Entropy of thermal CFTs on curved backgrounds

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We use holography in order to study the entropy of thermal conformal field theory (CFT) on $(1+1)$-dimensional curved backgrounds that contain horizons. Starting from the metric of the Bañados-Teitelboim-Zanelli (BTZ) black hole, we perform explicit coordinate transformations that set the boundary metric in de Sitter or black hole form. The dual picture describes a CFT at a temperature different from that of the horizon. We determine minimal surfaces that allow us to compute the entanglement entropy of a boundary region, as well as the temperature affecting the energy associated with a probe quark on the boundary. For an entangling surface that coincides with the horizon, we study the relation between entanglement and gravitational entropy through an appropriate definition of the effective Newton constant. We find that the leading contribution to the entropy is proportional to the horizon area, with a coefficient that exceeds the standard value for the Bekenstein-Hawking entropy. The difference is attributed to the divergence of the stress-energy tensor of the thermal CFT on the horizon. We demonstrate the universality of these findings by considering the most general metric in a $(2+1)$-dimensional anti–de Sitter bulk containing a nonrotating black hole and a static boundary with horizons.

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I. INTRODUCTION

The relation between entanglement and gravitational entropy in spaces that contain horizons can shed light on the fundamental nature of gravity. The entanglement entropy, a nonlocal quantity measuring the correlation between two subsystems, is UV divergent in continuum field theories. A crucial observation is that the divergent part scales with the area of the entangling surface [1]. The divergence is regulated through the presence of a physical cutoff in the theory. When the entangling surface is identified with a horizon, the connection of the entanglement entropy with the gravitational one must account for a relation between the cutoff and Newton constant. Such a connection may be possible if gravity is induced by quantum fluctuations of matter fields [2].

The AdS/CFT correspondence [3,4] has provided new insight on these issues. The most transparent picture emerges through the Ryu-Takayanagi conjecture [5,6], which states that the entanglement entropy of a part of the boundary enclosed by an entangling surface $\mathcal{A}$ is proportional to the area of a minimal surface γ_A extending from $\mathcal{A}$ into the bulk. The main difficulty one faces when trying to define the gravitational entropy is that the boundary metric is not dynamical, a feature that is equivalent to the vanishing of the effective Newton constant. A way out is suggested by the Randall-Sundrum construction [7], in which dynamical gravity is generated by cutting off the anti–de Sitter (AdS) space at a value $z=\epsilon$ of the bulk coordinates, before the boundary at $z=0$ is reached. A brane is introduced at $z=\epsilon$, with a symmetric copy of the truncated AdS space on its other side. In this context, the Newton constant can be seen as induced by quantum fluctuations of matter fields, represented by the bulk degrees of freedom that are integrated out [7,8]. The framework can be extended to any number of dimensions by considering the regulated form of the on-shell action in the context of holographic renormalization [9–12].

When the entanglement entropy is computed through the Ryu-Takayanagi conjecture [5,6], the area of the minimal surface γ_A diverges near the boundary, reflecting the short-distance entanglement of the local degrees of freedom on either side of the entangling surface $\mathcal{A}$. This makes the introduction of a cutoff necessary. The proportionality factor between the entanglement entropy and the area of the minimal surface involves the bulk Newton constant. A connection to the gravitational entropy of the boundary theory can be established only through an appropriate

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definition of the effective Newton constant for this theory. In the Randall-Sundrum construction [7] the brane is usually placed at a position of the order of the AdS length $l \gg \epsilon$. The effective Newton constant becomes a function of the bulk Newton constant and l. However, if the brane is located very close to the AdS boundary, the role of the cutoff becomes clearer. The cutoff now appears explicitly and becomes part of the definition of the effective couplings, such as the effective Newton constant. This approach is advocated in Refs. [2,13], and will form the basis of our analysis. Instead of using the Randall-Sundrum construction, we shall consider the regulated on-shell action in holographic renormalization [9,11], after extracting the leading cutoff dependence ϵ^{-2} from the induced metric. The cutoff is then absorbed in the effective couplings [14]. The procedure is described in Sec. II. C.

In recent work [14,15], following Refs. [2,13], we made use of the definition of the effective Newton constant that incorporates the cutoff. The leading contribution to the entropy has a universal form that depends only on the horizon area because the same degrees of freedom contribute to the entropy and Newton constant. The particular features of the underlying theory, such as the number of degrees of freedom, become apparent at the level of the subleading corrections to the entropy [14].

In this work we apply this approach to a new class of boundary theories. We focus on $(1+1)$-dimensional boundaries because of the formidable technical difficulties that arise when dealing with higher-dimensional spaces. However, several results can be obtained with interesting physical interpretation, which should be relevant for arbitrary dimensions. The new element that we consider is the presence of a bulk horizon with a characteristic temperature that may differ from the temperature of the boundary horizon. The simplest way to achieve this is by considering a bulk black hole metric with an arbitrary mass parameter. For a de Sitter boundary, such constructions have been studied in Refs. [16,17].

In our setup the dual thermal field theories live in curved spacetimes. The bulk metric of the dual description is constructed through appropriate slicings of a bulk space that includes a black hole. This gravity dual has two types of horizons. For a de Sitter boundary, one is the cosmological horizon,[1] associated with the fact that a timelike geodesic observer sees a thermal bath of particles $T_{dS} = H/(2\pi)$. The other is the bulk black hole horizon that specifies the temperature of the dual-field theory. While both of them indicate a thermal behavior, their crucial difference lies in the fact that the cosmological horizon is observer dependent and the effects of curvature cannot be easily disentangled from thermal effects. Our setup allows a special situation, where the bulk horizon has a different temperature than the horizon on the boundary. When the two temperatures coincide, the dual-field theory is in the canonical Bunch-Davies vacuum, which reduces to the standard Minkowski one as we take the limit $H \to 0$. It seems counterintuitive that an equilibrium configuration exists with the dual conformal field theory (CFT) at a different temperature than the one characterizing the horizon. However, this is made possible by the stress-energy tensor developing a singularity on the horizon [16].

For a de Sitter boundary, we compute the entanglement entropy and explore its relation to the gravitational entropy. In order to obtain a better understanding of the properties of the thermalized CFT, we also analyze a different minimal surface that corresponds to the world sheet of a string extending into the bulk from a single boundary point. Its area allows us to extract the thermal contribution to the energy of a heavy quark located at the boundary point [18] and read off the corresponding temperature, which can be identified with that of the thermalized CFT.

We generalize our findings by repeating the calculation of the entropy for a black hole boundary metric with a mass parameter different from that of the bulk black hole, starting from the black funnel solution of Ref. [19]. An extensive analysis of a similar geometry is presented in Ref. [20], where the emphasis is put on the possible thermal flows between the black hole and the surrounding environment. The bulk metric corresponds to a BTZ black hole with nonzero angular momentum and an unwrapped angular coordinate. The dual picture includes two characteristic temperatures. When these coincide, the angular momentum of the BTZ black hole vanishes and the configuration on the boundary corresponds to the Hartle-Hawking state of a black hole in thermal equilibrium with its surroundings. When one of the temperatures vanishes, one obtains the Unruh state on the boundary, characterized by energy flow away from the black hole. The parametrization of the bulk metric that we consider in Sec. IV corresponds to a generalization of the funnel solution in which the mass parameters of the bulk and boundary black holes are different, while the angular momentum of the bulk black hole vanishes. The dual theory is in a generalization of the Hartle-Hawking state with the environment at a higher temperature than the boundary black hole. Such a configuration is possible because the CFT stress-energy tensor diverges on the black hole horizon.

In the final section of the paper, we use the intuition developed in the particular cases of a de Sitter or black hole boundary in order to derive an expression for the entropy that applies to any boundary metric with a horizon. The resulting expression (5.9) is the main result of this work. Sections II and V contain the analysis of the general setting. The intervening Secs. III and IV, in which the particular cases of a de Sitter or black hole boundary are discussed, make the arguments used in Sec. V for the discussion of the general case clearer.

[1]When referring to the cosmological horizon, we include its extension into the bulk.

The plan of the paper is the following: In Sec. II we review the case of a bulk black hole for a flat AdS boundary. We determine the relevant minimal surfaces for the calculation of the entropy and the CFT temperature, and discuss the issue of the definition of Newton's constant. In Sec. III, through an appropriate choice of coordinates, we derive a metric with a dS boundary and a bulk black hole, dual to a thermal field theory in curved spacetime. We discuss analytically the entangling surface, the divergent terms, and the dependence of the entropy on the ratio of the thermal CFT temperature T and the de Sitter temperature T_{dS}. We also determine analytically the minimal surface that determines the energy associated with a heavy quark on the boundary, in order to deduce the temperature from the thermal contribution. The qualitative behavior of all minimal surfaces turns out to be related to the ratio T/T_{dS} that determines the relative strength of the gravitational potentials generated by the black hole and cosmological horizons. In Sec. IV we repeat the analysis for a boundary with a black hole horizon. Finally, in Sec. V we generalize our results for an arbitrary static boundary metric with horizons.

II. BULK BLACK HOLE

A. Interpretation

We consider solutions of the Einstein field equations in $2+1$ dimensions with a negative cosmological constant. All such solutions are locally isometric to AdS space. We normalize all dimensionful parameters with respect to the AdS length l, which is equivalent to setting $l=1$.

We start by considering a metric of the form

$$ds^2 = -(r^2-\mu)dt^2 + \frac{dr^2}{r^2-\mu} + r^2 d\phi^2, \tag{2.1}$$

with $r \in [\sqrt{\mu}, \infty]$. If μ vanishes, Eq. (2.1) describes part of the covering space of AdS. This becomes obvious if we observe that the definition of a new coordinate $u = 1/r$ turns Eq. (2.1) into the standard AdS metric in Poincaré coordinates. If μ is nonzero and the coordinate ϕ is taken to be periodic, with period equal to 2π, the metric describes a nonrotating BTZ black hole [21] with mass parameter μ and Hawking temperature $T = \sqrt{\mu}/(2\pi)$. In the following we allow ϕ to take values over the whole real axis. For the metrics we study, a μ-dependent bulk horizon always exists, which allows us to make the standard assumption that $\sqrt{\mu}$ is related to the temperature of the dual CFT.

An asymptotically AdS geometry can be related to a dual CFT on the boundary through the AdS/CFT correspondence [3,4]. An efficient way to establish the connection is through the use of Fefferman-Graham coordinates [22]. The most general $(2+1)$-dimensional metric that satisfies Einstein's equations with a negative cosmological constant is of the form [23]

$$ds^2 = \frac{1}{z^2}[dz^2 + g_{\mu\nu}dx^\mu dx^\nu], \tag{2.2}$$

where

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + z^2 g^{(2)}_{\mu\nu} + z^4 g^{(4)}_{\mu\nu}. \tag{2.3}$$

The stress-energy tensor of the dual CFT is [11]

$$\langle T_{\mu\nu}\rangle = \frac{1}{8\pi G_3}[g^{(2)} - \mathrm{tr}(g^{(2)})g^{(0)}]. \tag{2.4}$$

The metric of Eq. (2.1) can be put in the form of Eq. (2.2) by defining a new coordinate $z = (2/\mu)(r - \sqrt{r^2-\mu})$. It becomes

$$ds^2 = \frac{1}{z^2}\left[dz^2 - \left(1-\frac{\mu}{4}z^2\right)^2 dt^2 + \left(1+\frac{\mu}{4}z^2\right)^2 d\phi^2\right], \tag{2.5}$$

where $z \in [0, z_h := 2/\sqrt{\mu}]$ and the metric is of the desired form with a flat boundary,

$$ds_0^2 = g^{(0)}_{\mu\nu}dx^\mu dx^\nu = -dt^2 + d\phi^2. \tag{2.6}$$

From Eq. (2.4), we obtain the CFT energy density and pressure:

$$\rho = \frac{E}{V} = -\langle T^t_t\rangle = \frac{\mu}{16\pi G_3}, \tag{2.7}$$

$$p = \langle T^\phi_\phi\rangle = \frac{\mu}{16\pi G_3}. \tag{2.8}$$

B. Entanglement

We are interested in computing the entanglement entropy of a part of the boundary enclosed by an entangling surface $\mathcal{A}$. According to the Ryu-Takayanagi conjecture [5,6], the entanglement entropy is proportional to the area of a minimal surface γ_A extending from $\mathcal{A}$ into the bulk. For the metric (2.5) the area to be minimized is actually a length, given by

$$\mathrm{Area}(\gamma_A) = \int_{\phi_1}^{\phi_2} d\phi \frac{1}{z}\sqrt{\left(\frac{dz}{d\phi}\right)^2 + \left(1+\frac{\mu}{4}z^2\right)^2}. \tag{2.9}$$

The minimization results in the differential equation

$$z\left(1+\frac{1}{4}\mu z^2\right)z'' + \left(1-\frac{3}{4}\mu z^2\right)(z')^2 + \left(1-\frac{1}{4}\mu z^2\right)\left(1+\frac{1}{4}\mu z^2\right)^2 = 0. \tag{2.10}$$

Its solution with boundary conditions $z(\phi_1) = z(\phi_2) = 0$, with $\phi_2 > \phi_1$, is

$$z(\phi) = \frac{2}{\sqrt{\mu}} \sqrt{\frac{(e^{\sqrt{\mu}\phi} - e^{\sqrt{\mu}\phi_1})(e^{\sqrt{\mu}\phi_2} - e^{\sqrt{\mu}\phi})}{(e^{\sqrt{\mu}\phi} + e^{\sqrt{\mu}\phi_1})(e^{\sqrt{\mu}\phi} + e^{\sqrt{\mu}\phi_2})}}. \quad (2.11)$$

The solution (2.11) can also be expressed in coordinates (r, ϕ). It reads

$$r(\phi) = \frac{\sqrt{\mu} e^{\sqrt{\mu}\phi}(e^{\sqrt{\mu}\phi_1} + e^{\sqrt{\mu}\phi_2})}{\sqrt{(e^{2\sqrt{\mu}\phi} - e^{2\sqrt{\mu}\phi_1})(e^{2\sqrt{\mu}\phi_2} - e^{2\sqrt{\mu}\phi})}}. \quad (2.12)$$

Substituting Eq. (2.11) in Eq. (2.9) leads to a divergent integral. There are two identical divergences, arising from the points $\phi = \phi_1$ and $\phi = \phi_2$ at which the minimal curve approaches the boundary. The integral can be made finite by imposing a cutoff at $z = \epsilon$. The leading contribution is $\text{Area}(\gamma_A) = 2\log(1/\epsilon)$. The entanglement entropy is given by [5,6]

$$S = \frac{\text{Area}(\gamma_A)}{4G_3} = \frac{\log(1/\epsilon)}{2G_3}, \quad (2.13)$$

displaying the characteristic dependence on the UV cutoff.

C. Effective Newton constant

Expressions such as Eq. (2.13) for the entanglement entropy arise also in cases in which the boundary metric is nontrivial. We are especially interested in situations in which this metric has a horizon. If the entangling surface coincides with the horizon, one would expect the entropy to be related to the gravitational entropy. In order to assign a physical meaning to a divergent expression, such as Eq. (2.13), one must define an effective Newton constant for the boundary theory. Such a definition was used in Refs. [14,15], in the spirit of Ref. [13]. The effective Newton constant for a $(d+1)$-dimensional boundary theory is

$$G_{d+1} = (d-1)\epsilon^{d-1} G_{d+2}, \quad (2.14)$$

with $(d-1)\epsilon^{d-1}$ replaced by $1/\log(1/\epsilon)$ for $d = 1$. In four dimensions this definition can be justified in the context of the Randall-Sundrum (RS) model [7]. Notice that in the limit $\epsilon \to 0$ the constant vanishes and gravity becomes nondynamical. This makes it difficult to compute the gravitational entropy in the context of the AdS/CFT correspondence.

An alternative, more rigorous way to obtain the same result is through holographic renormalization [11,24], which produces the stress-energy tensor of the dual CFT discussed above. The bulk metric of an asymptotically AdS space is written in a Fefferman-Graham expansion [22] in terms of the bulk coordinate z. A solution is then obtained order by order. The on-shell gravitational action is regulated by restricting the bulk integral to the region $z > \epsilon$. The divergent terms are subtracted through the introduction of appropriate counterterms. In this way a renormalized effective action is obtained, expressed in terms of the induced metric γ_{ij} on the surface at $z = \epsilon$. In our approach the entropy is not renormalized. We assume that the cutoff ϵ is physical and we incorporate it in the effective couplings. This amounts to employing the regulated form of the effective action, without the subtraction of divergences. The leading terms, which would diverge for $\epsilon \to 0$, can be found in the counterterm action of holographic renormalization. They are expressed in terms of the induced metric γ_{ij}, which includes a factor ϵ^{-2}. Using the results of [11,24] and extracting the ϵ^{-2} factor from γ_{ij}, we can express the leading terms of the regulated action as

$$S = \frac{1}{16\pi G_3} \int d^2x \sqrt{-\gamma} \left[\frac{2}{\epsilon^2} - \log(\epsilon)\mathcal{R} \right]. \quad (2.15)$$

The first term corresponds to a cosmological constant, which must be (partially) canceled by vacuum energy localized on the surface at $z = \epsilon$, such as the brane tension in the RS model [7]. The second term is the standard Einstein term if the effective Newton constant G_2 is defined as

$$G_2 = \frac{G_3}{\log(1/\epsilon)}. \quad (2.16)$$

D. Thermal effects

In this section we examine the thermal effects of the strongly coupled fields sourced by a heavy probe quark located on the boundary.[2] The total energy of the quark can be obtained from the world-sheet area of a string attached to the quark and extending into the bulk [18]. One expects an infinite contribution from the region near the boundary, which needs to be regularized, and a contribution from the other endpoint that is attached to the bulk horizon, which carries the information on the temperature. The string solution with world-sheet parameters (τ, σ) corresponding to a quark in the coordinate system (2.1) is given by a radial-gauge parametrization of the form $(t, r) = (\tau, \sigma)$, where the string hangs from a fixed point with $\phi = \phi_b$ on the boundary at $r \to \infty$. For a flat boundary, the string remains straight as it extends into the bulk. The surface can be parametrized in a similar fashion in the coordinate system (2.5). In this case we have $(t, z) = (\tau, \sigma)$, where the string now hangs from a point $\phi = \phi_b$ on the boundary at $z \to 0$.

The minimal surface is a solution of the Nambu-Goto equations of motion. Its area includes a multiplicative time factor $\mathcal{T}$ and is given by

[2]For a study of the backreaction of a uniform distribution of heavy quarks on the entanglement entropy, see Ref. [25].

$$S = \frac{\mathcal{T}}{2\pi\alpha'}\int_{z_b}^{z_h} dz\sqrt{-g_{tt}g_{zz}} = \frac{\mathcal{T}}{2\pi\alpha'}\int_{z_b}^{z_h}\frac{dz}{4z^2}(4-\mu z^2), \tag{2.17}$$

where we impose a cutoff $z_b = \epsilon$, while $z_h = 2/\sqrt{\mu}$ is the deepest point that the string can reach in the bulk because of the presence of the black hole horizon. The (rescaled) energy E of the string, arising from the strongly coupled fields sourced by the quark, is given by

$$E \coloneqq \frac{2\pi\alpha'}{\mathcal{T}}S = \frac{1}{\epsilon} - \sqrt{\mu} + \mathcal{O}(\epsilon). \tag{2.18}$$

The thermal effects amount to a contribution proportional to $\sqrt{\mu}$ and, therefore, to the temperature.

III. STATIC DE SITTER BOUNDARY

In this section we study a thermal field theory of temperature T in a de Sitter spacetime with a different temperature T_{dS} associated with the cosmological horizon. The dual picture involves a metric with a de Sitter boundary in static coordinates in the presence of black hole configuration in the bulk. The metric (2.1) can be put in the form

$$ds^2 = \frac{1}{z^2}\left[dz^2 - (1-H^2\rho^2)\left(1-\frac{1}{4}\left[\frac{\mu-H^2}{1-H^2\rho^2}+H^2\right]z^2\right)^2 dt^2 + \left(1+\frac{1}{4}\left[\frac{\mu-H^2}{1-H^2\rho^2}-H^2\right]z^2\right)^2\frac{d\rho^2}{1-H^2\rho^2}\right], \tag{3.1}$$

with a de Sitter boundary metric

$$ds_0^2 = g_{\mu\nu}^{(0)}dx^\mu dx^\nu = -(1-H^2\rho^2)dt^2 + \frac{d\rho^2}{1-H^2\rho^2}. \tag{3.2}$$

We concentrate on the causally connected region $-1/H < \rho < 1/H$. The space contains two cosmological horizons (and their extensions into the bulk) at $\rho_h = \pm 1/H$, and a bulk horizon at

$$z_h(\rho) = \frac{2\sqrt{1-H^2\rho^2}}{\sqrt{\mu - H^4\rho^2}}. \tag{3.3}$$

It is apparent that the bulk horizon extends smoothly to the location of the cosmological horizon on the boundary only for $\mu \geq H^2$. For this reason, we consider only this parameter range in the following. The deepest point of the bulk horizon is at $z_{h,max} = 2/\sqrt{\mu}$ for $\rho = 0$. For $\mu > H^2$, the cosmological horizons are located behind the bulk horizon.

The metric of Eq. (3.1) can be derived from that of Eq. (2.1) with a coordinate transformation that does not affect the time coordinate. The transformation is

$$r(z,\rho) = \sqrt{\mu + \frac{[1-H^2\rho^2-(\mu-H^4\rho^2)z^2/4]^2}{z^2(1-H^2\rho^2)}}, \tag{3.4}$$

$$\phi(z,\rho) = \frac{1}{2H}\log\left[\frac{1+H\rho}{1-H\rho}\right] + \frac{1}{2\sqrt{\mu}}\log\left[\frac{1-H^2\rho^2+(\sqrt{\mu}-H^2\rho)^2z^2/4}{1-H^2\rho^2+(\sqrt{\mu}+H^2\rho)^2z^2/4}\right]. \tag{3.5}$$

The region near negative infinity for ϕ is mapped to the vicinity of $-1/H$ for $\rho < 0$, and the region near positive infinity for ϕ to the vicinity of $1/H$ for $\rho > 0$. It is apparent that ϕ cannot be considered to be periodic.

The stress-energy tensor corresponding to the metric (3.1) is equal to

$$\rho = -\langle T^t{}_t\rangle = \frac{1}{16\pi G_3}\left(\frac{\mu-H^2}{1-H^2\rho^2} - H^2\right), \tag{3.6}$$

$$p = \langle T^\rho{}_\rho\rangle = \frac{1}{16\pi G_3}\left(\frac{\mu-H^2}{1-H^2\rho^2} + H^2\right), \tag{3.7}$$

where the presence of the conformal anomaly is evident, as

$$\langle T^\mu{}_\mu\rangle = \frac{1}{8\pi G_3}H^2. \tag{3.8}$$

For $\mu > H^2$ the stress-energy tensor displays singularities at the locations of the horizons $\rho = \pm 1/H$ of static de Sitter space. As also discussed in Ref. [16], we interpret this configuration as a boundary CFT at a temperature $T = \sqrt{\mu}/(2\pi)$ larger than the de Sitter temperature $T_{dS} = H/(2\pi)$. The divergence of the energy density on the de Sitter horizon can be viewed as a heat bath localized at this point that keeps the system at a temperature different from the de Sitter temperature.

A. Entropy

We turn next to the discussion of the entropy associated with such a configuration. In the holographic approach the entanglement entropy is given by a spacelike minimal curve anchored at the two points on the boundary that act as the entangling surface. The calculation can be performed by minimizing the functional

$$\text{Area}(\gamma_A) = \int_{-1/H}^{1/H} d\rho\,\frac{1}{z}\sqrt{\left(\frac{dz}{d\rho}\right)^2 + \left(1+\frac{1}{4}\left[\frac{\mu-H^2}{1-H^2\rho^2}-H^2\right]z^2\right)^2\frac{1}{1-H^2\rho^2.}} \tag{3.9}$$

Finding an analytical solution of the resulting nonlinear second-order equation is a difficult problem.[3] However, it becomes tractable if one uses the fact that the minimal curve is a geometrical object which does not depend on the system of coordinates. This means that one can simply use the solution of Eq. (2.12) and transform the coordinates through Eqs. (3.4) and (3.5). The resulting implicit relation between the coordinates z and ρ can be solved explicitly in this case. By imposing the appropriate boundary conditions we obtain

$$\begin{aligned} z(\rho) = 2[&[(H^2\rho^2-1)(H^2\rho+\sqrt{\mu})^2F(\rho_1)F(\rho_2) \\ &+2\sqrt{\mu}(1-H^2\rho^2)F(\rho)^{\frac{1}{2}}(F(\rho_1)^{\frac{1}{2}}+F\left(\rho_2)^{\frac{1}{2}}\right)((H^2\rho+\sqrt{\mu})F(\rho_1)^{\frac{1}{2}}F(\rho_2)^{\frac{1}{2}}+(\sqrt{\mu}-H^2\rho)F(\rho)) \\ &+(H^2\rho^2-1)(\sqrt{\mu}-H^2\rho)^2F(\rho)^2 \\ &-(H^2\rho^2-1)F(\rho)((H^4\rho^2-\mu)F(\rho_1)+(H^4\rho^2-\mu)F(\rho_2)-4\mu F(\rho_1)^{\frac{1}{2}}F(\rho_2)^{\frac{1}{2}})]/ \\ &[((\sqrt{\mu}-H^2\rho)^2F(\rho)-(H^2\rho+\sqrt{\mu})^2F(\rho_1))((\sqrt{\mu}-H^2\rho)^2F(\rho)-(H^2\rho+\sqrt{\mu})^2F(\rho_2))]]^{\frac{1}{2}}. \end{aligned}$$

with

$$F(\rho) := \left(\frac{1-H\rho}{1+H\rho}\right)^{\frac{\sqrt{\mu}}{H}}. \tag{3.10}$$

The minimal curve approaches the boundary $z=0$ at $\rho=\rho_1$ and $\rho=\rho_2$. In Fig. 1 we present minimal curves for a range of entangling boundary segments, with lengths that increase until the whole region between the boundary horizons is covered. In this limit the minimal curve coincides with the bulk horizon.

The length of the minimal curve receives an infinite contribution from the region near the boundary and must be regulated through the introduction of an appropriate cutoff at $z=\epsilon$. It must be pointed out that the only element that differentiates the final result for the various metrics that we are considering is the way the divergence near the boundary is regulated. Our basic assumption is that the cutoff must be imposed on the bulk coordinate in the Fefferman-Graham parametrization. Even though this coordinate is always denoted by z, setting $z=\epsilon$ implements a different cutoff procedure for every choice of boundary metric. This is reflected in quantities that depend on the cutoff, such as the length of the minimal curve that determines the entanglement entropy.

We are interested in the limit that the entangling surface covers the whole region between the horizons of the boundary $(1+1)$-dimensional de Sitter metric. These are located at $\rho_1=-1/H$ and $\rho_2=1/H$. The resulting entropy results from the entanglement between the "North" and "South" static patches of the global geometry [14,15]. In this limit, the minimal curve (3.10) coincides with the bulk horizon of the metric of Eq. (3.1), which is given by Eq. (3.3) The length of the minimal curve can be computed by substituting this expression in Eq. (3.9). We focus on the two divergent contributions, arising from the vicinity of the boundary. Imposing a cutoff at $z=\epsilon$, we find

$$\text{Area}(\gamma_A) = 2\frac{\sqrt{\mu}}{H}\log\left(\frac{1}{\epsilon}\right) + \text{finite terms}. \tag{3.11}$$

The entanglement entropy is

$$S_{\text{dS}} = \frac{\text{Area}(\gamma_A)}{4G_3} = \frac{\sqrt{\mu}}{H}\frac{\log(1/\epsilon)}{2G_3}. \tag{3.12}$$

Making use of Eq. (2.16) allows us to write Eq. (3.12) as

$$S_{\text{dS}} = \frac{\sqrt{\mu}}{H}\frac{2}{4G_2}. \tag{3.13}$$

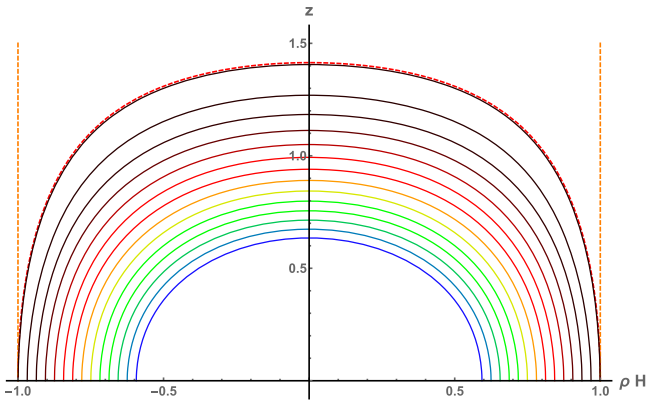

FIG. 1. The minimal curves for $\mu=2H^2$ and for different entangling boundary points. The bulk horizon of Eq. (3.3) is denoted by the curved dashed line, and the cosmological horizons at $\rho=\pm 1/H$ by the vertical dashed lines. When the endpoints approach the cosmological horizons, the entangling surface coincides with the bulk horizon.

[3]An alternative approach in order to determine the minimal surface would make use of generalized calibrations [26].

For $\sqrt{\mu} = H$ this is the expected gravitational entropy of the $(1+1)$-dimensional de Sitter space, whose horizons are two points [13]. The bulk black hole corresponds to a dual CFT at the same temperature as the horizons. For $\sqrt{\mu} > H$, the above expression seems to indicate a modification of Bekenstein's relation between the entropy and the area of the horizon [27]. This is clearly related to the divergence of the stress-energy tensor at the horizon, with a singular (positive) energy density given by Eq. (3.6). For this reason, the situation can also be viewed differently. In Ref. [14] it was shown, using holography, that the de Sitter entropy in $3+1$ dimensions receives subleading logarithmic corrections, regulated by the UV cutoff, which are proportional to the central charge of the dual CFT. These corrections arise through the higher curvature terms in the effective action responsible for the conformal anomaly. In $1+1$ dimensions we expect similar logarithmic corrections. However, these are now of the same order as the leading contribution to the gravitational entropy. In this spirit, Eq. (3.12) could be rewritten as

$$S_{\rm dS} = \frac{2}{4G_2} + \frac{1}{2G_3}\left(\frac{\sqrt{\mu}}{H} - 1\right)\log\left(\frac{1}{\epsilon}\right), \tag{3.14}$$

where the coefficient of the logarithm accounts for the additional degrees of freedom of a CFT thermalized above the horizon temperature. The divergence of the energy density on the horizon indicates that these degrees of freedom are localized there. For $\sqrt{\mu} < H$ the energy density is negative and its divergence indicates a pathology. The construction must be extended in order to account for the energy flow out of the horizon towards the environment. Such solutions are studied in Ref. [20] for the case of a boundary black hole.

As a side remark, we note that the first law of thermodynamics is satisfied for the thermalized CFT. The thermal energy density is given by the first term of the boundary stress-energy tensor (3.6), which includes the singularities at the horizons. The second term results from quantum effects that induce the conformal anomaly. The energy can be expressed as (see the corresponding discussion in [28])

$$E = \int_{\Sigma_t} d\rho \sqrt{h}\langle T_{\mu\nu}\rangle \xi^\mu n^\nu = -\int_{\Sigma_t} d\rho \sqrt{-g}\langle T^t{}_t\rangle, \tag{3.15}$$

where $\xi^\mu\partial_\mu = \partial_t$ is the time-translation Killing vector, Σ_t the surface (interval in this case) of constant t on the boundary, and $n^\mu\partial_\mu = \sqrt{|g^{tt}|}\partial_t$ the unit normal to Σ_t. From this expression we can derive the variation of E with respect to μ. The leading divergent contribution comes from the region near the two boundary horizons and is given by

$$dE = \frac{1}{16\pi G_3}\frac{d\mu}{H}\log\left(\frac{1}{\epsilon_\rho}\right). \tag{3.16}$$

The cutoff ϵ_ρ is imposed on the coordinate ρ in order to regulate the divergences near the horizons by stopping the integration at $\rho = \pm 1/H \mp \epsilon_\rho$. This cutoff is related to the cutoff ϵ imposed on the bulk coordinate z through Eq. (3.3), which gives

$$\epsilon_\rho = \frac{\mu - H^2}{8H}\epsilon^2. \tag{3.17}$$

Substitution in Eq. (3.16) results in

$$dE = \frac{1}{8\pi G_3}\frac{d\mu}{H}\log\left(\frac{1}{\epsilon}\right), \tag{3.18}$$

where we have kept the leading term. As we noted above, the second term in Eq. (3.14) can be interpreted as the entropy of the thermalized CFT. Variation with respect to μ and use of the relation $T = \sqrt{\mu}/(2\pi)$ reproduces correctly the first law of thermodynamics: $dS = dE/T$.

B. Thermal effects

In order to analyze the temperature of the CFT from a different perspective, we consider the thermal corrections to the energy associated with a heavy probe quark at some point on the boundary of a bulk space described by the metric (3.1). As we reviewed in Sec. II D, the energy can be obtained through the minimization of the Nambu-Goto action of a string with a single boundary endpoint at the location of the probe quark [18]. When the quark is placed at $\rho_b = 0$, the string extends along a straight line from the boundary to the bulk horizon. The energy is given by Eq. (2.18). The thermal effects are accounted for by a temperature $T = \sqrt{\mu}/(2\pi)$. As we assume that $\sqrt{\mu} \geq H$, this temperature satisfies $T \geq T_{dS}$, with the two coinciding only for $\sqrt{\mu} = H$.

If the string endpoint is placed at $\rho_b \neq 0$, a straight-line configuration does not minimize the Nambu-Goto action any more. The gravitational influence of the horizons is now uneven and the string bends toward the closest cosmological horizon. The equations for the minimization problem in this case are obtained from the action

$$S = \frac{\mathcal{T}}{2\pi\alpha'}\int d\rho \frac{1}{z^2}\sqrt{h(\rho)G_-^2(\rho,z)\left(z'(\rho)^2 + \frac{G_+^2(\rho,z)}{h(\rho)}\right)},$$
$$h(\rho) := 1 - H^2\rho^2, \quad G_\pm(\rho,z) := 1 \mp \frac{1}{4}\left(\frac{\mu - H^2}{h(\rho)} \pm H^2\right)z^2, \tag{3.19}$$

and are lengthy nonlinear differential equations of second order. We take advantage of the fact that we know the

straight string solution for the metric (2.1), which is given by the parametrization $(t, r, \phi) = (\tau, \sigma, \phi_b)$, with constant ϕ_b. From this we can obtain the corresponding solution for the metric (3.1) by using Eqs. (3.4) and (3.5). The surface that satisfies the equations of motion arising from minimizing the functional (3.19) reads

$$z(\rho) = \frac{2\sqrt{h(\rho)}\sqrt{F(\rho) - F(\rho_b)}}{\sqrt{-F(\rho)(\sqrt{\mu} + H^2\rho)^2 + (\sqrt{\mu} - H^2\rho)^2 F(\rho_b)}}, \tag{3.20}$$

where the function $F(\rho)$ has been defined in Eq. (3.10). In our coordinate system, ρ_b is the boundary position of the string, $\rho_b \in [0, 1/H]$, which implies that $F(\rho_b) \in [0, 1]$. The coordinates ρ_b and ϕ_b are related through $F(\rho_b) = e^{-2\sqrt{\mu}\phi_b}$. The point ρ_h at which the minimal surface hits the bulk horizon can be found by solving the equation $z(\rho_h) = z_h(\rho_h)$, with the horizon $z_h(\rho)$ given by Eq. (3.3).

We consider first the special case $\mu = H^2$, for which the CFT temperature is equal to the de Sitter temperature [29]. The minimal surface simplifies to

$$z(\rho) = \frac{2}{H}\sqrt{\frac{\rho - \rho_b}{\rho + \rho_b}}, \tag{3.21}$$

starting from the boundary point ρ_b and hitting the cosmological horizon at

$$z(H^{-1}) = \frac{2}{H}\sqrt{F(\rho_b)}. \tag{3.22}$$

The form of the surface is depicted in the left plot of Fig. 2. Surfaces that start from $\rho_b \simeq 0$ tend to approach quickly the bulk horizon before turning towards the cosmological one. On the other hand, surfaces with $\rho_b \simeq H^{-1}$ bend towards the cosmological horizon without going deep into the bulk. The energy can now be computed by performing the integration in Eq. (3.19) and cutting off the divergent contribution from the lower limit at $z = \epsilon$. We find

$$\begin{aligned} E &= -\frac{\sqrt{1 - H^2\rho_b^2}}{z(\rho)}\left(1 + \frac{1}{4}H^2 z(\rho)^2\right)\Big|_{\rho_b}^{H^{-1}} \\ &= \frac{\sqrt{1 - H^2\rho_b^2}}{\epsilon} - H + \mathcal{O}(\epsilon). \end{aligned} \tag{3.23}$$

The divergent part receives a correction corresponding to the redshift factor of the boundary metric. The finite term is proportional to the temperature T_{dS}, as expected.

We turn now to the general case $\mu > H^2$. The string described by Eq. (3.20) hits the bulk horizon at a point ρ_h satisfying the algebraic equation $z(\rho_h) = z_h(\rho_h)$, where $z_h(\rho)$ is given by Eq. (3.3). An implicit solution is given by the relation

$$F(\rho_h)\frac{\sqrt{\mu} + H^2\rho_h}{\sqrt{\mu} - H^2\rho_h} = F(\rho_b), \tag{3.24}$$

which can be solved explicitly for ρ_b. The leading terms of the above expression near the origin ($\rho_b \simeq 0$), and near the cosmological horizon ($\rho_b \simeq H^{-1}$), respectively, read

$$\begin{aligned} \rho_{h(0)} &= \frac{\mu}{\mu - H^2}\rho_b, \\ \rho_{h(H^{-1})} &= H^{-1} + \left(\frac{\sqrt{\mu} + H}{\sqrt{\mu} - H}\right)^{-\frac{H}{\sqrt{\mu}}}(\rho_b - H^{-1}). \end{aligned} \tag{3.25}$$

The integration of Eq. (3.19) can be performed analytically and, after some algebraic on-shell manipulations, gives

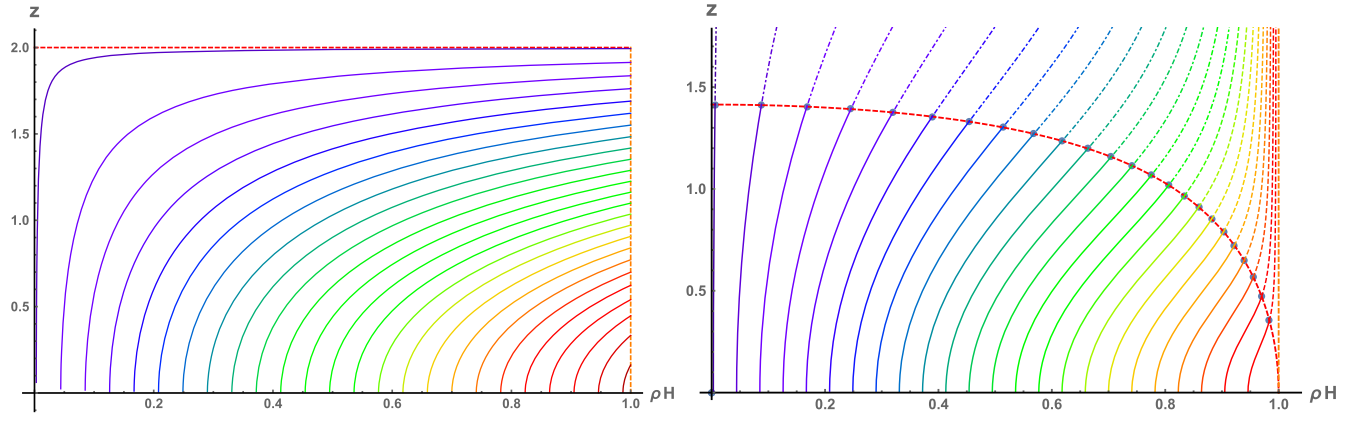

FIG. 2. The minimal surfaces for several boundary points between $\rho_b = 0$ and the cosmological horizon at $\rho_b = H^{-1}$. Left plot: $\mu = H^2$. The bulk horizon at $z = 2/\sqrt{\mu}$ is depicted as a horizontal dashed line and the cosmological horizon at $\rho = H^{-1}$ as a vertical dashed line. Right plot: $\mu \neq H^2$. The bulk horizon, given by Eq. (3.3), is depicted as a curved dashed line. The dotted-dashed extensions of the minimal surfaces beyond the bulk horizon are not relevant for our purposes.

$$E = -2H^2\rho\sqrt{\frac{\mu F(\rho)F(\rho_b)}{(F(\rho)-F(\rho_b))(-F(\rho)(\sqrt{\mu}+H^2\rho)^2+(\sqrt{\mu}-H^2\rho)^2F(\rho_b))}}\Bigg|_{\rho_b}^{\rho_h}. \tag{3.26}$$

The use of Eq. (3.24) demonstrates that the upper limit gives a finite contribution equal to $-\sqrt{\mu}$. The lower limit gives a divergent contribution, which is regulated by imposing a cutoff at $z=\epsilon$, using an expansion of the solution (3.20) around ρ_b. In this way we find

$$E = \frac{\sqrt{1-H^2\rho_b^2}}{\epsilon} - \sqrt{\mu} + \mathcal{O}(\epsilon). \tag{3.27}$$

The divergent term contains the same redshift factor found earlier, while the finite contribution arises through $\sqrt{\mu}$. This result confirms that the relevant scale for the CFT temperature is set by the mass of the bulk black hole.

IV. BOUNDARY BLACK HOLE

It is possible to define appropriate coordinates such that the metric on the boundary becomes that of a $(1+1)$-dimensional black hole. The form of the metric for an asymptotically AdS $(2+1)$-dimensional spacetime with an arbitrary static boundary metric has been derived in Ref. [23]. Exploiting this result, we consider a metric of the form

$$ds^2 = \frac{1}{z^2}\left[dz^2 - (1-e^{-2x})\left(1-\frac{\mu-e^{-4x}}{4(1-e^{-2x})}z^2\right)^2 dt^2 + \left(1-e^{-2x}z^2+\frac{\mu-e^{-4x}}{4(1-e^{-2x})}z^2\right)^2\frac{dx^2}{1-e^{-2x}}\right], \tag{4.1}$$

with a black hole boundary metric

$$ds_0^2 = g_{\mu\nu}^{(0)}dx^\mu dx^\nu = -(1-e^{-2x})dt^2 + \frac{dx^2}{1-e^{-2x}} \tag{4.2}$$

that has a horizon at $x=0$ and a mass parameter equal to 1.

Through the coordinate change

$$\tilde{r} = \operatorname{arccosh}(e^x), \tag{4.3}$$

the metric can be written as

$$ds^2 = \frac{1}{z^2}(dz^2 - f(\tilde{r},z)dt^2 + g(\tilde{r},z)d\tilde{r}^2]), \tag{4.4}$$

where

$$f(\tilde{r},z) = \tanh^2(\tilde{r})\left(1-z^2\frac{1}{4\tanh^2(\tilde{r})}\left(\mu-\frac{1}{\cosh^4(\tilde{r})}\right)\right)^2, \tag{4.5}$$

$$g(\tilde{r},z) = \left(1+z^2\frac{1}{4\tanh^2(\tilde{r})}\left(\mu-\frac{4}{\cosh^2(\tilde{r})}+\frac{3}{\cosh^4(\tilde{r})}\right)\right)^2. \tag{4.6}$$

For $\mu=1$ this metric reduces to the funnel configuration of Ref. [19]. It is apparent that covering the full range of positive and negative values of $\tilde{r}$ requires two copies of the metric (4.1) with x taking positive values.

The holographic stress-energy tensor for this metric is

$$\rho = -\langle T^t{}_t\rangle = \frac{1}{16\pi G_3}\frac{4-3e^{-2x}-\mu e^{2x}}{1-e^{2x}}, \tag{4.7}$$

$$p = \langle T^x{}_x\rangle = \frac{1}{16\pi G_3}\frac{e^{-2x}-\mu e^{2x}}{1-e^{2x}}, \tag{4.8}$$

displaying singularities at the location of the black hole horizon[4] at $x=0$. An expansion around this point gives

$$\rho = \frac{1}{32\pi G_3}\left(\frac{\mu-1}{x}+\mu-5\right)+\mathcal{O}(x), \tag{4.9}$$

$$p = \frac{1}{32\pi G_3}\left(\frac{\mu-1}{x}+\mu+3\right)+\mathcal{O}(x). \tag{4.10}$$

The conformal anomaly is

$$\langle T^\mu{}_\mu\rangle = \frac{e^{-2x}}{4\pi G_3}. \tag{4.11}$$

The transformations that connect the coordinates (z,x) with the coordinates (r,ϕ) of Eq. (2.1) are

$$r(z,x) = \sqrt{\mu + \frac{[1-e^{-2x}-(\mu-e^{-4x})z^2/4]^2}{z^2(1-e^{-2x})}}, \tag{4.12}$$

$$\phi(z,\rho) = \frac{1}{2}\log[-1+e^{2x}] - \frac{1}{2\sqrt{\mu}}\log\left[\frac{z^2-2e^{2x}(2+\sqrt{\mu}z^2)+e^{4x}(4+\mu z^2)}{z^2+2e^{2x}(-2+\sqrt{\mu}z^2)+e^{4x}(4+\mu z^2)}\right], \tag{4.13}$$

with the time coordinate remaining unaffected.

[4]Even though the stress-energy tensor is singular on the horizon of the boundary black hole, it is covariantly conserved, as opposed to the case of a bulk theory that includes a Chern-Simons term [30].

We are interested in an entangling surface, which covers the whole region $x > 0$ on the constant-time slice $t = 0$. The interior of this region is entangled with a symmetric region in the global geometry, again parametrized with $x > 0$ (but negative $\tilde{r}$). Both regions are located outside the future and past horizons. Similarly to the case of a de Sitter boundary metric, the minimal surface, which is a curve for a $(1+1)$-dimensional boundary, starts on the boundary $(z = 0)$ at the location of the horizon $x = 0$. However, as there is no other horizon on the boundary, the minimal surface must extend into the bulk indefinitely. As a result, it coincides with the bulk horizon

$$z(\rho) = 2\sqrt{\frac{1-e^{-2x}}{\mu - e^{-4x}}}. \tag{4.14}$$

In the context of the Ryu-Takayanagi proposal, the entanglement entropy is properly defined if the minimal curve returns to the boundary at a value $x_r \to \infty$. However, the contribution to the length from the return point is not relevant for our purposes, because it accounts for the entanglement with the region $x > x_r$ for any finite value of x_r. This has been already discussed in the context of a Rindler boundary in Ref. [15]. The upshot of these considerations is that the entanglement between the two regions of the $t = 0$ slice of the global geometry around the black hole horizon is accounted for by the length of the bulk horizon in the vicinity of $x = 0$. By substituting Eq. (4.14) in the expression

$$\text{Area}(\gamma_A) = \int_0 dx \frac{1}{z}\sqrt{\left(\frac{dz}{dx}\right)^2 + \left(1 - e^{-2x}z^2 + \frac{\mu - e^{-4x}}{4(1-e^{-2x})}z^2\right)^2 \frac{1}{1-e^{-2x}}}, \tag{4.15}$$

and keeping only the contribution from the lower limit, with a cutoff at $z = \epsilon$, we find

$$\text{Area}(\gamma_A) = \sqrt{\mu}\log\left(\frac{1}{\epsilon}\right). \tag{4.16}$$

The entanglement entropy is

$$S_{\text{bh}} = \frac{\text{Area}(\gamma_A)}{4G_3} = \sqrt{\mu}\frac{\log(1/\epsilon)}{4G_3}. \tag{4.17}$$

Similarly to the de Sitter entropy, the black hole entropy can be written, through use of Eq. (2.16), in two equivalent ways:

$$S_{\text{bh}} = \sqrt{\mu}\frac{1}{4G_2} = \frac{1}{4G_2} + \frac{1}{4G_3}(\sqrt{\mu} - 1)\log\left(\frac{1}{\epsilon}\right). \tag{4.18}$$

For $\mu = 1$, this expression reproduces correctly the black hole gravitational entropy, in agreement with Refs. [13,31,32]. For $\mu > 1$, Eq. (4.18) for the entropy can be given the same physical interpretation as the one discussed at the end of Sec. III A. The analysis of the thermal contributions to the energy of a probe quark leads to the conclusion that the relevant temperature scale is set by the mass term $\sqrt{\mu}$ of the bulk black hole, in agreement with the findings of Sec. III B. Moreover, the first law of thermodynamics can be verified to hold following the same procedure as in Sec. III A.

V. GENERAL ANALYSIS AND CONCLUSIONS

In the two previous sections we discussed specific examples of boundary metrics with horizons, which made it possible to derive explicit expressions for the minimal surfaces and the associated holographic entropy. The intuition that we developed can help us obtain a generalization of these expressions for an arbitrary boundary metric. The most general metric for a $(2+1)$-dimensional AdS bulk containing a nonrotating black hole and having a static boundary can be written in the form [23]

$$ds^2 = \frac{1}{z^2}\left[dz^2 - h(x)\left(1 + \frac{1}{16}\frac{h'^2(x) - 4\mu}{h(x)}z^2\right)^2 dt^2 + \left(1 + \frac{1}{4}h''(x)z^2 - \frac{1}{16}\frac{h'^2(x) - 4\mu}{h(x)}z^2\right)^2 \frac{dx^2}{h(x)}\right], \tag{5.1}$$

with a boundary metric

$$ds_0^2 = g_{\mu\nu}^{(0)}dx^\mu dx^\nu = -h(x)dt^2 + \frac{dx^2}{h(x)}. \tag{5.2}$$

The holographic stress-energy tensor is

$$\rho = -\langle T^t{}_t\rangle = \frac{1}{8\pi G_3}\left(\frac{4\mu - h'^2(x)}{8h(x)} + \frac{1}{2}h''(x)\right), \tag{5.3}$$

$$p = \langle T^\rho{}_\rho\rangle = \frac{1}{8\pi G_3}\left(\frac{4\mu - h'^2(x)}{8h(x)}\right), \tag{5.4}$$

and the conformal anomaly

$$\langle T^{\mu}{}_{\mu}\rangle = -\frac{1}{16\pi G_3}h''(x). \tag{5.5}$$

The indices for the above tensor are lowered and raised with the boundary metric (5.2). Notice that the stress-energy tensor diverges at a zero x_0 of $h(x)$, unless $\mu = h'^2(x_0)/4$.

In the parametrization of Eq. (5.1), horizons of the boundary metric correspond to zeros of the function $h(x)$. As we have seen in the previous sections, minimal curves, starting from the boundary horizon and extending into the bulk, coincide with the bulk horizon. They are given by the relation

$$z(x) = 4\sqrt{\frac{h(x)}{4\mu - h'^2(x)}}. \tag{5.6}$$

Calculating the length of the minimal curve through use of this expression gives

$$\mathrm{Area}(\gamma_A) = \sqrt{\mu}\int dx\left(\frac{1}{h(x)} + \frac{2h''(x)}{4\mu - h'^2(x)}\right). \tag{5.7}$$

In the vicinity of a zero of $h(x)$ at $x = x_0$, the first term gives the leading contribution. Expanding $h(x)$ around this point and using Eq. (5.6), the integration in Eq. (5.7) can be rewritten in terms of z near zero. Each horizon gives a contribution

$$\mathrm{Area}(\gamma_A) = \frac{2\sqrt{\mu}}{|h'(x_0)|}\int_{\epsilon}\frac{dz}{z} = \frac{2\sqrt{\mu}}{|h'(x_0)|}\log\left(\frac{1}{\epsilon}\right). \tag{5.8}$$

For a de Sitter boundary, the minimal curve approaches two horizons, while $|h'(x_0)| = 2H$, resulting in Eq. (3.11). For a black hole boundary, there is only one horizon, while $|h'(x_0)| = 2$, resulting in Eq. (4.16). Making use of Eq. (2.16) we arrive at the general result for the contribution to the entropy from every horizon:

$$S = \frac{\mathrm{Area}(\gamma_A)}{4G_3} = \frac{\sqrt{\mu}}{2G_2|h'(x_0)|}. \tag{5.9}$$

The parameter μ determines the Hawking temperature of the bulk black hole through $\sqrt{\mu} = 2\pi T$. The temperature is identified with that of the thermalized dual CFT. The quantity $|h'(x_0)|$ characterizes the temperature associated with the boundary horizon through the relation $|h'(x_0)| = 4\pi T_h$, as can be checked by imposing the periodicity of the time coordinate in the Euclidean metric. With these identifications, Eq. (5.9) can be written as $S = (T/T_h)/(4G_2)$. The inspection of Eq. (5.3) shows that the sign of the part of the energy density that diverges on the horizon is proportional to the quantity $4\mu/h'^2(x_0) - 1 = T^2/T_h^2 - 1$. Consequently, the dual picture is well defined for $T \geq T_h$. The entropy increase beyond the Hawking-Bekenstein value can be attributed to the localization of additional degrees of freedom on the horizon, which is implied by the divergence of the energy density on this surface. We discussed explicit examples of this situation in the previous two sections. For $T < T_h$ a stationary solution is possible only if it accounts for the energy flow from the horizon to the environment. The dual picture must take into account a nonzero angluar momentum for the bulk black hole. This situation is analyzed in Ref. [20].

As we have already pointed out, the element that differentiates the final result for the various metrics that we have considered is the way the divergence near the boundary is regulated. We assume that the physical cutoff must be imposed on the bulk coordinate in the Fefferman-Graham parametrization for every choice of boundary metric, consistently with the framework of holographic renormalization. Even though this coordinate is always denoted by z, setting $z = \epsilon$ implements a different cutoff procedure each time. This induces the dependence of the expression (5.9) on the boundary metric. We expect that our findings generalize for higher-dimensional AdS spaces with bulk black holes [16,17], as well as for similar holographic setups produced by nontrivial bulk potentials [33].

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