Decentralized Matching at Senior-Level: Stability and Incentives*

Ayşe Yazıcı†

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Abstract

We consider senior-level labor markets and study a decentralized game where firms can fire a worker whenever they wish to make an offer to another worker. The game starts with initial matching of firms and workers and proceeds with a random sequence of job offers. The outcome of the game depends on the random sequence according to which firms make offers and therefore is a probability distribution over the set of matchings. We provide theoretical support for the successful functioning of decentralized matching markets in a setup with myopic workers. We then identify a lower bound on outcomes that are achievable through strategic behavior. We find that in equilibrium either any sequence of offers leads to the same matching or workers (firms) do not agree on what matching is the worst (best) among all possible realizations of the outcome. This implies that workers can always act to avoid a possible realization that they unanimously find undesirable. Hence, a well-known result for centralized matching at the entry-level carries over to matching at the senior-level albeit without the intervention of a mediator.

KEYWORDS: Senior-level markets; Stability; Random matching
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†Durham University Business School, Department of Economics and Finance, Mill Hill Lane, Durham, UK, DH1 3LB. Tel: +44 (0) 191 3345498. E-mail: ayse.yazici@durham.ac.uk.

1 Introduction

Centralized job matching has received much attention in theory and practice since centralized procedures were introduced by market organizers to address market failures such as uncontrolled unraveling of appointment dates, recontracting and welfare losses; notable examples are medical residency matching and school choice. 'Stability' of outcomes is considered to be the main property that accounts for the success of centralized matching procedures. A matching is 'stable' if no agent and no firm-worker pair have an interest to deviate, i.e., no agent prefers being unmatched to being matched to her current partner and no firm-worker pair prefer each other to their current partners.[1]

It is generally thought that decentralized markets do not function well and will therefore benefit from improved coordination or centralization. However, it is not well understood why they remain decentralized.[2] One possible reason offered by theoretical studies is that these markets reach stable outcomes by means of decentralized decision-making. However, this result does not suggest much about the incentives of market participants to improve their prospects, in particular, the limits on successful (welfare enhancing) strategic behavior and the welfare implications of strategic interaction for agents. The theory of matching has clearly shown that there are systematic common/opposing interests among certain groups of agents over the set of stable matchings. Unlike in decentralized systems, the clearinghouse in centrally organized institutions decides which procedure to use and therefore can control the welfare consequences of strategic interaction via restricting achievable outcomes. To the best of our knowledge, this paper is the first study to understand the limits that agents face in strategic behavior in decentralized senior-level labor markets.

Due to the functional differences between institutions at the entry-level and the senior-level, studies of the former, albeit having advanced, do not apply to or address how equilibrium is reached at the senior-level professional markets. Entry-level job positions, e.g., the labor market for medical interns and residents, are initially vacant. Senior-level positions may not be initially vacant and become available when an incumbent vacates a position due to retirement or termination/expiration of contract.

Among examples of matching at the senior-level are the markets for CEO's and sports coaches. Each December, at the end of the football season, colleges make coach replacements prior to National Signing Day in February which is the last day for a high school senior to sign with the football team of an American college. Firing and hiring of head coaches mostly occur in December and the vacancy is filled in advance of the National Signing Day (Thomas and Van Horn, 2016). Nevertheless, such a short period of opportunity window for firing and hiring helps coordination in the market. The internal governance structure of some colleges may lead them to act faster than others; therefore some offers may reach candidates earlier than others. In the firing and

[1] In one-to-one matching, the set of stable matchings is equal to the core.

[2] In non-cooperative models of decentralized bilateral bargaining/trade in networks, Abreu and Manea (2012) show how an efficient matching can be achieved using a system of punishments and rewards. In a study of the formation of partnerships in social networks, Bloch et al. (2019) show that efficient matchings are achieved in a favor exchange game in which players need favors at random times and approach neighbors randomly to ask for them.

hiring season, when several senior positions become vacant, which vacancies are filled with which coaches depend on the order in which colleges make offers. Another source of uncertainty is the practice that colleges fire a coach whenever they wish to make an offer to another coach, without knowing whether the offer will be accepted or not.

To model decentralized senior-level matching where uncertainty over outcomes is accounted for, we study a sequential game that starts with an initial matching and proceeds with a random sequence of job offers.[3] The game begins with a lottery over sequences of firms. Each firm in the sequence is given the opportunity to offer its unique position to a worker. A worker who receives an offer compares it with the offer that she may be holding, and rejects one and keeps the other. We assume that no firm proposes to the same worker more than once and that no worker rejects the offer she is holding unless she receives a new one. Uncertainty over sequences of offers is translated to uncertainty over outcomes, therefore the outcome of the game is a lottery over matchings. Our decentralized procedure reduces to the firm-proposing Deferred Acceptance (DA) algorithm (Gale and Shapley, 1962) when each firm's position is initially vacant, in other words, the market in question is at the entry-level.[4] In such an instance, uncertainty over sequences of firms does not have any influence on outcomes, i.e., any sequence of firms leads to the same matching.

We show that the equilibrium outcome is a lottery over matchings such that, unless degenerate, workers (firms) do not agree on what is the worst (best) element in the support (Theorem 3). Thus, either each sequence of offers leads to the same matching in equilibrium or workers act to avoid a matching that they unanimously find undesirable. Based on the lessons learnt from two-sided matching at the entry-level, at a first glance firms seem to have a favored position in the game as proposers. Indeed, if each agent acts according to her/its true preference ordering and if each firm's position is initially vacant, then any execution of the procedure leads to the firm-optimal stable matching which firms (workers) unanimously find the best (worst) among all stable matchings.

The advantage of being proposers is not straightforward in our context. First, when strategic considerations are taken into account, our result shows that a matching that favors firms arises only if it is the unique realisation of the equilibrium outcome or else workers can always act to eliminate such a possibility. A similar result is also present in centralized entry-level professional markets. Consider a central agent who, upon receiving rank-order lists of preferences from market participants, applies the firm-optimal stable mechanism[5] to produce an outcome. When confronted with such a clearinghouse, workers can always eliminate their worst achievable partner by misreporting their preferences (Theorems 4.6 and 4.7, Roth and Sotomayor, 1990). Second, due to the existence of an initial matching situation, welfare of firms and of workers

[3]Theoretical studies of decentralized matching include Becker (1973), Blum et al. (1997), Cantala (2004), Diamantoudi et al. (2015), Haeringer and Wooders (2004), Niederle and Yariv (2009), Pais (2008), and Roth and Vande Vate (1991). Among the empirical/experimental studies of decentralized matching are Choo and Siow (2006), Echenique and Yariv (2009), and Menzel (2015).

[4]Our formulation indeed corresponds to the McVitie and Wilson (1970) version of the DA algorithm where at each step at most one firm makes an offer. However, for this special case a version that corresponds to the DA algorithm of Gale and Shapley (1962) can be formulated.

[5]This is the mechanism defined by the firm-proposing Deferred Acceptance algorithm.

does not monotonically decrease and increase respectively during the execution of the decentralized procedure, preventing a stable outcome occurring even when both sides act based on their true preference orderings.

There are two studies that are closely related to our paper. In a sightly different formulation of the game by Pais (2008), each firm keeps its initial partner until it makes a successful offer or until its initial partner receives and accepts an offer and vacates the position. In our formulation a firm fires its initial partner when it chooses to make an offer regardless of the outcome of the offer. The main result in Pais (2008) establishes that in an equilibrium where each firm acts according to its true preference ordering, any realized matching is *stable* at the *true* preferences. The same result holds for a similar decentralized game formulated by Blum et al. (1997) to study the vacancy chain problem. Starting from a *quasi-stable* matching (i.e., no blocking pair involves a matched firm), offers can only be made by a random sequence of firms with vacant positions. In decentralized matching, actions may be history dependent and therefore an agent's actions may not comply with a fixed preference ordering. Our first result is the counterpart of theirs in a more restricted but natural setting where workers are myopic and offers are not history dependent, that is, the strategy set of each agent is the class of preference orderings (Theorem 2). A myopic worker bases her decisions on a predetermined preference list and always accepts the offer that is ranked higher on a preference list.

Our second result is established for the general setup where the strategy space is not restricted to preference profiles. Different from the two papers, we study the structure of possible realizations of any equilibrium outcome and show that in equilibrium either any sequence of offers leads to the same matching or workers can always act to avoid the one that they unanimously find undesirable; unlike in centralized markets, it is achieved without the intervention of a mediator.

The organization of the paper is as follows. In section 2, we introduce the model. In section 3, we describe the decentralized game and define the equilibrium notions. In section 4, we address the strategic questions and present our results. In section 5, we conclude. We defer all proofs to the appendix.

2 The Model

Let $F = \{f_1, ..., f_n\}$ and $W = \{w_1, ..., w_m\}$ denote finite sets of firms and workers respectively. Let $V \equiv F \cup W$. Let $v \in V$ denote a generic agent, f a generic firm, and w a generic worker. For the rest of the paper, "it" refers to a firm and "she" refers to a worker. Each $v \in V$ has a **linear order** P_v over the agents on the other side and remaining unmatched.[6] Let $\mathcal{P}_v$ denote the set of such preferences for v. Let $P \equiv (P_v)_{v \in V}$ denote a preference profile. Let $\mathcal{P} \equiv \prod_{v \in V} \mathcal{P}_v$ denote the set of all preference profiles. Let P_{-v} denote profile $P_{V \setminus \{v\}}$. Let R_v denote the at-least-as-desirable-as relation associated with P_v. For each $v, v', v'' \in V$ $v' \; R_v \; v''$ means that either $v' = v''$ or $v' \; P_v \; v''$. We write v' is $\boldsymbol{P_v}$**-preferred than** v'' if $v' \; P_v \; v''$. A **problem** is a preference profile P.

[6]In other words, P_v is transitive, antisymmetric (strict) and total.

A **matching** is a function $\mu : V \longrightarrow V$ that satisfies the following: for each $f \in F$ and each $w \in W$, (i) $\mu(f) \neq f$ implies that $\mu(f) \in W$; (ii) $\mu(w) \neq w$ implies that $\mu(w) \in F$; and (iii) $\mu(f) = w$ if and only if $\mu(w) = f$. Let $\mathcal{M}$ denote the set of all matchings. We write that v is **unmatched at $\boldsymbol{\mu}$** if $\mu(v) = v$. We sometimes express an agent's partner at μ by 'μ-partner'. Let $\mu, \mu' \in \mathcal{M}$ be given. We write $\mu \ R_F \ \mu'$ when each firm finds its μ-partner at least as desirable as its μ'-partner. We similarly define $\mu \ R_W \ \mu'$. Let $V' \subseteq V$. Let $\mu(V') \equiv \bigcup_{v \in V'} \mu(v)$. Let $v, v' \in V$. Agent v' is **acceptable to $\boldsymbol{v}$ at $\boldsymbol{P}$** if v prefers v' to being unmatched, i.e., $v' \ P_v \ v$. We sometimes write v' is **$\boldsymbol{P_v}$-acceptable** to denote $v' \ P_v \ v$. Let $A(P_v)$ denote the set of acceptable partners at P_v. Let $\tilde{A}(P_v) \equiv A(P_v) \cup \{v\}$. Let $T(P_v)$ denote the top choice of v at P_v. For each $f \in F$ and each $v \in W \cup \{f\}$, let $U(v, P_f)$ denote the set of partners that f finds at least as desirable as v at P_f. Formally, $U(v, P_f) \equiv \{v' \in W \cup \{f\} : v' \ R_f \ v\}$. For each $w \in W$ and each $v \in F \cup \{w\}$, $U(v, P_w)$ is defined similarly.

Matching μ is **individually-rational at $\boldsymbol{P}$** if each agent finds her partner at μ at least as desirable as remaining unmatched, i.e., for each $v \in V$, $\mu(v) \ R_v \ v$. A pair (f, w) **blocks $\boldsymbol{\mu}$ at $\boldsymbol{P}$** if f and w are not matched at μ and would prefer to be matched to each other, i.e., $w \ P_f \ \mu(f)$ and $f \ P_w \ \mu(w)$. A matching is **stable at $\boldsymbol{P}$** if it is individually-rational and not blocked by any pair (f, w) at P. Let $IR(P)$ denote the set of individually-rational matchings at P. Let $S(P)$ denote the set of stable matchings at P. A firm f is **achievable** for worker w if there is a stable matching at P that matches them. For any matching problem, there is a **firm-optimal stable** matching μ_F which all firms find at least as desirable as any other stable matching and likewise a **worker-optimal stable** matching μ_W. We will use the following facts about stable matchings of a problem P.

Proposition 1. *[Theorem 2.22, Roth and Sotomayor, 1990] Let P be a matching problem. The set of unmatched agents is the same across stable matchings at P.*

Proposition 2. *[Lemma 2.20, Roth and Sotomayor, 1990] Let P be a matching problem and $\mu, \mu' \in S(P)$. Each firm finds its μ-partner at least as desirable as its μ'-partner if and only if each worker finds her μ'-partner at least as desirable as her μ-partner. That is, $\mu \ R_F \ \mu'$ if and only if $\mu' \ R_W \ \mu$.*

An immediate consequence of Proposition 2 is that the firm-optimal (worker-optimal) stable matching is the worker-pessimal (firm-pessimal) stable matching, that is, it assigns each worker (firm) her (its) least preferred achievable partner.

3 The Decentralized Game

3.1 Description of the Game

The decentralized game is defined by a problem P and an initial matching $\mu^I \in IR(P)$ which is known to all agents.

The game begins with nature choosing a sequence of firms at random according to which firms make offers. The first firm in the sequence is given the opportunity to make an offer. If unmatched at μ^I, then the firm has two options, namely (i) making an offer to a worker or (ii) passing its turn and remaining unmatched. If matched at μ^I, then the firm has three options, namely (i) firing its initial partner and making no offer, (ii) firing its initial partner and making an offer to another worker or (iii) passing its turn and keeping its initial partner.

If the firm passes its turn, whether be it initially matched or unmatched, the initial matching remains unchanged. If the firm fires its initial partner and makes no offer, a new matching in which the firm and its initial partner are unmatched is formed. If the firm fires its initial partner and makes an offer to another worker, the worker who receives the offer decides whether to accept or reject it. If she accepts, then a new matching is formed in which the worker and the offering firm are matched and their previous partners, if any, are left unmatched. If she rejects, a new matching in which the firm and its initial partner are unmatched is formed.

The game continues by allowing the next firm in the sequence to make an offer. Whenever a firm is given the opportunity to make an offer, its available options depend on whether it is currently matched or not. If unmatched, the firm may (i) make an offer to a worker to whom it has not proposed before or (ii) pass its turn and remain unmatched. Otherwise, it may (i) fire the worker it holds and make no offer, (ii) fire the worker it holds and make an offer to a worker different from the worker it holds and from workers to whom it has proposed before[7] or (iii) pass its turn and keep the worker it holds. Once a worker receives an offer, she may accept it and reject, if any, the firm she holds or reject the offer and keep, if any, the firm she holds.

The game continues until no firm wishes to make an offer or fire the worker it holds. The game ends when each firm sequentially passes its turn, at which point each worker is matched to the firm she holds. The fact that each firm is allowed to propose to the same worker only once ensures that the game ends after a finite number of offers. However, this restriction is introduced primarily for simplicity, and relaxing it would not change the results as long as a firm proposes to the same worker for a finite number of times.

During the course of the game, no worker is allowed to make an offer to any firm and a matched worker is allowed to reject the firm she holds only when she receives and accepts an alternative offer. Throughout the game, each agent is informed of events that have direct effects on her/it. Specifically, each firm learns only if its offer is accepted or rejected and if its position is made vacant whereas each worker learns only if she receives an offer from a firm. Thus, each information set of a firm is identified by the identity of its initial partner, whether its initial partner has resigned its position and of the ordered list of offers it has made together with the rejections it received. Each information set of a worker is identified by the identity of her initial partner, whether her initial partner has fired her and of the ordered list of offers she received together with her responses.

We now describe the random elements in the game. The game begins with a lottery over sequences of firms. We consider infinite sequences in which each firm appears

[7] This does not exclude the case that a firm may choose to propose to its initial partner whom it has fired earlier in the game.

infinitely many times. The sample space over which lotteries are considered is denoted by $\mathcal{O}$ and a sample point is denoted by o. Although there are uncountably many such sequences, many of these sequences are equivalent for all possible profile of strategies. Therefore, the set of resulting equivalence classes is finite.[8] Additionally, we assume that each such sequence has a positive probability of occurring. Since each firm is allowed to propose to the same worker only once and firing of a worker is possible only when a firm is matched to the worker, the fact that each firm appears in a sequence infinitely many times guarantees that at some point in the game each firm passes its turn. Thus, the game ends in finite time.[9]

We say that 'a worker is still holding her initial partner' at some point in the game if she is currently matched to and has not yet been fired by her initial partner. Thus, she has not accepted any of the offers that she has received so far. Also, her initial partner has not made any offer yet.

3.2 The Strategy Space

Actions of a worker (firm) may depend on the history of offers received (made) and therefore may not be compatible with a preference ordering. Nevertheless, a natural class of strategies is the set of 'preference strategies' (Blum et al., 1997). A preference strategy is one which dictates the action of an agent at each of her/its information sets to be consistent with a rank order list of preferences.

A worker who uses a preference strategy rejects the offer of a firm if the firm is unacceptable according to the preference list or else compares the offer she receives with the firm she holds based on the preference list, keeps the most preferred firm and rejects the other.

A firm who uses a preference strategy chooses an action depending on whether it is currently matched or not. An unmatched firm makes an offer to the most preferred acceptable worker (if any) in the preference list to whom it has not proposed before. If a matched firm currently holds the most preferred acceptable worker in the preference list to whom it has not proposed before, then the firm passes its turn. Otherwise, it fires the worker it holds and makes an offer to the most preferred acceptable worker (if any) in the preference list to whom it has not proposed before.

Remark 1. *When using a preference strategy, once a firm makes an offer to a worker and the worker accepts this offer in the game, the firm does not make any further offers unless the worker it holds accepts an offer from another firm.*

We consider a class of preference strategies called 'truncations'. Let $v \in V$, and $P_v \in \mathcal{P}_v$ contain k (≥ 0) acceptable partners. A truncation strategy Q_v is a list containing $k' \leq k$ acceptable partners of v such that the first k' elements of Q_v are the first k' elements of P_v, in the same order. Let $Tr(P_v)$ denote the set of all strategies that are truncations of P_v.

[8] We thank a referee for pointing this out.

[9] The assumption that each firm appears infinitely many times in a sequence is stronger than what is needed. It suffices to assume sufficiently long finite sequences so that the decentralized procedure terminates.

Remark 2. *If a worker is matched to a firm that is unacceptable according to her truncation strategy in an outcome of the game then she is matched to her initial partner. Furthermore, she has not accepted any offer from any firm and her initial partner has not proposed to any worker during the execution of the game.*

Study of senior-level labor markets, precisely having an initial matching situation, constrains consideration of more complex strategies and restricts analysis to the use of preference strategies in equilibrium. Example 1 below shows that the game may result in different matchings for different sequences of firms. The driving force behind this is the observation that w_2 rejects an offer in favor of her initial partner f_1 that she may otherwise accept had she not held her initial partner. In one sequence, the worker has already been fired by her initial partner f_1 when she receives the offer and consequently accepts it, while in the other, she still holds her initial partner f_1 when she receives the offer and consequently rejects it. However, right after the decision to reject the offer and to keep the initial partner f_1, w_2 is fired by f_1 to propose to another worker. Although her rejection of the offer benefits w_2 ultimately in the latter order in Example 1, it could well have harmed her by leaving her unmatched in the end. When using a preference strategy, the reason behind the rejection of an otherwise accepted offer is clearly known and detected. On the other hand, the rejection of this kind cannot be tracked when a general strategy is used because a worker may reject or accept an offer as she wishes.

The use of truncations is not an essential element of our findings. Our main results would remain intact if preference strategies are not truncations of the true preference lists but the rank order of the acceptable elements preserves the order of the true preferences. To keep matters simple, we consider truncations in equilibrium rather than this kind of preference strategies. Unless otherwise stated, results are established for broader strategy sets encompassing strategies that may not be consistent with any preference orderings. Thus, while agents use preference strategies in equilibrium, in particular truncations, deviations are general strategies which may not be identified by a preference ordering.

For each agent $v \in V$, s_v is a strategy of agent v and S_v is the set of all strategies. A strategy profile $s = (s_1, s_2, ..., s_n)$ is a list of strategies. The set of all strategy profiles is denoted by $S \equiv \prod_{v \in V} S_v$. A preference strategy of agent v is denoted by a preference ordering Q_v. A sequence of firms o and a strategy profile s define a play of the game (o, s).

3.3 Equilibrium Notions

We demonstrate via an example that with the same initial matching and the strategy profile, the game may result in different matchings for different sequences of firms.

Example 1. *Let the true preference profile P be as follows. Initial matching $\mu^I = \{(f_1, w_2)\}$ is shown in parenthesis.*

$$
\begin{array}{llll}
P_{f_1}: & w_1,\ (w_2), & P_{w_1}: & f_2,\ f_1,\ w_1 \\
P_{f_2}: & w_2,\ w_1, & P_{w_2}: & (f_1),\ f_3,\ f_2. \\
P_{f_3}: & w_2,\ f_3, & &
\end{array}
$$

Suppose that agents play the game according to the preference profile P. Consider an order where f_3, f_1 and f_2 make offers sequentially. The game starts with f_3's offer to w_2. Worker w_2 rejects f_3 in favor of her initial partner f_1. Next, firm f_1 fires w_2 and proposes to w_1 who is currently unmatched and hence accepts f_1. Then, f_2 proposes to w_2 who accepts it. In the subsequent moves, each firm passes its turn and thus the game terminates with the matching $\hat{\mu} = \{(f_1, w_1), (f_2, w_2)\}$.

Now, consider an order where f_2, f_1 and f_3 make offers sequentially and each moves twice before the next firm in the sequence does. Firm f_2 proposes to w_2. Worker w_2 rejects f_2 as she is initially matched to a firm she prefers. Then, f_2 proposes to w_1 who accepts it. Next, f_1 fires w_2 and proposes to w_1 who rejects it as she is currently matched to the most preferred firm in her preference list. Then f_1 proposes to w_2 who accepts it. When f_3 proposes to w_2, she rejects f_3 as she prefers the firm she currently holds to f_3. In the subsequent moves, each firm passes its turn and the game ends with the unique stable matching $\mu = \{(f_1, w_2), (f_2, w_1)\}$. $\diamondsuit$

Let $P \in \mathcal{P}$, $\mu^I \in IR(P)$ and $s \in S$ be given. The probability distribution over the sequences of firms is translated into a probability distribution over the set of matchings. Let a probability distribution over $\mathcal{O}$ be given. Let $G^{\mu^I}[s]$ denote the probability distribution over $\mathcal{M}$ induced by the game when agents act according to s. Since the initial matching is fixed throughout the paper, we suppress it and denote the probability distribution over $\mathcal{M}$ by $G[s]$. For each $f \in F$ and each $v \in W \cup \{f\}$, let $G_f[s]$ denote the probability distribution over $W \cup \{f\}$ induced by $G[s]$ and let $G_f[s](U(v, P_f))$ denote the probability that in the game, f is assigned a partner at least as desirable as v according to P_f. For each $w \in W$ and each $v \in F \cup \{w\}$, the probability distributions $G_w[s]$ and $G_w[s](U(v, P_w))$ are similarly defined.

To study strategic issues, we now define what constitutes a best strategy for an agent. Let $f \in F$. What follows can be similarly defined for a representative worker. Let $s \equiv (s_f, s_{-f})$ be a strategy profile. Let s'_f be an alternative strategy for f. We say that s_f is a better strategy than s'_f, given s_{-f}, if for each utility function compatible with f's true preferences, it yields a higher expected utility. The following is an equivalent statement in terms of stochastic dominance.

Strategy $\boldsymbol{s_f}$ **stochastically** $\boldsymbol{P_f}$**-dominates** $\boldsymbol{s'_f}$**, given** $\boldsymbol{s_{-f}}$, if for each $v \in W \cup \{f\}$, $G_f[s_f, s_{-f}](U(v, P_f)) \geq G_f[s'_f, s_{-f}](U(v, P_f))$.

Strategy profile s is an **sd-Nash equilibrium** if for each $v \in V$, s_v stochastically P_v-dominates each alternative strategy s'_v given s_{-v}.[10]

In words, strategy profile s is an sd-Nash equilibrium if no agent $v \in V$, given the strategies of all other agents s_{-v}, is able to increase the probability of receiving any partner v' and agents ranked higher than v' at P_v by using a strategy other than s_v. There always exists an sd-Nash equilibrium of the game (Pais, 2008). The following refinement of sd-Nash equilibrium takes into account the dynamic nature of the game.

Strategy profile s is a **subgame perfect sd-Nash equilibrium** of the game if it is an sd-Nash equilibrium in each subgame of the game.

There is no proper subgame of the game (Pais, 2008). Therefore, the two equilibrium

[10] This name is taken from Thomson (2011). The concept was introduced by d'Aspremont and Peleg (1988). It is referred to as ordinal Nash equilibrium in the literature.

concepts coincide.

Strategy profile $Q \in \mathcal{P}$ is an **sd-Nash equilibrium in truncations** if Q is an sd-Nash equilibrium and for each $v \in V$, Q_v is a truncation strategy.

4 Equilibrium Analysis

In this section, we present equilibrium results. Pais (2008) studies a slightly different version of the game that we consider. She allows a firm to keep its initial partner until it makes a successful offer or its position has been vacated by its initial partner. On the other hand, we require a firm to fire its initial partner when it chooses to make an offer to a worker (different from its initial partner). Our formulation is consistent with the functioning of senior-level professional markets where the hiring process starts with the announcement of a vacant position. A college's football coach hiring procedure starts with an announcement that the current head coach has been fired, resigned to start a head coach job elsewhere or decided to retire. Similarly, the search for a corporate CEO starts after the existing CEO dies, retires or is fired (Thomas and Van Horn, 2016).

She studies sd-Nash equilibria where each firm adopts a preference strategy and acts according to its true preference ordering. She proves that any realized equilibrium outcome is stable at the true preferences. Her result remains valid under our formulation.

Theorem 1. *[Pais,2008] Let $P \in \mathcal{P}$, $\mu^I \in IR(P)$ and $s \equiv (P_F, s_W)$ be an sd-Nash equilibrium of the game. Then, supp$G[s] \subseteq S(P)$.*

Our first result is the counterpart of Theorem 1 when workers are myopic and offers are not history dependent, therefore, the strategy space of each agent consists only of preference strategies. A myopic worker always accepts a higher ranked offer on a preference list and never rejects an offer on the presumption that a better offer may arrive in the future. We study sd-Nash equilibria in truncations and show that any realized equilibrium outcome is stable at the true preferences of workers and the strategies according to which firms act.

Theorem 2. *Let $P \in \mathcal{P}$ and $\mu^I \in IR(P)$. Let $S = \mathcal{P}$ and $Q \in \mathcal{P}$ be an sd-Nash equilibrium of the game in truncations. Then, supp$G[Q] \subseteq S(Q_F, P_W)$.*

As the true preference list is itself a truncation strategy, a direct implication is Corollary 1: any realized outcome of an equilibrium where each firm acts according to its true preference ordering, is stable at the true preferences.

Corollary 1. *Let $P \in \mathcal{P}$ and $\mu^I \in IR(P)$. Let $S = \mathcal{P}$ and $Q \equiv (P_F, Q_W) \in \mathcal{P}$ be an sd-Nash equilibrium of the game in truncations. Then, supp$G[Q] \subseteq S(P)$.*

Since the strategy set of each agent comprises of preference orderings, our result is also applicable to centralized institutions with an initial matching situation. Each market participant submits a preference list to a centralized authority which then randomly selects a sequence of firms and applies our decentralized procedure to produce an outcome.[11] As deterministic procedures inherently favor some agents over others, randomness can be introduced in centralized matching to achieve procedural fairness. This is felt most strongly in two-sided matching where the polarization of interests of agents on different sides is reflected in the structure of the set of stable matchings.

Theorem 2 has direct implications for centralized matching where professionals at all career stages not just entry-level are interested in new positions. One example of such centralized institutions is the placement of Conservative Rabbis which is controlled by a central authority. Upon receiving the list of vacancies circulated periodically, rabbis of all career levels who are interested in a new position submit their preferences over congregations with a vacant position to the placement committee. The committee recommends each congregation seeking a rabbi a list of three candidates from among those who have shown interest. Following interviews, each congregation seeking a rabbi may recruit one or ask the committee to recommend another list. The process continues until a rabbi is appointed (Granovetter, 1995).

The two results inform us of the stability of all possible realizations of the equilibrium outcome under certain restrictions. However, neither makes any statement about the structure of the possible realizations of the equilibrium outcome. Our next result addresses this question. Theorem 3 states that in an sd-Nash equilibrium in truncations either the support is a singleton so that each sequence of offers leads to the same matching or there is no matching in the support that all workers unanimously find worse than each other matching in the support. In other words, unless it is the unique realization of the outcome, workers coordinate their actions without the help of a mediator to avoid a matching that they unanimously find undesirable. The following example illustrates the essence of our result.

Example 2. *Let the true preference profile P be as follows. The initial matching $\mu^I = \{(f_1, w_2)\}$ is shown in parenthesis.*

$$\begin{array}{llll} P_{f_1}: & w_1,\ (w_2), & P_{w_1}: & f_2,\ f_1, \\ P_{f_2}: & w_2,\ w_1, & P_{w_2}: & (f_1),\ f_2. \end{array}$$

When the sequence of firms is $o: f_1, f_2, ..$, the game produces the firm-optimal stable matching $\mu_F = \{(f_1, w_1),\ (f_2, w_2)\}$. On the other hand, when the sequence of firms is $o': f_2, f_2, f_1, f_1, ...$, the game produces the worker-optimal stable matching $\mu_W = \{(f_1, w_2),\ (f_2, w_1)\}$. We show that P is not an sd-Nash equilibrium. If w_2 deviates and acts according to $Q_{w_2}: f_1, w_2$, the game produces μ_W for each sequence of firms. Thus, μ_W is obtained with probability 1. Unlike the case with strategy P_{w_2}, Q_{w_2} dictates w_2 to reject f_2 for each sequence of offers. Afterwards, f_2 proposes to w_1 who rejects f_1 in favor of f_2 for each sequence of offers. Firm f_1 then proposes to w_1 who accepts it. By rejecting f_2, w_2 triggers w_1's rejection of f_1 and ultimately receives an offer from her most preferred firm f_1. Indeed, (Q_{w_2}, P_{-w_2}) is an sd-Nash equilibrium in truncations. Precisely, each of the workers receives her most preferred firm under her true preferences and thus, has no interest in deviation. Worker w_2 never accepts f_2's offer therefore, for

[11]The formal description of the amended version of the DA algorithm based on our decentralized procedure can be found in the *working paper* version.

each sequence of firms w_1 ends up receiving an offer from f_2 hindering any chance of f_1 hiring w_1. $\Diamond$

We now present our main result.

Theorem 3. *Let $P \in \mathcal{P}$, $\mu^I \in IR(P)$ and $Q \in \mathcal{P}$ be an sd-Nash equilibrium in truncations. Then, either $|suppG[Q]| = 1$ or there is no matching $\mu \in suppG[Q]$ such that for each $\mu' \in suppG[Q] \backslash \{\mu\}$, $\mu' \; R_W \; \mu$.*

Suppose firms act based on their true preferences in equilibrium, $Q_F = P_F$. Since Q is an equilibrium when the strategy set of each agent is unrestricted, it remains to be an equilibrium when the strategy set of each agent consists only of preference orderings. Then by Theorem 2, each realization of the outcome is stable at the true preferences. Theorem 3 implies that whenever there is more than one stable matching in the support and one is considered worker-pessimal, at least one worker can deviate so as to eliminate the undesirable matching for all. This is reminiscent of a result well-known for centralized professional markets at the entry level where market organizers use the firm-optimal stable mechanism to match market participants. Whenever there is more than one stable outcome at the matching problem, at least one worker can profitably misrepresent her preferences and obtain her most preferred achievable partner, in particular, she can eliminate the worst achievable partner.

A consequence of Theorems 2 and 3 and Proposition 2 is that in equilibrium either the support is a singleton or no outcome in the support is firm-optimal.

We finally address the existence question. The following example shows that with no restrictions on the initial matching an sd-Nash equilibrium in truncations may not exist.

Example 3. *Let the true preference profile P be as follows. The initial matching $\mu^I = \{(f_1, w_2), (f_2, w_3)\}$ is shown in parenthesis.*

$$
\begin{array}{llllll}
P_{f_1}: & w_1,\ (w_2),\ f_1, & \qquad & P_{w_1}: & f_1,\ w_1, \\
P_{f_2}: & w_2,\ (w_3),\ f_2, & \qquad & P_{w_2}: & (f_1),\ f_2, \\
 & & & P_{w_3}: & (f_2),\ w_3.
\end{array}
$$

Let Q be an sd-Nash equilibrium in truncations. We first show that f_1 never proposes to w_2 at any play of the game. Suppose there is a play of the game at which f_1 proposes to w_2. Since a firm proposes to acceptable workers only, by $Q_{f_1} \in Tr(P_{f_1})$, f_1 must have proposed to w_1 who must have rejected the offer. By $Q_{w_1} \in Tr(P_{w_1})$ and $A(P_{w_1}) = \{f_1\}$, all firms must be listed unacceptable at Q_{w_1}. Since w_1 is initially unmatched, w_1 remains unmatched at any play of the game. Then Q_{w_1} cannot be part of an sd-Nash equilibrium as f_1 is the most preferred firm at her true preference ordering and rejected at some play of the game. Therefore, f_1 never proposes to w_2 at any play of the game. We next show that $Q_{f_2} = P_{f_2}$.

Case 1: $A(Q_{w_2}) = \emptyset$ or $A(Q_{w_2}) = \{f_1\}$. Firm f_2 is rejected by w_2 at any play of the game for any truncation strategy which dictates f_2 to propose to w_2. First, $A(Q_{w_3}) \neq \emptyset$. Otherwise, f_2 can be matched to its initial partner w_3 at a play of the game only if f_2 passes its turn and keeps its initial partner in its first move in the game. However, no truncation strategy for f_2 lists w_3 the highest ranked worker. Hence, f_2 is unmatched at any play of the game if it uses a truncation strategy. Let Q'_{f_2} be a deviation which lists w_3 the highest ranked worker. Firm f_2 passes its turn and remains matched to w_3 at any play of the game, contradicting our assumption that Q is an sd-Nash equilibrium. Thus, $A(Q_{w_3}) = A(P_{w_3}) = \{f_2\}$. Firm f_2 is matched to w_3 if $Q_{f_2} = P_{f_2}$ and remains unmatched if it uses other truncation strategies at any play of the game.

Case 2: $Q_{w_2} = P_{w_2}$. Consider any sequence of firms in which f_2 has its first move before f_1. Firm f_2 is rejected by w_2 in favor of f_1 for any truncation strategy which dictates f_2 to propose to w_2. We come up with a contradiction as in Case 1 if $A(Q_{w_3}) = \emptyset$. Then let $A(Q_{w_3}) = \{f_2\}$. Firm f_2 is matched to w_3 at any play of the game if $Q_{f_2} = P_{f_2}$ and remains unmatched if it uses other truncation strategies. Now consider any sequence in which f_1 has its first move before f_2. For any truncation strategy of f_1, f_1 fires w_2 before f_2 has its first move in the game. Also, we have shown that f_1 never proposes to w_2 at any play of the game. Therefore, f_2 is matched to w_2 at any play of the game if $Q_{f_2} = P_{f_2}$. Thus, f_2 is never matched to a less preferred partner when it uses P_{f_2} than any other truncation strategies but there is at least one sequence of firms in which f_2 is matched to a preferred worker when it uses P_{f_2} than any other truncation strategies.

We now show that given $Q_{f_2} = P_{f_2}$, no truncation strategy for w_2 can be part of an sd-Nash equilibrium. As shown before, w_2 is never matched to f_1 at any play of the game. If $A(Q_{w_2}) = \emptyset$ or $A(Q_{w_2}) = \{f_1\}$, then w_2 is unmatched at any play of the game. If $Q_{w_2} = P_{w_2}$, then consider any sequence in which f_2 has its first move before f_1. Firm f_2 is rejected by w_2 in favor of f_1. Therefore, w_2 is unmatched at such a play of the game if she acts according to Q. Let Q'_{f_2} be a deviation such that $A(Q'_{w_2}) = \{f_2\}$. Worker w_2 is matched to f_2 at any play of the game, contradicting the assumption that Q is an sd-Nash equilibrium.$\Diamond$

We now identify admissible initial matchings for which an sd-Nash equilibrium in truncations exists.

An initial matching μ^I is **admissible** if, $\mu^I \notin S(P)$ implies that $\mu^I \in IR(P)$ and there is $\mu \in S(P)$ such that

(a.1) for each $f \in F$ with $w \equiv \mu^I(f)$, if $\mu(f)\ P_f\ w$ then $\mu(w)\ P_w\ f$.

(a.2) for each $F' \subseteq F$ and each $W' \subseteq W$ such that μ^I and μ map F' onto W' and each $f' \in F'$, if for each $f \in F' \backslash \{f'\}$, $\mu^I(f) = T(P_f)$, then $\mu(f')\ R_{f'}\ \mu^I(f')$.

Admissibility requires an unstable initial matching to be individually-rational and necessitates the existence of a stable matching μ that satisfies conditions (a.1) and (a.2). Condition (a.1) says that if a firm f finds its initial partner w worse than its μ-partner then w also finds her initial partner f worse than her μ-partner. In other words, if f improves from its initial situation, then its initial partner does too. Condition (a.2) considers a group of firms F' mapped onto a group of workers W' by initial matching μ^I and stable matching μ. If each firm in F' is initially matched to its most favorite worker and if no worker in W' receives a better offer than its initial partner then each firm and worker in the group would end up being matched to her/its initial partner. This would prevent stable matching μ being achieved as an equilibrium outcome. Therefore, condition $(a.2)$ requires that if each firm but one is initially matched to its most favorite worker, the last one prefers its μ-partner to its initial partner. Let $\mathcal{M}^A(P)$ be the set of admissible initial matchings. Notice that a stable initial matching is vacuously admissible. Hence, $\mathcal{M}^A(P) \neq \emptyset$.

Our final result states that as long as the game starts with an admissible initial matching, an sd-Nash equilibrium in truncations exists.

Proposition 3. *Let P be a matching problem and $\mu^I \in \mathcal{M}^A(P)$. There is an sd-Nash equilibrium in truncations.*

5 Conclusion

We study a decentralized matching game in which starting from an initial matching firms sequentially offer their unique job positions to workers. We find that the realized equilibrium outcome is stable at the true preferences of workers and the strategies according to which firms act. This provides theoretical support for the success of senior-level professional markets when agents base their decisions on predetermined lists of preferences. We then show that in equilibrium either the realized outcome is unique or no realized outcome is worker-pessimal among all possible realizations. Hence, we re-establish a well-known fact for centralized entry-level labor markets that in equilibrium workers can act to eliminate a stable outcome that they all find undesirable with the exception that this is now achieved without the intervention of a mediator.

6 Appendix A

Proof of Theorem 2: Let Q be an sd-Nash equilibrium in truncations. Assume by contradiction that there is $\mu \in \text{supp}G[Q]$ such that $\mu \notin S(Q_F, P_W)$. We first show that $\mu \in IR(Q_F, P_W)$. Since no firm ever makes an offer to a worker who is unacceptable at Q, then for each $f \in F$, $\mu(f) \in \tilde{A}(Q_f)$. Suppose that there is a worker w such that $\mu(w) \notin \tilde{A}(P_w)$. Then $w\ P_w\ \mu(w)$. By $\mu^I \in IR(P)$, we have $\mu(w) \neq \mu^I(w)$. Let $\bar{Q}_w \in Tr(P_w)$ be such that $A(\bar{Q}_w) = \emptyset$.[12] By using $\bar{Q}_w$, w may end up unmatched or matched to her initial partner $\mu^I(w)$ but she is never matched to a firm that is unacceptable at P_w. Thus, Q_w does not stochastically P_w-dominate $\bar{Q}_w$. Hence, Q is not an sd-Nash equilibrium.

We have proved that $\mu \in IR(Q_F, P_W)$. Thus, there is a blocking pair for μ at (Q_F, P_W), i.e., there is a pair (f, w) such that $w\ Q_f\ \mu(f)$ and $f\ P_w\ \mu(w)$. Then f must have proposed to and been rejected by w at each play of the game that leads to μ.

Case 1: $\mu^I(w) = f$. We first show that $f \notin A(Q_w)$. Assume by contradiction that $f \in A(Q_w)$. Then w must have received and accepted an offer from a Q_w-preferred firm, say f', than f at each play of the game that leads to μ. Then either w rejects all further offers and is matched to f' or she receives a better offer and is matched to

[12]By the definition of a truncation strategy, the offer of an unacceptable firm is always rejected.

a Q_w-preferred firm than f'. In either case, $\mu(w)\ Q_w\ f$. As $Q_w \in Tr(P_w)$, $f \in A(Q_w)$ implies that $\mu(w)\ P_w\ f$, contradicting our assumption. Thus $f \notin A(Q_w)$. This, together with $f\ P_w\ \mu(w)$ implies that $\mu(w) = w$ and $A(Q_w) \subsetneq U(f, P_w)$.

Let $\bar{Q}_w \in \mathcal{P}_w$ be an alternative strategy which has the same ordering as Q_w except that f is the least preferred acceptable firm at $\bar{Q}_w$. Formally, $A(\bar{Q}_w) = A(Q_w) \cup \{f\}$, for each $f' \in A(\bar{Q}_w)\backslash\{f\}$, $f'\ \bar{Q}_w\ f$, and for each $v, v' \in (F \cup \{w\})\backslash\{f\}$, $v\ \bar{Q}_w\ v'$ if and only if $v\ Q_w\ v'$. By construction, $A(\bar{Q}_w) \subseteq U(f, P_w)$. Let $\bar{Q} \equiv (\bar{Q}_w, Q_{-w})$. We show that the probability that w achieves a partner in $U(f, P_w)$ is larger at $\bar{Q}$ than at Q. For each play of the game that leads to μ when agents act according to Q, f must have proposed to and been rejected by w. If w deviates and acts according to $\bar{Q}_w$, either she is matched to f or she receives a better offer and is matched to a $\bar{Q}_w$-preferred firm than f. Since $f \in A(\bar{Q}_w)$ and $A(\bar{Q}_w) \subseteq U(f, P_w)$, in either case w achieves a partner in $U(f, P_w)$, but she remains unmatched if she acts according to Q_w. Now consider any play of the game that does not lead to μ when agents act according to Q. Suppose w deviates and acts according to $\bar{Q}_w$. If f proposes to w, as before w achieves a partner in $U(f, P_w)$. Otherwise, w achieves the same partner as when she acts according to Q_w. This completes the proof that the probability that w achieves a partner in $U(f, P_w)$ is larger at $\bar{Q}$ than at Q. Hence, Q is not an sd-Nash equilibrium.

Case 2: $\mu^I(w) \neq f$. By Remark 2, either $f \notin A(Q_w)$ or w must have still been holding her initial partner when she rejected f. Let $\bar{Q}_w \in \mathcal{P}_w$ be an alternative strategy which has the same ordering as Q_w except that f is the top choice firm at $\bar{Q}_w$. Formally, $T(\bar{Q}_w) = f$ and for each $v, v' \in (F \cup \{w\})\backslash\{f\}$, $v\ \bar{Q}_w\ v'$ if and only if $v\ Q_w\ v'$. Let $\bar{Q} \equiv (\bar{Q}_w, Q_{-w})$. We show that the probability that w achieves a partner in $U(f, P_w)$ is larger at $\bar{Q}$ than at Q. For each play of the game that leads to μ when agents act according to Q, f must have proposed to and been rejected by w. If w deviates and acts according to $\bar{Q}_w$, she is matched to f, but she achieves a partner less preferred than f if she acts according to Q_w. Now consider any play of the game that does not lead to μ when agents act according to Q. Suppose w deviates and acts according to $\bar{Q}_w$. If f proposes to w, w is matched to f. Otherwise, w achieves the same partner as when she acts according to Q_w. This follows from the fact that w is not initially matched to f and therefore will never reject any offer that would be accepted had she acted according to Q_w. This completes the proof that the probability that w achieves a partner in $U(f, P_w)$ is larger at $\bar{Q}$ than at Q. Hence, Q is not an sd-Nash equilibrium. $\square$

Proof of Theorem 3: Let P be a matching problem, $\mu^I \in IR(P)$ be an initial matching and $Q \in \mathcal{P}$ be an sd-Nash equilibrium in truncations. Since Q is an equilibrium when there is no restriction on the strategy set of any agent, it continues to be an equilibrium when the strategy set consists only of preference orderings. By Theorem 2,

$$suppG[Q] \subseteq S(Q_F, P_W). \tag{1}$$

Assume by contradiction that $|suppG[Q]| \neq 1$ and there is $\mu \in suppG[Q]$ such that

$$\text{for each } \mu' \in suppG[Q]\backslash\{\mu\} \text{ and each } w \in W,\ \mu'(w)\ R_w\ \mu(w). \tag{2}$$

Let $\mu' \in suppG[Q] \backslash \{\mu\}$. Let (o, Q) and (o', Q) be two plays of the game that lead to matchings μ and μ' respectively. By (1) and Proposition 2,

$$\text{for each } f \in F, \text{ either } \mu(f) = \mu'(f) \text{ or } \mu(f)\ Q_f\ \mu'(f). \tag{3}$$

Step 1: We argue that no worker's truncation strategy ranks her μ-partner higher than her μ'-partner. Assume by contradiction that there is $w \in W$ such that $\mu(w) \neq \mu'(w)$ and $\mu(w)\ Q_w\ \mu'(w)$. By $\mu, \mu' \in S(Q_F, P_W)$ and Proposition 1, $\mu'(w), \mu(w) \in F$. By Remark 2, no worker accepts the offer of a firm unacceptable at her truncation strategy. As $Q_w \in Tr(P_w)$, (2) implies that $\mu'(w) = \mu^I(w)$ and $\mu(w) = w$, contradicting $\mu(w) \in F$. Thus,

$$\text{for each } w \in W, \text{ either } \mu(w) = \mu'(w) \text{ or } \mu'(w)\ Q_w\ \mu(w). \tag{4}$$

Before we move onto Step 2, we partition F and W. Let $F^{\mu} \equiv \{f \in F : \mu(f)\ Q_f\ \mu'(f)\}$ be the set of firms whose truncation strategy ranks their μ-partner higher than their μ'-partner. Let $F^{\mu=\mu'} \equiv \{f \in F : \mu(f) = \mu'(f)\}$ be the set of firms whose partners at μ and μ' are the same. We similarly define $W^{\mu'}$ and $W^{\mu=\mu'}$. By (3), F^{μ} and $F^{\mu=\mu'}$ form a partition of F and by (4), $W^{\mu'}$ and $W^{\mu=\mu'}$ form a partition of W. Since μ and μ' are one-to-one and F^{μ} and $W^{\mu'}$ are finite, μ and μ' map F^{μ} onto $W^{\mu'}$. Since F^{μ} and $F^{\mu=\mu'}$ form a partition of F, each firm must have proposed to its μ-partner in o'. By Step 1, each $w \in W$ with $\mu(w) \neq \mu'(w)$ must have rejected her μ-partner in o'.

Step 2: We show that there is a worker who must be still holding her initial partner when she rejects her μ-partner in o'. Assume by contradiction that no such worker exists. Let w' be the first worker who rejects her μ-partner in o'. By our assumption, w' rejects her μ-partner in favor of a firm $f' \neq \mu^I(w')$ in o'. Then $f'\ Q_{w'}\ \mu(w')$ and f' must have proposed to w' in o' who must have accepted it. By Remark 2, $f' \in A(Q_{w'})$. By $Q_{w'} \in Tr(P_{w'})$, $f'\ P_{w'}\ \mu(w')$. By $\mu \in S(Q_F, P_W)$, $\mu(f')\ Q_{f'}\ w'$. Then f' must have proposed to and been rejected by $\mu(f')$ before f' proposes to w'. This is a contradiction to the assumption that w' is the first worker who rejects her μ-partner in o'.

Step 3: We now provide a procedure that takes o' as an input and produces a new sequence of firms such that no firm is rejected by its μ-partner who is still holding her initial partner. Until we propose a profitable deviation, we fix the strategy profile at Q.
Procedure: Let $o^0 \equiv o'$.
Step $\boldsymbol{t \geq 1}$: Pick the first firm in o^{t-1} which is rejected by its μ-partner who is still holding her initial partner. Denote it by f^t. Let k^t denote the position of o^{t-1} in which f^t proposes to its μ-partner. Let $f^{I,t} \equiv \mu^I(\mu(f^t))$.

If $\mu(f^{I,t}) = f^{I,t}$, let $\bar{k}^t$ denote the first position of o^{t-1} in which $f^{I,t}$ passes its turn and remains unmatched. Otherwise, let $\bar{k}^t$ denote the position of o^{t-1} in which $f^{I,t}$ proposes to its μ-partner.

Let $k_1^t \equiv k^t$, $f_1^t \equiv f^t$ and $i \equiv 1$.
1. Check if each of the following conditions is satisfied.
(c.1) $\mu^I(f_i^t) \in W$. Let $w_i^t \equiv \mu^I(f_i^t)$.
(c.2) $\mu(w_i^t) \in F$.

(c.3) $\mu(w_i^t)$ proposes to w_i^t between positions k_i^t and $\bar{k}^t$ of o^{t-1}.
(c.4) $f_i^t \; Q_{w_i^t} \; \mu(w_i^t)$.
(c.5) f_i^t fires w_i^t to propose to its μ-partner in position k_i^t of o^{t-1}.

If each of the five conditions is satisfied, let k_{i+1}^t denote the position of o^{t-1} in which $\mu({w_i}^t)$ proposes to w_i^t, $f_{i+1}^t \equiv \mu(w_i^t)$ and $i \equiv i+1$, go to 1.
2. Otherwise, construct o^t as follows and go to step $t+1$.
For $l = 1, ..., i$, delete f_l^t between positions k_l^t and $\bar{k}^t$ of o^{t-1} and insert f_l^t in position $\bar{k}^t + l$ of o^{t-1}.

The procedure ends when no firm is rejected by its μ-partner who is still holding her initial partner.[13]

When constructing a new sequence o^t, the procedure defers the offer of f_1^t to its μ-partner in position k_1^t of o^{t-1} to a new position (position $\bar{k}^t + 1$ of o^{t-1}) where its μ-partner no longer holds her initial partner. If f_1^t has fired its initial partner w_1^t to propose to its μ-partner in position k_1^t of o^{t-1}, the deferral of its offer to its μ-partner may lead w_1^t to be still holding f_1^t after position k_1^t of the new sequence. While the procedure is taking a corrective action to avoid the rejection of f_1^t by its μ-partner who is still holding her initial partner, the deferral of f_1^t's offer may itself produce a rejection of this kind that would otherwise not be present. Precisely, if w_1^t receives an offer from her μ-partner between positions k_1^t and $\bar{k}_1^t$ of o^{t-1} (condition c.3) and she prefers her initial partner f_1^t to her μ-partner (condition c.4), the deferral of f_1^t's offer to its μ-partner and therefore the deferral of its firing of w_1^t will lead w_1^t to reject the offer of her μ-partner as she will still be holding her initial partner f_1^t. However, this offer would indeed have been accepted had the offer of f_1^t to its μ-partner not been deferred. To prevent this happening, the procedure also defers the offer that w_1^t receives from its μ-partner. The procedure iterates this process until no further rejections of this kind occur.

An assumption in our description of the game is that a firm fires its initial partner (if any) when it chooses to make an offer to a worker. If a firm fires its initial partner to make an offer to its μ-partner then its top choice in its preference strategy should be its μ-partner. The following remark points out this observation.

Remark 3. *Condition c.5 implies that f_i^t makes its first offer in position k_i^t of o^{t-1}. Hence, $T(Q_{f_i^t}) = \mu(f_i^t)$.*

The procedure is well-defined. We defer the proof to the Appendix B. For each step t of the procedure, let μ^t denote the outcome of the game in o^t. Let $\mu^0 \equiv \mu'$. Let $o^t\mid_k$ denote the finite sequence consisting of k elements which coincides with o^t for the first k elements. Let t_h be the final step of the procedure. The final sequence that the procedure returns is o^{t_h} and the outcome of the game in o^{t_h} is μ^{t_h}. Since each sequence has a positive probability of occurring, for each t with $t_h \geq t \geq 0$, $\mu^t \in suppG[Q]$. By (3),

$$\text{for each } f \in F, \text{either } \mu(f) = \mu^t(f) \text{ or } \mu(f) \; Q_f \; \mu^t(f). \tag{5}$$

Step 4: We show that the procedure returns a final sequence o^{t_h} in which no firm is rejected by its μ-partner. Assume by contradiction that there is a firm which is rejected

[13] An illustration of the procedure is available in the Appendix B of the *working paper* version.

by its μ-partner in o^{t_h}. Let f be the first such firm in o^{t_h}. Let $w \equiv \mu(f)$. Suppose f is rejected by w in favor of f'. Hence, $f' \; Q_w \; f = \mu(w)$. Since o^{t_h} is the final sequence returned by the procedure, no worker is holding her initial partner when she rejects her μ-partner in o^{t_h}. Therefore, $f' \neq \mu^I(w)$. Also, f' must have proposed to w in o^{t_h} who must have accepted it. By Remark 2, $f' \in A(Q_w)$. By $Q_w \in Tr(P_w)$, $f' \; P_w \; \mu(w)$. By $\mu \in S(Q_F, P_W)$, $\mu(f') \; Q_{f'} \; w$. This implies that f' has proposed to and been rejected by its μ-partner prior to its offer to w in o^{t_h}, contradicting the assumption that f is the first firm in o^{t_h} which is rejected by its μ-partner.

By (5) and Step 4, $\mu^{t_h} = \mu$. Thus we obtained an order o^{t_h} from o' where no firm is rejected by its μ-partner.

Step 5: We now create a new order $\tilde{o}$ from o^{t_h} by changing the position that f^1 in Step 1 of the procedure proposes to its μ-partner in o^{t_h}. Let $w^1 \equiv \mu(f^1)$. By the definition of the procedure, w^1 rejects f^1 in position k^1 of o' while she is still holding her initial partner. The procedure takes o' as an input and produces o^{t_h} where f^1's offer to its μ-partner w^1 is not rejected. Let $\tilde{o}$ be a sequence whose elements are the same as those in o^{t_h} but that differs from o^{t_h} in that the position in which f^1 proposes to its μ-partner w^1 in o^{t_h} is deleted and inserted in position k^1 of o^{t_h}. Since for each $t^h \geq t \geq 1$, $o^0 \mid_{k^1-1} = o^t \mid_{k^1-1}$, we have $\tilde{o} \mid_{k^1} = o^0 \mid_{k^1}$. Then, w^1 must be still holding her initial partner when she receives f^1's offer in position k^1 of $\tilde{o}$. Therefore f^1 must have been rejected by w^1 in $\tilde{o}$. Let $\tilde{\mu}$ denote the outcome of the game in $\tilde{o}$. Hence, $\mu(f^1) = w^1 \; Q_{f^1} \; \tilde{\mu}(f^1)$. By $\tilde{\mu} \in S(Q_F, P_W)$, $\tilde{\mu}(w^1) \; P_{w^1} \; f^1 = \mu(w^1)$. As $\mu(w^1) \in F$, by $\mu, \tilde{\mu} \in S(Q_F, P_W)$ and Proposition 1, $\tilde{\mu}(w^1) \in F$. Let $\tilde{f} \equiv \tilde{\mu}(w^1)$.

Step 6: We complete the proof by identifying a profitable deviation. Consider an alternative strategy s_{w^1} such that w^1 acts according to Q_{w^1} at each of her information sets except for the information set that leads to the acceptance of f^1's offer in (o^{t_h}, Q). Let $s \equiv (s_{w^1}, Q_{-w^1})$. When using s_{w^1}, w^1 rejects f^1's offer in position k^1 of o^{t_h} and acts according to Q_{w^1} in each other information set. Since for each $t \geq 1$, $o^0 \mid_{k^1-1} = o^t \mid_{k^1-1}$, f^1 has no decision nodes between position k^1 of o^{t_h} and the position in which it proposes to w^1 in o^{t_h}. Thus, by the construction of $\tilde{o}$, the same outcome is obtained in $(\tilde{o}, Q)$ and in $(o^{t_h}, (s_{w^1}, Q_{-w^1}))$. When nature chooses the sequence o^{t_h}, w^1 achieves $\tilde{f}$ if she acts according to s_{w^1} and she achieves f^1 if she acts according to Q_{w^1} and $\tilde{f} \; P_{w^1} \; f^1$. We next argue that for any other sequence where w^1 achieves a partner $f \in U(P_{w^1}, \tilde{f})$ in (o, Q) she cannot achieve a less preferred partner than f in (o, s). This will complete the proof as the probability of being assigned to a firm at least as desirable as $\tilde{f}$ is larger when she uses s_{w^1}, contradicting the assumption that Q is an sd-Nash equilibrium in truncations.

Let $o \in \mathcal{O}$ be such that w^1 achieves a partner $f \in U(P_{w^1}, \tilde{f})$ in (o, Q). Notice that $f \neq f^1$. If the information set where w^1 accepts f^1 in (o^{t_h}, Q) is not reached in (o, s) then s_{w^1} dictates the same actions as Q_{w^1} and w^1 is matched to f in (o, s) too. Suppose then that the information set where w^1 accepts f^1 in (o^{t_h}, Q) is reached in (o, s). In this information set, Q_{w^1} dictates w^1 to accept f^1 in (o, Q) and s_{w^1} dictates w^1 to reject f^1 in (o, s). Let k be the position of o in which w^1 accepts f^1 in (o, Q) and rejects f^1 in (o, s). Notice that Q_{w^1} and s_{w^1} dictate the same actions until position k of o. Since w^1 achieves a partner $f \in U(P_{w^1}, \tilde{f})$ in (o, Q) and $\tilde{f}\ Q_{w^1}\ f^1$, she must have received and accepted the offer of f after position k of o. Since $\mu^I(w^1)\ Q_{w^1}\ f^1$ and w^1 accepts f^1 in position k of o at Q, she must not be still holding her initial partner in position k of o at Q nor is she at s. We show that w^1 cannot be matched to a less preferred partner than f in (o, s).

Assume by contradiction that w^1 is matched to a less preferred firm than f in (o, s). 1) f is rejected by w^1 in (o, s). Since Q_{w^1} and s_{w^1} dictate the same actions until position k of o and $f \neq f^1$, w^1 must have received f's offer after position k in (o, s). Then w^1 must not be still holding her initial partner when she rejects f in (o, s). Also, s_{w^1} dictates w^1 to act according to Q_{w^1} after position k of o. Then f must have been rejected in favor of a Q_{w^1}-preferred firm. By $Q_{w^1} \in Tr(P_{w^1})$, and $f \in A(Q_{w^1})$, w^1 must be matched to a P_{w^1}-preferred firm than f in (o, s), contradicting our assumption. 2) f does not propose to w^1 in (o, s). Then there must be a position of o where f is rejected by a worker w at Q but held at s. Let k_0 be the first position of o where a firm, say f_0, is rejected by a worker at Q but held at s. Let w_0 denote the worker who rejects f_0 in position k_0 at Q. If f_0 is rejected by w_0 while she is still holding her initial partner then $\mu^I(w_0)$ makes its first offer after position k_0 of o. Then f_0 must have also been rejected by w_0 in position k_0 at s, contradicting the definition of f_0. Then f_0 must have been rejected in favor of a firm f' at Q whose offer w_0 receives in position k_0 of o. Since w_0 does not receive the offer in position k_0 at s, there must be a position prior to k_0 such that f' is rejected by a worker at Q but held at s, contradicting the definition of k_0. □

For the last result, we consider a subset of truncation strategies. Let $\mu \in \mathcal{M}$ and $P \in \mathcal{P}$. Let $v \in V$ be such that $\mu(v) \neq v$. We say Q_v is a **truncation strategy at $\boldsymbol{\mu(v)}$** if Q_v is a truncation strategy that has the same ordering as P_v up to the element $\mu(v)$ and ranks all other elements unacceptable. For $v \in V$ such that $\mu(v) = v$ any truncation strategy is a truncation strategy at $\mu(v)$. Let $Tr(P_v, \mu(v))$ denote the truncation of P_v at $\mu(v)$.

The following two lemmata will be used in the proof of Proposition 3. Let $P \in \mathcal{P}$ and $\mu \in S(P)$. The first one states that if an agent v prefers v' to her/its μ-partner and if each of v and v' adopts a truncation strategy at their respective μ-partners, then v must not be listed acceptable at v''s strategy.

Lemma 1. *Let $P \in \mathcal{P}$, $\mu^I \in \mathcal{M}$ and $\mu \in S(P)$. Let $v, v' \in V$ be such that $v'\ P_v\ \mu(v)$. If $Q_{v'} \in Tr(P_{v'}, \mu(v'))$, then $v \notin A(Q_{v'})$.*

Proof. By $\mu \in S(P)$, $\mu(v')\ P_{v'}\ v$. By $Q_{v'} \in Tr(P_{v'}, \mu(v'))$, $v \notin A(Q_{v'})$. □

Now, let $\mu \in S(P)$ satisfy conditions (a.1) and (a.2) in the definition of an admissible initial matching. If each agent $v \in V$ except one firm adopts a truncation strategy at $\mu(v)$, then the firm cannot receive a better partner than its μ-partner at any play of the game.

Lemma 2. *Let $f' \in F$. Let $\mu^I \in IR(P)$ and $\mu \in S(P)$ satisfy conditions (a.1) and (a.2). Let $Q \in \mathcal{P}$ be such that for each $v \in V \backslash \{f'\}$, $Q_v \in Tr(P_v, \mu(v))$. For each $\mu' \in suppG[Q]$, $\mu(f')\ \ R_{f'}\ \ \mu'(f')$.*

Proof. Assume by contradiction that there is $\mu' \in suppG[Q]$ such that $\mu'(f')\ P_{f'}\ \mu(f')$. Let $F_0 \equiv \{f \in F : \mu'(f)\ P_f\ \mu(f)\}$. Notice that $f' \in F_0$. By $\mu \in S(P)$,

$$\text{for each } f \in F_0,\ \mu'(f) \in W. \tag{6}$$

Each of the following numbered paragraphs begins with a statement and follows with its proof.

(p.1) For each $f \in F_0$, $\mu'(f) = \mu^I(f)$. Let $f \in F_0$. By (6), $\mu'(f) \in W$. Let $w \equiv \mu'(f)$. Then, $w\ P_f\ \mu(f)$. By Lemma 1, $f \notin A(Q_w)$. By Remark 2, $\mu'(w) = \mu^I(w) = f$.

(p.2) For each $f \in F_0 \backslash \{f'\}$, $\mu^I(f) = T(P_f)$. Let $f \in F_0 \backslash \{f'\}$ and $w \equiv \mu^I(f)$. By (p.1), $f \notin A(Q_w)$. By Remark 2, w never accepts f's offer at any play of the game. Then $\mu'(f) = w$ is possible only if $w = \mu^I(f) = T(P_f)$ and hence, f passes its turn and keeps w in its first move in the game.

Let $W_0 \equiv \{w \in W : \mu(w) \in F_0\}$.

(p.3) $\mu(W_0) \subseteq F_0$. This follows from the definition of W_0.

(p.4) $\mu^I(F_0) \subseteq W_0$. Let $f_0 \in F_0$. By the definition of F_0 and (p.1), $\mu'(f_0) = \mu^I(f_0)\ P_{f_0}\ \mu(f_0)$. Let $w \equiv \mu'(f_0)$. By $\mu \in S(P)$, $\mu(w)\ P_w\ f_0 = \mu^I(w)$. By $\mu^I \in IR(P)$, $\mu(w) \in F$. Let $f \equiv \mu(w)$. By $Q_w \in Tr(P_w, \mu(w))$, $\mu(w) = f\ Q_w\ f_0 = \mu'(w) = \mu^I(w)$. Then f must have made no offer to w at any play of the game that leads to μ'. Thus $\mu'(f)\ Q_f\ w$. We show that $f \in F_0$. If $f = f'$, then the result is immediate. Assume that $f \neq f'$. By $Q_f \in Tr(P_f, \mu(f))$, $\mu'(f)\ P_f\ w = \mu(f)$. Hence, $\mu(w) = f \in F_0$ and $\mu^I(f_0) = w \in W_0$. As f_0 is arbitrary, $\mu^I(F_0) \subseteq W_0$.

We complete the proof of lemma. As μ and μ^I are one-to-one, by (p.3) and (p.4), μ and μ' map F_0 onto W_0. By (p.2) for each $f \in F_0 \backslash \{f'\}$, $\mu^I(f) = T(P_f)$. But $\mu^I(f') = \mu'(f')\ P_{f'}\ \mu(f')$, contradicting (a.2). $\square$

Proof of Proposition 3: Let $\mu^I \in S(P)$. Let $Q \in \mathcal{P}$ be such that for each $v \in V$, $Q_v \in Tr(P_v, \mu^I(v))$. We show that Q is an sd-Nash equilibrium. Let $v, v' \in V$ be such that $v'\ P_v\ \mu^I(v)$. By Lemma 1, $v \notin A(Q_{v'})$. If $v \in F$, by Remark 2, v' never accepts v's offer at any play of the game. If $v \in W$, by Remark 1, v' never proposes to v at any play of the game. Thus each play of the game with profile Q leads to μ^I. This also shows that no agent v can unilaterally deviate and obtain a partner v' that satisfies $v'\ P_v\ \mu^I(v)$. Therefore, $Q \in \mathcal{P}$ is an sd-Nash equilibrium in truncations.

Now, let $\mu^I \notin S(P)$. Then $\mu^I \in IR(P)$ and there is $\mu \in S(P)$ that satisfies (a.1) and (a.2). Let $Q \in \mathcal{P}$ be such that for each $v \in V$, $Q_v \in Tr(P_v, \mu(v))$. We show that Q is an sd-Nash equilibrium. Lemma 2 is proved for an arbitrary firm f' and for a general strategy $Q_{f'}$. A direct implication is that for each $f \in F$, $\mu(f)\ R_f\ \mu'(f)$.

Step 1: We argue that $suppG[Q] = \{\mu\}$. Assume by contradiction that there is $\mu' \in suppG[Q]$ such that $\mu' \neq \mu$. Then there is $f \in F$ such that $\mu'(f) \neq \mu(f)$. By Lemma 1, $\mu(f)\ P_f\ \mu'(f)$. Since f never proposes to a worker unacceptable at Q_f, by $Q_f \in Tr(P_f, \mu(f))$, $\mu(f)\ Q_f\ \mu'(f) = f$ and $\mu(f) \in W$. Let $w \equiv \mu(f)$. Then f must have proposed to and been rejected by w at any play of the game that leads to μ'. By $Q_w \in Tr(P_w, \mu(w))$, any $f' \in F$ that satisfies $f'\ Q_w\ f = \mu(w)$ must also satisfy $f'\ P_w\ f = \mu(w)$. By Lemma 1, $w \notin A(Q_{f'})$. Thus, any $f' \in F$ that satisfies $f'\ Q_w\ f$ never proposes to w at any play of the game. Therefore, w must have been holding her initial partner when she has rejected f. Thus, $\mu^I(w)\ Q_w\ f = \mu(w)$. By $Q_w \in Tr(P_w, \mu(w))$, $\mu^I(w)\ P_w\ f = \mu(w)$. Let $f_0 \equiv \mu^I(w)$. By $\mu \in S(P)$, $\mu(f_0)\ P_{f_0}\ w$, contradicting condition (a.1).

Step 2: We now show that Q is an sd-Nash equilibrium. By Lemma 2, no firm can do better than being assigned to its μ-partner at any play of the game. Now, let $w \in W$ and $Q'_w \in \mathcal{P}_w$ be an alternative strategy. Let $\mu' \in suppG[Q'_w, Q_{-w}]$. We show that $\mu(w)\ R_w\ \mu'(w)$. Assume by contradiction that $\mu'(w)\ P_w\ \mu(w)$. By $\mu \in S(P)$, $\mu'(w) \in F$. Let $f \equiv \mu'(w)$. Thus, $f\ P_w\ \mu(w)$. Then, by Lemma 1, $w \notin A(Q_f)$. Thus f has never proposed to w at any play of the game, contradicting $f = \mu'(w)$. $\square$

7 Appendix B

Lemma 3. *The procedure in the proof of Theorem 3 is well-defined.*

Proof. The proof will proceed in steps.
Step 1: We show that $\bar{k}^t$ exists. Notice that for each step t of the procedure $\mu^{t-1} \in suppG[Q]$. Let $f \in F$. If $\mu(f) = f$, then by Theorem 2, $\mu^{t-1}, \mu \in S(Q_F, P_W)$ and by Proposition 1, $\mu(f) = \mu^{t-1}(f) = f$. Since the game ends when each firm sequentially passes its turn, there is a position of o^{t-1} where f passes its turn and remains unmatched. If $\mu(f) \neq f$, then by (5), either $\mu(f) = \mu^t(f)$ or $\mu(f)\ Q_f\ \mu^t(f)$. Thus f must have proposed to its μ-partner in o^{t-1}. Thus $\bar{k}^t$ exists.

Step 2: We show that for each step t of the procedure no firm is rejected by its μ-partner prior to position k^t of o^{t-1}. Let t be a step of the procedure. Assume by contradiction that there is a firm which is rejected by its μ-partner prior to position k^t of o^{t-1}. Let f be the first such firm in o^{t-1}. Let $w \equiv \mu(f)$. Suppose f is rejected by w in favor of f'. Hence, $f'\ Q_w\ f = \mu(w)$. By the definition of the procedure no worker is holding her initial partner when she rejects her μ-partner prior to position k^t of o^{t-1}. Therefore, $f' \neq \mu^I(w)$. Also, f' must have proposed to w in o^{t-1} who must have accepted it. By Remark 2, $f' \in A(Q_w)$. By $Q_w \in Tr(P_w)$, $f'\ P_w\ \mu(w)$. By $\mu \in S(Q_F, P_W)$, $\mu(f')\ Q_{f'}\ w$. This implies that f' has proposed to and been rejected by its μ-partner prior to its offer to w, contradicting the assumption that f is the first firm in o^{t-1} which is rejected by its μ-partner prior to position k^t of o^{t-1}.

Step 3: We show that step t ends in finite time. First we argue that $k^t < \bar{k}^t$. Assume by contradiction that $k^t > \bar{k}^t$. If $\mu(f^{I,t}) = f^{I,t}$, then $f^{I,t}$ passes its turn and remains unmatched in position $\bar{k}^t$. Otherwise, $f^{I,t}$ proposes to its μ-partner in position $\bar{k}^t$. By step 2, $f^{I,t}$ is not rejected by its μ-partner in position $\bar{k}^t$. In either case, $\mu(f^t)$ is not holding her initial partner $f^{I,t}$ in position k^t when she rejects f^t, contradicting the definition of the procedure. Thus, $k^t < \bar{k}^t$. Since for each i, $k_i^t < k_{i+1}^t$ and $\bar{k}^t$ is fixed, step t ends in finite time.

Step 4: We show that the procedure ends in finitely many steps. We show that for any two steps t_1 and t_2, $t_1 \neq t_2$ of the procedure, $f^{t_1} \neq f^{t_2}$. Since F is finite, this completes the step and the proof. The procedure returns a sequence in which no firm is rejected by its μ-partner who is still holding her initial partner.

Let t_1 and t_2, $t_1 < t_2$ be two steps of the procedure. We first show that in o^{t_1}, f^{I,t_1} has at least one decision node before f^{t_1} proposes to its μ-partner. Firm f^{I,t_1} has a decision node in position $\bar{k}^{t_1}$ of o^{t_1-1}. By part (2) of step t_1, the position in which f^{t_1} proposes to its μ-partner in o^{t_1-1} is moved to position $\bar{k}^{t_1}+1$ of o^{t_1-1}. Furthermore, all decision nodes of f^{t_1} between k^{t_1} and $\bar{k}^{t_1}$ are deleted. Thus, f^{I,t_1} has at least one decision node before f^{t_1} proposes to its μ-partner in o^{t_1}. We now show that f^{I,t_1} has at least one decision node before f^{t_1} proposes to its μ-partner in o^{t_2-1}. This will complete the proof because the μ-partner of f^{t_1} will have been fired by f^{I,t_1} before she will receive f^{t_1}'s offer in o^{t_2-1}. Thus, $f^{t_1} \neq f^{t_2}$.

Assume by induction that in $o^{t_1+1}, ..., o^{t_2-2}$, f^{I,t_1} has at least one decision node before f^{t_1} proposes to its μ-partner. We show that f^{I,t_1} also has at least one decision node before f^{t_1} proposes to its μ-partner in o^{t_2-1}.

Assume by contradiction that f^{I,t_1} has no decision nodes before f^{t_1} proposes to its μ-partner in o^{t_2-1}. Let $\tilde{k}$ be the position of o^{t_2-2} in which f^{t_1} proposes to its μ-partner. By the induction assumption f^{I,t_1} has at least one decision node prior to position $\tilde{k}$ of o^{t_2-2} but has no decision nodes before f^{t_1} proposes to its μ-partner in o^{t_2-1}. This implies that f^{I,t_1} proposes to its μ-partner prior to position $\tilde{k}$ of o^{t_2-2} and for some i, say r, in step t_2-1, we have $f_r^{t_2-1} = f^{I,t_1}$. Hence $k^{t_2-1} < \tilde{k}$. Since decision nodes of f^{I,t_1} before it proposes to its μ-partner in o^{t_2-2} are not deleted in the procedure and yet f^{I,t_1} has no decision nodes before f^{t_1} proposes to its μ-partner in o^{t_2-1}, then f^{I,t_1} makes its first offer to its μ-partner in o^{t_2-2}. Hence, we have Fact 1 below.
Fact 1: $T(Q_{f^{I,t_1}}) = \mu(f^{I,t_1})$.

For notational ease we suppress the superscript t_2-1 on firms and workers involved in part (1) of step t_2-1. We replace $f_1^{t_2-1}, , ..., f_r^{t_2-1}, w_1^{t_2-1}, ..., w_r^{t_2-1}$ by $f_1, ..., f_r, w_1, ..., w_r$ respectively.

Case 1: $\bar{k}^{t_2-1} = \tilde{k}$. By the definition of the procedure, the initial partner of $w_0 \equiv \mu(f_1)$ proposes to its μ-partner in position $\tilde{k}$ of o^{t_2-2}. Therefore, $\mu(f_1) = \mu^I(f^{t_1}) = w_0$. Also we have Fact 2 below.
Fact 2: f_1 is rejected by its μ-partner w_0 who is still holding her initial partner f^{t_1} in position k^{t_2-1} of o^{t_2-2}.

By Fact 2, $\mu^I(w_0) = f^{t_1}\ Q_{w_0}\ f_1 = \mu(w_0)$. By Remark 2, w_0 never accepts the offer of an unacceptable firm at Q_{w_0}. Then, $f_1 \in A(Q_{w_0})$. Since $Q_{w_0} \in Tr(P_{w_0})$, $f^{t_1}\ P_{w_0}\ f_1 = \mu(w_0)$. By $\mu \in S(Q_F, P_W)$, $\mu(f^{t_1})\ Q_{f^{t_1}}\ w_0 = \mu^I(f^{t_1})$. We show that $T(Q_{f^{t_1}}) = \mu(f^{t_1})$. Suppose not. Then there is $w \in W\backslash\{w_0\}$ such that $w\ Q_{f^{t_1}}\ \mu(f^{t_1})$. Since for each $t \geq t_1$, we have $o^{t_1-1} |_{k^{t_1}-1} = o^t |_{k^{t_1}-1}$ and since f^{t_1} proposes to its μ-partner in position k^{t_1} of o^{t_1-1}, f^{t_1} fires its initial partner w_0 and proposes to w prior to position k^{t_2-1} of o^{t_2-2}, contradicting Fact 2. Thus, $T(Q_{f^{t_1}}) = \mu(f^{t_1})$. Also, by Remark 3, for each

$i \in \{1, 2, ..., r\}$, $T(Q_{f_i}) = \mu(f_i)$.

$$\begin{array}{rccccc||cccccl}
 & \underline{Q_{f_1}} & \underline{Q_{f_2}} & \cdots & \underline{Q_{f_r}} & \underline{Q_{f^{t_1}}} & \underline{Q_{w_0}} & \underline{Q_{w_1}} & \cdots & \underline{Q_{w_{r-1}}} & \underline{Q_{w_r}} & \\
\boldsymbol{\mu} \rightarrow & \boldsymbol{w_0} & \boldsymbol{w_1} & & \boldsymbol{w_{r-1}} & \boldsymbol{w_r} & \vdots & \vdots & & \vdots & \vdots & \\
 & \vdots & \vdots & & \vdots & \vdots & f^{t_1} & f_1 & & f_{r-1} & f_r & \leftarrow \mu^I \\
\mu^I \rightarrow & w_1 & w_2 & & w_r & w_0 & \vdots & \vdots & & \vdots & \vdots & \\
 & \vdots & \vdots & & \vdots & \vdots & \boldsymbol{f_1} & \boldsymbol{f_2} & & \boldsymbol{f_r} & \boldsymbol{f^{t_1}} & \leftarrow \boldsymbol{\mu} \\
 & & & & & & \vdots & \vdots & & \vdots & \vdots &
\end{array}$$

Let $\hat{W} \equiv \bigcup_{i=0}^{r} w_i$ and $\hat{F} \equiv \bigcup_{i=1}^{r} f_i \cup \{f^{t_1}\}$. Consider the sequence o which leads to the matching μ. From the above profile of strategies we deduce that μ and μ^I map $\hat{F}$ onto $\hat{W}$. Thus, for each $f \in \hat{F}$, $\{\mu(f), \mu^I(f)\} \subseteq \hat{W}$. Each $f \in \hat{F}$ makes its first offer to and is rejected by its μ-partner who is still holding her initial partner. Therefore, no $f \in \hat{F}$ can be matched to its μ-partner when the sequence is o. This is a contradiction.

Case 2: $\bar{k}^{t_2-1} < \tilde{k}$. As $f_r = f^{I,t_1}$, the node in which f^{I,t_1} proposes to its μ-partner in o^{t_2-2} is deleted and inserted in position $\bar{k}^{t_2-1} + r$ of o^{t_2-2}. Notice that $k^{t_1} < k^{t_2-1} < \bar{k}^{t_2-1} < \tilde{k}$ and that $o^{t_2-2} \mid_{k^{t_1}-1} = o^{t_1} \mid_{k^{t_1}-1}$. Since f^{t_1} has no decision nodes between positions k^{t_1} and $\tilde{k}$ of o^{t_2-2}, then f^{I,t_1} has at least one decision node before f^{t_1} proposes to its μ-partner in o^{t_2-1}. This is a contradiction.

Case 3: $\tilde{k} < \bar{k}^{t_2-1}$. Notice that $f_r = f^{I,t_1}$ and $\mu(f^{t_1}) = \mu^I(f^{I,t_1})$. Thus $\mu(f^{t_1}) = \mu^I(f^{I,t_1}) = w_r$ (Conditions c.1 and c.2). The position of o^{t_2-2} in which f^{I,t_1} proposes to its μ-partner is denoted by $k_r^{t_2-1}$. By the induction assumption and Fact 1, $k_r^{t_2-1} < \tilde{k}$. Thus, $k_r^{t_2-1} < \tilde{k} < \bar{k}^{t_2-1}$. Hence f^{t_1} proposes to its μ-partner w_r between positions $k_r^{t_2-1}$ and $\bar{k}^{t_2-1}$ of o^{t_2-2} (Condition c.3). By the definition of the procedure, w_r rejects her μ-partner f^{t_1} and keeps her initial partner f^{I,t_1} in o^{t_1-1}. Thus $f^{I,t_1} = f_r \; Q_{w_r} \; \mu(w_r) = f^{t_1}$ (Condition c.4). By Fact 1, f^{I,t_1} fires its initial partner w_r to propose to its μ-partner in position $k_r^{t_2-1}$ of o^{t_2-2} (Condition c.5). Hence $f_{r+1} = f^{t_1}$.

By the construction of the procedure, the moves of f^{I,t_1} and of f^{t_1} in positions $k_r^{t_2-1}$ and $\tilde{k}$ of o^{t_2-2} respectively are deleted and inserted in positions $\bar{k}^{t_2-1} + r$ and $\bar{k}^{t_2-1} + r + 1$ of o^{t_2-2} respectively. Also, decision nodes of f^{I,t_1} between positions $k_r^{t_2-1}$ and $\bar{k}^{t_2-1}$ of o^{t_2-2} and of f^{t_1} between positions $\tilde{k}$ and $\bar{k}^{t_2-1}$ of o^{t_2-2} are deleted. Thus f^{I,t_1} has at least one decision node before f^{t_1} proposes to its μ-partner in o^{t_2-1}. This is a contradiction. $\square$

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