Multiparticle Solutions to Einstein's Equations

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In this Letter, we present the first multiparticle solutions to Einstein's field equations in the presence of matter. These solutions are iteratively obtained via the perturbiner method, which can circumvent gravity's infinite number of vertices with the definition of a multiparticle expansion for the inverse spacetime metric as well. Our construction provides a simple layout for the computation of tree level field theory amplitudes in D spacetime dimensions involving any number of gravitons and matter fields, with or without supersymmetry.

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Overview.—Gravity is still in many ways the least understood of the fundamental forces of nature, arguably at the macroscopic but definitely at the microscopic level. The former is splendidly described by the general theory of relativity, while the latter is hopefully made tangible by the long sought theory of quantum gravity.

As a first approximation, the Einstein-Hilbert action can be seen as a common denominator in this range of scales. It yields as classical equations of motion Einstein's field equations and offers a natural path for a (quantum) field theory of gravitons, the messengers of gravity.

From this field theory perspective, gravity contains an infinite number of vertices and is, in fact, nonrenormalizable. Even at tree level, the computation of graviton scattering amplitudes quickly becomes impractical using standard Feynman diagrams (e.g., [1]).

Modern scattering-amplitude techniques have overcome this problem. Among them the Britto-Cachazo-Feng-Witten (BCFW) recursion [2–4] and the double copy [5–8] are most successful. At their core, they are connected by a simple fact: cubic vertices are enough to describe any tree level graviton amplitude. In BCFW, we see this via on-shell recursions. In the double copy, graviton amplitudes are recast as two copies of gluon amplitudes with a special trivalent configuration using the color-kinematic duality [9]. Indeed, pure graviton amplitudes have been shown to be recursively described by a cubic action with auxiliary fields that is classically equivalent to the Einstein-Hilbert action [10]. This strictification has been formally demonstrated in [11] using L_∞ algebras.

Our Universe, on the other hand, is not pure gravity. Our interest resides in the study of interactions between gravitons and matter particles. In this case, results using BCFW recursions (e.g., [12,13]), double copy (e.g., [14]), or other diagrammatic techniques ([15]) are considerably scarce, subject to different subtleties and limitations that have so far eluded a more systematic and practical output. At the dawn of gravitational waves detection and black hole observation, any advance in the understanding and formulation of the scattering of gravitons by matter is very welcome. This Letter is a step in this direction.

To our avail, the tree level information of a given field theory can be elegantly extracted from its classical equations of motion [16]. This idea was further explored in [17,18] and later streamlined by the perturbiner method [19–21]. As it turns out, there is an inspired multiparticle ansatz for the solution of classical equations of motion that can be used to define an off-shell recursion for tree level amplitudes in terms of Berends-Giele currents [22]. This method recently regained interest [23–25] and has been since explored in different contexts [26–30]. Rather surprisingly, perturbiner methods have never been fully applied to gravity, except for the very early analysis of the self-dual case in [20] and a simplified version for conformal supergravity amplitudes in [31]. Naively, a proper recursive solution cannot be defined in a theory with an infinite number of vertices. As we will show, however, there is a way around this obstacle in gravity.

In this work, we propose a series of multiparticle solutions to Einstein's field equations based on the perturbiner method. These solutions encompass a broad class of interesting cases and can be applied to any two-derivative matter field theory coupled to gravity. We can then define n-point tree level scattering amplitudes between gravitons and matter particles using a similar prescription to the super Yang-Mills case [22]. In this prescription, diffeomorphism invariance is manifest, with a clear decoupling of pure gauge states. In addition, the analysis of the soft limit behavior at leading order is surprisingly transparent. First, we discuss pure gravity with a subsequent coupling to bosonic matter. We then recast the Einstein-Hilbert action in terms of the vielbein and the spin connection in order to introduce the coupling to fermionic matter and, consequently, supersymmetry. Our results are agnostic to the number of spacetime dimensions and can be easily automated. Whether or not there is an underlying worldsheet description, they provide a compact and efficient computation of the scattering of gravitons and matter at tree level.

Field equations and gravitons.—Einstein's field equations without cosmological constant can be cast as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}. \tag{1}$$

The right-hand side is the matter energy-momentum tensor $T_{\mu\nu}$ multiplied by the gravitational constant κ. On the left-hand side, $g_{\mu\nu}$ denotes the spacetime metric (with inverse $g^{\mu\nu}$), $R = g^{\mu\nu} R_{\mu\nu}$ is the scalar curvature, and $R_{\mu\nu}$ is the Ricci tensor. As usual, $R_{\mu\nu} \equiv R^{\rho}_{\mu\rho\nu}$, where $R^{\sigma}_{\mu\nu\rho}$ is the Riemann tensor,

$$R^{\sigma}_{\mu\nu\rho} = \partial_\nu \Gamma^{\sigma}_{\mu\rho} - \partial_\rho \Gamma^{\sigma}_{\mu\nu} + \Gamma^{\sigma}_{\nu\lambda}\Gamma^{\lambda}_{\mu\rho} - \Gamma^{\sigma}_{\rho\lambda}\Gamma^{\lambda}_{\mu\nu}, \tag{2}$$

and $\Gamma^{\sigma}_{\mu\nu} = g^{\rho\sigma}\Gamma_{\mu\nu\rho}$ is the Christoffel symbol, with

$$\Gamma_{\mu\nu\rho} = \frac{1}{2}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}). \tag{3}$$

The field equations [Eq. (1)] are covariant under general coordinate transformations ($\delta x^\mu = \lambda^\mu$), with the metric transforming as

$$\delta g_{\mu\nu} = g_{\mu\rho}\partial_\nu \lambda^\rho + g_{\nu\rho}\partial_\mu \lambda^\rho + \lambda^\rho \partial_\rho g_{\mu\nu}. \tag{4}$$

In the absence of matter, Eq. (1) reduces to

$$R_{\mu\nu} = 0, \tag{5}$$

which can be used to analyze linearized solutions around a given background, i.e., the gravitons. These single-particle solutions around flat space (with metric $\eta_{\mu\nu}$) are given by

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu} e^{ik\cdot x}, \tag{6}$$

with $k\cdot x = k_\mu x^\mu$. The graviton polarization $h_{\mu\nu}$ satisfies $\eta^{\nu\rho} k_\rho h_{\mu\nu} = \eta^{\mu\nu} h_{\mu\nu} = 0$. There is also a residual gauge transformation of the form $\delta h_{\mu\nu} = k_\mu \lambda_\nu + k_\nu \lambda_\mu$, with $k\cdot\lambda = 0$.

Multiparticle solutions and recursions.—We can now look at the multiparticle solutions of the graviton field $g_{\mu\nu}(x)$, satisfying Eq. (5). Consider

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \sum_P H_{P\mu\nu} e^{ik_P\cdot x}, \tag{7}$$

where $H_{P\mu\nu}$ represents the multiparticle currents. The word P denotes a sequence of ordered letters, $P = p_1 \ldots p_n$, where p_i is a single-particle label, with $k_P \equiv k_{p_1} + \cdots + k_{p_n}$.

In order to find the solutions for $H_{P\mu\nu}$, we have also to work with $g^{\mu\nu}(x)$ satisfying $g^{\mu\rho}g_{\rho\nu} = \delta^\mu_\nu$. For the expansion

$$g^{\mu\nu}(x) = \eta^{\mu\nu} - \sum_P I^{\mu\nu}_P e^{ik_P\cdot x}, \tag{8}$$

the inverse identity implies that the currents $I^{\mu\nu}_P$ are constrained to be

$$I^{\mu\nu}_P = \eta^{\mu\rho}\eta^{\nu\sigma} H_{P\rho\sigma} - \eta^{\nu\sigma} \sum_{P=Q\cup R} I^{\mu\rho}_Q H_{R\rho\sigma}, \tag{9}$$

where the sum goes over all deshuffles of P into ordered words Q, R (see, e.g., [28]). Although not explicitly, $I^{\mu\nu}_P = I^{\nu\mu}_P$ and this can be recursively demonstrated order by order in the subdeshuffles.

Multiparticle currents with one-letter words are simply associated to their single-particle equivalents (polarizations): $H_{p\mu\nu} = h_{p\mu\nu}$ and $I^{\mu\nu}_p = \eta^{\mu\rho}\eta^{\nu\sigma} h_{p\rho\sigma}$.

To every x-dependent object, we will associate a multiparticle expansion. For example, the Christoffel symbol can be expressed as $\Gamma_{\mu\nu\rho} = \sum_P \Gamma_{P\mu\nu\rho} e^{ik_P\cdot x}$, with

$$\Gamma_{P\mu\nu\rho} \equiv \frac{i}{2}(k_{P\mu} H_{P\nu\rho} + k_{P\nu} H_{P\mu\rho} - k_{P\rho} H_{P\mu\nu}). \tag{10}$$

The parameter of general coordinate transformations may also be cast as a multiparticle expansion as

$$\lambda^\mu = -i \sum_P \Lambda^\mu_P e^{ik_P\cdot x}. \tag{11}$$

This way, the gauge transformation [Eq. (4)] implies that

$$\begin{aligned}\delta H_{P\mu\nu} = &\sum_{P=Q\cup R} \Lambda^\rho_Q \{k_{Q\mu} H_{R\nu\rho} + k_{Q\nu} H_{R\mu\rho} + k_{R\rho} H_{R\mu\nu}\} \\ &+ k_{P\mu}\Lambda_{P\nu} + k_{P\nu}\Lambda_{P\mu}.\end{aligned} \tag{12}$$

We will choose the gauge $\eta^{\mu\nu}\Gamma_{\mu\nu\rho} = 0$. This is simpler than the de Donder gauge $g^{\mu\nu}\Gamma^\rho_{\mu\nu} = 0$, because its multiparticle version does not involve deshuffles, being neatly expressed as

$$\eta^{\mu\nu}\Gamma_{P\mu\nu\rho} = i\eta^{\mu\nu}\left(k_{P\mu} H_{P\nu\rho} - \frac{1}{2} k_{P\rho} H_{P\mu\nu}\right) = 0. \tag{13}$$

In this gauge, the multiparticle currents of the Ricci tensor $\mathcal{R}_{P\mu\nu}$ are computed to be

$$\begin{aligned}\mathcal{R}_{P\mu\nu} = &\frac{s_P}{2} H_{P\mu\nu} - i \sum_{P=Q\cup R} I^{\rho\sigma}_Q (k_{P\rho}\Gamma_{R\mu\nu\sigma} - k_{P\nu}\Gamma_{R\mu\rho\sigma}) \\ &- \eta^{\alpha\beta}\eta^{\rho\sigma} \sum_{P=Q\cup R} (\Gamma_{Q\nu\alpha\sigma}\Gamma_{R\mu\rho\beta} - \Gamma_{Q\rho\alpha\sigma}\Gamma_{R\mu\nu\beta}) \\ &+ \eta^{\alpha\beta} \sum_{P=Q\cup R\cup S} I^{\rho\sigma}_Q (\Gamma_{R\nu\rho\beta}\Gamma_{S\mu\alpha\sigma} - \Gamma_{R\alpha\rho\beta}\Gamma_{S\mu\nu\sigma}) \\ &+ \eta^{\alpha\beta} \sum_{P=Q\cup R\cup S} I^{\rho\sigma}_Q (\Gamma_{R\nu\alpha\sigma}\Gamma_{S\mu\rho\beta} - \Gamma_{R\rho\alpha\sigma}\Gamma_{S\mu\nu\beta}) \\ &- \sum_{P=Q\cup R\cup S\cup T} I^{\rho\sigma}_Q I^{\alpha\beta}_R (\Gamma_{S\nu\alpha\sigma}\Gamma_{T\mu\rho\beta} - \Gamma_{S\rho\alpha\sigma}\Gamma_{T\mu\nu\beta}),\end{aligned} \tag{14}$$

where $s_P \equiv \eta^{\mu\nu} k_{P\mu} k_{P\nu}$ denotes the generalized Mandelstam variables. The recursion relation for $H_{P\mu\nu}$ is then obtained using Eq. (5), i.e., $\mathcal{R}_{P\mu\nu} = 0$.

Tree level amplitudes.—Motivated by the Berends-Giele prescription [22], the tree level amplitude for the scattering of n gravitons is defined as

$$\begin{aligned}\mathcal{M}_n &\equiv \kappa \lim_{s_{2\ldots n}\to 0} s_{2\ldots n} h_{1\mu\nu} I^{\mu\nu}_{2\ldots n},\\ &= \kappa \lim_{s_{2\ldots n}\to 0} s_{2\ldots n} h_1^{\mu\nu} H_{2\ldots n\mu\nu}\end{aligned} \tag{15}$$

on the support of momentum conservation. Whenever convenient, we will raise or lower spacetime indices using the flat metric.

By construction, $H_{P\mu\nu}$ is symmetric in the exchange of any two single-particle labels. This symmetry is lifted to the amplitude $\mathcal{M}_n$, which is also symmetric in the exchange of any two graviton legs, although only $(n-1)$ are manifest through $H_{2\ldots n\mu\nu}$. The particle in the first leg can be thought of as an off-shell leg in the multiparticle recursion, then placed on shell in the definition of the amplitude in Eq. (15) via momentum conservation and the limit $s_{2\ldots n} \to 0$.

The amplitude $\mathcal{M}_n$ is invariant under the residual transformations of the graviton polarizations described after Eq. (6). In order to see this, we can examine the residual gauge transformations preserving Eq. (13). They lead to a recursion for the currents $\Lambda_{P\mu}$ in Eq. (11) given by

$$\Lambda_{P\mu} = -\frac{k_P^\nu}{s_P} \sum_{P=Q\cup R} \Lambda_Q^\rho (k_{Q\mu} H_{R\nu\rho} + k_{Q\nu} H_{R\mu\rho} + k_{R\rho} H_{R\mu\nu}). \tag{16}$$

It is then just an algebraic step to show the invariance of $\mathcal{M}_n$ under Eq. (12) with multiparticle parameters [Eq. (16)].

The three-point amplitude is given by the well-known result

$$\mathcal{M}_3(h_1, h_2, h_3) = \frac{\kappa}{4} h^1_{\mu\nu} h^2_{\alpha\beta} h^3_{\gamma\delta} V^{\mu\alpha\gamma} V^{\nu\beta\delta} \tag{17}$$

in terms of the three-point Yang-Mills vertex

$$V^{\mu\alpha\gamma} = (k_2^\mu - k_3^\mu)\eta^{\alpha\gamma} + (k_3^\alpha - k_1^\alpha)\eta^{\mu\gamma} + (k_1^\gamma - k_2^\gamma)\eta^{\mu\alpha}.$$

The current $H_{P\mu\nu}$ effectively describes interactions with vertices from three to five points, as can be seen from the number of deshuffles in Eq. (14). The four or higher point amplitudes will not be explicitly displayed here, as their size grows rapidly due to the nested deshuffles. We found it easier to perform most of the cross-checks numerically, since it is straightforward to implement the recursions for $H_{P\mu\nu}$ computationally.

The soft limit of graviton amplitudes has a universal behavior [32,33], constituting a natural test for our proposal in Eq. (15). As it turns out, its soft limit analysis is very simple at leading order.

We will take $h^1_{\mu\nu}$ as the soft graviton and parameterize its momentum as $k_1^\mu = \tau q^\mu$, with $q^2 = 0$ and parameter $\tau \to 0$. In the soft limit, we can directly identify the dominant contributions in $H_{23..n\mu\nu}$, for they come from the poles of the generalized Mandelstam variables with $(n-2)$ momenta. For example,

$$s_{3\ldots n} = (\tau q + k_2)^2 = 2\tau (q\cdot k_2) \tag{18}$$

is attached to the multiparticle current with $(n-2)$ particles, $H_{3..n\rho\sigma}$. We can then reexamine the recursion of the $(n-1)$-particle currents $H_{23..n\mu\nu}$ and readily express it as

$$\begin{aligned}s_{23..n} H_{23..n\mu\nu} = {}& k_{2\mu} k_{2\nu} h_2^{\rho\sigma} H_{3..n\rho\sigma} + \mathrm{sym}(2,3,\ldots,n)\\ &+ \mathcal{O}(\tau^0),\end{aligned} \tag{19}$$

where $\mathrm{sym}(2,3,\ldots,n)$ takes care of the symmetrization of the single-particle labels.

In terms of the amplitude, this parameterization leads to the leading order contribution

$$\lim_{\tau\to 0} \mathcal{M}_n = \frac{1}{\tau}\left(\sum_{a=2}^n \frac{k_{a\mu} h_1^{\mu\nu} k_{a\nu}}{2(q\cdot k_a)}\right) \mathcal{M}_{n-1}(h_2, \ldots, h_n), \tag{20}$$

manifesting the universal Weinberg pole. Since diffeomorphism invariance is inbuilt in our results, subleading soft limits should be directly reproduced [34,35].

Matter coupled to gravity.—The matter contributions to Eq. (1) come from the energy-momentum tensor

$$T_{\mu\nu} \equiv -2\frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} S_{\mathrm{matter}}, \tag{21}$$

where S_{matter} is the matter action. In terms of a multiparticle expansion, we have $T_{\mu\nu} = \sum_P \mathcal{T}_{P\mu\nu} e^{ik_P\cdot x}$, where the form of the currents $\mathcal{T}_{P\mu\nu}$ is particular to the model. In order for this to make sense, the single-particle solutions of the free equations of motion associated to the matter action must be described in terms of plane waves. These are our asymptotic states.

The recursion relations for the currents $H_{P\mu\nu}$ are obtained by plugging the corresponding multiparticle expansions in Eq. (1). The result is

$$\begin{aligned}\mathcal{R}_{P\mu\nu} = {}& \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}\mathcal{R}_{P\rho\sigma} + \kappa \mathcal{T}_{P\mu\nu}\\ &+ \frac{1}{2}\sum_{P=Q\cup R} (H_{Q\mu\nu}\eta^{\rho\sigma} - \eta_{\mu\nu} I_Q^{\rho\sigma})\mathcal{R}_{R\rho\sigma}\\ &- \frac{1}{2}\sum_{P=Q\cup R\cup S} H_{Q\mu\nu} I_R^{\rho\sigma} \mathcal{R}_{S\rho\sigma},\end{aligned} \tag{22}$$

where $\mathcal{R}_{P\mu\nu}$ is defined in Eq. (14). Naturally, we recover $\mathcal{R}_{P\mu\nu} = 0$ when $\mathcal{T}_{P\mu\nu} = 0$.

The amplitude prescription is the same as in Eq. (15), but now we are able to describe the scattering of matter bosons and gravitons.

Massive scalar: Our first example is the massive scalar coupled to gravity and otherwise free, with equation of motion

$$(g^{\mu\nu}\partial_\mu\partial_\nu - m^2)\phi = g^{\mu\nu}\Gamma^\rho_{\mu\nu}\partial_\rho\phi \tag{23}$$

and covariantly conserved energy-momentum tensor

$$T_{\mu\nu} = -\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}g_{\mu\nu}(g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi + m^2\phi^2). \tag{24}$$

Equation (23) leads to the following recursion for the scalar multiparticle currents Φ_P:

$$\begin{aligned}(s_P + m^2)\Phi_P = &\sum_{P=Q\cup R}(I_Q^{\mu\nu}k_{R\mu}k_{R\nu} + \eta^{\mu\nu}k_R^\rho\Gamma_{Q\mu\nu\rho})\Phi_R \\ &- \sum_{P=Q\cup R\cup S}\eta^{\mu\nu}I_Q^{\rho\sigma}\Phi_S(\Gamma_{R\mu\nu\sigma}k_{S\rho} + \Gamma_{R\rho\sigma\nu}k_{S\mu}) \\ &+ \sum_{P=Q\cup R\cup S\cup T}I_Q^{\mu\nu}I_R^{\rho\sigma}\Gamma_{S\mu\nu\sigma}k_{T\rho}\Phi_T.\end{aligned} \tag{25}$$

Similarly, Eq. (24) leads to

$$\begin{aligned}\mathcal{T}_{P\mu\nu} = &\sum_{P=Q\cup R}\left\{k_{Q\mu}k_{R\nu} + \frac{1}{2}\eta_{\mu\nu}[m^2 - (k_Q\cdot k_R)]\right\}\Phi_Q\Phi_R \\ &+ \frac{1}{2}\sum_{P=Q\cup R\cup S}H_{Q\mu\nu}\Phi_R\Phi_S[m^2 - (k_Q\cdot k_R)] \\ &+ \frac{1}{2}\sum_{P=Q\cup R\cup S}\eta_{\mu\nu}I_Q^{\rho\sigma}k_{R\rho}k_{S\sigma}\Phi_R\Phi_S \\ &+ \frac{1}{2}\sum_{P=Q\cup R\cup S\cup T}H_{Q\mu\nu}I_R^{\rho\sigma}k_{S\rho}k_{T\sigma}\Phi_S\Phi_T.\end{aligned} \tag{26}$$

These quantities are then used to compute the tree level scattering of gravitons and massive scalars. For example, the four-point amplitude with two gravitons (h_1, h_2) and two massive scalars (3, 4) is given by

$$\begin{aligned}\mathcal{M}_4 = &2\kappa^2(k_2\cdot h_1\cdot h_2\cdot k_1) - \frac{1}{2}\kappa^2 s_{34}(h_1\cdot h_2) + 2\kappa^2[(k_3\cdot h_1\cdot h_2\cdot k_4) + (k_4\cdot h_1\cdot h_2\cdot k_3)] \\ &+ \frac{4\kappa^2}{s_{34}}\left\{\frac{1}{2}(h_1\cdot h_2)[(k_1\cdot k_3)(k_2\cdot k_4) + (k_1\cdot k_4)(k_2\cdot k_3)] + (k_2\cdot h_1\cdot k_3)(k_4\cdot h_2\cdot k_1) + (k_2\cdot h_1\cdot k_4)(k_3\cdot h_2\cdot k_1)\right. \\ &- (k_3\cdot h_1\cdot k_4)(k_1\cdot h_2\cdot k_1) - (k_3\cdot h_2\cdot k_4)(k_2\cdot h_1\cdot k_2) - (k_2\cdot h_1\cdot h_2\cdot k_3)(k_1\cdot k_4) - (k_2\cdot h_1\cdot h_2\cdot k_4)(k_1\cdot k_3) \\ &\left.- (k_3\cdot h_1\cdot h_2\cdot k_1)(k_2\cdot k_4) - (k_4\cdot h_1\cdot h_2\cdot k_1)(k_2\cdot k_3)\right\} \\ &+ \frac{4\kappa^2}{(s_{23}+m^2)}(k_4\cdot h_1\cdot k_4)(k_3\cdot h_2\cdot k_3) + \frac{4\kappa^2}{(s_{24}+m^2)}(k_3\cdot h_1\cdot k_3)(k_4\cdot h_2\cdot k_4),\end{aligned} \tag{27}$$

matching known results in the literature, e.g., [36,37].

Yang-Mills theory: Here we provide the ingredients for computing the scattering of gravitons and gauge vectors.

The energy-momentum tensor of S_{YM} is given by

$$T_{\mu\nu} = \frac{2}{g^2_{\text{YM}}}\text{Tr}\left(g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} - \frac{1}{8}g_{\mu\nu}g^{\lambda\rho}g^{\delta\sigma}F_{\lambda\delta}F_{\rho\sigma}\right), \tag{28}$$

where g_{YM} is the coupling constant, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ is the field strength, and the trace Tr is taken over a given non-Abelian group or just $U(1)$ for Maxwell's theory. The equations of motion of the gauge field can be cast as

$$g^{\nu\rho}D_\nu F_{\mu\rho} = g^{\nu\rho}[A_\nu, F_{\mu\rho}], \tag{29}$$

where D_μ denotes the curved space covariant derivative

$$D_\nu F_{\mu\rho} = \partial_\nu F_{\mu\rho} - \Gamma^\sigma_{\nu\mu}F_{\alpha\rho} - \Gamma^\sigma_{\nu\rho}F_{\mu\alpha}. \tag{30}$$

It is then straightforward to plug the multiparticle expansion $A_\mu = \sum_P \mathcal{A}_{P\mu}e^{ik_P\cdot x}$ back in Eq. (29) and obtain a recursive definition for the currents $\mathcal{A}_{P\mu}$. The covariant gauge $g^{\mu\nu}D_\mu A_\nu = 0$ seems to be the simplest choice in this case.

These Einstein-Yang-Mills amplitudes can then be compared with other results in the literature obtained through different techniques, e.g., [38–41].

Fermions and supersymmetry.—In order to consider fermions, we turn the local Lorentz group into a gauge symmetry. The spacetime metric $g_{\mu\nu}$ is mapped to the (local) flat metric η_{ab} using the vielbein e^a_μ (with inverse e^μ_a) such that $g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu$. The gauge field of the Lorentz symmetry is the spin connection ω^{ab}_μ, and the "flattened" Riemann tensor $R^{ab}_{\mu\nu} \equiv \eta^{ac}e^b_\sigma e^\rho_c R^\sigma_{\rho\mu\nu}$ can be seen as its field strength, given by

$$R^{ab}_{\mu\nu} \equiv \partial_\mu\omega^{ab}_\nu + \eta_{cd}\omega^{ac}_\mu\omega^{db}_\nu - (\mu \leftrightarrow \nu), \tag{31}$$

with scalar curvature $R = e^\mu_a e^\nu_b R^{ab}_{\mu\nu}$.

Spinor couplings to the curved background are implemented by replacing spacetime derivatives by their Lorentz-covariant version. Given a spinor ψ, its covariant derivative is defined as

$$D_\mu \psi = \partial_\mu \psi + \frac{1}{4}\omega_\mu^{ab}\Gamma_{ab}\psi, \tag{32}$$

where $\Gamma_{ab} = \frac{1}{2}[\Gamma_a, \Gamma_b]$ and Γ_a denote the usual gamma matrices satisfying $\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}$.

Next, we rewrite Einstein's field equations in terms of the vielbein and, independently, the spin connection. This is known as the Palatini variation. In the presence of matter, they take the form

$$R_\mu^a = \kappa T_\mu^a + \frac{1}{2} e_\mu^a R, \tag{33}$$

$$\omega_\mu^{ab} = \kappa W_\mu^{ab} + \frac{1}{2} e^{\nu[a}(\partial_\mu e_\nu^{b]} - \Gamma_{\mu\nu\rho} e^{\rho b]}), \tag{34}$$

where $R_\mu^a \equiv e_b^\nu R_{\mu\nu}^{ab}$, $[ab] = ab - ba$, and $\Gamma_{\mu\nu\rho}$ is the zero-torsion Christoffel symbols in Eq. (3). The matter tensors are defined as

$$T_\mu^a \equiv e \frac{\delta}{\delta e_a^\mu} S_{\text{matter}}, \tag{35}$$

$$W_\mu^{ab} \equiv e P_{\mu\nu}^{ab,cd} \frac{\delta}{\delta \omega_\nu^{cd}} S_{\text{matter}}, \tag{36}$$

with $e = \det e_a^\mu$, and

$$P_{\mu\nu}^{ab,cd} \equiv \eta^{d[a}\left(e_\nu^{b]} e_\mu^c + \frac{1}{2}\eta^{b]c} g_{\mu\nu} + \frac{2}{(D-2)} e_\mu^{b]} e_\nu^c\right). \tag{37}$$

From here onward, the perturbiner method goes as usual. We define the multiparticle expansion for the vielbein and its inverse analogously to the metric expansions in Eqs. (7) and (8):

$$e_\mu^a = \delta_\mu^a + \sum_P E_{P\mu}^a e^{ik_P\cdot x}, \tag{38}$$

$$e_a^\mu = \delta_a^\mu - \sum_P F_{P\nu}^b e^{ik_P\cdot x}. \tag{39}$$

The mixed Kronecker deltas δ_μ^a and δ_a^μ indicate that the vielbeins are expanded around flat space. The inverse relations $e_\mu^a e_a^\nu = \delta_\mu^\nu$ and $e_\mu^a e_b^\nu = \delta_b^a$ constrain F_{Pa}^μ to satisfy

$$\begin{aligned} F_{Pa}^\mu &= \delta_a^\nu \delta_b^\mu E_{P\nu}^b - \delta_a^\nu \sum_{P=Q\cup R} E_{Q\nu}^b F_{Rb}^\mu, \\ &= \delta_a^\nu \delta_b^\mu E_{P\nu}^b - \delta_b^\mu \sum_{P=Q\cup R} E_{Q\nu}^b F_{Ra}^\nu. \end{aligned} \tag{40}$$

The proof of equivalence between the first and second lines follows the same logic of $I_P^{\mu\nu} = I_P^{\nu\mu}$ after Eq. (9).

We can then use general coordinate transformations and local Lorentz symmetry to fix a convenient gauge. We found the simplest one to be

$$\left(\eta^{\mu\nu}\eta_{ab} - \frac{1}{2}\delta_a^\mu \delta_b^\nu\right)\partial_\mu e_\nu^b = 0, \tag{41a}$$

$$\delta_a^\mu e_{\mu b} - \delta_b^\mu e_{\mu a} = 0. \tag{41b}$$

The first equation is a truncated version of Eq. (13), while the second is known as the symmetric gauge. In terms of the multiparticle currents, this gauge has a simple realization and does not involve deshuffles. The single-particle polarizations $e_{p\mu}^a$ satisfy $k_p^\mu e_{p\mu}^a = \delta_a^\mu e_{p\mu}^a = 0$, with residual gauge symmetry

$$\delta e_{p\mu a} = \delta_a^\nu (k_{p\mu}\lambda_{p\nu} + k_{p\nu}\lambda_{p\mu}) \tag{42}$$

and $k_p \cdot \lambda_p = 0$.

With these choices, the recursion for $E_{P\mu}^a$ can be written as

$$\begin{aligned} s_P E_{P\mu}^a &= \kappa\left(\delta_a^b \delta_\mu^\nu + \frac{1}{(2-D)}\delta_\mu^a \delta_b^\nu\right)\mathcal{T}_{P\nu}^b - i\kappa\delta_b^\nu (k_{P\mu}\mathcal{W}_{P\nu}^{ab} - k_{P\nu}\mathcal{W}_{P\mu}^{ab}) \\ &\quad + \frac{1}{2}\delta_b^\nu \sum_{P=Q\cup R} [k_{P\nu} F_Q^{\rho a}(k_{R\mu}E_{R\rho}^b - k_{R\rho}E_{R\mu}^b) - k_{P\mu}F_Q^{\rho a}(k_{R\nu}E_{R\rho}^b - k_{R\rho}E_{R\nu}^b) - (a \leftrightarrow b)] \\ &\quad + \sum_{P=Q\cup R} [iF_{Qb}^\nu (k_{R\mu}\Omega_{R\nu}^{ab} - k_{R\nu}\Omega_{R\mu}^{ab}) - \eta_{cd}\delta_b^\nu (\Omega_{Q\mu}^{ac}\Omega_{R\nu}^{db} - \Omega_{Q\nu}^{ac}\Omega_{R\mu}^{db}) + \frac{\kappa}{(2-D)}(E_{Q\mu}^a \delta_b^\nu - \delta_\mu^a F_{Qb}^\nu)\mathcal{T}_{R\nu}^b] \\ &\quad + \frac{1}{2}\delta_b^\nu \sum_{P=Q\cup R} [\eta^{\rho a}\eta^{\sigma b}(k_{P\mu}E_{Q\nu}^c - k_{P\nu}E_{Q\mu}^c) - (k_{P\mu}\delta_\nu^c - k_{P\nu}\delta_\mu^c)(F_Q^{\rho a}\eta^{\sigma b} - F_Q^{\rho b}\eta^{\sigma a})](k_{R\sigma}E_{R\rho c} - k_{R\rho}E_{R\sigma c}) \\ &\quad + \frac{1}{2}\delta_b^\nu \sum_{P=Q\cup R\cup S} [(k_{P\mu}\delta_\nu^c - k_{P\nu}\delta_\mu^c)F_Q^{\rho a}F_R^{\sigma b} - (k_{P\mu}E_{Q\nu}^c - k_{P\nu}E_{Q\mu}^c)(F_R^{\rho a}\eta^{\sigma b} - F_R^{\rho b}\eta^{\sigma a})](k_{S\sigma}E_{S\rho c} - k_{S\rho}E_{S\sigma c}) \\ &\quad + \sum_{P=Q\cup R\cup S}\left[\eta_{cd}F_{Qb}^\nu (\Omega_{R\mu}^{ac}\Omega_{S\nu}^{db} - \Omega_{R\nu}^{ac}\Omega_{S\mu}^{db}) - \frac{\kappa}{(2-D)}E_{Q\mu}^a F_{Rb}^\nu \mathcal{T}_{S\nu}^b\right] \\ &\quad + \frac{1}{2}\delta_b^\nu \sum_{P=Q\cup R\cup S\cup T}(k_{P\mu}E_{Q\nu}^c - k_{P\nu}E_{Q\mu}^c)F_R^{\rho a}F_S^{\sigma b}(k_{T\sigma}E_{T\rho c} - k_{T\rho}E_{T\sigma c}), \end{aligned} \tag{43}$$

where we have used the multiparticle currents of T^a_μ, W^{ab}_μ, and ω^{ab}_μ, respectively $\mathcal{T}^a_{P\mu}$, $\mathcal{W}^{ab}_{P\mu}$, and $\Omega^{ab}_{P\mu}$.

The n-point tree level amplitudes are defined as

$$\mathcal{M}_n \equiv \kappa \lim_{s_{2\ldots n}\to 0} s_{2\ldots n} e^\mu_{1a} E^a_{2\ldots n\mu}. \tag{44}$$

Similar to Eq. (15), the invariance of this amplitude under the residual gauge transformations [Eq. (42)] has a straightforward demonstration in the gauge [Eq. (41)].

Now that we are able to consistently account for fermionic degrees of freedom in the multiparticle solutions, it is just a small step to consider supersymmetric field theories, in particular supergravity.

Final remarks.—In this Letter, we have found multiparticle solutions to Einstein's field equations, with a compact recursive definition for graviton multiparticle currents in D-dimensional Minkowski space. These currents can then be used to compute any tree level scattering between gravitons and matter particles, with or without supersymmetry.

The key insight is the recursive definition of the inverse metric $g^{\mu\nu}$ in Eqs. (8) and (9), with an analogous expression for the inverse vielbein e^μ_a in Eqs. (39) and (40). Effectively, this recursion works as a truncation of the gravity action. It is yet another way of seeing that the infinite number of graviton vertices, though required by diffeomorphism invariance, play no role at the tree level dynamics.

The practical appeal of our formulas is that they can be easily computerized. For pure gravity amplitudes, this might not present an advantage over current methods, in particular the double copy construction using color-kinematics duality [7,8] and BCFW on-shell recursion [42,43]. Nevertheless, for the mixed scattering of gravitons and matter particles, the ingredients presented here constitute a versatile tool for computing tree level amplitudes in a broad class of theories. Our results do not require an underlying worldsheet theory (a "stringy" origin) and can be applied to more general field configurations. We believe our solutions can become a robust standard for such amplitudes, with a reliable framework to be used to test both (1) the extension of current methods and (2) possible new techniques for tree level scattering.

Toward a more efficient algorithm implementation, the recursions we presented can be recast in a "color-stripped" form. The name is inherited from the Yang-Mills perturbiner, where such a construction is more natural due to the color structure. For the graviton, a color-stripped perturbiner is not limited by the ordered words in the multiparticle expansion. In this case, the word splitting needed to recursively define the currents is greatly simplified (see, e.g., [28]). For example, the color-dressed deshuffle $P = Q \cup R$ of a n-letter word leads to $(2^n - 2)$ pairs of ordered words. This is to be compared with mere $(n-1)$ pairs of the deconcatenation $P = QR$ in the color-stripped form. We just have to be careful to properly symmetrize the final amplitudes with respect to the graviton legs, but this is computationally much less costly.

A more immediate extension of our results would be to consider multiparticle expansions in curved space. While this cannot be efficiently developed in general backgrounds, we found that the perturbiner extension to (anti) de Sitter spaces leads to an intuitive recursive definition of Witten diagrams [44] for different matter fields. This is being explored in an ongoing project [45].

Finally, a quick comment on loop computations. The perturbiner method seems to be intrinsically classical: it consists of solving equations of motion. However, recent results using homotopy algebras in quantum field theory [46] have uncovered that loop-level scattering amplitudes can also be recursively computed. The natural question then is this: Can we reformulate these recursions as solutions of some *quantum equation of motion*? Although this speculation looks a bit farfetched, based on very preliminary investigations we think the answer might be affirmative.

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