1}^{r} \frac{1}{a}\Big\{\frac{1}{1-p_i(t)}\Big\}_4 + \frac{1}{b}\{p_i(t)\}_4 + \frac{1}{c}\{1-p_i(t)^{-1}\}_4 \;=\; 0\,,$$

where $\{p_i(t)\}_{i=1}^r$ are the roots of $x^a(1-x)^b = t$, for fixed $a, b, c \in \mathbb{Z} \smallsetminus \{0\}$ with $a+b+c = 0$.

One can notice that the individual orbits are already Li_2 functional equations, since under the six-fold symmetry each reduces to a multiple of

$$\sum_{i=1}^{r} \{p_i(t)\}_2 \;=\; 0\,.$$

Hence $S_{5,2}$ of the same combination should be expressible in terms of Li_7. As was noted in Section 5, for the case $(a,b,c) = (1,2,-3)$, the roots of the equation can be rationally parametrised over $\mathbb{Q}$, giving a functional equation even with *rational* arguments.

Proposition 35. *Let $a,b,c \in \mathbb{Z}\smallsetminus\{0\}$, with $a+b+c=0$, and let $\{p_i(t)\}_{i=1}^r$ be the roots of $x^a(1-x)^b = t$. Then the following reduction holds on the mod-products symbol*

$$\sum_{i=1}^{r} \frac{1}{a} S^{\text{ш}}_{5,2}\Big(\frac{1}{1-p_i(t)}\Big) - \frac{1}{b} S^{\text{ш}}_{5,2}(p_i(t)^{-1}) + \frac{1}{c} S^{\text{ш}}_{5,2}(1-p_i(t)^{-1}) \overset{\text{ш}}{=}$$
$$\sum_{i=1}^{r} \frac{3a+b}{a^2}\,\mathrm{Li}^{\text{ш}}_7\Big(\frac{1}{1-p_i(t)}\Big) - \frac{3b+a}{b^2}\,\mathrm{Li}^{\text{ш}}_7(p_i(t)^{-1}) + \frac{3c+a}{c^2}\,\mathrm{Li}^{\text{ш}}_7(1-p_i(t)^{-1})\,.$$

Corollary 36. *Assuming $a > 0$, we have the following identity between clean single-valued functions*

$$\begin{aligned}
&\sum_{i=1}^{r} \frac{1}{a}\mathcal{S}^{\sqcup\!\sqcup}_{5,2}\Big(\frac{1}{1-p_i(t)}\Big) - \frac{1}{b}\mathcal{S}^{\sqcup\!\sqcup}_{5,2}(p_i(t)^{-1}) + \frac{1}{c}\mathcal{S}^{\sqcup\!\sqcup}_{5,2}(1-p_i(t)^{-1}) \\
&- \Big\{\sum_{i=1}^{r} \frac{3a+b}{a^2}\mathcal{L}^{\sqcup\!\sqcup}_7\Big(\frac{1}{1-p_i(t)}\Big) - \frac{3b+a}{b^2}\mathcal{L}^{\sqcup\!\sqcup}_7(p_i(t)^{-1}) + \frac{3c+a}{c^2}\mathcal{L}^{\sqcup\!\sqcup}_7(1-p_i(t)^{-1})\Big\} \\
&= \begin{cases} \dfrac{2a}{c}\zeta(7) & \text{if } b>0\,, \\ \dfrac{2(a^2b-a^2c-b^2c)}{abc}\zeta(7) & \text{if } -a<b<0\,, \\ -\dfrac{2(a^2+b^2)}{ab}\zeta(7) & \text{if } b<-a\,. \end{cases}
\end{aligned}$$

Proof. Consider the limit $t \to 0$, and use $\mathcal{L}^{\sqcup\!\sqcup}_7(0) = 0$, $\mathcal{L}^{\sqcup\!\sqcup}_7(1) = 2\zeta(7)$, $\mathcal{L}^{\sqcup\!\sqcup}_7(\infty) = 0$ and $\mathcal{S}^{\sqcup\!\sqcup}_{5,2}(0) = 0$, $\mathcal{S}^{\sqcup\!\sqcup}_{5,2}(1) = 6\zeta(7)$, $\mathcal{S}^{\sqcup\!\sqcup}_{5,2}(\infty) = 2\zeta(7)$.

If $b > 0$, we obtain roots $p_i = 0$ with multiplicity a and $p_i = 1$ with multiplicity b, giving constant $\frac{2a}{c}\zeta(7)$. If $-a < b < 0$, we obtain roots $p_i = 0$ with multiplicity a, and the constant is $\frac{2(a^2b-a^2c-b^2c)}{abc}\zeta(7)$. Otherwise $b < -a$ and we obtain roots $p_i = 0$ with multiplicity a and $p_i = \infty$ with multiplicity $-b-a$, giving the constant $-\frac{2(a^2+b^2)}{ab}\zeta(7)$. ☐

Proof of Proposition 35. The strategy is exactly the same as in Propositions 14 and 31. Expand out using the recursive definition of the mod-products symbol, and reduce to the algebraic functional equations in lower weight, plus the three-term relation for $S_{4,2}$. ☐

8.2. $\mathrm{Li}_{a_1} + \cdots + \mathrm{Li}_{a_n}$ **functional equations.** It is possible to construct simultaneous functional equations for Li_2 and Li_4, and more generally simultaneous functional equations for $\mathrm{Li}_{a_1}, \ldots, \mathrm{Li}_{a_n}$ in the following manner.

Consider the function $f(z) = \mu_1 \mathrm{Li}_{a_1}(z) + \cdots + \mu_n \mathrm{Li}_{a_n}(z)$, where $a_1 < \cdots < a_n$ are positive integers and $\mu_1, \ldots, \mu_n$ are arbitrary non-zero numbers. Then by the distribution relations of order N we have

$$f^{(N)}(z) \;\coloneqq\; \sum_{y^N = z} f(y) \;=\; \frac{\mu_1}{N^{a_1-1}}\,\mathrm{Li}_{a_1}(z) + \cdots + \frac{\mu_n}{N^{a_n-1}}\,\mathrm{Li}_{a_n}(z)\,.$$

Let $0 < N_1 < \cdots < N_n$ be positive integers, and denote $\underline{f}(z) = (f^{(N_1)}(z), \ldots, f^{(N_n)}(z))^T$, with T the transpose. Then collecting the various distribution relations we get the equation

$$\underline{f}(z) \;=\; V_{a,N}\, \underline{\mathrm{Li}_a}(z)\,,$$

where $\underline{\mathrm{Li}_a}(z) \;=\; (\mu_1 \mathrm{Li}_{a_1}(z), \ldots, \mu_n \mathrm{Li}_{a_n}(z))^T$, and $V_{a,N} \;=\; \big(N_j^{1-a_i}\big)_{i,j=1}^{n}$ is a generalised Vandermonde matrix. Since a_i and N_j are distinct, $\det(V_{a,N})$ is a non-zero multiple of an appropriate Schur polynomial $s_\lambda(N_1^{-1}, \ldots, N_n^{-1})$ (see [25, Equation 7, p. 75]), which is positive since $N_j > 0$ and s_λ is a sum of monomials with positive coefficients (see [25, p. 3]). Therefore, $V_{a,N}$ is an invertible matrix and we have

$$\underline{\mathrm{Li}_a}(z) \;=\; V_{a,N}^{-1}\underline{f}(z)\,.$$

The resulting combination $\mu_j \mathrm{Li}_{a_j}(z) = \sum_k \alpha_k \sum_{y^{N_k}=z} f(y)$, where $\alpha_k \in \mathbb{C}$ only depend on $N_1, \ldots, N_n$ and $a_1, \ldots, a_n$, then vanishes identically under any Li_{a_j} functional equation $\Lambda = \sum_\ell \gamma_\ell [x_\ell]$, so that $\sum_{k,\ell} \alpha_k \gamma_\ell \sum_{y^{N_k}=x_\ell} [y]$ is a functional equation for $\mathrm{Li}_{a_1}, \ldots, \mathrm{Li}_{a_n}$ simultaneously. For the special case when Λ is the distribution relation

$$\Lambda \;=\; \sum_{\ell=1}^{k} [z^M \zeta_k^\ell] - k^{1-a_j}[z^{Mk}]\,,$$

where $M = \mathrm{lcm}(N_1, \ldots, N_n)$, one obtains a rational $\mathrm{Li}_{a_1} + \cdots + \mathrm{Li}_{a_n}$ functional equation, but in general the functional equations constructed in this way will involve algebraic arguments.

8.3. **Bloch group identities.** Despite the scarcity of (rational) functional equations for $S_{5,2}$, we can still investigate experimentally, along the same lines as was done for the classical polylogarithms by Zagier in [53], whether combinations $S_{5,2}(\sum_j \nu_j [x_j])$ reduce to Li_7 whenever $\sum_j \nu_j \{x_j\}_k = 0$ for both $k = 2$ and $k = 4$.

Taking the algebraic identity $x^a(1-x)^b = t$ for $a = 1, b = 2$, and $t = \frac{4}{27}$ leads to three roots $p_i = \frac{1}{3}, \frac{1}{3}, \frac{4}{3}$. Proposition 35 then gives the following identity, after we apply the inversion formula from (14) to put the arguments of $\mathcal{S}_{5,2}^{ш}$ and $\mathcal{L}_7^{ш}$ into the interval $(-1, 1)$:

$$\begin{aligned} &\mathcal{S}_{5,2}^{ш}\bigg(\frac{2}{3}\Big[-\frac{1}{2}\Big] - \Big[-\frac{1}{3}\Big] - \frac{1}{3}\Big[\frac{1}{4}\Big] + \Big[\frac{1}{3}\Big] - 2\Big[\frac{2}{3}\Big] - \frac{1}{2}\Big[\frac{3}{4}\Big]\bigg) \;= \\ &\mathcal{L}_7^{ш}\bigg(\frac{14}{9}\Big[-\frac{1}{2}\Big] - \frac{8}{9}\Big[\frac{1}{4}\Big] + \frac{3}{2}\Big[\frac{1}{3}\Big] - \frac{7}{4}\Big[\frac{3}{4}\Big]\bigg) - \frac{10}{3}\zeta(7)\,. \end{aligned} \tag{39}$$

Note that the arguments of $\mathcal{S}_{5,2}^{ш}$ are exceptional $\{2,3\}$-units, i.e. they are numbers z such that both z and $1-z$ are of the form $\pm 2^k 3^\ell$, $k, \ell \in \mathbb{Z}$.

Remark 37. We can also give the underlying evaluations for Li_2 and Li_4 of this combination. The Li_2 reduction follows from application of the two-term relations (inversion, and $[x] + [1-x]$) and the order 2 distribution relation.

$$\begin{aligned} &\mathrm{Li}_2\bigg(\frac{2}{3}\Big[-\frac{1}{2}\Big] - \Big[-\frac{1}{3}\Big] - \frac{1}{3}\Big[\frac{1}{4}\Big] + \Big[\frac{1}{3}\Big] - 2\Big[\frac{2}{3}\Big] - \frac{1}{2}\Big[\frac{3}{4}\Big]\bigg) \\ &= \; -\frac{4}{3}\zeta(2) + \frac{4}{3}\log^2(2) - 2\log(2)\log(3) + \log^2(3)\,. \end{aligned}$$

The Li_4 evaluation can be obtained from the function L_4 in [53, Equation 31], which is a real-valued and non-zero function on $\mathbb{R}$. The function L_4 satisfies the algebraic Li_4 functional equation (from which the desired combination arises), as Zagier shows it satisfies the usual polylogarithm functional equation criterion, via $L_4(x_i(t)) \mapsto (1 - x_i(t)) \wedge x_i(t) \otimes x_i(t) \otimes x_i(t)$, although it is only *piecewise* constant on intervals between the roots of $x_i(t) \pm 1 = 0$. The jump in constant arises as L_4 is extended from the interval $[-1, 1]$ to $\mathbb{R}$ via $L_4(x) = -L_4(x^{-1})$. By keeping track of the jump in constant, one obtains the evaluation

$$\begin{aligned} &\mathrm{Li}_4\bigg(\frac{2}{3}\Big[-\frac{1}{2}\Big] - \Big[-\frac{1}{3}\Big] - \frac{1}{3}\Big[\frac{1}{4}\Big] + \Big[\frac{1}{3}\Big] - 2\Big[\frac{2}{3}\Big] - \frac{1}{2}\Big[\frac{3}{4}\Big]\bigg) \\ &= \; -\frac{19}{12}\zeta(4) - \zeta(2)\bigg(\frac{5}{3}\log^2(2) - 2\log(2)\log(3) + \frac{1}{2}\log^2(3)\bigg) \\ &\quad + \frac{19}{36}\log^4(2) - \frac{2}{3}\log^3(2)\log(3) + \frac{1}{2}\log^2(2)\log^2(3) - \frac{1}{3}\log(2)\log^3(3) + \frac{1}{12}\log^4(3)\,. \end{aligned}$$

Looking at all possible combinations of non-trivial exceptional $\{2,3\}$-units in $[-1,1)$ that define elements lying both in $\mathcal{B}_2(\mathbb{Q})$ and $\mathcal{B}_4(\mathbb{Q})$ (in this case it is equivalent to their vanishing in the pre-Bloch groups $B_2(\mathbb{Q})$ and $B_4(\mathbb{Q})$), we find that they form a 5-dimensional space, generated by

$$\begin{aligned}
\alpha_1 &= [-1]\,,\\
\alpha_2 &= 10\Big[-\frac{1}{2}\Big] + \Big[-\frac{1}{8}\Big] - 8\Big[\frac{1}{4}\Big] + 22\Big[\frac{1}{2}\Big]\,,\\
\alpha_3 &= 4\Big[-\frac{1}{2}\Big] - 6\Big[-\frac{1}{3}\Big] - 2\Big[\frac{1}{4}\Big] + 6\Big[\frac{1}{3}\Big] - 12\Big[\frac{2}{3}\Big] - 3\Big[\frac{3}{4}\Big]\,,\\
\alpha_4 &= \Big[-\frac{1}{8}\Big] - 8\Big[-\frac{1}{3}\Big] - 14\Big[-\frac{1}{2}\Big] + \Big[\frac{1}{9}\Big] - 5\Big[\frac{1}{4}\Big] - 8\Big[\frac{1}{3}\Big] - 2\Big[\frac{1}{2}\Big]\,,\\
\alpha_5 &= \Big[-\frac{1}{8}\Big] - 9\Big[-\frac{1}{3}\Big] + 4\Big[-\frac{1}{2}\Big] + \Big[\frac{1}{9}\Big] - 5\Big[\frac{1}{4}\Big] - 4\Big[\frac{1}{2}\Big] + 9\Big[\frac{3}{4}\Big] + \Big[\frac{8}{9}\Big]\,.
\end{aligned}$$

In each of these cases we expect $S_{5,2}(\alpha_j)$ to reduce to Li_7. For $\alpha_1 = [-1]$ we already gave the corresponding reduction for the analytic functions, the single-valued version of which is

$$\mathcal{S}^{ш}_{5,2}(-1) = -\frac{251}{64}\zeta(7)\,,$$

while the combination given as the argument of $S_{5,2}$ in (39) corresponds to a multiple of α_3. The remaining elements α_2, α_4, and α_5 appear to be a lot more difficult to reduce rigorously. However, in each case we can find a *candidate* combination which works numerically to high precision (we have verified them for the single-valued functions to 10,000 decimal places using PARI/GP [36]). For instance, for α_2 we have

$$\begin{aligned}
&\mathcal{S}^{ш}_{5,2}\big(10\big[-\tfrac{1}{2}\big] + \big[-\tfrac{1}{8}\big] - 8\big[\tfrac{1}{4}\big] + 22\big[\tfrac{1}{2}\big]\big) \overset{?}{=}\\
&\mathcal{L}^{ш}_{7}\Big(\tfrac{1}{1105}\big[-\tfrac{2048}{2187}\big] - \tfrac{77443}{195}\big[-\tfrac{3}{4}\big] + \tfrac{23501}{663}\big[-\tfrac{2}{3}\big] - \tfrac{32842}{9945}\big[-\tfrac{9}{16}\big] - \tfrac{1049696}{255}\big[-\tfrac{1}{2}\big]\\
&\quad + \tfrac{217}{34}\big[-\tfrac{4}{9}\big] + \tfrac{217}{765}\big[-\tfrac{27}{64}\big] - \tfrac{26449}{2210}\big[-\tfrac{3}{8}\big] + \tfrac{16321}{9945}\big[-\tfrac{1}{3}\big] - \tfrac{2420}{1989}\big[-\tfrac{8}{27}\big]\\
&\quad - \tfrac{51647}{884}\big[-\tfrac{1}{4}\big] + \tfrac{2648}{221}\big[-\tfrac{2}{9}\big] - \tfrac{3140}{663}\big[-\tfrac{1}{6}\big] - \tfrac{18}{1105}\big[-\tfrac{32}{243}\big] + \tfrac{3932}{1105}\big[-\tfrac{1}{8}\big]\\
&\quad - \tfrac{21139}{9945}\big[-\tfrac{1}{9}\big] - \tfrac{307}{1530}\big[-\tfrac{3}{32}\big] - \tfrac{217}{51}\big[-\tfrac{1}{12}\big] + \tfrac{83}{6630}\big[-\tfrac{27}{512}\big] - \tfrac{3167}{3978}\big[-\tfrac{1}{24}\big]\\
&\quad + \tfrac{9359}{9945}\big[-\tfrac{1}{27}\big] - \tfrac{88}{3315}\big[-\tfrac{1}{32}\big] + \tfrac{77}{3978}\big[-\tfrac{1}{48}\big] + \tfrac{328}{663}\big[-\tfrac{1}{54}\big] + \tfrac{217}{3060}\big[-\tfrac{1}{64}\big]\\
&\quad - \tfrac{61}{6630}\big[-\tfrac{2}{243}\big] + \tfrac{31}{1020}\big[-\tfrac{1}{324}\big] + \tfrac{12}{1105}\big[-\tfrac{1}{384}\big] - \tfrac{7}{2210}\big[-\tfrac{1}{4374}\big] - \tfrac{29}{1105}\big[\tfrac{1}{243}\big]\\
&\quad + \tfrac{23}{2210}\big[\tfrac{3}{128}\big] - \tfrac{217}{612}\big[\tfrac{1}{27}\big] + \tfrac{294}{221}\big[\tfrac{2}{27}\big] - \tfrac{5268}{1105}\big[\tfrac{1}{12}\big] - \tfrac{84341}{19890}\big[\tfrac{1}{8}\big]\\
&\quad + \tfrac{48827}{1989}\big[\tfrac{1}{6}\big] - \tfrac{217}{102}\big[\tfrac{3}{16}\big] + \tfrac{4895}{1989}\big[\tfrac{2}{9}\big] - \tfrac{985027}{39780}\big[\tfrac{1}{3}\big] + \tfrac{109586}{9945}\big[\tfrac{3}{8}\big]\\
&\quad + \tfrac{1253}{13260}\big[\tfrac{32}{81}\big] - \tfrac{1049557}{255}\big[\tfrac{1}{2}\big] - \tfrac{1174}{3315}\big[\tfrac{16}{27}\big] + \tfrac{67273}{663}\big[\tfrac{2}{3}\big] - \tfrac{4447459}{9945}\big[\tfrac{3}{4}\big]\\
&\quad + \tfrac{7859}{6630}\big[\tfrac{27}{32}\big] + \tfrac{643}{306}\big[\tfrac{8}{9}\big] - \tfrac{31}{1020}\big[\tfrac{243}{256}\big]\Big) + \tfrac{4241}{1105}\zeta(7)\,.
\end{aligned}$$

Remark 38. Notice that although each $x_j \in \mathbb{Q}$ that appears in this combination is a $\{2,3\}$-unit, we also have primes 5, 7, 11, and 13 appearing in factorisations of $1 - x_j$.

9. Identities in weight 8

In this section, we depth reduce $S_{4,4}(z)$ (Proposition 39) and we reduce $S_{5,3}$ evaluated on the same family of algebraic Li_2 functional equations (Proposition 43) as for $S_{3,2}$. A special case thereof allows to reduce $S_{5,3}(-1)$ to $S_{6,2}(-1)$ and $S_{6,2}(\frac{1}{2})$, modulo polylogarithms and products, and subsequently to match the coaction for $S_{6,2}(-1)$ and even arrive at a tentative evaluation (Proposition 41 and Appendix E).

Preconsiderations: Since $\lfloor (8+1)/3 \rfloor = 3$, Theorem 7 shows that we can at best reduce to depth 3, meaning $S_{5,3}$ is a new more complicated function in weight 8. On computing the 2-part of the motivic cobrackets, we find

$$\begin{aligned}
\delta^{\geq 2} S^{\mathfrak{L}}_{6,2}(z) &= \{1\}_3 \wedge \{z\}_5 - \{z\}_3 \wedge \{1\}_5\,,\\
\delta^{\geq 2} S^{\mathfrak{L}}_{5,3}(z) &= \{1\}_3 \wedge \big(\{z\}_5 + S^{\mathfrak{L}}_{3,2}(z)\big) + \big(\{1-z\}_3 - 2\{z\}_3\big) \wedge \{1\}_5 - \{1\}_3 \wedge \{1\}_5\,,\\
\delta^{\geq 2} S^{\mathfrak{L}}_{4,4}(z) &= \{1\}_3 \wedge \big(-\{z\}_5 - \{1-z\}_5 - \{1-z^{-1}\}_5 + 2S^{\mathfrak{L}}_{3,2}(z)\big)\\
&\quad + 2\,(\{1-z\}_3 - \{z\}_3) \wedge \{1\}_5 - \{1\}_3 \wedge \{1\}_5\,.
\end{aligned}$$

The second calculation requires $S_{3,2}(1) = 2\zeta(5)$ (mod products), while the third requires Proposition 12 to replace $S_{3,2}(1-x)$ by $-S_{3,2}(x)$.

We observe that $S_{5,3}(z)$ cannot reduce to $S_{6,2}$ motivically, even with more complicated arguments, since it contains a single Nielsen polylogarithm in its cobracket, which can never be matched by $S_{6,2}$ alone. Instead, we expect $S_{5,3}(z)$ to behave like Li_2 modulo $S_{6,2}$ and Li_8, as explained in Remark 8.

9.1. **Depth reduction of $S_{4,4}$.** We know that $S_{4,4}(z)$ reduces to $S_{5,3}$, so we can attempt to do this by explicitly killing the $S_{3,2}$ factor in the motivic cobracket.

Proposition 39. *The following reduction expresses $S_{4,4}$ in terms of lower depth Nielsen polylogarithms*

$$\begin{aligned} S_{4,4}(z) \;=\; & 2S_{5,3}(z) - S_{6,2}(1-z) - 3S_{6,2}(z) - S_{6,2}\Big(\frac{z}{z-1}\Big) \\ & + 2\,\mathrm{Li}_8(1-z) + 4\,\mathrm{Li}_8(z) + 4\,\mathrm{Li}_8\left(\frac{z}{z-1}\right) \text{ (mod products)}\,. \end{aligned}$$

Proof. The polynomial invariant of the difference of the left hand side and the right hand side is

$$\begin{aligned} 20X^3Y^3 - \Big(&30X^4Y^2 + 6XY^5 - 18X^5Y - 6(X-Y)^5Y \\ & - 2Y^6 + 4X^6 - 4(X-Y)^6\Big) \;=\; 0\,. \end{aligned}$$

In principle the (conjecturally) irreducible weight 8 MZV constant $\zeta(3,5)$ could play a role in this reduction. However, by Remark 3, we know that the constant in such a reduction must necessarily be a polynomial in Riemann zeta values, and since $\zeta(8) = \frac{24}{175}\zeta(2)^4$ is itself a product, no constant can appear in the mod-products reduction. □

9.2. **On the special values of $S_{6,2}(z)$ and $S_{5,3}(z)$ at $z=-1$ and $z=\frac{1}{2}$.** At $z=\frac{1}{2}$ or $z=-1$ we compute the cobracket as

$$\begin{aligned} \delta^{\geq 2} S^{\mathfrak{L}}_{6,2}(-1) &= -\frac{3}{16}\{1\}_3 \wedge \{1\}_5\,, \qquad & \delta^{\geq 2} S^{\mathfrak{L}}_{6,2}\Big(\frac{1}{2}\Big) &= -\frac{7}{8}\{1\}_3 \wedge \{1\}_5 + \{1\}_3 \wedge \Big\{\frac{1}{2}\Big\}_5\,, \\ \delta^{\geq 2} S^{\mathfrak{L}}_{5,3}(-1) &= -\frac{15}{32}\{1\}_3 \wedge \{1\}_5\,, \qquad & \delta^{\geq 2} S^{\mathfrak{L}}_{5,3}\Big(\frac{1}{2}\Big) &= -\frac{59}{32}\{1\}_3 \wedge \{1\}_5 + 2\{1\}_3 \wedge \Big\{\frac{1}{2}\Big\}_5\,. \end{aligned}$$

In order to match cobracket terms, we are thus led to investigating the following linear combination (on the left) and we find that it reduces to Riemann zeta values.

Proposition 40. *We have*

$$\frac{5}{2}S_{6,2}(-1) - S_{5,3}(-1) \;=\; -\frac{917}{768}\zeta(8) + \frac{1}{2}\zeta(3)\zeta(5) + \frac{1}{4}\zeta(2)\zeta(3)^2\,.$$

Proof. This follows from the MZV Data Mine [1], since each $S_{n,p}(-1)$ is an alternating MZV. □

A reduction of $S_{5,3}(\frac{1}{2})$ to $S_{6,2}(\frac{1}{2})$, $S_{6,2}(-1)$, polylogarithms and products also exists, and follows from the reduction of the reflection identity $S_{5,3}(z) + S_{5,3}(1-z)$ in Proposition 42 below. However we should not expect a reduction of $S_{6,2}(\frac{1}{2})$ to anything of lower depth, since the cobracket contains the factor $\{\frac{1}{2}\}_5 \neq 0$.

9.3. **Strategy for evaluating $S_{6,2}(-1)$.** Since $\delta^{\geq 2}\zeta^{\mathfrak{L}}(3,5) = -5\zeta^{\mathfrak{L}}(3) \wedge \zeta^{\mathfrak{L}}(5) = -5\{1\}_3 \wedge \{1\}_5$ it should be possible to reduce $S_{6,2}(-1)$ and $S_{5,3}(-1)$ individually to Li_8 and products, if we allow also the more familiar (conjecturally irreducible) constant $\zeta(3,5)$.

More precisely, the following combination, with trivial coboundary, should be expressible in terms of classical polylogarithms and products of lower weight terms

$$S_{6,2}(-1) - \frac{3}{80}\zeta(3,5) \overset{?}{=} 0 \text{ (mod } \mathrm{Li}_8\text{, products)}\,.$$

However, such a reduction is likely to be *much* more complicated than the corresponding reduction for $S_{4,2}(-1)$. The complicated part of the $S_{4,2}(-1)$ reduction stems from requiring terms $\sum_j \alpha_j \mathrm{Li}_6(x_j)$ such that the $(5,1)$-part of their coaction gives

$$\sum_j \alpha_j \, \mathrm{Li}^{\mathfrak{m}}_5(x_j) \otimes \log^{\mathfrak{u}}(x_j) \;=\; \zeta^{\mathfrak{m}}(5) \otimes \log^{\mathfrak{u}}(2)\,.$$

For weight 6, this was already possible using only arguments $\pm 2^j$, since one has the identity [53, p. 419]

$$\mathscr{L}_5\Big(-\frac{1}{8}\Big) - 162\mathscr{L}_5\Big(-\frac{1}{2}\Big) - 126\mathscr{L}_5\Big(\frac{1}{2}\Big) \;=\; \frac{403}{16}\zeta(5)\,.$$

To match the $\log^{\mathfrak{m}}(2) \otimes \zeta^{\mathfrak{u}}(7)$ term in

$$\Delta' S^{\mathfrak{m}}_{6,2}(-1) \;=\; -\frac{15}{16}\zeta^{\mathfrak{m}}(3) \otimes \zeta^{\mathfrak{u}}(5) - \frac{3}{4}\zeta^{\mathfrak{m}}(5) \otimes \zeta^{\mathfrak{u}}(3) + \frac{127}{64}\log^{\mathfrak{m}}(2) \otimes \zeta^{\mathfrak{m}}(7)\,,$$

one should try to find a Li_8 combination $\sum \alpha_j \, \mathrm{Li}_8^{\mathfrak{m}}(x_j)$ such that the $(7,1)$-part of their coaction simplifies to $\zeta^{\mathfrak{m}}(7) \otimes \log^{\mathfrak{u}}(2)$. (Then one can switch the order since $\Delta' \zeta^{\mathfrak{m}}(7)\log^{\mathfrak{m}}(2) = \zeta^{\mathfrak{m}}(7) \otimes \log^{\mathfrak{u}}(2) + \log^{\mathfrak{m}}(2) \otimes \zeta^{\mathfrak{u}}(7)$.) Unfortunately, the simplest such Li_7 combination which gives a non-zero multiple of $\zeta(7)$ already involves all 29 of the $\{2,3\}$-units x with $1-x$ only involving the factors $\{2,3,5,7\}$, [53, p. 420], and is only known numerically. In fact we simultaneously require, for $\nu_p(x_j)$ the p-adic valuation of x_j, that

$$\begin{aligned} \textstyle\sum_j \alpha_j \mathscr{L}_7(x_j)\nu_2(x_j) \;&\in\; \zeta(7)\mathbb{Q}^\times\,, \\ \textstyle\sum_j \alpha_j \mathscr{L}_7(x_j)\nu_p(x_j) \;&=\; 0\,, \quad p > 2\,, \end{aligned} \tag{40}$$

in order to match $\zeta^{\mathfrak{m}}(7) \otimes \log^{\mathfrak{u}}(2)$ in the coaction, and to avoid generating extraneous terms $\zeta^{\mathfrak{m}}(7) \otimes \log^{\mathfrak{u}}(p)$, $p > 2$.

To find such a combination, we can slightly adapt the procedure from [53] for inductively computing elements in the Bloch groups $\mathcal{B}_n(F)$. Take a set of elements $X = \{x_j\}$, each x_j of the form $\pm p_1^{k_1} \cdots p_\ell^{k_\ell}$. Firstly, we find the combinations $\sum_j n_j [x_j]$ in $\ker \beta_8$. Here $\beta_m \colon \mathbb{Z}[F] \to \mathrm{Sym}^{m-2}(F_{\mathbb{Q}}^\times) \otimes \bigwedge^2(F_{\mathbb{Q}}^\times)$ is as in [53] given by $[x] \mapsto [x]^{m-2} \otimes ([x] \wedge [1-x])$. We then can impose the conditions

$$\mathscr{L}_k\Big(\textstyle\sum_j n_j \nu_{p_1}(x_j)^{\mu_1} \cdots \nu_{p_\ell}(x_j)^{\mu_\ell} [x_j]\Big) = 0\,,$$

for $\mu_1 + \cdots + \mu_\ell = 8 - k$, with $k = 3, 5$, to obtain combinations which give $0 \cdot \zeta(3)$ and $0 \cdot \zeta(5)$ under $\mathscr{L}_3$ and $\mathscr{L}_5$ respectively. Assuming that $p_1 = 2$, we only need to impose the conditions

$$\mathscr{L}_7\Big(\textstyle\sum_j n_j \nu_{p_i}(x_j)[x_j]\Big) = 0\,,$$

for $i = 2, \ldots, \ell$, and then the combination $\Lambda = \sum_j n_j [x_j]$ has the property we desire (except that we cannot force a non-zero result in the case $i = 1$, i.e. we only obtain $\in \zeta(7)\mathbb{Q}$ rather than $\in \zeta(7)\mathbb{Q}^\times$ in (40)). The same observation as in [53] shows that it is possible to satisfy these conditions by taking $X = X(S)$ to be some set of S-units, for a sufficiently large set of primes S. Specifically, the number of conditions imposed grows polynomially in the size of S, but the Erdős-Stewart-Tijdeman Theorem (cf. [53], p.425) shows that the size of $X(S)$ grows exponentially in the size of S.

In the case where $x_j = \pm 2^a 3^b$, and $1 - x_j$ contains only factors $2, 3, 5, \ldots, 23$, (the original $p = 2$, plus $q = 7$ new extra factors) we are in fact guaranteed to find such a solution. The set of such x_j in $(-1, 1)$, excluding squares, consists of 75 elements. In weight $w = 8$, to be $\ker \beta_8$ we must impose

$$63 = q\binom{w+p-2}{p-1} + \binom{w+p-1}{p-1} - p$$

conditions. To force $\mathscr{L}_3$ and $\mathscr{L}_5$ images to be 0, we must impose a further

$$10 = 6 + 4 = \sum_{k \in \{3,5\}} \binom{w-k+p-1}{p-1}\,,$$

conditions. Finally, we have only $1 = p - 1$ more condition to force for the desired behaviour for the $\mathscr{L}_7$ image. In total we have 75 elements, and only 74 conditions, so the linear space of such combinations is (at least) 1-dimensional. Caveat: it might happen that the $\mathscr{L}_7 \cdot \nu_2$ image in (40) also 0, since we cannot force a non-zero multiple with linear algebra. Fortunately, this does not happen.

After performing the linear algebra, we find exactly one combination of 60 of these elements (the full expression is given in Appendix E)

$$\Lambda \;\coloneqq\; 50\,508\,755\,462\,288\,597\,796\Big[-\frac{2048}{2187}\Big] + \cdots + 2\,651\,619\,475\,018\,716\,827\,904\Big[\frac{243}{256}\Big]\,,$$

which satisfies

$$\begin{aligned} (\mathscr{L}_7 \cdot \nu_2)(\Lambda) \;&\overset{?}{=}\; -175\,442\,386\,671\,378\,179\,202\,538\,515\zeta(7)\,, \\ (\mathscr{L}_7 \cdot \nu_3)(\Lambda) \;&\overset{?}{=}\; 0\,. \end{aligned}$$

Here we write $(\mathscr{L}_7 \cdot \nu_p)(x) \;\coloneqq\; \mathscr{L}_7(x)\nu_p(x)$, and extend by linearity to formal linear combinations as usual.

Assuming these identities holds motivically, we can match the $\log^{\mathfrak{m}}(2) \otimes \zeta^{\mathfrak{u}}(7)$ term in $\Delta' S^{\mathfrak{m}}_{6,2}(-1)$, and we obtain a candidate reduction of the following form.

Proposition 41. *We have the following candidate evaluation of* $S_{6,2}(-1)$

$$\begin{aligned} S_{6,2}(-1) \overset{?}{=} & -\frac{127}{64} \operatorname{Li}_8\Big((-175\,442\,386\,671\,378\,179\,202\,538\,515)^{-1}\Lambda\Big) \\ & + \frac{3}{80}\zeta(3,5) + \frac{15}{16}\zeta(3)\zeta(5) + \frac{127}{64}\log(2)\zeta(7) \\ & + \sum_{2k+a+b=8} \lambda_{2k,a,b}\zeta(2k)\log^a(2)\log^b(3) \end{aligned}$$

for some $\lambda_{2k,a,b} \in \mathbb{Q}$ *which (for* $k < 4$*) come from the terms in the coaction of* $\operatorname{Li}^{\mathfrak{m}}_8(\lambda)$ *arising from the product terms in the analytic identities for* $(\operatorname{Li}^{\mathfrak{m}}_7 \cdot \nu_2)(\Lambda)$ *and* $(\operatorname{Li}^{\mathfrak{m}}_7 \cdot \nu_3)(\Lambda)$*, via* (37)*.*

The full candidate for this reduction is given in Appendix E, and has been verified to 20,000 decimal places in PARI/GP [36].

9.4. **Functional equations for** $S_{5,3}$**.** We expect that $S_{5,3}$ behaves like Li_2 modulo lower depth Nielsen polylogarithms. From Proposition 4 we have the inversion relation. The reflection relation for $S_{5,3}$ also holds, as the following shows.

Proposition 42. $S_{5,3}$ *satisfies the two-term reflection relation, modulo lower depth Nielsen polylogarithms*

$$\begin{aligned} S_{5,3}(1-z) + S_{5,3}(z) = {} & 2S_{6,2}(1-z) + 2S_{6,2}(z) + S_{6,2}\Big(\frac{z}{z-1}\Big) \\ & - 3\operatorname{Li}_8(1-z) - 3\operatorname{Li}_8(z) - 3\operatorname{Li}_8\Big(\frac{z}{z-1}\Big) \pmod{\text{products}}. \end{aligned}$$

Proof. The polynomial invariant is

$$\begin{aligned} -15X^2Y^4 + 15X^4Y^2 - \big(& -12XY^5 + 12X^5Y + 6(X-Y)^5Y \\ & + 3Y^6 - 3X^6 + 3(X-Y)^6\big) = 0. \end{aligned}$$

□

By working out the product terms, one obtains the following reduction, which is confirmed by the MZV Data Mine [1] via Remark 13.

$$\begin{aligned} S_{5,3}\Big(\frac{1}{2}\Big) = {} & S_{6,2}\Big(2\Big[\frac{1}{2}\Big] + \frac{1}{2}[-1]\Big) - 3\operatorname{Li}_8\Big(\frac{1}{2}\Big) - 2\operatorname{Li}_7\Big(\frac{1}{2}\Big)\log(2) + S_{5,2}\Big(\frac{1}{2}\Big)\log(2) \\ & - \frac{1}{2}\operatorname{Li}_6\Big(\frac{1}{2}\Big)\log^2(2) + \frac{2311}{768}\zeta(8) + \frac{1}{4}\zeta(2)\zeta(3)^2 - \frac{1}{2}\zeta(3)\zeta(5) \\ & - \Big(\frac{255}{128}\zeta(7) - \frac{1}{8}\zeta(4)\zeta(3) - \frac{1}{2}\zeta(2)\zeta(5)\Big)\log(2) + \frac{1}{2\cdot 2!}\Big(\frac{23}{16}\zeta(6) - \zeta(3)^2\Big)\log^2(2) \\ & - \frac{1}{3!}\Big(2\zeta(5) - \zeta(2)\zeta(3)\Big)\log^3(2) + \frac{5}{4\cdot 4!}\zeta(4)\log^4(2) \\ & - \frac{3}{5!}\zeta(3)\log^5(2) + \frac{3}{6!}\zeta(2)\log^6(2) - \frac{10}{8!}\log^8(2). \end{aligned}$$

This confirms the reduction suggested above following Proposition 40.

Naturally, one would hope to find a reduction for $S_{5,3}$ of the five-term relation. Using the result for $S_{3,2}$, we can eliminate the $\{1\}_3 \wedge S^{\mathfrak{L}}_{3,2}(z)$ component of the cobracket, from $S_{5,3}$(five-term). Unfortunately, we are still left with the non-trivial task of matching the remainder with $S_{6,2}$ terms, with rational arguments. The difficult part is to match the $\sum_i \{1\}_3 \wedge \{f_i(z)\}_5$ and $\sum_j \{1\}_5 \wedge \{g_j(z)\}_3$ components simultaneously with a combination of $S_{6,2}$ terms. One could apply the idea of Section 8.2 and use the duplication relation, to obtain

$$\delta^{\geq 2} S^{\mathfrak{L}}_{6,2}\Big(\frac{1}{4}[z^2] - [z] - [-z]\Big) = \frac{3}{16}\{1\}_3 \wedge \{z^2\}_5\,.$$

By substituting $\sqrt{f_i(z)}$ into this, one can match by brute force the full motivic cobracket of $S^{\mathfrak{L}}_{5,3}$(five-term). But then one is left with the more difficult task of matching the mod-products symbol by $\operatorname{Li}^{\mathfrak{u}}_8$ terms of arbitrary *algebraic* arguments.

On the other hand, $S_{3,2}$ of the algebraic Li_2 equation from Section 5 has a relatively simple expression in terms of Li_5. So matching the $S_{5,3}$ combination is more straightforward. We have, noting that Proposition 42 covers the special case $a = b = 1$ in more detail, that

Proposition 43. *Let $a, b, c \in \mathbb{Z} \smallsetminus \{0\}$, with $a+b+c=0$, and let $\{p_i(t)\}_{i=1}^r$ be the roots of $x^a(1-x)^b = t$. Then the following functional equation holds on the level of the mod-products symbol*

$$\sum_{i=1}^r S^{\mathfrak{u}}_{5,3}(p_i(t)) \overset{\sqcup\!\sqcup}{=} \sum_{i=1}^r \bigg\{ \frac{2b-a}{b} S_{6,2}(p_i(t)) + \frac{b}{a} S^{\mathfrak{u}}_{6,2}(1-p_i(t)) + \frac{b}{a+b} S^{\mathfrak{u}}_{6,2}(1-p_i(t)^{-1}) - \frac{a^2-2ab+3b^2}{b^2} \operatorname{Li}^{\mathfrak{u}}_8(p_i(t)) - \frac{2ab-b^2}{a^2} \operatorname{Li}^{\mathfrak{u}}_8(1-p_i(t)) - \frac{2ab+3b^2}{(a+b)^2} \operatorname{Li}^{\mathfrak{u}}_8(1-p_i(t)^{-1}) \bigg\} .$$

Corollary 44. *We have the clean single-valued identity*

$$\sum_{i=1}^r \mathcal{S}^{\sqcup\!\sqcup}_{5,3}(p_i(t)) = \sum_{i=1}^r \bigg\{ \frac{2b-a}{b} \mathcal{S}^{\sqcup\!\sqcup}_{6,2}(p_i(t)) + \frac{b}{a} \mathcal{S}^{\sqcup\!\sqcup}_{6,2}(1-p_i(t)) + \frac{b}{a+b} \mathcal{S}^{\sqcup\!\sqcup}_{6,2}(1-p_i(t)^{-1}) - \frac{a^2-2ab+3b^2}{b^2} \mathcal{L}^{\sqcup\!\sqcup}_8(p_i(t)) - \frac{2ab-b^2}{a^2} \mathcal{L}^{\sqcup\!\sqcup}_8(1-p_i(t)) - \frac{2ab+3b^2}{(a+b)^2} \mathcal{L}^{\sqcup\!\sqcup}_8(1-p_i(t)^{-1}) \bigg\} .$$

Proof. Consider the limit $t \to 0$ and use $\mathcal{L}^{\sqcup\!\sqcup}_8(0) = \mathcal{L}^{\sqcup\!\sqcup}_8(1) = \mathcal{L}^{\sqcup\!\sqcup}_8(\infty) = 0$, $\mathcal{S}^{\sqcup\!\sqcup}_{6,2}(0) = \mathcal{S}^{\sqcup\!\sqcup}_{6,2}(1) = \mathcal{S}^{\sqcup\!\sqcup}_{6,2}(\infty) = 0$, and $\mathcal{S}^{\sqcup\!\sqcup}_{5,3}(0) = \mathcal{S}^{\sqcup\!\sqcup}_{5,3}(1) = \mathcal{S}^{\sqcup\!\sqcup}_{5,3}(\infty) = 0$. Since both functions $\mathcal{S}^{\sqcup\!\sqcup}_{5,3}$ and $\mathcal{L}^{\sqcup\!\sqcup}_8$ vanish at all three points in $\{0, 1, \infty\}$, the constant in the clean single-valued identity is necessarily 0. ☐

Proof of Proposition 43. The proof strategy is the same as in the previous cases. It reduces to weight 7 functional equations, including the one in Proposition 35. ☐

9.5. **Nielsen ladders.** The concept of a 'ladder' in some given weight N (already used above) was introduced by Lewin in order to account for identities of the form $\operatorname{Li}_N\big(\sum_i \sum_{k=0}^1 n_{i,k}[(-1)^k\theta^i]\big) = 0$, modulo products of the same type, with $n_{i,k} \in \mathbb{Z}$, $i \geq 0$, for some algebraic number θ. Due to the duplication relation for Li_N one can actually reduce each such to a linear combination where all signs $(-1)^k$ have been dropped.

Definition 45. We call an identity of the type $\sum_j \sum_{k=0}^1 S_{N-j,j}\big(\sum_i \sum_{k=0}^1 n_{i,j,k}[(-1)^k\theta^i]\big) = 0$, modulo products of lower weight Nielsen polylogarithms (including polylogarithms and logarithms) evaluated at those same arguments, with $n_{i,j,k} \in \mathbb{Z}$, a *Nielsen ladder* of weight N.

Remark 46. We have a non-trivial example of a Nielsen ladder in weight 8 for the algebraic number $-\omega$ from Section 6.5. The exact same procedure as used there applies, except for the final use of the duplication relation which is not known for $S_{5,3}$, and we can depth reduce $S_{5,3}(2[\omega] + [-\omega])$ on the level of clean single-valued functions.

10. A family of depth reductions in general weight with arguments z, $1-z$ and $1-z^{-1}$

10.1. **Depth reduction in general weight.** We end with the following result generalising the Li_2-behaviour of $S_{3,2}$ and $S_{5,3}$ modulo lower depth from Propositions 12 and 42, and the Li_3-behaviour of $S_{4,2}$ modulo lower depth from Proposition 29. Moreover, it supports the claim about the behaviour of $S_{2m-\varepsilon,m}$ for $\varepsilon \in \{0, 1, 2\}$ alluded to in Remark 8. More precisely, we prove that $S_{2m-2,m}$ reduces to lower depth, and we expect that $S_{2m-1,m}$ behaves like $S_{1,1} = \operatorname{Li}_2$, and $S_{2m,m}$ behaves like $S_{2,1} = \operatorname{Li}_3$. For other cases, the cobracket potentially involves several terms of maximal depth.

Theorem 47. *For all $m \geq 1$ the following depth reductions, and two-term and three-term identities hold.*

(i) $$S^{\mathfrak{u}}_{2m,m}([z] + [1-z] + [1-z^{-1}]) \overset{\sqcup\!\sqcup}{=} \sum_{j=1}^{m-1} (-1)^{j+1} \binom{m-1+j}{j} S^{\mathfrak{u}}_{2m+j,m-j}\Big([z] + [1-z] + [1-z^{-1}]\Big) .$$

(ii) $$S^{\mathfrak{u}}_{2m-1,m}([z] + [1-z]) \overset{\sqcup\!\sqcup}{=} \sum_{j=1}^{m-1} (-1)^{j+1} \binom{m-2+j}{j} S^{\mathfrak{u}}_{2m-1+j,m-j}\Big([z] + [1-z] + \frac{j}{m-1}[1-z^{-1}]\Big) .$$

(iii) $S^{\mathfrak{u}}_{2m,m+1}(z) \overset{ш}{=}$
$$\sum_{j=1}^{m}(-1)^{j+1}\binom{m-1+j}{j}S^{\mathfrak{u}}_{2m+j,m+1-j}\Big([z]-\frac{j}{m+j-1}[1-z]-\frac{j}{m+j-1}[1-z^{-1}]\Big)\,.$$

Proof. Under the polynomial invariant from Section 3.3, Part (i) is equivalent to the identity
$$P_{m-1}(X,Y)-P_{m-1}(Y,X)+P_{m-1}(Y-X,-X)=0\,, \tag{41}$$
where
$$P_n(X,Y)\;\coloneqq\;\sum_{j=0}^{n}(-1)^j\binom{n+j}{j}\binom{3n+1}{n-j}X^{2n+1+j}Y^{n-j}\,.$$
A routine calculation shows that
$$P_n(X,Y)=X^{n+1}\sum_{a+b=n}\binom{n+a}{a}(XY)^a\binom{n+b}{b}(X(Y-X))^b\,,$$
which implies that
$$P_n(X,Y)=[w^n]\Big(\frac{X}{(1-XYw)(1-X(Y-X)w)}\Big)^{n+1}\,,$$
where we denote by $[z^n]f$ the coefficient of z^n in a power series f. Therefore, using the Lagrange inversion formula, we see that the generating series
$$U=U(X,Y,z)\;\coloneqq\;\sum_{n=0}^{\infty}P_n(X,Y)\frac{z^{n+1}}{n+1}$$
satisfies the cubic equation
$$U(1-XYU)(1-X(Y-X)U)\;=\;Xz\,.$$
This cubic equation in U has three solutions and it is easy to check that they are given by
$$\begin{aligned}U_1(X,Y,z)&=U(X,Y,z)\,,\\ U_2(X,Y,z)&=\frac{1}{XY}-U(Y,X,z)\,,\\ U_3(X,Y,z)&=\frac{1}{X(Y-X)}+U(Y-X,-X,z)\,.\end{aligned}$$
Since the coefficient of $-U^2$ in the associated monic cubic equation is $\frac{1}{XY}+\frac{1}{X(Y-X)}$ we get that
$$U(X,Y,z)-U(Y,X,z)+U(Y-X,-X,z)\;=\;0\,.$$
This proves (41).

The other two parts are proved in a similar way. We only outline the proof of (ii) which is slightly more complicated than (i). In this case we need to prove
$$Q_{m-1}(X,Y)-Q_{m-1}(Y,X)+\widetilde{Q}_{m-1}(Y-X,-X)=0\,, \tag{42}$$
where
$$\begin{aligned}Q_n(X,Y)\;&\coloneqq\;\sum_{j=0}^{n}(-1)^j\binom{n-1+j}{j}\binom{3n}{n-j}X^{2n+j}Y^{n-j}\,,\\ \widetilde{Q}_n(X,Y)\;&\coloneqq\;\sum_{j=0}^{n}(-1)^j\binom{n-1+j}{j}\binom{3n}{n-j}\frac{j}{n}X^{2n+j}Y^{n-j}\,,\quad n>0\,,\end{aligned}$$
and $\widetilde{Q}_0\;\coloneqq\;0$. We claim that the generating series
$$V(X,Y,z)\;\coloneqq\;\sum_{n=0}^{\infty}Q_n(X,Y)\frac{z^{n+1}}{n+1}\,,\qquad \widetilde{V}(X,Y,z)\;\coloneqq\;\sum_{n=0}^{\infty}\widetilde{Q}_n(X,Y)\frac{z^{n+1}}{n+1}\,,$$
can be expressed in terms of $U(X,Y,z)$ as
$$V(X,Y,z)\;=\;X^{-1}U+\frac{1}{2}(X-Y)U^2\,,\qquad \widetilde{V}(X,Y,z)=-\frac{1}{2}XU^2\,, \tag{43}$$
from which after a simple calculation we see that (42) is implied by
$$U_1^k+U_2^k+U_3^k\;=\;\frac{1}{(XY)^k}+\frac{1}{(X(Y-X))^k}\,,\quad k=1,2\,,$$

which again follows from Vieta's formulas for the cubic equation satisfied by U. To prove (43) we use Lagrange inversion in a more general form

$$H(g(z)) \;=\; \sum_{n\geq 0}[w^n](H'(w)\phi^{n+1}(w))\frac{z^{n+1}}{n+1}\,,$$

where $g(z)$ satisfies $g(z) = z\phi(g(z))$ and $H(w)$ is a formal power series without a constant term (here ϕ is a power series with $\phi(0) \neq 0$). To obtain (43) we use the following simple identities

$$Q_n(X,Y) \;=\; [w^n]\frac{X^{2n}(1+Yw)^{3n}}{(1+Xw)^n}\,, \qquad \widetilde{Q}_n(X,Y) \;=\; [w^n]\frac{-wX^{2n+1}(1+Yw)^{3n}}{(1+Xw)^{n+1}}\,,$$

and the analogous identity for P_n

$$P_n(X,Y) \;=\; [w^n]\frac{X^{2n+1}(1+Yw)^{3n+1}}{(1+Xw)^{n+1}}\,,$$

together with the Lagrange inversion formula for

$$\phi(w) \;=\; \frac{X^2(1+Yw)^3}{(1+Xw)}\,,$$

and the following three choices for H:

$$H_1(w) \;=\; \frac{w(2+(X+Y)w)}{2X^2(1+Yw)^2}\,, \quad H_2(w) \;=\; \frac{-w^2}{2X(1+Yw)^2}\,, \quad H_3(w) \;=\; \frac{w}{X(1+Yw)}\,.$$

Finally, (iii) is equivalent to

$$R_m(X,Y) + \widetilde{R}_m(Y,X) - \widetilde{R}_m(Y-X,-X) = 0\,,$$

where

$$R_n(X,Y) \;\coloneqq\; \sum_{j=0}^{n}(-1)^j\binom{n-1+j}{j}\binom{3n-1}{n-j}X^{2n+j-1}Y^{n-j}\,, \quad n\geq 1\,,$$

$$\widetilde{R}_n(X,Y) \;\coloneqq\; \sum_{j=0}^{n}(-1)^j\binom{n-1+j}{j}\binom{3n-1}{n-j}\frac{j}{n+j-1}X^{2n+j-1}Y^{n-j}\,, \quad n>1\,,$$

and we set $\widetilde{R}_1(X,Y) = -X^2$, $\widetilde{R}_0(X,Y) = -\frac{1}{3Y}$, and $R_0(X,Y) = \frac{2}{3X}$. This again follows by considering

$$W(X,Y,z) \;\coloneqq\; \sum_{n=0}^{\infty} R_n(X,Y)\frac{z^{n+1}}{n+1}\,, \qquad \widetilde{W}(X,Y,z) \;\coloneqq\; \sum_{n=0}^{\infty}\widetilde{R}_n(X,Y)\frac{z^{n+1}}{n+1}\,,$$

and showing, using Lagrange inversion, that

$$W(X,Y,z) \;=\; \frac{2}{3X^2}U + \frac{X-2Y}{6X}U^2\,, \qquad \widetilde{W}(X,Y,z) = -\frac{1}{3XY}U + \frac{Y-2X}{6Y}U^2\,. \qquad \square$$

Corollary 48. *There are Nielsen ladders in arbitrary weight.*

Proof. Denote by θ a root of $(1-x)\pm x^r$ for some $r\in\mathbb{Z}_{>0}$, so that $1-\theta = \mp\theta^r$ and $1-\theta^{-1} = \pm\theta^{r-1}$. Specialising to $z=\theta$ in the theorem, all the arguments become, up to sign, powers of θ. Terms with negative powers of θ can be replaced, via inversion (Proposition 4), by positive powers. $\square$

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Appendix A. Evaluation of $S_{3,2}$ at values involving the golden ratio

Recall the following evaluations involving the golden ratio $\phi = \frac{1}{2}(1+\sqrt{5})$ for Li_2 (see [44, Equations 1.20 and 1.21], or [55, Section 1.1]):

$$\mathrm{Li}_2(\phi^{-2}) = \frac{2}{5}\zeta(2) - \log^2(\phi)\,, \qquad \mathrm{Li}_2(\phi^{-1}) = \frac{3}{5}\zeta(2) - \log^2(\phi)\,,$$
$$\mathrm{Li}_2(-\phi^{-1}) = -\frac{2}{5}\zeta(2) + \frac{1}{2}\log^2(\phi)\,, \qquad \mathrm{Li}_2(-\phi) = -\frac{3}{5}\zeta(2) - \log^2(\phi)\,.$$

Corresponding to these Li_2 evaluations, we have the following evaluations for the clean single-valued Nielsen polylogarithm $\mathcal{S}^{ш}_{3,2}$:

$$\begin{aligned}
\mathcal{S}^{ш}_{3,2}(\phi^{-2}) &= \frac{1}{66}\mathcal{L}^{ш}_5\Big([\phi^{-6}] - 32[\phi^{-3}] + \frac{201}{2}[\phi^{-2}] - 48[\phi^{-1}]\Big) + \zeta(5)\,,\\
\mathcal{S}^{ш}_{3,2}(-\phi^{-1}) &= \frac{1}{6}\mathcal{L}^{ш}_5\Big(-[\phi^{-6}] + 32[\phi^{-3}] + \frac{159}{4}[\phi^{-2}] - 150[\phi^{-1}]\Big) + \zeta(5)\,,\\
\mathcal{S}^{ш}_{3,2}(\phi^{-1}) &= \frac{1}{66}\mathcal{L}^{ш}_5\Big(-[\phi^{-6}] + 32[\phi^{-3}] - \frac{243}{8}[\phi^{-2}] + 48[\phi^{-1}]\Big) + \zeta(5)\,,\\
\mathcal{S}^{ш}_{3,2}(-\phi) &= \frac{1}{66}\mathcal{L}^{ш}_5\Big([\phi^{-6}] - 32[\phi^{-3}] - \frac{219}{8}[\phi^{-2}] - 48[\phi^{-1}]\Big) + \zeta(5)\,.
\end{aligned}$$

For the complex analytic Nielsen polylogarithm $S_{3,2}$ we have:

$$\begin{aligned}
S_{3,2}(\phi^{-2}) = {} & \frac{1}{66}\mathrm{Li}_5\Big([\phi^{-6}] - 32[\phi^{-3}] + \frac{201}{2}[\phi^{-2}] - 48[\phi^{-1}]\Big) + \mathrm{Li}_4\left(\phi^{-2}\right)\log(\phi)\\
& + \frac{1}{2}\zeta(5) - \frac{2}{11}\zeta(4)\log(\phi) - \zeta(3)\,\mathrm{Li}_2\left(\phi^{-2}\right) - \frac{20}{33}\zeta(2)\log(\phi)^3 + \frac{79}{330}\log(\phi)^5
\end{aligned}$$

$$\begin{aligned}
S_{3,2}(-\phi^{-1}) = {} & \frac{1}{66}\mathrm{Li}_5\Big(-[\phi^{-6}] + 32[\phi^{-3}] + \frac{159}{4}[\phi^{-2}] - 150[\phi^{-1}]\Big) - \mathrm{Li}_4\left(-\phi^{-1}\right)\log(\phi)\\
& + \frac{1}{2}\zeta(5) - \frac{9}{11}\zeta(4)\log(\phi) - \zeta(3)\,\mathrm{Li}_2\left(-\phi^{-1}\right) + \frac{29}{66}\zeta(2)\log(\phi)^3 - \frac{19}{110}\log(\phi)^5
\end{aligned}$$

$$\begin{aligned}
S_{3,2}(\phi^{-1}) = {} & \frac{1}{66}\mathrm{Li}_5\Big(-[\phi^{-6}] + 32[\phi^{-3}] - \frac{243}{8}[\phi^{-2}] + 48[\phi^{-1}]\Big) + 2\,\mathrm{Li}_4\left(\phi^{-1}\right)\log(\phi)\\
& + \frac{1}{2}\zeta(5) - \frac{47}{44}\zeta(4)\log(\phi) - \zeta(3)\,\mathrm{Li}_2\left(\phi^{-1}\right) - \frac{5}{22}\zeta(2)\log(\phi)^3 + \frac{5}{264}\log(\phi)^5
\end{aligned}$$

$$\begin{aligned}
S_{3,2}(-\phi) = {} & \frac{1}{66}\mathrm{Li}_5\Big([\phi^{-6}] - 32[\phi^{-3}] - \frac{219}{8}[\phi^{-2}] - 48[\phi^{-1}]\Big) - 2\,\mathrm{Li}_4\left(-\phi\right)\log(\phi)\\
& + \frac{1}{2}\zeta(5) - \frac{195}{44}\zeta(4)\log(\phi) - \zeta(3)\,\mathrm{Li}_2\left(-\phi\right) - \frac{95}{66}\zeta(2)\log(\phi)^3 + \frac{43}{440}\log(\phi)^5
\end{aligned}$$

Appendix B. Reduction of $S_{3,2}([\frac{1}{9}] - 6[\frac{1}{3}])$ to Li_5 and products

By direct application of Corollary 20 to the five-term combination in (28), and simplifying the result via the two-term relations of $\mathcal{S}^{ш}_{3,2}$ and the inversion-relation of $\mathcal{L}^{ш}_5$, we obtain the following reduction

$$\begin{aligned}
\mathcal{S}^{ш}_{3,2}\Big(\big[\tfrac{1}{9}\big] - 6\big[\tfrac{1}{3}\big]\Big) = \tfrac{1}{66}\mathcal{L}^{ш}_5\Big(& 33\big[-\tfrac{3}{4}\big] + 27\big[-\tfrac{2}{3}\big] + \big[-\tfrac{81}{128}\big] + 7\big[-\tfrac{9}{16}\big] - 233\big[-\tfrac{1}{2}\big] - 30\big[-\tfrac{4}{9}\big]\\
& + \big[-\tfrac{27}{64}\big] + 18\big[-\tfrac{3}{8}\big] - 32\big[-\tfrac{1}{3}\big] - 10\big[-\tfrac{8}{27}\big] - 33\big[-\tfrac{1}{4}\big] + 48\big[-\tfrac{2}{9}\big]\\
& - 9\big[-\tfrac{3}{16}\big] - 48\big[-\tfrac{1}{6}\big] - \big[-\tfrac{32}{243}\big] + 22\big[-\tfrac{1}{8}\big] - 2\big[-\tfrac{1}{9}\big] - 9\big[-\tfrac{3}{32}\big]
\end{aligned}$$

$$
\begin{aligned}
&+30\big[-\tfrac{1}{12}\big] - 9\big[-\tfrac{1}{18}\big] + 9\big[-\tfrac{1}{24}\big] + 2\big[-\tfrac{1}{27}\big] - \big[-\tfrac{1}{32}\big] + 2\big[-\tfrac{1}{48}\big] \\
&-2\big[-\tfrac{1}{54}\big] + \big[-\tfrac{4}{243}\big] - \big[-\tfrac{1}{64}\big] + \big[-\tfrac{2}{243}\big] - \big[-\tfrac{1}{324}\big] + \big[-\tfrac{1}{384}\big] \\
&-\big[\tfrac{1}{162}\big] + \big[\tfrac{1}{96}\big] - \big[\tfrac{3}{256}\big] - \big[\tfrac{3}{128}\big] - 9\big[\tfrac{1}{36}\big] + 9\big[\tfrac{1}{24}\big] \\
&+\tfrac{15}{2}\big[\tfrac{1}{16}\big] + 11\big[\tfrac{2}{27}\big] - 12\big[\tfrac{1}{12}\big] + 90\big[\tfrac{1}{9}\big] - 19\big[\tfrac{1}{8}\big] + 9\big[\tfrac{4}{27}\big] \\
&+10\big[\tfrac{1}{6}\big] - 28\big[\tfrac{2}{9}\big] + \tfrac{129}{2}\big[\tfrac{1}{4}\big] + 9\big[\tfrac{9}{32}\big] - 9\big[\tfrac{8}{27}\big] + \big[\tfrac{81}{256}\big] \\
&-546\big[\tfrac{1}{3}\big] + 2\big[\tfrac{3}{8}\big] + \tfrac{33}{2}\big[\tfrac{4}{9}\big] + 35\big[\tfrac{1}{2}\big] - \big[\tfrac{128}{243}\big] - \tfrac{9}{2}\big[\tfrac{9}{16}\big] \\
&-3\big[\tfrac{16}{27}\big] + 189\big[\tfrac{2}{3}\big] - 69\big[\tfrac{3}{4}\big] + \big[\tfrac{27}{32}\big] - 22\big[\tfrac{8}{9}\big]\Big) - \tfrac{1655}{264}\zeta(5)\,.
\end{aligned}
$$

By applying the LLL lattice reduction algorithm to the above combination, we find the following shorter, but only numerically checked identity

$$
\begin{aligned}
\mathcal{S}^{\shuffle}_{3,2}\bigg(\bigg[\frac{1}{9}\bigg] - 6\bigg[\frac{1}{3}\bigg]\bigg) \overset{?}{=} \mathcal{L}^{\shuffle}_{5}\bigg(&\frac{1}{16}\bigg[\frac{1}{9}\bigg] + \frac{21}{2}\bigg[\frac{1}{4}\bigg] + 36\bigg[\frac{1}{3}\bigg] - 100\bigg[\frac{1}{2}\bigg] \\
&- 60\bigg[\frac{2}{3}\bigg] + \frac{69}{2}\bigg[\frac{3}{4}\bigg] - 2\bigg[\frac{8}{9}\bigg]\bigg) + \frac{1855}{12}\zeta(5)\,.
\end{aligned}
$$

Since $S_{3,2}([\frac{1}{9}] - 6[\frac{1}{3}])$ is already real, expanding out the definitions of $\mathcal{S}^{\shuffle}_{3,2}$ and $\mathcal{L}^{\shuffle}_{5}$ in the shorter identity leads to the following analytic identity

$$
\begin{aligned}
S_{3,2}\bigg(\bigg[\frac{1}{9}\bigg] - 6\bigg[\frac{1}{3}\bigg]\bigg) \overset{?}{=} {}& \mathrm{Li}_5\bigg(\frac{1}{16}\bigg[\frac{1}{9}\bigg] + \frac{21}{2}\bigg[\frac{1}{4}\bigg] + 36\bigg[\frac{1}{3}\bigg] - 100\bigg[\frac{1}{2}\bigg] - 60\bigg[\frac{2}{3}\bigg] + \frac{69}{2}\bigg[\frac{3}{4}\bigg] - 2\bigg[\frac{8}{9}\bigg]\bigg) \\
&+ \frac{1855}{24}\zeta(5) + 6\,\mathrm{Li}_4\Big(\frac{1}{3}\Big)\log\Big(\frac{2}{3}\Big) - \mathrm{Li}_4\Big(\frac{1}{9}\Big)\log\Big(\frac{8}{9}\Big) \\
&+ \zeta(2)\Big(\frac{128}{3}\log^3(2) - 84\log^2(2)\log(3) + 54\log(2)\log^2(3) - \frac{61}{6}\log^3(3)\Big) \\
&- \zeta(3)\,\mathrm{Li}_2\bigg(\bigg[\frac{1}{9}\bigg] - 6\bigg[\frac{1}{3}\bigg]\bigg) + \zeta(4)\Big(52\log(2) - \frac{239}{4}\log(3)\Big) \\
&- \frac{67}{6}\log^5(2) + 23\log^4(2)\log(3) - 23\log^3(2)\log^2(3) \\
&+ 17\log^2(2)\log^3(3) - \frac{33}{4}\log(2)\log^4(3) + \frac{19}{12}\log^5(3)\,.
\end{aligned}
$$

Appendix C. Reduction of $\mathrm{Li}_{5,1}(-x,-1)$ to $S_{4,2}$ and Li_6 modulo products

Introduce the notation $\{\pm, a; b, c, d\} := [\pm 2^a x^b (1-x)^c (1+x)^d]$. Then the following identity holds for the mod-products symbol.

$$
\begin{aligned}
&\mathrm{Li}^{\mathfrak{u}}_{5,1}(-x,-1) \overset{\shuffle}{=} \\
&S^{\mathfrak{u}}_{4,2}\Big(-\tfrac{1}{32}[x^2] + \tfrac{17}{4}[-x] - \tfrac{13}{4}[x] - \tfrac{33}{8}\big[\tfrac{1-x}{2}\big] + \tfrac{33}{8}\big[\tfrac{1-x}{1+x}\big] + \tfrac{33}{4}\big[\tfrac{2x}{1+x}\big] + \tfrac{33}{8}\big[\tfrac{1+x}{2}\big] + \tfrac{33}{16}\big[-\tfrac{4x}{(1-x)^2}\big] + \tfrac{33}{32}\big[\tfrac{(1-x)^2}{(1+x)^2}\big]\Big) \\
&+ \mathrm{Li}^{\mathfrak{u}}_6\Big(-\tfrac{81}{16}\{-,-1;0,0,1\} + \tfrac{81}{8}\{-,0;0,0,-1\} - \tfrac{261}{16}\{+,-1;0,0,1\} + \tfrac{57}{8}\{+,0;0,0,-1\} + \tfrac{81}{16}\{-,-1;0,1,0\} \\
&\quad + \tfrac{81}{8}\{-,0;0,1,0\} + \tfrac{261}{16}\{+,-1;0,1,0\} + \tfrac{57}{8}\{+,0;0,1,0\} - 9\{-,0;1,0,0\} + 8\{+,0;1,0,0\} \\
&\quad - \tfrac{1583}{6}\{-,0;0,1,-1\} - \tfrac{3265}{12}\{+,0;0,1,-1\} + \tfrac{81}{8}\{-,0;1,0,-1\} - \tfrac{5}{4}\{-,0;1,0,1\} + \tfrac{81}{16}\{-,1;1,0,-1\} \\
&\quad + \tfrac{57}{8}\{+,0;1,0,-1\} + \tfrac{5}{36}\{+,0;1,0,1\} - \tfrac{135}{16}\{+,1;1,0,-1\} - \tfrac{57}{8}\{-,0;1,-1,0\} - \tfrac{5}{36}\{-,0;1,1,0\} \\
&\quad - \tfrac{129}{16}\{-,1;1,-1,0\} - \tfrac{81}{8}\{+,0;1,-1,0\} + \tfrac{5}{4}\{+,0;1,1,0\} - \tfrac{81}{16}\{+,1;1,-1,0\} + \tfrac{1}{48}\{-,-3;0,0,3\} \\
&\quad - \tfrac{1}{24}\{-,0;0,0,-3\} - \tfrac{15}{8}\{-,-1;0,-1,2\} + \tfrac{15}{8}\{+,-1;0,-1,2\} - \tfrac{1}{24}\{+,0;0,-1,-2\} + \tfrac{15}{8}\{-,-1;0,2,-1\} \\
&\quad - \tfrac{15}{8}\{+,-1;0,2,-1\} - \tfrac{1}{24}\{+,0;0,2,1\} - \tfrac{1}{48}\{-,-3;0,3,0\} - \tfrac{1}{24}\{-,0;0,3,0\} + \tfrac{5}{9}\{+,0;1,0,-2\} \\
&\quad - \tfrac{21}{16}\{+,2;1,0,-2\} + \tfrac{1087}{192}\{-,0;1,-1,1\} + \tfrac{1}{16}\{-,0;1,1,1\} - \tfrac{1087}{192}\{+,0;1,1,-1\} - \tfrac{1}{16}\{+,0;1,1,1\} \\
&\quad - \tfrac{5}{9}\{-,0;1,-2,0\} - \tfrac{45}{16}\{-,2;1,-2,0\} + \tfrac{5}{4}\{-,0;2,0,-1\} - \tfrac{5}{36}\{+,0;2,0,-1\} + \tfrac{1}{8}\{+,0;2,0,1\} \\
&\quad - \tfrac{5}{4}\{-,0;2,-1,0\} + \tfrac{5}{36}\{+,0;2,-1,0\} - \tfrac{1}{8}\{+,0;2,1,0\} - \tfrac{259}{24}\{-,0;0,2,-2\} + \tfrac{1}{48}\{+,3;1,0,-3\} \\
&\quad - \tfrac{15}{8}\{-,1;1,1,-2\} - \tfrac{1}{24}\{+,0;1,-1,-2\} + \tfrac{15}{8}\{+,1;1,1,-2\} - \tfrac{3}{16}\{+,3;1,-1,-2\} + \tfrac{1}{24}\{-,0;1,-2,-1\} \\
&\quad - \tfrac{15}{8}\{-,1;1,-2,1\} + \tfrac{3}{16}\{-,3;1,-2,-1\} + \tfrac{15}{8}\{+,1;1,-2,1\} - \tfrac{1}{48}\{-,3;1,-3,0\} - \tfrac{1}{8}\{+,0;3,0,-1\} \\
&\quad + \tfrac{1}{8}\{-,0;3,-1,0\} + \tfrac{9}{16}\{+,-1;0,-2,3\} - \tfrac{9}{16}\{+,-1;0,3,-2\} + \tfrac{9}{32}\{-,2;1,1,-3\} - \tfrac{1}{288}\{+,0;1,1,3\} \\
&\quad + \tfrac{9}{32}\{+,2;1,1,-3\} + \tfrac{1}{288}\{-,0;1,3,1\} - \tfrac{9}{32}\{-,2;1,-3,1\} - \tfrac{9}{32}\{+,2;1,-3,1\} + \tfrac{1}{48}\{+,3;2,0,-3\} \\
&\quad - \tfrac{1}{24}\{-,0;2,-1,-2\} - \tfrac{3}{16}\{-,3;2,-1,-2\} + \tfrac{1}{24}\{-,0;2,-2,-1\} + \tfrac{3}{16}\{-,3;2,-2,-1\} - \tfrac{1}{48}\{+,3;2,-3,0\} \\
&\quad + \tfrac{1}{16}\{-,0;3,-1,-1\} + \tfrac{1}{240}\{-,0;3,1,-1\} - \tfrac{1}{16}\{+,0;3,-1,-1\} - \tfrac{1}{240}\{+,0;3,-1,1\} + \tfrac{1}{72}\{-,0;4,0,1\}
\end{aligned}
$$

$$
\begin{array}{lllll}
-\tfrac{1}{72}\{-,0;4,1,0\} & +\tfrac{5}{6}\{-,0;0,3,-3\} & +\tfrac{5}{6}\{+,0;0,3,-3\} & +\tfrac{1}{360}\{+,0;1,-2,-3\} & -\tfrac{9}{16}\{+,1;1,2,-3\} \\
-\tfrac{1}{360}\{-,0;1,-3,-2\} & +\tfrac{9}{16}\{-,1;1,-3,2\} & -\tfrac{1}{24}\{-,0;3,0,-3\} & -\tfrac{1}{48}\{-,3;3,0,-3\} & -\tfrac{1}{24}\{-,0;3,-1,-2\} \\
+\tfrac{1}{24}\{+,0;3,-2,-1\} & +\tfrac{1}{24}\{+,0;3,-3,0\} & +\tfrac{1}{48}\{+,3;3,-3,0\} & -\tfrac{1}{72}\{-,0;5,0,-1\} & +\tfrac{1}{72}\{+,0;5,-1,0\} \\
+\tfrac{1}{64}\{+,4;1,1,-5\} & +\tfrac{41}{480}\{-,0;1,3,-3\} & -\tfrac{41}{480}\{+,0;1,-3,3\} & -\tfrac{1}{64}\{-,4;1,-5,1\} & -\tfrac{1}{80}\{+,5;2,-1,-4\} \\
+\tfrac{1}{80}\{+,5;2,-4,-1\} & -\tfrac{1}{80}\{-,5;3,-1,-4\} & +\tfrac{1}{80}\{+,5;3,-4,-1\} & +\tfrac{1}{16}\{-,-1;0,-4,5\} & -\tfrac{1}{16}\{-,-1;0,5,-4\} \\
+\tfrac{1}{64}\{-,4;3,1,-5\} & -\tfrac{1}{64}\{+,4;3,-5,1\} & +\tfrac{1}{360}\{+,0;4,-2,-3\} & -\tfrac{1}{360}\{+,0;4,-3,-2\} & +\tfrac{1}{288}\{-,0;5,-1,-3\} \\
-\tfrac{1}{288}\{+,0;5,-3,-1\} & +\tfrac{1}{480}\{-,0;7,1,-1\} & -\tfrac{1}{480}\{+,0;7,-1,1\} & -\tfrac{1}{16}\{-,1;1,4,-5\} & +\tfrac{1}{16}\{+,1;1,-5,4\} \\
+\tfrac{3}{320}\{-,0;1,7,-7\} & -\tfrac{3}{320}\{+,0;1,-7,7\}\Big) . & & &
\end{array}
$$

Appendix D. Reduction of $S_{4,3}(z)$ to lower depth and products

For all $z \in \mathbb{C} \smallsetminus \big((-\infty,0] \cup [1,\infty)\big)$, the following reduction of $S_{4,3}$ to lower depth, and products holds

$$
\begin{aligned}
S_{4,3}(z) \; = \;& \\
& - S_{5,2}(1-z) + 2S_{5,2}(z) + S_{5,2}\Big(\frac{z}{z-1}\Big) + 2\operatorname{Li}_7(1-z) - 3\operatorname{Li}_7(z) - 3\operatorname{Li}_7\Big(\frac{z}{z-1}\Big) \\
& - S_{4,2}(z)\log(1-z) - \operatorname{Li}_6\Big(\frac{z}{z-1}\Big)\log(1-z) + 2\operatorname{Li}_6(z)\log(1-z) - \operatorname{Li}_6(1-z)\log(z) \\
& - \frac{1}{2!}\operatorname{Li}_5(z)\log^2(1-z) - \frac{1}{5!\,2!}\log^5(1-z)\log^2(z) + \frac{2}{6!}\log^6(1-z)\log(z) - \frac{3}{7!}\log^7(1-z) \\
& + \frac{1}{4!}\zeta(2)\log^4(1-z)\log(z) - \frac{2}{5!}\zeta(2)\log^5(1-z) + \frac{1}{3!}\zeta(3)\log^3(1-z)\log(z) - \frac{1}{4!}\zeta(3)\log^4(1-z) \\
& + \frac{1}{2!}\zeta(4)\log^2(1-z)\log(z) - \frac{7}{4!}\zeta(4)\log^3(1-z) + \zeta(5)\log(1-z)\log(z) - \frac{1}{2!}\zeta(2)\zeta(3)\log^2(1-z) \\
& + \zeta(6)\log(z) - \Big(\frac{5}{4}\zeta(6) + \frac{1}{2}\zeta(3)^2\Big)\log(1-z) + \Big(\zeta(7) - \zeta(3)\zeta(4) - \zeta(2)\zeta(5)\Big) .
\end{aligned}
$$

Appendix E. Reduction of $S_{6,2}(-1)$ to $\zeta(3,5)$ and lower depth

The following combination which we abbreviate as Λ below

$$
\begin{array}{lll}
50508755462288597796\,[-2^{11}3^{-7}] & +69841566365930200554764814\,[-2^{-2}3^{1}] & -775364232778811798418105642\,[-2^{1}3^{-1}] \\
+9614338651927197388368\,[-2^{-7}3^{4}] & +356655652241330545382160\,[-2^{-4}3^{2}] & +22509382601419271262985124160\,[-2^{-1}] \\
+126912035059272811134\,[-2^{10}3^{-7}] & -94164506374654687219920\,[-2^{2}3^{-2}] & -14944644124416655455996\,[-2^{-6}3^{3}] \\
+578363469392155525327836\,[-2^{-3}3^{1}] & -1389650271294609827123449194\,[-3^{-1}] & +142150983452642605772646\,[-2^{3}3^{-3}] \\
+234866506563215285097901896\,[-2^{-2}] & -2156235930824838852840480\,[-2^{1}3^{-2}] & +545472150324080280895440\,[-2^{-4}3^{1}] \\
+52935763185068637077963640\,[-2^{-1}3^{-1}] & -1637817842535871022208\,[-2^{5}3^{-5}] & -3832788189554116913056832\,[-2^{-3}] \\
-7534735430532974309850624\,[-3^{-2}] & +229760377972981805891088\,[-2^{-5}3^{1}] & +1824486564349437387795018\,[-2^{-2}3^{-1}] \\
+146288680291430373180960\,[-2^{-1}3^{-2}] & -1841986300588118314548\,[-2^{-9}3^{3}] & -449734750753601709254502\,[-2^{-3}3^{-1}] \\
+1026645718908856249515210\,[-3^{-3}] & +104005977148977093591408\,[-2^{-5}] & +98806808342364061998789\,[-2^{-4}3^{-1}] \\
-90845492233250820003624\,[-2^{-1}3^{-3}] & -856806635887547864148\,[-2^{2}3^{-5}] & -1618278846243184730952\,[-2^{-6}] \\
-19002715158472937734824\,[-2^{1}3^{-5}] & -865828810038222668088\,[-2^{-2}3^{-4}] & -6222083524060876926624\,[-2^{-7}3^{-1}] \\
-1110630706006093486416\,[-2^{-9}] & -738916774978949856954\,[-2^{-6}3^{-3}] & +256696765017519764574\,[-2^{-1}3^{-7}] \\
-538995368726709238620\,[2^{-3}3^{-6}] & +2659063284848174620104\,[3^{-5}] & +26750471369678154671328\,[2^{-5}3^{-1}] \\
-22869536297787068698224\,[2^{-7}3^{1}] & +26750471369678154671328\,[3^{-3}] & +1178782871405313104861940\,[2^{-1}3^{-2}] \\
+521062220884439910260592\,[2^{1}3^{-3}] & +2935816641298266366693024\,[2^{-2}3^{-1}] & -310513457035175924880\,[2^{-11}3^{5}] \\
-1295961764172934408392024\,[2^{-3}] & +47361088156862575216815120\,[2^{-1}3^{-1}] & +96143386519271973883680\,[2^{-4}3^{1}] \\
+6405316545334014067721724\,[2^{1}3^{-2}] & +25072646886079199648640\,[2^{3}3^{-3}] & -1384706396500391342516779656\,[3^{-1}] \\
+8128823582861345906142336\,[2^{-3}3^{1}] & -7000414961399434681344\,[2^{5}3^{-4}] & +22794254041869638225651427336\,[2^{-1}] \\
+116644982485749618488112\,[2^{4}3^{-3}] & -791724010495502232049202784\,[2^{1}3^{-1}] & +48384399276356552737368768\,[2^{-2}3^{1}] \\
-118427126740566034100976\,[2^{-5}3^{3}] & +42171788394774792880 1820\,[2^{3}3^{-2}] & +2651619475018716827904\,[2^{-8}3^{5}] ,
\end{array}
$$

satisfies

$$
\begin{aligned}
(\mathscr{L}_7 \cdot \nu_2)(\Lambda) \; &\overset{?}{=} \; -1754423866713781792025385 15\zeta(7) , \\
(\mathscr{L}_7 \cdot \nu_3)(\Lambda) \; &\overset{?}{=} \; 0 \cdot \zeta(7) ,
\end{aligned}
$$

where $(\mathscr{L}_7 \cdot \nu_p)(x) := \mathscr{L}_7(x)\nu_p(x)$, extended to formal linear combinations by linearity as usual.

The corresponding identities for the usual Li_7-function are as follows, where $(\operatorname{Li}_7 \cdot \nu_p)(x) := \operatorname{Li}_7(x)\nu_p(x)$ and extended to formal linear combinations. (These identities may be found by simplifying the $\mathscr{L}_7$ combination, via the lower weight identities required for constructing the element Λ, as in Section 9.3.)

$$
(\operatorname{Li}_7 \cdot \nu_2)(\Lambda) \; \overset{?}{=} \; -175442386671378179202538515\zeta(7)
$$

$$\begin{aligned}
&+ \Big(\tfrac{8660102386247332065470 3193}{2} \log(2) + 4329885842622522188336 2545 \log(3)\Big)\zeta(6) \\
&+ \Big(-2599897624420731693226 0482 \log^3(2) + 5093589850497028343582 0559 \log^2(2)\log(3) \\
&\qquad - 4486888822413056507544 9159 \log(2)\log^2(3) + 1082939923587375495689 6736 \log^3(3)\Big)\zeta(4) \\
&+ \Big(\tfrac{4383448906351379795178576}{5} \log^5(2) + 7599009263886799516961685 \log^4(2)\log(3) \\
&\qquad - 20383400950749825104250180 \log^3(2)\log^2(3) + 19516328962508895943069818 \log^2(2)\log^3(3) \\
&\qquad - 7657336820016538821970749 \log(2)\log^4(3) + 1060811632161976167107118 \log^5(3)\Big)\zeta(2) \\
&+ \Big(-\tfrac{6643933998483517187551296 1}{10} \log^7(2) + \tfrac{28178630425950960235675839 3}{10} \log^6(2)\log(3) \\
&\qquad - \tfrac{27324555335373362305400912 7}{5} \log^5(2)\log^2(3) + \tfrac{12013638714422129371915976 1}{2} \log^4(2)\log^3(3) \\
&\qquad - 3935922575762789167643227 8 \log^3(2)\log^4(3) + \tfrac{14974470937424485029160107 9}{10} \log^2(2)\log^5(3) \\
&\qquad - \tfrac{30171731893419043365973893}{10} \log(2)\log^6(3) + \tfrac{862648427095699972500049 8}{35} \log^7(3)\Big),
\end{aligned}$$

$$\begin{aligned}
&(\mathrm{Li}_7 \cdot \nu_3)(\Lambda) \overset{?}{=} 0 \cdot \zeta(7) \\
&+ \Big(43298858426225221883362545 \log(2) - \tfrac{47019048998647372592474736 9}{16} \log(3)\Big)\zeta(6) \\
&+ \Big(16978632834990094478606853 \log^3(2) - 44868888224130565075449159 \log^2(2)\log(3) \\
&\qquad + 32488197707621264870690208 \log(2)\log^2(3) - \tfrac{54219260939998101402163491}{8} \log^3(3)\Big)\zeta(4) \\
&+ \Big(15198018527773599033923 37 \log^5(2) - 10191700475374912552125090 \log^4(2)\log(3) \\
&\qquad + 19516328962508895943069818 \log^3(2)\log^2(3) - 15314673640033077643941498 \log^2(2)\log^3(3) \\
&\qquad + 5304058160809880835535590 \log(2)\log^4(3) - \tfrac{27704244501260460309276333}{40} \log^5(3)\Big)\zeta(2) \\
&+ \Big(\tfrac{28178630425950960235675839 3}{70} \log^7(2) - \tfrac{9108185111791120768466970 9}{5} \log^6(2)\log(3) \\
&\qquad + \tfrac{36040916143266388115747928 3}{10} \log^5(2)\log^2(3) - 3935922575762789167643227 8 \log^4(2)\log^3(3) \\
&\qquad + \tfrac{49914903124748283430533693}{2} \log^3(2)\log^4(3) - \tfrac{90515195680257130097921679}{10} \log^2(2)\log^5(3) \\
&\qquad + \tfrac{862648427095699972500049 8}{5} \log(2)\log^6(3) - \tfrac{10474550968563976414703313}{80} \log^7(3)\Big).
\end{aligned}$$

Using the idea from (37), we can recover the rest of the reduced coaction $\Delta' \mathrm{Li}_8^{\mathfrak{m}}(\Lambda)$ from a (hypothetical) motivic version of these identities, which would give the $\otimes \log^{\mathfrak{u}}(2)$ and $\otimes \log^{\mathfrak{u}}(3)$ components of $\Delta^{(7,1)} \mathrm{Li}_8^{\mathfrak{m}}(\Lambda)$, respectively. Moreover, we can obtain a combination whose reduced coaction is exactly $\zeta^{\mathfrak{m}}(7) \otimes \log^{\mathfrak{u}}(2)$, and so, together with $\zeta^{\mathfrak{m}}(7)\log^{\mathfrak{m}}(2)$, can match the term $\log^{\mathfrak{m}}(2) \otimes \zeta^{\mathfrak{u}}(7)$ in $\Delta S^{\mathfrak{m}}_{6,2}(-1)$. (Effectively, one integrates the $\otimes\log^{\mathfrak{u}}(2)$ component with respect to $\log^{\mathfrak{m}}(2)$, and the $\otimes\log^{\mathfrak{u}}(3)$ component with respect to $\log^{\mathfrak{m}}(3)$, except that terms that come from both will be doubly counted.)

We obtain the following candidate reduction of $S_{6,2}(-1)$ to $\zeta(3,5)$, polylogarithms and products, where the coefficient of $\zeta(8)$ is determined by numerical evaluation. Given a proven (motivic) version of the $(\mathrm{Li}_7 \cdot v_p)(\Lambda)$ identities above, we would know the coefficient $\zeta(8)$ is rational, nevertheless this coefficient does *appear* to be rational, numerically. We deliberately write the reduction in the following way, to highlight the structure of the reduction forced by the coaction. It has been verified to 20,000 decimal places in PARI/GP [36].

$$\begin{aligned}
&17544238667137817920253851 5 \cdot \frac{64}{127} \cdot S_{6,2}(-1) \overset{?}{=} \mathrm{Li}_8(\Lambda) - \frac{19402627481307724677394420487}{64}\zeta(8) \\
&+ 17544238667137817920253851 5 \cdot \frac{64}{127} \cdot \Big(\frac{3}{80}\zeta(3,5) + \frac{15}{16}\zeta(3)\zeta(5) + \frac{127}{64}\log(2)\zeta(7)\Big) \\
&+ \Big(-\tfrac{8660102386247332065470 3193}{4} \log^2(2) - 43298858426225221883362545 \log(2)\log(3) \\
&\qquad + \tfrac{47019048998647372592474736 9}{32} \log^2(3)\Big)\zeta(6) \\
&+ \Big(\tfrac{12999488122103658466130241 \log^4(2)}{2} - 16978632834990094478606853 \log^3(2)\log(3) \\
&\qquad + \tfrac{44868888224130565075449159}{2} \log^2(2)\log^2(3) - 10829399235873754956896736 \log(2)\log^3(3) \\
&\qquad + \tfrac{54219260939998101402163491 \log^4(3)}{32}\Big)\zeta(4)
\end{aligned}$$

$$
\begin{aligned}
&+ \Big(-\tfrac{730574817725229965863096}{5} \log^6(2) - 151980185277735990339 2337 \log^5(2)\log(3) \\
&\quad + 509585023768745627606 2545 \log^4(2)\log^2(3) - 650544298750296531435 6606 \log^3(2)\log^3(3) \\
&\quad + \tfrac{765733682001653882197 0749}{2} \log^2(2)\log^4(3) - 106081163216197616710 7118 \log(2)\log^5(3) \\
&\quad + \tfrac{923474816708682010309 2111}{80} \log^6(3)\Big)\zeta(2) \\
&+ \Big(\tfrac{664393399848351718755 12961}{80} \log^8(2) - \tfrac{281786304259509602356 758393}{70} \log^7(2)\log(3) \\
&\quad + \tfrac{910818511179112076846 69709}{10} \log^6(2)\log^2(3) - \tfrac{120136387144221293719 159761}{10} \log^5(2)\log^3(3) \\
&\quad + \tfrac{196796128788139458382 16139}{2} \log^4(2)\log^4(3) - \tfrac{499149031247482834305 33693}{10} \log^3(2)\log^5(3) \\
&\quad + \tfrac{301717318934190433659 73893}{20} \log^2(2)\log^6(3) - \tfrac{862648427095699972500 0498}{35} \log(2)\log^7(3) \\
&\quad + \tfrac{104745509685639764147 03313}{640} \log^8(3)\Big) \,.
\end{aligned}
$$

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