The geometry of random walk isomorphism theorems

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Abstract

The classical random walk isomorphism theorems relate the local times of a continuous-time random walk to the square of a Gaussian free field. A Gaussian free field is a spin system that takes values in Euclidean space, and this article generalises the classical isomorphism theorems to spin systems taking values in hyperbolic and spherical geometries. The corresponding random walks are no longer Markovian: they are the vertex-reinforced and vertex-diminished jump processes. We also investigate supersymmetric versions of these formulas.

Our proofs are based on exploiting the continuous symmetries of the corresponding spin systems. The classical isomorphism theorems use the translation symmetry of Euclidean space, while in hyperbolic and spherical geometries the relevant symmetries are Lorentz boosts and rotations, respectively. These very short proofs are new even in the Euclidean case.

Isomorphism theorems are useful tools, and to illustrate this we present several applications. These include simple proofs of exponential decay for spin system correlations, exact formulas for the resolvents of the joint processes of random walks together with their local times, and a new derivation of the Sabot–Tarrès formula for the limiting local time of the vertex-reinforced jump process.

Contents

1 Introduction

Random walk isomorphism theorems refer to a class of distributional identities that relate the local times of Markov processes to the squares of Gaussian fields. These theorems, which connect two different types of probabilistic objects, have their origins in the work of the physicist K.

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Symanzik [51]. Isomorphism theorems have been useful in the investigation of a variety of phenomena, and they can be used in two directions: to study field theoretic questions in terms of random walks, and to study random walks in terms of field theory. An incomplete list of topics investigated via isomorphism theorems includes: local times of Markov processes [35] and their large deviations [9, 14]; cover times and thick points of the simple random walk [1, 19, 29]; four-dimensional self-avoiding walk [4, 7]; ϕ^4 field theory [10, 11, 28]; and random walk loop soups [32, 52].

The purpose of this article is to expand the scope of isomorphism theorems beyond Gaussian fields. Namely, we describe, and make use of, isomorphism theorems that relate *non-Markovian* stochastic processes to *non-Gaussian* spin systems. Our proofs also provide a new perspective on isomorphism theorems: they are consequences of the symmetries of the underlying spin systems.

In Section 1.1 below we give an introduction to isomorphism theorems and the processes this article is concerned with. Before doing this, we briefly summarise the new results contained in this article:

- New and efficient proofs of the Brydges–Fröhlich–Spencer–Dynkin (BFS–Dynkin), Eisenbaum, and second generalised Ray–Knight isomorphism theorems for the simple random walk (SRW). These results are all derived in a few pages from a more general Ward identity for the Gaussian free field.

- New and efficient proofs of supersymmetric versions of the isomorphism theorems for the SRW. In particular, we prove a previously unknown supersymmetric version of the generalised second Ray–Knight isomorphism. For the reader's convenience we also present an introduction to supersymmetry directed towards probabilists in an appendix.

- New isomorphism theorems connecting the vertex-reinforced jump process (VRJP) with hyperbolic sigma models, and supersymmetric versions of these theorems. The analogue of the BFS–Dynkin isomorphism previously appeared in [5], and here we also establish analogues of the Eisenbaum and Ray–Knight isomorphism theorems. Our proofs are geometric and do not rely on any particular set of coordinates. In particular, we do not use horospherical coordinates.

- New isomorphism theorems for the vertex-diminished jump process (VDJP). The VDJP is connected to a spin model taking values in the hemisphere. It previously appeared in the context of the Ray–Knight isomorphism theorem for SRW in [43].

We also give several applications of these isomorphism theorems. In Section 6 we show that the Sabot–Tarrès limit formula for the local time of the VRJP [42] is a direct consequence of our supersymmetric Ray–Knight theorem for the $\mathbb{H}^{2|2}$ model. In Section 7 we show how isomorphism theorems yield fixed-time formulas and representations of the resolvents for the joint processes of the random walks together with their local times. Lastly, we prove some results concerning exponential decay of correlation functions for the associated spin models in Section 8.

1.1. Isomorphism theorems for hyperbolic and spherical geometries. Let X_t be a continuous-time stochastic process on a finite state space Λ with associated local times $\boldsymbol{L}_t = (L_t^i)_{i\in\Lambda}$. The processes considered in this paper are all of the form

$$\mathbb{P}[X_{t+dt} = j \mid (X_s)_{s\leqslant t}, X_t = i] = \beta_{ij}(1 + \varepsilon L_t^j)\, dt, \qquad \varepsilon \in \{-1, 0, 1\}, \tag{1.1}$$

where $\beta_{ij} \geqslant 0$ and $\beta_{ij} = \beta_{ji}$ for all $i, j \in \Lambda$.

The random walk models defined by (1.1) are defined more precisely below. The models have all appeared previously, though they have received varying amounts of attention. When $\varepsilon = 0$ the model is the *continuous-time simple random walk*; for $\varepsilon = 1$ it is the *vertex-reinforced jump process* (VRJP) first studied in [17, 18]; for $\varepsilon = -1$ it is the *vertex-diminished jump process* (VDJP) which appeared in [43]. As the names suggest, the VRJP is a random walk that is encouraged to revisit vertices it has visited in the past, while the VDJP is discouraged from doing so.

Let $\mathbb{R}^n$ denote n-dimensional Euclidean space, $\mathbb{H}^n$ denote n-dimensional hyperbolic space, and let $\mathbb{S}^n_+$ denote the upper hemisphere of the n-dimensional sphere. Below we will introduce spin systems that take values in these spaces, and then link these to the aforementioned random walks. The spin systems are the $\mathbb{R}^n$-valued *Gaussian free field* (GFF), corresponding to the SRW; the $\mathbb{H}^n$-valued *hyperbolic spin model*, corresponding to the VRJP; and the $\mathbb{S}^n_+$-valued *hemispherical spin model*, corresponding to the VDJP.

To give a flavour of the relationships that we will establish, recall Dynkin's formulation of an isomorphism linking the SRW and the $\mathbb{R}$-valued GFF [24]. Let $G = (\Lambda, E)$ be a finite graph, $h > 0$, and let $\langle\cdot\rangle$ denote the expectation of a GFF $(u_i)_{i\in\Lambda}$ with covariance $(-\Delta + h)^{-1}$. This is often called the *massive GFF* with *mass* $m = \sqrt{h}$. Let $\mathbb{E}_i$ denote the expectation of a continuous-time SRW X_t with associated local time field $\boldsymbol{L}_t = (L^i_t)_{i\in\Lambda}$, started from $i \in \Lambda$, with X_t independent of the GFF. Then for all bounded $g\colon \mathbb{R}^\Lambda \to \mathbb{R}$,

$$\Big\langle u_i u_j g(\frac{1}{2}\boldsymbol{u}^2)\Big\rangle = \Big\langle \int_0^\infty \mathbb{E}_i(g(\frac{1}{2}\boldsymbol{u}^2 + \boldsymbol{L}_t)1_{X_t=j})\, e^{-ht}\, dt\Big\rangle, \qquad \boldsymbol{u}^2 \equiv (u_i^2)_{i\in\Lambda}. \tag{1.2}$$

The left-hand side is a generalization of the spin-spin correlation between the spins u_i and u_j of the GFF. In particular, taking $g = 1$ in (1.2) reveals the well-known fact that the second moments of the massive GFF are given by the Green's function of a SRW killed at rate h.

In Theorems 3.3 and 4.4 we establish analogues of (1.2) for the hyperbolic and hemispherical spin models; the hyperbolic case first appeared in [5]. Our methods also allow us to establish other isomorphism theorems. In particular, we give new proofs of the Eisenbaum isomorphism theorem [26] and of the generalised second Ray–Knight theorem [27] for the GFF, and we establish analogues of these results for hyperbolic and hemispherical spin models. Our proofs apply to n-component spin systems for general $n \in \mathbb{N} = \{1, 2, \dots\}$ in all cases, and even for the GFF some of these results are new when $n > 1$. To ease our exposition we will refer to the generalised second Ray–Knight theorem as the Ray–Knight isomorphism in what follows.

1.2. Supersymmetric isomorphism theorems. There is another type of isomorphism that relates the simple random walk to a spin system, in which the GFF is replaced by the *supersymmetric Gaussian free field (SUSY GFF)*. These isomorphisms originated in work of McKane [37] and Parisi and Sourlas [41]. Supersymmetry has played a role in several interesting probabilistic problems [12, 13, 16, 23], and several of the applications we mentioned in the opening paragraph of this article involve the SUSY GFF [4, 7, 9, 14, 32].

The most important aspect of the SUSY isomorphism for the SRW is immediately apparent from the statement of the result, and hence we defer a careful definition of the SUSY GFF to Section 5. Let $\langle\cdot\rangle$ now denote the expectation with respect to the SUSY GFF. The SUSY isomorphism theorem is that for all smooth and bounded $g\colon \mathbb{R}^\Lambda \to \mathbb{R}$,

$$\Big\langle u_i^1 u_j^1 g(\frac{1}{2}|\boldsymbol{u}|^2)\Big\rangle = \int_0^\infty \mathbb{E}_i(g(\boldsymbol{L}_t)1_{X_t=j})\, e^{-ht}\, dt, \qquad |\boldsymbol{u}|^2 \equiv (|u_i|^2)_{i\in\Lambda}. \tag{1.3}$$

The key point of (1.3) is that the right-hand side only involves the simple random walk, while the left-hand side involves only the components $(u_i)_{i\in\Lambda}$ of the SUSY GFF. Thus questions about the local time of random walk can be rephrased purely in terms of the SUSY GFF.

The viewpoint that isomorphism theorems arise as a consequence of continuous symmetries applies equally well to supersymmetric spin systems. Beyond proving (1.3), Section 5 also establishes results analogous to (1.3) for the supersymmetric $\mathbb{H}^{2|2}$ and $\mathbb{S}^{2|2}_+$ models, and moreover we prove a SUSY variant of the Ray–Knight isomorphism. This is new even for the simple random walk. We emphasise that these theorems give direct access to the local times of the non-Markovian VRJP and VDJP in terms of the spin models. The analogue of (1.3) for $\mathbb{H}^{2|2}$ first appeared in [5].

1.3. Proof ideas. Our proofs of isomorphism theorems all follow a common strategy. The spin systems we consider possess continuous symmetries, and as a result satisfy integration by parts formulas that are called *Ward identities* in the physics literature. Isomorphism theorems are a direct consequence of these Ward identities.

A key step is to consider a random walk X_t to be a marginal of the joint process $(X_t, \boldsymbol{L}_t)$ of the walk and its local times together. Our Ward identities can be rephrased in terms of the infinitesimal generator of this joint process, and *all* of our isomorphism theorems follow quite quickly by choosing appropriate specializations of the Ward identities. In particular, this gives a unified set of proofs of the BFS–Dynkin, Eisenbaum, and Ray–Knight isomorphism theorems for the SRW.

1.4. Structure of this article. Section 2 gives our new proofs of the classical isomorphism theorems that link random walks to Gaussian fields. We present our arguments in detail in this familiar context as very similar ideas are used in Sections 3 and 4, which derive isomorphism theorems for the VRJP and VDJP. We derive supersymmetric isomorphisms for the SRW, the VRJP, and the VDJP in Section 5, and Sections 6 through 8 concern applications of our new isomorphisms.

To keep this article self-contained, Appendix A contains an introduction to the parts of supersymmetry needed to understand our supersymmetric isomorphisms and their applications. In Appendix B we discuss some further aspects of symmetries and supersymmetries that are not needed for our results, but that help place the results of this article in context.

1.5. Related literature and future directions.

Related literature. For monograph-length treatments of isomorphism theorems and related topics, e.g., loop soups, see [35, 52]. Many proofs of various isomorphism theorems have been given; here we mention only the recent [30, 43]. The major innovation in the present work is that we do not rely on Gaussian calculations. This is important both for obtaining results for $\mathbb{H}^n$ and $\mathbb{S}^n_+$, and for obtaining supersymmetric variants.

Future directions. This article describes isomorphism theorems that link spin systems on $\mathbb{R}^n$, $\mathbb{H}^n$, and $\mathbb{S}^n_+$ (and the supersymmetric versions when $n = 2$) to random walks. This provides a partial answer to a question of Kozma [31], who asked if there are other spin models (beyond the $\mathbb{H}^{2|2}$ model) with associated random walks. The development of a more systematic connection between spin models and random walks would be very interesting. In particular, it is natural to wonder if there are geometric spaces beyond $\mathbb{R}^n$, $\mathbb{H}^n$, and $\mathbb{S}^n_+$ that have associated isomorphism theorems.

Another interesting future direction would be to clarify the relation between our new isomorphism theorems and loop soups. In the setting of the SRW this connection is well-developed [35,52] — do these connections extend to the VRJP and VDJP? Similar questions can be asked about random interlacements; for recent progress in this direction see [39].

1.6. Notation and conventions. Λ will be a finite set and $\beta = (\beta_{ij})_{i,j\in\Lambda}$ will be a set of edge weights, i.e., $\beta_{ij} = \beta_{ji} \geqslant 0$. The edge weights induce a graph with vertices Λ and edge set $\{\{i,j\} \mid \beta_{ij} > 0\}$, and we will assume that this graph is connected. We also let $\boldsymbol{h} = (h_i)_{i\in\Lambda}$ denote a set of non-negative vertex weights; here we are setting a convention that bold symbols denote objects indexed by Λ. Both β and $\boldsymbol{h}$ will play the role of parameters in our models. For typographical reasons we will sometimes write h in place of $\boldsymbol{h}$ when there is no risk of confusion.

Suppose V is a set equipped with a binary operation $(x, y) \mapsto x \cdot y$. We write V^Λ for the set of maps from Λ to V, denote elements of this set by $\boldsymbol{u} = (u_i)_{i\in\Lambda}$, and let $|\boldsymbol{u}|^2 = (u_i \cdot u_i)_{i\in\Lambda}$. If elements of V are vectors, e.g., $u_i = (u_i^1, \dots, u_i^n) \in \mathbb{R}^n$, then we write $\boldsymbol{u}^\alpha = (u_i^\alpha)_{i\in\Lambda}$ for the collection of α^{th} components.

For a function $f\colon \mathbb{R} \to \mathbb{R}$ we often impose that f is smooth and has *rapid decay*. A sufficient condition is that f and its derivatives decay faster than any polynomial: for every p and k, there are constants $C_{p,k}$ such that the kth derivative satisfies $|f^{(k)}(u)| \leqslant C_{p,k}|u|^{-p}$. If $f\colon \mathbb{R}^n \to \mathbb{R}$, $(u_1, \dots, u_n) \mapsto f(u_1, \dots, u_n)$, then we say f has *rapid decay in* u_1 if $f(\cdot, u_2, \dots, u_n)$ has rapid decay with constants uniform in $u_2, \dots, u_n$. Rapid decay in u_j is defined analogously, and we say such an f has *rapid decay* if it has rapid decay in some coordinate. For a non-smooth function f, we say that f has rapid decay if the the above holds with $k = 0$.

Similarly, we often impose that $f\colon \mathbb{R}^n \to \mathbb{R}^m$ has *moderate growth.* A sufficient condition is that f has at most polynomial growth, i.e., there exists q and C_k such that $|\nabla^k f(u)| \leqslant C_k |u|^q$ for all k.

Given a function $f\colon \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$, $(i, \boldsymbol{\ell}) \mapsto f(i, \boldsymbol{\ell})$ we say f is *smooth*, *rapidly decaying*, etc. if it has this property with respect to its second coordinate $\boldsymbol{\ell}$. Throughout we will assume functions are Borel measurable without making this explicit.

2 Isomorphism theorems for flat geometry

In this section we introduce the simple random walk, the corresponding Gaussian free field, and several well-known isomorphism theorems relating these objects. The method of proof will be used repeatedly in the remainder of the paper when we consider other spin systems. An important aspect of the proofs is that they do not rely on explicit Gaussian computations; this is essential for the generalization of these theorems to non-Gaussian spin systems. Our proofs also show that these results are true for GFFs with any number of components.

2.1. Simple random walk and Gaussian free field.

Simple random walk. The continuous-time *simple random walk* (SRW) on Λ with symmetric edge weights $\beta \equiv (\beta_{ij})_{i,j\in\Lambda}$, i.e., $\beta_{ij} = \beta_{ji} \geqslant 0$, is the Markov jump process $(X_t)_{t\geqslant 0}$ with transition rates

$$\mathbb{P}[X_{t+dt} = i \mid X_t = j] = \beta_{ij}\, dt. \tag{2.1}$$

We write $\mathbb{P}_i$ and $\mathbb{E}_i$ for the law and expectation of X_t when it is started from the vertex i. Formally, X_t is a continuous-time Markov process with generator Δ_β, where the Laplacian Δ_β is the matrix indexed by Λ that acts on $f\colon \Lambda \to \mathbb{R}$ by

$$(\Delta_\beta f)(i) \equiv \sum_{j\in\Lambda} \beta_{ij}(f(j) - f(i)). \tag{2.2}$$

In what follows it will be useful to view X_t as a marginal of the Markov process $(X_t, \boldsymbol{L}_t)_{t\geqslant 0}$ consisting of X_t and its *local times* $\boldsymbol{L}_t \equiv (L_t^i)_{i\in\Lambda}$, which are defined by

$$L_t^i \equiv L_0^i + \int_0^t 1_{X_s = i}\, ds, \qquad i \in \Lambda, \tag{2.3}$$

where the vector $\boldsymbol{L}_0$ is a collection of free parameters called the *initial local time.* A short computation shows that the generator of $(X_t, \boldsymbol{L}_t)$ acts on smooth functions $f\colon \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ by

$$(\mathcal{L}f)(i, \boldsymbol{\ell}) = (\Delta_\beta f)(i, \ell) + \frac{\partial f(i, \boldsymbol{\ell})}{\partial \ell_i}, \quad \text{i.e.,} \quad \boldsymbol{\mathcal{L}f} = \Delta_\beta \boldsymbol{f} + \partial \boldsymbol{f}, \tag{2.4}$$

where Δ_β only acts on the first argument and the last equation uses the vector notation

$$\boldsymbol{f} \equiv (f(i, \boldsymbol{\ell}))_{i\in\Lambda}, \quad \partial \boldsymbol{f} \equiv (\frac{\partial f(i, \boldsymbol{\ell})}{\partial \ell_i})_{i\in\Lambda}. \tag{2.5}$$

We write $\mathbb{P}_{i,\boldsymbol{\ell}}$ for the law of $(X, \boldsymbol{L})$ started at $(i, \boldsymbol{\ell}) \in \Lambda \times \mathbb{R}^\Lambda$, and $\mathbb{E}_{i,\boldsymbol{\ell}}$ for its expectation. Note that $\mathbb{E}_{i,\boldsymbol{\ell}} f(X_t, \boldsymbol{L}_t) = \mathbb{E}_{i,\boldsymbol{0}} f(X_t, \boldsymbol{\ell} + \boldsymbol{L}_t)$, and in particular that $f_t(i, \boldsymbol{\ell}) \equiv \mathbb{E}_{i,\boldsymbol{\ell}} f(X_t, \boldsymbol{L}_t)$ is a smooth function with rapid decay in $\boldsymbol{\ell}$ if f is smooth with rapid decay.

Gaussian free field. The (n-component) *Gaussian free field* (*GFF* or $\mathbb{R}^n$ *model*) is a spin system taking values in $\mathbb{R}^n$. Its configurations are elements $\boldsymbol{u} \in (\mathbb{R}^n)^\Lambda$; by an abuse of notation we will write $\mathbb{R}^{n\Lambda}$ in place of $(\mathbb{R}^n)^\Lambda$. Let $\boldsymbol{h} = (h_i)_{i\in\Lambda}$, and assume $h_i \geqslant 0$. To define the probability of a configuration, let

$$H_\beta(\boldsymbol{u}) \equiv \frac{1}{2}(\boldsymbol{u}, -\Delta_\beta \boldsymbol{u}), \qquad H_{\beta,h}(\boldsymbol{u}) \equiv H_\beta(\boldsymbol{u}) + \frac{1}{2}(\boldsymbol{h}, |\boldsymbol{u}|^2), \tag{2.6}$$

where $(\boldsymbol{f}, \boldsymbol{g}) \equiv \sum_{i\in\Lambda} f_i g_i$, $|\boldsymbol{u}|^2 \equiv (u_i \cdot u_i)_{i\in\Lambda}$, and $\cdot$ is the Euclidean inner product. In (2.6) the Laplacian acts diagonally on the n components of $\boldsymbol{u}$, i.e., $\Delta_\beta \boldsymbol{u} = (\Delta_\beta \boldsymbol{u}^\alpha)_{\alpha=1}^n$, and hence (2.6) can be rewritten using

$$(\boldsymbol{u}, -\Delta_\beta \boldsymbol{u}) = \frac{1}{2}\sum_{i,j\in\Lambda} \beta_{ij}(u_i - u_j)^2, \qquad (\boldsymbol{h}, |\boldsymbol{u}|^2) = \sum_{i\in\Lambda} h_i u_i \cdot u_i, \tag{2.7}$$

where $(u_i - u_j)^2$ is shorthand for $(u_i - u_j)\cdot(u_i - u_j)$. Note that another common notation is $h_i = m_i^2 \geqslant 0$, and m_i is called the *mass* at the vertex i. Define the unnormalised expectation $[\cdot]_{\beta,h}$ on functions $F\colon \mathbb{R}^{n\Lambda} \to \mathbb{R}$ by

$$[F]_{\beta,h} \equiv \int_{\mathbb{R}^{n\Lambda}} F(\boldsymbol{u}) e^{-H_{\beta,h}(\boldsymbol{u})} d\boldsymbol{u}, \tag{2.8}$$

where the integral is with respect to Lebesgue measure $d\boldsymbol{u}$ on $\mathbb{R}^{n\Lambda}$. We set $[\cdot]_\beta \equiv [\cdot]_{\beta,0}$.

The *Gaussian free field* is the probability measure on $\mathbb{R}^{n\Lambda}$ defined by the normalised expectation

$$\langle F\rangle_{\beta,h} \equiv \frac{1}{Z_{\beta,h}}[F]_{\beta,h} = \frac{[Fe^{-\frac{1}{2}(\boldsymbol{h},|\boldsymbol{u}|^2)}]_\beta}{[e^{-\frac{1}{2}(\boldsymbol{h},|\boldsymbol{u}|^2)}]_\beta}, \qquad Z_{\beta,h} \equiv [1]_{\beta,h}. \tag{2.9}$$

Note that for the expectation in (2.9) to be well-defined we must have $Z_{\beta,h} < \infty$; this is the case if and only if $h_i > 0$ for some i. The divergence if $\boldsymbol{h} = \boldsymbol{0}$ is due to the invariance of $H_\beta(\boldsymbol{u})$ under the simultaneous translation $u_i \mapsto u_i + s$ for any $s \in \mathbb{R}^n$.

2.2. Fundamental integration by parts identity. For any differentiable $f\colon \mathbb{R}^{n\Lambda} \to \mathbb{R}$ we write

$$T_j f \equiv \frac{\partial f}{\partial u_j^1}, \qquad \boldsymbol{T} f \equiv (T_i f)_{i\in\Lambda}. \tag{2.10}$$

Thus T_j is the infinitesimal generator of translations of the j^{th} coordinate in the direction $e^1 = (1, 0, \dots, 0) \in \mathbb{R}^n$. The following lemma is a consequence of the translation invariance of Lebesgue measure, and we will derive all of our isomorphism theorems from this identity. In later sections of this paper we will derive analogous results by replacing the translation symmetry by different symmetries.

Lemma 2.1. *Let $[\cdot]_\beta$ be the unnormalised expectation of the $\mathbb{R}^n$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the SRW. Let $f\colon \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ be smooth with rapid decay, and let $\rho\colon \mathbb{R}^{n\Lambda} \to \mathbb{R}$ be smooth with moderate growth. Then:*

$$-\sum_{j\in\Lambda}\Big[\rho(\boldsymbol{u}) u_j^1 \mathcal{L} f(j, \frac{1}{2}|\boldsymbol{u}|^2)\Big]_\beta = \sum_{j\in\Lambda}\Big[(T_j\rho)(\boldsymbol{u}) f(j, \frac{1}{2}|\boldsymbol{u}|^2)\Big]_\beta. \tag{2.11}$$

In particular, the following integrated version holds for all $f\colon \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ with rapid decay:

$$\sum_{j\in\Lambda}\Big[\rho(\boldsymbol{u}) u_j^1 f(j, \frac{1}{2}|\boldsymbol{u}|^2)\Big]_\beta = \sum_{j\in\Lambda}\Big[(T_j\rho)(\boldsymbol{u}) \int_0^\infty \mathbb{E}_{j,\frac{1}{2}|\boldsymbol{u}|^2}(f(X_t, \boldsymbol{L}_t))\, dt\Big]_\beta. \tag{2.12}$$

Remark 2.2. Using (2.5) and with $(\boldsymbol{T}, \boldsymbol{f}) \equiv \sum_{i\in\Lambda} T_i f_i$, (2.11) can be restated compactly as

$$-\Big[(\rho(\boldsymbol{u})\boldsymbol{u}^1, (\mathcal{L}\boldsymbol{f})(\frac{1}{2}|\boldsymbol{u}|^2))\Big]_\beta = \Big[(\boldsymbol{T}\rho(\boldsymbol{u}), \boldsymbol{f}(\frac{1}{2}|\boldsymbol{u}|^2))\Big]_\beta. \tag{2.13}$$

Proof. We first prove (2.11) by integration by parts. If $f_1, f_2 \colon \mathbb{R}^{n\Lambda} \to \mathbb{R}$ are differentiable and have rapid decay, then integration by parts implies

$$[(T_j f_1) f_2]_\beta = \big[f_1(T_j^\star f_2)\big]_\beta, \tag{2.14}$$

where, for $f \colon \mathbb{R}^{n\Lambda} \to \mathbb{R}$ differentiable,

$$T_j^\star f(\boldsymbol{u}) \equiv -T_j f(\boldsymbol{u}) + (T_j H_\beta(\boldsymbol{u})) f(\boldsymbol{u}). \tag{2.15}$$

We now compute the right-hand side of (2.15). To simplify notation, let $x_i \equiv u_i^1$ and $\boldsymbol{x} \equiv (x_i)_{i\in\Lambda}$. By (2.6), (2.2), and using that T_j is the derivative in the x-component,

$$T_j H_\beta(\boldsymbol{u}) = \frac{1}{2} T_j \sum_{i\in\Lambda} u_i \cdot (-\Delta u)_i = (-\Delta_\beta \boldsymbol{x})_j, \tag{2.16}$$

so that for a function of the form $f(\frac{1}{2}|\boldsymbol{u}|^2)$,

$$-T_j^\star f(\frac{1}{2}|\boldsymbol{u}|^2) = (\Delta_\beta \boldsymbol{x})_j f(\frac{1}{2}|\boldsymbol{u}|^2) + x_j \frac{\partial f(\frac{1}{2}|\boldsymbol{u}|^2)}{\partial \ell_j}, \tag{2.17}$$

where the last term denotes a partial derivative with respect to the jth coordinate of the function f. By applying (2.17) to each of the functions $f(j, \frac{1}{2}|\boldsymbol{u}|^2)$ and using $(\boldsymbol{f}_1, \Delta_\beta \boldsymbol{f}_2) = (\Delta_\beta \boldsymbol{f}_1, \boldsymbol{f}_2)$,

$$-\sum_{j\in\Lambda} T_j^\star f(j, \frac{1}{2}|\boldsymbol{u}|^2) = \sum_{j\in\Lambda} x_j \left[\Delta_\beta f(j, \frac{1}{2}|\boldsymbol{u}|^2) + \frac{\partial f(j, \frac{1}{2}|\boldsymbol{u}|^2)}{\partial \ell_j}\right] = \sum_{j\in\Lambda} x_j (\mathcal{L} f)(j, \frac{1}{2}|\boldsymbol{u}|^2). \tag{2.18}$$

To verify (2.11), multiply (2.18) by ρ and use the result to rewrite the left-hand side of (2.11). The desired equation then follows by applying (2.14):

$$-\sum_{j\in\Lambda}\Big[\rho x_j \mathcal{L} f(j, \frac{1}{2}|\boldsymbol{u}|^2)\Big]_\beta = \sum_{j\in\Lambda}\Big[\rho T_j^\star f(j, \frac{1}{2}|\boldsymbol{u}|^2)\Big]_\beta = \sum_{j\in\Lambda}\Big[(T_j\rho) f(j, \frac{1}{2}|\boldsymbol{u}|^2)\Big]_\beta.$$

We now prove (2.12); it suffices to consider f smooth with rapid decay. Indeed, if f_ε is the convolution of f with a smooth mollifier in the second argument, one has $f_\varepsilon \to f$ pointwise and the f_ε are bounded uniformly in ε by a function with rapid decay, so by dominated convergence the result for f follows from the result for the f_ε. Let $f_t(i, \boldsymbol{\ell}) \equiv \mathbb{E}_{i,\boldsymbol{\ell}}(f(X_t, \boldsymbol{L}_t))$, and note that f_t is a smooth function with rapid decay since f has this property (see below (2.5)). Apply (2.11) to f_t and rewrite the left-hand side using Kolmogorov's backward equation, i.e., $\mathcal{L} f_t = \partial_t f_t$. The result is

$$-\frac{\partial}{\partial t}\sum_{j\in\Lambda}\Big[\rho(\boldsymbol{u}) u_j^1 \mathbb{E}_{j,\frac{1}{2}|\boldsymbol{u}|^2}(f(X_t, \boldsymbol{L}_t))\Big]_\beta = \sum_{j\in\Lambda}\Big[(T_j\rho)(\boldsymbol{u})\, \mathbb{E}_{j,\frac{1}{2}|\boldsymbol{u}|^2} f(X_t, \boldsymbol{L}_t)\Big]_\beta. \tag{2.19}$$

To conclude, integrate (2.19) over $(0, \infty)$. The result follows since the boundary term at infinity on the left-hand side vanishes. To see this last claim, recall that the graph induced by β is finite and connected, so $L_t^i \to \infty$ in probability for all vertices $i \in \Lambda$. When f has sufficient decay this implies

$$\lim_{T\to\infty} \mathbb{E}_{j,\frac{1}{2}|\boldsymbol{u}|^2} f(X_T, \boldsymbol{L}_T) = 0 \tag{2.20}$$

for all $\boldsymbol{u}$. If f has sufficient decay and ρ has moderate growth then (2.20) implies

$$\lim_{T\to\infty} [\rho(\boldsymbol{u}) \mathbb{E}_{j,\frac{1}{2}|\boldsymbol{u}|^2} f(X_T, \boldsymbol{L}_T)]_\beta = 0 \tag{2.21}$$

by dominated convergence, as desired. This completes the proof of (2.12). $\square$

Our proofs of the classical isomorphism theorems will apply Lemma 2.1 with the following choices of ρ and f; further details will be given in the proofs.

- BFS–Dynkin isomorphism: $\rho(\boldsymbol{u}) = u_a$ and $f(j, \boldsymbol{\ell}) = g(\boldsymbol{\ell})1_{j=b}$ with $a, b \in \Lambda$;
- Ray–Knight isomorphism: $T_a\rho(\boldsymbol{u}) \to \delta(u_a) - \delta(u_a - s)$ and $f(j, \boldsymbol{\ell}) \to g(\boldsymbol{\ell})\delta(\ell_a - \frac{s^2}{2})1_{j=a}$;
- Eisenbaum isomorphism: $\rho(\boldsymbol{u}) = \exp(s(\boldsymbol{h}, \boldsymbol{u}) - \frac{s^2}{2}(\boldsymbol{h}, \mathbf{1}))$ and $f(j, \boldsymbol{\ell}) = g(\boldsymbol{\ell})e^{-(h,\boldsymbol{\ell})}1_{j=a}$.

2.3. BFS–Dynkin isomorphism theorem. We now prove the BFS–Dynkin isomorphism theorem.

Theorem 2.3. *Let $[\cdot]_\beta$ be the unnormalised expectation of the $\mathbb{R}^n$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the SRW. Let $g\colon \mathbb{R}^\Lambda \to \mathbb{R}$ have rapid decay, and let $a, b \in \Lambda$. Then:*

$$\left[u_a^1 u_b^1\, g(\frac{1}{2}|\boldsymbol{u}|^2)\right]_\beta = \left[\int_0^\infty \mathbb{E}_{a,\frac{1}{2}|\boldsymbol{u}|^2}(g(\boldsymbol{L}_t))1_{X_t=b}\, dt\right]_\beta. \tag{2.22}$$

Proof. Apply Lemma 2.1 with $\rho(\boldsymbol{u}) = u_a^1$, $f(j, \boldsymbol{\ell}) = g(\boldsymbol{\ell})1_{j=b}$, and use $T_j\rho(\boldsymbol{u}) = 1_{j=a}$. □

If $\boldsymbol{h} \neq \mathbf{0}$, after replacing $g(\boldsymbol{\ell})$ by $g(\boldsymbol{\ell})e^{-(\boldsymbol{h},\boldsymbol{\ell})}$ in (2.22) the unnormalised expectation can be normalised using (2.9). Since $\mathbb{E}_{a,\boldsymbol{\ell}}(g(\boldsymbol{L}_t)) = \mathbb{E}_a(g(\boldsymbol{L}_t + \boldsymbol{\ell}))$ for the simple random walk, we immediately obtain Dynkin's formulation of this theorem as stated, e.g., in [52, Theorem 2.8].

Corollary 2.4. *Let $\langle\cdot\rangle_\beta$ be the expectation of the $\mathbb{R}^n$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the SRW. Let $g\colon \mathbb{R}^\Lambda \to \mathbb{R}$ be bounded, $a, b \in \Lambda$, and suppose $\boldsymbol{h} \neq \mathbf{0}$. Then*

$$\left\langle u_a^1 u_b^1\, g(\frac{1}{2}|\boldsymbol{u}|^2)\right\rangle_{\beta,h} = \left\langle \int_0^\infty \mathbb{E}_a\Big(g(\boldsymbol{L}_t + \frac{1}{2}|\boldsymbol{u}|^2)e^{-(\boldsymbol{h},\boldsymbol{L}_t)}1_{X_t=b}\Big)\, dt\right\rangle_{\beta,h}. \tag{2.23}$$

We have rebranded this the BFS–Dynkin isomorphism because a version of Corollary 2.4 first appeared in the work of Brydges, Fröhlich, and Spencer [8, Theorem 2.2].

2.4. Ray–Knight isomorphism. The Ray–Knight isomorphism (i.e., the generalised second Ray–Knight theorem) is also a quick consequence of Lemma 2.1. Several other proofs of this identity exist for the 1-component GFF, see [27, 43] and references therein. For an explanation of the name, see [52, Remark 2.19].

We introduce the following notation for translations to emphasise the analogy between the classical Ray–Knight isomorphism and its hyperbolic and spherical versions. Let θ_s be the translation of all coordinates by $s \in \mathbb{R}$ in the direction $e^1 = (1, 0, \dots, 0) \in \mathbb{R}^n$, i.e., $\theta_s f(\boldsymbol{u}) \equiv f(\boldsymbol{u} + s\boldsymbol{e}^1)$ for $\boldsymbol{e}^1 = (e^1, \dots, e^1) \in \mathbb{R}^{n\Lambda}$. In particular, $\theta_s \boldsymbol{u} = \boldsymbol{u} + s\boldsymbol{e}^1$. Note that θ_s is the group action associated to the diagonal translation symmetry, which has infinitesimal generator $\sum_{j\in\Lambda} T_j$.

We will write

$$[\delta_{u_0}(u_a)F]_\beta \tag{2.24}$$

for the expectation of the spin model in which the spin at vertex a is fixed to $u_0 = (0, \dots, 0) \in \mathbb{R}^n$.

Theorem 2.5. *Let $[\cdot]_\beta$ be the unnormalised expectation of the $\mathbb{R}^n$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the SRW. Let $g\colon \mathbb{R}^\Lambda \to \mathbb{R}$ be a smooth compactly supported function, let $a \in \Lambda$, and let $s \in \mathbb{R}$. Then*

$$\left[g(\frac{1}{2}|\theta_s \boldsymbol{u}|^2)\delta_{u_0}(u_a)\right]_\beta = \left[\mathbb{E}_{a,\frac{1}{2}|\boldsymbol{u}|^2} g(\boldsymbol{L}_{\tau(\frac{s^2}{2})})\delta_{u_0}(u_a)\right]_\beta \tag{2.25}$$

where $\tau(\gamma) \equiv \inf\{t \mid L_a^t \geq \gamma\}$ and $u_0 = (0, \dots, 0) \in \mathbb{R}^n$.

Proof of Theorem 2.5. Since the identity is trivial if $s = 0$, assume $s \neq 0$. The proof is by applying Lemma 2.1 with $\rho_\varepsilon(\boldsymbol{u}) \equiv \rho_\varepsilon(u_a)$, $f(j, \boldsymbol{\ell}) \equiv g(\boldsymbol{\ell})\eta_\varepsilon(\ell_a)1_{j=a}$, and the functions $\rho_\varepsilon \colon \mathbb{R}^n \to \mathbb{R}$ and $\eta_\varepsilon \colon \mathbb{R} \to \mathbb{R}$ chosen such that $T_a\rho_\varepsilon$ and η_ε are smooth compactly supported approximations to $\delta_{u_0} - \delta_{\theta_s u_0}$ and $\delta_{\frac{1}{2}s^2}$ subject to $\rho_\varepsilon(v)\eta_\varepsilon(\frac{1}{2}|v|^2) = 0$ for all $v \in \mathbb{R}^n$. Explicitly, with $\delta^{(k)}_{\tilde{u},\varepsilon}(x)$ denoting a smooth approximation to a delta function at $\tilde{u} \in \mathbb{R}^k$ with support in $|x| < \varepsilon/2$, we may take

$$\rho_\varepsilon(u_a) = \int_0^{s-\varepsilon} \delta^{(n)}_{u_0,\varepsilon}(\theta_{-r}u_a)\,dr, \quad \eta_\varepsilon(\ell) = \delta^{(1)}_{0,\varepsilon}\Big(\ell - \frac{1}{2}s^2 - \frac{\varepsilon}{2}\Big). \tag{2.26}$$

By Lemma 2.1, since $\rho_\varepsilon(u_a)\eta_\varepsilon(\frac{1}{2}|u_a|^2) = 0$,

$$\Big[T_a\rho_\varepsilon(u_a)\int_0^\infty \mathbb{E}_{a,\frac{1}{2}|\boldsymbol{u}|^2}(g(\boldsymbol{L}_t)\eta_\varepsilon(L^a_t)1_{X_t=a})\,dt\Big]_\beta = \Big[\rho_\varepsilon(u_a)u^1_a g(\frac{1}{2}|\boldsymbol{u}|^2)\eta_\varepsilon(\frac{1}{2}|u_a|^2)\Big]_\beta = 0. \tag{2.27}$$

Let $dL^a = 1_{X_t=a}\,dt$. By the continuity[1] of $s \mapsto \mathbb{E}_{a,\boldsymbol{\ell}}g(\boldsymbol{L}_{\tau(\frac{1}{2}s^2)})$ and the definition of η_ε,

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathbb{E}_{a,\boldsymbol{\ell}}\int_0^\infty g(\boldsymbol{L}_t)\eta_\varepsilon(L^a_t)1_{X_t=a}\,dt &= \lim_{\varepsilon\to 0}\mathbb{E}_{a,\boldsymbol{\ell}}\int_0^\infty g(\boldsymbol{L}_{\tau(L^a)})\eta_\varepsilon(L^a)\,dL^a \\ &= \lim_{\varepsilon\to 0}\int_0^\infty \mathbb{E}_{a,\boldsymbol{\ell}}(g(\boldsymbol{L}_{\tau(\gamma)}))\eta_\varepsilon(\gamma)\,d\gamma = \mathbb{E}_{a,\boldsymbol{\ell}}g(\boldsymbol{L}_{\tau(\frac{1}{2}s^2)}),\end{aligned} \tag{2.28}$$

uniformly in $\boldsymbol{\ell}$ with $\ell_a \leqslant \frac{1}{2}s^2$. Since $T_a\rho_\varepsilon(u_a) = \delta^{(n)}_{u_0,\varepsilon}(u_a) - \delta^{(n)}_{u_0\varepsilon}(\theta_{-(s-\varepsilon)}u_a)$, taking the limit $\varepsilon \to 0$ in (2.27) yields, by (2.28),

$$\Big[\mathbb{E}_{a,\frac{1}{2}|\boldsymbol{u}|^2}(g(\boldsymbol{L}_{\tau(\frac{s^2}{2})}))\delta_{u_0}(u_a)\Big]_\beta = \Big[\mathbb{E}_{a,\frac{1}{2}|\theta_s\boldsymbol{u}|^2}(g(\boldsymbol{L}_{\tau(\frac{s^2}{2})}))\delta_{u_0}(u_a)\Big]_\beta \tag{2.29}$$

where we have used the invariance of $[\cdot]_\beta$ under θ_s, i.e., $[F]_\beta = [\theta_s F]_\beta$. To conclude, observe

$$\Big[\mathbb{E}_{a,\frac{1}{2}|\theta_s\boldsymbol{u}|^2}(g(\boldsymbol{L}_{\tau(\frac{s^2}{2})}))\delta_{u_0}(u_a)\Big]_\beta = \Big[g(\frac{1}{2}|\theta_s\boldsymbol{u}|^2)\delta_{u_0}(u_a)\Big]_\beta \tag{2.30}$$

since $\tau(\frac{1}{2}s^2) = 0$ if $L^a_0 = \frac{s^2}{2}$. $\square$

2.5. Eisenbaum isomorphism theorem. The Eisenbaum isomorphism theorem involves a continuous-time random walk with killing. Thus let X_t be a killed random walk with killing rates $\boldsymbol{h}$, and let $\boldsymbol{L}_t$ be its local times. To be precise, the generator of the joint process $(X_t, \boldsymbol{L}_t)_{t\geqslant 0}$ is given by

$$(\mathcal{L}^h f)(i, \boldsymbol{\ell}) \equiv \mathcal{L}f(i, \boldsymbol{\ell}) - h_i f(i, \boldsymbol{\ell}), \quad \text{i.e.,} \quad \mathcal{L}^h = \mathcal{L} - \boldsymbol{h}. \tag{2.31}$$

for $f \colon \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ smooth. We let $\mathbb{E}^h_{i,\boldsymbol{\ell}}$ denote the corresponding (deficient) expectation, i.e., integration with respect to the density of the killed random walk, which may have measure less than 1. Note that the killing does not depend on the initial local times, i.e.,

$$\mathbb{E}^h_{i,\boldsymbol{\ell}}\big(g(X_t, \boldsymbol{L}_t)\big) = \mathbb{E}_{i,\boldsymbol{\ell}}\big(g(X_t, \boldsymbol{L}_t)e^{-\sum_{j\in\Lambda}h_j(L^j_t - \ell_j)}\big), \tag{2.32}$$

and we can hence write

$$\mathbb{E}_{i,\boldsymbol{\ell}}(g(X_t, \boldsymbol{L}_t)e^{-\sum_{j\in\Lambda}h_j L^j_t}) = \mathbb{E}^h_{i,\boldsymbol{\ell}}(g(X_t, \boldsymbol{L}_t))e^{-\sum_{j\in\Lambda}h_j\ell_j}. \tag{2.33}$$

---

[1] To see continuity, since g is compactly supported, it suffices to show that for a sufficiently large T, $s \mapsto \mathbb{E}_{a,\boldsymbol{\ell}}g(\boldsymbol{L}_{\tau(\frac{1}{2}s^2)\wedge T})$ is continuous. Since g is Lipschitz, it suffices to show $\mathbb{E}_{a,\boldsymbol{\ell}}|\boldsymbol{L}_{\tau(\frac{1}{2}s^2-\delta)\wedge T} - \boldsymbol{L}_{\tau(\frac{1}{2}s^2+\delta)\wedge T}|_1 \to 0$ as $\delta \to 0$, $|\cdot|_1$ the 1-norm. Let J_δ be the event that a jump occurs in the interval $[\frac{1}{2}s^2 - \delta, \frac{1}{2}s^2 + \delta]$. Then

$$\mathbb{E}_{a,\boldsymbol{\ell}}|\boldsymbol{L}_{\tau(\frac{1}{2}s^2-\delta)\wedge T} - \boldsymbol{L}_{\tau(\frac{1}{2}s^2+\delta)\wedge T}|_1 \leqslant \delta + T\mathbb{P}_{a,\ell}(J_\delta) = O_T(\delta).$$

Probabilistically, the deficient law can be realised as a Markov process with state space $(\Lambda \cup \{\dagger\}) \times \mathbb{R}^{\Lambda\cup\{\dagger\}}$, where $\dagger \notin \Lambda$ is an absorbing 'cemetery' state. The walk jumps from i to $\dagger$ with rate h_i. The generator acts on functions that are identically zero at $\dagger$, and we identify such functions with functions on $\Lambda \times \mathbb{R}^\Lambda$. We denote the time of the one and only jump to $\dagger$ by ζ.

The following theorem is a version of Eisenbaum's isomorphism [26].

Theorem 2.6. *Suppose $\boldsymbol{h} \neq \boldsymbol{0}$. Let $\langle\cdot\rangle_{\beta,h}$ be the expectation of the $\mathbb{R}^n$ model, and let $\mathbb{E}^h_{i,\boldsymbol{\ell}}$ be the expectation of the killed SRW. Let $g\colon \mathbb{R}^\Lambda \to \mathbb{R}$ have moderate growth, let $a \in \Lambda$, and let $s \in \mathbb{R}$. Then*

$$\Big\langle (\theta_s u_a^1) g(\frac{1}{2}|\theta_s \boldsymbol{u}|^2) \Big\rangle_{\beta,h} = s \sum_{i\in\Lambda} h_i \Big\langle \int_0^\infty \mathbb{E}^h_{i,\frac{1}{2}|\theta_s\boldsymbol{u}|^2}(g(\boldsymbol{L}_t) 1_{X_t=a})\, dt \Big\rangle_{\beta,h}. \tag{2.34}$$

Proof. We apply Lemma 2.1 with

$$\rho(\boldsymbol{u}) \equiv e^{s(\boldsymbol{h},\boldsymbol{u}) - \frac{s^2}{2}(\boldsymbol{h},\boldsymbol{1})} = e^{\frac{1}{2}(\boldsymbol{h},|\boldsymbol{u}|^2)}(e^{-\frac{1}{2}(\boldsymbol{h},|\theta_{-s}\boldsymbol{u}|^2)}), \tag{2.35}$$

$$f(j,\boldsymbol{\ell}) \equiv g(\boldsymbol{\ell}) e^{-(\boldsymbol{h},\boldsymbol{\ell})} 1_{j=a}. \tag{2.36}$$

While ρ does not have moderate growth in the sense of our conventions, the very rapid (Gaussian) decay of f is sufficient for the lemma to hold. We then use that $(T_j\rho)(\boldsymbol{u}) = s h_j \rho(\boldsymbol{u})$ to obtain

$$\begin{aligned} s\sum_{j\in\Lambda} h_j \Big[\rho(\boldsymbol{u}) \int_0^\infty \mathbb{E}_{j,\frac{1}{2}|\boldsymbol{u}|^2}(g(\boldsymbol{L}_t) 1_{X_t=a} e^{-(\boldsymbol{h},\boldsymbol{L}_t)})\, dt\Big]_\beta &= \sum_{j\in\Lambda} \Big[\rho(\boldsymbol{u}) u_j^1 g(\frac{1}{2}|\boldsymbol{u}|^2) 1_{j=a} e^{-\frac{1}{2}(\boldsymbol{h},|\boldsymbol{u}|^2)}\Big]_\beta \\ &= \Big[u_a^1 g(\frac{1}{2}|\boldsymbol{u}|^2) e^{-\frac{1}{2}(\boldsymbol{h},|\theta_{-s}\boldsymbol{u}|^2)}\Big]_\beta, \end{aligned} \tag{2.37}$$

by inserting the definition (2.35). Using (2.33) to substitute

$$\rho(\boldsymbol{u}) \mathbb{E}_{j,\frac{1}{2}|\boldsymbol{u}|^2}(g(\boldsymbol{L}_t) e^{(-\boldsymbol{h},\boldsymbol{L}_t)}) = \mathbb{E}^h_{j,\frac{1}{2}|\boldsymbol{u}|^2}(g(\boldsymbol{L}_t)) e^{-\frac{1}{2}(\boldsymbol{h},|\theta_{-s}\boldsymbol{u}|^2)} \tag{2.38}$$

and by the translation invariance of $[\cdot]_\beta$, i.e., $[\theta_s F]_\beta = [F]_\beta$, we can rewrite (2.37) as

$$s\sum_{j\in\Lambda} h_j \Big[\Big(\int_0^\infty \mathbb{E}^h_{j,\frac{1}{2}|\theta_s\boldsymbol{u}|^2}(g(\boldsymbol{L}_t) 1_{X_t=a})\, dt\Big) e^{-\frac{1}{2}(\boldsymbol{h},|\boldsymbol{u}|^2)}\Big]_\beta = \Big[(\theta_s u_a^1) g(\frac{1}{2}|\theta_s\boldsymbol{u}|^2) e^{-\frac{1}{2}(\boldsymbol{h},|\boldsymbol{u}|^2)}\Big]_\beta. \tag{2.39}$$

This can be re-written in terms of $[\cdot]_{\beta,h}$ as

$$s\sum_{j\in\Lambda} h_j \Big[\int_0^\infty \mathbb{E}^h_{j,\frac{1}{2}|\theta_s\boldsymbol{u}|^2}(g(\boldsymbol{L}_t) 1_{X_t=a})\, dt\Big]_{\beta,h} = \Big[(\theta_s u_a^1) g(\frac{1}{2}|\theta_s\boldsymbol{u}|^2)\Big]_{\beta,h}, \tag{2.40}$$

and normalising gives (2.34). □

We will now derive the usual formulation of the Eisenbaum isomorphism as a corollary. For notational simplicity, suppose $n = 1$, and let $u_i = u_i^1$. Writing the translations explicitly, Theorem 2.6 yields, for $\boldsymbol{s} = (s, s, \dots, s) \in \mathbb{R}^\Lambda$, $s \neq 0$,

$$\begin{aligned} \Big\langle \frac{u_a + s}{s} g(\frac{1}{2}|\boldsymbol{u}+\boldsymbol{s}|^2)\Big\rangle_{\beta,h} &= \sum_{i\in\Lambda} h_i \Big\langle \mathbb{E}^h_{i,\frac{1}{2}|\boldsymbol{u}+\boldsymbol{s}|^2} \int_0^\infty g(\boldsymbol{L}_t) 1_{X_t=a}\, dt\Big\rangle_{\beta,h} \\ &= \sum_{i\in\Lambda} h_i \Big\langle \mathbb{E}_i \int_0^\infty g(\frac{1}{2}|\boldsymbol{u}+\boldsymbol{s}|^2 + \boldsymbol{L}_t) 1_{X_t=a} e^{-\sum_{j\in\Lambda} h_j L_t^j}\, dt\Big\rangle_{\beta,h} \\ &= \sum_{i\in\Lambda} h_i \Big\langle \mathbb{E}_a \int_0^\infty g(\frac{1}{2}|\boldsymbol{u}+\boldsymbol{s}|^2 + \boldsymbol{L}_t) 1_{X_t=i} e^{-\sum_{j\in\Lambda} h_j L_t^j}\, dt\Big\rangle_{\beta,h} \end{aligned} \tag{2.41}$$

where in the last line we have used the reversibility of the killed random walk. Bringing the sum inside the Gaussian expectation, we recognise the conditional density that X jumps from i to $\dagger$ at time t, proving the following corollary. Recall ζ is the time of the jump to the cemetery state.

Corollary 2.7. *Suppose $\boldsymbol{h} \neq \boldsymbol{0}$. Let $\langle \cdot \rangle_{\beta,h}$ be the expectation of the $\mathbb{R}^n$ model, and let $\mathbb{E}^h_{i,\boldsymbol{\ell}}$ be the expectation of the killed SRW. Suppose $g \colon \mathbb{R}^\Lambda \to \mathbb{R}$ has moderate growth, $a \in \Lambda$, and $\boldsymbol{s} = (s, s, \dots, s) \in \mathbb{R}^\Lambda$ with $s \neq 0$. Then*

$$\Big\langle \frac{u_a + s}{s} g(\frac{1}{2}|\boldsymbol{u} + \boldsymbol{s}|^2) \Big\rangle_{\beta,h} = \Big\langle \mathbb{E}^h_a \big(g(\frac{1}{2}|\boldsymbol{u} + \boldsymbol{s}|^2 + \boldsymbol{L}_\zeta)\big) \Big\rangle_{\beta,h}. \tag{2.42}$$

3 Isomorphism theorems for hyperbolic geometry

In this section we describe spin models with hyperbolic symmetry, the associated vertex-reinforced jump processes, and isomorphism theorems that link these objects. The proofs follow closely those of Section 2, but with the translation symmetry of $\mathbb{R}^n$ replaced by the boost symmetry of $\mathbb{H}^n$.

3.1. The vertex-reinforced jump process. The *vertex-reinforced jump process* (VRJP) X_t with *initial local time* $\boldsymbol{L}_0 \in (0,\infty)^\Lambda$ and *initial vertex* $v \in \Lambda$ is the process X_t with $X_0 = v$ and jump rates

$$\mathbb{P}_{v,\boldsymbol{L}_0}[X_{t+dt} = j \mid (X_s)_{s \leqslant t}, X_t = i] = \beta_{ij} L^j_t \, dt, \tag{3.1}$$

where the local times $\boldsymbol{L}_t$ of X_t are defined as in (2.3). Note that (1.1) with $\varepsilon = 1$ is the special case of (3.1) in which $\boldsymbol{L}_0 = \boldsymbol{1}$. The construction of a VRJP with given initial local times is straightforward, see [18, Section 2]. Our assumption that the graph induced by the edge weights β is connected implies that $L^j_t \to \infty$ as $t \to \infty$ in probability for all j and all sets of initial local times, see [18, Lemma 1].

As in Section 2, it will be helpful to view X_t as the marginal of the process $(X_t, \boldsymbol{L}_t)$ that includes the local times $\boldsymbol{L}_t$. For convenience we will also call this joint process a VRJP. Unlike X_t, the joint process $(X_t, \boldsymbol{L}_t)$ is a Markov process. The generator $\mathcal{L}$ of the joint process acts on smooth functions $g \colon \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ by

$$(\mathcal{L}g)(i, \boldsymbol{\ell}) = \sum_{j \in \Lambda} \beta_{ij} \ell_j (g(j, \boldsymbol{\ell}) - g(i, \boldsymbol{\ell})) + \frac{\partial g(i, \boldsymbol{\ell})}{\partial \ell_i}. \tag{3.2}$$

We note that $g_t(i, \boldsymbol{\ell}) = \mathbb{E}_{i,\boldsymbol{\ell}} \, g(X_t, \boldsymbol{L}_t)$ is smooth in ℓ for any $t > 0$ if g is smooth. This can be seen, for example, from the explicit construction of the VRJP in [18, Section 2].

3.2. Hyperbolic symmetry. The VRJP will be seen to be closely related with hyperbolic symmetry, i.e., the Lorentz group $O(n, 1)$. In this subsection we discuss the relevant aspects of this group and its action on Minkowski and hyperbolic space.

Minkowski space. Minkowski space $\mathbb{R}^{n,1}$ is the vector space $\mathbb{R}^{n+1}$ equipped with the indefinite *Minkowski inner product*

$$u_1 \cdot u_2 \equiv -u^0_1 u^0_2 + \sum_{\alpha=1}^n u^\alpha_1 u^\alpha_2, \tag{3.3}$$

where each $u_i = (u^0_i, u^1_i, \dots, u^n_i) \in \mathbb{R}^{n,1}$. The points $u \in \mathbb{R}^{n,1}$ with $u \cdot u < 0$ are called *time-like*. The set of time-like vectors with $u^0 > 0$ is called the *causal future*; schematically this is the shaded area in Figure 3.1. In what follows, for $u \in \mathbb{R}^{n,1}$ it will be notationally convenient to write $z = u^0$ and $x = u^1$.

The group preserving the quadratic form $u \cdot u$ given by (3.3) is the *Lorentz group* $O(n, 1)$. The *restricted Lorentz group* $SO^+(n, 1)$ is the subgroup of $T \in O(n, 1)$ with $\det T = 1$ and $T_{00} > 0$. $SO^+(n, 1)$ preserves the causal future, see Figure 3.1. The elements of $SO^+(n, 1)$ can be written as compositions of rotations and boosts. We briefly review the aspects of these transformations needed for what follows. Rotations act on the coordinates $u^1, \dots, u^n$ exactly as in Euclidean space, while a boost θ_s by $s \in \mathbb{R}$ in the xz-plane acts by

$$\theta_s z = x \sinh s + z \cosh s, \quad \theta_s x = x \cosh s + z \sinh s, \quad \theta_s u^\alpha = u^\alpha, \quad (\alpha = 2, \dots, n), \tag{3.4}$$

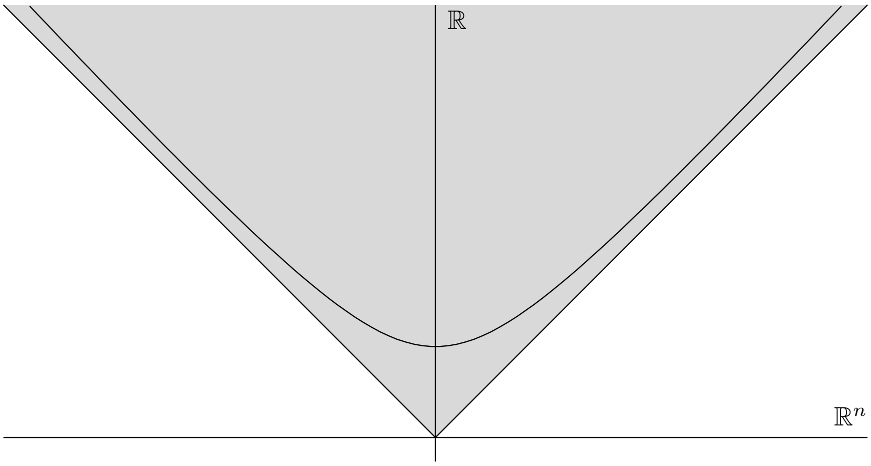

Figure 3.1. Minkowski space $\mathbb{R}^{n,1}$. The shaded area is the causal future and the hyperboloid is $\mathbb{H}^n$.

and similarly for boosts in other planes. From (3.4) it follows that the infinitesimal generator T of boosts in the xz-plane is the linear differential operator satisfying

$$Tz = x, \quad Tx = z, \quad Tu^\alpha = 0, \quad (\alpha = 2, \dots, n), \tag{3.5}$$

i.e.,

$$T \equiv z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}. \tag{3.6}$$

Hyperbolic space. When given the metric induced by the Minkowski inner product, the set

$$\mathbb{H}^n \equiv \{u \in \mathbb{R}^{n,1} \mid u \cdot u = -1, z > 0\} \tag{3.7}$$

is a model for n-dimensional hyperbolic space. Note that (3.7) implies $z \geqslant 1$. For $u, v \in \mathbb{H}^n$, $-u\cdot v = \cosh(d(u,v))$, where $d(u,v)$ is the geodesic distance from u to v. In particular, $-u\cdot v \geqslant 1$. For details on why this is indeed hyperbolic space see, e.g. [15].

$\mathbb{H}^n$ is the orbit under $SO^+(n,1)$ of the point $u_0 = (1, 0, \dots, 0)$, and the stabiliser of u_0 is the subgroup $SO(n)$. Thus $\mathbb{H}^n$ can be identified with $SO^+(n,1)/SO(n)$. It is parameterised by $(u^1, \dots, u^n) \in \mathbb{R}^n$:

$$\mathbb{H}^n = \{u \in \mathbb{R}^{n,1} \mid (u^1, \dots, u^n) \in \mathbb{R}^n, z = \sqrt{1 + (u^1)^2 + \dots (u^n)^2}\}. \tag{3.8}$$

In these coordinates, the $SO^+(n,1)$-invariant Haar measure on $\mathbb{H}^n$ can be written as

$$du = \frac{du^1 \dots du^n}{z(u)}, \quad z(u) \equiv \sqrt{1 + (u^1)^2 + \dots + (u^n)^2}. \tag{3.9}$$

Note that the Lorentz boost (3.4) maps $\mathbb{H}^n$ to $\mathbb{H}^n$, and that in the parameterization of $\mathbb{H}^n$ by $(u^1, \dots, u^n)$, the infinitesimal Lorentz boost in the xz-plane is given by

$$T \equiv z\frac{\partial}{\partial x}. \tag{3.10}$$

This is because T satisfies the defining equations (3.5): $Tz = x$, $Tx = z$, and $Tu^\alpha = 0$ for $\alpha \geqslant 2$. In the last calculation we have used the definition (3.8) of $z(u)$. The invariance of the measure du under Lorentz boosts implies that for differentiable $f\colon \mathbb{H}^n \to \mathbb{R}$ with sufficient decay,

$$\int_{\mathbb{H}^n} Tf \, du = 0. \tag{3.11}$$

3.3. Hyperbolic sigma model. Hyperbolic spin models are analogues of the Gaussian free field defined in terms of the Minkowski inner product instead of the Euclidean inner product. While it is possible to define a spin model associated to the entire causal future of Minkowski space, see Figure 3.1, for now we restrict ourselves to the *sigma model* version of this model in which spins are constrained to lie in $\mathbb{H}^n$. We will later consider (the supersymmetric version of) a spin model taking values in the causal future in Section 7.2.

In the $\mathbb{H}^n$ sigma model there is a spin $u_i \in \mathbb{H}^n$ for each $i \in \Lambda$. We again let β be a non-negative collection of edge weights and $\boldsymbol{h} \geqslant \boldsymbol{0}$ be a collection of non-negative vertex weights. For a spin configuration $\boldsymbol{u}$ we consider the energy

$$H_\beta(\boldsymbol{u}) \equiv \frac{1}{2}(\boldsymbol{u}, -\Delta_\beta \boldsymbol{u}) = \frac{1}{4}\sum_{i,j\in\Lambda} \beta_{ij}(u_i - u_j)^2, \quad H_{\beta,h}(\boldsymbol{u}) = H_\beta(\boldsymbol{u}) + (\boldsymbol{h}, \boldsymbol{z} - \boldsymbol{1}), \tag{3.12}$$

analogous to (2.6), except that the inner product in $(u_i - u_j)^2 = (u_i - u_j)\cdot(u_i - u_j)$ is now given by the Minkowski inner product. The mass term has also been replaced by the term $(\boldsymbol{h}, \boldsymbol{z} - \boldsymbol{1})$ since $z_i \geqslant 1$ for all i.

Note that $H_\beta(\boldsymbol{u})$ is invariant under the diagonal action of $SO^+(n,1)$, analogous to the invariance of (2.6) by the Euclidean group. Moreover, since $u_i \cdot u_i = -1$, we have $(u_i - u_j)^2 = -2 - 2u_i \cdot u_j$, we can thus rewrite $H_\beta(\boldsymbol{u})$ in terms of $\tilde{u} \equiv (u^1, \dots, u^n) \in \mathbb{R}^n$ as

$$H_\beta(\boldsymbol{u}) = -\frac{1}{2}\sum_{i,j\in\Lambda} \beta_{ij}\left(\sum_{\alpha=1}^n u_i^\alpha u_j^\alpha - z_i z_j + 1\right), \tag{3.13}$$

where we recall that $z_i = z_i(\tilde{u}_i)$ is given by (3.8). Define an unnormalised expectation $[\cdot]_{\beta,h}$ on functions $F\colon \mathbb{H}^{n\Lambda} \to \mathbb{R}$ by

$$[F]_{\beta,h} \equiv \int_{\mathbb{H}^{n\Lambda}} F(\boldsymbol{u}) e^{-H_{\beta,h}(\boldsymbol{u})}\, d\boldsymbol{u} = \int_{\mathbb{R}^{n\Lambda}} F(\boldsymbol{u}) e^{-H_{\beta,h}(\boldsymbol{u})} \prod_{i\in\Lambda} \frac{d\tilde{u}_i}{z(\tilde{u}_i)}, \tag{3.14}$$

where $d\boldsymbol{u}$ is the Λ-fold product of the invariant measure on $\mathbb{H}^n$. In the second equality we have written this integral using the parametrization by $\mathbb{R}^n$ in (3.9). When $\boldsymbol{h} = \boldsymbol{0}$ we set $[\cdot]_\beta \equiv [\cdot]_{\beta,h}$.

The $\mathbb{H}^n$*-model* is the probability measure on $\mathbb{H}^{n\Lambda}$ defined by the normalised expectation

$$\langle F\rangle_{\beta,h} \equiv \frac{1}{Z_{\beta,h}}[F]_{\beta,h}, \qquad Z_{\beta,h} \equiv [1]_{\beta,h}. \tag{3.15}$$

Note that for (3.15) to be well-defined we must have $Z_{\beta,h} < \infty$. This is the case if and only if $h_i > 0$ for some i due to the invariance of $H_\beta(\boldsymbol{u})$ under the non-compact boost symmetry of $\mathbb{H}^n$.

Remark 3.1. This model was studied in [49] as a toy model for some aspects of random band matrices. See Remark 5.8 below for further details on this connection.

3.4. Fundamental integration by parts identity. The statement of the following lemma is formally identical to that of Lemma 2.1. However, the objects in its statement are now hyperbolic versions: $\mathcal{L}$ is the generator of the VRJP, $[\cdot]_\beta$ is the unnormalised expectation from (3.14), T_j is the infinitesimal Lorentz boost in the xz-plane in the jth coordinate specified by (3.5), and $\frac{1}{2}|\boldsymbol{u}|^2$ is replaced by $\boldsymbol{z}$.

Lemma 3.2. *Let $[\cdot]_\beta$ be the unnormalised expectation of the $\mathbb{H}^n$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the VRJP. Let $f\colon \Lambda\times\mathbb{R}^\Lambda \to \mathbb{R}$ be a smooth function with rapid decay, and let $\rho\colon \mathbb{H}^{n\Lambda} \to \mathbb{R}$ be smooth with moderate growth. Then:*

$$-\sum_{j\in\Lambda}[\rho(\boldsymbol{u}) x_j \mathcal{L} f(j, \boldsymbol{z})]_\beta = \sum_{j\in\Lambda}[(T_j\rho)(\boldsymbol{u}) f(j,\boldsymbol{z})]_\beta. \tag{3.16}$$

In particular, the following integrated version holds for all $f\colon \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ with rapid decay:

$$\sum_{j\in\Lambda} [\rho(\boldsymbol{u}) x_j f(j,\boldsymbol{z})]_\beta = \sum_{j\in\Lambda} \Big[(T_j\rho)(\boldsymbol{u}) \int_0^\infty \mathbb{E}_{j,\boldsymbol{z}}(f(X_t,\boldsymbol{L}_t))\, dt \Big]_\beta. \tag{3.17}$$

Proof. The proof is again by integration by parts and closely follows that of Lemma 2.1. Indeed, using that $[\cdot]_\beta$ has density e^{-H_β} with respect to the Lorentz invariant measure on $\mathbb{H}^{n\Lambda}$, the identity (3.11) implies that for $f_1, f_2\colon \mathbb{H}^{n\Lambda} \to \mathbb{R}$ smooth and with sufficient decay,

$$[(T_i f_1) f_2]_\beta = [f_1 (T_i^\star f_2)]_\beta, \tag{3.18}$$

where

$$T_i^\star f(\boldsymbol{u}) = -T_i f(\boldsymbol{u}) + (T_i H_\beta(\boldsymbol{u})) f(\boldsymbol{u}). \tag{3.19}$$

Using (3.13) and (3.5) yields

$$T_i H_\beta(\boldsymbol{u}) = -\frac{1}{2} \sum_{j,k\in\Lambda} \beta_{jk} T_i (x_j x_k - z_j z_k) = \sum_{j\in\Lambda} \beta_{ij} (x_i z_j - x_j z_i) \tag{3.20}$$

and hence, using (3.5) and the chain rule to compute $T_i f$,

$$-T_i^\star f(\boldsymbol{z}) = \sum_{j\in\Lambda} \beta_{ij} (x_j z_i - x_i z_j) f(\boldsymbol{z}) + x_i \frac{\partial f(\boldsymbol{z})}{\partial \ell_i}. \tag{3.21}$$

Applying (3.21) to each function $f(i,\boldsymbol{z})$ and summing over i yields

$$-\sum_{i\in\Lambda} T_i^\star f(i,\boldsymbol{z}) = \sum_{i\in\Lambda} x_i \big(\sum_{j\in\Lambda} \beta_{ij} z_j (f(j,\boldsymbol{z}) - f(i,\boldsymbol{z})) + \frac{\partial f(i,\boldsymbol{z})}{\partial \ell_i} \big) = \sum_{i\in\Lambda} x_i (\mathcal{L} f)(i,\boldsymbol{z}) \tag{3.22}$$

by the formula (3.2) for $\mathcal{L}$. The remainder of the proof follows the proof of Lemma 2.1. □

3.5. Hyperbolic isomorphism theorems. The following theorems are analogues of the BFS–Dynkin, Ray–Knight, and Eisenbaum isomorphism theorems. Their proofs are analogous to those in Section 2, using Lemma 3.2 in place of Lemma 2.1, and using hyperbolic versions of ρ and f. We begin with the hyperbolic version of the BFS–Dynkin isomorphism, i.e., Theorem 2.3. It first appeared in [5] and was proven there using horospherical coordinates. Here we give a more intrinsic proof that avoids horospherical coordinates.

Theorem 3.3. *Let $[\cdot]_\beta$ be the unnormalised expectation of the $\mathbb{H}^n$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the VRJP. Let $g\colon \mathbb{R}^\Lambda \to \mathbb{R}$ have rapid decay, and let $a, b \in \Lambda$. Then*

$$[x_a x_b g(\boldsymbol{z})]_\beta = \Big[z_a \int_0^\infty \mathbb{E}_{a,\boldsymbol{z}}(g(\boldsymbol{L}_t) 1_{X_t=b})\, dt \Big]_\beta. \tag{3.23}$$

Proof. Apply Lemma 3.2 with $\rho(\boldsymbol{u}) = x_a$, $f(j,\boldsymbol{\ell}) = g(\boldsymbol{\ell}) 1_{j=b}$, and use $T_j\rho(\boldsymbol{u}) = 1_{j=a} z_j$. □

The next theorem is a hyperbolic version of the Ray–Knight isomorphism, i.e., Theorem 2.5. Recall the definition of a boost θ_s by $s \in \mathbb{R}$ in the xz-plane from (3.4). In what follows we let θ_s act diagonally on $\boldsymbol{u} \in \mathbb{H}^{n\Lambda}$, and we write $\theta_s \boldsymbol{z}$ to denote the first component of $\theta_s \boldsymbol{u}$. We also write $[f \delta_{u_0}(u_a)]_\beta$ for the expectation of the spin model in which the spin u_a is fixed at $u_0 \in \mathbb{H}^n$.

Theorem 3.4. *Let $[\cdot]_\beta$ be the unnormalised expectation of the $\mathbb{H}^n$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the VRJP. Let $g\colon \mathbb{R}^\Lambda \to \mathbb{R}$ be a smooth compactly supported function, let $a \in \Lambda$, and let $s \in \mathbb{R}$. Then*

$$[g(\theta_s \boldsymbol{z}) \delta_{u_0}(u_a)]_\beta = \big[\mathbb{E}_{a,\boldsymbol{z}} g(\boldsymbol{L}_{\tau(\cosh s)}) \delta_{u_0}(u_a) \big]_\beta \tag{3.24}$$

where $\tau(\gamma) = \inf\{t \,|\, L_a^t \geq \gamma\}$ and $u_0 = (1, 0, \dots, 0) \in \mathbb{H}^n$.

Proof of Theorem 3.4. Since the identity is trivial if $s=0$, assume $s\neq 0$. We begin by applying Lemma 3.2 with $\rho_\varepsilon(\boldsymbol{u})=\rho_\varepsilon(u_a)$, $f(j,\boldsymbol{\ell})=g(\boldsymbol{\ell})\eta_\varepsilon(\ell_a)1_{j=a}$, with the functions $\rho_\varepsilon\colon \mathbb{H}^n\to\mathbb{R}$ and $\eta_\varepsilon\colon\mathbb{R}\to\mathbb{R}$ chosen such that $T_a\rho_\varepsilon$ and η_ε are smooth compactly supported approximations to $\delta_{u_0}(u_a)-\delta_{\theta_s u_0}(u_a)$ and $\delta_{\cosh s}(\ell_a)$ subject to $\rho_\varepsilon(u_a)\eta_\varepsilon(z_a)=0$ for all $u_a\in\mathbb{H}^n$. Since $s\neq 0$, these conditions can be shown to be satisfiable by explicit construction. Exactly as in the proof of Theorem 2.5 this yields

$$\left[T_a\rho_\varepsilon(u_a)\int_0^\infty \mathbb{E}_{a,\boldsymbol{z}}(g(\boldsymbol{L}_t)\eta_\varepsilon(L_t^a)1_{X_t=a})\,dt\right]_\beta=0, \tag{3.25}$$

i.e.,

$$\left[\delta_{\theta_{s-\varepsilon}u_0,\varepsilon}(u_0)\int_0^\infty \mathbb{E}_{a,\boldsymbol{z}}(g(\boldsymbol{L}_t)\eta_\varepsilon(L_t^a)1_{X_t=a}\,dt\right]_\beta=\left[\delta_{u_0,\varepsilon}(u_0)\int_0^\infty \mathbb{E}_{a,\boldsymbol{z}}(g(\boldsymbol{L}_t)\eta_\varepsilon(L_t^a)1_{X_t=a})\,dt\right]_\beta. \tag{3.26}$$

As in (2.28), by the continuity[2] of $s\mapsto \mathbb{E}_{a,\boldsymbol{\ell}}g(\boldsymbol{L}_{\tau(\cosh s)})$ and the definition of η_ε,

$$\lim_{\varepsilon\to0}\mathbb{E}_{a,\boldsymbol{\ell}}\int_0^\infty g(\boldsymbol{L}_t)\eta_\varepsilon(L_t^a)1_{X_t=a}\,dt=\lim_{\varepsilon\to0}\int_0^\infty \mathbb{E}_{a,\boldsymbol{\ell}}(g(\boldsymbol{L}_{\tau(\gamma)})\eta_\varepsilon(\gamma)\,d\gamma=\mathbb{E}_{a,\boldsymbol{\ell}}g(\boldsymbol{L}_{\tau(\cosh s)}), \tag{3.27}$$

uniformly in $\boldsymbol{\ell}$ with $\ell_a\leqslant\cosh s$.

To conclude, we use (3.27) to take $\varepsilon\to0$ in (3.26). More precisely, we use that $\delta_{\theta_s u_0}$ concentrates the u_a integral at $z_a=\cosh s$ on the left-hand side, and hence the time integral at $t=0$. By the boost invariance of $[\cdot]_\beta$, this term produces the left-hand side of (3.24):

$$\left[\delta_{\theta_s u_0}(u_a)\mathbb{E}_{a,\boldsymbol{z}}(g(\boldsymbol{L}_{\tau(\cosh s)}))\right]_\beta=\left[\delta_{u_0}(u_a)\mathbb{E}_{a,\theta_s\boldsymbol{z}}(g(\boldsymbol{L}_{\tau(\cosh s)}))\right]_\beta=[\delta_{u_0}(u_a)g(\theta_s\boldsymbol{z})]_\beta. \tag{3.28}$$

Again by (3.27), the δ_{u_0} on the right-hand side of (3.26) concentrates the time integral at $\tau(\cosh s)$, which gives the right-hand side of (3.24). $\square$

Finally, we prove a hyperbolic version of the Eisenbaum isomorphism theorem, i.e., Theorem 2.6. This concerns a killed VRJP. The generator of this killed process $(X_t,\boldsymbol{L}_t)_{t\geqslant0}$ acts on smooth functions $f\colon\Lambda\times\mathbb{R}^\Lambda\to\mathbb{R}$ as

$$(\mathcal{L}^h f)(i,\boldsymbol{\ell})\equiv\mathcal{L}f(i,\boldsymbol{\ell})-h_if(i,\boldsymbol{\ell}),\quad\text{i.e.,}\quad \mathcal{L}^h=\mathcal{L}-\boldsymbol{h}, \tag{3.29}$$

where $\mathcal{L}$ is now the generator of the VRJP and h_i are the killing rates. We let $\mathbb{E}^h_{i,\boldsymbol{\ell}}$ denote the corresponding deficient expectation. As for the SRW, the killing does not depend on the initial local times, i.e.,

$$\mathbb{E}^h_{i,\boldsymbol{\ell}}\big(g(X_t,\boldsymbol{L}_t)\big)=\mathbb{E}_{i,\boldsymbol{\ell}}\big(g(X_t,\boldsymbol{L}_t)e^{-\sum_{j\in\Lambda}h_j(L_t^j-\ell_j)}\big), \tag{3.30}$$

and we can thus write

$$\mathbb{E}_{i,\boldsymbol{\ell}}(g(X_t,\boldsymbol{L}_t)e^{-\sum_{j\in\Lambda}h_j(L_t^j-1)})=\mathbb{E}^h_{i,\boldsymbol{\ell}}(g(X_t,\boldsymbol{L}_t))e^{-\sum_{j\in\Lambda}h_j(\ell_j-1)}=\mathbb{E}^h_{i,\boldsymbol{\ell}}(g(X_t,\boldsymbol{L}_t))e^{-(\boldsymbol{h},\boldsymbol{\ell}-\boldsymbol{1})}. \tag{3.31}$$

Theorem 3.5. *Let $\langle\cdot\rangle_{\beta,h}$ be the expectation of the $\mathbb{H}^n$ model, and let $\mathbb{E}^h_{i,\boldsymbol{\ell}}$ be the expectation of the killed VRJP with $\boldsymbol{h}\neq\boldsymbol{0}$. Let $g\colon\Lambda\times\mathbb{R}^\Lambda\to\mathbb{R}$ be of moderate growth, and let $s\in\mathbb{R}$. Then*

$$\sum_{i\in\Lambda}\langle(\theta_s x_i)g(i,\theta_s\boldsymbol{z})\rangle_{\beta,h}=\sum_{i\in\Lambda}h_i\left\langle(\theta_s x_i-x_i)\int_0^\infty\mathbb{E}^h_{i,\theta_s\boldsymbol{z}}(g(X_t,\boldsymbol{L}_t))\,dt\right\rangle_{\beta,h}. \tag{3.32}$$

[2]Continuity can be proven by an argument similar to the one we gave for simple random walk near (2.28): after restricting to times at most T using compact support, the claim follow from the fact that $\mathbb{P}(J_\delta)=O_T(\delta)$ since the jump rates up to time T are bounded by $O(T)$.

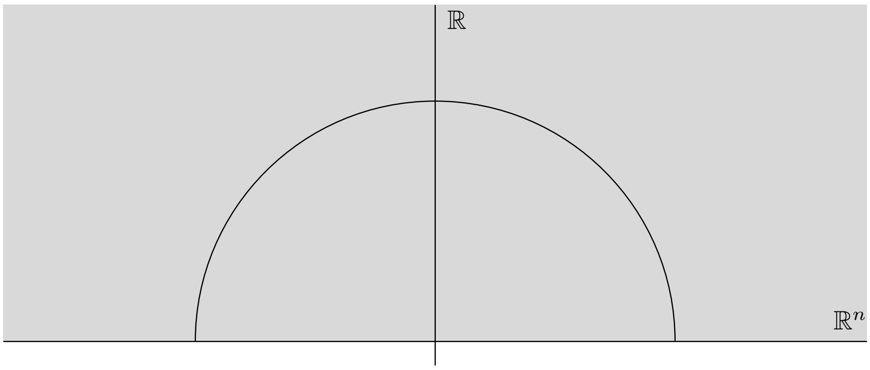

Figure 4.1. The upper half-plane in Euclidean space $\mathbb{R}^{n+1}$ (shaded) and the upper hemisphere $\mathbb{S}^n_+$.

Proof. Analogously to the proof of Theorem 2.6, we apply Lemma 3.2 with

$$\rho(\boldsymbol{u}) \equiv e^{(\boldsymbol{h},\boldsymbol{z}-\theta_{-s}\boldsymbol{z})} = e^{(\boldsymbol{h},\boldsymbol{z}-\boldsymbol{1})}(e^{-(\boldsymbol{h},\theta_{-s}\boldsymbol{z}-\boldsymbol{1})}) \tag{3.33}$$

$$f(j,\boldsymbol{\ell}) \equiv g(\boldsymbol{\ell})e^{-(\boldsymbol{h},\boldsymbol{\ell}-\boldsymbol{1})}1_{j=a}, \tag{3.34}$$

and use that $(T_j\rho)(\boldsymbol{u}) = h_j(x_j - \theta_{-s}x_j)\rho(\boldsymbol{u})$ to obtain

$$\sum_{j\in\Lambda} h_j\Big[(x_j - \theta_{-s}x_j)\rho(\boldsymbol{u})\int_0^\infty \mathbb{E}_{j,\boldsymbol{z}}(g(\boldsymbol{L}_t)1_{X_t=a}e^{-(\boldsymbol{h},\boldsymbol{L_t}-\boldsymbol{1})})\,dt\Big]_\beta$$
$$= \sum_{j\in\Lambda}\Big[\rho(\boldsymbol{u})x_j g(\boldsymbol{z})1_{j=a}e^{-(\boldsymbol{h},\boldsymbol{z}-\boldsymbol{1})}\Big]_\beta = \Big[x_a^1 g(\boldsymbol{z})e^{-(\boldsymbol{h},\theta_{-s}\boldsymbol{z}-\boldsymbol{1})}\Big]_\beta. \tag{3.35}$$

Using (3.31) to substitute

$$\rho(\boldsymbol{u})\mathbb{E}_{j,\boldsymbol{z}}(g(\boldsymbol{L}_t)1_{X_t=a}e^{-(\boldsymbol{h},\boldsymbol{L}_t-\boldsymbol{1})}) = \mathbb{E}^h_{j,\boldsymbol{z}}(g(\boldsymbol{L}_t)1_{X_t=a})e^{-(\boldsymbol{h},\theta_{-s}\boldsymbol{z}-\boldsymbol{1})}, \tag{3.36}$$

and the boost invariance of the spin expectation $[\theta_s\cdot]_\beta = [\cdot]_\beta$, we can rewrite (3.35) as

$$\sum_{j\in\Lambda} h_j\Big[(\theta_s x_j - x_j)\int_0^\infty \mathbb{E}^h_{j,\theta_s\boldsymbol{z}}(g(\boldsymbol{L}_t)1_{X_t=a})\,dt\Big]_{\beta,h} = [(\theta_s x_a)g(\theta_s\boldsymbol{z})]_{\beta,h} \tag{3.37}$$

where we have absorbed the magnetic terms $e^{-(\boldsymbol{h},\boldsymbol{z}-\boldsymbol{1})}$ into the measures. Normalising gives (3.32). □

4 Isomorphism theorems for spherical geometry

In this section we describe analogues of the theorems of Sections 2 and 3 for spherical geometry.

4.1. The vertex-diminished jump process. The *vertex-diminished jump process* (VDJP) $(X_t, \boldsymbol{L}_t)$ with initial conditions $(v, \boldsymbol{L}_0) \in \Lambda \times (0,1]^\Lambda$ is the Markov process with conditional jump rates

$$\mathbb{P}_{v,\boldsymbol{L}_0}[X_{t+dt} = j \mid (X_s)_{s\leqslant t}, X_t = i] = \beta_{ij}L^j_t\,dt \tag{4.1}$$

that is stopped at the time $\zeta \equiv \inf\{s \mid \text{exists } j\in\Lambda \text{ s.t. } L^j_s \leqslant 0\}$. Here $\boldsymbol{L}_t$ is the collection of local times of X_t defined by

$$L^j_t \equiv L^j_0 - \int_0^t 1_{X_s=j}\,ds, \tag{4.2}$$

and $L^j_0 > 0$ is the *initial local time* at j. It is straightforward to see that $(X_t, \boldsymbol{L}_t)$ is well-defined up to ζ by a step-by-step construction as is done for the VRJP in [18]. Note that (1.1) with $\varepsilon = -1$ describes the VDJP with $\boldsymbol{L}_0 = \boldsymbol{1}$.

The generator $\mathcal{L}$ of the VDJP acts on smooth functions $g\colon \Lambda \times (0,1]^{\Lambda} \to \mathbb{R}$ by

$$(\mathcal{L}g)(i,\boldsymbol{\ell}) = \sum_{j\in\Lambda} \beta_{ij}\ell_j(g(j,\ell) - g(i,\ell)) - \frac{\partial g(i,\ell)}{\partial \ell_i}. \tag{4.3}$$

We write $\mathbb{P}_{i,\boldsymbol{L_0}}$ and $\mathbb{E}_{i,\boldsymbol{L}_0}$ for the law and expectation of the VDJP with initial condition $(i, \boldsymbol{L}_0)$.

4.2. Rotational symmetry. We consider the space $\mathbb{R}^{n+1}$ equipped with the Euclidean inner product $u \cdot v = u^0v^0 + \cdots + u^nv^n$, which is preserved by the orthogonal group $O(n+1)$. In the next section we will define an unnormalised expectation exactly as in Section 2, but we will investigate the consequences of rotational symmetries instead of translational symmetries.

4.3. The hemispherical spin model $\mathbb{S}^n_+$.

4.3.1. Hemispherical space. In this section we discuss a spin system that takes values in $\mathbb{S}^n_+$, the open upper hemisphere of the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. See Figure 4.1. For notational convenience we write $u = (u^0, \dots, u^n) \in \mathbb{R}^{n+1}$ and let $z = u^0$, and we will also often write $x = u^1$. Then

$$\mathbb{S}^n_+ \equiv \{u \in \mathbb{R}^{n+1} \mid u \cdot u = 1, z > 0\}, \tag{4.4}$$

where the inner product is Euclidean. $\mathbb{S}^n_+$ is parametrised by the open unit ball in $\mathbb{R}^n$, i.e., by

$$\mathbb{B}^n = \{(u^1, \dots, u^n) \in \mathbb{R}^n \mid (u^1)^2 + \cdots + (u^n)^2 < 1\}. \tag{4.5}$$

4.3.2. Symmetries. In the flat and hyperbolic settings we considered the Euclidean group $O(n) \ltimes \mathbb{R}^n$ and the restricted Lorentz group $SO^+(n,1)$. Unlike in these settings, the orthogonal group $O(n+1)$ does not preserve the hemisphere. Our results, however, were based on the *infinitesimal* symmetries of flat and hyperbolic space, and the hemisphere still possesses useful infinitesimal symmetries. This section briefly explains this; the key identity is (4.9).

The infinitesimal symmetries of the hemisphere form a representation of the Lie algebra $\mathfrak{so}(n+1)$, see Appendix B.3. The associated invariant measure du on $\mathbb{S}^n_+$ can be written in coordinates as

$$du = \frac{du^1 \dots du^n}{z(u)}, \quad z(u) = \sqrt{1 - (u^1)^2 - \cdots - (u^n)^2}. \tag{4.6}$$

This is the invariant measure on the full sphere restricted to $\mathbb{S}^n_+$. We let θ_s denote a rotation by $s \in \mathbb{R}$ in the xz-plane. Note that in the coordinates $(x, u^2, \dots, u^n)$ the infinitesimal generator of rotations in the xz-plane is

$$T \equiv z\frac{\partial}{\partial x}, \tag{4.7}$$

which acts on the coordinate functions as

$$Tz = -x, \qquad Tx = z, \quad Tu^{\alpha} = 0, \quad (\alpha = 2, \dots, n). \tag{4.8}$$

A consequence of T being an infinitesimal symmetry of the hemisphere is that for compactly supported smooth $f\colon \mathbb{S}^n_+ \to \mathbb{R}$,

$$\int_{\mathbb{S}^n_+} Tf \, du = 0, \tag{4.9}$$

an identity which is also easily proven by rewriting the integral as an integral over $\mathbb{S}^n$ and using the rotational invariance of the full sphere.

4.3.3. The $\mathbb{S}^n_+$ model. By a by now familiar abuse of notation, we write $\mathbb{S}^{n\Lambda}_+$ in place of $(\mathbb{S}^n_+)^\Lambda$. Define, for $\boldsymbol{u} \in \mathbb{S}^{n\Lambda}_+$,

$$H_\beta(\boldsymbol{u}) \equiv \frac{1}{2}(\boldsymbol{u}, -\Delta_\beta \boldsymbol{u}), \quad H_{\beta,h}(\boldsymbol{u}) \equiv H_\beta(\boldsymbol{u}) + (\boldsymbol{h}, \boldsymbol{1} - \boldsymbol{z}), \tag{4.10}$$

where as before β and $\boldsymbol{h}$ are collections of non-negative edge and vertex weights, respectively. For $F : \mathbb{S}^{n\Lambda}_+ \to \mathbb{R}$ define the unnormalised expectation

$$[F]_{\beta,h} \equiv \int_{\mathbb{S}^{n\Lambda}_+} F(\boldsymbol{u}) e^{-H_{\beta,h}(\boldsymbol{u})} \, d\boldsymbol{u} = \int_{\mathbb{B}^{n\Lambda}} F(\boldsymbol{u}) e^{-H_{\beta,h}(\boldsymbol{u})} \prod_{i \in \Lambda} \frac{du_i^1 \dots du_i^n}{z(u_i)} \tag{4.11}$$

where $d\boldsymbol{u} \equiv \prod_{i\in\Lambda} du_i$, and each du_i is a copy of the invariant measure on $\mathbb{S}^n_+$. The $\mathbb{S}^n_+$ *model* is the probability measure defined by the normalised expectation

$$\langle F \rangle_{\beta,h} \equiv \frac{[F]_{\beta,h}}{Z_{\beta,h}}, \quad Z_{\beta,h} \equiv [1]_{\beta,h}. \tag{4.12}$$

Unlike the GFF and $\mathbb{H}^n$-model, the $\mathbb{S}^n_+$ model is well-defined if $\boldsymbol{h} = \boldsymbol{0}$, and we will omit the subscripts h to indicate $\boldsymbol{h} = \boldsymbol{0}$.

Remark 4.1. The *spherical $O(n)$ models* are obtained by removing the restriction that spins lie in the upper hemisphere in (4.11). See Remark 4.3 below.

4.4. Isomorphism theorems. The following isomorphism theorems are analogues of those in Section 2 and 3. We again start with a fundamental integration by parts identity, with the change that now $\mathcal{L}$ is the generator of the VDJP, $[\cdot]_\beta$ is the unnormalised expectation of (4.11), and T_j is the infinitesimal rotation in the xz-plane in the jth coordinate specified by (4.7).

Lemma 4.2. *Let $[\cdot]_\beta$ be the unnormalised expectation of the $\mathbb{S}^n_+$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the VDJP. Let $f : \Lambda \times (0,1]^\Lambda \to \mathbb{R}$ be a smooth compactly supported function and let $\rho : \mathbb{S}^{n\Lambda}_+ \to \mathbb{R}$ be smooth. Then:*

$$-\sum_{j\in\Lambda} [\rho(\boldsymbol{u}) x_j \mathcal{L} f(j, \boldsymbol{z})]_\beta = \sum_{j\in\Lambda} [(T_j \rho)(\boldsymbol{u}) f(j, \boldsymbol{z})]_\beta. \tag{4.13}$$

In particular, the following integrated version holds for compactly supported $f : \Lambda \times (0,1]^\Lambda \to \mathbb{R}$:

$$\sum_{j\in\Lambda} [\rho(\boldsymbol{u}) x_j f(j, \boldsymbol{z})]_\beta = \sum_{j\in\Lambda} \left[(T_j\rho)(\boldsymbol{u}) \int_0^\infty \mathbb{E}_{j,\boldsymbol{z}}(f(X_t, \boldsymbol{L}_t)) \, dt \right]_\beta. \tag{4.14}$$

Proof. By (4.9) we can integrate by parts. The proof is almost identical to that of Lemma 3.2, the only differences being $\mathbb{H}^{n\Lambda}$ is replaced $\mathbb{S}^{n\Lambda}$, and $T_i = z_i \frac{\partial}{\partial x_i}$ is the infinitesimal generator of a rotation in the xz-plane at i instead of a Lorentz boost. This introduces a sign, i.e.,

$$T_i f(\boldsymbol{z}) = -x_i \frac{\partial f(\boldsymbol{z})}{\partial \ell_i} \tag{4.15}$$

where the hyperbolic model had a factor of $+1$ in (3.21), producing the VDJP generator instead of the VRJP generator. The remainder of the proof is essentially unchanged. □

Remark 4.3. Analytically, (4.13) holds for the spherical $O(n)$ model, although it is no longer obvious how to interpret $\mathcal{L}$ as the generator of a Markov process since 'jump rates' become negative. In particular, it is unclear how to obtain a formula like (4.14). A probabilistic interpretation of $\mathcal{L}$ for the $O(n)$ model, without restricting to the hemisphere, would be very interesting.

The hemispherical BFS–Dynkin isomorphism theorem for the VDJP is as follows:

Theorem 4.4. *Let $[\cdot]_\beta$ be the unnormalised expectation of the $\mathbb{S}^n_+$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the VDJP. Suppose $g\colon (0,1]^\Lambda \to \mathbb{R}$ is compactly supported. Then for $a, b \in \Lambda$,*

$$[x_a x_b g(\boldsymbol{z})]_\beta = \left[z_a \int_0^\infty \mathbb{E}_{a,\boldsymbol{z}}(g(\boldsymbol{L}_t) 1_{X_t=b})\, dt \right]_\beta. \tag{4.16}$$

Proof. Apply Lemma 4.2 with $\rho(\boldsymbol{u}) = x_a$, $f(j, \boldsymbol{\ell}) = g(\boldsymbol{\ell})1_{j=b}$, and use $T_j\rho(\boldsymbol{u}) = 1_{j=a}z_j$. □

The fact that finite symmetries do not preserve the hemisphere leads to slightly different formulations of the Eisenbaum and Ray–Knight isomorphism theorems as compared to the GFF and $\mathbb{H}^n$ models. We let $[F(\boldsymbol{u})\delta_{u_0}(u_a)]_\beta$ denote the unnormalised expectation for the spin model in which the spin at u_a is fixed to be $u_0 \in \mathbb{S}^n_+$.

Theorem 4.5. *Let $[\cdot]_\beta$ be the unnormalised expectation of the $\mathbb{S}^n_+$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the VDJP. Let $g\colon (0,1]^\Lambda \to \mathbb{R}$ be a smooth compactly supported function, let $a \in \Lambda$, and let $s \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then*

$$\left[\mathbb{E}_{a,\boldsymbol{z}}(g(\boldsymbol{L}_{\tau(\cos s)})1_{\{\tau(\cos s)<\zeta\}})\delta_{u_0}(u_a)\right]_\beta = [g(\boldsymbol{z})\delta_{\theta_s u_0}(u_a)]_\beta \tag{4.17}$$

where $\tau(\gamma) = \inf\{t \,|\, L^a_t \le \gamma\}$ and $u_0 = (1, 0, \dots, 0) \in \mathbb{S}^n_+$.

Proof. The proof is analogous to the proof of Theorem 2.5. Since the identity is trivial if $s = 0$, assume $s \neq 0$. We begin by applying Lemma 4.2 with $\rho(\boldsymbol{u}) \equiv \rho_\varepsilon(u_a)$, $f(j, \boldsymbol{\ell}) \equiv g(\boldsymbol{\ell})\eta_\varepsilon(\ell_a)1_{j=a}$, with the functions $\rho_\varepsilon\colon \mathbb{S}^n_+ \to \mathbb{R}$ and $\eta_\varepsilon\colon (0,1] \to \mathbb{R}$ chosen such that $T_a\rho$ and η are smooth compactly supported approximations to $\delta_{u_0}(u_a) - \delta_{\theta_s u_0}(u_a)$ and $\delta_{\cos s}(\ell_a)$ subject to $\rho_\varepsilon(u_a)\eta_\varepsilon(z_a) = 0$ for all $u_a \in \mathbb{S}^n_+$. Since $s \neq 0$, these conditions can be shown to be satisfiable by explicit construction. Exactly as in the proof of Theorem 2.5 this yields

$$\left[T_a\rho_\varepsilon(u_a) \int_0^\infty \mathbb{E}_{a,\boldsymbol{z}}(g(\boldsymbol{L}_t)\eta_\varepsilon(L^a_t)1_{X_t=a})\, dt \right]_\beta = 0. \tag{4.18}$$

To conclude, we use that $\theta_s u_0$ has z-coordinate $\cos s$, so the term with $\delta_{\theta_s u_0}(u_a)$ concentrates the u_a integral at $z_a = \cos s$, and hence the time integral at $t = 0$. This gives the right-hand side of (3.24). The term with $\delta_{u_0}(u_a)$ concentrates the time integral at $\tau(\cos s)$ and gives the left-hand side of (3.24) as the integrand is non-zero only if $\tau(\cos s) < \zeta$. □

The hemispherical Eisenbaum isomorphism theorem concerns a killed VDJP. The generator of this killed process $(X_t, \boldsymbol{L}_t)_{t\geqslant 0}$ acts on smooth compactly supported $f\colon \Lambda \times (0,1]^\Lambda \to \mathbb{R}$ by

$$(\mathcal{L}^h f)(i, \boldsymbol{\ell}) \equiv \mathcal{L}f(i, \boldsymbol{\ell}) - h_i f(i, \boldsymbol{\ell}), \quad \text{i.e.,} \quad \mathcal{L}^h = \mathcal{L} - \boldsymbol{h}, \tag{4.19}$$

where $\mathcal{L}$ is the generator of the VDJP and $h_i \geqslant 0$ are the killing rates. We let $\mathbb{E}^h_{i,\boldsymbol{\ell}}$ denote the corresponding deficient expectation. As for the SRW, the killing does not depend on the initial local times, i.e.,

$$\mathbb{E}^h_{i,\boldsymbol{\ell}}\big(g(X_t, \boldsymbol{L}_t)\big) = \mathbb{E}_{i,\boldsymbol{\ell}}\big(g(X_t, \boldsymbol{L}_t)e^{-\sum_{j\in\Lambda} h_j(\ell_j - L^j_t)}\big). \tag{4.20}$$

Notice that the sign in the killing term $e^{-\sum_{j\in\Lambda} h_j(\ell_j - L^j_t)}$ is reversed: this because the local times of the VDJP are decreasing rather than increasing by (4.2). We can rewrite (4.20) as

$$\mathbb{E}_{i,\boldsymbol{\ell}}(g(X_t, \boldsymbol{L}_t)e^{-\sum_{j\in\Lambda} h_j(1-L^j_t)})) = \mathbb{E}^h_{i,\boldsymbol{\ell}}(g(X_t, \boldsymbol{L}_t))e^{-\sum_{j\in\Lambda} h_j(1-\ell_j)}. \tag{4.21}$$

Theorem 4.6. *Let $[\cdot]_\beta$ be the unnormalised expectation of the $\mathbb{S}^n_+$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the killed VDJP. Suppose that $g\colon (0,1]^\Lambda \to \mathbb{R}$ is compactly supported, and $s \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then*

$$\left[x_a g(\boldsymbol{z}) e^{-(\boldsymbol{h}, \boldsymbol{1}-\theta_{-s}\boldsymbol{z})} \right]_\beta = \sum_{i\in\Lambda} h_i \left[(x_i - \theta_{-s}x_i) \int_0^\infty \mathbb{E}^h_{i,\boldsymbol{z}}(g(X_t, \boldsymbol{L}_t))\, dt\, e^{(\boldsymbol{h}, \boldsymbol{1}-\theta_{-s}\boldsymbol{z})} \right]_\beta. \tag{4.22}$$

Proof. We apply Lemma 4.2 with

$$\rho(\boldsymbol{u}) \equiv e^{(\boldsymbol{h},\theta_{-s}\boldsymbol{z}-\boldsymbol{z})} = e^{(\boldsymbol{h},\boldsymbol{1}-\boldsymbol{z})}(e^{-(\boldsymbol{h},\boldsymbol{1}-\theta_{-s}\boldsymbol{z})}) \tag{4.23}$$

$$f(j,\boldsymbol{\ell}) \equiv g(\boldsymbol{\ell})e^{-(\boldsymbol{h},\boldsymbol{1}-\boldsymbol{\ell})}1_{j=a}, \tag{4.24}$$

and use that $(T_j\rho)(\boldsymbol{u}) = h_j(x_j - \theta_{-s}x_j)\rho(\boldsymbol{u})$ to obtain

$$\begin{aligned}\sum_{j\in\Lambda} h_j\Big[(x_j - \theta_{-s}x_j)\rho(\boldsymbol{u})\int_0^\infty \mathbb{E}_{j,\boldsymbol{z}}(g(\boldsymbol{L}_t)1_{X_t=a}e^{-(\boldsymbol{h},\boldsymbol{1}-\boldsymbol{L}_t)})\,dt\Big]_\beta &= \sum_{j\in\Lambda}\Big[\rho(\boldsymbol{u})x_j g(\boldsymbol{z})1_{j=a}e^{-(\boldsymbol{h},\boldsymbol{1}-\boldsymbol{z})}\Big]_\beta \\ &= \Big[x_a g(\boldsymbol{z})e^{-(\boldsymbol{h},\boldsymbol{1}-\theta_{-s}\boldsymbol{z})}\Big]_\beta. \end{aligned} \tag{4.25}$$

Using (4.21) to substitute

$$\rho(\boldsymbol{u})\mathbb{E}_{j,\boldsymbol{z}}(g(\boldsymbol{L}_t)1_{X_t=a}e^{-(\boldsymbol{h},\boldsymbol{1}-\boldsymbol{L}_t)}) = \mathbb{E}^h_{j,\boldsymbol{z}}(g(\boldsymbol{L}_t)1_{X_t=a})e^{-(\boldsymbol{h},\boldsymbol{1}-\theta_{-s}\boldsymbol{z})} \tag{4.26}$$

on the left hand side of (4.25) gives the desired result. □

5 Isomorphism theorems for supersymmetric spin models

In this section we introduce the supersymmetric $\mathbb{R}^{2|2}$, $\mathbb{H}^{2|2}$, and $\mathbb{S}_+^{2|2}$ spin models and derive isomorphism theorems that relate them to the SRW, the VRJP, and the VDJP. Readers who are not familiar with the mathematics of supersymmetry may consult Appendix A, which contains an introduction to supersymmetry as used in this article, before reading this section.

5.1. Supersymmetric Gaussian free field.

5.1.1. Super-Euclidean space and the SUSY GFF. The *supersymmetric Gaussian free field* (*SUSY GFF* or $\mathbb{R}^{2|2}$ *model*) is defined in terms of the algebra of observables $\Omega^{2\Lambda}(\mathbb{R}^{2\Lambda}) \equiv \Omega^{2|\Lambda|}(\mathbb{R}^{2|\Lambda|})$, see Appendix A. This algebra replaces the algebra of observables $C^\infty(\mathbb{R}^{n\Lambda})$ of the usual n-component Gaussian free field.

Concretely, let $(\xi_i)_{i\in\Lambda}$ and $(\eta_i)_{i\in\Lambda}$ be the generators of the Grassmann algebra $\Omega^{2\Lambda}$, let $(x_i, y_i)_{i\in\Lambda}$ be coordinates for $\mathbb{R}^{2\Lambda}$, and let $\Omega^{2\Lambda}(\mathbb{R}^{2\Lambda})$ be the algebra with coefficients in $C^\infty(\mathbb{R}^{2\Lambda})$ generated by $(\xi_i)_{i\in\Lambda}$ and $(\eta_i)_{i\in\Lambda}$ as in Appendix A. We call elements F of $\Omega^{2\Lambda}(\mathbb{R}^{2\Lambda})$ *forms*, and say that a form is smooth, rapidly decaying, compactly supported, etc., if all of its coefficient functions have this property.

We think of $\Omega^{2\Lambda}(\mathbb{R}^{2\Lambda})$ as the smooth functions on a putative superspace $(\mathbb{R}^{2|2})^\Lambda$, though $(\mathbb{R}^{2|2})^\Lambda$ has no formal meaning, i.e., we will only work with the algebra $\Omega^{2\Lambda}(\mathbb{R}^{2\Lambda})$. There are two ordinary (even) coordinates and two anticommuting (odd) coordinates for each element $i \in \Lambda$, and by analogy with the familiar representation of a vector $u_i \in \mathbb{R}^2$ in terms of its coordinate functions $u_i = (x_i, y_i)$, we will abuse notation and write $u_i \in \mathbb{R}^{2|2}$ to refer to a *supervector* $u_i = (x_i, y_i, \xi_i, \eta_i)$, i.e., a tuple of of even and odd coordinates. Further, we define the super-Euclidean 'inner product' on $\mathbb{R}^{2|2}$ by

$$u_i \cdot u_j \equiv x_i x_j + y_i y_j - \xi_i\eta_j + \eta_i\xi_j. \tag{5.1}$$

Note that the 'inner product' (5.1) defines a form, and is not an inner product in the standard sense of the term. Similarly, we write $\boldsymbol{u} = (u_i)_{i\in\Lambda}$ to denote the collection of the u_i, and define $(\boldsymbol{u}, -\Delta_\beta\boldsymbol{u})$ analogously, i.e., by

$$\begin{aligned}(\boldsymbol{u}, -\Delta_\beta\boldsymbol{u}) &\equiv \sum_{i\in\Lambda}\sum_{j\in\Lambda}\beta_{ij}(x_i(x_i - x_j) + y_i(y_i - y_j) - \xi_i(\eta_i - \eta_j) + \eta_i(\xi_i - \xi_j)) && (5.2)\\ &= \frac{1}{2}\sum_{i,j\in\Lambda}\beta_{ij}(u_i\cdot u_i + u_j\cdot u_j - u_i\cdot u_j - u_j\cdot u_i), && (5.3)\end{aligned}$$

where the second equality is a calculation. The formal rules introduced above imply the last quantity is $\frac{1}{4}\sum_{i,j\in\Lambda}\beta_{ij}(u_i-u_j)^2$ if we interpret u_i-u_j as $(x_i-x_j, y_i-y_j, \xi_i-\xi_j, \eta_i-\eta_j)$ and use (5.1) to compute $(u_i-u_j)^2 \equiv (u_i-u_j)\cdot(u_i-u_j)$.

For $F \in \Omega^{2\Lambda}(\mathbb{R}^{2\Lambda})$, the normalised Berezin integral is denoted

$$\int_{(\mathbb{R}^{2|2})^\Lambda} F \equiv \frac{1}{(2\pi)^{|\Lambda|}}\int d\boldsymbol{x}\,d\boldsymbol{y}\,\partial_{\boldsymbol{\eta}}\,\partial_{\boldsymbol{\xi}}\,F, \tag{5.4}$$

where $\partial_{\boldsymbol{\eta}}\partial_{\boldsymbol{\xi}}$ is defined by $\partial_{\eta_{|\Lambda|}}\partial_{\xi_{|\Lambda|}}\dots\partial_{\eta_1}\partial_{\xi_1}$, $d\boldsymbol{x} = dx_{|\Lambda|}\dots dx_1$, and $d\boldsymbol{y} = dy_{|\Lambda|}\dots dy_1$ for some fixed ordering of the $i\in\Lambda$ from 1 to $|\Lambda|$.

To define the supersymmetric GFF, suppose $\boldsymbol{h}\geqslant \boldsymbol{0}$ and let

$$H_\beta(\boldsymbol{u}) \equiv \frac{1}{2}(\boldsymbol{u}, -\Delta_\beta \boldsymbol{u}), \quad H_{\beta,h}(\boldsymbol{u}) \equiv H_\beta(\boldsymbol{u}) + \frac{1}{2}(\boldsymbol{h}, |\boldsymbol{u}|^2), \tag{5.5}$$

where $|\boldsymbol{u}|^2 \equiv (u_i\cdot u_i)_{i\in\Lambda}$, and hence $(\boldsymbol{h},|\boldsymbol{u}|^2) = \sum_{i\in\Lambda} h_i u_i\cdot u_i$. Both H_β and $H_{\beta,h}$ are elements of $\Omega^{2\Lambda}(\mathbb{R}^{2\Lambda})$. The superexpectation of the *supersymmetric Gaussian free field* is the linear map that assigns to each $F\in\Omega^{2\Lambda}(\mathbb{R}^{2\Lambda})$ the value

$$[F]_{\beta,h} \equiv \int_{(\mathbb{R}^{2|2})^\Lambda} F e^{-H_{\beta,h}}, \tag{5.6}$$

and we write $[F]_\beta$ when $\boldsymbol{h}=\boldsymbol{0}$. For $\boldsymbol{h}\neq\boldsymbol{0}$, this superexpectation is indeed normalised; see the paragraph below (5.13).

5.1.2. Symmetries. In this section we describe the two aspects of the symmetries of the SUSY GFF that we require. Further details about these symmetries, which form a Lie superalgebra, can be found in Appendix B.4.

As for the GFF, the infinitesimal generator of translation in the x-direction at $i\in\Lambda$ is

$$T_i \equiv \frac{\partial}{\partial x_i}, \tag{5.7}$$

and T_i acts on coordinates as

$$T_i x_j = 1_{i=j}, \quad T_i y_j = 0, \quad T_i\eta_j = 0, \quad T_i\xi_j = 0, \qquad i,j\in\Lambda. \tag{5.8}$$

Thus it is analogous to the operators T_i for the ordinary GFF, and it leads to analogous Ward identities, i.e., for forms F with sufficient decay,

$$\int_{(\mathbb{R}^{2|2})^\Lambda} (T_i F) = 0. \tag{5.9}$$

For $s\in\mathbb{R}$ the finite symmetry associated to $\sum_{i\in\Lambda} T_i$ will be denoted θ_s, which acts by

$$\theta_s x_i = x_i + s, \quad \theta_s y_i = y_i, \quad \theta_s \eta_i = \eta_i, \quad \theta_s\xi_i = \xi_i, \qquad i\in\Lambda. \tag{5.10}$$

The second symmetry of importance is the *supersymmetry generator*

$$Q \equiv \sum_{i\in\Lambda} Q_i \qquad Q_i \equiv \xi_i\frac{\partial}{\partial x_i} + \eta_i\frac{\partial}{\partial y_i} - x_i\frac{\partial}{\partial \eta_i} + y_i\frac{\partial}{\partial \xi_i}, \tag{5.11}$$

which acts on coordinates as

$$Qx_i = \xi_i, \quad Qy_i = \eta_i, \quad Q\xi_i = -y_i, \quad Q\eta_i = x_i, \qquad i\in\Lambda. \tag{5.12}$$

This supersymmetry generator is responsible for a very powerful Ward identity known as the *localisation lemma*: for any smooth function $f\colon \mathbb{R}^{\Lambda\times\Lambda}\to\mathbb{R}$ with sufficient decay,

$$\int_{(\mathbb{R}^{2|2})^\Lambda} f(\boldsymbol{u}\boldsymbol{u}^T) = f(\boldsymbol{0}), \tag{5.13}$$

where $\boldsymbol{u}\boldsymbol{u}^T$ denotes the collection of forms $(u_i\cdot u_j)_{i,j\in\Lambda}$; see Theorem A.8 and Corollary A.10. In particular, the expectation (5.6) is normalised if $\boldsymbol{h}\neq\boldsymbol{0}$, i.e., $[1]_{\beta,h}=1$.

5.1.3. Isomorphism theorems for the SUSY GFF. This section presents isomorphism theorems for the SUSY GFF. The statement of the following fundamental Ward identity is formally identical to that of Lemma 2.1, but now the expectation $[\cdot]_\beta$ is that of a SUSY GFF.

Lemma 5.1. *Let $[\cdot]_\beta$ be the superexpectation of the $\mathbb{R}^{2|2}$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the SRW. Let $f\colon \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ be a smooth function with rapid decay, and let $\rho \in \Omega^{2\Lambda}(\mathbb{R}^{2\Lambda})$ have moderate growth. Then:*

$$-\sum_{j\in\Lambda}\Big[\rho(\boldsymbol{u})x_j\mathcal{L}f(j,\frac{1}{2}|\boldsymbol{u}|^2)\Big]_\beta = \sum_{j\in\Lambda}\Big[(T_j\rho)(\boldsymbol{u})f(j,\frac{1}{2}|\boldsymbol{u}|^2)\Big]_\beta. \tag{5.14}$$

In particular, the following integrated version holds for all smooth $f\colon \Lambda\times\mathbb{R}^\Lambda \to \mathbb{R}$ with rapid decay:

$$\sum_{j\in\Lambda}\Big[\rho(\boldsymbol{u})x_jf(j,\frac{1}{2}|\boldsymbol{u}|^2)\Big]_\beta = \sum_{j\in\Lambda}\Big[(T_j\rho)(\boldsymbol{u})\int_0^\infty \mathbb{E}_{j,\frac{1}{2}|\boldsymbol{u}|^2}(f(X_t,\boldsymbol{L}_t))\,dt\Big]_\beta. \tag{5.15}$$

Proof. Starting from (5.9), the proof is identical to that of Lemma 2.1. □

As a consequence, we obtain the same isomorphism theorems for the supersymmetric GFF as for the non-supersymmetric one. However, for the supersymmetric model, we may in addition use *localisation* to greatly simplify the left-hand side of (5.15) when $T_j\rho(\boldsymbol{u})$ is supersymmetric.

Theorem 5.2. *Let $[\cdot]_\beta$ be the superexpectation of the $\mathbb{R}^{2|2}$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the SRW. Let $g\colon \mathbb{R}^\Lambda \to \mathbb{R}$ be a smooth function with rapid decay, and let $a,b\in\Lambda$. Then*

$$\Big[x_ax_bg(\frac{1}{2}|\boldsymbol{u}|^2)\Big]_\beta = \int_0^\infty \mathbb{E}_{a,0}(g(\boldsymbol{L}_t)1_{X_t=b})\,dt. \tag{5.16}$$

Proof. Apply Lemma 5.1 with $\rho(\boldsymbol{u}) = x_a$, $f(j,\boldsymbol{\ell}) = g(\boldsymbol{\ell})1_{j=b}$, and note $T_j\rho(\boldsymbol{u}) = 1_{j=a}$. Thus the integrand on the right-hand side of (5.15) is a function of $|\boldsymbol{u}|^2$, and hence is supersymmetric. By applying localisation, i.e., (5.13), we conclude

$$\Big[\int_0^\infty \mathbb{E}_{a,\frac{1}{2}|\boldsymbol{u}|^2}(1_{X_t=b}g(\boldsymbol{L}_t))\,dt\Big]_\beta = \int_0^\infty \mathbb{E}_{a,0}(1_{X_t=b}g(\boldsymbol{L}_t))\,dt. \qquad □$$

Remark 5.3. Theorem 5.2 has its origins in physics [33,34,37,41]. A formulation similar to the one presented here was given in [14], see also [32].

The Ray–Knight isomorphism theorem applies to spin models in which the spin at vertex a is fixed; in the supersymmetric version the constraint is $u_a = (0,0,0,0)$. We write the corresponding unnormalised expectation of an observable F as

$$[F\delta_{u_0}(u_a)]_\beta. \tag{5.17}$$

Theorem 5.4. *Let $[\cdot]_\beta$ be the superexpectation of the $\mathbb{R}^{2|2}$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the SRW. Let $g\colon \mathbb{R}^\Lambda\to\mathbb{R}$ be smooth and compactly supported, let $a\in\Lambda$, and let $s\in\mathbb{R}$. Then*

$$\Big[g(\frac{1}{2}|\theta_s\boldsymbol{u}|^2)\delta_{u_0}(u_a)\Big]_\beta = \mathbb{E}_{a,0}g(\boldsymbol{L}_{\tau(\frac{s^2}{2})}) \tag{5.18}$$

where $\tau(\gamma) \equiv \inf\{t \mid L_a^t \geq \gamma\}$ and $u_0 = (0,0,0,0)$.

Proof. The proof is by applying Lemma 5.1 with $\rho(\boldsymbol{u}) \equiv \rho_\varepsilon(u_a)$, $f(j,\boldsymbol{\ell}) \equiv g(\boldsymbol{\ell})\eta_\varepsilon(\ell_a)1_{j=a}$, and the form $\rho_\varepsilon \in \Omega^2(\mathbb{R}^2)$ and function $\eta_\varepsilon\colon \mathbb{R}\to\mathbb{R}$ chosen such that $T_a\rho_\varepsilon$ and η_ε are smooth compactly supported approximations to $\delta_{u_0}(u_a) - \delta_{u_0}(\theta_{-s}u_a)$ and $\delta_{\frac{1}{2}s^2}$ subject to $\rho_\varepsilon(u_a)\eta_\varepsilon(\frac{1}{2}|u_a|^2) = 0$. We refer to Appendix B.5 for smooth approximations to $\delta_{u_0}(u_a)$.

An argument identical to the one in the proof of Theorem 2.5 shows

$$\left[\delta_{u_0,\varepsilon}(\theta_{-(s-\varepsilon)}u_a)\int_0^\infty \mathbb{E}_{a,\frac{1}{2}|\boldsymbol{u}|^2}(g(\boldsymbol{L}_t)\eta_\varepsilon(L_t^a)1_{X_t=a}\,dt\right]_\beta$$
$$= \left[\delta_{u_0,\varepsilon}(u_a)\int_0^\infty \mathbb{E}_{a,\frac{1}{2}|\boldsymbol{u}|^2}(g(\boldsymbol{L}_t)\eta_\varepsilon(L_t^a)1_{X_t=a})\,dt\right]_\beta. \quad (5.19)$$

By choosing $\delta_{u_0,\varepsilon}(u_a)$ to be supersymmetric, i.e., $Q\delta_{u_0,\varepsilon}=0$, the integrand on the right-hand side is a product of supersymmetric forms and is therefore supersymmetric. Applying supersymmetric localisation (i.e., (5.13)) hence shows

$$\left[\delta_{u_0,\varepsilon}(\theta_{-(s-\varepsilon)}u_a)\int_0^\infty \mathbb{E}_{a,\frac{1}{2}|u|^2}(g(\boldsymbol{L}_t)\eta_\varepsilon(L_t^a)1_{X_t=a})\,dt\right]_\beta = \int_0^\infty \mathbb{E}_{a,0}(g(\boldsymbol{L}_t)\eta_\varepsilon(L_t^a)1_{X_t=a})\,dt. \quad (5.20)$$

Applying a global translation $\theta_{s-\varepsilon}$ on the left-hand side and then taking $\varepsilon\to 0$ as in the proof of Theorem 2.5 gives the desired result

$$\left[g(\frac{1}{2}|\theta_s\boldsymbol{u}|^2)\delta_{u_0}(u_a)\right]_\beta = \mathbb{E}_{a,0}g(\boldsymbol{L}_{\tau(\frac{s^2}{2})}). \qquad \square$$

The preceding two theorems are analogues of the BFS–Dynkin and Ray–Knight isomorphisms for the SUSY GFF. While calculations analogous to those leading to the Eisenbaum isomorphism can be carried out for the SUSY GFF, it is not possible to apply localisation, because the form $\frac{1}{2}|\theta_s\boldsymbol{u}|^2$ that arises (recall (2.34)) is not supersymmetric.

5.2. SUSY hyperbolic model $\mathbb{H}^{2|2}$. In this section we introduce the supersymmetric analogue of the $\mathbb{H}^2$ model, and then obtain the associated isomorphism theorems.

5.2.1. Super-Minkowski space $\mathbb{R}^{3|2}$ and the super-Minkowski model. Let $(\xi_i,\eta_i)_{i\in\Lambda}$ be the generators of the Grassmann algebra $\Omega^{2\Lambda}$. The algebra of observables $\Omega^{2\Lambda}(\mathbb{R}^{3\Lambda})$ is the algebra generated by $(\xi_i,\eta_i)_{i\in\Lambda}$ with coefficients in $C^\infty(\mathbb{R}^{3\Lambda})$. Choosing orthonormal coordinates $(z_i,x_i,y_i)_{i\in\Lambda}$ for $\mathbb{R}^{3\Lambda}$, a supervector $u_i\in\mathbb{R}^{3|2}$ is a tuple of even and odd coordinates $u_i=(z_i,x_i,y_i,\xi_i,\eta_i)$, and we say that $\mathbb{R}^{3|2}$ is a *super-Minkowski space* when equipped with the 'inner product'

$$u_i\cdot u_j \equiv -z_iz_j + x_ix_j + y_iy_j - \xi_i\eta_j + \eta_i\xi_j. \quad (5.21)$$

We have written 'inner product' to emphasise that $u_i\cdot u_j$ is a form, and hence this is not an inner product in the standard sense of the term.

5.2.2. $\mathbb{H}^{2|2}$ sigma model. To define a supersymmetric analogue of $\mathbb{H}^2$, define the even form

$$z = z(x,y,\xi,\eta) \equiv \sqrt{1+x^2+y^2-2\xi\eta} = \sqrt{1+x^2+y^2} - \frac{\xi\eta}{\sqrt{1+x^2+y^2}}. \quad (5.22)$$

Using the definition (5.21), a short calculation shows that $u_i\cdot u_i=-1$, just as for $\mathbb{H}^2$. The algebra of forms $\Omega^2(\mathbb{H}^2)$ is the algebra over $C^\infty(\mathbb{H}^2)$ generated by two Grassmann generators ξ and η. In coordinates, we have $F(u)=F(z,x,y,\xi,\eta)=F(\sqrt{1+x^2+y^2-2\xi\eta},x,y,\xi,\eta)$, and hence every form $F\in\Omega^2(\mathbb{H}^2)$ can be identified with a form in $\Omega^2(\mathbb{R}^2)$. Using this correspondence we define the Berezin integral for $F\in\Omega^2(\mathbb{H}^2)$ as

$$\int_{\mathbb{H}^{2|2}} F \equiv \int_{\mathbb{R}^{2|2}} \frac{1}{z}F = \frac{1}{2\pi}\int dx\,dy\,\partial_\xi\,\partial_\eta\frac{1}{z}F \quad (5.23)$$

where on the right-hand side we are viewing F as a form in $\Omega^2(\mathbb{R}^2)$. Similarly,

$$\int_{(\mathbb{H}^{2|2})^\Lambda} F \equiv \int_{(\mathbb{R}^{2|2})^\Lambda} \frac{1}{\prod_{i\in\Lambda} z_i}F = \frac{1}{(2\pi)^{|\Lambda|}}\int d\boldsymbol{x}\,d\boldsymbol{y}\,\partial_{\boldsymbol{\eta}}\,\partial_{\boldsymbol{\xi}}\,\frac{1}{\prod_{i\in\Lambda} z_i}F \quad (5.24)$$

where we note there is no ambiguity in the product of the z_i as they are even forms.

Define, for $\boldsymbol{h} \geqslant \boldsymbol{0}$,

$$H_\beta(\boldsymbol{u}) \equiv \frac{1}{2}(\boldsymbol{u}, -\Delta_\beta \boldsymbol{u}), \quad H_{\beta,h}(\boldsymbol{u}) \equiv H_\beta(\boldsymbol{u}) + (\boldsymbol{h}, \boldsymbol{z} - \boldsymbol{1}), \tag{5.25}$$

where

$$\begin{aligned}(\boldsymbol{u}, -\Delta_\beta \boldsymbol{u}) &\equiv \frac{1}{2}\sum_{i,j\in\Lambda} \beta_{ij}(u_i \cdot u_i + u_j \cdot u_j - u_i \cdot u_j - u_j \cdot u_i) = \frac{1}{2}\sum_{i,j\in\Lambda} \beta_{ij}(-2 - 2u_i \cdot u_j),\\ (\boldsymbol{h}, \boldsymbol{z} - \boldsymbol{1}) &\equiv \sum_{i\in\Lambda} h_i(z_i - 1),\end{aligned} \tag{5.26}$$

and each $u_i \cdot u_j$ is defined as in (5.21). The equality in the first line holds because $u_i \cdot u_i = -1$. We define the $\mathbb{H}^{2|2}$ *model* superexpectation for $F \in \Omega^{2\Lambda}(\mathbb{H}^{2\Lambda})$ by

$$[F]_{\beta,h} \equiv \int_{(\mathbb{H}^{2|2})^\Lambda} F e^{-H_{\beta,h}}, \tag{5.27}$$

and we write $[F]_\beta$ in the case $\boldsymbol{h} = \boldsymbol{0}$. For $\boldsymbol{h} \neq \boldsymbol{0}$, the superexpectation is normalised, i.e., $[1]_{\beta,h} = 1$. This is a consequence of supersymmetry, see (5.32) below.

5.2.3. Symmetries. There are two symmetries necessary for what follows, and we introduce them in this section. For a further discussion of the Lie superalgebra of symmetries associated to the $\mathbb{H}^{2|2}$ model see Appendix B.4.

The first relevant symmetry is the infinitesimal Lorentz boost in the xz plane at $i \in \Lambda$:

$$T_i \equiv z_i \frac{\partial}{\partial x_i} = \sqrt{1 + x^2 + y^2 - 2\xi\eta}\,\frac{\partial}{\partial x_i}, \tag{5.28}$$

which acts on coordinates as

$$T_i z_j = x_j 1_{i=j}, \quad T_i x_j = z_j 1_{i=j}, \quad T_i y_j = 0, \quad T_i \xi_j = 0, \quad T_i \eta_j = 0 \qquad i,j \in \Lambda. \tag{5.29}$$

As for the SUSY GFF, this leads to a Ward identity for forms F with rapid decay:

$$\int_{(\mathbb{H}^{2|2})^\Lambda} (T_i F) = 0. \tag{5.30}$$

For $s \in \mathbb{R}$ the finite symmetry associated to $\sum_{i\in\Lambda} T_i$ will be denoted θ_s, and acts as (for $j \in \Lambda$)

$$\theta_s z_j = z_j \cosh s + x_j \sinh s, \;\; \theta_s x_j = x_j \cosh s + z_j \sinh s, \;\; \theta_s y_j = y_j, \;\; \theta_s \xi_j = \xi_j, \;\; \theta_s \eta_j = \eta_j. \tag{5.31}$$

The second relevant symmetry is the supersymmetry generator Q, which is defined by (5.11). Note that z_i can be written as $z_i = \sqrt{1 + |\tilde{u}_i|^2}$, where $\tilde{u}_i \equiv (x_i, y_i, \xi_i, \eta_i) \in \mathbb{R}^{2|2}$. Thus, z_i is supersymmetric, i.e., $Qz_i = 0$. This implies the same localisation Ward identity applies for $\mathbb{H}^{2|2}$ as for $\mathbb{R}^{2|2}$, i.e., for smooth functions $f\colon \mathbb{R}^\Lambda \times \mathbb{R}^{\Lambda\times\Lambda} \to \mathbb{R}$ with sufficient decay,

$$\int_{(\mathbb{H}^{2|2})^\Lambda} f(\boldsymbol{z}, \tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}^T) = f(\boldsymbol{1}, \boldsymbol{0}) \tag{5.32}$$

where $\boldsymbol{0}$ is the matrix indexed by Λ with all entries 0, and we have written $\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}^T$ to denote the set of forms $(\tilde{u}_i \cdot \tilde{u}_j)_{i,j\in\Lambda}$.

5.2.4. Isomorphism theorems for the $\mathbb{H}^{2|2}$ model. Let $\mathbb{E}_{i,\boldsymbol{\ell}}$ denote the expectation for a VRJP started from initial conditions $(i, \boldsymbol{\ell})$. We begin with the SUSY analogue of Lemma 3.2.

Lemma 5.5. *Let $[\cdot]_\beta$ be the superexpectation of the $\mathbb{H}^{2|2}$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the VRJP. Let $f\colon \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ be a smooth function with rapid decay, and let $\rho \in \Omega^{2\Lambda}(\mathbb{H}^{2\Lambda})$ have moderate growth. Then:*

$$-\sum_{j\in\Lambda}[\rho(\boldsymbol{u})x_j\mathcal{L}f(j,\boldsymbol{z})]_\beta = \sum_{j\in\Lambda}[(T_j\rho)(\boldsymbol{u})f(j,\boldsymbol{z})]_\beta. \tag{5.33}$$

In particular, the following integrated version holds for all smooth $f\colon \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ with rapid decay:

$$\sum_{j\in\Lambda}[\rho(\boldsymbol{u})x_j f(j,\boldsymbol{z})]_\beta = \sum_{j\in\Lambda}\left[(T_j\rho)(\boldsymbol{u})\int_0^\infty \mathbb{E}_{j,\boldsymbol{z}}(f(X_t,\boldsymbol{L}_t))\,dt\right]_\beta. \tag{5.34}$$

Proof. The proof is identical to that of Lemma 3.2. □

The SUSY analogue of Theorem 3.3 is the following.

Theorem 5.6. *Let $[\cdot]_\beta$ be the superexpectation of the $\mathbb{H}^{2|2}$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the VRJP. Let $g\colon \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ be a smooth function with rapid decay, and let $a, b \in \Lambda$. Then*

$$[x_a x_b g(\boldsymbol{z})]_\beta = \int_0^\infty \mathbb{E}_{a,\mathbf{1}}(g(\boldsymbol{L}_t)1_{X_t=b})\,dt. \tag{5.35}$$

Proof. Apply Lemma 5.5 with $\rho(\boldsymbol{u}) = x_a$ and $f(j,\boldsymbol{\ell}) = g(\boldsymbol{\ell})1_{j=b}$. Thus $T_j\rho(\boldsymbol{u}) = 1_{j=a}z_a$. By applying localisation, i.e., (5.32), we obtain

$$[x_a x_b g(\boldsymbol{z})]_\beta = \left[z_a\int_0^\infty \mathbb{E}_{a,\boldsymbol{z}}(g(\boldsymbol{L}_t)1_{X_t=b})\,dt\right]_\beta = \int_0^\infty \mathbb{E}_{a,\mathbf{1}}(g(\boldsymbol{L}_t)1_{X_t=b})\,dt. \qquad \square$$

Theorem 5.7. *Let $[\cdot]_\beta$ be the superexpectation of the $\mathbb{H}^{2|2}$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the VRJP. Let $g\colon \mathbb{R}^\Lambda \to \mathbb{R}$ be a smooth compactly supported function, let $a \in \Lambda$, and let $s \in \mathbb{R}$. Then*

$$[g(\theta_s\boldsymbol{z})\delta_{u_0}(u_a)]_\beta = \mathbb{E}_{a,\mathbf{1}}g(\boldsymbol{L}_{\tau(\cosh s)}) \tag{5.36}$$

where $\tau(\gamma) = \inf\{t \mid L_a^t \geq \gamma\}$ and $u_0 = (1,0,0,0,0)$.

Proof. Applying Lemma 5.5 with $\rho(\boldsymbol{u}) \equiv \rho_\varepsilon(u_a)$, $f(j,\boldsymbol{\ell}) \equiv g(\boldsymbol{\ell})\eta_\varepsilon(\ell_a)1_{j=a}$, and the form $\rho_\varepsilon \in \Omega^2(\mathbb{H}^2)$ and function $\eta_\varepsilon\colon \mathbb{R}_+ \to \mathbb{R}$ chosen such that $T_a\rho_\varepsilon$ and η_ε are smooth compactly supported approximations to $\delta_{u_0}(u_a) - \delta_{\theta_s u_0}(u_a)$ and $\delta_{\cosh s}$ subject to $\rho_\varepsilon(u_a)\eta_\varepsilon(z_a) = 0$, an argument identical to the proof of Theorem 3.4 shows

$$\left[\delta_{u_0,\varepsilon}(\theta_{-(s-\varepsilon)}u_a)\int_0^\infty \mathbb{E}_{a,\boldsymbol{z}}(g(\boldsymbol{L}_t)\eta_\varepsilon(L_t^a)1_{X_t=a})\,dt\right]_\beta = \left[\delta_{u_0,\varepsilon}(u_a)\int_0^\infty \mathbb{E}_{a,\boldsymbol{z}}(g(\boldsymbol{L}_t)\eta_\varepsilon(L_t^a)1_{X_t=a})\,dt\right]_\beta. \tag{5.37}$$

As in the proof of Theorem 5.4, $\delta_{u_0,\varepsilon}(u_a)$ is chosen to be supersymmetric. The claim follows by applying localisation to the right-hand side, boosting the left-hand side by $\theta_{s-\varepsilon}$, and then taking $\varepsilon \to 0$ as in the proof of Theorem 3.4:

$$\left[\mathbb{E}_{a,\boldsymbol{z}}g(\boldsymbol{L}_{\tau(\cosh s)})\delta_{u_0}(u_a)\right]_\beta = \mathbb{E}_{a,\mathbf{1}}g(\boldsymbol{L}_{\tau(\cosh s)}). \qquad \square$$

Remark 5.8. The $\mathbb{H}^{2|2}$ model was introduced in [54]; it serves as a toy model for Efetov's supersymmetric approach to studying random band matrices [25]. The connection between random band matrices and hyperbolic symmetry goes back to Wegner and Schäfer [46, 53], and Efetov made use of supersymmetry to avoid the use of the replica trick. For further discussion see [23], and for other uses of supersymmetry in the study of random matrices see, e.g., [20, 21, 47].

Remark 5.9. Unlike the $\mathbb{H}^n$ models, the $\mathbb{H}^{2|2}$ model captures the phenomenology of a localisation/delocalisation transition [23, 48].

5.3. SUSY hemispherical model $\mathbb{S}_+^{2|2}$. In this section we introduce the supersymmetric analogue of the $\mathbb{S}_+^2$ model, and then obtain the associated isomorphism theorems.

5.3.1. Integrals over $\mathbb{S}_+^{2|2}$. In this subsection we work with smooth compactly supported forms in $\Omega^{2\Lambda}(\mathbb{S}_+^{2\Lambda})$, which we denote $\Omega_c^{2\Lambda}(\mathbb{S}_+^{2\Lambda})$. Concretely, we will identify such forms with compactly supported forms in $\Omega^{2\Lambda}(\mathbb{B}^{2\Lambda})$, where $\mathbb{B}^2$ is the open unit ball, by setting

$$z = z(x, y, \xi, \eta) \equiv \sqrt{1 - x^2 - y^2 + 2\xi\eta} = \sqrt{1 - x^2 - y^2} + \frac{\xi\eta}{\sqrt{1 - x^2 - y^2}}, \tag{5.38}$$

By considering $\mathbb{B}^2$ as a subset of $\mathbb{R}^2$, a compactly supported form in $\Omega^{2\Lambda}(\mathbb{B}^{2\Lambda})$ can be trivially extended to a form in $\Omega^{2\Lambda}(\mathbb{R}^{2\Lambda})$, and we may therefore apply the results of Appendix A.

Similarly to the notation introduced in Section 5.2.2, let $u_i = (z_i, x_i, y_i, \xi_i, \eta_i)$, and let

$$u_i \cdot u_j \equiv z_i z_j + x_i x_j + y_i y_j - \xi_i \eta_j + \eta_i \xi_j, \qquad i, j \in \Lambda. \tag{5.39}$$

With these definitions, $u_i \cdot u_i = 1$, just as for $\mathbb{S}_+^2$. We define, for $F \in \Omega_c^2(\mathbb{S}_+^2)$,

$$\int_{\mathbb{S}_+^{2|2}} F \equiv \frac{1}{2\pi} \int dx\, dy\, \partial_\xi\, \partial_\eta \frac{1}{z} F, \tag{5.40}$$

and similarly, for $F \in \Omega_c^{2\Lambda}(\mathbb{S}_+^{2\Lambda})$,

$$\int_{(\mathbb{S}_+^{2|2})^\Lambda} F \equiv \frac{1}{(2\pi)^{|\Lambda|}} \int d\boldsymbol{x}\, d\boldsymbol{y}\, \partial_{\boldsymbol{\xi}}\, \partial_{\boldsymbol{\eta}} \frac{1}{\prod_{i\in\Lambda} z_i} F, \tag{5.41}$$

where we note there is no ambiguity in the product of the z_i as they are even forms.

5.3.2. $\mathbb{S}_+^{2|2}$ model. Define, for $\boldsymbol{h} \geqslant \boldsymbol{0}$,

$$H_\beta(\boldsymbol{u}) \equiv \frac{1}{2}(\boldsymbol{u}, -\Delta_\beta \boldsymbol{u}), \quad H_{\beta,h}(\boldsymbol{u}) \equiv H_\beta(\boldsymbol{u}) + (\boldsymbol{h}, \boldsymbol{1} - \boldsymbol{z}), \tag{5.42}$$

where

$$\begin{aligned} (\boldsymbol{u}, -\Delta_\beta \boldsymbol{u}) &\equiv \frac{1}{2} \sum_{i,j\in\Lambda} \beta_{ij}(u_i \cdot u_i + u_j \cdot u_j - u_i \cdot u_j - u_j \cdot u_i) = \frac{1}{2} \sum_{i,j\in\Lambda} \beta_{ij}(2 - 2u_i \cdot u_j), \\ (\boldsymbol{h}, \boldsymbol{1} - \boldsymbol{z}) &\equiv \sum_{i\in\Lambda} h_i(1 - z_i) \end{aligned} \tag{5.43}$$

and $u_i \cdot u_j$ is defined as in (5.39). We define the $\mathbb{S}_+^{2|2}$ *model* superexpectation of $F \in \Omega_c^{2\Lambda}(\mathbb{S}_+^{2\Lambda})$ by

$$[F]_\beta \equiv \int_{(\mathbb{S}_+^{2|2})^\Lambda} F e^{-H_\beta}, \quad [F]_{\beta,h} \equiv \int_{(\mathbb{S}_+^{2|2})^\Lambda} F e^{-H_{\beta,h}}. \tag{5.44}$$

5.3.3. Symmetries. As in the previous sections, there are two symmetries of relevance to the following discussion. For details on the Lie superalgebra associated to $\mathbb{S}_+^{2|2}$, see Appendix B.4. The first symmetry of relevance is an infinitesimal rotation in the xz-plane at $i \in \Lambda$, which has generator

$$T_i \equiv z_i \frac{\partial}{\partial x_i} = \sqrt{1 - x_i^2 - y_i^2 + 2\xi_i\eta_i} \frac{\partial}{\partial x_i}, \tag{5.45}$$

and acts on coordinates as

$$T_i z_j = -x_j 1_{i=j}, \quad T_i x_j = z_j 1_{i=j}, \quad T_i y_j = 0, \quad T_i \xi_j = 0, \quad T_i \eta_j = 0, \qquad i, j \in \Lambda. \tag{5.46}$$

As for the SUSY GFF, this leads to a Ward identity for all sufficiently rapidly decaying forms F:

$$\int_{(\mathbb{S}_+^{2|2})^\Lambda} (T_i F) = 0. \tag{5.47}$$

For $s \in \mathbb{R}$ the finite rotation associated to $\sum_{i\in\Lambda} T_i$ is denoted θ_s, and acts as, for $j \in \Lambda$,

$$\theta_s z_j = z_j \cos s - x_j \sin s, \quad \theta_s x_j = x_j \cos s + z_j \sin s, \quad \theta_s y_j = y_j, \quad \theta_s \xi_j = \xi_j, \quad \theta_s \eta_j = \eta_j. \tag{5.48}$$

The second symmetry of importance is the supersymmetry generator Q defined by (5.11). Note that z_i can be written as $z_i = \sqrt{1 - |\tilde{u}_i|^2}$, where $\tilde{u}_i \equiv (x_i, y_i, \xi_i, \eta_i) \in \mathbb{R}^{2|2}$. It follows that z_i is supersymmetric, i.e., $Qz_i = 0$. This implies the same localisation Ward identity applies for $\mathbb{S}_+^{2|2}$ as for $\mathbb{R}^{2|2}$, i.e., for $f\colon (0,1]^\Lambda \times [-1,1]^{\Lambda\times\Lambda} \to \mathbb{R}$ that are smooth and compactly supported,

$$\int_{(\mathbb{S}_+^{2|2})^\Lambda} f(\boldsymbol{z}, \tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}^T) = f(\boldsymbol{1}, \boldsymbol{0}), \tag{5.49}$$

where $\boldsymbol{0}$ is the matrix indexed by Λ with all entries 0 and $\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}^T \equiv (\tilde{u}_i \cdot \tilde{u}_j)_{i,j\in\Lambda}$.

5.3.4. Isomorphism theorems for the $\mathbb{S}_+^{2|2}$ model. Let $\mathbb{E}_{i,\boldsymbol{\ell}}$ denote the expectation for a VDJP started from initial conditions $(i, \boldsymbol{\ell}) \in \Lambda \times (0,1]^\Lambda$.

Lemma 5.10. *Let $[\cdot]_\beta$ be the superexpectation of the $\mathbb{S}_+^{2|2}$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the VDJP. Let $f\colon \Lambda \times (0,1]^\Lambda \to \mathbb{R}$ be a smooth compactly supported function and let $\rho \in \Omega_c^{2\Lambda}(\mathbb{S}_+^{2\Lambda})$. Then:*

$$-\sum_{j\in\Lambda} [\rho(\boldsymbol{u}) x_j \mathcal{L} f(j, \boldsymbol{z})]_\beta = \sum_{j\in\Lambda} [(T_j\rho)(\boldsymbol{u}) f(j, \boldsymbol{z})]_\beta. \tag{5.50}$$

In particular, the following integrated version holds for smooth and compactly supported $f\colon \Lambda \times (0,1]^\Lambda \to \mathbb{R}$:

$$\sum_{j\in\Lambda} [\rho(\boldsymbol{u}) x_j f(j, \boldsymbol{z})]_\beta = \sum_{j\in\Lambda} \left[(T_j\rho)(\boldsymbol{u}) \int_0^\infty \mathbb{E}_{j,\boldsymbol{z}}(f(X_t, \boldsymbol{L}_t))\, dt \right]_\beta. \tag{5.51}$$

Proof. The proof is identical to that of Lemma 4.2. □

The SUSY analogue of Theorem 4.4 is the following.

Theorem 5.11. *Let $[\cdot]_\beta$ be the superexpectation of the $\mathbb{S}_+^{2|2}$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the VDJP. Let $g\colon (0,1]^\Lambda \to \mathbb{R}$ be a smooth compactly supported function, and let $a, b \in \Lambda$. Then*

$$[x_a x_b g(\boldsymbol{z})]_\beta = \int_0^\infty \mathbb{E}_{a,\boldsymbol{1}}(g(\boldsymbol{L}_t) 1_{X_t = b})\, dt. \tag{5.52}$$

Proof. The proof is essentially identical to that of Theorem 5.6. □

Theorem 5.12. *Let $[\cdot]_\beta$ be the superexpectation of the $\mathbb{S}_+^{2|2}$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the VDJP. Let $g\colon (0,1]^\Lambda \to \mathbb{R}$ be a smooth compactly supported function, let $a \in \Lambda$, and let $s \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then*

$$[g(\boldsymbol{z}) \delta_{\theta_s u_0}(u_a)]_\beta = \mathbb{E}_{a,\boldsymbol{1}}(g(\boldsymbol{L}_{\tau(\cos s)}) 1_{\tau(\cos s) < \zeta}) \tag{5.53}$$

where $\tau(\gamma) = \inf\{t \mid L_t^a \le \gamma\}$ and $\theta_s u_0 = (\cos s, \sin s, 0, 0, 0) \in \mathbb{S}_+^{2|2}$.

Proof. The proof is, *mutatis mutandis*, identical to that of Theorem 5.7. □

6 Application to limiting local times: the Sabot–Tarrès limit

In [42], Sabot and Tarrès established the first connection between the vertex-reinforced jump process and the SUSY hyperbolic sigma model. Their result relates the asymptotic local time distribution of a time-changed VRJP to a certain *horospherical* marginal of the $\mathbb{H}^{2|2}$ model. In this section we derive their result (as stated in [44, Appendix B]) from the Ray–Knight isomorphism for the $\mathbb{H}^{2|2}$ model. The essence of the result is the following corollary of Theorem 5.7. Recall that we write $(z, x, y, \xi, \eta) \in \mathbb{R}^{3|2}$.

Corollary 6.1. *Let $[\cdot]_\beta$ be the superexpectation of the $\mathbb{H}^{2|2}$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the VRJP. For $g\colon \mathbb{R}^\Lambda \to \mathbb{R}$ smooth and compactly supported,*

$$\lim_{\gamma\to\infty} \mathbb{E}_{a,\mathbf{1}}\Big(g(\frac{1}{\gamma}\boldsymbol{L}_{\tau(\gamma)})\Big) = [g(\boldsymbol{z}+\boldsymbol{x})\delta_{u_0}(u_a)]_\beta \tag{6.1}$$

where $\tau(\gamma) = \inf\{t | L_a^t \geqslant \gamma\}$ and $u_0 = (1, 0, 0, 0, 0)$.

Proof. We write $\gamma = \cosh s$. Then by Theorem 5.7 applied to $g(\boldsymbol{L}_{\tau(\cosh s)}/\cosh s)$,

$$\begin{aligned}\mathbb{E}_{a,\mathbf{1}}\Big(g(\frac{1}{\cosh s}\boldsymbol{L}_{\tau(\cosh s)})\Big) &= \Big[g(\frac{1}{\cosh s}\theta_s\boldsymbol{z})\delta_{u_0}(u_a)\Big]_\beta \\ &= [g(\boldsymbol{z}+\boldsymbol{x}\tanh s)\delta_{u_0}(u_a)]_\beta,\end{aligned} \tag{6.2}$$

by using (5.31) to compute $\theta_s\boldsymbol{z} = \cosh s\boldsymbol{z} + \sinh s\boldsymbol{x}$. Since $\tanh s \to 1$ as $s \to \infty$, by dominated convergence we obtain

$$\lim_{s\to\infty} \mathbb{E}_{a,\mathbf{1}}\Big(g(\frac{1}{\cosh s}\boldsymbol{L}_{\tau(\cosh s)})\Big) = [g(\boldsymbol{z}+\boldsymbol{x})\delta_{u_0}(u_a)]_\beta. \qquad \square$$

We now recover [42, Theorem 2] as stated in [45, Theorem B]. Write $\log(\boldsymbol{v}) = (\log(v_i))_{i\in\Lambda}$. Applying Corollary 6.1 to a function $g \circ \log$ yields

$$\lim_{\gamma\to\infty} \mathbb{E}_{a,\mathbf{1}}\big(g(\log(\boldsymbol{L}_{\tau(\gamma)}) - \log\gamma)\big) = [g(\log(\boldsymbol{z}+\boldsymbol{x}))\delta_{u_0}(u_a)]_\beta \tag{6.3}$$

where $\log\gamma = (\log\gamma)_{i\in\Lambda}$. To recover [42, Theorem 2] we rewrite the right-hand side of (6.3). To do this, recall, e.g., from [23, Section 2.2], that horospherical coordinates for the $\mathbb{H}^{2|2}$ model are given by the change of generators from (x, y, ξ, η) to $(s, t, \psi, \bar\psi)$, where

$$\begin{gathered} x \equiv \sinh t - \frac{1}{2}(s^2 + 2\psi\bar\psi)e^t, \quad y \equiv se^t, \quad z \equiv \cosh t + \frac{1}{2}(s^2+2\psi\bar\psi)e^t, \\ \xi \equiv \psi e^t, \quad \eta \equiv \bar\psi e^t. \end{gathered} \tag{6.4}$$

Let

$$H_1(\boldsymbol{t}) \equiv \frac{1}{2}\sum_{i,j\in\Lambda} \beta_{ij}(\cosh(t_i - t_j) - 1). \tag{6.5}$$

The right-hand side of (6.3) can be written explicitly in horospherical coordinates as

$$[g(\log(\boldsymbol{z}+\boldsymbol{x}))\delta(u_a)]_\beta = \frac{1}{\sqrt{2\pi}^{|\Lambda|-1}}\int_{\mathbb{R}^{|\Lambda|-1}} g(\boldsymbol{t})e^{-H_1(\boldsymbol{t})}\sqrt{\det D(\beta,\boldsymbol{t})}\prod_{i\neq a} e^{-t_i}\,dt_i, \tag{6.6}$$

where $D(\beta, \boldsymbol{t})$ is the $(|\Lambda| - 1) \times (|\Lambda| - 1)$ matrix with entries

$$D_{ij}(\beta,\boldsymbol{t}) \equiv \begin{cases} -\beta_{ij}e^{t_i+t_j}, & i \neq j \\ \sum_{k\neq a}\beta_{ik}e^{t_i+t_k} + \beta_{ai}e^{t_i} & i = j \end{cases} \tag{6.7}$$

indexed by $i, j \in \Lambda\backslash\{a\}$. This is [42, Theorem 2] as stated in [45, Theorem B]. In obtaining this formula we have used Theorem A.12 to perform the change of generators and then integrated out s, ψ and $\bar\psi$, which can be done explicitly as conditioned on the t-variables these are Gaussian integrals, see [23, Section 2.3].

Remark 6.2. Qualitatively, the appearance of horospherical coordinates can be explained as follows. The hyperbolic Ray–Knight isomorphism relates the time evolution of the VRJP by $\cosh s$ to the Lorentz boost by s in the xz-plane. Since the asymptotics of Lorentz boosts in the xz-plane are captured by the t marginal in horospherical coordinates, the formulation of the asymptotic local time distribution in terms of the t marginal is quite geometrically natural.

The Sabot–Tarrès limit formula [42, Theorem 2] can also be derived from the hyperbolic BFS–Dynkin isomorphism. More precisely, this can be done by using Corollary 7.2 below, see [50]. In this derivation the role of horospherical coordinates can be seen even more explicitly.

For another explanation of the relation of horospherical coordinates to the VRJP, see [38].

7 Time changes and resolvent formulas

In this section we describe some useful variations and reformulations of our theorems. For the sake of simplicity we only consider the VRJP, but analogous results also hold for the SRW and the VDJP.

7.1. Time-changed and fixed-time formulas. In the literature on the VRJP time changes have played an important role; see, for example, [42]. For comparision with these references, this section briefly explains how isomorphism theorems can be translated to these time-changes.

For a Markov process $(X_s, \boldsymbol{L}_s)$ on $\Lambda \times \mathbb{R}^\Lambda$, let $V\colon [\min_{i\in\Lambda} L_0^i, \infty) \to [\min_{i\in\Lambda} V(L_0^i), \infty)$ be an increasing diffeomorphism and define a random function $A\colon [0,\infty) \to [0,\infty)$ by

$$A(s) \equiv \int_0^s V'(L_u^{X_u})\, du = \sum_{i\in\Lambda} V(L_s^i) - V(L_0^i). \tag{7.1}$$

We define $(\tilde{X}_t, \tilde{\boldsymbol{L}}_t)$, the *time-change by V* of $(X_t, \boldsymbol{L}_t)$, by

$$\tilde{X}_t \equiv X_{A^{-1}(t)}, \quad \tilde{L}_t^i \equiv V(L^i_{A^{-1}(t)}) = V(L_0^i) + \int_0^t 1_{\tilde{X}_u = i}\, du\,. \tag{7.2}$$

Note that $A(0) = A^{-1}(0) = 0$, $\tilde{X}_0 = X_0$ and $\tilde{L}_0^i = V(L_0^i)$.

In this section we will write $V(\boldsymbol{1}) \equiv (V(1))_{i\in\Lambda}$. The next corollary is an example of an isomorphism theorem for a time-changed process.

Corollary 7.1. *Let $[\cdot]_\beta$ be the superexpectation of the $\mathbb{H}^{2|2}$ model, and let $(\tilde{X}_t, \tilde{\boldsymbol{L}}_t)$ be the time-change by V of the VRJP with expectation $\mathbb{E}_{i,\boldsymbol{\ell}}$. Then*

$$\int_0^\infty \mathbb{E}_{a,V(\boldsymbol{1})}(g(\tilde{X}_t, \tilde{\boldsymbol{L}}_t))\, dt = \sum_{i\in\Lambda} [x_a x_i V'(z_i)\, g(i, V(\boldsymbol{z}))]_\beta. \tag{7.3}$$

Proof. By (7.2) and the change of variable $s = A^{-1}(t)$,

$$\begin{aligned}\int_0^\infty \mathbb{E}_{\tilde{X}_0,\tilde{\boldsymbol{L}}_0}(g(\tilde{X}_t, \tilde{\boldsymbol{L}}_t))\, dt &= \int_0^\infty \mathbb{E}_{X_0,\boldsymbol{L}_0}\big(g(X_{A^{-1}(t)}, V(\boldsymbol{L}_{A^{-1}(t)}))\big)\, dt \\ &= \int_{A^{-1}(0)}^{A^{-1}(\infty)} \mathbb{E}_{X_0,\boldsymbol{L}_0}\big(g(X_s, V(\boldsymbol{L}_s))A'(s)\big)\, ds \\ &= \int_0^\infty \mathbb{E}_{X_0,\boldsymbol{L}_0}\big(g(X_s, V(\boldsymbol{L}_s))V'\big(L_s^{X_s}\big)\big)\, ds. \end{aligned} \tag{7.4}$$

The claim now follows from Theorem 5.6 in the case that $g(i, \boldsymbol{\ell})$ is of the form $\delta_{i,j} f(\boldsymbol{\ell})$. The result for more general functions follows by summing (or by using the second part of Lemma 5.5). □

The next corollary shows that supersymmetric isomorphism theorems also give formulas for the local time distribution at *fixed* times.

Corollary 7.2. *Let $[\cdot]_\beta$ be the superexpectation of the $\mathbb{H}^{2|2}$ model, and let $(\tilde{X}_t, \tilde{\boldsymbol{L}}_t)$ be the time-change by V of the VRJP with expectation $\mathbb{E}_{i,\boldsymbol{\ell}}$. Let $\delta_\varepsilon \colon \mathbb{R} \to \mathbb{R}$ be a smooth and compactly supported approximation to δ_0. Then for $g \colon \mathbb{R}^\Lambda \to \mathbb{R}$ smooth and rapidly decaying and any $T > 0$,*

$$\mathbb{E}_{a,V(\mathbf{1})} g\Big(\tilde{\boldsymbol{L}}_T - \frac{T}{N}\Big) = \lim_{\varepsilon \to 0} \sum_{i \in \Lambda} \Big[x_a x_i V'(z_i) g\Big(V(\boldsymbol{z}) - \frac{T}{N}\Big) \delta_\varepsilon\Big(\sum_{i\in\Lambda}\Big(V(z_i) - V(1) - \frac{T}{N}\Big)\Big)\Big]_\beta .$$

Proof. The left-hand side can be written as

$$\begin{aligned}
\mathbb{E}_{a,V(\mathbf{1})}(g(\tilde{\boldsymbol{L}}_T - \frac{T}{N})) &= \sum_{i\in\Lambda} \mathbb{E}_{a,V(\mathbf{1})}(g(\tilde{\boldsymbol{L}}_T - \frac{T}{N})1_{X_T=i}) \\
&= \lim_{\varepsilon\to 0} \sum_{i\in\Lambda} \int_0^\infty dt\, \mathbb{E}_{a,V(\mathbf{1})}\Big(g(\tilde{\boldsymbol{L}}_t - \frac{T}{N})1_{X_t=i}\Big)\, \delta_\varepsilon(t-T) \\
&= \lim_{\varepsilon\to 0} \sum_{i\in\Lambda} \int_0^\infty dt\, \mathbb{E}_{a,V(\mathbf{1})}\Big(g(\tilde{\boldsymbol{L}}_t - \frac{T}{N})1_{X_t=i}\delta_\varepsilon(\sum_{i\in\Lambda}(\tilde{L}_t^i - V(1)) - T)\Big) \\
&= \lim_{\varepsilon\to 0} \sum_{i\in\Lambda} \Big[x_a x_i V'(z_i) g\Big(V(\boldsymbol{z}) - \frac{T}{N}\Big)\delta_\varepsilon\Big(\sum_{i\in\Lambda}(V(z_i) - V(1) - \frac{T}{N})\Big)\Big]_\beta .
\end{aligned} \tag{7.5}$$

The second equality used that $t \mapsto \mathbb{E}_{a,V(\mathbf{1})}(g(\tilde{\boldsymbol{L}}_t - T/N)1_{X_t=i})$ is continuous, the third equality used that $\sum_{i\in\Lambda}(L_t^i - V(1)) = t$ for any $t \geqslant 0$, and the fourth equality is Corollary 7.1. □

By making use of an appropriate time-change, Corollary 7.2 is the starting point for an alternative derivation of the Sabot–Tarrès limit formula (6.6), see Remark 6.2. Similar results have also been used as the starting point for the study of large deviations of the local time of the SRW [9, Theorem 1].

7.2. Resolvent of the joint local time process. The supersymmetric isomorphism theorems for the VRJP in Section 5.2 concern fixed initial local times for the joint process $(X_t, \boldsymbol{L}_t)$, i.e., $\boldsymbol{L}_0 = \mathbf{1}$. This initial condition arises from supersymmetric localisation at $(z, x, y, \xi, \eta) = (1, 0, 0, 0, 0)$ due to the sigma model constraint $u \cdot u = -1$. A more general and geometrically instructive formulation can be obtained by considering the joint process $(X_t, \boldsymbol{L}_t)$ with a general initial condition. This formulation involves the super-Minkowski space from Section 5.2.1 as opposed to the space $\mathbb{H}^{2|2}$.

7.2.1. Super-Minkowski model. Recall super-Minkowski space $\mathbb{R}^{3|2}$ from Section 5.2.1. We define the Berezin integral for an observable $F \in \Omega^{2\Lambda}(\mathbb{R}^{3\Lambda})$ by

$$\int_{(\mathbb{R}^{3|2})^\Lambda} F \equiv \frac{1}{(2\pi)^{|\Lambda|}} \int d\boldsymbol{x}\, d\boldsymbol{y}\, d\boldsymbol{z}\, \partial_{\boldsymbol{\eta}}\, \partial_{\boldsymbol{\xi}}\, F, \tag{7.6}$$

where $\partial_{\boldsymbol{\eta}}\, \partial_{\boldsymbol{\xi}}$ is defined by $\partial_{\eta_{|\Lambda|}}\partial_{\xi_{|\Lambda|}} \dots \partial_{\eta_1}\partial_{\xi_1}$, $d\boldsymbol{x} = dx_{|\Lambda|} \dots dx_1$, $d\boldsymbol{y} = dy_{|\Lambda|} \dots dy_1$, and $d\boldsymbol{z} = dz_{|\Lambda|} \dots dz_1$ for some fixed ordering of the $i \in \Lambda$ from 1 to $|\Lambda|$.

For $u \in \mathbb{R}^{3|2}$, we write $u \cdot u < 0$ if the degree 0 part of the form $u \cdot u$ is negative, where here $u \cdot u$ denotes the super-Minkowski inner product (5.21). For a spin configuration $\boldsymbol{u} \in (\mathbb{R}^{3|2})^\Lambda$ we write $\boldsymbol{u} \cdot \boldsymbol{u} < 0$ if $u_i \cdot u_i < 0$ for all $i \in \Lambda$ and we then define

$$r_i \equiv \sqrt{-u_i \cdot u_i}, \tag{7.7}$$

and let $\boldsymbol{r} = (r_i)_{i\in\Lambda}$. For such a spin configuration we consider the Hamiltonian

$$H_\beta(\boldsymbol{u}) \equiv \frac{1}{2}(\boldsymbol{u}, -\Delta_\beta \boldsymbol{u}) + \frac{1}{2}(\boldsymbol{r}, -\Delta_\beta \boldsymbol{r}), \tag{7.8}$$

where the inner product for the u_i is the one from (5.21) and the r_i are forms that are multiplied in the ordinary way: $(\boldsymbol{r}, -\Delta_\beta \boldsymbol{r}) = \sum_{i\in\Lambda} r_i(-\Delta_\beta r)_i$. Let $F \in \Omega^{2\Lambda}(\mathbb{R}^{3\Lambda})$ be a smooth form compactly supported on $\{\boldsymbol{u}\cdot\boldsymbol{u} < 0, \boldsymbol{z} > 0\}$, i.e., whose coefficient functions vanish when the degree 0 part of any form $u_i \cdot u_i$ is non-negative or when $z_i \leqslant 0$ for any i. We define an unnormalised superexpectation by

$$[F]_\beta \equiv \int_{(\mathbb{R}^{3|2})^\Lambda} F(\boldsymbol{u}) e^{-H_\beta(\boldsymbol{u})} 1_{\boldsymbol{u}\cdot\boldsymbol{u}<0} 1_{\boldsymbol{z}>0}, \tag{7.9}$$

with $\boldsymbol{u}\cdot\boldsymbol{u} < 0$ as defined above. The assumption that F has compact support ensures the integrand is smooth. We call this the *super-Minkowski model.* Note that $\{\boldsymbol{u}\cdot\boldsymbol{u} < 0, \boldsymbol{z} > 0\}$ is a version of the causal future for super-Minkowski space; see Figure 3.1.

7.2.2. Symmetries and localisation. Let

$$T = x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}. \tag{7.10}$$

Then T represents an infinitesimal Lorentz boost in the xz-plane since

$$Tx = z, \qquad Tz = x, \tag{7.11}$$

and $Ty = T\xi = T\eta = 0$. Note also that $Tr = 0$.

The Hamiltonian H_β is invariant under T, i.e., $\sum_{i\in\Lambda} T_i H_\beta(\boldsymbol{u}) = 0$. Here we have written T_i for the version of T applying to the i-th coordinate. Moreover the integral (7.6) is invariant under T. In addition, the model is supersymmetric with supersymmetry generator Q as in (5.11), and the following localisation statement holds for all smooth $f\colon (0,\infty)^{2\Lambda} \to \mathbb{R}$ with compact support:

$$[f(\boldsymbol{z},\boldsymbol{r})]_\beta = \int_{(0,\infty)^\Lambda} d\boldsymbol{z}\, f(\boldsymbol{z},\boldsymbol{z}). \tag{7.12}$$

This can be seen by integrating over $\boldsymbol{z}$ last when computing the superexpectation, and using localisation for (x, y, η, ξ), i.e., Corollary A.10.

7.2.3. Resolvent formula. The super-Minkowski model is related to the resolvent of the VRJP.

Theorem 7.3. *Let $[\cdot]_\beta$ be the superexpectation of the super-Minkowski model, and let $\pi = (\pi(i,\boldsymbol{r}))$ be a smooth compactly support probability measure on $\Lambda\times(0,\infty)^\Lambda$. For all smooth $f\colon \Lambda\times\mathbb{R}^\Lambda \to \mathbb{R}$ with rapid decay,*

$$\int_0^\infty \mathbb{E}_{\boldsymbol{\pi}} f(X_t, \boldsymbol{L}_t)\, dt = \sum_{i,j\in\Lambda} \left[\frac{\pi(i,\boldsymbol{r})}{r_i} x_i x_j f(j,\boldsymbol{z})\right]_\beta \tag{7.13}$$

where we have written $\mathbb{E}_{\boldsymbol{\pi}}$ to denote the expectation of a VRJP with initial condition $(X_0, \boldsymbol{L}_0)$ distributed according to π.

Remark 7.4. In the notation of Remark 2.2, Theorem 7.3 can be compactly rewritten as

$$\int_0^\infty \mathbb{E}_{\boldsymbol{\pi}} f(X_t, \boldsymbol{L}_t)\, dt = \left[(\boldsymbol{x}, \frac{\boldsymbol{\pi}(\boldsymbol{r})}{\boldsymbol{r}})(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{z}))\right]_\beta. \tag{7.14}$$

The proof of Theorem 7.3 uses that Lemma 5.5 remains true if $[\cdot]_\beta$ is interpreted as the expectation of the super-Minkowski model, and then follows the standard route as follows.

Proof. Let $\rho(\boldsymbol{u}) \equiv \sum_{i\in\Lambda} \pi(i,\boldsymbol{r}) x_i / r_i$, and let T_i be the infinitesimal boost given by (5.28). Since $T_i r_i = 0$ and $T_i x_i = z_i$ we have $T_j \rho = \pi(j,\boldsymbol{r}) z_j / r_j$. Since Lemma 5.5 holds for the super-Minkowski model, we apply (5.34) to obtain

$$\sum_{i,j\in\Lambda} \left[\frac{\pi(i,\boldsymbol{r})}{r_i} x_i x_j f(j,\boldsymbol{z})\right]_\beta = \sum_{j\in\Lambda} \left[\frac{z_j}{r_j}\pi(j,\boldsymbol{r}) \int_0^\infty \mathbb{E}_{j,\boldsymbol{z}}(f(X_t, \boldsymbol{L}_t))\, dt\right]_\beta. \tag{7.15}$$

By localisation, i.e., (7.12), the right-hand side equals

$$\int_{\mathbb{R}_+^\Lambda} d\boldsymbol{z} \sum_{j\in\Lambda} \pi(j,\boldsymbol{z}) \int_0^\infty \mathbb{E}_{j,\boldsymbol{z}}(f(X_t, \boldsymbol{L}_t))\, dt = \int_0^\infty \mathbb{E}_{\boldsymbol{\pi}}(f(X_t, \boldsymbol{L}_t))\, dt. \qquad \square$$

8 Application to exponential decay of correlations in spin systems

In this section we prove theorems about the exponential decay of spin-spin correlations. Let $d(i,j)$ denote the graph distance between vertices i and j in the graph induced by the edge weights β; this distance is finite since the induced graph is finite and connected by assumption.

We first consider the $\mathbb{H}^{2|2}$ model with constant and non-zero external field.

Theorem 8.1. *Consider the $\mathbb{H}^{2|2}$ model with $\sup_{i\in\Lambda}\sum_{j\in\Lambda}\beta_{ij} \leqslant \beta_*$ and $h_i = h > 0$ for all $i \in \Lambda$. Let $c(\beta_*, h) \equiv \log(1 + h/\beta_*)$. Then for all $i, j \in \Lambda$,*

$$[x_i x_j]_{\beta,h} \leqslant \frac{1}{h} e^{-c(\beta_*,h)d(i,j)}. \tag{8.1}$$

Proof. Let τ_j be the hitting time of j, i.e., $\tau_j \equiv \inf\{s \geqslant 0 \mid X_s = j\}$. Then by choosing g an exponential in Theorem 5.6,

$$[x_i x_j]_{\beta,h} = \mathbb{E}_{i,\mathbf{1}} \int_0^\infty 1_{X_s=j} e^{-hs}\, ds = \mathbb{E}_{i,\mathbf{1}} 1_{\tau_j<\infty} \int_{\tau_j}^\infty 1_{X_s=j} e^{-hs}\, ds \leqslant \frac{1}{h}\mathbb{P}_{i,\mathbf{1}}(\tau_j < \infty). \tag{8.2}$$

The inequality follows as the integral is at most $\int_0^\infty e^{-hs} ds = h^{-1}$.

If $\tau_j < \infty$ then there are at least $d(i,j)$ times at which a rate h exponential clock does *not* ring before a rate β_* clock, as there are at least $d(i,j)$ jumps to previously unvisited vertices on any path from i to j. The probability of a rate h clock ringing only after some rate β_{ij} clock is at most $\beta_*/(\beta_*+h)$. Each of these events are independent by the memorylessness of the exponential, and hence

$$\mathbb{P}_{i,\mathbf{1}}(\tau_j < \infty) \leqslant \left(\frac{\beta_*}{\beta_* + h}\right)^{d(i,j)} = e^{-c(\beta_*,h)d(i,j)}$$

Combined with (8.2) this proves the theorem. □

Remark 8.2. Theorem 8.1 gives a positive rate $\log(1 + h/\beta_*) \sim ch$ of exponential decay for some $c > 0$ for any value of β. For small β, i.e., high temperatures, it is known that the rate stays uniformly bounded away from 0 as $h \downarrow 0$ [2, 22]. The rate is expected to be bounded away from 0 for any β when the graph Λ tends to $\mathbb{Z}^2$. On the other hand, for $\Lambda \uparrow \mathbb{Z}^d$ with $d \geqslant 3$ it is conjectured that the rate behaves asymptotically as $\sim c\sqrt{h}$ as $h \downarrow 0$.

It would be interesting to obtain an analogue of Theorem 8.1 for the $\mathbb{H}^n$ model by using Theorem 3.3. This would require an appropriate estimate on the z-field to control the initial local times of the VRJP. We do not pursue this direction here.

For the hemispherical spin models, the estimates on the z-field are trivial because $|z_i| \leqslant 1$, and we thus consider both the $\mathbb{S}^n_+$ model and the $\mathbb{S}^{2|2}_+$ model. For $\mathbb{S}^{2|2}_+$ we have only defined the superexpectation of compactly supported observables. To define the superexpectation of non-compactly supported observables requires a treatment of superintegrals with boundaries; since we do not need this general treatment we instead define the two-point function $[x_i x_j]_{\beta,h}$ for the $\mathbb{S}^{2|2}_+$ model by $[x_i x_j]_{\beta,h} \equiv \lim_{n\to\infty}[x_i x_j f_n(\boldsymbol{z})]_{\beta,h}$ where f_n is a sequence of smooth and bounded approximations to $1_{\boldsymbol{z}>0}$. The proof of the following theorem shows that this limit exists.

Theorem 8.3. *Consider the $\mathbb{S}^n_+$ model with $\sup_{i\in\Lambda}\sum_{j\in\Lambda}\beta_{ij} \leqslant \beta_*$, and let $c(\beta_*) = \log(1 - e^{-\beta_*})$. Then for all $i, j \in \Lambda$,*

$$\langle x_i x_j \rangle_{\beta,h} \leqslant e^{-c(\beta_*)d(i,j)}. \tag{8.3}$$

The same result holds for the superexpectation $[x_i x_j]_{\beta,h}$ of the $\mathbb{S}^{2|2}_+$ model.

Proof. We first consider $\mathbb{S}_+^{2|2}$. Let f_n be a sequence of smooth and bounded approximations to $1_{\boldsymbol{z}>0}$. Letting $\mathbb{E}_{i,\mathbf{1}}$ be the expectation for a VDJP with initial local time $\mathbf{1}$, Theorem 5.11 implies

$$[x_i x_j]_{\beta,h} = \lim_{n\to\infty} [x_i x_j f_n(\boldsymbol{z})]_{\beta,h} = \lim_{n\to\infty} \mathbb{E}_{i,\mathbf{1}} \int_0^\infty f_n(\boldsymbol{L}) 1_{X_t=j} e^{-\sum_v h_v L_v^t}\, dt,$$

To obtain upper bounds we may assume, without loss of generality, that $\boldsymbol{h} = \mathbf{0}$. By definition, X_t dies once the local time at any vertex reaches 0. Since f_n is asymptotically bounded above by one, it therefore suffices to bound the probability that X_t reaches j.

By the definition of the VDJP, for each $r \in \Lambda$ the jump rate out of r is bounded above by β_*. Thus for each $k \in \mathbb{N}$ there is probability at least $e^{-\beta_*}$ the walk X_t dies after its kth jump and before its $(k+1)$st jump. The probability X_t reaches j is at most the probability that X_t does not die before taking $d(i,j)$ steps, and hence

$$[x_i x_j]_{\beta,h} \leqslant (1 - e^{-\beta_*})^{d(i,j)} = e^{-c(\beta_*)d(i,j)}.$$

This completes the proof for $\mathbb{S}_+^{2|2}$. For $\mathbb{S}_+^n$, we use (the normalised form of) Theorem 4.4 in place of Theorem 5.11. The argument above applies pointwise in the initial local time, so using $0 \leqslant z_i \leqslant 1$ we obtain the same conclusion. □

Remark 8.4. A result closely related to Theorem 8.3 is given in [36, Theorem 2].

A Introduction to supersymmetric integration

This appendix gives a self-contained introduction to the mathematics of supersymmetry that is relevant for this article. For complementary treatments, see in particular [6,13,40]. In Appendix B we discuss some further aspects of supersymmetry that are relevant to this article, but that are not needed to understand the main text.

A.1. Integration of differential forms. We begin by reviewing the important example of integration of differential forms on Euclidean space $\mathbb{R}^N$. Let $x_1, \dots, x_N$ be coordinates on $\mathbb{R}^N$. A *differential form* on $\mathbb{R}^N$ can be written as

$$F = F_0 + \cdots + F_N \tag{A.1}$$

where $F_0 \in C^\infty(\mathbb{R}^N)$ is a 0*-form,* i.e., an ordinary function, and F_p is a *p-form*, i.e., a nonzero sum of terms of the form

$$f_{i_1,\dots,i_p}(x_1,\dots,x_N)\, dx_{i_1} \wedge \cdots \wedge dx_{i_p}, \qquad 1 \leqslant i_j \leqslant N,\ \ 1 \leqslant j \leqslant p, \tag{A.2}$$

where $f_{i_1,\dots,i_p} \in C^\infty(\mathbb{R}^N)$, the coordinates are viewed as functions $x_i \colon \mathbb{R}^N \to \mathbb{R}$ in $C^\infty(\mathbb{R}^N)$, and the differentials dx_i are the generators of a Grassmann algebra. This means that the dx_i are formal variables that are multiplied with the anti-commuting wedge product:

$$dx_i \wedge dx_j = -dx_j \wedge dx_i. \tag{A.3}$$

In particular, $dx_i \wedge dx_i = 0$. Later, the $\wedge$ will often be omitted. By extending the wedge product to differential forms by linearity, we obtain a unital associative algebra over $C^\infty(\mathbb{R}^N)$. This is the exterior algebra of differential forms on $\mathbb{R}^N$, which we denote $\Omega(\mathbb{R}^N)$.

The form F_p in (A.1) is the *degree p* part of F. We say F has *degree p* or *is a p-form* if $F = F_p$. Since $dx_i \wedge dx_i = 0$, there are no forms of degree greater than N. A form F of degree N is said to be of *top degree* and such an F can be written as

$$F(\boldsymbol{x}) = f(\boldsymbol{x})\, dx_1 \wedge \cdots \wedge dx_N \tag{A.4}$$

for some $f \in C^\infty(\mathbb{R}^N)$, where we abbreviate $\boldsymbol{x} = (x_1, \ldots, x_N)$. The anticommutativity of the wedge product implies that the order of the differentials determines an overall sign in (A.4). Keeping this in mind, the integral of a top degree form F is defined by

$$\int_{\mathbb{R}^N} F \equiv \int_{\mathbb{R}^N} f(\boldsymbol{x})\, dx_1 \cdots dx_N \tag{A.5}$$

where the right-hand side is an ordinary integral with respect to Lebesgue measure. For $p < N$ the integral of a p-form F_p is defined to be zero: $\int_{\mathbb{R}^N} F_p \equiv 0$. Having defined the integral on p-forms for all p, we extend the definition of the integral to the entire algebra $\Omega(\mathbb{R}^N)$ of differential forms by linearity.

Example A.1 (Change of variables)**.** The differential notation and the use of the wedge product is consistent with, and motivated by, the following change of variable formula. Let $\Phi\colon \mathbb{R}^N \to \mathbb{R}^N$ be an orientation preserving diffeomorphism. Then by the change of variables formula from calculus

$$\begin{aligned}\int f(x_1, \ldots, x_N)\, dx_1 \wedge \cdots \wedge dx_N &= \int f(\Phi_1(\boldsymbol{x}), \ldots, \Phi_N(\boldsymbol{x}))(\det D\Phi)\, dx_1 \wedge \cdots \wedge dx_N \\ &= \int f(\Phi_1(\boldsymbol{x}), \ldots, \Phi_N(\boldsymbol{x}))\, d\Phi_1(\boldsymbol{x}) \wedge \cdots \wedge d\Phi_N(x)\end{aligned} \tag{A.6}$$

where $D\Phi$ is the Jacobian matrix of Φ and the second equality has made use of the definition

$$d\Phi_i(\boldsymbol{x}) = \sum_{j=1}^{N} \frac{\partial \Phi_i(\boldsymbol{x})}{\partial x_j} dx_j, \tag{A.7}$$

which leads, by a calculation, to the identity

$$d\Phi_1(\boldsymbol{x}) \wedge \cdots \wedge d\Phi_N(\boldsymbol{x}) = (\det D\Phi)\; dx_1 \wedge \cdots \wedge dx_N. \tag{A.8}$$

A.2. Odd and even forms. A differential form is *even* if it is a sum of p-forms with all p even and it is *odd* if it is a sum of p-forms with all p odd. We say a form is *homogeneous* if it is either even or odd. We can decompose a general form F as

$$F = F_{\text{even}} + F_{\text{odd}}, \qquad \Omega(\mathbb{R}^N) = \Omega_{\text{even}}(\mathbb{R}^N) \oplus \Omega_{\text{odd}}(\mathbb{R}^N), \tag{A.9}$$

where F_{even} is the sum of the degree p parts of F with p even, and similarly for F_{odd}. As the wedge product of a p-form with a q-form is either 0 or a $(p+q)$-form, the exterior algebra equipped with the wedge product is a $\mathbb{Z}_2$-graded algebra. $\mathbb{Z}_2$-graded algebras are also called *superalgebras.* Formally, this means that if we define the parity of a homogeneous form as

$$\alpha(F) \equiv \begin{cases} 0 \in \mathbb{Z}_2, & F = F_{\text{even}} \\ 1 \in \mathbb{Z}_2, & F = F_{\text{odd}} \end{cases} \tag{A.10}$$

then $\alpha(F \wedge G) = \alpha(F) + \alpha(G) \bmod 2$. A calculation shows that for homogeneous F, G

$$F \wedge G = (-1)^{\alpha(F)\alpha(G)} G \wedge F, \tag{A.11}$$

and in particular, even elements commute with all other elements by linearity.

A.3. Berezin integral. In this section we introduce Grassmann algebras and the Berezin integral. Integration of differential forms as introduced in the previous sections constitute a special case.

A.3.1. Grassmann algebras. Let Ω^M be a Grassmann algebra with generators $\xi_1, \dots, \xi_M$; as the subscripts suggest we will always assume there is a fixed (but arbitrary) order on the generators. Thus Ω^M is the unital associative algebra generated by the $(\xi_i)_{i=1}^M$ subject to the anticommutation relations

$$\xi_i \xi_j + \xi_j \xi_i = 0, \quad 1 \leqslant i \leqslant j \leqslant M. \tag{A.12}$$

Let $\Omega^M(\mathbb{R}^N)$ be the algebra over $C^\infty(\mathbb{R}^N)$ generated by the $(\xi_i)_{i=1}^M$. Elements of this algebra can be written as

$$\sum_{\substack{I \subset \{1,\dots,M\} \\ I=\{i_1,\dots,i_p\}}} f_I(\boldsymbol{x})\, \xi_{i_1} \cdots \xi_{i_p} \tag{A.13}$$

where $f_I \in C^\infty(\mathbb{R}^N)$ for each $I \subset \{1, \dots, M\}$, and we have arranged the product of generators according to the given fixed order: $i_1 < i_2 < \cdots < i_p$.

Example A.2. The differentials $\xi_i = dx_i$ are an instance of a Grassmann algebra, and the algebra of differential forms on $\mathbb{R}^N$ can be identified with $\Omega^N(\mathbb{R}^N)$.

We continue to use the term *form* for elements of $\Omega^M(\mathbb{R}^N)$ when $N \neq M$. The notion of the degree of a form and the $\mathbb{Z}_2$-grading that we defined for differential forms extends to this more general context.

A.3.2. Integration. For $i \in \{1, 2, \dots, M\}$ the *left-derivative* $\frac{\partial}{\partial \xi_i} : \Omega^M \to \Omega^M$ is the unique linear map determined by

$$\frac{\partial}{\partial \xi_i}(\xi_i F) = F \quad \text{if } \xi_i F \neq 0, \qquad \frac{\partial}{\partial \xi_i} 1 = 0. \tag{A.14}$$

We sometimes write $\partial_{\xi_i} = \frac{\partial}{\partial \xi_i}$. Note that ∂_{ξ_i} is an *anti-derivation*: if F is a homogeneous form, then

$$\partial_{\xi_i}(FG) = (\partial_{\xi_i} F)G + (-1)^{\alpha(F)} F(\partial_{\xi_i} G). \tag{A.15}$$

The left-derivative extends naturally to an anti-derivation on $\Omega^M(\mathbb{R}^N)$ by defining

$$\partial_{\xi_i}(f(\boldsymbol{x})\xi_{i_1} \dots \xi_{i_p}) = f(\boldsymbol{x})\partial_{\xi_i}(\xi_{i_1} \dots \xi_{i_p}). \tag{A.16}$$

Example A.3. The left-derivative gives a convenient formulation of the integral of a differential form. Let $F \in \Omega^N(\mathbb{R}^N)$ be a differential form and write $\xi_i = dx_i$. Then

$$\int F = \int_{\mathbb{R}^N} dx_1 \cdots dx_N\, \partial_{\xi_N} \cdots \partial_{\xi_1} F = \int_{\mathbb{R}^N} d\boldsymbol{x}\, \partial_{\boldsymbol{\xi}}\, F \tag{A.17}$$

where the left-hand side is the integral as a differential form in the sense of Section A.1, and the last equality made use of the definition $\partial_{\boldsymbol{\xi}} \equiv \partial_{\xi_N} \dots \partial_{\xi_1}$. Note that the order used in defining $\partial_{\boldsymbol{\xi}}$ matters.

The notation on the right-hand side of (A.17) is called the *Berezin integral.* This is a useful notion because it is possible to change variables in $\boldsymbol{x}$ and $\boldsymbol{\xi}$ separately, as will be discussed below in Section A.5. The Berezin integral generalises to $N \neq M$ as follows.

Definition A.4. *For $F \in \Omega^M(\mathbb{R}^N)$, the* Berezin integral *of F is*

$$\int F \equiv \int_{\mathbb{R}^N} dx_1 \cdots dx_N\, \partial_{\xi_M} \cdots \partial_{\xi_1} F = \int_{\mathbb{R}^N} d\boldsymbol{x}\, \partial_{\boldsymbol{\xi}}\, F, \tag{A.18}$$

where the last equality is by the definitions $d\boldsymbol{x} = dx_1 \dots dx_N$ and $\partial_{\boldsymbol{\xi}} \equiv \partial_{\xi_M} \dots \partial_{\xi_1}$. We say a form F is integrable *if it can be written as a finite sum of forms of the type $f(\boldsymbol{x})\, \xi_{i_1} \dots \xi_{i_p}$ with f integrable on $\mathbb{R}^N$.*

The expression $d\boldsymbol{x}\, \partial_{\boldsymbol{\xi}}$ on the right-hand side of (A.18) is an example of a *superintegration form.* More generally a superintegration form is given by $d\boldsymbol{x}\, \partial_{\boldsymbol{\xi}}\, F$ for F an even integrable form, and integration with respect to this superintegration form is defined by $\int G = \int_{\mathbb{R}^N} d\boldsymbol{x}\, \partial_{\boldsymbol{\xi}}\, FG$.

A.3.3. Functions of forms. Suppose $g \in C^\infty(\mathbb{R}^k)$. We will use $\alpha = (\alpha_1, \dots, \alpha_k)$ to denote multi-indices, and we will also use the notation

$$g^{(\alpha)}(\boldsymbol{x}) \equiv \frac{\partial}{\partial x_1^{\alpha_1}} \cdots \frac{\partial}{\partial x_k^{\alpha_k}} g(x), \qquad x^\alpha \equiv x_1^{\alpha_1} \cdots x_k^{\alpha_k}.$$

Definition A.5. *Let $g \in C^\infty(\mathbb{R}^k)$ and $F^1, \dots F^k \in \Omega^M(\mathbb{R}^N)$ be even forms. Then $g(F^1, \dots, F^k) \in \Omega^M(\mathbb{R}^N)$ is defined by the following formula, where the sum runs over all multiindices α:*

$$g(F^1, \dots, F^k) \equiv \sum_\alpha \frac{1}{\alpha!} g^{(\alpha)}(F_0^1, \dots, F_0^k)(F - F_0)^\alpha. \tag{A.19}$$

Note that the product defining $(F - F_0)^\alpha$ is the wedge product, i.e., this is shorthand for $(F^1 - F_0^1)^{\alpha_1} \wedge \cdots \wedge (F^k - F_0^k)^{\alpha_k}$, and $(F^1 - F_0^1)^{\alpha_1}$ is the α_1-fold wedge product of this form with itself. There is no ambiguity in the ordering since all forms are assumed even. The formal Taylor expansion in (A.19) is finite because forms of degree greater than N do not exist. As a simple example of a function of a form, the reader may wish to verify that

$$e^{-x_1^2 - \xi_1\xi_2} = e^{-x_1^2}(1 - \xi_1\xi_2). \tag{A.20}$$

A.4. Gaussian integrals and localisation. Let $A \in \mathbb{R}^{N\times N}$ be positive definite. The $O(2)$-invariant Gaussian measure on $\mathbb{R}^{2N}$ associated to the matrix A has density

$$e^{-\frac{1}{2}(\boldsymbol{x}, A\boldsymbol{x}) - \frac{1}{2}(\boldsymbol{y}, A\boldsymbol{y})} (\det A) \prod_{i=1}^N \frac{dx_i\, dy_i}{2\pi}. \tag{A.21}$$

Let $\xi_1, \dots, \xi_N, \eta_1, \dots, \eta_N$ be generators of the Grassmann algebra Ω^{2N}, and define

$$\partial_{\boldsymbol{\eta}} \partial_{\boldsymbol{\xi}} \equiv \partial_{\eta_N} \partial_{\xi_N} \cdots \partial_{\eta_1} \partial_{\xi_1} \qquad (\boldsymbol{\xi}, A\boldsymbol{\eta}) \equiv \sum_{i=1}^N A_{ij} \xi_i \eta_j. \tag{A.22}$$

A computation shows that

$$\partial_{\boldsymbol{\eta}} \partial_{\boldsymbol{\xi}} e^{(\boldsymbol{\xi}, A\boldsymbol{\eta})} = \partial_{\boldsymbol{\eta}} \partial_{\boldsymbol{\xi}} \frac{1}{N!} (\sum_{i=1}^N A_{ij} \xi_i \eta_j)^N = \det A. \tag{A.23}$$

Remark A.6. The form $e^{(\boldsymbol{\xi}, A\boldsymbol{\eta})} = e^{\frac{1}{2}(\boldsymbol{\xi}, A\boldsymbol{\eta}) - \frac{1}{2}(\boldsymbol{\eta}, A\boldsymbol{\xi})} \in \Omega^{2N}$ is called a *Grassmann Gaussian.* The corresponding Grassmann Gaussian expectation $\langle F \rangle \equiv [F]/[1]$ where $[F] \equiv \partial_{\boldsymbol{\eta}} \partial_{\boldsymbol{\xi}} (e^{(\boldsymbol{\xi}, A\boldsymbol{\eta})} F) \in \mathbb{R}$ for $F \in \Omega^{2N}$, and hence $[1] = \det A$ by (A.23), behaves in many ways like a Gaussian integral.

Using (A.23), the Gaussian density (A.21) can be written as

$$\prod_{i=1}^N \frac{dx_i\, dy_i\, \partial_{\eta_i} \partial_{\xi_i}}{2\pi} e^{-\frac{1}{2}(\boldsymbol{x}, A\boldsymbol{x}) - \frac{1}{2}(\boldsymbol{y}, A\boldsymbol{y}) + \frac{1}{2}(\boldsymbol{\xi}, A\boldsymbol{\eta}) - \frac{1}{2}(\boldsymbol{\eta}, A\boldsymbol{\xi})}. \tag{A.24}$$

The form given by $(2\pi)^{-N}$ times the exponential in (A.24) is called the *super-Gaussian form.* Thus the Gaussian density is the coefficient of the top degree part of the super-Gaussian form.

To lighten the notation, we will now write $u_i \equiv (x_i, y_i, \xi_i, \eta_i)$ and call u_i a *supervector.* For supervectors u_i and u_j define a form

$$u_i \cdot u_j \equiv x_i x_j + y_i y_j - \xi_i \eta_j + \eta_i \xi_j. \tag{A.25}$$

We unite the supervectors u_i into $\boldsymbol{u} \equiv (u_i)_{i=1}^N$ and introduce the following shorthand notation for the form that occurs in the exponent of (A.24):

$$(\boldsymbol{u}, A\boldsymbol{u}) \equiv \sum_{i,j=1}^N A_{ij} u_i \cdot u_j. \tag{A.26}$$

For a form F we define *the superintegral of* F by

$$\int_{(\mathbb{R}^{2|2})^N} F \equiv \frac{1}{(2\pi)^N} \int_{\mathbb{R}^{2N}} d\boldsymbol{x}\, d\boldsymbol{y}\, \partial_{\boldsymbol{\eta}}\, \partial_{\boldsymbol{\xi}}\, F, \tag{A.27}$$

where $d\boldsymbol{x} \equiv dx_N \dots dx_1$ and similarly for $d\boldsymbol{y}$. Then, since the coefficient of the top degree part of (A.24) is the density of a Gaussian,

$$\int_{(\mathbb{R}^{2|2})^N} e^{-\frac{1}{2}(\boldsymbol{u}, A\boldsymbol{u})} = 1. \tag{A.28}$$

The fact that this superintegral is one is a simple example of *localisation* for superintegrals of supersymmetric forms. The rest of this section describes this phenomenon.

The *supersymmetry generator* $Q\colon \Omega^{2N}(\mathbb{R}^{2N}) \to \Omega^{2N}(\mathbb{R}^{2N})$ is defined as

$$Q \equiv \sum_{i=1}^{N} Q_i\,, \qquad Q_i \equiv \xi_i \frac{\partial}{\partial x_i} + \eta_i \frac{\partial}{\partial y_i} - x_i \frac{\partial}{\partial \eta_i} + y_i \frac{\partial}{\partial \xi_i}. \tag{A.29}$$

Thus Q formally exchanges the even and odd generators of $\Omega^{2N}(\mathbb{R}^{2N})$:

$$Qx_i = \xi_i, \quad Qy_i = \eta_i, \quad Q\xi_i = -y_i, \quad Q\eta_i = x_i. \tag{A.30}$$

A form $F \in \Omega^{2N}(\mathbb{R}^{2N})$ is defined to be *supersymmetric* if $QF = 0$. Note that Q is an anti-derivation, and hence $Q(F_1F_2) = 0$ if F_1 and F_2 are both supersymmetric forms.

Example A.7. The following forms are supersymmetric:

$$u_i \cdot u_j = x_i x_j + y_i y_j - \xi_i \eta_j + \eta_i \xi_j. \tag{A.31}$$

Much of the magic of supersymmetry is due to the fundamental *localisation theorem*:

Theorem A.8. *Suppose* $F \in \Omega^{2N}(\mathbb{R}^{2N})$ *is supersymmetric and integrable. Then*

$$\int_{(\mathbb{R}^{2|2})^N} F = F_0(0) \tag{A.32}$$

*where the right-hand side is the degree-*0 *part of* F *evaluated at* 0.

To keep this introduction to supersymmetry self-contained, we provide the beautiful and instructive proof of this theorem in Appendix B.2. To prove an important corollary of the theorem we need the following chain rule, proven in [40, p.59] or [3, Solution to Exercise 11.4.3].

Lemma A.9. *The supersymmetry generator* Q *obeys the chain rule for even forms, in the sense that if* $K = (K_j)_{j=1}^J$ *is a finite collection of even forms, and if* $f\colon \mathbb{R}^J \to \mathbb{C}$ *is* C^∞, *then*

$$Q(f(K)) = \sum_{j=1}^{J} f_j(K) QK_j, \tag{A.33}$$

where f_j *denotes the partial derivative of* f *with respect to the* j*th coordinate.*

Let $\boldsymbol{u}\boldsymbol{u}^T$ denote the collection $(u_i \cdot u_j)_{i,j=1}^N$ of forms defined in (A.31).

Corollary A.10. *For any smooth function* $f\colon \mathbb{R}^{N\times N} \to \mathbb{R}$ *with sufficient decay,*

$$\int_{(\mathbb{R}^{2|2})^N} f(\boldsymbol{u}\boldsymbol{u}^T) = f(\boldsymbol{0}). \tag{A.34}$$

Proof. Let $F = f(\boldsymbol{u}\boldsymbol{u}^T)$. Then $F_0(\boldsymbol{0}) = f(\boldsymbol{0})$ and $QF = \sum_{ij} f_{ij}(\boldsymbol{u}\boldsymbol{u}^T) Q(u_i \cdot u_j) = 0$ by the chain rule of Lemma A.9, where f_{ij} denotes the partial derivative of f with respect to the ij-th coordinate. The claim follows from Theorem A.8. □

A.5. Change of generators. Recall the general expression (A.13) for a form $F \in \Omega^M(\mathbb{R}^N)$. We will sometimes write $F(\boldsymbol{x}, \boldsymbol{\xi})$ or $F(x_1, \dots, x_N, \xi_1, \dots, \xi_M)$ to denote a form written in this way.

Definition A.11. *A collection of even elements* $(x_i)_{i=1}^N$ *and odd elements* $(\xi_j)_{j=1}^M$ *is a* set of generators *for* $\Omega^M(\mathbb{R}^N)$ *if every* $F \in \Omega^M(\mathbb{R}^N)$ *can be written in the form* (A.13)*.*

Note that Example A.1 provided an example of a change of generators

$$y_i = \Phi_i(x_1, \dots, x_N), \quad \eta_i = dy_i = \sum_{j=1}^N \frac{\partial \Phi_i}{\partial x_j}(x_1, \dots, x_N)\, dx_j \tag{A.35}$$

along with a corresponding change of variables formula.

It is both possible and useful to change between sets of generators in the sense of Definition A.11 without the even and odd generators changing together. Moreover, there is an extension of the usual change of variables formula that applies in this setting. This formula relies on the notion of *superdeterminant* (or Berezinian) of a *supermatrix* M:

$$\mathrm{sdet} M \equiv \det(A - BD^{-1}C) \det D^{-1} \quad \text{for } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{A.36}$$

where the entries of M are elements of a Grassmann algebra, the entries of the blocks A and D are even, the entries of the blocks B and C are odd, and D is invertible. Invertibility means invertibility in the (commutative) algebra of even elements of the Grassmann algebra. The next result is [6, Theorem 2.1]. In the theorem rapid decay means each of the coefficient functions of F have rapid decay.

Theorem A.12. *Suppose* $y_i = y_i(\boldsymbol{x}, \boldsymbol{\xi})$ *and* $\eta_i = \eta_i(\boldsymbol{x}, \boldsymbol{\xi})$ *are a set of generators. Then for any* F *with sufficiently rapid decay,*

$$\int d\boldsymbol{y}\, \partial_{\boldsymbol{\eta}}\, F(\boldsymbol{y}, \boldsymbol{\eta})\, \mathrm{sdet}(M) = \int d\boldsymbol{x}\, \partial_{\boldsymbol{\xi}}\, F(\boldsymbol{x}, \boldsymbol{\xi}), \tag{A.37}$$

where M *is of the form in* (A.36) *with entries* $A_{ij} = \frac{\partial y_i}{\partial x_j}$, $B_{ij} = \frac{\partial y_i}{\partial \xi_j}$, $C_{ij} = \frac{\partial \eta_i}{\partial x_j}$, $D_{ij} = \frac{\partial \eta_i}{\partial \xi_j}$*.*

Implicit in Theorem A.12 is that a change of generators always results in an invertible D, so the superdeterminant is well-defined.

Example A.13. Let x, ξ_1, ξ_2 be generators for $\Omega^2(\mathbb{R})$. Then the set of forms $\{x + g(x)\xi_1\xi_2, \xi_1, \xi_2\}$ is also a set of generators, and

$$\int dx\, \partial_{\xi_1}\partial_{\xi_2}\, F(x, \xi_1, \xi_2) = \int dx\, \partial_{\xi_1}\partial_{\xi_2}\, F(x + g(x)\xi_1\xi_2, \xi_1, \xi_2)(1 + g'(x)\xi_1\xi_2). \tag{A.38}$$

It is instructive to verify the claims of the previous example by hand, and we briefly do so. To see the claim that these forms are a set of generators, recall that by definition

$$F(x + g(x)\xi_1\xi_2, \xi_1, \xi_2) = F(x, \xi_1, \xi_2) + F'(x, \xi_1, \xi_2)g(x)\xi_1\xi_2. \tag{A.39}$$

Letting $y \equiv g(x)\xi_1\xi_2$, a general form of $\{x + g(x)\xi_1\xi_2, \xi_1, \xi_2\}$ is thus, for some functions a, b, c, d,

$$a(x+y) + b(x+y)\xi_1 + c(x+y)\xi_2 + d(x+y)\xi_1\xi_2 = a(x) + b(x)\xi_1 + c(x)\xi_2 + (d(x) + a'(x)g(x))\xi_1\xi_2,$$

which clearly shows a general form in $\{x, \xi_1, \xi_2\}$ can be expressed as a form in $\{x + g(x)\xi_1\xi_2, \xi_1, \xi_2\}$.

To verify (A.38) integrate (A.39). Integrating the term containing F' by parts yields

$$\int dx\, \partial_{\xi_1}\partial_{\xi_2}\, F(x + g(x)\xi_1\xi_2, \xi_1, \xi_2) = \int dx\, \partial_{\xi_1}\partial_{\xi_2}\, F(x, \xi_1, \xi_2)(1 - g'(x)\xi_1\xi_2). \tag{A.40}$$

Since $F(x + g(x)\xi_1\xi_2, \xi_1, \xi_2)g'(x)\xi_1\xi_2 = F(x, \xi_1, \xi_2)g'(x)\xi_1\xi_2$, (A.38) follows. This can alternately be verified by computing the superdeterminant of

$$M = \begin{pmatrix} 1 + g'(x)\xi_1\xi_2 & \xi_2 & -\xi_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.41}$$

B Further aspects of symmetries and supersymmetry

This appendix discusses some additional aspects of supersymmetry. First, we briefly introduce complex coordinates, which have often been used in the literature (see, e.g., [13]). Second, we prove Theorem A.8. The remaining sections discuss symmetries and Ward identities, and in particular, highlight how Theorem A.8 is an example of a Ward identity arising from an infinitesimal supersymmetry.

B.1. Complex coordinates. In Appendix A we introduced Grassmann algebras over $\mathbb{R}$ and forms given by smooth functions with values in $\mathbb{R}$. Sometimes it is convenient to work with Grassmann algebras over $\mathbb{C}$ and complex-valued functions, and many discussions of supersymmetry do so, see [13] and references therein. To facilitate comparisons with the literature we briefly introduce complex coordinates and relate them to the presentation of Appendix A.

To introduce complex coordinates we set

$$z = \frac{1}{\sqrt{2}}(x+iy), \quad \bar{z} = \frac{1}{\sqrt{2}}(x-iy), \quad \zeta = \frac{1}{\sqrt{2i}}(\xi+i\eta), \quad \bar{\zeta} = \frac{1}{\sqrt{2i}}(\xi-i\eta). \tag{B.1}$$

Correspondingly, define

$$\frac{\partial}{\partial z_i} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\right), \quad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}\right), \tag{B.2}$$

and define ∂_{ζ_i} and $\partial_{\bar{\zeta}_i}$ to be the antiderivations on Ω^{2N} such that

$$\frac{\partial}{\partial \zeta_i}\zeta_j = \frac{\partial}{\partial \bar{\zeta}_i}\bar{\zeta}_j = \delta_{ij}, \qquad \frac{\partial}{\partial \zeta_i}\bar{\zeta}_j = \frac{\partial}{\partial \bar{\zeta}_i}\zeta_j = 0. \tag{B.3}$$

Up to an irrelevant factor of $\sqrt{i}$ (a constant factor plays no role in determining if a form is supersymmetric), the supersymmetry generator can be written in complex coordinates as

$$Q = \sum_{i=1}^{N} Q_i, \quad Q_i = \zeta_i\frac{\partial}{\partial z_i} + \bar{\zeta}_i\frac{\partial}{\partial \bar{z}_i} - z_i\frac{\partial}{\partial \zeta_i} + \bar{z}_i\frac{\partial}{\partial \bar{\zeta}_i}. \tag{B.4}$$

Hence it acts on the complex generators by

$$Qz_i = \zeta_i, \quad Q\bar{z}_i = \bar{\zeta}_i, \quad Q\zeta_i = -z_i, \quad Q\bar{\zeta}_i = \bar{z}_i. \tag{B.5}$$

Writing $u_i = (z_i, \zeta_i)$ for $i = 1, \dots, N$, the following forms are supersymmetric:

$$u_i \cdot \bar{u}_j \equiv z_i\bar{z}_j + \zeta_i\bar{\zeta}_j. \tag{B.6}$$

Realisation by differential forms. Complex coordinates can be realised in terms of differential forms as follows. Denote the coordinates of $\mathbb{R}^2$ by x and y with differentials dx and dy, and set

$$z = \frac{1}{\sqrt{2}}(x+iy), \quad \bar{z} = \frac{1}{\sqrt{2}}(x-iy), \quad dz = \frac{1}{\sqrt{2i}}(dx+idy), \quad d\bar{z} = \frac{1}{\sqrt{2i}}(dx-idy). \tag{B.7}$$

B.2. Proof of Theorem A.8. The proof of Theorem A.8 will use the complex coordinates introduced in Appendix B.1, and will also make use of the following terminology and facts. A form is called *Q-closed* (*supersymmetric*) if $QF = 0$ and it is called *Q-exact* if $F = QG$ for some form $G \in \Omega^{2N}(\mathbb{R}^{2N})$. The Q-closed forms $u_i \cdot u_j$ from Example A.7 are also Q-exact, as can be verified by checking

$$z_i\bar{z}_j + z_j\bar{z}_i + \zeta_i\bar{\zeta}_j - \bar{\zeta}_i\zeta_j = Q\lambda_{ij}, \quad \lambda_{ij} \equiv z_i\bar{\zeta}_j + z_j\bar{\zeta}_i. \tag{B.8}$$

Proof of Theorem A.8. Any integrable form F can be written as $K = \sum_\alpha F^\alpha \zeta^\alpha$ with (i) ζ^α a monomial in $\{\zeta_i, \bar{\zeta}_i\}_{i=1}^N$ and (ii) F^α an integrable function of $\{z_i, \bar{z}_i\}_{i=1}^N$. To emphasise this, we write $K = K(\boldsymbol{z}, \bar{\boldsymbol{z}}, \boldsymbol{\zeta}, \bar{\boldsymbol{\zeta}})$. To simplify notation we write $\int$ in place of $\int_{(\mathbb{R}^{2|2})^N}$.

Step 1. Let $S = \sum_{i=1}^N (z_i \bar{z}_i + \zeta_i \bar{\zeta}_i)$. We prove the following version of Laplace's Principle:

$$\lim_{t\to\infty} \int e^{-tS} F = F_0(\mathbf{0}). \tag{B.9}$$

Let $t > 0$. We make the change of generators $z_i = \frac{1}{\sqrt{t}} z_i'$ and $\zeta_i = \frac{1}{\sqrt{t}} \zeta_i'$. This transformation has unit Berezinian. Let $\omega \equiv -\sum_{i=1}^N \zeta_i \bar{\zeta}_i$. After dropping the primes, we obtain

$$\int e^{-tS} F = \int e^{-\sum_{i=1}^N z_i \bar{z}_i + \omega} F(\tfrac{1}{\sqrt{t}} \boldsymbol{z}, \tfrac{1}{\sqrt{t}} \bar{\boldsymbol{z}}, \tfrac{1}{\sqrt{t}} \boldsymbol{\zeta}, \tfrac{1}{\sqrt{t}} \bar{\boldsymbol{\zeta}}), \tag{B.10}$$

where $\frac{1}{\sqrt{t}} \boldsymbol{z} \equiv \{\frac{1}{\sqrt{t}} z_i\}_{i=1}^N$, and similarly for the other generators. To evaluate the right-hand side, we expand e^ω and and obtain

$$\int e^{-tS} F = \sum_{n=0}^N \int e^{-\sum_{i=1}^N z_i \bar{z}_i} \frac{1}{n!} \omega^n F(\tfrac{1}{\sqrt{t}} \boldsymbol{z}, \tfrac{1}{\sqrt{t}} \bar{\boldsymbol{z}}, \tfrac{1}{\sqrt{t}} \boldsymbol{\zeta}, \tfrac{1}{\sqrt{t}} \bar{\boldsymbol{\zeta}}). \tag{B.11}$$

We write $K = K^0 + G$, where K^0 is the degree zero part of K. The contribution of K^0 to (B.11) involves only the $n = N$ term and equals

$$\int e^{-tS} F^0 = \int e^{-\sum_{i=1}^N z_i \bar{z}_i} \frac{1}{N!} \omega^N F^0(\tfrac{1}{\sqrt{t}} \boldsymbol{z}, \tfrac{1}{\sqrt{t}} \bar{\boldsymbol{z}}), \tag{B.12}$$

so by the continuity of F_0,

$$\lim_{t\to\infty} \int e^{-tS} F_0 = F_0(\mathbf{0}) \int e^{-\sum_{i=1}^N z_i \bar{z}_i} \frac{1}{N!} \omega^N = F_0(\mathbf{0}) \int e^{-S}. \tag{B.13}$$

By (A.28) with A the identity matrix, this proves that

$$\lim_{t\to\infty} \int e^{-tS} F_0 = F_0(0). \tag{B.14}$$

To complete the proof of (B.9), it remains to show that $\lim_{t\to\infty} \int e^{-tS} G = 0$. As above,

$$\int e^{-tS} G = \sum_{n=0}^N \int e^{-\sum_{i=1}^N z_i \bar{z}_i} \frac{1}{n!} \omega^n \, G\Big(\tfrac{1}{\sqrt{t}} \boldsymbol{z}, \tfrac{1}{\sqrt{t}} \bar{\boldsymbol{z}}, \tfrac{1}{\sqrt{t}} \boldsymbol{\zeta}, \tfrac{1}{\sqrt{t}} \bar{\boldsymbol{\zeta}}\Big). \tag{B.15}$$

Since G has no degree-zero part, the term with $n = N$ is zero. Terms with smaller values of n require factors $\zeta_i \bar{\zeta}_i$ for some i from G, and these factors carry inverse powers of t. They therefore vanish in the limit, and the proof of (B.9) is complete.

Step 2. The Laplace approximation is exact:

$$\int e^{-tS} F \quad \text{is independent of } t \geq 0. \tag{B.16}$$

To prove this, recall that $S = Q\lambda$. Also, $Qe^{-S} = 0$ by the chain rule of Lemma A.9, and $QF = 0$ by assumption. Therefore,

$$\frac{d}{dt} \int e^{-tS} F = -\int e^{-tS} S F = -\int e^{-tS} (Q\lambda) F = -\int Q\big(e^{-tS} \lambda F\big) = 0, \tag{B.17}$$

since the integral of any Q-exact form is zero, because it can be written as a sum of derivatives (whose integral vanishes due to the assumption of rapid decay) and a form of degree lower than the top degree (whose integral vanishes by definition).

Step 3. Finally, we combine Laplace's Principle (B.9) and the exactness of the Laplace approximation (B.16), to obtain the desired result

$$\int F = \lim_{t\to\infty} \int e^{-tS} F = F_0(\mathbf{0}). \qquad \square$$

B.3. Symmetries. This appendix briefly reviews symmetries in the context of smooth manifolds, to prepare the way for a discussion of symmetries of superalgebras.

B.3.1. Infinitesimal symmetries. For a smooth manifold M, infinitesimal symmetries are described by the infinite-dimensional Lie algebra of smooth vector fields, $\mathrm{Vect}(M)$. Vector fields act on functions through the Lie derivative, which associates to every vector field $X \in \mathrm{Vect}(M)$ a *derivation* $T_X : C^\infty(M) \to C^\infty(M)$. We recall that a derivation is a linear map that obeys the Leibniz rule $T_X(fg) = T_X(f)g + fT_X(g)$. Concretely, if M is n-dimensional and X is represented in local coordinates as $X = \sum_{\alpha=1}^n g(u^1, \dots, u^n)\frac{\partial}{\partial u^\alpha}$, then $T_X(f) = \sum_{\alpha=1}^n g(u^1, \dots, u^n)\frac{\partial f}{\partial u^\alpha}$.

In fact, every derivation on $C^\infty(M)$ arises from a vector field, and hence there is an isomorphism $\mathrm{Vect}(M) \simeq \mathrm{Der}(C^\infty(M))$. Thus we can replace geometric objects (vector fields) with algebraic objects (derivations). The perspective will be useful for superspaces, as their definition is fundamentally algebraic rather than geometric.

B.3.2. Integral symmetries. Rather than examining the entire Lie algebra $\mathrm{Der}(C^\infty(M))$, it is often useful to consider subalgebras that respect additional structures on the manifold. We will be interested in the following case where M carries a measure μ. Let $\int_M f$ denote the integral of a function $f : M \to \mathbb{R}$ with respect to the measure μ. We call $\int_M$ an *integral on* M.

Definition B.1. *Let $\int_M$ be an integral on a smooth manifold M. A derivation $T \in \mathrm{Der}(C^\infty(M))$ is an* infinitesimal symmetry *of the integral if for all $f \in C^\infty(M)$ with rapid decay*

$$\int_M Tf = 0. \tag{B.18}$$

Infinitesimal symmetries lead to integration by parts formulas, otherwise known as *Ward identities*: suppose T is a symmetry of $\int_M$, and that $f, g \in C^\infty(M)$ have rapid decay. Then

$$\int_M T(fg) = 0, \tag{B.19}$$

since fg has rapid decay. Since T acts as a derivation, we obtain the Ward identity

$$\int_M (Tf)g = -\int_M f(Tg). \tag{B.20}$$

For spin systems, different infinitesimal symmetries are obtained depending on whether we examine the Gibbs measure $e^{-H_\beta}\, d\boldsymbol{u}$ or the underlying measure $d\boldsymbol{u}$. Ward identities for one lead to (anomalous) Ward identities for the other. For instance, letting $[f]_\beta = \int_{M^\Lambda} fe^{-H_\beta}\, d\boldsymbol{u}$ denote an unnormalised expectation, and letting T be an infinitesimal symmetry of $d\boldsymbol{u}$,

$$\int_{M^\Lambda} T(fe^{-H_\beta}) = 0, \quad \text{i.e.,} \quad \int_{M^\Lambda} (Tf - f(TH_\beta))e^{-H_\beta} = 0 \tag{B.21}$$

and hence

$$[Tf]_\beta = [f(TH_\beta)]_\beta. \tag{B.22}$$

B.3.3. Global symmetries. For spin system Gibbs measures $[F]_\beta = \int_{M^\Lambda} F\, e^{-H_\beta} d\boldsymbol{u}$, an important role is played by derivations $T \in \mathrm{Der}(C^\infty(M^\Lambda))$ which can be written in the form

$$T \equiv \sum_{i\in\Lambda} T_i, \tag{B.23}$$

where each T_i is a copy of a single site derivation

$$T_i = \sum_{\alpha=1}^n f_\alpha(u_i)\frac{\partial}{\partial u_i^\alpha} \tag{B.24}$$

with f_α independent of $i \in \Lambda$. We call these *diagonal derivations.* If a diagonal derivation is an infinitesimal symmetry of the Gibbs measure, then we say that it is a *global symmetry.* The spin system Hamiltonians in this paper are of the form $H_\beta(\boldsymbol{u}) = \frac{1}{4}\sum_{i,j\in\Lambda}\beta_{ij}(u_i - u_j)^2$ with $(u_i - u_j)^2 \equiv (u_i - u_j)\cdot(u_i - u_j)$ for some inner product. Hence the global symmetries are equivalently those diagonal derivations which satisfy

$$T(u_i - u_j)^2 = 0 \tag{B.25}$$

for all $i, j \in \Lambda$. These correspond to the infinitesimal isometries of the target space, and form a representation of a finite dimensional Lie algebra.

For the GFF on $\mathbb{R}^n$, the global symmetries are of the form

$$T \equiv \sum_{i\in\Lambda} T_i, \quad T_i = \sum_{\alpha,\beta=1}^{n} R_{\alpha\beta} u_i^\alpha \frac{\partial}{\partial u_i^\beta} + \sum_{\gamma=1}^{n} S_\gamma \frac{\partial}{\partial u_i^\gamma}, \tag{B.26}$$

where R is an $n \times n$ real skew-symmetric matrix and S is a real vector in $\mathbb{R}^n$. The global symmetries of $\mathbb{R}^n$ hence form a representation of the Euclidean Lie algebra $\mathfrak{so}(n) \ltimes \mathbb{R}^n$ under the Lie bracket of derivations. Global symmetries of Minkowski space $\mathbb{R}^{n,1}$ are of the same form as (B.26), but R is now skew-symmetric with respect to the Minkowski inner product, i.e.,

$$R^T J + JR = 0, \quad J = \operatorname{diag}(-1, 1, \dots, 1). \tag{B.27}$$

This gives a representation of the Poincare Lie algbera $\mathfrak{so}(n,1) \ltimes \mathbb{R}^{n,1}$.

Global symmetries of the $\mathbb{H}^n$ and $\mathbb{S}^n_+$ spin models are induced from Lorentz/orthogonal symmetries of $\mathbb{R}^{n,1}$ and $\mathbb{R}^{n+1}$ respectively, i.e., global symmetries have the form

$$T \equiv \sum_{i\in\Lambda} T_i, \quad T_i = \sum_{\alpha,\beta} R_{\alpha\beta} u_i^\alpha \frac{\partial}{\partial u_i^\beta}. \tag{B.28}$$

For the $\mathbb{H}^n$ model these form a representation of the Lorentzian Lie algebra $\mathfrak{so}(n,1)$, and for the $\mathbb{S}^n_+$ model these form a representation of the orthogonal Lie algebra $\mathfrak{so}(n+1)$. In coordinates, these symmetries can be written as

$$T \equiv \sum_{i\in\Lambda} T_i, \quad T_i = \sum_{\alpha,\beta=1}^{n} R_{\alpha\beta} u_i^\alpha \frac{\partial}{\partial u_i^\beta} + \sum_{\gamma=1}^{n} S_\gamma z_i \frac{\partial}{\partial u_i^\gamma} \tag{B.29}$$

where $S_\gamma = R_{0\gamma}$ and $z = \sqrt{1 + (u^1)^2 + \cdots + (u^n)^2}$ for $\mathbb{H}^n$, while $S_\gamma = R_{(n+1)\gamma}$ and $z = \sqrt{1 - (u^1)^2 - \cdots - (u^n)^2}$ for $\mathbb{S}^n_+$.

B.4. Symmetries of supersymmetric spaces. Infinitesimal symmetries of Berezin integrals and the global symmetries of supersymmetric spaces have descriptions similar to those of the previous section. The primary difference is that all objects are graded.

B.4.1. Superderivations and supersymmetries. Let A be a $\mathbb{Z}^2$-graded algebra (or superalgebra) such as $A = \Omega^n(\mathbb{R}^m)$. Thus $A = A_0 \oplus A_1$ where elements in A_0 are even and elements in A_1 are odd. Using this decomposition, a linear map $T\colon A \to A$ can be written in blocks as

$$Tf = \begin{bmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \end{bmatrix}. \tag{B.30}$$

A linear map is *even* if $T_{01} = T_{10} = 0$, and *odd* if $T_{00} = T_{11} = 0$. As for functions, a *homogeneous* linear map is one that is even or odd. We extend the parity function to homogeneous maps by

$$\alpha(T) = \begin{cases} 0 \in \mathbb{Z}_2, & T \text{ is even} \\ 1 \in \mathbb{Z}_2, & T \text{ is odd} \end{cases}, \tag{B.31}$$

and for homogeneous f we have $\alpha(Tf) = \alpha(T) + \alpha(f)$. A *homogeneous superderivation* is then defined as a homogeneous linear map $T\colon A \to A$ that obeys the super-Leibniz rule

$$T(fg) = (Tf)g + (-1)^{\alpha(T)\alpha(f)} f(Tg). \tag{B.32}$$

Thus even and odd superderivations are derivations and antiderivations, respectively. A general superderivation is a sum of an even and an odd superderivation. The collection of superderivations on A forms a Lie superalgebra $\mathrm{SDer}(A)$ with the supercommutator defined on homogeneous superderivations by

$$[T_1, T_2] = T_1 \circ T_2 - (-1)^{\alpha(T_1)\alpha(T_2)} T_2 \circ T_1, \tag{B.33}$$

and extended to all superderivations by linearity. If $A = \Omega^n(M)$ is a superalgebra of forms on an m-dimensional manifold M, then every superderivation $T \in \mathrm{SDer}(A)$ can be realised in coordinates $(x^1, \dots, x^m, \xi^1, \dots, \xi^n)$ as

$$T = \sum_{\alpha=1}^{m} F_\alpha \frac{\partial}{\partial x^\alpha} + \sum_{\alpha=1}^{n} G_\alpha \frac{\partial}{\partial \xi^\alpha} \tag{B.34}$$

where $F_\alpha, G_\alpha \in A$. If T is an even/odd superderivation then F_α are even/odd forms and G_α are odd/even forms.

Berezin integral symmetries and global symmetries. We define a *Berezin integral* $\int_M$ on a superalgebra $\Omega^n(M)$ to be a linear map defined by integrating forms F against an even Berezin integral form $d\boldsymbol{x}\, \partial_{\boldsymbol{\xi}}\, \rho(\boldsymbol{x}, \boldsymbol{\xi})$, i.e.,

$$\int_M F \equiv \int_{\mathbb{R}^{m|n}} d\boldsymbol{x}\, \partial_{\boldsymbol{\xi}}\, \rho(\boldsymbol{x}, \boldsymbol{\xi}) F(\boldsymbol{x}, \boldsymbol{\xi}). \tag{B.35}$$

Definition B.2. *Let $\int_M$ be a Berezin integral on a superalgebra $\Omega^n(M)$. A superderivation $T \in \mathrm{SDer}(\Omega^n(M))$ is an* infinitesimal symmetry *of $\int_M$ if for all $F \in \Omega^n(M)$ with rapid decay*

$$\int_M TF = 0. \tag{B.36}$$

This leads to Ward identities in the same manner as the non-supersymmetric case, the only difference coming from the super-Leibniz rule: for homogeneous superderivations $T \in \mathrm{SDer}(\Omega^n(M))$ and forms $F, G \in \Omega^n(M)$ we have

$$\int_M TF = (-1)^{\alpha(T)\alpha(F)+1} \int_M TG. \tag{B.37}$$

Global symmetries of supersymmetric spin systems are infinitesimal symmetries of the form

$$T \equiv \sum_{i \in \Lambda} T_i, \tag{B.38}$$

i.e., they are diagonal infinitesimal symmetries. For the spin systems considered in this paper, which are defined in terms of quadratic Hamiltonians $\frac{1}{4} \sum_{i,j\in\Lambda} \beta_{ij} (u_i - u_j)^2$, global symmetries are those that annihilate the appropriate super-Euclidean or super-Minkowski inner product

$$T(u_i - u_j)^2 = 0 \tag{B.39}$$

for all $i, j \in \Lambda$. Here we have written $(u_i - u_j)^2$ for the form $(u_i - u_j) \cdot (u_i - u_j)$. The following subsections briefly discuss this condition for the $\mathbb{R}^{2|2}, \mathbb{H}^{2|2}$, and $\mathbb{S}_+^{2|2}$ models.

B.4.2. $\mathbb{R}^{2|2}$ *model.* The inner product associated to the SUSY GFF is

$$u_i \cdot u_j = x_i x_j + y_i y_j - \xi_i \eta_j + \eta_i \xi_j, \tag{B.40}$$

giving the global symmetries as diagonal superderivations $T \in \mathrm{SDer}(\Omega^{2\Lambda}(\mathbb{R}^{2\Lambda}))$ satisfying

$$T(u_i - u_j)^2 = T\big((x_i - x_j)^2 + (y_i - y_j)^2 - 2(\xi_i - \xi_j)(\eta_i - \eta_j)\big) = 0 \tag{B.41}$$

for all $i, j \in \Lambda$.

Concretely, letting $u_i = (u_i^1, \dots, u_i^4) = (x_i, y_i, \xi_i, \eta_i)$, these are derivations of the form

$$T \equiv \sum_{i\in\Lambda} T_i, \quad T_i = \sum_{\alpha,\beta=1}^{4} R_{\alpha\beta} u_i^\alpha \frac{\partial}{\partial u_i^\beta} + \sum_{\gamma=1}^{4} S_\gamma \frac{\partial}{\partial u_i^\gamma} \tag{B.42}$$

where R is a real 4×4 matrix (independent of $i \in \Lambda$) such that

$$R^{ST} J + JR = 0, \tag{B.43}$$

where R^{ST}, the *supertranspose* of R, and J are given by

$$R^{ST} \equiv \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{ST} = \begin{bmatrix} A^T & C^T \\ -B^T & D^T \end{bmatrix}, \qquad J \equiv \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right], \tag{B.44}$$

and S is a real vector. With the supercommutator of superderivations, these form a representation of the super-Euclidean Lie superalgebra $\mathfrak{osp}(2|2) \ltimes \mathbb{R}^{2|2}$. In particular, the supersymmetry generator

$$Q \equiv \sum_{i\in\Lambda} Q_i = \sum_{i\in\Lambda} \xi_i \frac{\partial}{\partial x_i} + \eta_i \frac{\partial}{\partial y_i} - x_i \frac{\partial}{\partial \eta_i} + y_i \frac{\partial}{\partial \xi_i} \tag{B.45}$$

and the infinitesimal global translation

$$T \equiv \sum_{i\in\Lambda} T_i = \sum_{i\in\Lambda} \frac{\partial}{\partial x_i} \tag{B.46}$$

are global symmetries.

A short computation shows that the individual T_i and Q_i are symmetries of the flat Berezin–Lebesgue measure $d\boldsymbol{x}\, d\boldsymbol{y}\, \partial_{\boldsymbol{\xi}}\, \partial_{\boldsymbol{\eta}}$. For instance, if F is a compactly supported form with top degree component $F_{2\Lambda}(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{\xi\eta}$,

$$\int_{(\mathbb{R}^{2|2})^\Lambda} (T_i F) = \int_{\mathbb{R}^{2\Lambda}} d\boldsymbol{x}\, d\boldsymbol{y}\, \partial_{\boldsymbol{\xi}}\, \partial_{\boldsymbol{\eta}} (T_i F) = \int_{\mathbb{R}^{2\Lambda}} d\boldsymbol{x}\, d\boldsymbol{y} \frac{\partial}{\partial x_i} F_{2\Lambda}(\boldsymbol{x}, \boldsymbol{y}) = 0 \tag{B.47}$$

where in the last step we have used the translation invariance of the usual Lebesgue measure. A particular case of this is formula (5.9).

B.4.3. Super-Minkowski space $\mathbb{R}^{3|2}$*.* The inner product associated to the super-Minkowski model is the super-Minkowski inner product

$$u_i \cdot u_j = -z_i z_j + x_i x_j + y_i y_j - \xi_i \eta_j + \eta_i \xi_j, \tag{B.48}$$

giving the global symmetries as diagonal superderivations $T \in \mathrm{SDer}(\Omega^{2\Lambda}(\mathbb{R}^{3\Lambda}))$ satisfying

$$T(u_i - u_j)^2 = T\big(-(z_i - z_j)^2 + (x_i - x_j)^2 + (y_i - y_j)^2 - 2(\xi_i - \xi_j)(\eta_i - \eta_j)\big) = 0 \tag{B.49}$$

for all $i, j \in \Lambda$. Concretely, letting $u_i = (u_i^0, u_i^1, u_i^2, u_i^3, u_i^4) = (z_i, x_i, y_i, \xi_i, \eta_i)$, these are derivations of the form

$$T \equiv \sum_{i\in\Lambda} T_i, \quad T_i = \sum_{\alpha,\beta=0}^{4} R_{\alpha\beta} u_i^\alpha \frac{\partial}{\partial u_i^\beta} + \sum_{\gamma=1}^{5} S_\gamma \frac{\partial}{\partial u_i^\gamma} \tag{B.50}$$

where R is a real 5×5 matrix such that

$$R^{ST} J + JR = 0 \tag{B.51}$$

with J now the 5×5 matrix

$$J = \left[\begin{array}{ccc|cc} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right], \tag{B.52}$$

and S a real vector. These global symmetries form a representation of the super-Poincare Lie superalgebra $\mathfrak{osp}(2,1|2) \ltimes \mathbb{R}^{3|2}$ with the supercommutator of superderivations. In particular, the supersymmetry generator

$$Q \equiv \sum_{i\in\Lambda} Q_i = \sum_{i\in\Lambda} \left(\xi_i \frac{\partial}{\partial x_i} + \eta_i \frac{\partial}{\partial y_i} - x_i \frac{\partial}{\partial \eta_i} + y_i \frac{\partial}{\partial \xi_i} \right) \tag{B.53}$$

and the global Lorentz boost

$$T \equiv \sum_{i\in\Lambda} T_i = \sum_{i\in\Lambda} \left(z_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial z_i} \right). \tag{B.54}$$

are global symmetries of the super-Minkowski spin model. As for the $\mathbb{R}^{2|2}$ model, the individual T_i and Q_i are symmetries of the Berezin–Lebesgue measure $d\boldsymbol{x}\, d\boldsymbol{y}\, d\boldsymbol{z}\, \partial_{\boldsymbol{\xi}}\, \partial_{\boldsymbol{\eta}}$.

B.4.4. $\mathbb{S}_+^{2|2}$ and $\mathbb{H}^{2|2}$ models. As for their standard counterparts, the global symmetries of the $\mathbb{S}_+^{2|2}$ and $\mathbb{H}^{2|2}$ models are induced from the ambient super-Euclidean and super-Minkowski spaces. In both cases, the global symmetries in ambient coordinates are

$$T \equiv \sum_{i\in\Lambda} T_i, \quad T_i = \sum_{\alpha,\beta=0}^{4} R_{\alpha\beta} u_i^\alpha \frac{\partial}{\partial u_i^\beta}, \tag{B.55}$$

which form a representation of $\mathfrak{osp}(2,1|2)$ for the $\mathbb{H}^{2|2}$ model, and a representation of $\mathfrak{osp}(3|2)$ for $\mathbb{S}_+^{2|2}$. In coordinates, the T_i are written

$$T_i = \sum_{\alpha,\beta=1}^{4} R_{\alpha\beta} u_i^\alpha \frac{\partial}{\partial u_i^\beta} + \sum_{\gamma=1}^{4} S_\gamma z_i \frac{\partial}{\partial u_i^\gamma} \tag{B.56}$$

with $z_i = \sqrt{1 + x_i^2 + y_i^2 - 2\xi\eta}$ for $\mathbb{H}^{2|2}$ and $z_i = \sqrt{1 - x_i^2 - y_i^2 + 2\xi\eta}$ for $\mathbb{S}_+^{2|2}$ and $S_\gamma = R_{3\gamma}$ in both cases. As before, the supersymmetry generator

$$Q \equiv \sum_{i\in\Lambda} Q_i = \sum_{i\in\Lambda} \xi_i \frac{\partial}{\partial x_i} + \eta_i \frac{\partial}{\partial y_i} - x_i \frac{\partial}{\partial \eta_i} + y_i \frac{\partial}{\partial \xi_i} \tag{B.57}$$

is a global symmetry of both the $\mathbb{H}^{2|2}$ and $\mathbb{S}_+^{2|2}$ models, as is the global Lorentz boost/rotation

$$T \equiv \sum_{i\in\Lambda} T_i = \sum_{i\in\Lambda} z_i \frac{\partial}{\partial x_i}. \tag{B.58}$$

A short computation also shows that the individual T_i and Q_i are symmetries of the Berezin–Haar measure $d\boldsymbol{x}\, d\boldsymbol{y}\, \partial_{\boldsymbol{\xi}}\, \partial_{\boldsymbol{\eta}} \frac{1}{\prod_{i\in\Lambda} z_i}$.

B.5. SUSY delta functions. We begin by defining Dirac delta functions to integrate against forms F in $\Omega^2(\mathbb{R}^2)$. We will assume F is given by a smooth function of an even form. Let $u_0 = (0,0,0,0) \in \mathbb{R}^{2|2}$, and let $G \in \Omega^2(\mathbb{R}^2)$ be a smooth compactly supported form with $\int_{\mathbb{R}^{2|2}} G = 1$. For $\varepsilon > 0$ define smooth forms

$$\delta_{u_0}^{(\varepsilon)}(u) \equiv G(\frac{1}{\varepsilon}u), \qquad \frac{1}{\varepsilon}u = (\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{\xi}{\varepsilon}, \frac{\eta}{\varepsilon}). \tag{B.59}$$

We then define

$$\int_{\mathbb{R}^{2|2}} F(u)\delta_{u_0} \equiv \lim_{\varepsilon\to 0} \int_{\mathbb{R}^{2|2}} F(u)\delta_{u_0}^{(\varepsilon)}(u). \tag{B.60}$$

The change of generators that rescales each generator by ε^{-1} has unit Berezinian, and hence

$$\int_{\mathbb{R}^{2|2}} F(u)\delta_{u_0} = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^{2|2}} F(\varepsilon u)\delta_{u_0}^{(1)}(u) = F_0(0) \int_{\mathbb{R}^{2|2}} \delta_{u_0}^{(1)}(u) = F_0(0), \tag{B.61}$$

where we recall F_0 is the degree zero part of F. In the third equality we have used that the degree p parts of F for $p \geqslant 1$ carry factors of ε, and hence vanish in the limit. The last equality follows since $\int_{\mathbb{R}^{2|2}} \delta_{u_0}^{(1)} = \int_{\mathbb{R}^{2|2}} G = 1$.

Suppose $\theta_s \colon (x, y, \xi, \eta) \mapsto (\theta_s x, \theta_s y, \theta_s \xi, \theta_s \eta)$ is invertible with inverse θ_{-s}, and that $\theta_s u_0$ only has non-zero even components. In this setting we define $\delta_{\theta_s u_0}(u)$ by $\delta_{u_0}(\theta_{-s}u)$. If the transformation θ_s has unit Berezinian, then we obtain

$$\int_{\mathbb{R}^{2|2}} F(u)\delta_{\theta_s u_0}(u) = \int_{\mathbb{R}^{2|2}} F(u)\delta_{u_0}(\theta_{-s}u) = \int_{\mathbb{R}^{2|2}} F(\theta_s u)\delta_{u_0}(u) = F_0(\theta_s u_0). \tag{B.62}$$

It is often convenient to choose G as a supersymmetric form. For $\mathbb{R}^{2|2}$, this can be achieved by choosing any smooth compactly supported function $g \colon \mathbb{R} \to \mathbb{R}$ with $g(0) = 1$, and setting $G = g(|u|^2)$.

The definition of delta functions on $\Omega^{2N}(\mathbb{R}^{2N})$ is analogous, but now based on a smooth compact form $G \in \Omega^{2N}(\mathbb{R}^{2N})$.

For $\mathbb{H}^{2|2}$ and $\mathbb{S}_+^{2|2}$, we define delta functions by making using of the definition on $\mathbb{R}^{2|2}$. Namely, for $\mathbb{H}^{2|2}$ in the coordinates $\tilde{u} = (x, y, \xi, \eta)$ with $z(\tilde{u}) = \sqrt{1 + x^2 + y^2 - 2\xi\eta}$, we set

$$\delta_{u_0}^{(\varepsilon, \mathbb{H}^{2|2})}(u) = z(\tilde{u})\delta_{\tilde{u}_0}^{(\varepsilon)}(\tilde{u}) \tag{B.63}$$

where $u_0 = (1, 0, 0, 0, 0) \in \mathbb{H}^{2|2}$, $\delta_{\tilde{u}_0}^{(\varepsilon)}(\tilde{u})$ is a delta function for $\mathbb{R}^{2|2}$ as constructed above, and $\tilde{u}_0 = (0, 0, 0, 0) \in \mathbb{R}^{2|2}$. Then

$$\lim_{\varepsilon\to 0} \int_{\mathbb{H}^{2|2}} F\delta_{u_0}^{(\varepsilon, \mathbb{H}^{2|2})} = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^{2|2}} F(z(\tilde{u}), x, y, \xi, \eta)\delta_{\tilde{u}_0}^{(\varepsilon)}(\tilde{u}) = F_0(1, 0, 0), \tag{B.64}$$

i.e., the zero-degree part of F evaluated at the point $(z, x, y) = (1, 0, 0) \in \mathbb{H}^2$. The construction for $\mathbb{S}_+^{2|2}$ is analogous.

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References

[1] Y. Abe and M. Biskup. Exceptional points of two-dimensional random walks at multiples of the cover time. Preprint, arXiv:1903.04045.

[2] O. Angel, N. Crawford, and G. Kozma. Localization for linearly edge reinforced random walks. *Duke Mathematical Journal*, 163(5):889–921, 2014.

[3] R. Bauerschmidt, D.C. Brydges, and G. Slade. Introduction to a renormalisation group method. Book. Available at `http://www.statslab.cam.ac.uk/~rb812/`.

[4] R. Bauerschmidt, D.C. Brydges, and G. Slade. Logarithmic correction for the susceptibility of the 4-dimensional weakly self-avoiding walk: a renormalisation group analysis. *Comm. Math. Phys.*, 337(2):817–877, 2015.

[5] R. Bauerschmidt, T. Helmuth, and A. Swan. Dynkin isomorphism and Mermin–Wagner theorems for hyperbolic sigma models and recurrence of the two-dimensional vertex-reinforced jump process. Preprint, arXiv:1802.02077, to appear in Ann. Probab.

[6] F.A. Berezin. *Introduction to superanalysis*, volume 9 of *Mathematical Physics and Applied Mathematics*. D. Reidel Publishing Co., Dordrecht, 1987. Edited and with a foreword by A. A. Kirillov, With an appendix by V. I. Ogievetsky, Translated from the Russian by J. Niederle and R. Kotecký.

[7] D. Brydges, S.N. Evans, and J.Z. Imbrie. Self-avoiding walk on a hierarchical lattice in four dimensions. *Ann. Probab.*, 20(1):82–124, 1992.

[8] D. Brydges, J. Fröhlich, and T. Spencer. The random walk representation of classical spin systems and correlation inequalities. *Comm. Math. Phys.*, 83(1):123–150, 1982.

[9] D. Brydges, R. van der Hofstad, and W. König. Joint density for the local times of continuous-time Markov chains. *Ann. Probab.*, 35(4):1307–1332, 2007.

[10] D.C. Brydges, J. Fröhlich, and A.D. Sokal. The random-walk representation of classical spin systems and correlation inequalities. II. The skeleton inequalities. *Comm. Math. Phys.*, 91(1):117–139, 1983.

[11] D.C. Brydges, T. Helmuth, and M. Holmes. The continuous time lace expansion. *In preparation.*

[12] D.C. Brydges and J.Z. Imbrie. Branched polymers and dimensional reduction. *Ann. of Math. (2)*, 158(3):1019–1039, 2003.

[13] D.C. Brydges, J.Z. Imbrie, and G. Slade. Functional integral representations for self-avoiding walk. *Probab. Surv.*, 6:34–61, 2009.

[14] D.C. Brydges and I. Muñoz Maya. An application of Berezin integration to large deviations. *J. Theoret. Probab.*, 4(2):371–389, 1991.

[15] J.W. Cannon, W.J. Floyd, R. Kenyon, and W.R. Parry. Hyperbolic geometry. In *Flavors of Geometry*, volume 31 of *Math. Sci. Res. Inst. Publ.*, pages 59–115. Cambridge Univ. Press, Cambridge, 1997.

[16] S. Caracciolo, A.D. Sokal, and A. Sportiello. Spanning forests and $OSP(N|2M)$-invariant σ-models. *J. Phys. A*, 50(11):114001, 52, 2017.

[17] B. Davis and S. Volkov. Continuous time vertex-reinforced jump processes. *Probab. Theory Related Fields*, 123(2):281–300, 2002.

[18] B. Davis and S. Volkov. Vertex-reinforced jump processes on trees and finite graphs. *Probab. Theory Related Fields*, 128(1):42–62, 2004.

[19] J. Ding, J.R. Lee, and Y. Peres. Cover times, blanket times, and majorizing measures. *Ann. of Math. (2)*, 175(3):1409–1471, 2012.

[20] M. Disertori. Density of states for GUE through supersymmetric approach. *Rev. Math. Phys.*, 16(9):1191–1225, 2004.

[21] M. Disertori, M. Lohmann, and A. Sodin. The density of states of 1D random band matrices via a supersymmetric transfer operator. *Journal of Spectral Theory*, 2019.

[22] M. Disertori and T. Spencer. Anderson localization for a supersymmetric sigma model. *Comm. Math. Phys.*, 300(3):659–671, 2010.

[23] M. Disertori, T. Spencer, and M.R. Zirnbauer. Quasi-diffusion in a 3D supersymmetric hyperbolic sigma model. *Comm. Math. Phys.*, 300(2):435–486, 2010.

[24] E.B. Dynkin. Gaussian and non-Gaussian random fields associated with Markov processes. *J. Funct. Anal.*, 55(3):344–376, 1984.

[25] K.B. Efetov. Supersymmetry and theory of disordered metals. *Adv. in Phys.*, 32(1):53–127, 1983.

[26] N. Eisenbaum. Une version sans conditionnement du théorème d'isomorphisms de Dynkin. In *Séminaire de Probabilités, XXIX*, volume 1613 of *Lecture Notes in Math.*, pages 266–289. Springer, Berlin, 1995.

[27] N. Eisenbaum, H. Kaspi, M.B. Marcus, J. Rosen, and Z. Shi. A Ray-Knight theorem for symmetric Markov processes. *Ann. Probab.*, 28(4):1781–1796, 2000.

[28] J. Fröhlich. On the triviality of $\lambda\varphi_d^4$ theories and the approach to the critical point in $d \underset{(-)}{>} 4$ dimensions. *Nuclear Phys. B*, 200(2):281–296, 1982.

[29] A. Jego. Thick points of random walk and the gaussian free field. Preprint, arXiv:1809.04369.

[30] A. Kassel and T. Lévy. Covariant symanzik identities. Preprint, arXiv:1607.05201.

[31] G. Kozma. Reinforced random walk. In *European Congress of Mathematics*, pages 429–443. Eur. Math. Soc., Zürich, 2013.

[32] Y. Le Jan. Temps local et superchamp. In *Séminaire de Probabilités, XXI*, volume 1247 of *Lecture Notes in Math.*, pages 176–190. Springer, Berlin, 1987.

[33] J.M. Luttinger. A new method for the asymptotic evaluation of a class of path integrals. *J. Math. Phys.*, 23(6):1011–1016, 1982.

[34] J.M. Luttinger. The asymptotic evaluation of a class of path integrals. II. *J. Math. Phys.*, 24(8):2070–2073, 1983.

[35] M.B. Marcus and J. Rosen. *Markov processes, Gaussian processes, and local times*, volume 100 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006.

[36] O.A. McBryan and T. Spencer. On the decay of correlations in SO(n)-symmetric ferromagnets. *Comm. Math. Phys.*, 53(3):299–302, 1977.

[37] A.J. McKane. Reformulation of $n \to 0$ models using anticommuting scalar fields. *Phys. Lett. A*, 76(1):22–24, 1980.

[38] F. Merkl, S.W. Rolles, and P. Tarrès. Convergence of vertex-reinforced jump processes to an extension of the supersymmetric hyperbolic nonlinear sigma model. *Probability Theory and Related Fields*, pages 1–39, 2016.

[39] F. Merkl, S.W.W. Rolles, and P. Tarrés. Random interlacements for vertex-reinforced jump processes. Preprint, arXiv:1903.07910.

[40] A.D. Mirlin. Statistics of energy levels and eigenfunctions in disordered and chaotic systems: supersymmetry approach. In *New directions in quantum chaos (Villa Monastero, 1999)*, volume 143 of *Proc. Internat. School Phys. Enrico Fermi*, pages 223–298. IOS, Amsterdam, 2000.

[41] G. Parisi and N. Sourlas. Random magnetic fields, supersymmetry, and negative dimensions. *Phys. Rev. Lett.*, 43:744–745, Sep 1979.

[42] C. Sabot and P. Tarrès. Edge-reinforced random walk, vertex-reinforced jump process and the supersymmetric hyperbolic sigma model. *Journal of the European Mathematical Society (JEMS)*, 17(9):2353–2378, 2015.

[43] C. Sabot and P. Tarres. Inverting Ray-Knight identity. *Probab. Theory Related Fields*, 165(3-4):559–580, 2016.

[44] C. Sabot, P. Tarrès, and X. Zeng. The vertex reinforced jump process and a random Schrödinger operator on finite graphs. *Ann. Probab.*, 45(6A):3967–3986, 2017.

[45] C. Sabot and X. Zeng. A random Schrödinger operator associated with the Vertex Reinforced Jump Process on infinite graphs. *J. Amer. Math. Soc.*, 32(2):311–349, 2019.

[46] L. Schäfer and F. Wegner. Disordered system with n orbitals per site: Lagrange formulation, hyperbolic symmetry, and Goldstone modes. *Z. Phys. B*, 38(2):113–126, 1980.

[47] M. Shcherbina and T. Shcherbina. Universality for 1d random band matrices: sigma-model approximation. *J. Stat. Phys.*, 172(2):627–664, 2018.

[48] T. Spencer. SUSY statistical mechanics and random band matrices. In *Quantum many body systems*, volume 2051 of *Lecture Notes in Math.*, pages 125–177. Springer, Heidelberg, 2012.

[49] T. Spencer and M.R. Zirnbauer. Spontaneous symmetry breaking of a hyperbolic sigma model in three dimensions. *Comm. Math. Phys.*, 252(1-3):167–187, 2004.

[50] A. Swan. PhD thesis, University of Cambridge, 2020.

[51] K. Symanzik. Euclidean quantum field theory. In R. Jost, editor, *Local Quantum Field Theory*, New York, 1969. Academic Press.

[52] A.-S. Sznitman. *Topics in occupation times and Gaussian free fields.* Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2012.

[53] F. Wegner. Disordered electronic system as a model of interacting matrices. *Phys. Rep.*, 67(1):15–24, 1980. Common trends in particle and condensed matter physics (Proc. Winter Adv. Study Inst., Les Houches, 1980).

[54] M.R. Zirnbauer. Fourier analysis on a hyperbolic supermanifold with constant curvature. *Comm. Math. Phys.*, 141(3):503–522, 1991.