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Ball covering property and number of ends of CD spaces with non-negative curvature outside a compact set

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Abstract. In this paper, we adapt work of Z.-D. Liu to prove a ball covering property for non-branching CD spaces with non-negative curvature outside a compact set. As a consequence, we obtain uniform bounds on the number of ends of such spaces.

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1. Introduction. In [10,11], Z.-D. Liu proved that Riemannian manifolds with non-negative Ricci curvature outside a compact set satisfy a certain *ball covering property*. In the following, we denote the metric ball of radius r centered at $p \in M$ by $B_r(p)$ and the closed metric ball with the same radius and center by $\overline{B}_r(p)$.

Theorem A (Z.-D. Liu). *Let M^n be a complete Riemannian manifold with non-negative Ricci curvature outside a compact set B. Assume that $\mathrm{Ric}_M \geq (n-1)H$ and that $B \subset B_{D_0}(p_0)$ for some $p_0 \in M$ and $D_0 > 0$. Then for any $\mu > 0$, there exists $C = C(n, HD_0^2, \mu) > 0$ such that, for any $r > 0$, the following property is satisfied: If $S \subset \overline{B}_r(p_0)$, there exist $p_1, \dots, p_k \in S$ with $k \leq C$ such that*

$$S \subset \bigcup_{j=1}^{k} B_{\mu \cdot r}(p_j).$$

We state and prove this result in the more general context of non-branching metric measure spaces satisfying the curvature-dimension condition introduced by Lott–Sturm–Villani [12,14,15] (see the section on preliminaries below for

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the definitions). This class of spaces contains the class of RCD spaces, as it was recently shown that the latter spaces are non-branching (see [6, Theorem 1.3]), so, a fortiori, it also includes Alexandrov spaces [13,17] and weighted Riemannian manifolds. More precisely, we prove the following theorem.

Theorem B. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a non-branching metric measure space satisfying the $\mathsf{CD}(K,N)$ condition for some $N > 1$ and $K \in \mathbb{R}$. Assume that B is a compact subset of X with $B \subset B_{D_0}(p_0)$ for some $p_0 \in M$ and $D_0 > 0$ and such that the $\mathsf{CD}_{loc}(0,N)$ condition is satisfied on $X \setminus B$ (see Definition 2.6). Then for any $\mu > 0$, there exists $C = C(N, KD_0^2, \mu) > 0$ such that, for any $r > 0$, the following property is satisfied: If $S \subset \overline{B}_r(p_0)$, there exist $p_1, \ldots, p_k \in S$ with $k \leq C$ and such that*

$$S \subset \bigcup_{j=1}^{k} B_{\mu \cdot r}\left(p_j\right).$$

The proof follows the arguments of [10,11] almost verbatim, albeit with some needed adaptations to account for the more general hypotheses. The main tools we need are a version of the local-to-global theorem for the CD condition (see Lemma 2.9) and a Bishop–Gromov inequality for certain star-shaped sets (see Theorem 2.11). In general, one can prove that $\mathsf{CD}(K,N)$ spaces support a Bishop–Gromov inequality for star-shaped sets following the proof of [15, Theorem 2.3], just as is done in [5, Proposition 3.5] to get a timelike Bishop-Gromov inequality in the context of Lorentzian synthetic spaces. However, it is important to notice that Lemma 2.11 is not a direct consequence of this fact. Namely, since the $\mathsf{CD}(K,N)$ condition implies a Bishop–Gromov inequality with parameters K, N and we are interested in the corresponding inequality with parameters $0, N$, we need to follow the original proof of the Bishop-Gromov inequality in [15] and make sure that all optimal transports involved remain in the region where $\mathsf{CD}_{\mathrm{loc}}(0,N)$ holds.

Finally, a direct consequence is that spaces satisfying the hypotheses of Theorem B have a uniformly bounded number of ends (see Definition 3.3).

Corollary C. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space satisfying the $\mathsf{CD}(K,N)$ condition with $N \geq 1$ and $K \in \mathbb{R}$. Assume that B is a compact subset of X with $B \subset B_{D_0}(p_0)$ for some $p_0 \in M$ and $D_0 > 0$ and such that the $\mathsf{CD}_{loc}(0,N)$ condition is satisfied on $X \setminus B$. Then there exists $C = C(N, KD_0^2) > 0$ such that (X, d) has at most C ends.*

In [3], a result bounding the number of ends of manifolds with non-negative Ricci curvature outside a compact set was obtained using different techniques. This argument was extended in [16] to the case of smooth metric measure spaces with non-negative Bakry–Émery Ricci curvature outside of a compact set. A related result bounding the number of ends of Alexandrov spaces with non-negative sectional curvature outside a compact set was obtained in [9]. More recently, a result bounding the number of ends of $\mathsf{RCD}(0,N)$ spaces was obtained in [8].

2. Preliminaries. In this section, we provide a brief overview of the definitions and results we will need to prove Theorem B. Throughout the article, we consider complete and geodesic metric measure spaces $(X, \mathsf{d}, \mathfrak{m})$ such that $\mathfrak{m}$ is finite on bounded sets and $\operatorname{supp}(\mathfrak{m}) = X$. We begin by recalling the definition of the *Wasserstein space*.

Definition 2.1. Let $\mathcal{P}(X, \mathsf{d})$ be the set of Borel probability measures on X and $\mathcal{P}_2(X, \mathsf{d}, \mathfrak{m}) \subset \mathcal{P}(X, \mathsf{d})$ the space of those probability measures that are absolutely continuous with respect to $\mathfrak{m}$ and have finite second moment, i.e., for some (and therefore for any) $x_0 \in X$, the following holds:

$$\int d^2(x, x_0)\ d\mathfrak{m}(x) < \infty.$$

This set $\mathcal{P}_2(X, \mathsf{d}, \mathfrak{m})$ is endowed with the 2-Wasserstein metric

$$W_2(\mu_0, \mu_1) = \inf \left(\int\limits_{X \times X} \mathsf{d}^2\left(x_0, x_1\right)\ d\pi\left(x_0, x_1\right) \right)^{1/2},$$

where the infimum is taken over all couplings $\pi \in \mathcal{P}(X \times X)$ from μ_0 to μ_1, i.e., probability measures on $X \times X$ having first and second marginals equal to μ_0 and μ_1 respectively.

Remark 2.2. It turns out that $(\mathcal{P}_2(X), W_2)$ is also a complete separable geodesic space. Moreover, in this case, the distance $W_2(\mu_0, \mu_1)$ can be characterized as

$$W_2^2\left(\mu_0, \mu_1\right) = \min_{\pi} \int \int\limits_0^1 |\gamma_t|^2\ dt\ d\pi(\gamma),$$

where the minimum is taken among all $\pi \in \mathcal{P}(C([0,1], X))$ such that $(e_i)_{\#}\pi = \mu_i$, $i = 0, 1$. Here e_t denotes the usual evaluation map at time t. The set of minimizers is denoted by $\operatorname{OptGeo}(\mu_0, \mu_1)$, and minimizers, which are always supported in $\operatorname{Geo}(X)$ (the set of geodesics of (X, d)), are called *optimal plans*. It is known that $(\mu_t)_{t\in[0,1]}$ is a geodesic connecting μ_0 to μ_1 if and only if there exists $\pi \in \operatorname{OptGeo}(\mu_0, \mu_1)$ such that $\mu_t = (e_t)_{\#}\pi$ (see [1]).

In order to recall the definition of the CD condition, we now recall the *volume distortion coefficients*:

$$\mathfrak{s}_\kappa(\theta) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin\left(\sqrt{\kappa}\theta\right) & \text{if } \kappa > 0, \\ \theta & \text{if } \kappa = 0, \\ \frac{1}{\sqrt{\kappa}} \sinh\left(\sqrt{-\kappa}\theta\right) & \text{if } \kappa < 0, \end{cases}$$

$$\sigma_{K,N}^{(t)}(\theta) = \begin{cases} +\infty & \text{if } K\theta^2 \geq N\pi^2, \\ t & \text{if } K\theta^2 = 0 \text{ or } K\theta^2 < 0 \text{ and } N = 0, \\ \frac{\mathfrak{s}_{K/N}(t\theta)}{\mathfrak{s}_{K/N}(\theta)} & \text{if } K\theta^2 < N\pi^2 \text{ and } K\theta^2 \neq 0, \end{cases}$$

$$\tau_{K,N}^{(t)}(\theta) = t^{1/N} \sigma_{K,N-1}^{(t)}(\theta)^{1-1/N}.$$

Definition 2.3. Given parameters $K \in \mathbb{R}$ and $N \geq 1$, $(X, \mathsf{d}, \mathfrak{m})$ *satisfies the* $\mathsf{CD}(K, N)$ *condition* if for any $\mu_0, \mu_1 \in \mathcal{P}_2(X, \mathsf{d}, \mathfrak{m})$, there exists an optimal plan $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ such that, for any $t \in [0,1]$ and any $N' \geq N$,

$$\int \rho_t^{1-1/N'}(x)\ d\mathfrak{m}(x) \geq \int \tau_{K,N'}^{(1-t)}\left(\mathsf{d}\left(\gamma_0, \gamma_1\right)\right) \rho_0^{-1/N'}(\gamma_0) + \tau_{K,N'}^{(t)}\left(\mathsf{d}\left(\gamma_0, \gamma_1\right)\right) \rho_1^{-1/N'}(\gamma_1)\ d\pi(\gamma), \tag{2.1}$$

where ρ_t is the density of the absolutely continuous part of $(e_t)_{\#}\pi$ with respect to $\mathfrak{m}$.

Let us also recall the definition of the reduced curvature-dimension condition CD^* due to Bacher–Sturm [2].

Definition 2.4. Given parameters $K \in \mathbb{R}$ and $N \geq 1$, $(X, \mathsf{d}, \mathfrak{m})$ *satisfies the* $\mathsf{CD}^*(K, N)$ *condition* if for any $\mu_0, \mu_1 \in \mathcal{P}_2(X, \mathsf{d}, \mathfrak{m})$, there exists an optimal plan $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ such that, for any $t \in [0,1]$ and any $N' \geq N$,

$$\int \rho_t^{1-1/N'}(x)\ d\mathfrak{m}(x) \geq \int \sigma_{K,N'}^{(1-t)}\left(\mathsf{d}\left(\gamma_0, \gamma_1\right)\right) \rho_0^{-1/N'}(\gamma_0) + \sigma_{K,N'}^{(t)}\left(\mathsf{d}\left(\gamma_0, \gamma_1\right)\right) \rho_1^{-1/N'}(\gamma_1)\ d\pi(\gamma), \tag{2.2}$$

where ρ_t is the density of the absolute continuous part of $(e_t)_{\#}\pi$ with respect to $\mathfrak{m}$.

Let us point out that, by work of Cavalletti and Milman [4], an essentially non-branching metric measure space (see the Definition on Page 50 in [4]) satisfies the $\mathsf{CD}(K, N)$ condition if and only if it satisfies the $\mathsf{CD}^*(K, N)$ condition (see [4, Corollary 13.6]).

Definition 2.5. Given parameters $K \in \mathbb{R}$ and $N > 1$, $(X, \mathsf{d}, \mathfrak{m})$ *satisfies the* $\mathsf{CD}_{\mathrm{loc}}(K, N)$ *condition* if each point $x \in X$ has a neighborhood $M(x)$ such that, for each $\mu_0, \mu_1 \in \mathcal{P}_2(X, \mathsf{d}, \mathfrak{m})$ supported in $M(x)$, there exists an optimal plan $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ satisfying (2.1) for all $t \in [0,1]$ and $N' \geq N$.

We will assume that Definition 2.5 holds outside a compact set $B \subset X$ in the following sense.

Definition 2.6. The $\mathsf{CD}_{\mathrm{loc}}(K, N)$ *condition holds in an open set* $\Omega \subset X$ if each point $x \in \Omega$ has a neighborhood $M(x) \subset \Omega$ such that, for each $\mu_0, \mu_1 \in \mathcal{P}_2(X, \mathsf{d}, \mathfrak{m})$ supported in $M(x)$, there exists an optimal plan $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ satisfying (2.1) for all $t \in [0,1]$ and $N' \geq N$.

From this point on, we will also assume that $(X, \mathsf{d}, \mathfrak{m})$ is non-branching in the following sense.

Definition 2.7. A metric space (X, d) is *non-branching* if whenever we have a quadruple (z, x_0, x_1, x_2) such that z is a midpoint of x_0, x_1 and of x_0, x_2, this implies that $x_1 = x_2$.

In [7], it was proved that non-branching CD spaces have unique optimal plans in the following sense.

Theorem 2.8. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a complete, separable, non-branching $\mathsf{CD}(K, N)$ space for some $K \in \mathbb{R}$ and $N \geq 1$. Then for any $\mu_0, \mu_1 \in \mathcal{P}_2(X, \mathsf{d}, \mathfrak{m})$, there is a unique optimal plan $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ and this π is induced by a map, i.e., there exists a μ_0-measurable map $F : X \to \mathrm{Geo}(X)$ such that $\pi = F_{\#}\mu_0$.*

In [2], it was proved that the $\mathsf{CD}_{\mathrm{loc}}(K, N)$ condition implies the reduced curvature-dimension condition $\mathsf{CD}^*(K, N)$. In a similar fashion, and emulating the arguments in [2], we prove Lemma 2.9, that allows us to obtain, in Theorem 2.10, a version of the Brunn–Minkowski inequality. In turn, Theorem 2.10 implies Theorem 2.11, a Bishop–Gromov-type inequality. This result will be instrumental in generalizing Theorem A.

Below, we let $\mathcal{P}_\infty(X, \mathsf{d}, \mathfrak{m})$ denote the space of probability measures which are absolutely continuous with respect to $\mathfrak{m}$ and have bounded support. For the next lemma, note that we cannot directly apply the local-to-global property since we are using the restricted metric on Ω and this might not be a geodesic space unless Ω is geodesically convex, for example. However, the proof of [2, Theorem 5.1] applies verbatim as we are assuming that all the measures involved are connected by a geodesic in $\mathcal{P}_\infty(X, \mathsf{d}, \mathfrak{m})$ [1].

Lemma 2.9. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a $\mathsf{CD}(K_0, N)$ space and $\Omega \subset X$ an open set such that the $\mathsf{CD}_{loc}(K, N)$ condition holds in Ω. If $\mu_0, \mu_1 \in \mathcal{P}_\infty(X, \mathsf{d}, \mathfrak{m})$ are supported on Ω and the optimal plan $\pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ given by Theorem 2.8 is supported on geodesics contained in Ω, then π satisfies the condition (2.2) for K and for all $t \in [0, 1]$ and $N' \geq N$.*

Now we can follow the arguments in [15] to prove a generalized Bishop–Gromov result for star-shaped sets outside a compact set. To this end, we need the following modified version of the Brunn-Minkowski inequality.

Theorem 2.10. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a $\mathsf{CD}(K_0, N)$ space and $\Omega \subset X$ an open set such that the $\mathsf{CD}_{loc}(K, N)$ condition holds in Ω. Then for all measurable sets $A_0, A_1 \subset \Omega$ such that $\mathfrak{m}(A_0)\mathfrak{m}(A_1) > 0$ and $A_t \subset \Omega$ for all $t \in [0, 1]$,*

$$\mathfrak{m}(A_t)^{1/N'} \geq \sigma^{(1-t)}_{K,N'}(\theta)\mathfrak{m}(A_0)^{1/N'} + \sigma^{(t)}_{K,N'}(\theta)\mathfrak{m}(A_1)^{1/N'} \tag{2.3}$$

holds for all $t \in [0, 1]$ and all $N' \geq N$, where A_t denotes the set of points which divide geodesics starting in A_0 and ending in A_1 with ratio $t : (1 - t)$ and where θ denotes the minimal/maximal length of such geodesics, that is,

$$A_t := \{y \in X : \exists (x_0, x_1) \in A_0 \times A_1 \ s.t.\ \mathsf{d}(y, x_0) = t\mathsf{d}(x_0, x_1), \\ \mathsf{d}(y, x_1) = (1 - t)\mathsf{d}(x_0, x_1)\}$$

and

$$\theta := \begin{cases} \inf_{x_0 \in A_0, x_1 \in A_1} \mathsf{d}(x_0, x_1) & \text{if } K \geq 0, \\ \sup_{x_0 \in A_0, x_1 \in A_1} \mathsf{d}(x_0, x_1) & \text{if } K < 0. \end{cases}$$

In particular, if $K \leq 0$, then

$$\mathfrak{m}(A_t)^{1/N'} \geq (1 - t)\mathfrak{m}(A_0)^{1/N'} + t\mathfrak{m}(A_1)^{1/N'}.$$

[1] In fact, the proof of [2, Claim 5.2] simplifies in our case as by Theorem 2.8, the geodesic joining μ_0 and μ_1 is unique, so there is no need to construct the sequence $\Gamma^{(i)}$.

Proof. Assuming that $0 < \mathfrak{m}(A_0)\mathfrak{m}(A_1) < \infty$, and thanks to Lemma 2.9, we can apply $\mathsf{CD}^*(K,N)$ to $\mu_i := (1/\mathfrak{m}(A_i))1_{A_i}\mathfrak{m}$ for $i = 0,1$ and proceed as in [15, Proposition 2.1], just replacing the coefficients $\tau^{(t)}_{K,N'}(\cdot)$ by $\sigma^{(t)}_{K,N'}(\cdot)$. Observe that we cannot use the equivalence between the $\mathsf{CD}(K,N)$ and $\mathsf{CD}^*(K,N)$ conditions due to F. Cavalletti and E. Milman (see [4]) in order to get condition (2.1) in Lemma 2.9, even though the space X is non-branching, since we are assuming the $\mathsf{CD}_{\mathrm{loc}}(K,N)$ only in an open subset of X. The general case follows by approximation of A_i by sets of finite volume. $\square$

Recall that, given a geodesic metric space (X,d) and $x \in X$, a set $W_x \subset X$ is *star-shaped* at x if $x \in W_x$ and for any $y \in W_x$ not in the cut-locus of x, the minimal geodesic joining x and y is contained in W_x. In that case, we set

$$\begin{aligned} v(W_x,r) &= \mathfrak{m}\left(\overline{B}_r(x)\cap W_x\right) \quad \text{and} \\ s(W_x,r) &= \limsup_{\eta\to 0}\frac{1}{\eta}\mathfrak{m}\left(\left(\overline{B}_{r+\eta}(x)\setminus B_r(x)\right)\cap W_x\right). \end{aligned}$$

Theorem 2.11 (Bishop–Gromov theorem for star-shaped sets). *Let $(X,\mathsf{d},\mathfrak{m})$ be a $\mathsf{CD}(K_0,N)$ space and $\Omega \subset X$ an open set such that the $\mathsf{CD}_{loc}(K,N)$ condition holds in Ω. Let $x \in \Omega$ and $W_x \subset \Omega$ be a star-shaped set at x such that, for some $\epsilon_0 > 0$, every geodesic connecting points in $B_{\epsilon_0}(x)$ with points in W_x is contained in Ω. Then*

$$\frac{s(W_x,r)}{s(W_x,R)} \ge \left(\frac{\mathfrak{s}_{K/N}(r)}{\mathfrak{s}_{K/N}(R)}\right)^N \quad \textit{and} \quad \frac{v(W_x,r)}{v(W_x,R)} \ge \frac{\int_0^r \mathfrak{s}_{K/N}(t)^N\,dt}{\int_0^R \mathfrak{s}_{K/N}(t)^N\,dt}$$

for all $0 < r \le R \le \mathrm{rad}_x(W_x) := \sup\{\mathsf{d}(x,y) : y \in W_x\}$. Moreover, for $K = 0$, we obtain the sharp inequalities

$$\frac{s(W_x,r)}{s(W_x,R)} \ge \left(\frac{r}{R}\right)^{N-1} \quad \textit{and} \quad \frac{v(W_x,r)}{v(W_x,R)} \ge \left(\frac{r}{R}\right)^{N}.$$

Proof. Let $0 < r \le R \le \mathrm{rad}_x(W_x)$, $0 < \epsilon \le \epsilon_0$, and $\eta > 0$ and let

$$A_0 := B_\epsilon(x)\cap W_x \quad \text{and} \quad A_1 := \left(\overline{B}_{(1+\eta)R}(x)\setminus B_R(x)\right)\cap W_x.$$

In particular, the (r/R)-intermediate set $A_{r/R}$ between A_0 and A_1 is contained in Ω, so we can apply Theorem 2.10 to get

$$\begin{aligned} \mathfrak{m}\left(A_{r/R}\right)^{1/N} \ge{}& \sigma^{(1-r/R)}_{K,N}\left(R\mp\eta R\mp\epsilon\right)\mathfrak{m}(A_0)^{1/N} \\ &+\sigma^{(r/R)}_{K,N}\left(R\mp\eta R\mp\epsilon\right)\mathfrak{m}\left(A_1\right)^{1/N}. \end{aligned}$$

We can take $\epsilon \to 0$, which yields

$$\mathfrak{m}\left(\left(\overline{B}_{(1+\eta)r}(x)\setminus B_r(x)\right)\cap W_x\right) \ge \sigma^{(r/R)}_{K,N}\left((1\mp\eta)R\right)^N \mathfrak{m}\left(\left(\overline{B}_{(1+\eta)R}(x)\setminus B_R(x)\right)\cap W_x\right)$$

which implies

$$\begin{aligned} &\frac{1}{\eta r}\mathfrak{m}\left(\left(\overline{B}_{(1+\eta)r}(x)\setminus B_r(x)\right)\cap W_x\right) \\ &\quad\ge \frac{1}{\eta R}\mathfrak{m}\left(v\left(\overline{B}_{(1+\eta)R}(x)\setminus B_R(x)\right)\cap W_x\right)\sigma^{(r/R)}_{K,N}\left((1\mp\eta)\,R\right)^N. \end{aligned}$$

In the case $K = 0$, we actually have $\sigma_{K,N}^{(r/R)}((1 \mp \eta)R)^N = (r/R)^N$. The preceding inequality in turn implies

$$\begin{aligned}\frac{1}{\eta r}\mathfrak{m}\left(\left(\overline{B}_{(1+\eta)r}(x) \setminus B_r(x)\right) \cap W_x\right) \\ \geq \frac{1}{\eta R}\mathfrak{m}\left(\left(\overline{B}_{(1+\eta)R}(x) \setminus B_R(x)\right) \cap W_x\right)\left(\frac{r}{R}\right)^{N-1}.\end{aligned}$$

We thus can conclude just as in the proof of [15, Theorem 2.3]. □

3. Proofs. In this section, we prove Theorem B. The proof follows almost verbatim the arguments in [11]. For the convenience of the reader, we elaborate on this argument and stress the needed changes due to the more general hypotheses. We proceed with the following technical lemma assuming the hypothesis of Theorem B.

Lemma 3.1. *Assume that $p \in X$ and $R > 0$ are such that $B_{2R}(p) \subset X \setminus B_{2D}(p_0)$. Then for every $m > 0$, there exists $\delta = \delta(m) \in (0, 1)$ such that whenever a subset $W \subset B_R(p)$ satisfies*

$$\mathfrak{m}(W) \geq \frac{1}{m} \cdot \mathfrak{m}\left(B_R(p)\right), \tag{3.1}$$

then there exists $q \in W$ such that $\mathsf{d}(q, p) \leq \delta R$. In particular, $B_{(1-\delta)R}(q) \subset B_R(p)$.

Proof. Let $\delta \in (0, 1)$ and $W \subset B_R(p) \setminus \overline{B}_{\delta R}(p)$ be such that (3.1) holds. Since $B_{2R}(p) \cap B_{2D}(p_0) = \varnothing$, the optimal plan between any two probability measures supported on $B_{(1+\eta)R}(p)$ for sufficiently small $\eta > 0$ is concentrated in geodesics outside $B_D(p_0)$. Therefore, applying Theorem 2.11, we obtain

$$\frac{1}{m} \leq \frac{\mathfrak{m}(W)}{\mathfrak{m}(B_R(p))} \leq \frac{\mathfrak{m}\left(B_R(p) \setminus B_{\delta R}(p)\right)}{\mathfrak{m}(B_R(p))} \leq \frac{\int_{\delta R}^R t^{N-1}\, dt}{\int_0^R t^{N-1}\, dt} = 1 - \delta^N,$$

that is, $\delta \leq (1 - 1/m)^{1/N}$. Thus, if we take $\delta = (1 - 1/(2m))^{1/N}$ and any W satisfying (3.1), then $W \cap \overline{B}_{\delta R}(p) \neq \varnothing$, i.e., there is some $q \in W$ such that $\mathsf{d}(q, p) \leq \delta R$. In particular, for such $q \in W$ and any $x \in B_{(1-\delta)R}(q)$, we have

$$\mathsf{d}(x, p) \leq \mathsf{d}(x, q) + \mathsf{d}(q, p) < (1 - \delta)R + \delta R = R,$$

so $B_{(1-\delta)R}(q) \subset B_R(p)$. □

Proof of Theorem B. Clearly we can assume that $K < 0$. Moreover, by rescaling the metric in X by $\sqrt{-K}$, we can further assume that $(X, \mathsf{d}, \mathfrak{m})$ satisfies the $\mathsf{CD}(-1, N)$ condition. In particular, we get that $B \subset B_D(p_0)$ where $D = \sqrt{-K}D_0$.

For $\mu > 2$, the result follows from the fact that $\overline{B}_r(p_0) \subset B_{\mu \cdot r}(p)$ for any $p \in \overline{B}_r(p_0)$, so we can set $C(N, KD_0^2, \mu) = 1$. Therefore, we will assume from now on that $0 < \mu \leq 2$.

We now divide S into the union of $S_1 = S \cap B_{\mu r/2}(p_0)$ and $S_2 = S \setminus S_1$. If $S_1 \neq \varnothing$, then S_1 can be covered by just one $B_{\mu r}(p)$ with $p \in S_1$. In any case, we only need to estimate the covering number of S_2. We will actually estimate the number of $(\mu r/4)$-balls needed to cover S_2, so for simplicity let us denote $t = \mu/4$.

Now, fix some $\lambda > 2$. The case when $tr \leq \lambda D$ follows exactly as in [11, Page 11] (where instead of λ it suffices to consider 2). Therefore we assume that $tr > \lambda D$. In particular, for $q \in S_2$,

$$B_{tr}(q) \cap B_{\lambda D}(p_0) = \varnothing.$$

Write $\partial B_{\lambda D}(p_0)$ as the union of subsets $\{U_1, \ldots, U_m\}$ such that $\mathsf{d}(x, y) < 2D$ for any $x, y \in U_a$. This can be done as follows. Take a maximal set of points $\{q_1, \ldots, q_m\} \subset \partial B_{\lambda D}(p_0)$ such that $\mathsf{d}(q_a, q_b) \geq D$, $a \neq b$. Then

$$\partial B_{\lambda D}(p_0) \subset \bigcup_{j=1}^{m} B_D(p_0),$$
$$B_{D/2}(q_a) \cap B_{D/2}(q_b) = \varnothing, \quad a \neq b.$$

Suppose $B_{D/2}(q_s)$ has the smallest volume among all $B_{D/2}(q_j)$. Since $\bigcup_{i=1}^{m} B_{D/2}(q_j) \subset B_{(1/2+2\lambda)D}(q_s)$, the Bishop-Gromov inequality corresponding to the condition $\mathsf{CD}(-1, N)$ [15, Theorem 2.3] yields

$$m < \frac{\int_0^{(1/2+2\lambda)D} \sinh\left(t\sqrt{1/(N-1)}\right)^{N-1} dt}{\int_0^{D/2} \sinh\left(t\sqrt{1/(N-1)}\right)^{N-1} dt} =: m(N, D, \lambda). \tag{3.2}$$

We define $U_a = B_D(q_a) \cap \partial B_{\lambda D}(p_0)$ for $a = 1, \ldots, m$.

Let M_r be the subset of M consisting of all points on any minimal geodesic emanating from p_0 that is no shorter than r. Note that $M - B_r(p_0) \subset M_r$, and M_r is star-shaped at p_0.

We now divide $M_{\lambda D}$ into m cones K_a by defining K_a to be the subset consisting of all points on any minimal geodesic emanating from p_0 that intersects U_a. Observe that, by the triangle inequality, if $\mathsf{d}(x_i, p_0) > \lambda D$, $x_i \in K_a$, $i = 1, 2$, then any minimal geodesic connecting x_1 and x_2 will not pass through $B_{\lambda D/2}(p_0)$. Indeed, let γ_i be a minimal geodesic from p_0 to x_i with $\gamma_i(\lambda D) \in U_a$, $i = 1, 2$. Then the broken geodesic from x_1 to $\gamma_1(\lambda D)$ to $\gamma_2(\lambda D)$ to x_2 has length no greater than $\mathsf{d}(x_1, p_0) + \mathsf{d}(x_2, p_0) - 2\lambda D + 2D$. On the other hand, if a minimal geodesic connecting x_1 and x_2 intersects $B_{\lambda D/2}(p_0)$, then it would have a length greater than $\mathsf{d}(x_1, p_0) + \mathsf{d}(x_2, p_0) - \lambda D$, which is a contradiction.

Now we can estimate the covering number just as in [10,11]. For the convenience of the reader, we will repeat some of the constructions.

Take a maximal set of points $\{p_1, \ldots, p_k\}$ in S_2 such that $\mathsf{d}(p_i, p_j) > tr$, $i \neq j$. Then

$$S_2 \subset \bigcup_j B_{tr}(p_j),$$
$$B_{tr/2}(p_i) \cap B_{tr/2}(p_j) = \varnothing, \quad i \neq j.$$

We then divide the points p_j into m families as follows: for each ball $B_{tr/2}(p_j)$, look at $\mathfrak{m}(B_{tr/2}(p_j) \cap K_a)$, $a = 1, \ldots, m$. Fix an a_j such that $\mathfrak{m}(B_{tr/2}(p_j) \cap K_{a_j})$

is maximal. Then

$$\mathfrak{m}(B_{tr/2}(p_j)\cap K_{a_j}) \geq \frac{1}{m}\mathfrak{m}(B_{tr/2}(p_j)) \geq \frac{1}{m(N,D,\lambda)}\mathfrak{m}(B_{tr/2}(p_j)), \quad (3.3)$$

where the first inequality follows from the fact that the intersections $B_{tr/2}(p_j)\cap K_a$ cover the ball $B_{tr/2}(p_j)$ and $B_{tr/2}(p_j)\cap K_{a_j}$ is the intersection of maximal measure, whereas the second inequality follows from equation (3.2). We denote

$$B^{L,a_j}_{p_j} := B_{tr/2}(p_j)\cap K_{a_j},$$

and place p_j in the a_j-th family, call it $\mathcal{F}_{a_j}$. Fix a K_a. Suppose $B^{L,a}_p$ has the smallest volume among all $B^{L,a}_{p_j}$ in this cone. By Lemma 3.1, we can find a $q \in B^{L,a}_p$ such that

$$B_{(1-\delta)tr/2}(q) \subset B_{tr/2}(p)$$

where $\delta = \delta(m(N,D,\lambda))$. Let W_q be the star-shaped set such that $y \in W_q$ if and only if there is a point x belonging to either $B_{(1-\delta)tr/2}(q)$ or $B^{L,a}_{p_j}$ for some $p_j \in \mathcal{F}_a$ and there is a minimal geodesic γ connecting q and x which passes y.

Observe that, for $\epsilon_0 = \min\{(1-\delta)tr, (\lambda-2)D/2\} > 0$, $z \in B_{\epsilon_0}(q)$, and $y \in W_q$, any geodesic joining z with y is outside $B_{\lambda D/2}(p_0)$. Indeed, if $x \in B_{(1-\delta)tr/2}(q)\cup\bigcup B^L_{p_j}$ is such that y is in a geodesic γ joining q with x and γ_1 is a geodesic joining z with y and passing through $B_{\lambda D/2}(p_0)$, then the broken geodesic α from q to z to y to x will have length greater than

$$\mathsf{d}(q,p_0) + \mathsf{d}(x,p_0) - \lambda D.$$

However, α also has length no greater than

$$\epsilon_0 + \epsilon_0 + \mathsf{d}(q,y) + \mathsf{d}(y,x) = 2\epsilon_0 + L(\gamma) \leq 2\epsilon_0 + \mathsf{d}(q,p_0) + \mathsf{d}(x,p_0) - 2\lambda D + 2D$$

which is a contradiction.

By a simple triangle inequality, we get that $\mathsf{d}(q,y) \leq (2+t)r$ for all $y \in W_q$. Therefore, applying Theorem 2.11 with $K = 0$, we get

$$\frac{\mathfrak{m}(W_q)}{\mathfrak{m}\left(B_{(1-\delta)tr/2}(q)\right)} \leq 2^N(2+t)^N(1-\delta)^{-N}t^{-N}.$$

However,

$$\frac{\mathfrak{m}(W_q)}{\mathfrak{m}\left(B_{(1-\delta)tr/2}(q)\right)} \geq \sum_{p_j\in\mathcal{F}_a}\frac{\mathfrak{m}\left(B^{L,a}_{p_j}\right)}{\mathfrak{m}\left(B_{tr/2}(p)\right)} \geq \frac{\#\mathcal{F}_a}{m},$$

where $\#\mathcal{F}_a$ denotes the cardinality of $\mathcal{F}_a$. Thus we get

$$\#\mathcal{F}_a \leq 2^N(2+t)^N(1-\delta)^{-N}t^{-N}m.$$

Adding up the contributions from the m families $\mathcal{F}_a$ and combining with (3.2), we get that

$$k \leq 2^N(2+t)^N(1-\delta)^{-N}t^{-N}m(N,D,\lambda)^2. \quad (3.4)$$

The right hand side of (3.4) depends on N, D (which in turn depends on KD_0^2), μ, and λ. Taking $\lambda \searrow 2$, we get the constant $C = C(N, KD_0^2, \mu) > 0$. $\square$

Remark 3.2. Note that in Theorem B, we could require the $\mathsf{CD}(K,N)$ condition in X and the $\mathsf{CD}_{\mathrm{loc}}(0,N')$ condition in $X\setminus B$ with $N\neq N'$ and following the same argument, we would get a constant $C=C(N,N',KD_0^2,\mu)$. We stated the result as is for the sake of simplicity.

Definition 3.3. Let $k\in\mathbb{N}$. A metric space (X,d) *has k ends* if the following conditions hold:

1. For any K compact, $X\setminus K$ has at most k unbounded connected components.
2. There exists K' compact such that $X\setminus K'$ has exactly k unbounded connected components.

Proof of Corollary C. If the result is false, we take r large enough so that $X\setminus\overline{B}_r(p_0)$ has $n>C(N,KD_0^2,1/2)$ unbounded connected components. It is clear that each such unbounded connected component E requires at least one ball of radius r to cover $E\cap\partial\overline{B}_{2r}(p_0)$. This contradicts Theorem B. □

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