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Conformal manifolds and 3d mirrors of (D_n, D_m) theories

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Abstract: The Argyres-Douglas (AD) theories of type (D_n, D_m), realized by type IIB geometrical engineering on a single hypersurface singularity, are studied. We analyze their conformal manifolds and propose the 3d mirror theories of all theories in this class upon reduction on a circle. A subclass of the AD theories in question that admits marginal couplings is found to be SO or USp gaugings of certain $D_p(\mathrm{SO}(2N))$ and $D_p(\mathrm{USp}(2N))$ theories. For such theories, we develop a method to derive this weakly-coupled description from the Newton polygon associated to the singularity. We further find that the presence of crepant resolutions of the geometry is reflected in the presence of a (non-abelian) symplectic-type gauge node in the quiver description of the 3d mirror theory. The other important results include the 3d mirrors of all $D_p(\mathrm{SO}(2N))$ theories, as well as certain properties of the $D_p(\mathrm{USp}(2N))$ theories that admit Lagrangian descriptions.

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Contents

1 Introduction

Since the discovery of the Seiberg-Witten (SW) solution [1, 2], four-dimensional $\mathcal{N} = 2$ theories have attracted a lot of attention. This is often due to the fact that the large amount of supersymmetry constrains the dynamics enough to make these models a useful theoretical laboratory for the exploration of nonperturbative dynamics. Remarkably, the physics on the Coulomb branch (CB) of these theories is accessible even for nonlagrangian models and this fact led to the discovery of a large class of intrinsically strongly-coupled theories, first as low-energy theories at singular points of the Coulomb branch of $\mathcal{N} = 2$ gauge theories [3–6] and then more abstractly via geometric methods, either as compactifications of higher dimensional superconformal theories (SCFTs) [7–11] or from the compactification of superstring theory on local Calabi-Yau (CY) 3-folds [12, 13]. In this latter case the geometric engineering in Type IIB is particularly convenient since the complex structure moduli of the geometry (i.e. classical properties) encode the information about the quantum-corrected Coulomb branch physics.

A natural question is then how the stringy geometry encodes the information about the Higgs branch (HB) of the four-dimensional theory. This is harder to address since a classical Type IIB analysis is not enough to provide the answer. An effective strategy to make progress in this direction is to describe the Higgs branch of the 4d theory as the Coulomb branch of a three-dimensional theory with eight supercharges, the so-called 3d mirror dual [14] or magnetic quiver [15–25] in more modern terminology. Following previous work (see e.g. [26–40]) our goal is to construct the 3d mirror theory in the case of local Calabi-Yau geometries described by hypersurface singularities in $\mathbb{C}^4$. More specifically, in this note we consider hypersurfaces given by the sum of two ADE singularities (usually referred to as (G, G') models [41]), where both singularities are of type D.

The analysis of the present work represents a natural continuation of [37, 38], where (A, A) and (A, D) theories have been considered, and with respect to the cases already discussed in the literature presents a new hurdle: the (D, D) singularities are not terminal and we have crepant resolutions, which imply a mismatch between the dimension of the Higgs branch and the number of mass parameters since crepant divisors contribute to the former but not to the latter (see the discussion in the Introduction of [35]). As a result, the 3d mirror cannot simply be an abelian gauge theory as it was in the (A, A) and (A, D) cases. This would indeed immediately imply that the Higgs branch dimension and the rank of the global symmetry of the 4d theory agree. We find that the difference is accounted for by the presence in the 3d mirror of a balanced $\mathrm{USp}(2n)$ gauge node, whose topological symmetry [42, 43] contributes one to the rank of the global symmetry of the 4d theory and whose rank contributes n to the dimension of the Higgs branch of the 4d theory. This represents a new conceptual step towards a systematic understanding of the Higgs branch of 4d SCFTs from hypersurface singularities.

The strategy we apply in this work to extract the 3d mirrors of (D_n, D_m) theories is to start from the already understood (A, A) and (A, D) cases and to construct the 3d quivers step by step for a large set of examples. This then allows us to guess the general answer. In order to implement this program, we first need to study the conformal manifold of these theories and identify weakly-coupled cusps (see also [44–46]), which turn out to involve gaugings of D_p(SO) or D_p(USp) theories introduced in [47–50] (see also [38]). This therefore reduces the problem to gauging building blocks whose 3d mirror is already known. The key feature for identifying the structure of the conformal manifold is the fact that from the geometry we can easily extract the Seiberg-Witten curve of the theory, at least in a certain limit. This is due to the specific structure of the family of hypersurface singularities which engineers the (D_n, D_m) theories, which allows us to plot on a plane the various deformations of the theory.

Let us describe salient features of the 3d mirrors of (D_n, D_m) theories. For generic values of n and m, such a theory consists of a collection of free hypermultiplets, along with an interacting 3d $\mathcal{N} = 4$ SCFT that admits a quiver description. We first discuss the latter. The quiver consists of a balanced central node of the USp-type, which is surrounded by a collection of $\mathrm{SO}(2) \cong D_1$ gauge nodes, possibly with certain number of flavors of hypermultiplets. A subset of such D_1 gauge nodes are connected together by lines to form a complete graph. The central USp gauge node is connected to the surrounding D_1 nodes in a highly non-trivial way, described in detail in the main text. We now describe the origin and properties of such free hypermultiplets. As described in the context of the (A_n, A_m) and (A_n, D_m) theories [37, 38, 51], the free sector arises from dimensional reduction of the SCFTs with no Higgs branch, also known as the non-Higgsable SCFTs, that are present at a generic point of the Higgs branch of the 4d theory. The readers are referred to ([38], appendix C) for an extensive list of the non-Higgsable SCFTs and their properties. In particular, the difference between the rank of the 4d theory and the Higgs branch of the aforementioned quiver is equal to the number of the free hypermultiplets and thus the rank of the non-Higgsable SCFTs in question. Moreover, the difference between the value $24(c - a)$ of the 4d theory and the Coulomb branch dimension of the aforementioned 3d quiver (which is equal to the Higgs branch dimension of the 4d theory given by (3.26) below) is equal to the total value of $24(c-a)$ of the non-Higgsable SCFTs in question. We use these two conditions as a non-trivial test of the proposed 3d mirror theories throughout the paper.

The paper is organized as follows: in section 2 we summarize our notation and conventions, in section 3 we review (D_n, D_m) theories and study their conformal manifold. We also compute the number of mass parameters and the dimension of the Higgs branch by counting crepant divisors. In section 4 we determine all the 3d mirrors of $D_p(\mathrm{SO}(2N))$ theories with $p < 2N-2$, which is needed for the analysis of (D_n, D_m) models. This result complements the analysis carried out in [38]. The main result is contained in section 5, where we describe the 3d mirrors of all (D_n, D_m) theories. We conclude with appendix A which includes new results about $D_p(\mathrm{USp}(2N))$ theories.

2 Notation and convention

Throughout the paper, we use the following abbreviations in the quiver diagrams:
$\mathrm{SO}(2N) = D_N$, $\mathrm{USp}(2N) = C_N$ and $\mathrm{SO}(2N+1) = B_N$. We denote by $/\mathbb{Z}_2$ the diagonal $\mathbb{Z}_2$ quotient of the gauge symmetry.

We follow the same terminology as in [43] to characterize the orthosymplectic gauge groups in 3d $\mathcal{N}=4$ gauge theories. A USp($2N$) gauge group with N_f flavors in the fundamental representation is said to be balanced, overbalanced and underbalanced if $N_f =, >, < 2N+1$, respectively. An SO(N) gauge group with N_f flavors in the vector representation is said to be balanced, overbalanced and underbalanced if $N_f =, >, < N-1$, respectively. On the contrary, in 4d $\mathcal{N}=2$ gauge theories, the condition for a USp($2N$) gauge group with N_f to have a zero beta-function is $N_f = 2N+2$, and that for an SO(N) gauge group with N_f flavors to have a zero beta-function is $N_f = N-2$. It is worth noting that a USp gauge group that satisfies the zero-beta function condition in 4d is overbalanced in 3d, whereas an SO gauge group that satisfies the zero-beta function condition in 4d is underbalanced in 3d.

We also adopt the following notations for the quiver diagrams.

- The R copies of half-hypermultiplets in the representation $[\mathbf{2N};\mathbf{2}]$ of the gauge group USp($2N$) $\times$ SO(2) are denoted by

$$C_N \overset{R}{\text{———}} D_1\,. \tag{2.1}$$

It gives rise to an SU(R) flavor symmetry. To make the Cartan elements of SU(R) manifest, we should interpret (2.1) as denoting the half-hypermultiplets in the following representation of $\{\mathrm{USp}(2N)\times \mathrm{U}(1)\}\times \mathrm{SU}(R)$, where the quantity in $\{\cdots\}$ denotes the gauge factors and $\mathrm{U}(1)\cong \mathrm{SO}(2)$:

$$[\mathbf{2N};+1;\overline{\mathbf{R}}]\oplus[\mathbf{2N};-1;\mathbf{R}]\,. \tag{2.2}$$

- The F flavors of hypermultiplets carrying charge 2 under $\mathrm{U}(1)\cong\mathrm{SO}(2)$ are denoted by

$$D_1 \sim\!\sim\!\sim [F]_2\,, \tag{2.3}$$

where the wiggle line and subscript 2 emphasize the charge 2 under the U(1) gauge group. This gives rise to an SU(F) flavor symmetry. In other words, (2.3) denotes the chiral multiplets in the following representation of $\mathrm{U}(1)\times\mathrm{SU}(F)$:

$$[+2;\overline{\mathbf{F}}]\oplus[-2;\mathbf{F}]\,. \tag{2.4}$$

- An edge connecting two SO(2) gauge nodes with multiplicity M is denoted by

$$D_1 \overset{M}{\text{———}} D_1\,. \tag{2.5}$$

This represents M copies of half-hypermultiplets in the representation $[\mathbf{2};\mathbf{2}]$ of the gauge group SO(2) $\times$ SO(2). It gives rise to a $\mathrm{U}(M)^2/\mathrm{U}(1)$ flavor symmetry, whose algebra is isomorphic to $\mathrm{SU}(M)\times\mathrm{SU}(M)\times\mathrm{U}(1)$. To make the Cartan elements of the latter manifest, we should interpret (2.5) as denoting the half-hypermultiplets in the following representation of $\{\mathrm{U}(1)\times\mathrm{U}(1)\}\times\mathrm{SU}(M)\times\mathrm{SU}(M)\times\mathrm{U}(1)$, where each of the first two U(1) factors are isomorphic to each SO(2) gauge group:

$$\begin{aligned}&[+1;+1;\overline{\mathbf{M}};\mathbf{1};-1]\oplus[-1;-1;\mathbf{M};\mathbf{1};+1]\\ &\oplus[+1;-1;\mathbf{1};\mathbf{M};+1]\oplus[-1;+1;\mathbf{1};\overline{\mathbf{M}};-1]\,.\end{aligned} \tag{2.6}$$

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- An edge connecting an SO(2) gauge node to an SO($2N$) gauge node (for $N \geq 2$) with multiplicity M is denoted by

$$D_N \overset{M}{\text{——}} D_1\,. \tag{2.7}$$

This represents M copies of half-hypermultiplets in the representation $[\mathbf{2N};\mathbf{2}]$ of the gauge group $\mathrm{SO}(2N)\times\mathrm{SO}(2)$. This gives rise to an SU(M) flavor symmetry. To make the Cartan elements of SU(M) manifest, we should interpret (2.1) as denoting the half-hypermultiplets in the following representation of $\{\mathrm{SO}(2N)\times \mathrm{U}(1)\}\times\mathrm{SU}(M)$, where the quantity in $\{\cdots\}$ denotes the gauge factors and $\mathrm{U}(1)\cong\mathrm{SO}(2)$:

$$[\mathbf{2N};+1;\overline{\mathbf{M}}]\oplus[\mathbf{2N};-1;\mathbf{M}]\,. \tag{2.8}$$

3 The conformal manifold for (D, D) Argyres-Douglas theories

The Calabi-Yau hypersurface singularity which engineers in Type IIB the (D_n, D_m) SCFT, with $m, n \geq 3$, reads as follows

$$F(u,x,y,z) = x^{n-1} + xu^2 + y^{m-1} + yz^2\,;\;\Omega = \frac{dudxdydz}{dF}\,. \tag{3.1}$$

Because of the equivalence $(D_n, D_m) = (D_m, D_n)$ we can without loss of generality assume $m \geq n$. By assigning scaling dimension 1 to the holomorphic 3-form and imposing, as usual, homogeneity of the hypersurface singularity, we can easily determine the dimension of the various coordinates:

$$[x] = \frac{2m-2}{n+m-2}\,;\;[y] = \frac{2n-2}{n+m-2}\,;\;[u] = \frac{(n-2)(m-1)}{n+m-2}\,;\;[z] = \frac{(n-1)(m-2)}{n+m-2}\,. \tag{3.2}$$

The allowed deformations, which describe expectation values of Coulomb branch operators, mass parameters and relevant/marginal couplings can be parametrized as follows:

$$x^{n-1} + xu^2 + y^{m-1} + yz^2 + P(x,y) + uQ(y) + zS(x) + Muz = 0\,, \tag{3.3}$$

where P, Q and S are polynomials. Notice that M has always dimension 1 and is therefore a mass parameter for all values of n and m, as can be easily seen from (3.2). We can also notice that the term xy has always dimension 2.

In order to determine the dimension of the conformal manifold, we should count parameters of dimension 0 appearing in (3.3). In order to do this, it is convenient to introduce the parameters

$$a \equiv \mathrm{GCD}(n-1, m-1)\,;\; p \equiv \frac{n-1}{a}\,;\; q = \frac{m-1}{a}\,. \tag{3.4}$$

It is easy to see that a parameter appearing in $P(x, y)$ is marginal if and only if it multiplies a term of the form $x^{kp}y^{(a-k)q}$ with $1 \leq k \leq a-1$ and therefore there are $a-1$ of them. A marginal parameter in $Q(y)$ and $S(x)$ respectively can instead appear when either $\frac{(m-1)n}{2n-2}$ or $\frac{m(n-1)}{2m-2}$ are integers. Notice that, since n and $n-1$ are coprime (and analogously for m

and $m-1$), the two conditions above are mutually exclusive unless n and m are equal and even. Notice that for $m > n$ only the former can be satisfied.

In conclusion, we find that if $n = m$, the conformal manifold has dimension

$$\begin{cases} n-2 & \text{if } n \text{ is odd,} \\ n & \text{if } n \text{ is even.} \end{cases} \tag{3.5}$$

If instead, $n < m$, we introduce

$$f \equiv \frac{n(m-1)}{2(n-1)}\,, \tag{3.6}$$

and the dimension of the conformal manifold is[1]

$$\begin{cases} a & \text{if } f \text{ is an integer,} \\ a-1 & \text{otherwise.} \end{cases} \tag{3.7}$$

We would now like to study the cusps of the conformal manifold and identify the gauge groups becoming weakly-coupled there. In order to do that, it is convenient to set M to zero in (3.3) and introduce, generalizing the analysis of [52], two Lagrange multipliers λ and μ as follows:

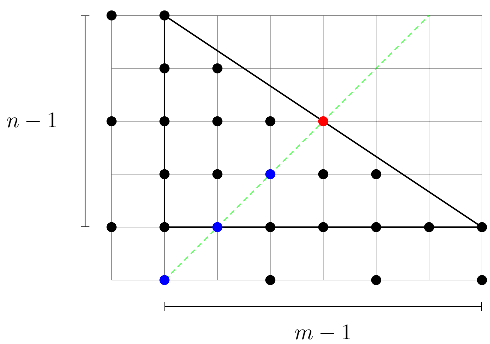
$$x^{n-1} + xu^2 + y^{m-1} + yz^2 + P(x,y) + uQ(y) + zS(x) + \lambda u + \mu z = 0\,. \tag{3.8}$$

Viewing this expression as a superpotential, we can integrate out the massive variables u and z using the equations of motion. As long as the term Muz (which would couple the equations of motion) is absent, we can rewrite (3.8) in terms of x and y only and therefore obtain a Seiberg-Witten curve describing the (D_n, D_m) theory, in the limit $M \to 0$. This can be written in the form

$$x^{n-1} + y^{m-1} + P(x,y) + \frac{Q(y)^2}{x} + \frac{S(x)^2}{y} = 0\,, \tag{3.9}$$

where the polynomials P, Q and S are as in (3.3).

The parametrization (3.9) is particularly convenient since we can plot the various deformations, which are all monomials of the form $x^a y^b$, on a plane. The coordinates of the corresponding point are given by the powers (b, a). Let us give an example for ease of the reader. In the case $n = 5$ and $m = 7$ we can represent the allowed deformations in (3.9) on a plane as follows:



(3.10)

[1]The fact that for $a = 1$ the theory is isolated was already noticed in [46].

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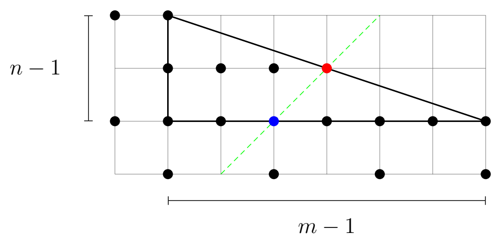
We refer to a planar plot like (3.10) as the Newton polygon of the singularity. All the solid dots represent allowed deformations. Those inside the triangle (including those on the boundary) appear in $P(x,y)$ in (3.9). Those below the triangle arise from $Q(y)$ and those on the left come from $S(x)$. We should remember that all dots outside the triangle represent terms in (3.9) which are squares of more fundamental parameters.

The representation (3.10) is particularly convenient for identifying marginal deformations in $P(x,y)$, since they arise as dots sitting on the diagonal edge of the triangle. In the case at hand of the (D_5, D_7) theory, we clearly see a single marginal parameter (denoted in red) corresponding to the deformation x^2y^3 in (3.9). In order to identify one weakly-coupled cusp in the conformal manifold, namely providing a description of the theory involving a vector multiplet coupled to two or more matter sectors, it is convenient to recall that $[x] + [y] = 2$. Therefore, if we consider a marginal deformation appearing in $P(x,y)$ of the form $x^\alpha y^\beta$ with

$$\alpha = kp\,, \quad \beta = (a-k)q\,, \tag{3.11}$$

we can then notice that the parameter multiplying the term $x^{\alpha-i}y^{\beta-i}$ has dimension $2i$, therefore suggesting these are the Casimir invariants of a gauge group, either of USp or SO type. In (3.10) these correspond to the blue dots, which have dimension 2, 4 and 6 respectively. The parameter of dimension 6, which lies outside the triangle, is actually the square of a parameter of dimension 3 and we therefore see that the dots in blue represent the Casimirs of a SO(6) group.

Generalizing this observation, we are led to the conclusion that whenever we have a marginal deformation of the form $x^\alpha y^\beta$ with $\beta - \alpha$ odd, there is a SO($2N$) vector multiplet (we will be more specific about the value of N momentarily). If instead $\beta - \alpha$ is even, we do not find a deformation corresponding to the Pfaffian and our guess is that the gauge group is of type USp($2N$). For example, in the case $n = 3$ and $m = 7$ we have the marginal deformation xy^3 and only one blue dot corresponding to a Casimir of dimension 2. There is no Casimir associated with a deformation term belonging to $Q(y)$ in (3.9):



(3.12)

Regarding the matter sectors coupled to the vector multiplet, we find it convenient to exploit once again the Newton polygon and consider the straight line passing through the red and blue dots. This divides the plane in two half-planes, each containing the deformations associated with one matter sector coupled to the vector multiplet. Exploiting this guiding principle, we find that the Coulomb branch spectrum is compatible with the following description of the (D_n, D_m) SCFT.

When $\alpha - \beta$ is odd, the gauge group is of SO type, as we have already explained. More precisely,

1. For $\beta > \alpha$ we have

$$D_{n-1+\beta-\alpha}(\mathrm{SO}(2\beta+2)) - \mathrm{SO}(2\alpha+2) - D_{m-1+\alpha-\beta}(\mathrm{SO}(2\alpha+2))\,. \tag{3.13}$$

The matter sector on the left has a partially closed regular puncture labeled by partition $[\beta-\alpha, \beta-\alpha, 1^{2\alpha+2}]$.

2. For $\beta < \alpha$ we have

$$D_{n-1+\beta-\alpha}(\mathrm{SO}(2\beta+2)) - \mathrm{SO}(2\beta+2) - D_{m-1+\alpha-\beta}(\mathrm{SO}(2\alpha+2))\,. \tag{3.14}$$

Now the matter sector on the right has a partially closed regular puncture labeled by partition $[\alpha-\beta, \alpha-\beta, 1^{2\beta+2}]$.

When instead $\alpha - \beta$ is even the gauge group is of USp type. Specifically

3. For $\beta > \alpha$ we have

$$D_{n-1+\beta-\alpha}(\mathrm{USp}(2\beta)) - \mathrm{USp}(2\alpha) - D_{m-1+\alpha-\beta}(\mathrm{USp}(2\alpha))\,. \tag{3.15}$$

The matter sector on the left has a partially closed regular puncture labeled by partition $[\beta-\alpha, \beta-\alpha, 1^{2\alpha}]$.

4. For $\beta < \alpha$ we have

$$D_{n-1+\beta-\alpha}(\mathrm{USp}(2\beta)) - \mathrm{USp}(2\beta) - D_{m-1+\alpha-\beta}(\mathrm{USp}(2\alpha))\,. \tag{3.16}$$

Now the matter sector on the right has a partially closed regular puncture labeled by partition $[\alpha-\beta, \alpha-\beta, 1^{2\beta}]$.

5. For $\beta = \alpha$ we have

$$\begin{array}{c} D_{n-1}(\mathrm{USp}(2\alpha)) \;-\; \mathrm{USp}(2\alpha) \;-\; D_{m-1}(\mathrm{USp}(2\alpha))\,. \\ | \\ [\mathrm{SO}(2)] \end{array} \tag{3.17}$$

Both matter sectors have a full regular puncture in this case, but we also have a hypermultiplet in the fundamental of $\mathrm{USp}(2\alpha)$.

Notice that in all cases described above the gauging is conformal as expected. A natural question at this stage is how the parameter M appearing in (3.3) enters the above descriptions of the (D_n, D_m) theory, since it does not fit in the curve (3.9). The answer is the following: in the first four cases, one of the matter sectors features a puncture of the form $[b, b, 1^c]$. The symmetry carried by the $[1^c]$ part is always gauged, but we also have a rank 1 factor for the global symmetry carried by the $[b, b]$ part. The corresponding mass parameter is identified with M. In the fifth case, instead, the $[b, b]$ part is missing and M is identified with the mass of the hypermultiplet in the fundamental of the USp group.

A careful reader might also wonder about the role of the marginal couplings appearing in $Q(y)$ or $S(x)$, since we have never discussed those. It is easy to see that $Q(y)$ includes a marginal parameter only if a term of the form xy^β with β even is marginal. According to our analysis, such a term implies the presence of a SO(4) gauge group, and therefore the marginal term in $Q(y)$ can be interpreted as providing the second marginal coupling of the SO(4) vector multiplet. Analogously, a marginal parameter in $S(x)$ implies marginality of a term of the form $x^\alpha y$ with α even, which also leads to a SO(4) gauging.

3.1 Counting mass deformations

By the same token, we can compute the number of mass parameters, that is given by the deformation parameters of dimension 1 in (3.3). This equals the rank of the flavor symmetry group of (D_n, D_m) theory.
It can be checked that $Q(y)$ includes a dimension 1 parameter only if $m-1$ is odd, and analogously $S(y)$ includes a mass parameter if $n-1$ is odd. On the other hand, $P(x,y)$ includes mass parameters if p and q are both odd, and in this case we find a extra mass parameters. Overall, taking also into account the parameter M we find the following result, depending on the parity of n and m, we find that the number of mass parameters is:

$$\begin{cases} 1 & \text{if } n-1 \text{ and } m-1 \text{ are even and } p \text{ or } q \text{ is even,} \\ 2 & \text{if } n-1 \text{ is odd and } m-1 \text{ is even,} \\ 2 & \text{if } n-1 \text{ is even and } m-1 \text{ is odd,} \\ a+1 & \text{if } n-1 \text{ and } m-1 \text{ are even and } p \text{ and } q \text{ are odd,} \\ a+3 & \text{if } n-1 \text{ and } m-1 \text{ are odd.} \end{cases} \tag{3.18}$$

3.2 Counting crepant divisors

The dimension of the Higgs Branch of a (G, G') theory can be computed adding to the number of masses, the number of crepant divisors of the canonical singularity describing the geometry of the theory [35, 36].

The number of crepant divisors for a canonical singularity can be computed following the algorithm described in [53, 54] that we are now going to briefly review.[2] Such algorithm can be also easily implemented, for instance, in `SageMath` [55].

Let us define a polynomial

$$f = \sum_i a_i \prod_j x_j^{m_i^j}\,, \tag{3.19}$$

where a_i are integer coefficients, and $m_i = (m^1, \ldots, m^n)$ are the exponents associated to the i-th monomial in f. The locus of $f = 0$ gives an isolated canonical hypersurface singularity. Let us define M to be a free abelian group $\mathbb{Z}^n$, and $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$. From f, it is possible to define its Newton polyhedron, $\Gamma_+(f)$, which is the convex hull in $M_\mathbb{R}$ of the set given by the union of the m_i and the positive quadrant in $\mathbb{R}$.

[2]We thank C. Closset for pointing out this technique to compute the number of crepant divisors.

We now introduce a set of vectors $\alpha = (\alpha_1, \dots \alpha_n)$ that belongs to $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, which is the dual of M, and for each α, we define $\alpha(m) = \alpha \cdot m_i$ and

$$\alpha(f) = \min_{m_i \in f} \alpha \cdot m_i \,. \tag{3.20}$$

The vectors α are called *weightings*, and from this set of vectors, we call *crepant weightings* those that satisfies

$$|\alpha| = \alpha(f) + 1 \,, \tag{3.21}$$

where $|\alpha|$ is the sum of the components of α.

For each crepant weighting we also introduce

$$g(\alpha) = \min_{m_i \in \Gamma_+(f)} \alpha \cdot m_i \,, \tag{3.22}$$

and we call the face of $\Gamma(f)$ corresponding to α as the set

$$\Gamma_\alpha = \{ m_i \in \Gamma_+(f) \, : \, \alpha(m) = g(\alpha) \} \,. \tag{3.23}$$

Finally, we define the length Γ_α as the number of vectors composing the face Γ_α minus 1. The number of crepant divisors for an isolated canonical singularity X is [53, 54]

$$c(X) = \sum_\alpha c(\alpha) \text{ with } c(\alpha) = \begin{cases} \text{length } \Gamma_\alpha & \text{if } \dim \Gamma_\alpha = 1 \,, \\ 1 & \text{if } \dim \Gamma_\alpha \geq 2 \,. \end{cases} \tag{3.24}$$

The HB dimension of a (D_n, D_m) AD theory is given by the sum of the number of masses in (3.18) and the number of crepant divisors. We find that for given a (D_n, D_m) theory X, for $m \geq n$, the number of its crepant divisors is

$$c(X) = \left\lfloor \frac{n}{2} \right\rfloor - 1 \,. \tag{3.25}$$

We will see in subsequent sections that the rank of the balanced symplectic gauge group in the mirror theory for a given (D_n, D_m) theory is, in fact, equal to one plus the number of crepant divisors of the corresponding theory.[3]

[3] We assume that the gauge symmetry in the mirror theory consists of a collection of abelian gauge groups, together with a single balanced symplectic gauge group. This statement follows from the fact that the HB dimension of the (D_n, D_m) theory is equal to the sum of the rank of all gauge groups in the mirror theory, and that the number of mass parameters of the 4d theory is equal to the total number of FI parameters of the mirror theory, where the latter comes from the abelian gauge groups and the balanced symplectic gauge group. This assumption fits all special cases that can be cross-checked, e.g. with those that admit class $\mathcal{S}$ descriptions. Moreover, the fact that the balanced gauge group should be symplectic and not (special) orthogonal can be seen in a special class of theories as follows. Consider the last case of (3.18) such that $m-1$ is a multiple of $n-1$ (m and n are even), so that we have $a = n-1$ and $n+2$ mass parameters. We have a balanced $\mathrm{USp}(n)$ gauge group with $n+1$ $\mathrm{U}(1)$ gauge groups, which act as $n+1$ flavors for the balanced gauge group; altogether we have $n+2$ FI parameters, as required. On the other hand, if the balanced gauge group were $\mathrm{SO}(n)$ or $\mathrm{O}(n)$, we would require number of $\mathrm{U}(1)$ gauge groups to be $n-1$, acting as $n-1$ flavors of the balanced gauge group, but this would not saturate the required number $n+2$ of FI parameters.

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The HB dimension for a (D_n, D_m), using (3.18), can thus be written as follows:

$$\begin{cases} \left\lfloor \frac{n}{2} \right\rfloor & \text{if } n-1 \text{ and } m-1 \text{ are even and } p \text{ or } q \text{ is even,} \\ \left\lfloor \frac{n}{2} \right\rfloor + 1 & \text{if } n-1 \text{ is odd and } m-1 \text{ is even,} \\ \left\lfloor \frac{n}{2} \right\rfloor + 1 & \text{if } n-1 \text{ is even and } m-1 \text{ is odd,} \\ a + \left\lfloor \frac{n}{2} \right\rfloor & \text{if } n-1 \text{ and } m-1 \text{ are even and } p \text{ and } q \text{ are odd,} \\ a + \left\lfloor \frac{n}{2} \right\rfloor + 2 & \text{if } n-1 \text{ and } m-1 \text{ are odd.} \end{cases} \tag{3.26}$$

4 The 3d mirror of $D_p(\mathrm{SO}(2N))$ theories with $p < 2N - 2$

In [38] we have determined the 3d mirror of $D_p(\mathrm{SO}(2N))$ theories with $p < 2N - 2$ only in some cases. Here we would like to fill in this gap since this will be useful for determining the 3d mirrors of (D_n, D_m) theories. We will divide the analysis in two cases; p odd and p even. The analysis will be largely based on an analogy with the $\mathrm{SU}(N)$ case, which is well understood and is discussed in detail in [37].

4.1 $D_p(\mathrm{SO}(2N))$ theories with p odd

These models are the simplest since they do not include any mass parameters apart from those associated with the $\mathrm{SO}(2N)$ global symmetry. It turns out that in this case the 3d mirror is given by two orthosymplectic quiver tails with flavors attached at some of the gauge nodes, which are all balanced.

For convenience, we define the parameter x as follows:

$$x \equiv \left\lfloor \frac{2N-2}{p} \right\rfloor \ . \tag{4.1}$$

For x odd or for $2N - 2$ divisible by p (where, for the latter, x is even), we propose that the 3d mirror of $D_p(\mathrm{SO}(2N))$ is identified with the theory $T^{\sigma}_{[1^{2N}]}[\mathrm{SO}(2N)]$, where

$$\sigma = \left[(x+1)^{2N-1-px}, x^{xp+p+1-2N}, 1\right] \ . \tag{4.2}$$

Notice that the above σ is always an even partition of $2N$ regardless of the parity of x. If $2N - 2$ is divisible by p, then σ can be written as $\sigma = [x + 1, x^{p-1}, 1]$. The explicit quivers can be described as follows:

$$\begin{array}{ccccccccccccc} & & & & & & \left[C_{\frac{2N-1-px}{2}}\right] & & & & & & \\ & & & & & & | & & & & & & \\ C_{\frac{p-1}{2}} & - B_{p-1} - \cdots - & C_{\frac{p-1}{2}x} & - & D_{\frac{2N-1-x}{2}} & - & C_{\frac{2N-3-x}{2}} & - \cdots - & C_1 & - & D_1 \\ | & & | & & & & & & & & \\ B_0 & & \left[B_{\frac{xp+p-2N}{2}}\right] & & & & & & & & \end{array} \tag{4.3}$$

The tail on the left has alternating gauge groups of type C and B, with increasing ranks, which are multiples of $\frac{p-1}{2}$. The tail on the right is instead equivalent to $T[\mathrm{SO}(2N-1-x)]$. The mirror theory also comes with

$$H_{\mathrm{free}} = X\left(\frac{p-1}{2} - X\right) , \tag{4.4}$$

free hypermultiplets,[4] where

$$X = \left(N + \frac{x-1}{2}\right) \bmod \left(\frac{p+1}{2}\right) . \tag{4.5}$$

Let us discuss the case in which p divides $2N-2$. Here $X=0$ and there are no free hypermultiplets. Since p is odd, $x=(2N-2)/p$ is even and, according to ([48], appendix C), the corresponding $D_p(\mathrm{SO}(2N))$ theory in 4d admits a Lagrangian description. As we commented in ([38], section 8.4), the Lagrangian description of the 4d theory is the same as that of the quiver description of the 3d $T_\sigma[\mathrm{SO}(2N)]$ theory, with σ given by (4.2), namely

$$[D_{\mathfrak{m}p+1}] - C_{\mathfrak{m}(p-1)} - D_{\mathfrak{m}(p-2)+1} - \cdots - C_{2\mathfrak{m}} - D_{\mathfrak{m}+1} \tag{4.6}$$

where $\mathfrak{m} \equiv x/2$. Each C and D gauge group in the 4d quiver theory has zero beta-function, whereas in the 3d quiver the C-gauge group is overbalanced and the D gauge group is underbalanced, rendering the theory "bad" in the sense of [43]. At this stage, it is not clear whether the quiver for $T_\sigma[\mathrm{SO}(2N)]$ describes the reduction of such a $D_p(\mathrm{SO}(2N))$ theory to 3d. If we *assume*[5] that this is true, then the mirror theory is as described above. We hope to gain better understanding of this case in the future.

For x even and $2N-2$ not divisible by p, we propose that the 3d mirror for $D_p(\mathrm{SO}(2N))$ is identified with the theory $T^{\sigma'}_{[1^{2N}]}[\mathrm{SO}(2N)]$, where the partition σ' can be obtained from the partition σ by "lifting a box up", i.e.

$$\sigma' = [(x+1)^{2N-px}, x^{xp+p-1-2N}, x-1, 1] . \tag{4.7}$$

Note that if the box is not liftable, i.e. $xp+p-1-2N<0$, the partition σ' is identical to the partition σ, e.g. $N=5$, $p=3$, we have $\sigma=\sigma'=[3,3,3,1]$. The theory $T^{\sigma'}_{[1^{2N}]}[\mathrm{SO}(2N)]$

[4] We remark that the expression for the number of the free hypermultiplets H_{free} in the mirror theory for $D_p(\mathrm{SO}(2N))$ with p odd and $p \geq 2N-2$, given by ([38], (5.4)), can also be written as $H_{\mathrm{free}} = \left(\frac{p-2N+1}{2}\right) N = \left[\frac{p-1}{2} - (N-1)\right] N$. We see that (4.4) indeed has a similar form to such an expression.

[5] This assumption should be taken cautiously. We observe that only when p divides $2N-2$, the Coulomb branch dimension of the mirror theory is $(p-1)/2$ larger than the value $24(c-a)$ of the corresponding 4d theory. We have checked that all the other cases do not have this problem, namely the Coulomb branch dimension of the mirror theory is always less than or equal to the integer part of $24(c-a)$. In this case, it is not clear whether $24(c-a)$ is equal to the Higgs branch of the 4d theory, since the orthogonal gauge groups may not be completely Higgsed at a generic point on the Higgs branch. This is due to the fact that the orthogonal gauge group which is conformal in 4d is underbalanced in 3d. Nevertheless, we emphasize that the Higgs branch dimension of the mirror theory (4.2) is exactly equal to the rank of the corresponding 4d theory, as it should be.

admits a quiver description of the form

$$
\begin{array}{ccccccccccccccccc}
 & & & & & & & & [C_{\frac{px+p-1-2N}{2}}] & & & & & & & & \\
 & & & & & & & & | & & & & & & & & \\
C_{\frac{p-1}{2}} & - & B_{p-1} & - & \cdots & - & C_{\frac{p-1}{2}(x-1)} & - & D_{\frac{p-1}{2}x} & - & C_{\frac{2N-2-x}{2}} & - & D_{\frac{2N-2-x}{2}} & - & \cdots & - & C_1 \; - \; D_1 \\
| & & & & & & | & & & & | & & & & & & \\
B_0 & & & & & & B_0 & & & & [D_{\frac{2N-px}{2}}] & & & & & &
\end{array}
\tag{4.8}
$$

The number of free hypermultiplets in this case (i.e. x even and p does not divide $2N-2$) is

$$H_{\text{free}} = Y\left(\frac{p+1}{2} - Y\right) , \tag{4.9}$$

where

$$Y = \left(N + \frac{x}{2}\right) \bmod \left(\frac{p+1}{2}\right) . \tag{4.10}$$

Let us mention some consistency checks for the proposed mirror theory. It is convenient to discuss these via examples.

- **The case of** $N = 9$ **and** $p = 7$. Here $x = 2$. The 4d theory has $24(c-a) = 459/7 = 65 + (4/7)$ and it has rank 27. The proposed mirror theory has CB dimension 65 and HB dimension 23. We thus have $27 - 23 = 4$ free hypermultiplets, in agreement with the fact that the non-Higgsable SCFT has $24(c-a)$ equal to $4/7$, which is expected to be that of the $(A_1, A_4)^{\otimes 2}$ theory whose rank is $(2 \times 2) = 4$.
- **The case of** $N = 12$ **and** $p = 5$. Here $x = 4$. The 4d theory has $24(c-a) = 552/5 = 110 + (2/5)$ and it has rank 24. The proposed mirror theory $T^{\sigma'}[\mathrm{SO}(2N)]$ has CB dimension 110 and HB dimension 22. We thus have $24 - 22 = 2$ free hypermultiplets, in agreement with the fact that the non-Higgsable SCFT has $24(c-a)$ equal to $2/5$, which is expected to be that of the $(A_1, A_2)^{\otimes 2}$ theory whose rank is $(2 \times 1) = 2$.
- **The case of** $N = 12$ **and** $p = 7$. Here $x = 3$. The 4d theory has $24(c-a) = 828/7 = 118 + (2/7)$ and it has rank 36. The proposed mirror theory $T^{\sigma}[\mathrm{SO}(2N)]$ has CB dimension 118 and HB dimension 34. We thus have $36 - 34 = 2$ free hypermultiplets, in agreement with the fact that the non-Higgsable SCFT has $24(c-a)$ equal to $2/7$, which is expected to be that of the (A_1, A_4) theory whose rank is 2.

Let us also briefly comment on the procedure of "lifting a box up" in the case of x even and $2N - 2$ not divisible by p. The reason for this is two-fold. First, this procedure leads to the desired Higgs branch and Coulomb branch dimensions of the mirror theory that pass the above checks. Secondly, this procedure cures the "badness" of the $T_\sigma[\mathrm{SO}(2N)]$ theory (see also [56] for a related discussion). To illustrate this point, we take an example of $N = 12$ and $p = 5$, where $\sigma = [5, 5, 5, 4, 4, 1]$, $\sigma' = [5, 5, 5, 5, 3, 1]$, and

$$
\begin{aligned}
T_\sigma[\mathrm{SO}(24)]: &\quad [D_{12}] - C_9 - D_7 - C_4 - D_3 \\
T_{\sigma'}[\mathrm{SO}(24)]: &\quad [D_{12}] - C_9 - D_7 - C_4 - D_2
\end{aligned}
\tag{4.11}
$$

Observe that the former contains an underbalanced D_3 gauge node, rendering the theory bad, whereas the latter does not contain any underbalanced gauge nodes.

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4.2 $D_p(\mathrm{SO}(2N))$ theories with p even

The difference with respect to the previous case is that for p even, we have extra mass parameters besides those associated with the $\mathrm{SO}(2N)$ global symmetry. Their number depends on the parity of $\frac{2N-2}{\mathrm{GCD}(2N-2,p)}$ and we will therefore consider the two cases separately.

4.2.1 The case $\frac{2N-2}{\mathrm{GCD}(2N-2,p)}$ even

In this case we have only one mass parameter besides those of $\mathrm{SO}(2N)$. The 3d mirror is given by a orthosymplectic quiver with $2N-2$ balanced gauge groups and one overbalanced D_1 group whose FI parameter accounts for the extra mass parameter of the 4d theory. We introduce again the parameter

$$x = \left\lfloor \frac{2N-2}{p} \right\rfloor \tag{4.12}$$

and we discuss the cases in which p does and does not divide $2N-2$ separately.

If p does not divide $2N-2$ (here x can be even or odd), we propose that the 3d mirror is given by a $T_\rho^\sigma[\mathrm{SO}(2N+2p-2)]$ theory with

$$\rho = [(p-1)^2, 1^{2N}]\,;\ \sigma = [(x+3)^{2N-2-xp}, (x+2)^{xp+p-2N+2}]\,. \tag{4.13}$$

For x odd, the quiver reads

$$\begin{array}{c}
\qquad\qquad\qquad\left[D_{\frac{xp+p-2N+2}{2}}\right] \\
\qquad\qquad\qquad | \\
D_1 - C_{p/2} - \cdots - C_{xp/2-(x-1)/2} - D_{N-1-(x-1)/2} - C_{\frac{2N-3-x}{2}} - \cdots - C_1 - D_1 \\
\qquad\qquad\qquad\qquad\qquad\qquad | \\
\qquad\qquad\qquad\qquad\qquad\qquad \left[C_{\frac{2N-2-xp}{2}}\right]
\end{array} \tag{4.14}$$

Again, all the nodes not indicated explicitly are of C or D type and balanced. On the other hand, for x even, the quiver reads

$$\begin{array}{c}
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\left[D_{\frac{2N-2-xp}{2}}\right] \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad | \\
D_1 - C_{p/2} - \cdots - C_{(x-1)p/2-(x-2)/2} - D_{x(p-1)/2+1} - C_{\frac{2N-2-x}{2}} - \cdots - C_1 - D_1 \\
\qquad\qquad\qquad\qquad | \\
\qquad\qquad\qquad\qquad \left[C_{\frac{px+p-(2N-2)}{2}}\right]
\end{array} \tag{4.15}$$

This mirror theory also comes with a number of free hypermultiplets given by[6]

$$H_{\text{free}} = \begin{cases} (p-N_p+2)\left(N_p - \frac{p}{2} + \frac{1}{2}\right) - p - \frac{3}{2}\,, & \text{if } x \text{ odd,} \\ \left(\frac{p}{2} - N_p + \frac{5}{2}\right) N_p - p - \frac{3}{2}\,, & \text{if } x \text{ even,} \end{cases} \tag{4.16}$$

[6]We remark that the expression for the number of the free hypermultiplets H_{free} in the mirror theory for $D_p(\mathrm{SO}(2N))$ with p even, $p \geq 2N-2$ and $\frac{2N-2}{\mathrm{GCD}(2N-2,p)}$ even, given by ([38], (6.4)), can also be written as $H_{\text{free}} = \left(\frac{p}{2} - N + \frac{5}{2}\right) N - p - \frac{3}{2}$. We see that (4.16), for x even, indeed has the same form as the former expression with N replaced by N_p.

with N_p defined as

$$N_p = N \text{ mod } p \ . \tag{4.17}$$

If p divides $2N-2$,[7] the 4d theory $D_p(\mathrm{SO}(2N))$ admits a Lagrangian description, which turns out to coincide with the quiver for $T_\sigma^\rho[\mathrm{SO}(2N+2p-2)]$ theory, with

$$\begin{aligned} &\rho = [(p-1)^2, 1^{2N}] = [(p-1)^2, 1^{xp+2}] \ ; \\ &\sigma = [(x+3)^{2N-1-xp}, (x+2)^{xp+p-2N}, x+1] = [x+3, (x+2)^{p-2}, x+1] \end{aligned} \tag{4.18}$$

namely

$$[D_{\mathfrak{m}p+1}] - C_{\mathfrak{m}(p-1)} - D_{\mathfrak{m}(p-2)+1} - \cdots - D_{2\mathfrak{m}+1} - C_{\mathfrak{m}} - [D_1] \tag{4.19}$$

where $\mathfrak{m} \equiv x/2$. The 3d mirror of this quiver is given by a $T_\rho^\sigma[\mathrm{SO}(2N+2p-2)]$ theory

$$\begin{array}{c} \qquad\qquad\qquad\qquad\qquad B_0 \qquad\qquad\qquad\qquad\qquad\quad B_0 \\ \qquad\qquad\qquad\qquad\qquad | \qquad\qquad\qquad\qquad\qquad\qquad | \\ D_1 - C_{p/2} - \cdots - C_{\frac{(x-1)p}{2}-\frac{x-2}{2}} - B_{\frac{x(p-1)}{2}} - C_{\frac{x(p-1)}{2}} - \cdots - C_1 - D_1 \\ | \\ \left[C_{\frac{p-2}{2}}\right] \end{array} \tag{4.20}$$

All the gauge groups not indicated explicitly are alternating of C and D type. The ranks are fixed by requiring all groups to be balanced. It can be checked that in this case, the Higgs branch dimension of the mirror theory (4.20) is exactly equal to the rank of the corresponding 4d theory $D_p(\mathrm{SO}(2N))$. We conclude that there are no free (twisted) hypermultiplets upon reduction of such a 4d theory on a circle. Similarly to Footnote 4, we remark that the Coulomb branch dimension of the mirror theory (4.20) is $(p-2)/2$ larger than the value of $24(c-a)$ of the corresponding 4d theory. In the special case of $p=2$, the 4d theory is simply the $\mathrm{USp}(x)$ SQCD with $x+2$ flavors. The corresponding 3d mirror theory (4.20), with $p=2$, turns out to be coincident with ([57], figure 11) with $k=x/2$ and $N=x+2$, as it should be.

We can perform a similar consistency check for the proposed mirror theory, as in the previous subsection.

- **The case of** $N=7$ **and** $p=10$. Here $x=1$. The 4d theory has $24(c-a) = 211/5 = 42+(1/5)$ and it has rank 31. The proposed mirror theory $T_\rho^\sigma[\mathrm{SO}(2N+2p-2)]$ has CB dimension 42 and HB dimension 30. We thus have $31-30=1$ free hypermultiplets, in agreement with the fact that the non-Higgsable SCFT has $24(c-a)$ equal to $1/5$, which is expected to be that of the (A_1, A_2) theory whose rank is 1.

- **The case of** $N=13$ **and** $p=10$. Here $x=2$. The 4d theory has $24(c-a) = 742/5 = 148+(2/5)$ and it has rank 58. The proposed mirror theory $T_\rho^\sigma[\mathrm{SO}(2N+2p-2)]$ has CB dimension 148 and HB dimension 56. We thus have $58-56=2$ free hypermultiplets, in agreement with the fact that the non-Higgsable SCFT has $24(c-a)$ equal to $2/5$, which is expected to be that of the $(A_1, A_2)^{\otimes 2}$ theory whose rank is 2.

[7]In this case $x = \frac{2N-2}{p}$ is even by our assumption that $\frac{2N-2}{\mathrm{GCD}(2N-2,p)} = \frac{2N-2}{p} = x$ is even.

4.2.2 The case $\frac{2N-2}{\mathrm{GCD}(2N-2,p)}$ odd

We recall that the only way in which we used the fact that $\frac{2N-2}{\mathrm{GCD}(2N-2,p)}$ is even in section 4.2.1 is in requiring that the flavors attached to the central C-type node in (4.14) and (4.15) are not gauged. This has to be the case in order to match the rank of the global symmetry of the 4d theory discussed in section 4.2.1.

We shall shortly see that the main part of the mirror quiver in the previous section can be carried through in the case of $\frac{2N-2}{\mathrm{GCD}(2N-2,p)}$ odd as well. We claim that the structure of the two tails in the mirror dual is the same as in section 4.2.1, which is also supported by the known form of the mirrors presented in ([38], section 8.3). The only differences are that

1. The D-type flavor nodes in (4.14) and (4.15) are now gauged to account for the larger number of mass parameters and they form a complete graph, whose structure we can try to guess using as a guidance the special cases we have already worked out.

2. The C-type flavor node that is attached to a central D-type gauge node in (4.14) and (4.15) is replaced by lines with appropriate multiplicities that connect such a D-type gauge node to the vertices of the complete graph.

Let us now be explicit about the above description.

As before, we use the notation

$$x = \left\lfloor \frac{2N-2}{p} \right\rfloor \,. \tag{4.21}$$

We start from $T^{\sigma}_{\rho}[\mathrm{SO}(2N+2p-2)]$ theory with the same partitions as in (4.13), namely

$$\rho = [(p-1)^2, 1^{2N}]\,;\ \sigma = [(x+3)^{2N-2-xp}, (x+2)^{xp+p-2N+2}]\,. \tag{4.22}$$

Let us first consider the case x odd. The quiver description of such a theory is given by (4.14). For convenience, we reproduce the diagram here:

$$\begin{array}{c} \qquad\qquad\qquad\qquad \left[D_{\frac{xp+p-2N+2}{2}}\right] \\ \qquad\qquad\qquad\qquad | \\ D_1 - C_{p/2} - \cdots - C_{xp/2-(x-1)/2} - D_{N-1-(x-1)/2} - C_{\frac{2N-3-x}{2}} - \cdots - C_1 - D_1 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad | \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left[C_{\frac{2N-2-xp}{2}}\right] \end{array} \tag{4.23}$$

Since we should now introduce $\mathrm{GCD}(2N-2,p)/2$ new mass parameters, we replace the $\left[D_{\frac{xp+p-2N+2}{2}}\right]$ flavor node with a collection of $\mathrm{GCD}(2N-2,p)/2$ D_1 gauge nodes. We assume they are all identical. By analogy with ([37], section 5), we propose the following candidate structure for the mirror theory:

1. Keep the middle line of quiver (4.23) as it is.

2. Replace the $\left[D_{\frac{xp+p-2N+2}{2}}\right]$ flavor node with a collection of $\mathrm{GCD}(2N-2,p)/2$ D_1 gauge nodes.

3. Each D_1 group is connected to the node $C_{xp/2-(x-1)/2}$ with an edge of multiplicity

$$m_A \equiv \frac{(x+1)p-(2N-2)}{\mathrm{GCD}(2N-2,p)} \ . \tag{4.24}$$

4. Remove the $[C_{\frac{2N-2-xp}{2}}]$ from (4.23) and connect the $D_{N-1-(x-1)/2}$ gauge node to each of the D_1 gauge nodes in Step 2 by an edge with multiplicity

$$m_B \equiv \frac{(2N-2)-xp}{\mathrm{GCD}(2N-2,p)} \ . \tag{4.25}$$

5. Each pair of D_1 nodes is connected by an edge whose multiplicity is equal to

$$m_G \equiv m_A m_B = \frac{[(x+1)p-(2N-2)][(2N-2)-xp]}{\mathrm{GCD}(2N-2,p)^2} \ . \tag{4.26}$$

   These form a complete graph of $\mathrm{GCD}(2N-2,p)/2$ nodes, with all edge multiplicity equal to m_G.

6. There are

$$\frac{1}{2}(m_A-1)m_B \tag{4.27}$$

   hypermultiplets of charge 2 charged under each $\mathrm{U}(1) \cong D_1$ gauge group.

7. The quiver constructed above has an overall $\mathbb{Z}_2$ that needs to be decoupled.

Assuming there are no other ingredients, the dimension of the HB of the quiver matches the dimension of the CB of the 4d SCFT provided that the number of free hypermultiplets is

$$\begin{aligned} H_{\mathrm{free}} &= \frac{1}{4}(m_A-1)(m_B-1)\mathrm{GCD}(2N-2,p) \\ &= \frac{[(x+1)p-(2N-2)-\mathrm{GCD}(2N-2,p)]\,[(2N-2)-xp-\mathrm{GCD}(2N-2,p)]}{4\mathrm{GCD}(2N-2,p)} \ . \end{aligned} \tag{4.28}$$

It is worth pointing out that the above expression of H_{free} takes the same form as that for the $D_p(\mathrm{SU}(N))$ theory with $p \leq N$, where the latter is given by ([37], (5.5)). The only difference between the two expressions are the prefactors, where they are 1/4 in the former and 1/2 in the latter.

We now turn to the case x even. The procedure is very similar to that of the case x odd, with the roles of m_A and m_B interchanged. Explicitly, we start from the theory (4.22), whose quiver description is given by (4.15). For convenience, we reproduce it here again:

$$\begin{array}{ccccccccccccc} & & & & & & & & \left[D_{\frac{2N-2-xp}{2}}\right] & & & & \\ & & & & & & & & | & & & & \\ D_1 & - & C_{p/2} & - \cdots - & C_{(x-1)p/2-(x-2)/2} & - & D_{x(p-1)/2+1} & - & C_{\frac{2N-2-x}{2}} & - \cdots - & C_1 & - & D_1 \\ & & & & & & | & & & & & & \\ & & & & & & \left[C_{\frac{px+p-(2N-2)}{2}}\right] & & & & & & \end{array} \tag{4.29}$$

We then follow the subsequent steps:

1. Keep the middle line of quiver (4.29) as it is.

2. Replace the $\left[D_{\frac{2N-2-xp}{2}}\right]$ flavor node with a collection of $\mathrm{GCD}(2N-2,p)/2$ D_1 gauge nodes.

3. Each D_1 group is connected to the node $C_{(2N-2-x)/2}$ with an edge of multiplicity m_B.

4. Remove the $\left[C_{\frac{px+p-(2N-2)}{2}}\right]$ from (4.29) and connect the $D_{x(p-1)/2+1}$ gauge node to each of the D_1 gauge nodes in step 2 by an edge with multiplicity m_A.

5. Each pair of D_1 nodes is connected by an edge whose multiplicity is equal to m_G. These form a complete graph of $\mathrm{GCD}(2N-2,p)/2$ nodes, with all edge multiplicity equal to $m_G = m_A m_B$.

6. There are

$$\frac{1}{2}(m_B-1)m_A \tag{4.30}$$

hypermultiplets of charge 2 charged under each $\mathrm{U}(1) \cong D_1$ gauge group.

7. The quiver constructed above has an overall $\mathbb{Z}_2$ that needs to be decoupled.

There are also H_{free} free hypermultiplets, given by (4.28).

Examples.

- Let us consider the case in which p divides $2N-2$, so that

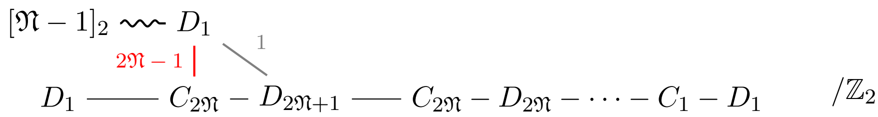
$$2N-2 = xp\,, \quad x \text{ is odd}\,. \tag{4.31}$$

The mirror theory for $D_p(\mathrm{SO}(xp+2))$ is then

$$\begin{array}{c} \overbrace{D_1 \ \cdots \ D_1}^{p/2 \text{ nodes}} \\ \diagdown \ \diagup \\ D_1 - C_{p/2} - \cdots - C_{\frac{xp-x+1}{2}} - D_{\frac{xp-x+1}{2}} - C_{\frac{px-x-1}{2}} - D_{\frac{px-x-1}{2}} - \cdots - C_1 - D_1 \quad /\mathbb{Z}_2 \end{array} \tag{4.32}$$

When $x=1$, namely $p = 2N-2$, we recover the quiver described in ([38], section 6.2) as expected.

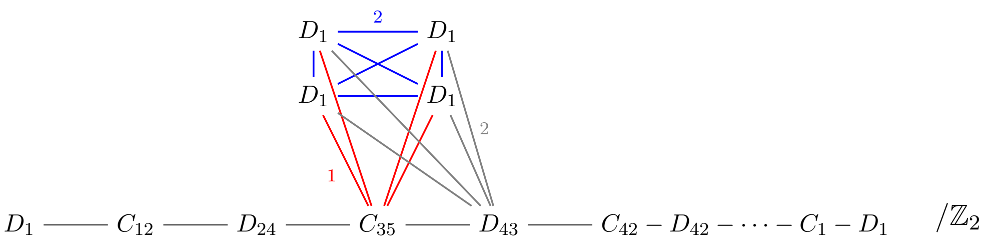
- The mirror theory of $D_{4\mathfrak{N}}(\mathrm{SO}(4\mathfrak{N}+4))$, for which we have $N = 2\mathfrak{N}+2$, $2N-2 = 4\mathfrak{N}+2$, $p = 4\mathfrak{N}$, $\mathrm{GCD}(2N-2,p) = 2$, $x = 1$, $m_A = 2\mathfrak{N}-1$, $m_B = 1$, is described by[8]



(4.33)

  There are no free hypermultiplets in this example.

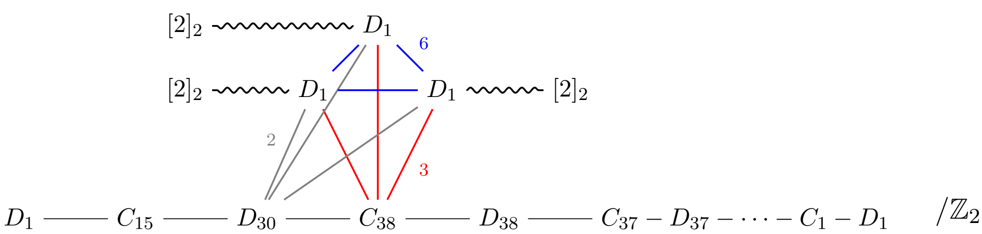
- The mirror theory of $D_{24}(\mathrm{SO}(90))$, for which we have $N = 45$, $2N-2 = 88$, $p = 24$, $\mathrm{GCD}(2N-2,p) = 8$, $x = 3$, $m_A = 1$, $m_B = 2$, is described by



(4.34)

  There are no free hypermultiplets in this example.

- The mirror theory of $D_{30}(\mathrm{SO}(80))$, for which we have $N = 40$, $2N-2 = 78$, $p = 30$, $\mathrm{GCD}(2N-2,p) = 6$, $x = 2$, $m_A = 2$, $m_B = 3$, is described by



(4.35)

  There are 3 free hypermultiplets in this example.

Consistency checks. We can perform a similar consistency check of the proposed mirror theory as in precedent subsections. Let us demonstrate this in two examples:

- **Let us take** $N = 8$ **and** $p = 10$. Here $x = 1$. The 4d theory has $24(c-a) = 281/5 = 56 + (1/5)$ and it has rank 35. The proposed mirror theory has CB dimension 56 and HB dimension 34. We thus have $35 - 34 = 1$ free hypermultiplets, in agreement with the fact that the non-Higgsable SCFT has $24(c-a)$ equal to $1/5$, which is expected to be that of the (A_1, A_2) theory whose rank is 1.

[8]This quiver has

$$\dim_{\mathbb{H}} \mathrm{HB}\,(4.33) = 4\mathfrak{N}^2 + 3\mathfrak{N} - 2\,,$$
$$\dim_{\mathbb{H}} \mathrm{CB}\,(4.33) = 4\mathfrak{N}^2 + 6\mathfrak{N} + 3\,,$$

which are equal to the CB and HB dimensions of the 4d $D_{4\mathfrak{N}}(\mathrm{SO}(4\mathfrak{N}+4))$ theory, respectively. Moreover, it has the CB symmetry $\mathrm{SO}(4\mathfrak{N}+4) \times \mathrm{SO}(2)^2$, which is the same as the flavor symmetry of the 4d theory.

- **Let us take** $N = 40$ **and** $p = 30$. Here $x = 2$. The 4d theory has $24(c-a) = 7658/5 = 1531 + (3/5)$ and it has rank 578. The proposed mirror theory has CB dimension 1531 and HB dimension 575. We thus have $578 - 575 = 3$ free hypermultiplets, in agreement with the fact that the non-Higgsable SCFT has $24(c-a)$ equal to 3/5, which is expected to be that of the $(A_1, A_2)^{\otimes 3}$ theory whose rank is 3.

5 The 3d mirror of (D_n, D_m) theories

In this section, we discuss the 3d mirror of (D_n, D_m) theories where, without loss of generality, we assume that $m \geq n$. It is convenient to arrange the discussion according to the number of mass parameters of the (D_n, D_m) theories. In terms of 3d mirror theories, such parameter corresponds to the rank of topological symmetry that arises from D_1 gauge nodes as well as the balanced C-type gauge nodes. All information regarding CB dimension, central charges, and number of mass parameters for all (D_n, D_m) theories can be computed, for instance, using the code given in [58].

5.1 Theories with 1 mass parameter

The (D_n, D_m) theories with one mass parameter have both n and m odd and either $\frac{n-1}{\mathrm{GCD}(n-1,m-1)}$ or $\frac{m-1}{\mathrm{GCD}(n-1,m-1)}$ is even; see (3.18). For definiteness, we take $m > n$. According to (3.26), such a theory has Higgs branch dimension $(n-1)/2$. Indeed, this is in agreement with the observation that the value of $24(c-a)$ of the 4d theory is at least $(n-1)/2$, with a possibility of an additional fractional number. The latter corresponds to the value of $24(c-a)$ of the non-Higgsable sector present on the Higgs branch of the 4d theory. On the other hand, the rank of the 4d theory in this class is $\frac{1}{2}(mn-1)$.

Since the number of mass parameters of the 4d theory (which is one) corresponds to the rank of the topological symmetry of the 3d mirror theory, we conclude that the latter should contain either one D_1 gauge group or one balanced C-type gauge group, but not both. Given that the Higgs branch dimension of the 4d theory is $(n-1)/2$, the 3d mirror must have the Coulomb branch dimension $(n-1)/2$. A more plausible option would be the latter. We thus propose that the mirror theory should be described by the $C_{(n-1)/2}$ SQCD with n flavors, namely

$$C_{(n-1)/2} - [D_n]\,, \tag{5.1}$$

together with

$$H_{\text{free}} = \frac{1}{2}(mn - n^2 + n - 1) \tag{5.2}$$

free hypermultiplets. This number of free hypermultiplets precisely coincides with the rank of the non-Higgsable SCFT along the Higgs branch of the 4d theory. In the next subsection, we provide a derivation of such a proposal for the $(D_{4\mathfrak{n}-1}, D_{4\mathfrak{m}+4\mathfrak{n}-3})$ theory, with $\mathfrak{m}, \mathfrak{n} \geq 1$.

Let us consider the special case of $n = 3$ and $m = 4\mathfrak{m}+1$, i.e. the $(D_3, D_{4\mathfrak{m}+1}) \cong (A_3, D_{4\mathfrak{m}+1})$ theory. The 3d mirror is given below ([38], (6.7)), with $\mu \to 2$, $\mathfrak{m} \to 1$, $\mathfrak{N} \to \mathfrak{m}$, and is described by the SQED with 4 flavors with $6\mathfrak{m}-2$ free hypermultiplets. This theory is indeed dual to the USp(2) SQCD with 3 flavors, given by (5.1), with $H_{\text{free}} = 6\mathfrak{m} - 2$ free hypermultiplets, given by (5.2), as expected.

In the special case that $m - n$ divides $m - 1$, we conjecture that the non-Higgsable SCFT can be identified as

$$\left(A_{m-n}, D_{\frac{m-1}{m-n}+n-1}\right) . \tag{5.3}$$

The value of $24(c-a)$ of this theory is precisely the difference between the value of $24(c-a)$ of the (D_n, D_m) in question and $(n-1)/2$. Moreover, such a non-Higgsable SCFT has rank equal to H_{free}, given by (5.2), as it should be. Let us demonstrate this in the example of $n = 7$ and $m = 9$, i.e. the (D_7, D_9) theory. The value of $24(c - a)$ is $23/7 = 3 + (2/7)$. According to (3.26), the Higgs branch dimension of the (D_7, D_9) theory is 3. Hence, the value of $24(c - a)$ of the non-Higgsable SCFT is $2/7$. We identify the latter as the (A_2, D_{10}) theory, as claimed in (5.3). This theory has rank 10, corresponding to 10 free hypermultiplets as proposed in (5.2).

If $m - n$ does not divide $m - 1$, the structure of non-Higgsable SCFT could be more complicated. Let us consider the example of the (D_5, D_{11}) theory. The value of $24(c - a)$ is $2 + (5/7)$ and the Higgs branch dimension is 2, so we expect the non-Higgsable SCFT to have $24(c - a)$ equal to $5/7$. Since (5.2) gives 17 free hypermultiplets, we expect also that the non-Higgsable SCFT has rank 17. One of the possibilities that fits these data is to identify the non-Higgsable SCFT in question as $(A_2, A_3) \otimes (A_2, D_4) \otimes (A_2, D_{10})$, although we cannot confirm this. We leave the identification of the non-Higgsable SCFT in this case as an open problem for future work.

5.1.1 Derivation of the 3d mirror for $(D_{4\mathfrak{n}-1}, D_{4\mathfrak{m}+4\mathfrak{n}-3})$

In this section we focus on a subclass of theories with one mass parameter, i.e. those with $n = 4\mathfrak{n} - 1$ and $m = 4\mathfrak{m} - 2 + n$, namely $(D_{4\mathfrak{n}-1}, D_{4\mathfrak{m}+4\mathfrak{n}-3})$, with $\mathfrak{m}, \mathfrak{n} \geq 1$.

Following the notation of (3.11), these theories correspond to $a = 2$, $k = 1$, $p = 2\mathfrak{n} - 1$, $q = 2\mathfrak{m} + 2\mathfrak{n} - 2$, and so

$$\alpha = 2\mathfrak{n} - 1\,, \quad \beta = 2\mathfrak{m} + 2\mathfrak{n} - 2\,. \tag{5.4}$$

Notice that $\beta - \alpha$ is odd and $\beta > \alpha$. According to (3.13), such a theory can be written as

$$(D_{4\mathfrak{n}-1}, D_{4\mathfrak{m}+4\mathfrak{n}-3}) = D_{2\mathfrak{m}+4\mathfrak{n}-3}\,(\mathrm{SO}(4\mathfrak{m} + 4\mathfrak{n} - 2)) - \mathrm{SO}(4\mathfrak{n}) - D_{2\mathfrak{m}+4\mathfrak{n}-3}\,(\mathrm{SO}(4\mathfrak{n}))\,. \tag{5.5}$$

where the full puncture of the theory on the left is partially closed to $[(2\mathfrak{m} - 1)^2, 1^{4\mathfrak{n}}]$.

The 3d mirror of $D_{2\mathfrak{m}+4\mathfrak{n}-3}\,(\mathrm{SO}(4\mathfrak{m} + 4\mathfrak{n} - 2))$ is described in section 4.1 with $x = 1$ and $X = \mathfrak{m}$, namely $T^{[2^{2\mathfrak{m}},1^{4\mathfrak{n}-2}]}_{[1^{4\mathfrak{m}+4\mathfrak{n}-2}]}[\mathrm{SO}(4\mathfrak{m}+4\mathfrak{n}-2)]$ with $2\mathfrak{m}(\mathfrak{n}-1)$ free hypermultiplets. The partial closure of the puncture leads to the mirror theory $T^{[2^{2\mathfrak{m}},1^{4\mathfrak{n}-2}]}_{[(2\mathfrak{m}-1)^2,1^{4\mathfrak{n}}]}[\mathrm{SO}(4\mathfrak{m} + 4\mathfrak{n} - 2]$, whose quiver is given by

$$\begin{array}{c} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [C_{\mathfrak{m}}] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad | \\ D_1 - C_1 - D_2 - C_2 - \cdots - C_{2\mathfrak{n}-1} - D_{2\mathfrak{n}} - C_{2\mathfrak{n}-1} - [D_{2\mathfrak{n}-1}] \end{array} \tag{5.6}$$

together with $2\mathfrak{m}(\mathfrak{n} - 1)$ free hypermultiplets.

On the other hand, the 3d mirror of $D_{2\mathfrak{m}+4\mathfrak{n}-3}(\mathrm{SO}(4\mathfrak{n}))$ is given by ([38], (5.4)), with $\mathfrak{m} \to \mathfrak{m}$, $N \to 2\mathfrak{n}$ and $\mu \to 1$. Explicitly, this is the $T[\mathrm{SO}(4\mathfrak{n})]$ theory, whose quiver is

$$D_1 - C_1 - \cdots - D_{2\mathfrak{n}-1} - C_{2\mathfrak{n}-1} - [D_{2\mathfrak{n}}] \,, \tag{5.7}$$

together with

$$H_{\text{free}} = 2\mathfrak{n}(\mathfrak{m} - 1) \tag{5.8}$$

free hypermultiplets.

We now gauge the common $\mathrm{SO}(4\mathfrak{n})$ Coulomb branch symmetry of the two aforementioned mirror theories, as indicated in (5.5). As a result, we obtain

$$[C_{\mathfrak{m}}] - [D_{2\mathfrak{n}}] - C_{2\mathfrak{n}-1} - [D_{2\mathfrak{n}-1}] \tag{5.9}$$

with $2\mathfrak{m}(\mathfrak{n}-1) + 2\mathfrak{n}(\mathfrak{m}-1)$ free hypermultiplets. This theory can be rewritten as

$$C_{2\mathfrak{n}-1} - [D_{4\mathfrak{n}-1}] \tag{5.10}$$

with $2\mathfrak{m}(\mathfrak{n}-1) + 2\mathfrak{n}(\mathfrak{m}-1) + 4\mathfrak{m}\mathfrak{n} = 2(4\mathfrak{m}\mathfrak{n} - \mathfrak{m} - \mathfrak{n})$ free hypermultiplets. This is in agreement as the proposal (5.1) and (5.2), as it should be.

5.2 Theories with 2 mass parameters

According to (3.18), each (D_n, D_m) theory with two mass parameters falls into one of the two categories: either n is even and m is odd, or n is odd and m is even. In the following discussion, we assume that $m > n$.

Let us first discuss the case of n even and m odd. We propose that the mirror theory is

$$T_{[(n-1)^2,1^2]}[\mathrm{SO}(2n)] : \quad [D_n] - C_{n/2} - D_1 \tag{5.11}$$

together with the following number of free hypermultiplets:

$$H_{\text{free}} = \frac{1}{2} n(m - n - 1) \;. \tag{5.12}$$

On the other hand, for n odd and m even, we propose that the mirror theory is

$$[C_F] - D_1 - C_{(n-1)/2} - [D_{n-1}] \tag{5.13}$$

with

$$F = \frac{1}{2}(m - n + 1) \,, \tag{5.14}$$

together with the following number of free hypermultiplets:

$$H_{\text{free}} = (n-2)F = \frac{1}{2}(n-2)(m-n+1) \;. \tag{5.15}$$

In each case, the two mass parameters of the 4d theory correspond to two topological symmetries of the 3d mirror theory; one arises from the D_1 gauge group and the other arises from the balanced C-type gauge group. It can be checked that the Coulomb branch dimension of the mirror theory is in agreement with the Higgs branch of the 4d theory given by (3.26), and that the Higgs branch dimension of the mirror theory is in agreement with the rank of the 4d theory. Moreover, the difference between the value of $24(c-a)$ and the Higgs branch dimension of the 4d theory is in agreement with the value of $24(c-a)$ of the non-Higgsable SCFT, whose rank is in agreement with H_{free} in each case. As an example, we find that for the $(D_{2\mathfrak{n}+2}, D_{4\mathfrak{n}+3})$ theories, with $\mathfrak{n} \geq 1$, the non-Higgsable SCFT can be identified as $(A_{2\mathfrak{n}}, D_{2\mathfrak{n}+2})$, whose central charges satisfy $a = c$ and whose rank is equal to the number of free hypermultiplets, namely $2\mathfrak{n}(\mathfrak{n}+1)$, as expected.

5.2.1 The special case of $(D_3, D_{2\mathfrak{n}+2}) \cong (A_3, D_{2\mathfrak{n}+2})$ theories

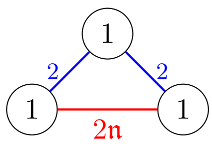
This is a special case where $n = 3$ and $m = 2\mathfrak{n}+2$. In this case, from (5.1) and (5.2), the mirror theory of $(D_3, D_{2\mathfrak{n}+2}) \cong (A_3, D_{2\mathfrak{n}+2})$ is described by

$$\begin{aligned} &[C_{\mathfrak{n}}] - D_1 - C_1 - [D_2] \\ &\text{with } H_{\text{free}} = \mathfrak{n} \text{ hypermultiplets} \ . \end{aligned} \tag{5.16}$$

However, as discussed in [38], each theory in this class can be obtained from closing the maximal puncture of the $D^{4\mathfrak{n}+2}_{4\mathfrak{n}+6}(\mathrm{SO}(4\mathfrak{n}+4))$ theory. The mirror theory for $(D_3, D_{2\mathfrak{n}+2}) = (A_3, D_{2\mathfrak{n}+2})$ admits the following two descriptions. One is discussed explicitly in ([38], section 6.1.2), namely

$$\begin{aligned} &D_1 \overset{2}{\text{———}} D_1 \rightsquigarrow [2\mathfrak{n}]_2 \qquad /\mathbb{Z}_2 \\ &+ \mathfrak{n} \text{ free hypermultiplets} \end{aligned} \tag{5.17}$$

The other description is a simple modification of ([38], (4.17)), namely



+ $\mathfrak{n}$ free hypermultiplets (5.18)

The $\mathfrak{n}$ free hypermultiplets arise from the non-Higgsable SCFT

$$(A_1, A_{2\mathfrak{n}}) \ . \tag{5.19}$$

We thus have an isomorphism between the following three theories:

$$[C_{\mathfrak{n}}] - D_1 - C_1 - [D_2] \tag{5.20}$$

$$D_1 \overset{2}{\text{———}} D_1 \rightsquigarrow [2\mathfrak{n}]_2 \qquad /\mathbb{Z}_2 \tag{5.21}$$

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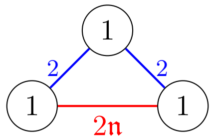

(5.22)

It can be checked, for example using the Hilbert series,[9] that the Higgs branches and the Coulomb branches of the three theories agree with each other.

5.2.2 Derivation of the 3d mirror of $(D_{11}, D_{10\mathfrak{m}+6})$

Let us consider the $(D_{11}, D_{10\mathfrak{m}+6})$ theory, with $\mathfrak{m} \geq 1$. Such a theory admits the following decomposition:

$$(D_{11}, D_{10\mathfrak{m}+6}) = D_{6\mathfrak{m}+9}(\mathrm{SO}(12\mathfrak{m}+8)) - \mathrm{SO}(10) - D_{4\mathfrak{m}+6}(\mathrm{SO}(10))\,, \tag{5.23}$$

where $D_{6\mathfrak{m}+9}(\mathrm{SO}(12\mathfrak{m}+8))$ has a partially closed regular puncture labeled by the partition $[(6\mathfrak{m}-1)^2, 1^{10}]$.

The mirror of $D_{4\mathfrak{m}+6}(\mathrm{SO}(10))$ has been found in ([38], (6.7)), with $\mu = 1$ and $\mathfrak{N} = 2$, to be $T^\rho_\sigma[\mathrm{SO}(8\mathfrak{m}+20)]$ with

$$\rho = \left[3^8, 2^{4\mathfrak{m}-2}\right]\,, \quad \sigma = \left[(4\mathfrak{m}+5)^2, 1^{10}\right]\,, \tag{5.24}$$

whose quiver description is

$$\begin{array}{ccccccccccccccccc} D_1 & - & C_1 & - & D_2 & - & C_2 & - & D_3 & - & C_3 & - & D_4 & - & C_4 & - & D_1 \\ &&&&&&&&&&&&&& | && | \\ &&&&&&&&&&&&&& [D_4] && [C_{2\mathfrak{m}-1}] \end{array} \tag{5.25}$$

together with $H_{\text{free}} = 6\mathfrak{m} - 5$ free hypermultiplets.

The mirror of $D_{6\mathfrak{m}+9}(\mathrm{SO}(12\mathfrak{m}+8))$ has been found using the procedure in section 4.1 to be $T^\sigma[\mathrm{SO}(12\mathfrak{m}+8)]$ with $\sigma = [2^{6\mathfrak{m}-2}, 1^{12}]$ and $H_{\text{free}} = 15\mathfrak{m} - 5$ free hypermultiplets. Partially closing the full puncture to $[(6\mathfrak{m}-1)^2, 1^{10}]$ yields the mirror theory $T^{[2^{6\mathfrak{m}-2},1^{12}]}_{\rho'}[\mathrm{SO}(12\mathfrak{m}+8)]$ where $\rho' = [(6\mathfrak{m}-1)^2, 1^{10}]$, whose quiver description is

$$\begin{array}{ccccccccccccccccccc} D_1 & - & C_1 & - & D_2 & - & C_2 & - & D_3 & - & C_3 & - & D_4 & - & C_4 & - & D_5 & - & C_5 \\ &&&&&&&&&&&&&&&& | && | \\ &&&&&&&&&&&&&&&& [C_{3\mathfrak{m}-1}] && [D_6] \end{array} \tag{5.26}$$

[9]In general, the Higgs branch symmetry of each of these three theories is $\mathrm{U}(2\mathfrak{n}) \times \mathrm{SO}(4)$, and the Coulomb branch symmetry of each theory is $\mathrm{SO}(2)^2$. In the duality frame (5.20), the adjoint representation of the $\mathrm{U}(2\mathfrak{n})$ factor of the Higgs branch symmetry arises from the representation $[0,\ldots,0] \oplus [2,0,\ldots,0] \oplus [0,1,0,\ldots,0]$ of the $C_\mathfrak{n}$ flavour symmetry. As an example, for $\mathfrak{n} = 3$, the unrefined Coulomb branch Hilbert series of these three theories reads

$$1 + 2t^2 + 5t^4 + 8t^6 + 17t^8 + 26t^{10} + 41t^{12} + \ldots\,,$$

where 2 is the dimension of Coulomb branch symmetry $\mathrm{SO}(2)^2$. The unrefined Higgs branch Hilbert series of these three theories reads

$$1 + 42t^2 + 48t^3 + 676t^4 + 1200t^5 + 6888t^6 + 13920t^7 + 52048t^8 + \ldots\,,$$

where 42 is the dimension of the Higgs branch symmetry $\mathrm{U}(6) \times \mathrm{SO}(4)$. These computations can be performed as shown in ([38], appendix A).

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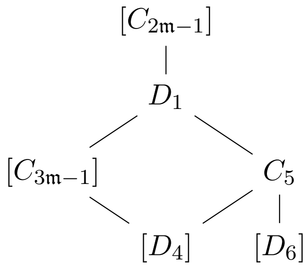
Finally, the mirror of $(D_{11}, D_{10\mathfrak{m}+6})$ is obtained gauging the common SO(10) Coulomb branch symmetry between (5.25) and (5.26). This can be done by fusing the tails of the two theories together and splitting the D_5 node in the latter into D_1 and $[D_4]$, we obtain the following result



(5.27)

together with $(6\mathfrak{m}-5)+(15\mathfrak{m}-5) = 21\mathfrak{m}-10$ free hypermultiplets. This can be rewritten as[10]

$$[C_{5\mathfrak{m}-2}] - D_1 - C_5 - [D_{10}] \tag{5.28}$$

with $H_{\text{free}} = (24\mathfrak{m}-8)+(21\mathfrak{m}-10) = 45\mathfrak{m}-18 = 9(5\mathfrak{m}-2)$ free hypermultiplets.

5.3 Theories with $2M+1$ mass parameters, with $M \geq 1$

In this section, we focus on the (D_n, D_m) theories with $2M+1$ mass parameters, where $M \geq 1$. They can be parametrized as

$$\begin{aligned} &n = 4M\mathfrak{n} - (2M-1)\,, \quad m = n + 4M\mathfrak{m} \\ &\text{GCD}(2\mathfrak{n}-1, 2\mathfrak{m}) = 1\ . \end{aligned} \tag{5.29}$$

where $\mathfrak{n}, \mathfrak{m} \geq 1$ and we allow $\mathfrak{m}$ and $\mathfrak{n}$ to also take the values $(\mathfrak{m}=0, \mathfrak{n}=1)$. We first discuss the latter case and then proceed with the general case.

5.3.1 The (D_{2M+1}, D_{2M+1}) theory

Let us consider the special case $\mathfrak{m}=0$ and $\mathfrak{n}=1$, namely, the (D_{2M+1}, D_{2M+1}) theory. It has rank $M(2M+1)$. The value of $24(c-a)$, which is also equal to the Higgs branch dimension, is $3M$. There are $2M+1$ mass parameters and $2M-1$ marginal operators. The central charges are:

$$a = \frac{1}{24}M(4M+1)(4M+5)\,, \quad c = \frac{1}{3}M(M+1)(2M+1)\,. \tag{5.30}$$

This theory can also be realized using (3.17) with $a = 2M$, $k = M$, $p = q = 1$ and $\alpha = \beta = M$:

$$\begin{array}{c} \hspace{1.5cm}[\text{SO}(2)] \\ \hspace{1.5cm}| \\ (D_{2M+1}, D_{2M+1}) = D_{2M}(\text{USp}(2M)) \; - \; \text{USp}(2M) \; - \; D_{2M}(\text{USp}(2M)) \end{array} \tag{5.31}$$

where $[\text{SO}(2)]$ means a *full hypermultiplet* in the fundamental of $\text{USp}(2M)$.

[10]This quiver has

$$\dim_{\mathbb{H}}\ \text{CB} = 6\,, \quad \dim_{\mathbb{H}}\ \text{HB} = 10\mathfrak{m} + 50 + H_{\text{free}} = 55\mathfrak{m} + 32\ .$$

They agree with the Higgs branch dimension and the rank of the 4d theory in question, respectively.

We, in fact, observe that the (D_{2M+1}, D_{2M+1}) also admits the following description:

$$\begin{aligned}&\text{a class } \mathcal{S} \text{ theory associated with the } A_{2M} \text{ twisted sphere}\\ &\text{with } 2N \text{ minimal untwisted punctures (each labelled by } [2M,1]),\\ &\text{and 2 minimal twisted punctures (each labelled by } [2M]_t),\end{aligned} \tag{5.32}$$

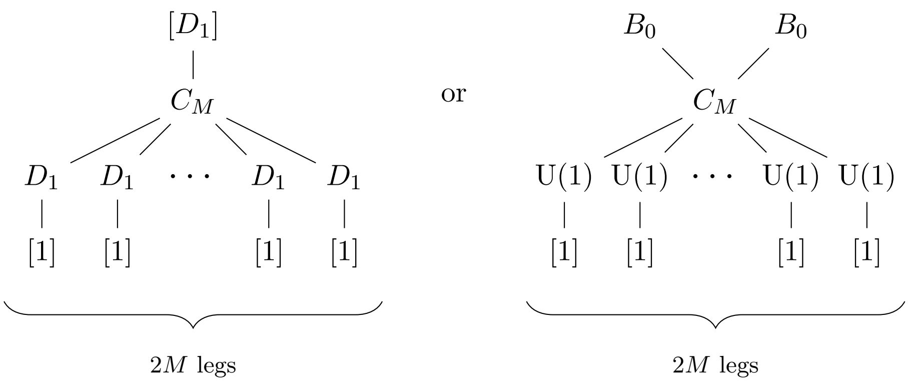
It can indeed be checked that the (a, c) central charges[11] of the two theories are equal. This provides a non-trivial test for the following duality:

$$(5.31) \longleftrightarrow (5.32)\ . \tag{5.33}$$

It was pointed out in ([36], (6.16)) that, upon reduction to 3d, the (D_{2M+1}, D_{2M+1}) theory also admits the following quiver description with mixed unitary and special unitary gauge groups:

$$\begin{array}{ccccccccccccc} & & & & \mathrm{SU}(1) & & & & & & & & \\ & & & & | & & & & & & & & \\ & & & & \mathrm{U}(M) & & & & & & & & \\ & & & & | & & & & & & & & \\ \mathrm{SU}(1) & - & \mathrm{U}(M) & - & \mathrm{SU}(2M) & - & \mathrm{SU}(2M-1) & - & \cdots & - & \mathrm{SU}(2) & - & \mathrm{SU}(1) \end{array} \tag{5.34}$$

Since the class $\mathcal{S}$ description of the (D_{2M+1}, D_{2M+1}) theory is known to be (5.32), we can apply the prescription provided in [59, 60] to find the mirror theory, which can be described as follows:



(5.35)

Note that the quivers on the left and on the right are equivalent. Each component in the right quiver in (5.35) comes from

$$\begin{aligned} T_{[2M,1]}[\mathrm{SU}(2M+1)] :&\quad \mathrm{U}(1) - [2M+1] \quad \text{or} \quad [2M] - \mathrm{U}(1) - [1] \\ T_{[2M]}[\mathrm{USp}(2M)] :&\quad [C_M] - B_0 \end{aligned} \tag{5.36}$$

[11]The contribution to the effective number of hypermultiplets and vector multiplets (n_h, n_v) of the punctures $[2M, 1]$ and $[2M]_t$ are respectively $\left((2M+1)^2,\ (2M+2)(2M)\right)$ and $\left(\frac{2}{3}M\left(4M^2+9M+5\right), \frac{1}{6}M\left(16M^2+36M+23\right)\right)$. Adding these together with the contribution of the A_{2N} sphere $\left(-\frac{4}{3}(2M+1)[(2M+1)^2-1], -\frac{4}{3}(2M+1)[(2M+1)^2-1]-2M\right)$, we obtain $(n_h, n_v) = \left(\frac{2}{3}M\left(4M^2+6M+5\right), \frac{1}{3}M\left(8M^2+12M+1\right)\right)$ of (5.32). Using the relation $(a, c) = \left(\frac{2n_v+n_h}{12}, \frac{5n_v+n_h}{24}\right)$, we obtain the central charges of (5.32) to be $(a, c) = \left(\frac{1}{24}M(4M+1)(4M+5), \frac{1}{3}M(M+1)(2M+1)\right)$. This is indeed equal to those of the (D_{2M+1}, D_{2M+1}) theory; see (5.30).

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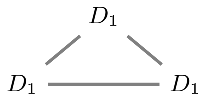
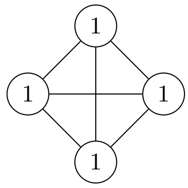
such that the C_M symmetry from each theory is commonly gauged. In conclusion, we have found a new mirror pair, namely

$$(5.34) \overset{\text{mirror}}{\longleftrightarrow} (5.35)\ . \tag{5.37}$$

Note that for $M = 1$, (5.37) is self-mirror. However, due to ([60], section 4.1) and ([38], (4.16) with $\mathfrak{m} = 1$), we have the following dual descriptions:

$(5.35)_{M=1} \longleftrightarrow$  $/\mathbb{Z}_2 \longleftrightarrow$  (5.38)

5.3.2 General result: the 3d mirror for theories with $2M + 1$ mass parameters

Based on (5.35), we now propose a prescription to construct the 3d mirror for the (D_n, D_m) theories with $2M + 1$ mass parameters.

Let us adopt the parametrization (5.29). The quiver description for the mirror theory in question contains the balanced $C_{(n-1)/2} = C_{M(2\mathfrak{n}-1)}$ central gauge node connected to one flavor $[D_1]$ node and to $2M$ D_1 gauge nodes in the following way.

1. Connect the $C_{M(2\mathfrak{n}-1)}$ central node to each D_1 gauge node with a red line with multiplicity $2\mathfrak{n} - 1$.

2. Connect each pair of D_1 gauge nodes by a blue edge with multiplicity $\mathfrak{m}(2\mathfrak{n} - 1)$. These form a complete graph with $2M$ nodes such that each node is connected by a blue line.

3. Each D_1 gauge group in the complete graph has

$$F = \mathfrak{m}(\mathfrak{n} + 1) + (2\mathfrak{n} - 1) \tag{5.39}$$

hypermultiplets with charge 1 under the corresponding $\mathrm{U}(1) \cong D_1$ gauge group.

4. There are

$$H_{\text{free}} = 2M(\mathfrak{m} - 1)(\mathfrak{n} - 1) \tag{5.40}$$

free hypermultiplets. We conjecture that the non-Higgsable SCFTs are

$$(A_{\mathfrak{m}-1}, A_{2\mathfrak{n}-2})^{\otimes(2M)}\ . \tag{5.41}$$

The quiver that we just described has

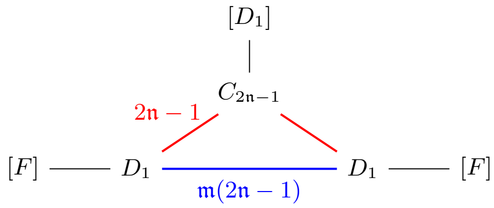
$$\begin{aligned} \dim_{\mathbb{H}} \text{CB} &= M + 2M\mathfrak{n}\,, \\ \dim_{\mathbb{H}} \text{HB} &= M(2M - 5) - [4M(M - 1)]\mathfrak{m} - [2M(4M - 3)]\mathfrak{n} \\ &\quad + [2M(4M - 1)]\mathfrak{m}\mathfrak{n} + (8M^2)\mathfrak{n}^2 + H_{\text{free}}\ . \end{aligned} \tag{5.42}$$

These are indeed in agreement with the Higgs branch dimension and the rank of the 4d theory, respectively. Moreover, the value of $24(c-a)$ of the aforementioned non-Higgsable SCFTs plus the above $\dim_{\mathbb{H}}$ CB indeed gives the value of $24(c-a)$ of the corresponding 4d theory, as it should be. The $2M+1$ mass parameters in 4d theory corresponds to the $2M$ FI parameters associated with $2M$ D_1 gauge groups, and one hidden FI parameter [42] associated with the balanced $C_{M(2\mathfrak{n}-1)}$ gauge group.

Let us now consider this quiver theory in various special cases. Some of these also serve as a non-trivial test of the above proposal.

5.3.3 Example: $M = 1$, i.e. three mass parameters

We parametrize the theory in this class as $(D_{4\mathfrak{n}-1}, D_{4\mathfrak{n}-1+4\mathfrak{m}})$ with $\mathrm{GCD}(2\mathfrak{n}-1, 2\mathfrak{m}) = 1$. The proposed mirror theory is



(5.43)

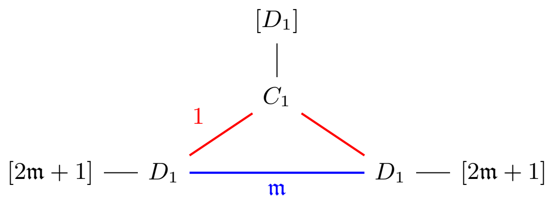
with $F = \mathfrak{m}(\mathfrak{n}+1) + (2\mathfrak{n}-1)$

together with

$$H_{\text{free}} = 2(\mathfrak{m}-1)(\mathfrak{n}-1) \tag{5.44}$$

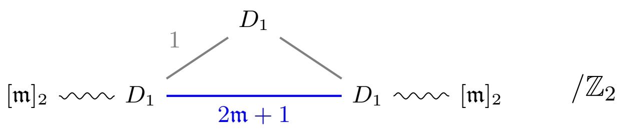
free hypermultiplets. We emphasize that $[F]$ denotes F hypermultiplets of charge 1 under the corresponding $\mathrm{U}(1) \cong D_1$ in the quiver.

The special case of $\mathfrak{n} = 1$, namely the $(D_3, D_{4\mathfrak{m}+3}) = (A_3, D_{4\mathfrak{m}+3})$, is particularly interesting. In this case, (5.43) reduces to



(5.45)

In fact, there is another description for the mirror theory for $(A_3, D_{4\mathfrak{m}+3})$, given by ([38], (6.49))[12] with $\mathfrak{m}$ and $\mathfrak{N}$ in that reference set to 1 and $\mathfrak{m}+1$ respectively:



(5.46)

[12]There is a minor typo in ([38], (6.49)) (version 2). The correction should be as follows: the blue edge with multiplicity $M = \mathfrak{m}(2\mathfrak{N}-1)$ should be in between two D_1 nodes attached to the wiggle lines, whereas the remaining edges should be gray with multiplicity $\mathfrak{m}$.

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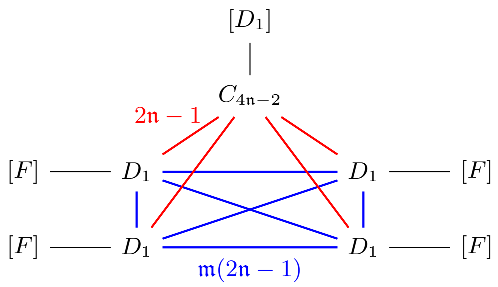
where $[\mathfrak{m}]_2$ attached to the wiggle line denote $\mathfrak{m}$ hypermultiplets with charge 2 under the corresponding $\mathrm{U}(1) = D_1$ node. Note that (5.45) and (5.46) have the same Coulomb branch dimension and the same Higgs branch dimension. The Higgs branch symmetry of each of these theories is $\mathrm{U}(2\mathfrak{m}+1)^2 \times \mathrm{U}(\mathfrak{m})^2 \times \mathrm{U}(1)$, as it is manifest in each description. The Coulomb branch symmetry of each theory is $\mathrm{U}(1)^3$; in (5.45) this arises from the U(1) topological symmetry of each of the two D_1 nodes and the emergent U(1) topological symmetry of the balanced C_1 node [43], whereas in (5.46) such as Coulomb branch symmetry arises from the U(1) topological symmetry of each of the three D_1 nodes. It can also be checked that the Coulomb branch Hilbert series and the Higgs branch Hilbert series of these two theories are equal.[13] We thus claim the duality:

$$(5.45) \quad \longleftrightarrow \quad (5.46)\ . \tag{5.47}$$

This also provides a non-trivial check of the proposal (5.43).

5.3.4 Example: $M = 2$, i.e. five mass parameters

Each theory in this class can be written as $(D_{8\mathfrak{n}-3}, D_{8\mathfrak{n}-3+8\mathfrak{m}})$ with $\mathrm{GCD}(2\mathfrak{n}-1, 2\mathfrak{m}) = 1$. The proposed mirror theory is



(5.48)

with $F = \mathfrak{m}(\mathfrak{n}+1) + (2\mathfrak{n}-1)$

together with

$$H_{\text{free}} = 4(\mathfrak{m}-1)(\mathfrak{n}-1) \tag{5.49}$$

free hypermultiplets.

Let us provide a check for the number of free hypermultiplets via the example of $\mathfrak{n} = 2$ and $\mathfrak{m} = 4$, i.e. the (D_{13}, D_{45}) theory. The value of $24(c-a)$ of this theory is

[13] As an example, for $\mathfrak{m} = 2$, the unrefined Higgs branch Hilbert series of (5.45) and (5.46) are

$$1 + 59t^2 + 248t^3 + 2070t^4 + 10440t^5 + 54650t^6 + \dots,$$

where 59 is the dimension of the Higgs branch symmetry $\mathrm{U}(5)^2 \times \mathrm{U}(2)^2 \times \mathrm{U}(1)$, and the Coulomb branch Hilbert series of both theories are

$$1 + 3t^2 + 8t^4 + 16t^6 + 29t^8 + 47t^{10} + \dots,$$

where 3 is the dimension of the Coulomb branch symmetry $\mathrm{U}(1)^3$. These computations can be performed as shown in ([38], appendix A).

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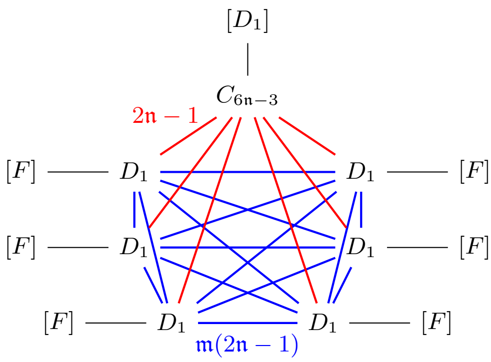
$82/7 = 10 + (12/7)$, where 10 is the Higgs branch dimension of the 4d theory, according to (3.26), which is in agreement with the Coulomb branch dimension of the mirror theory. The fraction $12/7$ can be identified as the value of $24(c - a)$ of the non-Higgsable SCFT $(A_2, A_3)^{\otimes 4} \cong (A_3, A_2)^{\otimes 4}$, in agreement with the claim (5.41). Since this theory has rank 12, we expect to have 12 free hypermultiplets, as stated above.

5.3.5 Example: $M = 3$, i.e. seven mass parameters

Each theory in this class can be written as $(D_{12\mathfrak{n}-5}, D_{12\mathfrak{n}-5+12\mathfrak{m}})$ with $\mathrm{GCD}(2\mathfrak{n}-1, 2\mathfrak{m}) = 1$. The proposed mirror theory is



$$\text{with } F = \mathfrak{m}(\mathfrak{n}+1) + (2\mathfrak{n}-1) \tag{5.50}$$

together with

$$H_{\text{free}} = 6(\mathfrak{m}-1)(\mathfrak{n}-1) \tag{5.51}$$

free hypermultiplets.

Let us provide a check for the number of free hypermultiplets via the example of $\mathfrak{n} = 2$ and $\mathfrak{m} = 2$, i.e. the (D_{19}, D_{43}) theory. The value of $24(c - a)$ of this theory is $81/5 = 15 + (6/5)$, where 15 is the Higgs branch dimension of the 4d theory, according to (3.26), which is in agreement with the Coulomb branch dimension of the mirror theory. The fraction $6/5$ can be identified as the value of $24(c - a)$ of the non-Higgsable SCFT $(A_1, A_2)^{\otimes 6}$, in agreement with the claim (5.41). Since this theory has rank 6, we expect to have 6 free hypermultiplets, as stated above.

5.4 Theories with $2M + 2$ mass parameters, with $M \geq 1$

A theory in this class can be written as (D_n, D_m), such that

$$\begin{aligned} &n = (4M-2)\mathfrak{n} - (2M-2)\,, \quad m = n + (4M-2)\mathfrak{m}\,, \\ &\mathrm{GCD}(2\mathfrak{n}-1, 2\mathfrak{m}) = 1 \end{aligned} \tag{5.52}$$

where $\mathfrak{n}, \mathfrak{m} \geq 1$ and we allow $\mathfrak{m}$ and $\mathfrak{n}$ to also take the values $(\mathfrak{m} = 0, \mathfrak{n} = 1)$. We first discuss the latter case and then proceed with the general case.

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5.4.1 The (D_{2M}, D_{2M}) theory

Let us focus on the case of $\mathfrak{m} = 0$ and $\mathfrak{n} = 1$, i.e. the (D_{2M}, D_{2M}) theory. As pointed out in ([36], (6.6)), this theory admits the following Lagrangian description

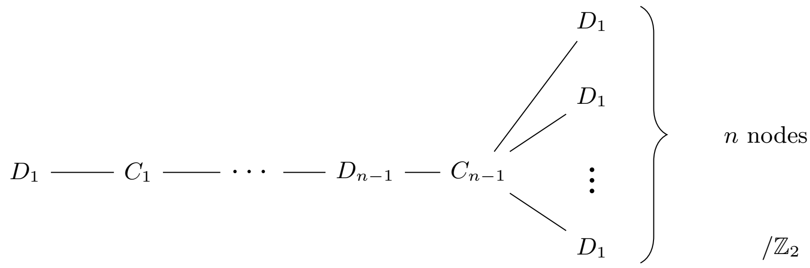
$$\begin{array}{c} \mathrm{SU}(1) \\ | \\ \mathrm{SU}(M) \\ | \\ \mathrm{SU}(1) - \mathrm{SU}(M) - \mathrm{SU}(2M-1) - \mathrm{SU}(2M-2) - \cdots - \mathrm{SU}(2) - \mathrm{SU}(1) \end{array} \tag{5.53}$$

where each SU(1) should be treated as one flavor of hypermultiplets transforming under the fundamental representation of the node next to it. It can be checked that the Coulomb branch spectrum, the rank, the Higgs branch dimension, and the (a, c) central charges of this quiver theory[14] match perfectly with those of the (D_{2M}, D_{2M}) theory.

According to section 3, with $a = 2M - 1$, $p = q = 1$, $k = M - 1$, $\alpha = M - 1$, $\beta = M$, there is another description of the (D_{2M}, D_{2M}) theory, namely that given by (3.13):

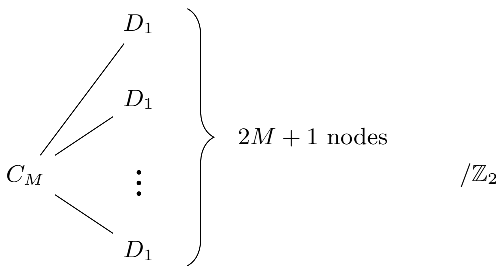
$$(D_{2M}, D_{2M}) = D_{2M}(\mathrm{SO}(2M + 2)) - \mathrm{SO}(2M) - D_{2M-2}(\mathrm{SO}(2M)) \ . \tag{5.55}$$

We can use this description to find the mirror theory for the (D_{2M}, D_{2M}) theory. Recall that the mirror theory of $D_{2n-2}(\mathrm{SO}(2n))$ is described by ([38], section 6.2), namely



(5.56)

Considering such a theory for $n = M$ and for $n = M+1$ and gauging the common Coulomb branch symmetry $\mathrm{SO}(2M)$, as indicated in (5.55), by fusing the two quiver tails together, we obtain the following mirror theory for (D_{2M}, D_{2M}):



(5.57)

[14] The central charges are

$$a = \frac{1}{24}\left(16M^3 - 7M - 5\right) \ , \quad c = \frac{1}{6}\left(4M^3 - M - 1\right) \ . \tag{5.54}$$

The Coulomb branch dimension (i.e. the rank) is $(2M + 1)(M - 1)$, while the Higgs branch dimension is $3M + 1$. There are $2M + 2$ masses and $2M$ marginal operators.

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The $2M+2$ mass parameters of the 4d theory corresponds to one hidden FI parameter of the balanced C_M gauge node [42] and $2M+1$ FI parameters associated with each D_1 gauge node.

Since each gauge node in the quiver (5.53) has zero beta-function, the reduction to 3d yields a 3d $\mathcal{N}=4$ gauge theory with the same quiver description [36, 37]. We have thus established a new mirror pair, namely

$$(5.53)_{3d} \overset{\text{mirror}}{\longleftrightarrow} (5.57)\ . \tag{5.58}$$

5.4.2 General result: the 3d mirror for theories with $2M+2$ mass parameters

We now provide a prescription to construct the quiver description for the corresponding 3d mirror theory as follows. The quiver in question contain the balanced $C_{n/2}$ central gauge node connected to $2M+1$ D_1 gauge nodes in the following way.

1. Choose any two of the D_1 nodes (call them B_1 and B_2). Connect each of them to the central $C_{n/2}$ node with a black line with multiplicity 1.
2. Connect each of the rest of the D_1 nodes (call them A_1, A_2, $\cdots$, A_{2M-1}) to the central $C_{n/2}$ node with a red line with multiplicity $2\mathfrak{n}-1$.
3. Connect any two A_i and A_j nodes with a blue line with multiplicity $\mathfrak{m}(2\mathfrak{n}-1)$. These form a complete graph with $2M-1$ nodes such that each node is connected by a blue line.
4. Each of the A_i nodes has $\mathfrak{m}(\mathfrak{n}-1)$ flavors of hypermultiplets with carrying 2 under $\mathrm{U}(1)=D_1$. Each of the B_i nodes has no flavor charged under it.
5. Connect the node B_1 to each of the A_1, A_2, $\cdots$, A_{2M-1} nodes by a gray line with multiplicity $\mathfrak{m}$.
6. Quotient the above theory by an overall $\mathbb{Z}_2$ symmetry.
7. There are

$$H_{\text{free}} = (2M-1)(\mathfrak{m}-1)(\mathfrak{n}-1) \tag{5.59}$$

free hypermultiplets. We conjecture that the non-Higgsable SCFTs are

$$(A_{\mathfrak{m}-1}, A_{2\mathfrak{n}-2})^{\otimes(2M-1)}\ . \tag{5.60}$$

Note that the above procedure is very similar to that described in ([38], section 6.1.2). The quiver that we just described has

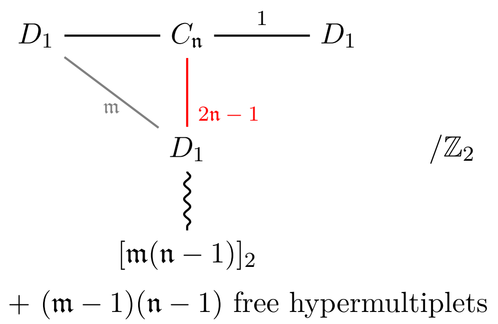
$$\begin{aligned}
\dim_{\mathbb{H}} \mathrm{CB} &= \mathfrak{n}(2M-1)+M+2\,,\\
\dim_{\mathbb{H}} \mathrm{HB} &= \mathfrak{m}(2M-1)[(M+1)(4\mathfrak{n}-2)-7\mathfrak{n}+5]+2(M+1)^2(1-2\mathfrak{n})^2\\
&\quad +(M+1)[6(5-4\mathfrak{n})\mathfrak{n}-11]+9\mathfrak{n}(2\mathfrak{n}-3)+11+H_{\text{free}}\\
&= 2\mathfrak{m}(2M-3)[(M+1)(2\mathfrak{n}-1)-3\mathfrak{n}+2]+2(M+1)^2(1-2\mathfrak{n})^2\\
&\quad +(M+1)[4(7-6\mathfrak{n})\mathfrak{n}-9]+2(2-3\mathfrak{n})^2\,.
\end{aligned} \tag{5.61}$$

These quantities are indeed in agreement with the Higgs branch dimension and the rank of the 4d theory, respectively. Moreover, the value of $24(c-a)$ of the aforementioned non-Higgsable SCFTs plus the above $\dim_{\mathbb{H}}$ CB indeed gives the value of $24(c-a)$ of the corresponding 4d theory, as it should be. The $2M+2$ mass parameters in 4d theory corresponds to the $2M+1$ FI parameters associated with $2M+1$ D_1 gauge groups, and one hidden FI parameter associated with the balanced $C_{n/2}$ gauge group.

Let us now consider this quiver theory in various special cases. Some of these also serve as a non-trivial test of the above proposal.

5.4.3 Example: $M=1$, i.e. four mass parameters

The mirror theory for $(D_{2\mathfrak{n}}, D_{2\mathfrak{n}+2\mathfrak{m}})$ with $\mathrm{GCD}(2\mathfrak{n}-1, 2\mathfrak{m}) = 1$ is



(5.62)

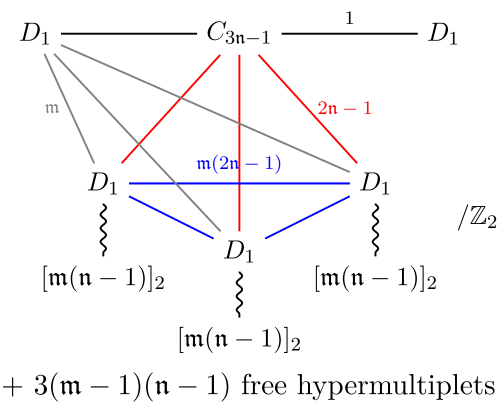
This theory has

$$\begin{aligned} \dim_{\mathbb{H}} \mathrm{HB}\,(5.62) &= \mathfrak{m}\mathfrak{n} + \mathfrak{m} + 2\mathfrak{n}^2 + \mathfrak{n} - 3 + H_{\mathrm{free}} = 2\mathfrak{n}(\mathfrak{m}+\mathfrak{n}) - 2 \\ \dim_{\mathbb{H}} \mathrm{CB}\,(5.62) &= \mathfrak{n} + 3\ . \end{aligned} \tag{5.63}$$

Note that the special case of $\mathfrak{n} = 1$ corresponds to $(D_2, D_{2+2\mathfrak{m}}) = D_{2\mathfrak{m}+2}(\mathrm{SO}(4))$; the theory (5.62) reduces to the mirror theory of $D_{2\mathfrak{m}+2}(\mathrm{SO}(4))$ described by ([38], (6.34)) with $\mathfrak{N} = 1$.

5.4.4 Example: $M=2$, i.e. six mass parameters

The mirror theory for $(D_{6\mathfrak{n}-2}, D_{6\mathfrak{n}+6\mathfrak{m}-2})$ with $\mathrm{GCD}(2\mathfrak{n}-1, 2\mathfrak{m}) = 1$ is



(5.64)

This theory has

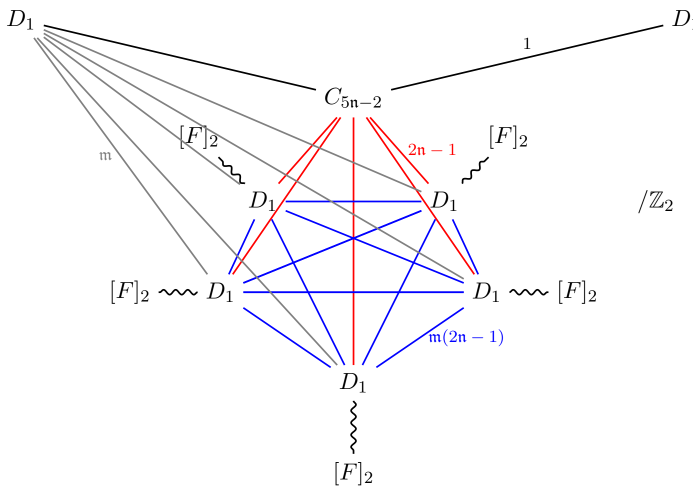
$$\begin{aligned}\dim_{\mathbb{H}} \mathrm{HB}\,(5.64) &= 3\mathfrak{m}(5\mathfrak{n}-1)+9\mathfrak{n}(2\mathfrak{n}-1)-4+H_{\text{free}} \\ &= 6\mathfrak{m}(3\mathfrak{n}-1)+6\mathfrak{n}(3\mathfrak{n}-2)-1 \\ \dim_{\mathbb{H}} \mathrm{CB}\,(5.64) &= 3\mathfrak{n}+4\ .\end{aligned} \tag{5.65}$$

In section 5.4.6, we shall derive the mirror theory (5.64) for a subclass of theories with $\mathfrak{n} = \mathfrak{N}$ and $\mathfrak{m} = 1$, i.e. $(D_{6\mathfrak{N}-2}, D_{6\mathfrak{N}+4})$. This serves as a highly non-trivial test of the proposal (5.64).

Let us also provide a check for the number of free hypermultiplets via the example of $\mathfrak{n} = 2$ and $\mathfrak{m} = 4$, i.e. the (D_{10}, D_{34}) theory. The value of $24(c-a)$ of this theory is $79/7 = 10 + (9/7)$, where 10 is the Higgs branch dimension of the 4d theory, according to the last case of (3.26), which is in agreement with the Coulomb branch dimension of the mirror theory. The fraction 9/7 can be identified as the value of $24(c-a)$ of the non-Higgsable SCFT $(A_2, A_3)^{\otimes 3} \cong (A_3, A_2)^{\otimes 3}$, in agreement with the claim (5.60). Since this theory has rank 9, we expect to have 9 free hypermultiplets, as stated above.

5.4.5 Example: $M = 3$, i.e. eight mass parameters

The mirror theory for $(D_{10\mathfrak{n}-4}, D_{10\mathfrak{n}+10\mathfrak{m}-4})$ with $\mathrm{GCD}(2\mathfrak{n}-1, 2\mathfrak{m}) = 1$ is



(5.66)

$+\ 5(\mathfrak{m}-1)(\mathfrak{n}-1)$ free hypermultiplets, and with $F = \mathfrak{m}(\mathfrak{n}-1)$

This theory has

$$\begin{aligned}\dim_{\mathbb{H}} \mathrm{HB}\,(5.66) &= 15\mathfrak{m}(3\mathfrak{n}-1)+5\mathfrak{n}(10\mathfrak{n}-7)-1+H_{\text{free}} \\ &= 10\mathfrak{m}(5\mathfrak{n}-2)+10\mathfrak{n}(5\mathfrak{n}-4)+4 \\ \dim_{\mathbb{H}} \mathrm{CB}\,(5.66) &= 5\mathfrak{n}+5\ .\end{aligned} \tag{5.67}$$

Let us also provide a check for the number of free hypermultiplets via the example of $\mathfrak{n} = 2$ and $\mathfrak{m} = 4$, i.e. the (D_{16}, D_{56}) theory. The value of $24(c-a)$ of this theory is $120/7 = 15 + (15/7)$, where 15 is the Higgs branch dimension of the 4d theory, according to (3.26), which is in agreement with the Coulomb branch dimension of the mirror theory.

The fraction 15/7 can be identified as the value of $24(c-a)$ of the non-Higgsable SCFT $(A_2, A_3)^{\otimes 5} \cong (A_3, A_2)^{\otimes 5}$, in agreement with the claim (5.60). Since this theory has rank 15, we expect to have 15 free hypermultiplets, as stated above.

5.4.6 Derivation of (5.64) in a special case of $(D_{6\mathfrak{N}-2}, D_{6\mathfrak{N}+4})$

Let use take $\mathfrak{n} = \mathfrak{N}$ and $\mathfrak{m} = 1$, i.e. $(D_{6\mathfrak{N}-2}, D_{6\mathfrak{N}+4})$. This theory admits two descriptions. One description corresponds to (3.13) with $(\alpha = 2\mathfrak{N}-1, \beta = 4\mathfrak{N}+2)$:

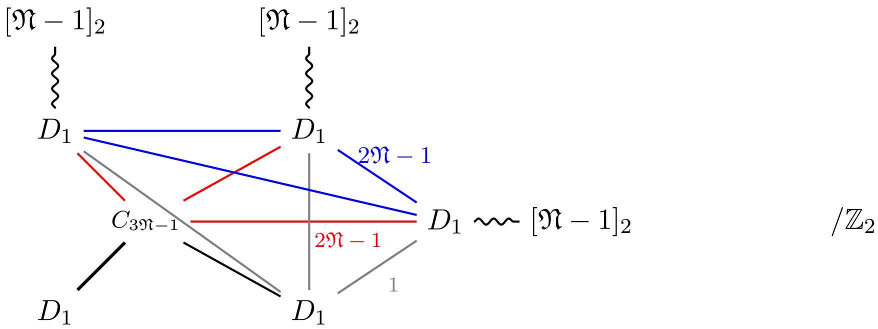
$$(D_{6\mathfrak{N}-2}, D_{6\mathfrak{N}+4}) = D_{8\mathfrak{N}}(\mathrm{SO}(8\mathfrak{N}+6)) - \mathrm{SO}(4\mathfrak{N}) - D_{4\mathfrak{N}}(\mathrm{SO}(4\mathfrak{N}))\,, \tag{5.68}$$

with the left matter sector has a full puncture partially closed to $[(2\mathfrak{N}+3)^2, 1^{4\mathfrak{N}}]$. The other description corresponds to (3.14) with $(\alpha = 4\mathfrak{N}-2, \beta = 2\mathfrak{N}+1)$:

$$(D_{6\mathfrak{N}-2}, D_{6\mathfrak{N}+4}) = D_{4\mathfrak{N}}(\mathrm{SO}(4\mathfrak{N}+4)) - \mathrm{SO}(4\mathfrak{N}+4) - D_{8\mathfrak{N}}(\mathrm{SO}(8\mathfrak{N}-2))\,, \tag{5.69}$$

with the right matter sector has a full puncture partially closed to $[(2\mathfrak{N}-3)^2, 1^{4\mathfrak{N}+4}]$.

For this theory, the mirror theory is given by (5.64) with $\mathfrak{n} = \mathfrak{N}$ and $\mathfrak{m} = 1$. We redraw it as follows.

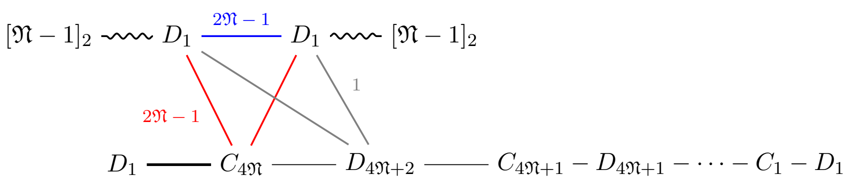


(5.70)

There are no free hypermultiplets for this mirror theory.

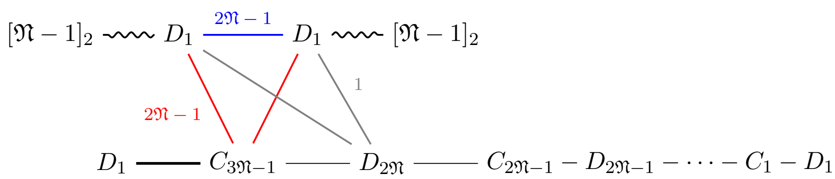
We now derive (5.70) by gluing the mirror theories of those indicated in (5.68) and (5.69).

Gluing via (5.68). Let us first consider (5.68). The mirror theory for $D_{8\mathfrak{N}}(\mathrm{SO}(8\mathfrak{N}+6))$ is given in section 4.2.2, with $N = 4\mathfrak{N}+3$, $p = 8\mathfrak{N}$, $2N-2 = 8\mathfrak{N}+4$, $\mathrm{GCD}(2N-2, p) = 4$, $x = 1$:



(5.71)

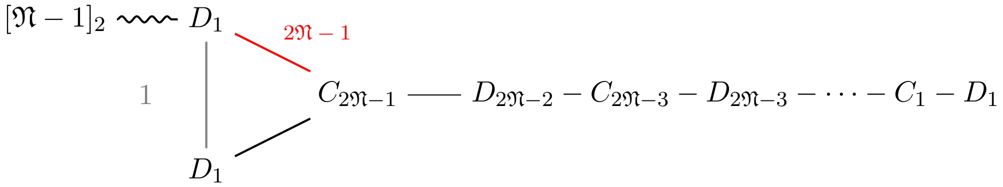
Upon partially closing the full puncture of $D_{8\mathfrak{N}}(\mathrm{SO}(8\mathfrak{N}+6))$ to $[(2\mathfrak{N}+3)^2, 1^{4\mathfrak{N}}]$, the mirror theory becomes



(5.72)

where the tail $D_1 - C_1 - \cdots - D_{2\mathfrak{N}-1} - C_{2\mathfrak{N}-1}$ gives rise to the SO($4\mathfrak{N}$) CB symmetry corresponding to the part $1^{4\mathfrak{N}}$ of the partition, whereas the balanced gauge group $C_{3\mathfrak{N}-1}$ gives rise to the SO(2) CB symmetry corresponding to the part $(2\mathfrak{N}-3)^2$ of the partition. Note that the $D_{2\mathfrak{N}}$ gauge node is overbalanced. Indeed, the tail on the right is determined by the $T_{[(2\mathfrak{N}+3)^2,1^{4\mathfrak{N}}]}[\mathrm{SO}(8\mathfrak{N}+6)]$ theory.

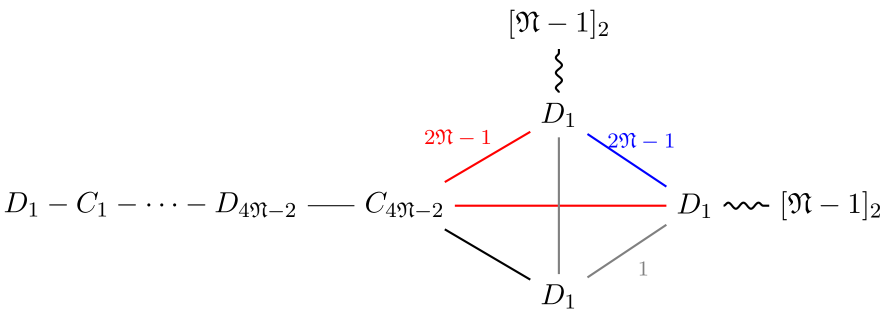
On the other hand, the mirror theory for $D_{4\mathfrak{N}}(\mathrm{SO}(4\mathfrak{N}))$ is given by ([38], (6.35)) with $\mathfrak{m} = 1$:



(5.73)

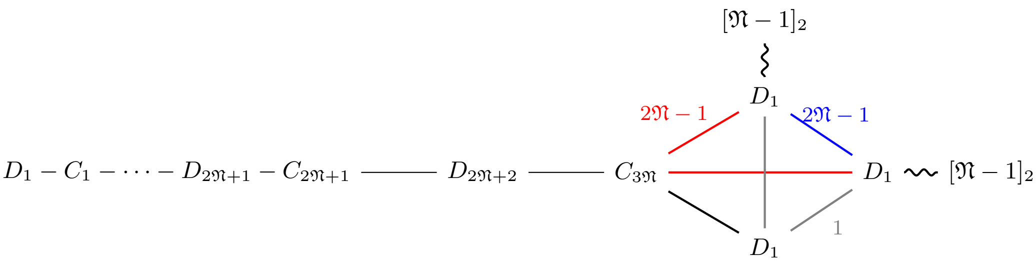
Now we glue (5.72) and (5.73) together by fusing the tails $D_1 - C_1 - \cdots - D_{2\mathfrak{N}-1} - C_{2\mathfrak{N}-1}$ of the two quivers. The latter corresponds to commonly gauging the CB symmetry SO($4\mathfrak{N}$) of the two theories, as instructed by (5.68). In doing so, we split the $D_{2\mathfrak{N}}$ into two D_1 nodes, with the connections as depicted in the left most part of (5.73). These two D_1 nodes become those linked by the gray line with label "1" in (5.72). The leftmost D_1 node in (5.72) becomes the lower left D_1 node in (5.70). This part of gluing indeed explains the $D_1 - C_{3\mathfrak{n}-1}$ tail in (5.64). The top two D_1 nodes linked by the blue line in (5.72) become the top two D_1 nodes in (5.70). Each of the three D_1 nodes attached to the wiggle line are connected together by the blue line, as in (5.72). We thus arrive at (5.70) as expected.

Gluing via (5.69). The mirror theory for $D_{8\mathfrak{N}}(\mathrm{SO}(8\mathfrak{N}-2))$ is given by ([38], (6.48)) with $\mathfrak{m} = 1$:



(5.74)

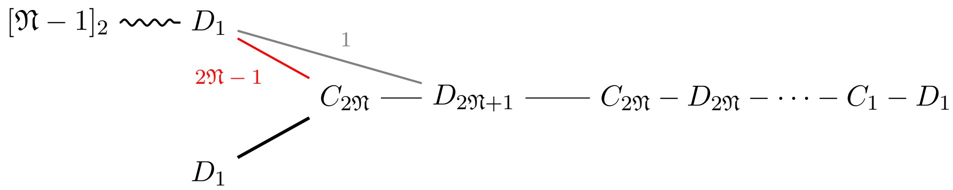
Upon partially closing the full puncture of $D_{8\mathfrak{N}}(\mathrm{SO}(8\mathfrak{N}-2))$ to $[(2\mathfrak{N}-3)^2, 1^{4\mathfrak{N}+4}]$, the mirror theory becomes



(5.75)

where the tail $D_1 - C_1 - \cdots - D_{2\mathfrak{N}+1} - C_{2\mathfrak{N}+1}$ gives rise to the $\mathrm{SO}(4\mathfrak{N}+4)$ CB symmetry corresponding to the part $1^{4\mathfrak{N}+4}$ of the partition, whereas the balanced gauge group $C_{3\mathfrak{N}}$ gives rise to the $\mathrm{SO}(2)$ CB symmetry corresponding to the part $(2\mathfrak{N}-3)^2$ of the partition. Note that the $D_{2\mathfrak{N}+2}$ gauge node is overbalanced. Indeed, the tail on the left is determined by the $T_{[(2\mathfrak{N}-3)^2,1^{4\mathfrak{N}+4}]}[\mathrm{SO}(8\mathfrak{N}-2)]$ theory.

On the other hand, the mirror theory for $D_{4\mathfrak{N}}(\mathrm{SO}(4\mathfrak{N}+4))$ is described by (4.33):



(5.76)

Now we glue (5.76) and (5.75) together by fusing the tails $D_1 - C_1 - \cdots - D_{2\mathfrak{N}+1} - C_{2\mathfrak{N}+1}$ of the two quivers. This corresponds to commonly gauging the CB symmetry $\mathrm{SO}(4\mathfrak{N}+4)$ of the two theories, as instructed by (5.69). However, there is no explicit $C_{2\mathfrak{N}+1}$ gauge group in (5.76), but the $D_{2\mathfrak{N}+1}$ node is connected to the $C_{2\mathfrak{N}}$ node and a D_1 node (connected to the wiggle line). In other word, the aforementioned $C_{2\mathfrak{N}+1}$ group should be viewed as split into $C_{2\mathfrak{N}}$ and C_1, where the D_1 subgroup of the latter is gauged. Thus, upon gluing the two theories, we also need to split the $C_{3\mathfrak{N}}$ node of (5.75) into $C_{3\mathfrak{N}-1}$ and C_1, where the latter C_1 is then identified with the C_1 group mentioned earlier whose D_1 subgroup is gauged. The central node thus becomes $C_{3\mathfrak{N}-1}$ as in (5.70). The mirror theory in question thus acquires the left part of (5.76). The lower left D_1 node in (5.76) becomes the lower left D_1 node in (5.70). This part of gluing indeed explains the $D_1 - C_{3\mathfrak{n}-1}$ tail in (5.64). Moreover, each D_1 node connected to the wiggle lines are connected together as in the previous case of gluing. We thus arrive at (5.70), as required.

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A Some properties of $D_p(\mathrm{USp}(2N))$

In this appendix, we collect some properties of the $D_p(\mathrm{USp}(2N))$ that we have found. This theory was defined and discussed in ([38], (2.11)–(2.14)).

First, similarly to eqs. (3.13) to (3.17), we find that a certain subclass of the $D_p(\mathrm{USp}(2N))$ theories admits a weakly coupling cusp in the conformal manifold:

$$\begin{aligned} &D_{2\mu+2}(\mathrm{USp}((2\mu+2)(2\mathfrak{m}-1))) = \\ &\quad = D_{2\mu}(\mathrm{USp}(2\mu(2\mathfrak{m}-1))) - \mathrm{USp}(2\mu(2\mathfrak{m}-1)) - D_2(\mathrm{USp}((4\mu+2)(2\mathfrak{m}-1)))\,. \end{aligned} \tag{A.1}$$

where $\mu \geq 1$ and $\mathfrak{m} \geq 1$.

Secondly, we find that whenever N is a multiple of p, the $D_p(\mathrm{USp}(2N))$ theory admits a Lagrangian description. Let us analyze two cases according to the parity of p.

The case of p even. In this case, we write

$$p = 2\mu\,, \quad N = \mathfrak{m}p = 2\mathfrak{m}\mu\,, \qquad \mu,\, \mathfrak{m} \geq 1\,. \tag{A.2}$$

The Lagrangian description of $D_{2\mu}(\mathrm{USp}(4\mathfrak{m}\mu))$ can be written as

$$D_{\mathfrak{m}+1} - C_{2\mathfrak{m}} - D_{3\mathfrak{m}+1} - \cdots - D_{(2\mu-1)\mathfrak{m}+1} - [C_{2\mu\mathfrak{m}}]\,. \tag{A.3}$$

As we commented around Footnote 5, it is not clear whether reduction of this theory on a circle to 3d yields a 3d $\mathcal{N}=4$ gauge theory with the same quiver description. Nevertheless, if we view (A.3) as a 3d $\mathcal{N}=4$ gauge theory, this is a quiver description of the $T^\sigma_\rho[\mathrm{USp}(4\mathfrak{m}\mu)]$ theory, with

$$\sigma = \left[1^{4\mu\mathfrak{m}}\right] \quad \text{and} \quad \rho = \left[(2\mathfrak{m})^{2\mu}, 1\right]\,, \tag{A.4}$$

whose mirror theory is $T^\rho_\sigma[\mathrm{SO}(4\mathfrak{m}\mu+1)]$ and can be described by

$$\begin{array}{ccccccccccccccc} C_\mu - B_{2\mu+1} - \cdots - B_{(2\mu-1)\mathfrak{m}-3} - C_{(2\mu-1)\mathfrak{m}-1} - & B_{(2\mu-1)\mathfrak{m}} & - C_{(2\mu-1)\mathfrak{m}N} - B_{(2\mu-1)\mathfrak{m}N-1} - C_{(2\mu-1)\mathfrak{m}-1} - \cdots - B_1 - C_1 - B_0 \\ | \qquad\qquad\qquad\qquad\qquad\qquad & | & \\ [B_0] \qquad\qquad\qquad\qquad\qquad\qquad & [C_\mu] & \end{array} \tag{A.5}$$

We observe that the Higgs branch dimension of the mirror theory (4.2) is exactly equal to the rank of the corresponding 4d theory. However, the Coulomb branch dimension of the mirror theory is larger than the value $24(c-a)$ of the corresponding 4d theory. As we commented in Footnote 5, it is not clear whether $24(c-a)$ is equal to the Higgs branch of the 4d theory, since the orthogonal gauge groups may not be completely Higgsed at a generic point on the Higgs branch. This is due to the fact that the orthogonal gauge group which is conformal in 4d is underbalanced in 3d; see an explicit example in (A.6) below.

As an example, for $\mu=1$, the 4d theory is simply an $\mathrm{SO}(2\mathfrak{m}+2)$ SQCD with $2\mathfrak{m}$ flavors:

$$D_2(\mathrm{USp}(4\mathfrak{m}))\,:\; D_{\mathfrak{m}+1} - [C_{2\mathfrak{m}}]\,. \tag{A.6}$$

Viewing this as a 3d $\mathcal{N}=4$ gauge theory, the mirror theory is given by ([57], figure 17)

$$\begin{array}{ccc} C_1 - B_1 - \cdots - B_{\mathfrak{m}-1} - C_{\mathfrak{m}} - & B_{\mathfrak{m}} & - C_{\mathfrak{m}} - B_{\mathfrak{m}-1} - C_{\mathfrak{m}-1} - \cdots - B_1 - C_1 - B_0 \\ | \qquad\qquad\qquad\qquad\qquad & | & \\ [B_0] \qquad\qquad\qquad\qquad\qquad & [C_1] & \end{array} \tag{A.7}$$

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The case of p odd. In this case, we write

$$p = 2\mu + 1\,, \quad N = \mathfrak{m}p = \mathfrak{m}(2\mu+1)\,, \qquad \mu,\, \mathfrak{m} \geq 1\,. \tag{A.8}$$

The Lagrangian description of $D_{2\mu+1}(\mathrm{USp}(2(2\mu+1)\mathfrak{m}))$ can be written as

$$[D_2] - C_{\mathfrak{m}} - D_{2\mathfrak{m}+1} - C_{3\mathfrak{m}} - \cdots - D_{2\mu\mathfrak{m}+1} - [C_{(2\mu+1)\mathfrak{m}}]\,. \tag{A.9}$$

Viewing it a 3d $\mathcal{N} = 4$ gauge theory, this is in fact a quiver description of the $T^{\sigma}_{\rho}\,[\mathrm{USp}((4\mu+2)\mathfrak{m}+8\mu)]$ theory, with

$$\sigma = \left[(2\mu)^4, 1^{2(2\mu+1)\mathfrak{m}}\right] \quad \text{and} \quad \rho = \left[(2\mathfrak{m}+4)^{2\mu}, 2\mathfrak{m}+1\right]\,. \tag{A.10}$$

whose mirror theory is $T^{\rho}_{\sigma}[\mathrm{SO}((4\mu+2)\mathfrak{m}+8\mu+1)]$. The same comments below (A.5) apply here.

As an example, for $\mu = 1$, we have

$$D_3(\mathrm{USp}(6\mathfrak{m}))\;:\;[D_2] - C_{\mathfrak{m}} - D_{2\mathfrak{m}+1} - [C_{3\mathfrak{m}}]\,. \tag{A.11}$$

Viewing this as a 3d $\mathcal{N} = 4$ gauge theory, the mirror theory can be described as

$$\begin{array}{c} D_1 - C_1 - D_2 - C_3 - D_4 - C_5 - \cdots - D_{2\mathfrak{m}-2} - \underset{\underset{\displaystyle [B_0]}{|}}{C_{2\mathfrak{m}-1}} - B_{2\mathfrak{m}-1} - C_{2\mathfrak{m}} - \underset{\underset{\displaystyle [C_1]}{|}}{B_{2\mathfrak{m}}} - C_{2\mathfrak{m}} - B_{2\mathfrak{m}-1} - \cdots - C_1 - B_0 \end{array} \tag{A.12}$$

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