Differentiable maps with isolated critical points are not necessarily open in infinite dimensional spaces

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Abstract. Saint Raymond asked whether continuously differentiable maps with isolated critical points are necessarily open in infinite dimensional (Hilbert) spaces. We answer this question negatively by constructing counterexamples in various settings including all weakly separable spaces.

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1. Introduction

It is well known [1, 5, 17] that C^1 (continuously differentiable) maps without critical points between Banach spaces are open. Saint Raymond [22] asked whether such phenomenon still occurs if the given maps are relaxed to having isolated critical points in infinite dimensional (Hilbert) spaces. The purpose of the paper is to answer this question negatively by constructing counterexamples in various real Banach spaces including all weakly separable ones.

Back to finite dimensional spaces, Saint Raymond [22] proved that C^1 vector fields with countably many critical points are open provided that the dimension of the ambient space is higher than 1. This result was rediscovered by the second-listed author [14] and is implied by Theorem 1 or 2 in [24] by Titus and Young. Furthermore, the second-listed author [15] showed that C^1 vector fields with isolated critical points are local homeomorphisms given that the dimension of the ambient space is higher than 2. For the interest of readers, we refer to [2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 18, 20, 21, 23] for some relevant works in Euclidean spaces.

Throughout the paper Banach spaces are assumed to be over the field $\mathbb{R}$ of real numbers, and differentiability always means Fréchet differentiability. Our main result reads as follows.

Theorem 1.1. *Suppose that on the ambient space X_0 of some Banach space $X = (X_0, \|\cdot\|)$, there exists another norm μ strictly weaker than $\|\cdot\|$ such that*

$$x \in X\backslash\{0\} \mapsto \mu(x) \in \mathbb{R}$$

is everywhere differentiable. Then for any $s \geq 1$, the map

$$x \mapsto x \cdot \mu(x)^s$$

defined on X is C^1, non-open, and has 0 as its only critical point.

To be precise, a map F from an open subset U of X to another Banach space Y is said to be differentiable at some point $x_0 \in U$ if there exists a bounded linear operator $J_F(x_0) : X \to Y$, called the Jacobian or Fréchet derivative of F at x_0, such that

$$F(x_0 + h) = F(x_0) + J_F(x_0)h + o(\|h\|) \quad (h \to 0),$$

and x_0 is said to be a regular point of F if the inverse of $J_F(x_0)$ exists as a bounded linear map from Y to X, otherwise x_0 is called a critical point of F. Moreover, F is said to be C^1 if $x \mapsto J_F(x)$ is well defined on U and it is continuous from U to the space of all bounded linear maps from X to Y.

As an immediate application of Theorem 1.1, on the square-summable sequence space l^2 one can first set another norm

$$\mu((x_k)_{k=1}^{\infty}) := \left(\sum_{k=1}^{\infty} \frac{x_k^2}{k}\right)^{\frac{1}{2}},$$

then apply Theorem 1.1 to answer Saint Raymond's question negatively in the current setting of infinite dimensional real separable Hilbert spaces. More examples will be given in the last section of the paper.

2. Proof of Theorem 1.1

This section is devoted to proving Theorem 1.1. In the following we denote the given map $x \mapsto x \cdot \mu(x)^s$ on X by F.

(i) Continuous differentiability. We first point out that the norm μ on X_0 actually is a C^1 function on $X\backslash\{0\}$, which should be a standard result and follows essentially from [19, Thm. 2(c)]. A short proof of this fact will be included in the appendix for the sake of completeness. With this property available, we can easily deduce from Differential Calculus (see e.g. [5, 17]) that F is a C^1 map on $X\backslash\{0\}$ with Jacobian explicitly given by

$$J_F(x)y = y \cdot \mu(x)^s + x \cdot s\mu(x)^{s-1} \cdot J_\mu(x)y, \tag{2.1}$$

where $x \in X\backslash\{0\}$, $y \in X$. Note then for any $x \in X\backslash\{0\}$, one has

$$\|J_\mu(x)\| \leq \limsup_{\|y\|\to 0} \frac{|\mu(x+y) - \mu(x)|}{\|y\|} \leq \sup_{y \neq 0} \frac{\mu(y)}{\|y\|} < \infty,$$

where the finiteness of the supremum, denoted by β, is due to the fact that the norm μ on X_0 is weaker than $\|\cdot\|$. Thus for any $x \in X\backslash\{0\}$, we have

$$\begin{aligned}\|J_F(x)\| &\leq \mu(x)^s + \|x\| \cdot s\mu(x)^{s-1} \cdot \beta \\ &\leq (\beta\|x\|)^s + \|x\| \cdot s(\beta\|x\|)^{s-1} \cdot \beta \\ &= (1+s)\beta^s \cdot \|x\|^s,\end{aligned}$$

where the second inequality is due to $s \geq 1$. On the other hand, it is easy to see that F is differentiable at the origin with $J_F(0) = 0$, hence the recent upper bound for $\|J_F(x)\|$ implies that $x \mapsto J_F(x)$ is continuous at the origin as well. Thus we have proved that F is a C^1 map on X.

(ii) *Non-openness*. We argue by contradiction and suppose F is an open map. Since $F(0) = 0$, one can fix a $\delta \in (0,1)$ such that for any y in the unit sphere of X, there exists an element x_y in the unit open ball of X satisfying $\delta y = F(x_y)$, or equivalently

$$\delta y = x_y \cdot \mu(x_y)^s. \tag{2.2}$$

Obviously, x_y must be of the form $x_y = r_y y$ for some $r_y \in (0,1)$. Thus taking $\|\cdot\|$-norm on both sides of (2.2) yields

$$\delta = r_y \cdot \mu(r_y y)^s \leq \mu(y)^s, \tag{2.3}$$

where the last inequality is due to $r_y \in (0,1)$. Note then (2.3) contradicts the assumption that μ is strictly weaker than $\|\cdot\|$. Therefore, F is not an open map on X.

(iii) *Unique critical point*. It is known in part (i) that 0 is a critical point of F. In the rest part we let x be an arbitrary non-zero element of X_0, and are going to prove that $J_F(x)$ is bijective on X_0. If so, one can then deduce from Banach's isomorphism theorem that the inverse of $J_F(x)$ is a bounded linear operator on X, or equivalently x is a regular point of F. To establish the bijectivity of $J_F(x)$, we need to show that there exists a unique element $y \in X_0$ depending on an arbitrarily prescribed $z \in X_0$ such that $J_F(x)y = z$, which, in terms of (2.1), is equivalent to

$$y \cdot \mu(x)^s + x \cdot s\mu(x)^{s-1} \cdot J_\mu(x)y = z. \tag{2.4}$$

Considering $\mu(x)^s \neq 0$, we see that y must be a linear combination of x and z. Actually, y is of the form

$$y = \frac{z}{\mu(x)^s} + \gamma x \tag{2.5}$$

for some $\gamma \in \mathbb{R}$. By substituting (2.5) into (2.4) and recalling the well-known fact $J_\mu(x)x = \mu(x)$, we get

$$x \cdot \left(\gamma\mu(x)^s + s\mu(x)^{s-1} \cdot \left[\frac{J_\mu(x)z}{\mu(x)^s} + \gamma\mu(x)\right]\right) = 0,$$

or equivalently

$$\gamma(1+s)\mu(x)^s + s \cdot \frac{J_\mu(x)z}{\mu(x)} = 0 \tag{2.6}$$

because x is a non-zero element of X_0. Hence the solution γ to (2.6) exists and is uniquely given by

$$\gamma = -\frac{s \cdot J_\mu(x)z}{(s+1)\mu(x)^{s+1}}. \tag{2.7}$$

Consequently, the solution y to (2.4) exists and is uniquely given by

$$y = \frac{z}{\mu(x)^s} - \frac{s \cdot J_\mu(x)z}{(s+1)\mu(x)^{s+1}} \cdot x. \tag{2.8}$$

This finishes the proof of the bijectivity claim, and thus concludes the proof of the whole theorem.

3. Examples

In this section we construct two examples so as to apply Theorem 1.1.

(i) l^p spaces. For any $p \in (1, \infty)$, let $l^p = (l^p, \|\cdot\|_p)$ denote the standard p-summable sequence space. It is known [16, 25] that

$$x = (x_k)_{k=1}^\infty \in l^p \backslash \{0\} \mapsto \|x\|_p \in \mathbb{R}$$

is a differentiable function. Let $J : l^p \mapsto l^p$ be the bounded linear map

$$(x_k)_{k=1}^\infty \mapsto (\frac{x_k}{k})_{k=1}^\infty,$$

and define $\mu : l^p \to \mathbb{R}$ as $\mu(x) = \|J(x)\|_p$. It is straightforward to see that μ is a strictly weaker norm than $\|\cdot\|_p$ on l^p, and the function

$$x \in l^p \backslash \{0\} \mapsto \|J(x)\|_p \in \mathbb{R}$$

is everywhere differentiable because of the composition rule. Thus for any $s \geq 1$, we can deduce from Theorem 1.1 that the map $x \mapsto x \cdot \|J(x)\|_p^s$ defined on l^p is C^1, non-open, and has 0 as its only critical point.

(ii) *Weakly separable spaces*. We call an infinite dimensional real Banach space $X = (X, \|\cdot\|)$ *weakly separable* if there exists a sequence of continuous linear functionals $\{l_k \in X^*\}_{k=1}^\infty$ such that

$$x = 0 \text{ in } X \Longleftrightarrow l_k(x) = 0 \text{ for all } k \in \mathbb{N}. \tag{3.1}$$

Given such a space, we may assume without loss of generality that $\|l_k\|_{X^*} = 1$ for all $k \in \mathbb{N}$. Then it is straightforward to verify that

$$\mu : x \in X \mapsto \left(\sum_{k=1}^\infty \frac{l_k(x)^2}{2^k} \right)^{\frac{1}{2}} \in \mathbb{R}$$

is another norm on X, and $x \mapsto \mu(x)^2$ is a C^1 function on X with Jacobian explicitly given by

$$J_{\mu^2}(x)y = \sum_{k=1}^\infty \frac{l_k(x)l_k(y)}{2^{k-1}},$$

where $x, y \in X$. Note that the latter fact implies that

$$x \in X \backslash \{0\} \mapsto \mu(x) \in \mathbb{R}$$

is a C^1 function as well. For each $q \in \mathbb{N}$,

$$X_q := \bigcap_{k=1}^{q} \{x \in X : l_k(x) = 0\}$$

is a closed subspace of $(X, \|\cdot\|)$ with codimension $\leq q$. Since X is infinite dimensional, one can pick a non-zero element x_q of X_q for each $q \in \mathbb{N}$. We then note

$$\mu(x_q) = \Big(\sum_{k=q+1}^{\infty} \frac{l_k(x_q)^2}{2^k} \Big)^{\frac{1}{2}} \leq \frac{\|x_q\|}{\sqrt{2^q}} \quad (q \in \mathbb{N}),$$

which implies that μ is strictly weaker than $\|\cdot\|$ on X. Thus for any $s \geq 1$, we can deduce from Theorem 1.1 that the map

$$x \in X \mapsto x \cdot \mu(x)^s \in X$$

is C^1, non-open, and has 0 as its only critical point. Typical weakly separable spaces include all infinite dimensional real separable Banach spaces and their dual spaces[1], thus l^p and $L^p(\mathbb{R}^d)$ $(d \in \mathbb{N})$ spaces are weakly separable for all $p \in [1, \infty]$.

Appendix

In this appendix we provide an elementary proof of the following proposition.
Proposition: Suppose that on the ambient space X_0 of some normed space $X = (X_0, \|\cdot\|)$, there exists another norm μ such that it is everywhere differentiable on $X\backslash\{0\}$. Then μ is a C^1 function on $X\backslash\{0\}$.

Proof: Let $x \in X\backslash\{0\}$ and $\epsilon > 0$ be arbitrary. Since μ is differentiable at x (in the normed space X), there exists a $\delta \in (0, \|x\|)$ such that

$$|\mu(x+h) - \mu(x) - J_\mu(x)h| \leq \epsilon\|h\|$$

for all $h \in X$ with $\|h\| < \delta$. Thus fixing any $y, z \in X$ with $\|y\| < \|z\| = \frac{\delta}{2}$ gives

$$\mu(x+y+z) - \mu(x) \leq J_\mu(x)(y+z) + \epsilon\|y+z\|. \tag{3.2}$$

Considering $t \in [0,1] \mapsto \mu(x+ty)$ is a differentiable convex function whose derivative at $t = 0$ is $J_\mu(x)y$, we get (see e.g. [10, (3.18.5)])

$$J_\mu(x)y \leq \mu(x+y) - \mu(x). \tag{3.3}$$

In much the same way, we have

$$J_\mu(x+y)z \leq \mu(x+y+z) - \mu(x+y). \tag{3.4}$$

Adding the last three inequalities together yields

$$J_\mu(x+y)z - J_\mu(x)z \leq \epsilon\|y+z\|.$$

By symmetry we should also have

$$J_\mu(x)z - J_\mu(x+y)z \leq \epsilon\|z-y\|.$$

[1] Since a proof of this claim can be obtained by applying the Hahn-Banach theorem suitably, we intend to leave the details to the interested reader. Hint: for separable Banach spaces, apply Corollary 7 in [1, p.51]; while for their dual spaces, use Theorem 10 in [1, p.52].

Combining the last two inequalities and recalling $\|y\| < \|z\| = \frac{\delta}{2}$, we get

$$\|J_\mu(x+y) - J_\mu(x)\| \leq 2\epsilon$$

for all $y \in X$ with $\|y\| < \frac{\delta}{2}$. This suffices to prove the proposition.

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