

# Disjoint Paths and Connected Subgraphs for $H$ -Free Graphs<sup>\*</sup>

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**Abstract.** The well-known DISJOINT PATHS problem is to decide if a graph contains  $k$  pairwise disjoint paths, each connecting a different terminal pair from a set of  $k$  distinct vertex pairs. We determine, with an exception of two cases, the complexity of the DISJOINT PATHS problem for  $H$ -free graphs. If  $k$  is fixed, we obtain the  $k$ -DISJOINT PATHS problem, which is known to be polynomial-time solvable on the class of all graphs for every  $k \geq 1$ . The latter does no longer hold if we need to connect vertices from terminal sets instead of terminal pairs. We completely classify the complexity of  $k$ -DISJOINT CONNECTED SUBGRAPHS for  $H$ -free graphs, and give the same almost-complete classification for DISJOINT CONNECTED SUBGRAPHS for  $H$ -free graphs as for DISJOINT PATHS. Moreover, we give exact algorithms for DISJOINT PATHS and DISJOINT CONNECTED SUBGRAPHS on graphs with  $n$  vertices and  $m$  edges that have running times of  $O(2^n n^2 k)$  and  $O(3^n km)$ , respectively.

**Keywords:** disjoint paths;  $H$ -free graph; complexity dichotomy

## 1 Introduction

A path from a vertex  $s$  to a vertex  $t$  in a graph  $G$  is an  $s$ - $t$  path of  $G$ , and  $s$  and  $t$  are called its *terminals*. Two pairs  $(s_1, t_1)$  and  $(s_2, t_2)$  are *disjoint* if  $\{s_1, t_1\} \cap \{s_2, t_2\} = \emptyset$ . In 1980, Shiloach [20] gave a polynomial-time algorithm for testing if a graph with disjoint terminal pairs  $(s_1, t_1)$  and  $(s_2, t_2)$  has vertex-disjoint paths  $P^1$  and  $P^2$  such that each  $P^i$  is an  $s_i$ - $t_i$  path. This problem can be generalized as follows.

DISJOINT PATHS

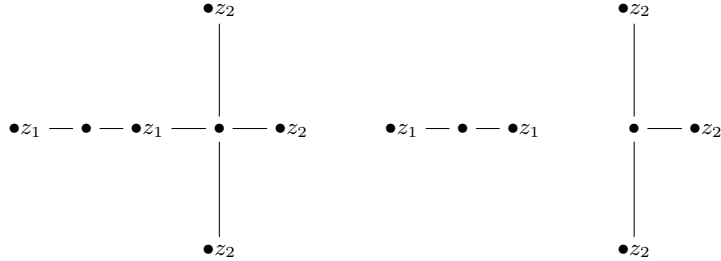
*Instance:* a graph  $G$  and pairwise disjoint terminal pairs  $(s_1, t_1) \dots, (s_k, t_k)$ .

*Question:* Does  $G$  have pairwise vertex-disjoint paths  $P^1, \dots, P^k$  such that  $P^i$  is an  $s_i$ - $t_i$  path for  $i \in \{1, \dots, k\}$ ?

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**Fig. 1.** An example of a yes-instance  $(G, Z_1, Z_2)$  of (2-)DISJOINT CONNECTED SUBGRAPHS (left) together with a solution (right).

Karp [12] proved that DISJOINT PATHS is NP-complete. If  $k$  is fixed, that is, not part of the input, then we denote the problem as  $k$ -DISJOINT PATHS. For every  $k \geq 1$ , Robertson and Seymour proved the following celebrated result.

**Theorem 1 ([19]).** *For all  $k \geq 2$ ,  $k$ -DISJOINT PATHS is polynomial-time solvable.*

The running time in Theorem 1 is cubic. This was later improved to quadratic time by Kawarabayashi, Kobayashi and Reed [13].

As DISJOINT PATHS is NP-complete, it is natural to consider special graph classes. The DISJOINT PATHS problem is known to be NP-complete even for graph of clique-width at most 6 [8], split graphs [9], interval graphs [16] and line graphs. The latter result can be obtained by a straightforward reduction (see, for example, [8, 9]) from its edge variant, EDGE DISJOINT PATHS, proven to be NP-complete by Even, Itai and Shamir [5]. On the positive side, DISJOINT PATHS is polynomial-time solvable for cographs, or equivalently,  $P_4$ -free graphs [8].

We can generalize the DISJOINT PATHS problem by considering terminal sets  $Z_i$  instead of terminal pairs  $(s_i, t_i)$ . We write  $G[S]$  for the subgraph of a graph  $G = (V, E)$  induced by  $S \subseteq V$ , where  $S$  is *connected* if  $G[S]$  is connected.

**DISJOINT CONNECTED SUBGRAPHS**

*Instance:* a graph  $G$  and pairwise disjoint terminal sets  $Z_1, \dots, Z_k$ .

*Question:* Does  $G$  have pairwise disjoint connected sets  $S_1, \dots, S_k$  such that  $Z_i \subseteq S_i$  for  $i \in \{1, \dots, k\}$ ?

If  $k$  is fixed, then we write  $k$ -DISJOINT CONNECTED SUBGRAPHS. We refer to Figure 1 for a simple example of an instance  $(G, Z_1, Z_2)$  of 2-DISJOINT CONNECTED SUBGRAPHS. Robertson and Seymour [19] proved in fact that  $k$ -DISJOINT CONNECTED SUBGRAPHS is cubic-time solvable as long as  $|Z_1| + \dots + |Z_k|$  is fixed (this result implies Theorem 1). Otherwise, van 't Hof et al. [23] proved that already 2-DISJOINT CONNECTED SUBGRAPHS is NP-complete even if  $|Z_1| = 2$  (and  $|Z_2|$  may have arbitrarily large size). The same authors also proved that 2-DISJOINT



**Fig. 2.** The graph  $H = 3P_1 + P_4$ .

CONNECTED SUBGRAPHS is NP-complete for split graphs. Afterwards, Gray et al. [7] proved that 2-DISJOINT CONNECTED SUBGRAPHS is NP-complete for planar graphs. Hence, Theorem 1 cannot be extended to hold for  $k$ -DISJOINT CONNECTED SUBGRAPHS.

We note that in recent years a number of exact algorithms were designed for  $k$ -DISJOINT CONNECTED SUBGRAPHS. Cygan et al. [4] gave an  $O^*(1.933^n)$ -time algorithm for the case  $k = 2$  (see [18, 23] for faster exact algorithms for special graph classes). Telle and Villanger [21] improved this to time  $O^*(1.7804^n)$ . Recently, Agrawal et al. [1] gave an  $O^*(1.88^n)$ -time algorithm for the case  $k = 3$ . Moreover, the 2-DISJOINT CONNECTED SUBGRAPHS problem plays a crucial role in graph contractibility: a connected graph can be contracted to the 4-vertex path if and only if there exist two vertices  $u$  and  $v$  such that  $(G - \{u, v\}, N(u), N(v))$  is a yes-instance of 2-DISJOINT CONNECTED SUBGRAPHS (see, e.g. [15, 23]).

A class of graphs that is closed under vertex deletion is called *hereditary*. Such a graph class can be characterized by a unique set  $\mathcal{F}$  of minimal forbidden induced subgraphs. Hereditary graphs enable a systematic study of the complexity of a graph problem under input restrictions: by starting with the case where  $|\mathcal{F}| = 1$ , we may already obtain more general methodology and a better understanding of the complexity of the problem. If  $|\mathcal{F}| = 1$ , say  $\mathcal{F} = \{H\}$  for some graph  $H$ , then we obtain the class of  $H$ -free graphs, that is, the class of graphs that do not contain  $H$  as an induced subgraph (so, an  $H$ -free graph cannot be modified to  $H$  by vertex deletions only). In this paper, we start such a systematic study for DISJOINT PATHS and DISJOINT CONNECTED SUBGRAPHS, both for the case when  $k$  is part of the input and when  $k$  is fixed.

## Our Results

By combining some of the aforementioned known results with a number of new results, we prove the following two theorems in Sections 3 and 4, respectively. In particular, we generalize the polynomial-time result for DISJOINT PATHS on  $P_4$ -free graphs to hold even for DISJOINT CONNECTED SUBGRAPHS. See Figure 2 for an example of a graph  $H = sP_1 + P_4$ ; we refer to Section 2 for undefined terminology.

**Theorem 2.** *Let  $H$  be a graph. If  $H \subseteq_i sP_1 + P_4$ , then for every  $k \geq 2$ ,  $k$ -DISJOINT CONNECTED SUBGRAPHS on  $H$ -free graphs is polynomial-time solvable; otherwise even 2-DISJOINT CONNECTED SUBGRAPHS is NP-complete.*

**Theorem 3.** *Let  $H$  be a graph not in  $\{3P_1, 2P_1 + P_2, P_1 + P_3\}$ . If  $H \subseteq_i P_4$ , then DISJOINT CONNECTED SUBGRAPHS is polynomial-time solvable for  $H$ -free graphs; otherwise even DISJOINT PATHS is NP-complete.*

Theorem 2 completely classifies, for every  $k \geq 2$ , the complexity of  $k$ -DISJOINT CONNECTED SUBGRAPHS on  $H$ -free graphs. Theorem 3 determines the complexity of DISJOINT PATHS and DISJOINT CONNECTED SUBGRAPHS on  $H$ -free graphs for every graph  $H$  except if  $H \in \{3P_1, 2P_1 + P_2, P_1 + P_3\}$ . In Section 5 we reduce the number of open cases from six to *three* by showing some equivalencies.

In Section 6 we complement the above results by giving exact algorithms for both problems based on Held-Karp type dynamic programming techniques [10, 2]. In Section 7 we give some directions for future work. In particular we prove that both problems are polynomial-time solvable for co-bipartite graphs, which form a subclass of the class of  $3P_1$ -free graphs.

## 2 Preliminaries

We use  $H \subseteq_i H'$  to indicate that  $H$  is an induced subgraph of  $H'$ , that is,  $H$  can be obtained from  $H'$  by a sequence of vertex deletions. For two graphs  $G_1$  and  $G_2$  we write  $G_1 + G_2$  for the *disjoint union*  $(V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$ . We denote the disjoint union of  $r$  copies of a graph  $G$  by  $rG$ . A graph is said to be a linear forest if it is a disjoint union of paths.

We denote the path and cycle on  $n$  vertices by  $P_n$  and  $C_n$ , respectively. The *girth* of a graph that is not a forest is the number of edges of a smallest induced cycle in it.

The *line graph*  $L(G)$  of a graph  $G$  has vertex set  $E(G)$  and there exists an edge between two vertices  $e$  and  $f$  in  $L(G)$  if and only if  $e$  and  $f$  have a common end-vertex in  $G$ . The claw  $K_{1,3}$  is the 4-vertex star. It is readily seen that every line graph is claw-free. Recall that a graph is  $H$ -free if it does not contain  $H$  as induced subgraph. For a set of graphs  $\{H_1, \dots, H_r\}$ , we say that a graph  $G$  is  $(H_1, \dots, H_r)$ -free if  $G$  is  $H_i$ -free for every  $i \in \{1, \dots, r\}$ .

A *clique* is a set of pairwise adjacent vertices and an *independent set* is a set of pairwise non-adjacent vertices. A graph is *split* if its vertex set can be partitioned into two (possibly empty) sets, one of which is a clique and the other is an independent set. A graph is split if and only if it is  $(C_4, C_5, P_4)$ -free [6]. A graph is a *cograph* if it can be defined recursively as follows: any single vertex is a cograph, the disjoint union of two cographs is a cograph, and the join of two cographs  $G_1, G_2$  is a cograph (the *join* adds all edges between the vertices of  $G_1$  and  $G_2$ ). A graph is a cograph if and only if it is  $P_4$ -free [3].

A graph  $G = (V, E)$  is *multipartite*, or more specifically, *r-partite* if  $V$  can be partitioned into  $r$  (possibly empty) sets  $V_1, \dots, V_r$ , such that there is an edge between two vertices  $u$  and  $v$  if and only if  $u \in V_i$  and  $v \in V_j$  for some  $i, j$  with  $i \neq j$ . If  $r = 2$ , we also say that  $G$  is *bipartite*. If there exist an edge between every vertex of  $V_i$  and every vertex of  $V_j$  for every  $i \neq j$ , then the multipartite graph  $G$  is *complete*.

The *complement* of a graph  $G = (V, E)$  is the graph  $\overline{G} = (V, \{uv \mid u, v \in V, u \neq v \text{ and } uv \notin E\})$ . The complement of a bipartite graph is a *cobipartite* graph. A set  $W \subseteq V$  is a *dominating set* of a graph  $G$  if every vertex of  $V \setminus W$  has a neighbour in  $W$ , or equivalently,  $N[W]$  (the closed neighbourhood of  $W$ ) is

equal to  $V$ . We say that  $W$  is a *connected dominating set* if  $W$  is a dominating set and  $G[W]$  is connected.

### 3 The Proof of Theorem 2

We consider  $k$ -DISJOINT CONNECTED SUBGRAPHS for fixed  $k$ . First, we show a polynomial-time algorithm on  $H$ -free graphs when  $H \subseteq_i sP_1 + P_4$  for some fixed  $s \geq 0$ . Then, we prove the hardness result.

For the algorithm, we need the following lemma for  $P_4$ -free graphs, or equivalently, cographs. This lemma is well known and follows immediately from the definition of a cograph: in the construction of a connected cograph  $G$ , the last operation must be a join, so there exists cographs  $G_1$  and  $G_2$ , such that  $G$  obtained from adding an edge between every vertex of  $G_1$  and every vertex of  $G_2$ . Hence, the spanning complete bipartite graph of  $G$  has non-empty partition classes  $V(G_1)$  and  $V(G_2)$ .

**Lemma 1.** *Every connected  $P_4$ -free graph on at least two vertices has a spanning complete bipartite subgraph.*

Two instances of a problem  $\Pi$  are *equivalent* when one of them is a yes-instance of  $\Pi$  if and only if the other one is a yes-instance of  $\Pi$ . We note that if two adjacent vertices will always appear in the same set of every solution  $(S_1, \dots, S_k)$  for an instance  $(G, Z_1, \dots, Z_k)$ , then we may contract the edge between them at the start of any algorithm. This takes linear time. Moreover,  $H$ -free graphs are readily seen (see e.g. [15]) to be closed under edge contraction if  $H$  is a linear forest. Hence, we can make the following observation.

**Lemma 2.** *For  $k \geq 2$ , from every instance of  $(G, Z_1, \dots, Z_k)$  of  $k$ -DISJOINT CONNECTED SUBGRAPHS we can obtain in polynomial time an equivalent instance  $(G', Z'_1, \dots, Z'_k)$  such that every  $Z'_i$  is an independent set. Moreover, if  $G$  is  $H$ -free for some linear forest  $H$ , then  $G'$  is also  $H$ -free.*

We can now prove the following lemma.

**Lemma 3.** *Let  $H$  be a graph. If  $H \subseteq_i sP_1 + P_4$ , then for every  $k \geq 1$ ,  $k$ -DISJOINT CONNECTED SUBGRAPHS on  $H$ -free graphs is polynomial-time solvable.*

*Proof.* Let  $H \subseteq_i sP_1 + P_4$  for some  $s \geq 0$ . Let  $(G, Z_1, \dots, Z_k)$  be an instance of  $k$ -DISJOINT CONNECTED SUBGRAPHS, where  $G$  is an  $H$ -free graph. By Lemma 2, we may assume without loss of generality that  $G$  is connected and moreover that  $Z_1, \dots, Z_k$  are all independent sets.

We first analyze the structure of a solution  $(S_1, \dots, S_k)$  (if it exists). For  $i \in \{1, \dots, k\}$ , we may assume that  $S_i$  is inclusion-wise minimal, meaning there is no  $S'_i \subset S_i$  that contains  $Z_i$  and is connected. Consider a graph  $G[S_i]$ . Either  $G[S_i]$  is  $P_4$ -free or  $G[S_i]$  contains an induced  $rP_1 + P_4$  for some  $0 \leq r \leq s - 1$ . We will now show that in both cases,  $S_i$  is the (not necessarily disjoint) union of  $Z_i$  and a connected dominating set of  $G[S_i]$  of constant size.

First suppose that  $G[S_i]$  is  $P_4$ -free. As  $G[S_i]$  is connected and  $Z_i$  is independent, we apply Lemma 1 to find that  $S_i \setminus Z_i$  contains a vertex  $u$  that is adjacent to every vertex of  $Z_i$ . Hence, by minimality,  $S_i = Z_i \cup \{u\}$  and  $\{u\}$  is a connected dominating set of  $G[S_i]$  of size 1.

Now suppose that  $G[S_i]$  has an induced  $rP_1 + P_4$  for some  $r \geq 0$ , where we choose  $r$  to be maximum. Note that  $r \leq s - 1$ . Let  $W$  be the vertex set of the induced  $rP_1 + P_4$ . Then, as  $r$  is maximum,  $W$  dominates  $G[S_i]$ . Note that  $G[W]$  has  $r + 1 \leq s$  connected components. Then, as  $G[S_i]$  is connected and  $W$  is a dominating set of  $G[S_i]$  of size  $r + 4 \leq s + 3$ , it follows from folklore arguments (see e.g. [22, Prop. 6.3.24]) that  $G[S_i]$  has a connected dominating set  $W'$  of size at most  $3s + 1$ . Moreover, by minimality,  $S_i = Z_i \cup W'$ .

Hence, in both cases we find that  $S_i$  is the union of  $Z_i$  and a connected dominating set of  $G[S_i]$  of size at most  $t = 3s + 1$ ; note that  $t$  is a constant, as  $s$  is a constant.

Our algorithm now does as follows. We consider all options of choosing a connected dominating set of each  $G[S_i]$ , which from the above has size at most  $t$ . As soon as one of the guesses makes every  $Z_i$  connected, we stop and return the solution. The total number of options is  $O(n^{tk})$ , which is polynomial as  $k$  and  $t$  are fixed. Moreover, checking the connectivity condition can be done in polynomial time. Hence, the total running time of the algorithm is polynomial.  $\square$

The proof our next result is inspired by the aforementioned NP-completeness result of [23] for instances  $(G, Z_1, Z_2)$  where  $|Z_1| = 2$  but  $G$  is a general graph.

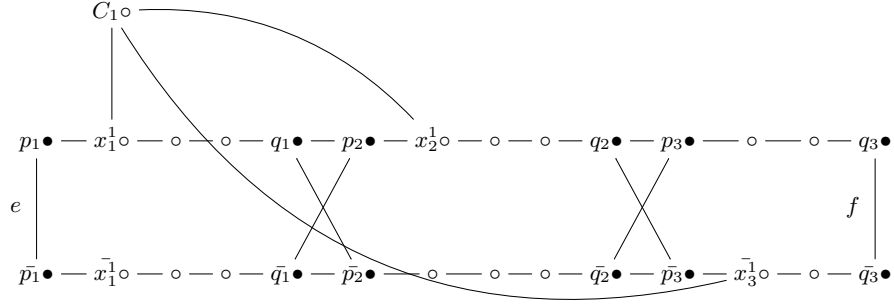
**Lemma 4.** *The 2-DISJOINT CONNECTED SUBGRAPHS problem is NP-complete even on instances  $(G, Z_1, Z_2)$  where  $|Z_1| = 2$  and  $G$  is a line graph.*

*Proof.* Note that the problem is in NP. We reduce from 3-SAT. Let  $\phi = \phi(x_1, \dots, x_n)$  be an instance of 3-SAT with clauses  $C_1, \dots, C_m$ . We construct a corresponding graph  $G = (V, E)$  as follows. We start with two disjoint paths  $P$  and  $\bar{P}$  on vertices  $p_i, x_i, q_i$  and  $\bar{p}_i, \bar{x}_i, \bar{q}_i$ , respectively, where  $x_i, \bar{x}_i$  correspond to the positive and negative literals in  $\phi$ , respectively. To be more precise, we define:

$$P = p_1, x_1, q_1, p_2, x_2, q_2, \dots, p_n, x_n, q_n, \text{ and } \bar{P} = \bar{p}_1, \bar{x}_1, \bar{q}_1, \dots, \bar{p}_n, \bar{x}_n, \bar{q}_n,$$

We add the two edges  $e = p_1\bar{p}_1$ , and  $f = q_n\bar{q}_n$ . For  $i = 1, \dots, n - 1$ , we also add edges  $q_i\bar{p}_{i+1}$  and  $\bar{q}_i p_{i+1}$ . We now replace each  $x_i$  by vertices  $x_i^{j_1}, x_i^{j_2}, \dots, x_i^{j_r}$ , where  $j_1, \dots, j_r$  are the indices of the clauses  $C_j$  that contain  $x_i$ . That is, we replace the subpath  $p_i, x_i, q_i$  of  $P$  by the path  $p_i, x_i^{j_1}, x_i^{j_2}, \dots, x_i^{j_r}, q_i$ . We do the same path replacement operation on  $\bar{P}$  with respect to every  $\bar{x}_i$ . Finally, we add every clause  $C_j$  as a vertex and add an edge between  $C_j$  and  $x_i^j$  if and only if  $x_i \in C_j$ , and between  $C_j$  and  $\bar{x}_i^j$  if and only if  $\bar{x}_i \in C_j$ . This completes the description of  $G = (V, E)$ . We refer to Figure 3 for an illustration of our construction.

We now focus on the line graph  $L = L(G)$  of  $G$ . Let  $Z_1 = \{e, f\} \subseteq E = V(L)$  and let  $Z_2$  consist of all vertices of  $L$  that correspond to edges in  $G$  that are incident to some  $C_j$ . Note that  $Z_1$  and  $Z_2$  are disjoint. Moreover, each clause  $C_j$



**Fig. 3.** The construction described with edges added for the clause  $C_1 = (x_1 \vee x_2 \vee \bar{x}_3)$ .

corresponds to a clique of size at most 3 in  $L$ , which we call the clause clique of  $C_j$ . We claim that  $\phi$  is satisfiable if and only if the instance  $(L, Z_1, Z_2)$  of 2-DISJOINT CONNECTED SUBGRAPHS is a yes-instance.

First suppose that  $\phi$  is satisfiable. Let  $\tau$  be a satisfying truth assignment for  $\phi$ . In  $G$ , we let  $P^1$  denote the unique path whose first edge is  $e$  and whose last edge is  $f$  and that passes through all  $x_i^j \in V$  if  $x_i = 0$  and through all  $\bar{x}_i^j$  if  $x_i = 1$ . In  $L$  we let  $S_1$  consist of all vertices of  $L(P^1)$ ; note that  $Z_1 = \{e, f\}$  is contained in  $S_1$  and that  $S_1$  is connected. We let  $P^2$  denote the “complementary” path in  $G$  whose first edge is  $e$  and whose last edge is  $f$  but that passes through all  $x_i^j$  if and only if  $P^1$  passes through all  $\bar{x}_i^j$ , and conversely ( $i = 1, \dots, n$ ). In  $L$ , we put all vertices of  $L(P^2)$ , except  $e$  and  $f$ , together with all vertices of  $Z_2$  in  $S_2$ . As  $\tau$  satisfies  $\phi$ , some vertex of each clause clique is adjacent to a vertex of  $P^2$ . Hence, as  $P^2$  is a path,  $S_2$  is connected and we found a solution for  $(L, Z_1, Z_2)$ .

Now suppose that  $(L, Z_1, Z_2)$  is a yes-instance of 2-DISJOINT CONNECTED SUBGRAPHS. Then  $V(L)$  can be partitioned into two vertex-disjoint connected sets  $S_1$  and  $S_2$  such that  $Z_1 \subseteq S_1$  and  $Z_2 \subseteq S_2$ . In particular,  $L[S_1]$  contains a path  $P^1$  from  $e$  to  $f$ . In fact, we may assume that  $S_1 = V(P^1)$ , as we can move every other vertex of  $S_1$  (if they exist) to  $S_2$  without disconnecting  $S_2$ .

Note that  $P^1$  corresponds to a connected subgraph that contains the adjacent vertices  $p_1$  and  $\bar{p}_1$  as well as the adjacent vertices  $q_n$  and  $\bar{q}_n$ . Hence, we can modify  $P^1$  into a path  $Q$  in  $G$  that starts in  $p_1$  or  $\bar{p}_1$  and that ends in  $q_n$  or  $\bar{q}_n$ . Note that  $Q$  contains no edge incident to a clause vertex  $C_j$ , as those edges correspond to vertices in  $L$  that belong to  $Z_2$ . Hence, by construction,  $Q$  “moves from left to right”, that is,  $Q$  cannot pass through both some  $x_i^j$  and  $\bar{x}_i^j$  (as then  $Q$  needs to pass through either  $x_i^j$  or  $\bar{x}_i^j$  again implying that  $Q$  is not a path).

Moreover, if  $Q$  passes through some  $x_i^j$ , then  $Q$  must pass through all vertices  $x_i^{j_h}$ . Similarly, if  $Q$  passes through some  $\bar{x}_i^j$ , then  $Q$  must pass through all vertices  $\bar{x}_i^{j_h}$ . As  $Q$  connects the edges  $p_1\bar{p}_1$  and  $q_n\bar{q}_n$ , we conclude that  $Q$  must pass, for  $i = 1, \dots, n$ , through either every  $x_i^{j_h}$  or through every  $\bar{x}_i^{j_h}$ . Thus we may define

a truth assignment  $\tau$  by setting

$$x_i = \begin{cases} 1 & \text{if } Q \text{ passes through all } \bar{x}_i^j \\ 0 & \text{if } Q \text{ passes through all } x_i^j. \end{cases}$$

We claim that  $\tau$  satisfies  $\phi$ . For contradiction, assume some clause  $C_j$  is not satisfied. Then  $Q$  passes through all its literals. However, then in  $S_2$ , the vertices of  $Z_2$  that correspond to edges incident to  $C_j$  are not connected to other vertices of  $Z_2$ , a contradiction. This completes the proof of the lemma.  $\square$

A straightforward modification of the reduction of Lemma 5 gives us Lemma 6. We can also obtain Lemma 6 by subdividing the graph  $G$  in the proof of Lemma 4 twice (to get a bipartite graph) or  $p$  times (to get a graph of girth at least  $p$ ).

**Lemma 5 ([23]).** 2-DISJOINT CONNECTED SUBGRAPHS is NP-complete for split graphs, or equivalently,  $(2P_2, C_4, C_5)$ -free graphs.

**Lemma 6.** 2-DISJOINT CONNECTED SUBGRAPHS is NP-complete for bipartite graphs and for graphs of girth at least  $p$ , for every integer  $p \geq 3$ .

We are now ready to prove Theorem 2.

**Theorem 2 (restated)** Let  $H$  be a graph. If  $H \subseteq_i sP_1 + P_4$ , then for every  $k \geq 1$ ,  $k$ -DISJOINT CONNECTED SUBGRAPHS on  $H$ -free graphs is polynomial-time solvable; otherwise even 2-DISJOINT CONNECTED SUBGRAPHS is NP-complete.

*Proof.* If  $H$  contains an induced cycle  $C_s$  for some  $s \geq 3$ , then we apply Lemma 6 by setting  $p = s + 1$ . Now assume that  $H$  contains no cycle, that is,  $H$  is a forest. If  $H$  has a vertex of degree at least 3, then  $H$  is a superclass of the class of claw-free graphs, which in turn contains all line graphs. Hence, we can apply Lemma 4. In the remaining case  $H$  is a linear forest. If  $H$  contains an induced  $2P_2$ , we apply Lemma 5. Otherwise  $H$  is an induced subgraph of  $sP_1 + P_4$  for some  $s \geq 0$  and we apply Lemma 3.  $\square$

## 4 The Proof of Theorem 3

We first prove the following result, which generalizes the corresponding result of DISJOINT PATHS for  $P_4$ -free graphs due to Gurski and Wanke [8]. We show that we can use the same modification to a matching problem in a bipartite graph.

**Lemma 7.** DISJOINT CONNECTED SUBGRAPHS is polynomial-time solvable for  $P_4$ -free graphs.

*Proof.* For some integer  $k \geq 2$ , let  $(G, Z_1, \dots, Z_k)$  be an instance of DISJOINT CONNECTED SUBGRAPHS where  $G$  is a  $P_4$ -free graph. By Lemma 2 we may assume that every  $Z_i$  is an independent set. Now suppose that  $(G, Z_1, \dots, Z_k)$  has a solution  $(S_1, \dots, S_k)$ . Then  $G[S_i]$  is a connected  $P_4$ -free graph. Hence, by Lemma 1,  $G[S_i]$  has a spanning complete bipartite graph on non-empty partition



classes  $A_i$  and  $B_i$ . As every  $Z_i$  is an independent set, it follows that either  $Z_i \subseteq A_i$  or  $Z_i \subseteq B_i$ . If  $Z_i \subseteq A_i$ , then every vertex of  $B_i$  is adjacent to every vertex of  $Z_i$ . Similarly, if  $Z_i \subseteq B_i$ , then every vertex of  $A_i$  is adjacent to every vertex of  $Z_i$ . We conclude that in every set  $S_i$ , there exists a vertex  $y_i$  such that  $Z_i \cup \{y_i\}$  is connected.

The latter enables us to construct a bipartite graph  $G' = (X \cup Y, E')$  where  $X$  contains vertices  $x_1, \dots, x_k$  corresponding to the set  $Z_1, \dots, Z_k$  and  $Y$  is the set of non-terminal vertices of  $G$ . We add an edge between  $x_i \in X$  and  $y \in Y$  if and only if  $y$  is adjacent to every vertex of  $Z_i$ . Then  $(G, Z_1 \dots Z_k)$  is a yes-instance of DISJOINT CONNECTED SUBGRAPHS if and only if  $G'$  contains a matching of size  $k$ . It remains to observe that we can find a maximum matching in polynomial time, for example, by using the Hopcroft-Karp algorithm for bipartite graphs [11].  $\square$

The first lemma of a series of four is obtained by a straightforward reduction from the EDGE DISJOINT PATHS problem (see, e.g. [8, 9]), which was proven to be NP-complete by Even, Itai and Shamir [5]. The second lemma follows from the observation that an edge subdivision of the graph  $G$  in an instance of DISJOINT PATHS results in an equivalent instance of DISJOINT PATHS; we apply this operation a sufficiently large number of times to obtain a graph of large girth. The third lemma is due to Heggernes et al. [9]. We modify their construction to prove the fourth lemma.

**Lemma 8.** DISJOINT PATHS is NP-complete for line graphs.

**Lemma 9.** For every  $g \geq 3$ , DISJOINT PATHS is NP-complete for graphs of girth at least  $g$ .

**Lemma 10 ([9]).** DISJOINT PATHS is NP-complete for split graphs, or equivalently,  $(C_4, C_5, 2P_2)$ -free graphs.

**Lemma 11.** DISJOINT PATHS is NP-complete for  $(4P_1, P_1 + P_4)$ -free graphs.

*Proof.* We reduce from DISJOINT PATHS on split graphs, which is NP-complete by Lemma 10. By inspection of this result (see [9, Theorem 3]), we note that the instances  $(G, \{(s_1, t_1), \dots, (s_k, t_k)\})$  have the following property: the split graph  $G$  has a split decomposition  $(C, I)$ , where  $C$  is a clique,  $I$  an independent set,  $C$  and  $I$  are disjoint, and  $C \cup I = V(G)$ , such that  $I = \{s_1, \dots, s_k, t_1, \dots, t_k\}$ . Now let  $G'$  be obtained from  $G$  by, for each terminal  $s_i$ , adding edges to  $s_j$  and  $t_j$  for all  $j \neq i$ . Then consider the instance  $(G', \{(s_1, t_1), \dots, (s_k, t_k)\})$ .

We note that  $G'[C]$  is still a complete graph, while  $G'[I]$  is a complete graph minus a matching. It is immediate that  $G'$  is  $4P_1$ -free. Moreover, any induced subgraph  $H$  of  $G'$  that is isomorphic to  $P_4$  must contain at least two vertices of  $I$  and at least one vertex of  $C$ . If  $H$  contains two vertices of  $C$ , then as  $G'[C]$  is a clique,  $H$  contains two non-adjacent vertices in  $I$ . Similarly, if  $H$  contains one vertex of  $C$  (and thus three vertices of  $I$ ), then  $H$  contains two non-adjacent vertices in  $I$ . Since  $C$  is a clique in  $G'$  and every (other) vertex of  $I$  is adjacent in  $G'$  to any pair of non-adjacent vertices of  $I$ , it follows that  $G'$  is  $P_1 + P_4$ -free as well.

We claim that  $(G, \{(s_1, t_1), \dots, (s_k, t_k)\})$  is a yes-instance if and only if  $(G', \{(s_1, t_1), \dots, (s_k, t_k)\})$  is a yes-instance. This is because the edges that were added to  $G$  to obtain  $G'$  are only between terminal vertices of different pairs. These edges cannot be used by any solution of DISJOINT PATHS for  $(G', \{(s_1, t_1), \dots, (s_k, t_k)\})$ , and thus the feasibility of the instance is not affected by the addition of these edges.  $\square$

We are now ready to prove Theorem 3.

**Theorem 3 (restated)** *Let  $H$  be a graph not in  $\{3P_1, 2P_1 + P_2, P_1 + P_3\}$ . If  $H \subseteq_i P_4$ , then DISJOINT CONNECTED SUBGRAPHS is polynomial-time solvable for  $H$ -free graphs; otherwise even DISJOINT PATHS is NP-complete.*

*Proof.* First suppose that  $H$  contains a cycle  $C_r$  for some  $r \geq 3$ . Then DISJOINT PATHS is NP-complete for the class of  $H$ -free graphs, as DISJOINT PATHS is NP-complete on the subclass consisting of graphs of girth  $r + 1$  by Lemma 9. Now suppose that  $H$  contains no cycle, that is,  $H$  is a forest. If  $H$  contains a vertex of degree at least 3, then the class of  $H$ -free graphs contains the class of claw-free graphs, which in turn contains the class of line graphs. Hence, we can apply Lemma 8. It remains to consider the case where  $H$  is a forest with no vertices of degree at least 3, that is, when  $H$  is a linear forest.

If  $H$  contains four connected components, then the class of  $H$ -free graphs contains the class of  $4P_1$ -free graphs, and we can use Lemma 11. If  $H$  contains an induced  $P_5$  or two connected components that each have at least one edge, then  $H$  contains the class of  $2P_2$ -free graphs, and we can use Lemma 10. If  $H$  contains two connected components, one of which has at least four vertices, then  $H$  contains the class of  $(P_1 + P_4)$ -free graphs, and we can use Lemma 11 again. As  $H \notin \{3P_1, 2P_1 + P_2, P_1 + P_3\}$ , this means that in the remaining case  $H$  is an induced subgraph of  $P_4$ . In that case even DISJOINT CONNECTED SUBGRAPHS is polynomial-time solvable on  $H$ -free graphs, due to Lemma 7.  $\square$

## 5 Reducing the Number of Open Cases to Three

Theorem 3 shows that we have the same three open cases for DISJOINT PATHS and DISJOINT CONNECTED SUBGRAPHS, namely when  $H \in \{3P_1, P_1 + P_3, 2P_1 + P_2\}$ . We show that instead of six open cases, we have in fact only three.

**Proposition 1.** *DISJOINT PATHS and DISJOINT CONNECTED SUBGRAPHS are equivalent for  $3P_1$ -free graphs.*

*Proof.* Every instance of DISJOINT PATHS is an instance of DISJOINT CONNECTED SUBGRAPHS. Let  $(G, Z_1, \dots, Z_k)$  be an instance of DISJOINT CONNECTED SUBGRAPHS where  $G$  is a  $3P_1$ -free graph. By Lemma 2 we may assume that each  $Z_i$  is an independent set. Then, as  $G$  is  $3P_1$ -free, each  $Z_i$  has size at most 2. So we obtained an instance of DISJOINT PATHS.  $\square$

**Proposition 2.** DISJOINT PATHS on  $(P_1 + P_3)$ -free graphs and DISJOINT CONNECTED SUBGRAPHS on  $(P_1 + P_3)$ -free graphs are polynomially equivalent to DISJOINT PATHS on  $3P_1$ -free graphs.

*Proof.* We prove that we can solve DISJOINT CONNECTED SUBGRAPHS in polynomial time on  $(P_1 + P_3)$ -free graphs if we have a polynomial-time algorithm for DISJOINT PATHS on  $3P_1$ -free graphs. Showing this suffices to prove the theorem, as DISJOINT PATHS is a special case of DISJOINT CONNECTED SUBGRAPHS and  $3P_1$ -free graphs form a subclass of  $(P_1 + P_3)$ -free graphs.

Let  $(G, Z_1, \dots, Z_k)$  be an instance of DISJOINT CONNECTED SUBGRAPHS, where  $G$  is a  $(P_1 + P_3)$ -free graph. Olariu [17] proved that every connected  $\overline{P_1 + P_3}$ -free graph is either triangle-free or complete multipartite. Hence, the vertex set of  $G$  can be partitioned into sets  $D_1, \dots, D_p$  for some  $p \geq 1$  such that

- every  $G[D_i]$  is  $3P_1$ -free or the disjoint union of complete graphs, and
- for every  $i, j$  with  $i \neq j$ , every vertex of  $D_i$  is adjacent to every vertex of  $D_j$ .

Using this structural characterization, we first argue that we may assume that each  $Z_i$  has size 2, making the problem an instance of DISJOINT PATHS. Then we show that we can either solve the instance outright or can alter  $G$  to be  $3P_1$ -free.

First, we argue about the size of each  $Z_i$ . By Lemma 2 we may assume that every  $Z_i$  is an independent set and is thus contained in the same set  $D_j$ . If  $G[D_j]$  is  $3P_1$ -free, then this implies that any  $Z_i$  that is contained in  $D_j$  has size 2. If  $G[D_j]$  is a disjoint union of complete graphs, then each vertex of a  $Z_i$  that is contained in  $D_j$  belongs to a different connected component of  $D_j$  and  $Z_i \cup \{v\}$  is connected for every vertex  $v \notin D_j$ . As at least one vertex  $v \notin D_j$  is needed to make such a set  $Z_i$  connected, we may therefore assume that for a solution  $(S_1, \dots, S_k)$  (if it exists),  $S_i = Z_i \cup \{v\}$  for some  $v \notin D_j$ . The latter implies that we may assume without loss of generality that every such  $Z_i$  has size 2 as well.

If  $p = 1$ , then each connected component of  $G$  is  $3P_1$ -free, and we are done. Hence, we assume that  $p \geq 2$ . In fact, since any two distinct sets  $D_i$  and  $D_j$  are complete to each other, the union of any two  $3P_1$ -free graphs induces a  $3P_1$ -free graph. Therefore we may assume without loss of generality that only  $G[D_1]$  might be  $3P_1$ -free, whereas  $G[D_2], \dots, G[D_p]$  are disjoint unions of complete graphs.

Recall that  $Z_i = \{s_i, t_i\}$  for every  $i \in \{1, \dots, k\}$  and we search for a solution  $(P^1, \dots, P^k)$  where each  $P^i$  is a path from  $s_i$  to  $t_i$ . First suppose  $s_i$  and  $t_i$  belong to  $D_1$ . Then  $P^i$  has length 2 or 3 and in the latter case,  $V(P^i) \subseteq D_1$ . Now suppose that  $s_i$  and  $t_i$  belong to  $D_h$  for some  $h \in \{2, \dots, k\}$ . Then  $P^i$  has length exactly 2, and moreover, the middle (non-terminal) vertex of  $P^i$  does not belong to  $D_h$ .

We will now check if there is a solution  $(P^1, \dots, P^k)$  such that every  $P^i$  has length exactly 2. We call such a solution to be of *type 1*. In a solution of type 1, every  $P^i = s_i u t_i$  for some non-terminal vertex  $u$  of  $G$ . If  $s_i$  and  $t_i$  belong to  $D_h$  for some  $h \in \{2, \dots, p\}$ , then  $u \in D_j$  for some  $j \neq i$ . If  $s_i$  and  $t_i$  belong to  $D_1$ , then  $u \in D_j$  for some  $j \neq 1$  but also  $u \in D_1$  is possible, namely when  $u$  is adjacent to both  $s_i$  and  $t_i$ .

Verifying the existence of a type 1 solution is equivalent to finding a perfect matching in a bipartite graph  $G' = A \cup B$  that is defined as follows. The set  $A$  consists of one vertex  $v_i$  for each pair  $\{s_i, t_i\}$ . The set  $B$  consists of all non-terminal vertices  $u$  of  $G$ . For  $\{s_i, t_i\} \subseteq D_1$ , there exists an edge between  $u$  and  $v_i$  in  $G'$  if and only if in  $G$  it holds that  $u \in D_h$  for some  $h \in \{2, \dots, p\}$  or  $u \in D_1$  and  $u$  is adjacent to both  $s_i$  and  $t_i$ . For  $\{s_i, t_i\} \subseteq D_h$  with  $h \in \{2, \dots, p\}$ , there exists an edge between  $u$  and  $v_i$  in  $G'$  if and only if in  $G$  it holds that  $u \in D_j$  for some  $j \in \{1, \dots, p\}$  with  $h \neq j$ . We can find a perfect matching in  $G'$  in polynomial time by using the Hopcroft-Karp algorithm for bipartite graphs [11].

Suppose that we find that  $(G, \{s_1, t_1\}, \dots, \{s_k, t_k\})$  has no solution of type 1. As a solution can be assumed to be of type 1 if  $G[D_1]$  is the disjoint union of complete graphs, we find that  $G[D_1]$  is not of this form. Hence,  $G[D_1]$  is  $3P_1$ -free. Recall that  $G[D_j]$  is the disjoint union of complete graphs for  $2 \leq i \leq p$ . It remains to check if there is a solution that is of *type 2* meaning a solution  $(P^1, \dots, P^k)$  in which at least one  $P^i$ , whose vertices all belong to  $D_1$ , has length 3.

To find a type 2 solution (if it exists) we construct the following graph  $G^*$ . We let  $V(G^*) = A_1 \cup A_2 \cup B_1 \cup B_2$ , where

- $A_1$  consists of all terminal vertices from  $D_1$ ;
- $A_2$  consists of all non-terminal vertices from  $D_1$ ;
- $B_1$  consists of all terminal vertices from  $D_2 \cup \dots \cup D_p$ ; and
- $B_2$  consists of all non-terminal vertices from  $D_2 \cup \dots \cup D_p$ .

Note that  $V(G^*) = V(G)$ . To obtain  $E(G^*)$  from  $E(G)$  we add some edges (if they do not exist in  $G$  already) and also delete some edges (if these existed in  $G$ ):

- (i) for each  $\{s_i, t_i\} \subseteq B_1$ , add all edges between  $s_i$  and vertices of  $B_2$ , and delete any edges between  $t_i$  and vertices of  $B_2$ ;
- (ii) add an edge between every two terminal vertices in  $B_1$  that belong to different terminal pairs; and
- (iii) add an edge between every two vertices of  $B_2$ .

We note that  $G^*[D_1]$  is the same graph as  $G[D_1]$  and thus  $G^*[D_1]$  is  $3P_1$ -free. Moreover,  $G^*[B_1 \cup B_2]$  is  $3P_1$ -free by part (i) of the construction. Hence, as there exists an edge between every vertex of  $A_1 \cup A_2$  and every vertex of  $B_1 \cup B_2$  in  $G$  and thus also in  $G^*$ , this means that  $G^*$  is  $3P_1$ -free. It remains to prove that  $(G, \{s_1, t_1\}, \dots, \{s_k, t_k\})$  and  $(G^*, \{s_1, t_1\}, \dots, \{s_k, t_k\})$  are equivalent instances.

First suppose that  $(G, \{s_1, t_1\}, \dots, \{s_k, t_k\})$  has a solution  $(P^1, \dots, P^k)$ . Assume that the number of paths of length 3 in this solution is minimum over all solutions for  $(G, \{s_1, t_1\}, \dots, \{s_k, t_k\})$ . We note that  $(P^1, \dots, P^k)$  is a solution for  $(G^*, \{s_1, t_1\}, \dots, \{s_k, t_k\})$  unless there exists some  $P^i$  that contains an edge of  $E(G) \setminus E(G^*)$ . Suppose this is indeed the case. As  $G^*[D_1] = G[D_1]$  and every edge between a vertex of  $A_1 \cup A_2$  and a vertex of  $B_1 \cup B_2$  also exists in  $G^*$ , we find that the paths connecting terminals from pairs in  $D_1$  are paths in  $G^*$ . Hence,  $s_i$  and  $t_i$  belong to  $D_h$  for some  $h \in \{2, \dots, p\}$  and thus  $P^i = s_i u t_i$  where  $u$  is a vertex of  $D_j$  for some  $j \in \{2, \dots, p\}$  with  $j \neq h$ .

As we already found that  $(G, \{s_1, t_1\}, \dots, \{s_k, t_k\})$  has no type 1 solution, there is at least one  $P^{i'}$  with length 3, so  $P^{i'} = s_{i'} v v' t_{i'}$  is in  $G[D_1]$ . However, we

can now obtain another solution for  $(G, \{s_1, t_1\}, \dots, \{s_k, t_k\})$  by changing  $P^i$  into  $s_i v t_i$  and  $P^{i'}$  into  $s_i u t_{i'}$ , a contradiction, as the number of paths of length 3 in  $(P^1, \dots, P^k)$  was minimum. We conclude that every  $P^i$  only contains edges from  $E(G) \cap E(G^*)$ , and thus  $(P^1, \dots, P^k)$  is a solution for  $(G^*, \{s_1, t_1\}, \dots, \{s_k, t_k\})$ .

Now suppose that  $(G^*, \{s_1, t_1\}, \dots, \{s_k, t_k\})$  has a solution  $(P^1, \dots, P^k)$ . Consider a path  $P^i$ . First suppose that  $s_i$  and  $t_i$  both belong to  $B_1$ . Then we may assume without loss of generality that  $P^i = s_i u t_i$  for some  $u \in A_2$ . As  $B_1$  only contains terminals from pairs in  $D_2 \cup \dots \cup D_p$ , the latter implies that  $P^i$  is a path in  $G$  as well. Now suppose that  $s_i$  and  $t_i$  both belong to  $A_1$ . Then we may assume without loss of generality that  $P^i = s_i u t_i$  for some non-terminal vertex of  $V(G) = V(G^*)$  or  $P^i = s_i u u' t_i$  for two vertices  $u, u'$  in  $A_2 \subseteq D_1$ . Hence,  $P^i$  is a path in  $G$  as well. We conclude that  $(P^1, \dots, P^k)$  is a solution for  $(G, \{s_1, t_1\}, \dots, \{s_k, t_k\})$ . This completes our proof.  $\square$

## 6 Exact Algorithms

In this section, we briefly mention exact algorithms. Using Held-Karp type dynamic programming techniques [2, 10], we can obtain exact algorithms for DISJOINT PATHS and DISJOINT CONNECTED SUBGRAPHS running in time  $O(2^n n^2 k)$  and  $O(3^n k m)$ , respectively.

**Theorem 4.** DISJOINT PATHS can be solved in  $O(2^n n^2 k)$  time.

*Proof.* We devise a Held-Karp type [10, 2] dynamic programming algorithm. Given a set  $S \subseteq V(G)$ , a vertex  $v \in S$ , and an integer  $i \in \{1, \dots, k\}$ , let  $D[S, v, i]$  be true if and only if  $S$  can be partitioned into vertex-disjoint paths  $P^1, \dots, P^i$  such that  $P^i$  starts in  $s_i$  and ends in  $v$  and  $P^j$  is an  $s_j$ - $t_j$  path for each  $j \in \{1, \dots, i-1\}$ . Then we set  $D[S, v, 1]$  to true if and only if  $S$  is equal to the vertex set of an  $s_1$ - $v$  path. The correctness of the base case is immediate from the definition. Beyond the base case, we set  $D[S, s_i, i] = D[S \setminus \{s_i\}, t_{i-1}, i-1]$  and for all  $v \neq s_i$ ,  $D[S, v, i]$  is set to true if and only if there is a neighbour  $w \in S$  of  $v$  for which  $D[S \setminus \{v\}, w, i]$  is true. Indeed, if  $S$  can be partitioned into vertex-disjoint paths  $P^1, \dots, P^i$  such that  $P^i$  starts in  $s_i$  and ends in  $v$  and  $P^j$  is an  $s_j$ - $t_j$  path for each  $j \in \{1, \dots, i-1\}$ , then

- if  $v = s_i$ , then  $P^i$  is a single-vertex path and thus  $S \setminus \{s_i\}$  can be partitioned into vertex-disjoint paths  $P^1, \dots, P^{i-1}$  such that  $P^j$  is an  $s_j$ - $t_j$  path for each  $j \in \{1, \dots, i-1\}$ , and thus  $D[S \setminus \{s_i\}, t_{i-1}, i-1]$  is true;
- otherwise, let  $w$  be the vertex preceding  $v$  on  $P^i$ , and thus  $S \setminus \{v\}$  can be partitioned into vertex-disjoint paths  $P^1, \dots, P^{i-1}, Q^i$  such that  $Q^i$  starts in  $s_i$  and ends in  $w$  ( $Q^i$  is the part of  $P^i$  from  $s_i$  to  $w$ ) and  $P^j$  is an  $s_j$ - $t_j$  path for each  $j \in \{1, \dots, i-1\}$ , and thus  $D[S \setminus \{v\}, w, i]$  is true.

Conversely, if  $v = s_i$  and  $D[S \setminus \{s_i\}, t_{i-1}, i-1]$  is true, then  $S \setminus \{s_i\}$  can be partitioned into vertex-disjoint paths  $P^1, \dots, P^{i-1}$  such that  $P^j$  is an  $s_j$ - $t_j$  path for each  $j \in \{1, \dots, i-1\}$ , and thus  $S$  can be partitioned into vertex-disjoint paths  $P^1, \dots, P^i$  such that  $P^i$  starts and ends in  $s_i$  and  $P^j$  is an  $s_j$ - $t_j$  path

for each  $j \in \{1, \dots, i-1\}$ . Hence,  $D[S, s_i, i]$  is true. If  $v \neq s_i$  and there is a neighbour  $w \in S$  of  $v$  for which  $D[S \setminus \{v\}, w, i]$  is true, meaning that  $S \setminus \{v\}$  can be partitioned into vertex-disjoint paths  $P^1, \dots, P^i$  such that  $P^i$  starts in  $s_i$  and ends in  $w$  and  $P^j$  is an  $s_j$ - $t_j$  path for each  $j \in \{1, \dots, i-1\}$ , then  $S$  can be partitioned into vertex-disjoint paths  $P^1, \dots, P^{i-1}, Q^i$  such that  $Q^i$  starts in  $s_i$ , follows  $P^i$  and ends in  $v$ , and  $P^j$  is an  $s_j$ - $t_j$  path for each  $j \in \{1, \dots, i-1\}$ . Hence,  $D[S, v, i]$  is true.

Finally, the given instance of DISJOINT PATHS is a yes-instance if and only if there is a set  $S \subseteq V(G)$  for which  $D[S, t_k, k]$  is true. The correctness follows by definition.

It is immediate that the running time of the algorithm is  $O(2^n n^2 k)$ , as there are  $2^n n k$  table entries that each require at most  $O(n)$  time to fill.  $\square$

**Theorem 5.** DISJOINT CONNECTED SUBGRAPHS *can be solved in  $O(3^n k m)$  time.*

*Proof.* We propose a similar, but slightly more crude algorithm as the one before. Given a set  $S \subseteq V(G)$  and an integer  $i \in \{1, \dots, k\}$ , let  $D[S, i]$  be true if and only if  $S$  can be partitioned into vertex-disjoint set  $S_1, \dots, S_i$  such that  $S_j$  is connected and  $Z_j \subseteq S_j$  for each  $j \in \{1, \dots, i\}$ . We set  $D[S, 1]$  to true if and only if  $Z_1 \subseteq S$  and  $S$  is connected. Beyond the base case, we set  $D[S, i]$  to true if and only if there is a set  $S' \subset S$  for which  $Z_i \subseteq S'$ ,  $S'$  is connected, and  $D[S \setminus S', i-1]$  is true. Finally, the given instance of DISJOINT CONNECTED SUBGRAPHS is a yes-instance if and only if there is a set  $S \subseteq V(G)$  for which  $D[S, k]$  is true. The proof of correctness is similar (but simpler) to the proof of Theorem 4.

It is immediate that the running time is  $O(3^n k m)$ . Each table entry  $D[S, i]$  requires  $O(2^{|S|} m)$  time to fill. Hence, the running time to fill all table entries where  $S$  has size  $\ell$  is  $k \binom{n}{\ell} 2^\ell m$ . This means that the total running time is  $\sum_{\ell=0}^n \binom{n}{\ell} 2^\ell m k = O(3^n k m)$ , where the latter equality follows from the Binomial Theorem.  $\square$

## 7 Conclusions

We first gave a dichotomy for DISJOINT  $k$ -CONNECTED SUBGRAPHS in Theorem 2: for every  $k$ , the problem is polynomial-time solvable on  $H$ -free graphs if  $H \subseteq_i sP_1 + P_4$  for some  $s \geq 0$  and otherwise it is NP-complete even for  $k = 2$ . Two vertices  $u$  and  $v$  are a  $P_4$ -suitable pair if  $(G - \{u, v\}, N(u), N(v))$  is a yes-instance of 2-DISJOINT CONNECTED SUBGRAPHS. Recall that a graph  $G$  can be contracted to  $P_4$  if and only if  $G$  has a  $P_4$ -suitable pair. Deciding if a pair  $\{u, v\}$  is a suitable pair is polynomial-time solvable for  $H$ -free graphs if  $H$  is an induced subgraph of  $P_2 + P_4$ ,  $P_1 + P_2 + P_3$ ,  $P_1 + P_5$  or  $sP_1 + P_4$  for some  $s \geq 0$ ; otherwise it is NP-complete [15]. Hence, we conclude from our new result that the presence of the two vertices  $u$  and  $v$  that are connected to the sets  $Z_1 = N(u)$  and  $Z_2 = N(v)$ , respectively, yield exactly three additional polynomial-time solvable cases.

We also classified, in Theorem 3, the complexity of DISJOINT PATHS and DISJOINT CONNECTED SUBGRAPHS for  $H$ -free graphs. Due to Propositions 1 and 2, there are three non-equivalent open cases left and we ask the following:

**Open Problem 1.** Determine the computational complexity of DISJOINT PATHS on  $H$ -free graph for  $H \in \{3P_1, 2P_1 + P_2\}$  and the computational complexity of DISJOINT CONNECTED SUBGRAPHS on  $H$ -free graphs for  $H = 2P_1 + P_2$ .

The three open cases seem challenging. We were able to prove the following positive result for a subclass of  $3P_1$ -free graphs, namely cobipartite graphs, or equivalently,  $(3P_1, C_5, \overline{C_7}, \overline{C_9}, \dots)$ -free graphs.

**Theorem 6.** DISJOINT PATHS is polynomial-time solvable for cobipartite graphs.

*Proof.* Let  $G = (A \cup B, E)$ , with cliques  $A$  and  $B$ , be the given cobipartite graph. If  $s_i$  and  $t_i$  are adjacent in  $G$ , then use the direct edge between them as the path  $P^i$ . We can then reduce the instance by removing  $s_i$  and  $t_i$ . We now assume the instance has thus been reduced and (by abuse of notation) all terminal pairs are nonadjacent in  $G$ .

We now construct a bipartite graph  $G'$  by removing each edge within the cliques  $A$  and  $B$  as well as any edge  $s_i t_j$  both of whose endpoints are terminals. We then obtain a new graph  $G''$  by deleting each terminal vertex and adding for each terminal pair  $(s_i, t_i)$ , a new vertex  $x_i$  whose neighbourhood is the union of the neighbourhoods of  $s_i$  and  $t_i$  in  $G'$ . We claim that  $G$  contains the required  $k$  disjoint paths  $P^1 \dots P^k$  if and only if  $G''$  contains a matching of size at least  $k$ . We can check the latter in polynomial time by using the Hopcroft-Karp algorithm for bipartite graphs [11].

We first assume that  $G$  contains the disjoint paths  $P^1 \dots P^k$ . Note that, since  $G$  is  $3P_1$ -free, we may assume each path has length at most 3. A matching  $M$  of size  $k$  is obtained as follows. For each  $i = 1 \dots k$ , if  $P^i$  has length 2 we add the edge  $x_i v_i$  to  $M$  where  $v_i$  is the interior vertex of  $P^i$ . If  $P^i$  has length 3 then we add its interior edge  $u_i v_i$  to  $M$ .

Next assume  $G''$  contains a matching  $M$  of size  $k$ . For each edge of  $M$  which includes a vertex  $x_i$  corresponding to a terminal pair  $(s_i, t_i)$  we set  $P^i$  to be  $s_i v_i t_i$  where  $v_i$  is the vertex matched to  $x_i$ . Note that any edge  $uv$  in  $G$  which contains no terminal vertex and has one endpoint in each of  $A$  and  $B$  lies on a path of length 3 between any two terminal vertices. Therefore, for each  $i$  such that the vertex  $x_i$  is not matched in  $M$ , we can choose a distinct edge  $u_i v_i$  in  $M$  to obtain the path  $s_i u_i v_i t_i$  in  $G$ .  $\square$

Finally, in Section 6 we obtained exact algorithms for DISJOINT PATHS and DISJOINT CONNECTED SUBGRAPHS running in time  $O(2^n n^2 m)$  and  $O(3^n km)$ , respectively. Faster exact algorithms are known for  $k$ -DISJOINT CONNECTED SUBGRAPHS for  $k = 2$  and  $k = 3$  [4, 21, 1], but we are unaware if there exist faster algorithms for general graphs.

**Open Problem 2.** Is there an exact algorithm for DISJOINT PATHS or DISJOINT CONNECTED SUBGRAPHS on general graphs where the exponential factor is  $(2-\epsilon)^n$  or  $(3-\epsilon)^n$ , respectively, for some  $\epsilon > 0$ ?

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