




Continuous dependence and convergence for a Kelvin–Voigt fluid of order one

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Abstract

It is shown that the solution to the boundary - initial value problem for a Kelvin–Voigt fluid of order one depends continuously upon the Kelvin–Voigt parameters, the viscosity, and the viscoelastic coefficients. Convergence of a solution is also shown.

Keywords Continuous dependence · Kelvin–Voigt · A priori bound · Viscoelasticity

1 Introduction

The equations for a viscoelastic fluid have been increasingly occupying attention. Such fluids occur everywhere in real life and differ from Navier - Stokes fluids in that the stress depends on the history of the velocity gradient. As such, the equations for such fluids present many mathematical challenges, see e.g. [1–15].

A particular class of viscoelastic fluids of interest here are those associated with the names of Kelvin and of Voigt, see e.g. [16–21]. Much of the interest in these fluids stems from work of Russian writers in this field and analytical studies of Kelvin–Voigt fluids are contained in [22,23], with generalizations of these models and analytical results to encompass the non - isothermal situation in [24–26]. A lucid account of viscoelastic fluids associated with the names of Maxwell, of Oldroyd, and of Kelvin and Voigt, is contained in [27], where the solution existence question is analysed, see also [28].

Kelvin–Voigt fluids are being increasingly employed in real life applications especially in industrial and engineering contexts. Many of these are reviewed in [19], but we highlight here their employment in viscous dampers in large buildings, cf. [29,30]. For example, a large viscous damper is utilized in the 1667 feet high tower Taipei 101 in the city of Taipei. This building has been constructed to withstand earthquakes and typhoons and the large viscous damper is essential.

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The goal of this article is to analyse continuous dependence of a solution to the equations for a Kelvin–Voigt fluid of order one. Such questions are important and belong to the general area of structural stability. [31], p. 304, pose the problem of what effect does changing the parameters in a differential equation have upon the solution to such an equation. They introduce this as the concept of structural stability. In this article we concentrate on continuous dependence on parameters in the equations for a Kelvin–Voigt fluid of order one. This is continuous dependence on the model itself which is structural stability of the model. We point out that continuous dependence on the model has been the subject of much recent attention in continuum mechanics, see e.g. [13,16,32–54].

In the next section we introduce the Kelvin–Voigt equations of order one. The following section establishes continuous dependence on the Kelvin–Voigt parameter, λ . This is important as this coefficient multiplies the highest derivative term in the equations. After this we establish continuous dependence upon the remaining coefficients in the governing equations.

2 The Kelvin–Voigt equations of order one

Throughout this article we employ standard indicial notation in conjunction with the Einstein summation convention. Hence, partial differentiation with respect to x_i is written as $_{,i} \equiv \partial/\partial x_i$, and Δ denotes the Laplacian in \mathbb{R}^3 .

Let $v_i(\mathbf{x}, t)$ be the velocity, $p(\mathbf{x}, t)$ be the pressure, and let $f_i(\mathbf{x}, t)$ be the body force at position \mathbf{x} and time t . Now, let $q_i^m(\mathbf{x}, t)$, $m = 1, \dots, L$, be viscoelastic variables. Then [27] define a hierarchy of viscoelastic fluid models. They write that a Maxwell fluid of order L , $L \in \mathbb{N}$, satisfies the equations

$$\begin{aligned} v_{i,t} + v_j v_{i,j} - \sum_{m=1}^L \beta_m^{(1)} \Delta q_i^m + p_{,i} &= f_i, \\ v_{i,i} &= 0, \\ q_{i,t}^m + \gamma_m q_i^m &= v_i, \quad m = 1, \dots, L. \end{aligned} \tag{1}$$

An Oldroyd fluid of order L , $L \in \mathbb{N}$, satisfies the equations

$$\begin{aligned} v_{i,t} + v_j v_{i,j} - \mu^{(2)} \Delta v_i - \sum_{m=1}^L \beta_m^{(2)} \Delta q_i^m + p_{,i} &= f_i, \\ v_{i,i} &= 0, \\ q_{i,t}^m + \gamma_m q_i^m &= v_i, \quad m = 1, \dots, L. \end{aligned} \tag{2}$$

A Kelvin–Voigt fluid of order L , $L \in \mathbb{N}$, satisfies the equations

$$v_{i,t} + v_j v_{i,j} - \lambda \Delta v_{i,t} - \mu^{(3)} \Delta v_i - \sum_{m=1}^L \beta_m^{(3)} \Delta q_i^m + p_{,i} = f_i,$$

$$\begin{aligned}
 v_{i,i} &= 0, \\
 q_{i,t}^m + \gamma_m q_i^m &= v_i, \quad m = 1, \dots, L.
 \end{aligned}
 \tag{3}$$

In equations (1) - (3), the coefficients $\beta_m^{(k)}$, $k = 1, 2, 3$, $m = 1, \dots, L$, γ_m , $m = 1, \dots, L$, $\mu^{(2)}$, $\mu^{(3)}$ and λ are positive constants, and it should be noted that there is no sum on m in the terms $\gamma_m q_i^m$.

The nonlinearity in equations (1) - (3) consists of the $v_j v_{i,j}$ term. An equation which has been studied is one for the so called Navier - Stokes - Voigt fluid, cf. the very interesting results on attractors and regularity by [55], [56]. The Navier - Stokes - Voigt equations, which are also known as the Kelvin–Voigt equations of order zero, are

$$\begin{aligned}
 v_{i,t} + v_j v_{i,j} - \lambda \Delta v_{i,t} - \mu \Delta v_i + p_{,i} &= f_i, \\
 v_{i,i} &= 0.
 \end{aligned}
 \tag{4}$$

These equations also contain the nonlinear term $v_j v_{i,j}$ and a very interesting article establishing the existence of weak solutions and of strong solutions in appropriate $L^q(\Omega)$ spaces for $1 < q < \infty$ with Ω a bounded domain in \mathbb{R}^d , $d \geq 2$, is due to [57]. These writers raise the question of how equations (4) might be derived from the balance of linear momentum and continuity equations for a viscous incompressible fluid, namely,

$$\begin{aligned}
 v_{i,t} + v_j v_{i,j} &= T_{ji,j} + f_i, \\
 v_{i,i} &= 0,
 \end{aligned}
 \tag{5}$$

where T_{ij} is the symmetric Cauchy stress tensor. They argue that some writers present for equations (4) a constitutive equation of form

$$T_{ij} = -p\delta_{ij} + 2\mu d_{ij} + 2\lambda d_{ij,t},
 \tag{6}$$

where $d_{ij} = (v_{i,j} + v_{j,i})/2$, and they point out that this is not correct since the tensor $d_{ij,t}$ is not objective. Thus, the term $-\lambda \Delta v_{i,t}$ in (4) should be regarded as a regularization term for the Navier - Stokes equations.

One may use the [57] argument also for equations (3). However, a fully nonlinear model for a Navier - Stokes - Voigt fluid is derived by [58]. This model is now referred to as a Walters fluid, see e.g. [59]. The model of [58] recognizes the fact that $d_{ij,t}$ is not objective as [57] point out and they replace it with an objective derivative of form

$$d_{ij}^\nabla = d_{ij,t} + v_k d_{ij,k} - v_{j,k} d_{ik} - v_{i,k} d_{kj},$$

see also [60], [59]. This requires one to analyse instead of (4) the momentum equation

$$\begin{aligned} v_{i,t} + v_j v_{i,j} = & -p_{,i} + \mu \Delta v_i + \lambda \Delta v_{i,t} \\ & + \lambda [v_k (v_{i,j} + v_{j,i}),_{k}],_j \\ & - \lambda [v_{j,k} (v_{i,k} + v_{k,i}),_j] \\ & - \lambda [v_{i,k} (v_{k,j} + v_{j,k}),_j]. \end{aligned} \quad (7)$$

Here the divergence of \mathbf{v} is also zero. If we employ this argument for a Kelvin–Voigt fluid of order one then instead of equation (3)₁ with $L = 1$ we should use (7) with a $-\beta \Delta q_i$ term added to the right hand side.

Of particular interest to the present article are equations (1) - (3) for fluids of order one, i.e. when $L = 1$. [13] study continuous dependence and convergence for a linear system arising from the Maxwell equations (1) and we develop linear equations here. In the linear case we write the analogous systems to (1) - (3) of order one as, with $f_i = 0$,

Maxwell,

$$\begin{aligned} v_{i,t} - \beta \Delta q_i + p_{,i} &= 0, \\ v_{i,i} &= 0, \\ q_{i,t} + \gamma q_i &= v_i, \end{aligned} \quad (8)$$

Oldroyd,

$$\begin{aligned} v_{i,t} - \mu \Delta v_i - \beta \Delta q_i + p_{,i} &= 0, \\ v_{i,i} &= 0, \\ q_{i,t} + \gamma q_i &= v_i, \end{aligned} \quad (9)$$

Kelvin–Voigt,

$$\begin{aligned} v_{i,t} - \lambda \Delta v_{i,t} - \mu \Delta v_i - \beta \Delta q_i + p_{,i} &= 0, \\ v_{i,i} &= 0, \\ q_{i,t} + \gamma q_i &= v_i. \end{aligned} \quad (10)$$

It is sometimes convenient to eliminate q_i in these equations and derive the following equations for v_i and a generalized pressure, of form,

Maxwell,

$$\begin{aligned} v_{i,tt} - \beta \Delta v_i + \gamma v_{i,t} &= -\phi_{,i}, \\ v_{i,i} &= 0, \end{aligned} \quad (11)$$

Oldroyd,

$$\begin{aligned} v_{i,tt} - \mu \Delta v_{i,t} + \gamma v_{i,t} - (\beta + \mu \gamma) \Delta v_i &= -\phi_{,i}, \\ v_{i,i} &= 0, \end{aligned} \quad (12)$$

Kelvin–Voigt,

$$\begin{aligned} v_{i,tt} - \lambda \Delta v_{i,tt} - (\gamma \lambda + \mu) \Delta v_{i,t} + \gamma v_{i,t} - (\beta + \mu \gamma) \Delta v_i &= -\phi_{,i}, \\ v_{i,i} &= 0, \end{aligned} \tag{13}$$

where $\phi = p_{,t} + \gamma p$.

We observe that formally, as $\lambda \rightarrow 0$ the Kelvin–Voigt system (13) tends to the Oldroyd equations (12). As $\mu \rightarrow 0$ the Oldroyd equations (12) tend to the Maxwell equations (11). Furthermore, if we put $\gamma = \beta = 1/\zeta$ and rescale the pressure ϕ as $\phi = \psi/\zeta$, then the Maxwell system (11) becomes

$$\begin{aligned} \zeta v_{i,tt} - \Delta v_i + v_{i,t} &= -\psi_{,i}, \\ v_{i,i} &= 0. \end{aligned} \tag{14}$$

When $\zeta \rightarrow 0$, equations (14) tend to those of Stokes flow, cf. [13].

In this article we concentrate on equations (10) or (13) and we establish continuous dependence of the solution upon the parameters λ, β, γ and μ .

3 Continuous dependence upon the Kelvin–Voigt coefficient λ

As remarked at the end of the introduction, the Kelvin–Voigt coefficient multiplies the highest derivative term in (10) or (13), and as such it is highly important that the solution depends continuously on changes in this parameter. To establish continuous dependence upon λ we let (u_i, q_i^1, p^1) and (v_i, q_i^2, p^2) be solutions to equations (10) for fixed constants μ, β and γ , but for λ_1 and λ_2 , respectively. In each case the equations are defined on a bounded region Ω in \mathbb{R}^3 with boundary Γ which is sufficiently regular to allow application of the divergence theorem. We here restrict attention to a bounded domain in \mathbb{R}^3 although the methods work for a bounded domain in $\mathbb{R}^n, n \geq 2$.

Thus, (u_i, q_i^1, p^1) satisfies the boundary - initial value problem

$$\begin{aligned} u_{i,t} - \lambda_1 \Delta u_{i,t} - \mu \Delta u_i - \beta \Delta q_i^1 &= -p_{,i}^1, \\ u_{i,i} &= 0, \\ q_{i,t}^1 + \gamma q_i^1 &= u_i, \end{aligned} \tag{15}$$

on $\Omega \times (0, T)$, for some $T > 0$, with the boundary and initial conditions,

$$\begin{aligned} u_i &= \ell_i(\mathbf{x}), \quad \mathbf{x} \in \Gamma, \\ u_i(\mathbf{x}, 0) &= f_i(\mathbf{x}), \quad q_i^1(\mathbf{x}, 0) = h_i(\mathbf{x}), \quad \mathbf{x} \in \Omega, \end{aligned} \tag{16}$$

for prescribed functions f_i, h_i and ℓ_i . In general, one should like $\ell_i(\mathbf{x})$ to also depend on t . The method we employ requires us to allow ℓ_i to depend only on \mathbf{x} , which was also the case in [13].

The solution (v_i, q_i^2, p^2) satisfies the boundary - initial value problem

$$\begin{aligned} v_{i,t} - \lambda_2 \Delta v_{i,t} - \mu \Delta v_i - \beta \Delta q_i^2 &= -p_{,i}^2, \\ v_{i,i} &= 0, \\ q_{i,t}^2 + \gamma q_i^2 &= v_i, \end{aligned} \quad (17)$$

on $\Omega \times (0, T)$, for some $T > 0$, with the boundary and initial conditions,

$$\begin{aligned} v_i &= \ell_i(\mathbf{x}), \quad \mathbf{x} \in \Gamma, \\ v_i(\mathbf{x}, 0) &= f_i(\mathbf{x}), \quad q_i^2(\mathbf{x}, 0) = h_i(\mathbf{x}), \quad \mathbf{x} \in \Omega. \end{aligned} \quad (18)$$

Define the difference variables w_i, q_i, π and λ as

$$w_i = u_i - v_i, \quad q_i = q_i^1 - q_i^2, \quad \pi = p^1 - p^2, \quad \lambda = \lambda_1 - \lambda_2.$$

Then (w_i, q_i, π) satisfies the boundary - initial value problem

$$\begin{aligned} w_{i,t} - \lambda \Delta w_{i,t} - \lambda_2 \Delta w_{i,t} - \mu \Delta w_i - \beta \Delta q_i &= -\pi_{,i}, \\ w_{i,i} &= 0, \\ q_{i,t} + \gamma q_i &= w_i, \end{aligned} \quad (19)$$

on $\Omega \times (0, T)$, for some $T > 0$, with the boundary and initial conditions,

$$\begin{aligned} w_i &= 0, \quad \mathbf{x} \in \Gamma, \\ w_i(\mathbf{x}, 0) &= 0, \quad q_i(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in \Omega. \end{aligned} \quad (20)$$

Let now (\cdot, \cdot) and $\|\cdot\|$ denote the inner product and norm on $L^2(\Omega)$.

We commence the continuous dependence analysis by multiplying (19)₁ by w_i and integrating over Ω . After use of the boundary conditions one obtains

$$\frac{d}{dt} \left(\frac{1}{2} \|\mathbf{w}\|^2 + \frac{\lambda}{2} \|\nabla \mathbf{w}\|^2 \right) + \mu \|\nabla \mathbf{w}\|^2 = -\lambda (u_{i,tj}, w_{i,j}) - \beta (w_{i,j}, q_{i,j}). \quad (21)$$

Next we differentiate equation (19)₃ with respect to x_j and t , multiply by $q_{i,j}$ and integrate over Ω to find

$$\frac{d}{dt} \frac{1}{2} \|\nabla \mathbf{q}\|^2 + \gamma \|\nabla \mathbf{q}\|^2 = (w_{i,j}, q_{i,j}). \quad (22)$$

Upon forming (21) + $\beta \times$ (22) one derives

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|\mathbf{w}\|^2 + \frac{\lambda_2}{2} \|\nabla \mathbf{w}\|^2 + \frac{\beta}{2} \|\nabla \mathbf{q}\|^2 \right) + \beta \lambda \|\nabla \mathbf{q}\|^2 + \mu \|\nabla \mathbf{w}\|^2 \\ = -\lambda (u_{i,tj}, w_{i,j}) \end{aligned}$$

$$\leq \frac{\lambda^2}{2\epsilon} \|\nabla \mathbf{u}_{,t}\|^2 + \frac{\epsilon}{2} \|\nabla \mathbf{w}\|^2, \tag{23}$$

for $\epsilon > 0$, where we have employed the arithmetic - geometric mean inequality. Pick $\epsilon = \mu$ and integrate over $(0, t)$ to find

$$\begin{aligned} & \frac{1}{2} \|\mathbf{w}\|^2 + \frac{\lambda_2}{2} \|\nabla \mathbf{w}\|^2 + \frac{\beta}{2} \|\nabla \mathbf{q}\|^2 + \beta\gamma \int_0^t \|\nabla \mathbf{q}\|^2 ds \\ & + \frac{\mu}{2} \int_0^t \|\nabla \mathbf{w}\|^2 ds \leq \frac{\lambda^2}{2\mu} \int_0^t \|\nabla \mathbf{u}_{,s}\|^2 ds. \end{aligned} \tag{24}$$

We now proceed to bound the right hand side of (24) in terms of data. To do this we differentiate (15)_{1,3} to obtain

$$\begin{aligned} u_{i,tt} - \lambda_1 \Delta u_{i,tt} - \mu \Delta u_{i,t} - \beta \Delta q_{i,t}^1 &= -p_{,it}^1, \\ q_{i,ttj}^1 + \gamma q_{i,tj}^1 &= u_{i,tj}. \end{aligned} \tag{25}$$

To employ (25)₁ we additionally prescribe $u_{i,t}(\mathbf{x}, 0) = g_i(\mathbf{x})$. Multiply (25)₁ by $u_{i,t}$ and integrate over Ω noting $u_{i,t} = 0$ on Γ . Multiply (25)₂ by $q_{i,tj}^1$ and integrate over Ω . After some manipulation one may obtain

$$\frac{d}{dt} \left(\frac{1}{2} \|\mathbf{u}_{,t}\|^2 + \frac{\lambda_1}{2} \|\nabla \mathbf{u}_{,t}\|^2 + \frac{\beta}{2} \|\nabla \mathbf{q}_{,t}^1\|^2 \right) + \mu \|\nabla \mathbf{u}_{,t}\|^2 + \beta\gamma \|\nabla \mathbf{q}_{,t}^1\|^2 = 0.$$

This equation is integrated over $(0, t)$ to find using the initial data

$$\begin{aligned} & \frac{1}{2} \|\mathbf{u}_{,t}\|^2 + \frac{\lambda_1}{2} \|\nabla \mathbf{u}_{,t}\|^2 + \frac{\beta}{2} \|\nabla \mathbf{q}_{,t}^1\|^2 + \mu \int_0^t \|\nabla \mathbf{u}_{,s}\|^2 ds \\ & + \beta\gamma \int_0^t \|\nabla \mathbf{q}_{,s}^1\|^2 ds = \frac{1}{2} \|\mathbf{g}\|^2 + \frac{\lambda_1}{2} \|\nabla \mathbf{g}\|^2 + \|\nabla \mathbf{f} - \gamma \nabla \mathbf{g}\|^2 \\ & = d_0, \end{aligned} \tag{26}$$

where d_0 is the data term defined in (26). From this expression one deduces

$$\int_0^t \|\nabla \mathbf{u}_{,s}\|^2 ds \leq \frac{1}{\mu} d_0.$$

Employ this in (24) to obtain the inequality

$$\begin{aligned} & \frac{1}{2} \|\mathbf{w}\|^2 + \frac{\lambda_2}{2} \|\nabla \mathbf{w}\|^2 + \frac{\beta}{2} \|\nabla \mathbf{q}\|^2 + \beta\gamma \int_0^t \|\nabla \mathbf{q}\|^2 ds \\ & + \frac{\mu}{2} \int_0^t \|\nabla \mathbf{w}\|^2 ds \leq \frac{d_0}{2\mu^2} \lambda^2. \end{aligned} \tag{27}$$

Inequality (27) establishes continuous dependence of a solution on λ in the measures $\|\mathbf{w}\|$, $\|\nabla \mathbf{w}\|$, and $\|\nabla \mathbf{q}\|$.

Remark If we set $\lambda_2 = 0$ and put $\lambda_1 = \lambda$ then (27) yields a convergence result indicating how u_i converges to v_i , a solution to the Oldroyd order one equations.

4 Continuous dependence upon the parameters μ, β, γ

Let now (u_i, q_i^1, p^1) and (v_i, q_i^2, p^2) be solutions to the Kelvin–Voigt order one equations (10) for the same coefficient λ , but with values μ_1, β_1, γ_1 , and μ_2, β_2, γ_2 , respectively. Thus, these solutions satisfy the boundary - initial value problems

$$\begin{aligned} u_{i,t} - \lambda \Delta u_{i,t} - \mu_1 \Delta u_i - \beta_1 \Delta q_i^1 &= -p_{,i}^1, \\ u_{i,i} &= 0, \\ q_{i,t}^1 + \gamma_1 q_i^1 &= u_i, \end{aligned} \quad (28)$$

on $\Omega \times (0, T)$, with

$$\begin{aligned} u_i &= \ell_i(\mathbf{x}), \quad \mathbf{x} \in \Gamma, \\ u_i(\mathbf{x}, 0) &= f_i(\mathbf{x}), \quad q_i^1(\mathbf{x}, 0) = h_i(\mathbf{x}), \quad \mathbf{x} \in \Omega, \end{aligned} \quad (29)$$

and

$$\begin{aligned} v_{i,t} - \lambda \Delta v_{i,t} - \mu_2 \Delta v_i - \beta_2 \Delta q_i^2 &= -p_{,i}^2, \\ v_{i,i} &= 0, \\ q_{i,t}^2 + \gamma_2 q_i^2 &= v_i, \end{aligned} \quad (30)$$

on $\Omega \times (0, T)$, with

$$\begin{aligned} v_i &= \ell_i(\mathbf{x}), \quad \mathbf{x} \in \Gamma, \\ v_i(\mathbf{x}, 0) &= f_i(\mathbf{x}), \quad q_i^2(\mathbf{x}, 0) = h_i(\mathbf{x}), \quad \mathbf{x} \in \Omega. \end{aligned} \quad (31)$$

Define the difference variables by

$$\begin{aligned} w_i &= u_i - v_i, \quad q_i = q_i^1 - q_i^2, \quad \pi = p^1 - p^2, \\ \mu &= \mu_1 - \mu_2, \quad \beta = \beta_1 - \beta_2, \quad \gamma = \gamma_1 - \gamma_2. \end{aligned}$$

One then finds that the difference solution satisfies the boundary - initial value problem

$$\begin{aligned} w_{i,t} - \lambda \Delta w_{i,t} - \mu \Delta u_i - \mu_2 \Delta w_i - \beta \Delta q_i^1 - \beta_2 \Delta q_i &= -\pi_{,i}, \\ w_{i,i} &= 0, \\ q_{i,t} + \gamma q_i^1 + \gamma_2 q_i &= w_i, \end{aligned} \quad (32)$$

on $\Omega \times (0, T)$, with

$$\begin{aligned} w_i &= 0, \quad \mathbf{x} \in \Gamma, \\ w_i(\mathbf{x}, 0) &= 0, \quad q_i(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in \Omega. \end{aligned} \tag{33}$$

Next, differentiate (32)₃ with respect to x_j , and multiply the result by $q_{i,j}$ and integrate over Ω . Multiply (32)₁ by w_i and integrate over Ω . After some integration by parts and use of the boundary conditions one may combine the results to derive

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|\mathbf{w}\|^2 + \frac{\lambda}{2} \|\nabla \mathbf{w}\|^2 + \frac{\beta_2}{2} \|\nabla \mathbf{q}\|^2 \right) + \mu_2 \|\nabla \mathbf{w}\|^2 + \gamma_2 \beta_2 \|\nabla \mathbf{q}\|^2 \\ = -\mu(u_{i,j}, w_{i,j}) - \beta(q_{i,j}^1, w_{i,j}) - \beta_2 \gamma(q_{i,j}^1, q_{i,j}). \end{aligned}$$

We next employ the arithmetic - geometric mean inequality on the right hand side of this equation to see that

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|\mathbf{w}\|^2 + \frac{\lambda}{2} \|\nabla \mathbf{w}\|^2 + \frac{\beta_2}{2} \|\nabla \mathbf{q}\|^2 \right) + \mu_2 \|\nabla \mathbf{w}\|^2 + \gamma_2 \beta_2 \|\nabla \mathbf{q}\|^2 \\ \leq \frac{2}{\mu_2} \mu^2 \|\nabla \mathbf{u}\|^2 + \left(\frac{2}{\mu_2} \beta^2 + \frac{1}{\beta_2 \gamma_2} \gamma^2 \right) \|\nabla \mathbf{q}^1\|^2. \end{aligned} \tag{34}$$

We need to bound the right hand side of (34) in terms of μ^2 , β^2 and γ^2 . To do this we observe that u_i satisfy the equations (13) for γ_1, μ_1, β_1 , and we impose $u_{i,t}(\mathbf{x}, 0) = g_i(\mathbf{x})$. We multiply the appropriate version of (13)₁ by $u_{i,t}$ and integrate over Ω to obtain after integration by parts and integration over $(0, t)$,

$$\begin{aligned} \frac{1}{2} \|\mathbf{u}_{,t}\|^2 + \frac{\lambda}{2} \|\nabla \mathbf{u}_{,t}\|^2 + \left(\frac{\beta_1 + \gamma_1 \mu_1}{2} \right) \|\nabla \mathbf{u}\|^2 \\ + (\gamma_1 \lambda + \mu_1) \int_0^t \|\nabla \mathbf{u}_{,s}\|^2 ds + \gamma_1 \int_0^t \|\mathbf{u}_{,s}\|^2 ds \\ = \frac{1}{2} \|\mathbf{g}\|^2 + \frac{\lambda}{2} \|\nabla \mathbf{q}\|^2 + \left(\frac{\beta_1 + \gamma_1 \mu_1}{2} \right) \|\nabla \mathbf{f}\|^2. \end{aligned} \tag{35}$$

Furthermore, differentiate (28)₃ to find

$$q_{i,jt}^1 + \gamma_1 q_{i,j}^1 = u_{i,j}.$$

Multiply this by $q_{i,j}^1$ and integrate over Ω to find after use of the arithmetic - geometric mean inequality

$$\frac{d}{dt} \|\nabla \mathbf{q}^1\|^2 + \gamma_1 \|\nabla \mathbf{q}^1\|^2 \leq \frac{1}{\gamma_1} \|\nabla \mathbf{u}\|^2. \tag{36}$$

Let d_1 be the data term

$$d_1 = \frac{1}{(\beta_1 + \gamma_1 \mu_1)} \|\mathbf{g}\|^2 + \frac{\lambda}{(\beta_1 + \gamma_1 \mu_1)} \|\nabla \mathbf{g}\|^2 + \|\nabla \mathbf{f}\|^2.$$

Then from (35) we deduce

$$\|\nabla \mathbf{u}\|^2 \leq d_1. \quad (37)$$

We employ this estimate in (36) and integrate to obtain

$$\|\nabla \mathbf{q}^1\|^2 \leq \frac{1}{\gamma_1^2} d_1 + \|\nabla \mathbf{g}\|^2 e^{-\gamma_1 t} = d_2, \quad (38)$$

where d_2 is the indicated data term. Upon employment of (37) and (38) in (34) and a further integration we may obtain

$$\begin{aligned} \frac{1}{2} \|\mathbf{w}\|^2 + \frac{\lambda}{2} \|\nabla \mathbf{w}\|^2 + \frac{\beta_2}{2} \|\nabla \mathbf{q}\|^2 \\ + \frac{\mu_2}{2} \int_0^t \|\nabla \mathbf{w}\|^2 ds + \frac{\gamma_2 \beta_2}{2} \int_0^t \|\nabla \mathbf{q}\|^2 ds \leq d_3, \end{aligned} \quad (39)$$

where d_3 is the data term (involving μ^2, β^2, γ^2)

$$d_3 = \frac{2d_1 t}{\mu_2} \mu^2 + \left(\frac{2}{\mu_2} \beta^2 + \frac{1}{\beta_2 \gamma_2} \gamma^2 \right) \left(\frac{d_1 t}{\gamma_1^2} + \frac{1}{\gamma_1} \|\nabla \mathbf{g}\|^2 \right).$$

Inequality (39) demonstrates continuous dependence of the solution upon the parameters μ, β and γ .

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Compliance with ethical standards

Conflicts of interest There are no conflicts of interest.

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