



Communications in Partial Differential Equations

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/lpde20

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Megan Griffin-Pickering & Alpár R. Mészáros

To cite this article: Megan Griffin-Pickering & Alpár R. Mészáros (2022) A variational approach to first order kinetic mean field games with local couplings, Communications in Partial Differential Equations, 47:10, 1945-2022, DOI: 10.1080/03605302.2022.2101003

To link to this article: https://doi.org/10.1080/03605302.2022.2101003

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A variational approach to first order kinetic mean field games with local couplings

Megan Griffin-Pickering and Alpár R. Mészáros

Department of Mathematical Sciences, Durham University, Durham, UK

ABSTRACT

First order kinetic mean field games formally describe the Nash equilibria of deterministic differential games where agents control their acceleration, asymptotically in the limit as the number of agents tends to infinity. The known results for the well-posedness theory of mean field games with control on the acceleration assume either that the running and final costs are regularizing functionals of the density variable, or the presence of noise, i.e. a second-order system. In this article we construct global in time weak solutions to a first order mean field games system involving kinetic transport operators, where the costs are local (hence non-regularizing) functions of the density variable with polynomial growth. We show the uniqueness of these solutions on the support of the agent density. This is achieved by characterizing solutions through two convex optimization problems in duality. As part of our approach, we develop tools for the analysis of mean field games on a non-compact domain by variational methods. We introduce a notion of 'reachable set', built from the initial measure, that allows us to work with initial measures with or without compact support. In this way we are able to obtain crucial estimates on minimizing sequences for merely bounded and continuous initial measures. These are then carefully combined with L¹-type averaging lemmas from kinetic theory to obtain pre-compactness for the minimizing sequence. Finally, under stronger convexity and monotonicity assumptions on the data, we prove higher order Sobolev estimates of the solutions.

ARTICLE HISTORY

Received 23 December 2021 Accepted 9 July 2022

2020 MATHEMATICS SUBJECT CLASSIFICATION 49N80; 91A16; 49L12; 35Q91

KEYWORDS

Mean field games; optimal control of acceleration; kinetic equations; well-posedness theory

1. Introduction

The aim of the theory of mean field games (MFG for short) is to characterize limits of Nash equilibria of stochastic or deterministic differential games when the number of agents tends to infinity. Such models were first proposed about 15 years ago, simultaneously by Lasry-Lions [1–3] and Huang-Malhamé-Caines [4].

This theory turned out to be extremely rich in applications and it provided excellent mathematical questions. Its literature has witnessed a huge increase in the last decade. From the theoretical viewpoint, there are two main approaches to the study of MFG.

CONTACT Megan Griffin-Pickering 🔯 megan.k.griffin-pickering@durham.ac.uk 🗈 Department of Mathematical Sciences, Durham University, Durham DH1 3LE, UK

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One is based on analytical and PDE techniques, while the other is a probabilistic approach. The first approach goes back to the original works of Lasry-Lions and has been extended in a great variety of directions in the subsequent years by many authors. If a non-degenerate idiosyncratic noise is present in the models, this typically yields a parabolic structure for the corresponding PDEs and one can expect (strong) classical solutions or a suitable regularity for weak solutions to the corresponding PDE systems, even when the corresponding Lagrangians are local functions of the density variable. For a non-exhaustive list of works in this direction we refer the reader to [5-12]. The probabilistic approach proved to be equally successful for problems involving Lagrangians that are nonlocal functions of the measure variable. This approach seems to be very powerful for handling different kinds of noises in combination with the non-degenerate idiosyncratic one, such as the common noise. For a non-exhaustive collection of works in this direction we refer to [13-16].

When the model lacks a non-degenerate idiosyncratic noise, this clearly poses technical difficulties in the analysis. Typically, it means that additional structural assumptions need to be imposed on the data to be able to hope for (weak) solutions. Such conditions are, for instance, suitable notions of convexity/monotonicity (cf. [17,18]), or the presence of a suitable variational structure, as in the case of potential games ([19–24]). In the case of local couplings, it was pointed out by Lions in [25] that the MFG system (including the planning problem) can be transformed into a degenerate elliptic system in space-time with oblique boundary conditions. Relying on this idea, in a quite general setting, under suitable assumptions on the data (such as strict monotonicity and strong convexity of the Hamiltonians in the measure and momentum variables, respectively; regularity and positivity conditions on the initial data), it has been proven recently in [26,27] that the corresponding first order MFG systems have smooth classical solutions.

For an excellent, relatively complete account on the subject and a summary of results to date we refer the reader to the collection [17].

In this article we study a class of first order *kinetic* MFG systems, involving Lagrangians that are local functions of the density variable and that possess a variational structure, in the sense of [19–21].

In our setting, the MFG system can be formally written as

$$\begin{cases} -\partial_t u(t,x,v) - v \cdot D_x u(t,x,v) + H(x,v,D_v u(t,x,v)) = f(x,v,m), & \text{in } (0,T) \times \mathbb{M} \times \mathbb{R}^d, \\ \partial_t m(t,x,v) + v \cdot D_x m(t,x,v) - \operatorname{div}_v (mD_p H(x,v,D_v u(t,x,v))) = 0, & \text{in } (0,T) \times \mathbb{M} \times \mathbb{R}^d, \\ m(0,x,v) = m_0(x,v), & u(T,x,v) = g(x,v,m_T), & \text{in } \mathbb{M} \times \mathbb{R}^d. \end{cases}$$

$$(1.1)$$

Here \mathbb{M} denotes either that *d*-dimensional flat torus \mathbb{T}^d or the whole *d*-dimensional Euclidean space \mathbb{R}^d and is the physical space for the position *x* of the agents, while the velocity vector *v* of the agents lies in \mathbb{R}^d . T > 0 is an arbitrary time horizon, $H : \mathbb{M} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is the Hamiltonian function, while $f, g : \mathbb{M} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ stand for the running and final costs of the agents, respectively.

Under suitable assumptions on the data, we obtain the global in time existence, uniqueness and Sobolev regularity of weak solutions to (1.1), relying on two convex

optimization problems in duality. One of these problems can be seen as an optimal control problem for the Hamilton-Jacobi equation, while its dual is an optimal control problem for the continuity equation (cf. [19–21]).

Review of the literature in connection to our work

MFG systems of type (1.1) have been introduced in the context of models when agents control their acceleration. It seems that such a model can be traced back to the work [28] (in the engineering community), where the authors proposed a MFG model where agents control their acceleration. In the mathematical community, the first works in this framework seem to be the ones [29-31]. These works consider Hamiltonians (with our notation H - f and final cost functions that are nonlocal regularizing functions in the measure variable. Moreover, the Hamiltonians need to be either purely quadratic or have quadratic growth in the momentum variable. In addition, in [29,30] further conditions on the initial measure m_0 are also imposed. In [29] m_0 is taken to be compactly supported and Hölder continuous, while in [30] m_0 is taken to be compactly supported. These two works construct weak solutions to the corresponding MFG system in the sense that the Hamilton-Jacobi equation has to be understood in the viscosity sense, while the continuity equation is understood in the sense of distributions. In [31] the initial measure m_0 can be quite general and the corresponding Hamiltonian does not need to have the so-called 'separable structure' which was assumed in [29,30] and is also assumed in this article. These more general hypotheses come at the price of obtaining a weaker notion of solution to the MFG system: the so-called mild solutions. However, the authors show that, under the additional separability assumption on the Hamiltonian, mild solutions become more standard weak solutions in the sense described above.

Several interesting new works are built on the models introduced in [29–31]. In [32] the authors study the ergodic behavior of MFG systems, for the case of Hamiltonians that are purely quadratic in the momentum variable and nonlocal regularizing coupling functions f, g, with additional growth assumption on f in the v variable. In [33] the authors obtain mild solutions to MFG under acceleration control and state constraints, under assumptions similar to the ones in [29] on the Hamiltonians, with the possibility to consider Hamiltonians that are power-like functions in the momentum variable. Lastly, in [34] the author studies a perturbation problem associated to MFG under acceleration vanishes.

MFG models with *degenerate* diffusion share some common features with kinetic type problems. In this context we can mention several works. In [35] and [36] the authors study time independent MFG systems with purely quadratic Hamiltonians and nonlocal regularizing coupling functions, where the diffusion operator is hypoelliptic or satisfies a suitable Hörmander condition. It is also worth mentioning that our system (1.1) shares some similarities with MFG models where agents interact also through their velocities. In this direction we refer to the works [37–41].

Finally, a second order MFG system of type (1.1) has been recently studied in [42]. In this work the author obtains weak and renormalized solutions (in the spirit of [12]) to a MFG system that involves a non-degenerate diffusion in the v direction. This seems

to be the only work in the context of kinetic type MFG models where the coupling functions f and g are taken to be local functions of the density variable m. Here the Hamiltonian H is assumed to depend only on the momentum variable and either to be globally Lipschitz continuous or to have quadratic/sub-quadratic growth. There are several summability properties and moment bounds imposed on the initial density m_0 . In the case of Lipschitz continuous Hamiltonians, the coupling functions f, g are supposed to fulfill several further assumptions: a strong uniform increasing property in the mvariable and their derivatives in the (x, v) variable must have a linear growth condition in the m variable.

In [42] the presence of the diffusion in the v direction allows the author to use suitable De Giorgi type arguments to show that the solution to the Fokker-Planck equation is bounded and has fractional Sobolev regularity. These estimates seem to be instrumental to set up a fixed point scheme and to show that the MFG system has a weak solution. Furthermore, the presence of this diffusion allows to obtain second order Sobolev estimates for the MFG system.

Description of our results

As highlighted above, in this work we are inspired by [19–21] and we obtain existence and uniqueness of weak solutions to (1.1) (in the sense of Definition 2.3) via two convex optimization problems in duality (Problem 3.1 and Problem 3.3). Compared to these works, several major differences arise which require new ideas. A first obvious difference is that in our setting (in contrast to the compact setting of the flat torus which is considered in the mentioned references) the velocity variable v lives in the non-compact space \mathbb{R}^d . This clearly introduces technical issues in the analysis.

To prove our main results, the general outline of our programme is the same as the one of [19-21]: prove the duality for Problem 3.1 and Problem 3.3; suitably relax Problem 3.1 (this will be Problem 3.8) and show that the value of this is the same as the original one; show existence of optimizers for the relaxed problem and apply the duality result again to obtain existence of solutions in a suitable weak sense. In this article *H* is supposed to have a superlinear growth in the momentum variable, and *f* and *g* are supposed to have polynomial growth in their last variables. The growth of *f*, *g* may be taken independently of the growth of the Hamiltonian (we refer to the next section for the precise assumptions).

To show that the value of the relaxed problem is the same as the original one, a standard approach used in [19–21] is to test the Hamilton-Jacobi inequality of any competitor by competitors of the dual problem (i.e. solutions to the continuity equation). To justify this computation a mollification argument was applied for solutions to the continuity equation. In our case, this mollification alone is not enough because of the non-compact setting. Therefore a delicate cutoff argument has to be also implemented.

The most delicate part, however, is to obtain existence of optimizers to the relaxed problem and in particular to obtain proper compactness results for the minimizing sequences. First, in our case the time trace of the solutions to the Hamilton-Jacobi inequality constraint in Problem 3.8 is quite weak: $u(t, \cdot)$ has to be understood as a locally finite signed Radon measure. Since in this work m_0 may have non-compact

support, it takes additional effort to give a meaning to $\int_{\mathbb{M}\times\mathbb{R}^d} m_0 u_0(dx dv)$ (a term that appears in the objective functional present in Problem 3.8). Our construction, although completely different, has some similarities in spirit with the one in [43], to define similar time boundary traces.

In order to obtain suitable estimates for the minimizing sequence of the relaxed problem, in [19-21] a typical trick was to test the Hamilton-Jacobi inequality constraint by the initial measure m_0 . For this reason, it was necessary to impose enough regularity, and more importantly a uniform positive lower bound of this density everywhere. Because of this, estimates on the quantity $\int_{\mathbb{T}^d} m_0 u_0 dx$, would readily yield summability estimates on u_0 solely. We emphasize that in this article we assume that m_0 is merely a bounded and continuous probability density and so we take a completely different route when obtaining such estimates. We introduce the *reachable set* \mathcal{U}_{m_0} , a set of points in time, space and velocity that can be reached from $spt(m_0)$ with arbitrary smooth admissible controls (cf. Definition 2.2). In fact, by the controllability of the underlying ODE system, which satisfies the Kalman rank condition, we have $\mathcal{U}_{m_0} = (\{0\} \times \operatorname{spt}(m_0)) \cup$ $((0, T) \times \mathbb{M} \times \mathbb{R}^d)$. In order to obtain our crucial estimates on the corresponding minimizing sequence we use well chosen test functions that are supported in U_{m_0} . This construction seems to be new in the literature on variational MFG and we believe that it could be instrumental also in other settings, to possibly relax regularity, positivity or compact support assumptions on m_0 .

As there is no Hopf-Lax type representation formula available for solutions to our Hamilton-Jacobi equations (which was the case in [19,20]), first, we obtain estimates on truncations of the solutions. These are similar in flavor to the corresponding estimates in [21], and such ideas date back to [44]. As our terminal data typically have merely local summability, this will be the source of additional technical issues (in contrast to [21], where the terminal data was taken to be regular enough).

Let us underline that the ideas and constructions that we have described so far allow us to obtain summability estimates on u and $D_{v}u$, using the structure of the problem. This is not sufficient to yield weak precompactness for minimizing sequences due to the lack of regularity estimates in x. To recover the necessary compactness we make use of averaging lemmas available in kinetic theory. Averaging lemmas go back to the works [45,46] and provide improved regularity and compactness properties for velocity averages of solutions of kinetic transport equations (see Subsection 6.1 for the precise definitions). For more details and a survey of results we refer the reader to the review [47] and the references cited therein. When regularity with respect to v is additionally available, similar properties can be deduced for the full density function: we refer for instance to [48] for regularity results in the L^p case for 1 . We carefully tailorthis approach to our setting, combining our estimates on $D_{\nu}u$ with L^{1} averaging lemmas [49-51] to deduce precompactness for minimizing sequences. In this way we prove Theorem 6.8 on the existence of a minimizer of Problem 3.8. This in turn implies Theorem 2.4, that system (1.1) has a (unique) weak solution. As was similarly obtained in [19–21], we show the uniqueness of *m* and the uniqueness of *u* on $\{m > 0\}$.

A natural question that arises in the context of variational MFG is whether the variational structure and further strong monotonicity and convexity assumptions on the data would yield higher order Sobolev estimates on weak solutions. Such estimates were recently obtained in more classical frameworks in [23,24, 39, 52–54]. In this article we pursue similar Sobolev estimates, implied by taking stronger assumptions on the data. In comparison with the works [23,24], in our setting we need to work with a considerably weaker notion of time trace of u, which is not stable under perturbations of the initial measure m_0 . Therefore, our Sobolev estimates remain local in time on (0, T]. Another delicate difference is due to the presence of the kinetic transport term. Because of this, a careful choice of perturbations need to be used, which take into account the kinetic nature of the problem. As a result of this, interestingly, first we obtain estimates on differential operators of the form $(tD_x + D_v)$ applied to m and $D_v u$. For the precise results in this direction we refer to Theorem 8.2, Corollary 8.4 and Corollary 8.5.

The structure of the article is as follows. In Section 2 we state our standing assumptions and main results. In Section 3 we present the two variational problems in duality along with the relaxed problem of the primal problem. In Section 4 we have collected some preliminary estimates on weak solutions of the Hamilton-Jacobi inequality obtained on the reachable set U_{m_0} . In Section 5 we show that the relaxed problem has the same value as the primal problem and hence the duality result holds. Section 6 contains the existence result of a solution to the relaxed problem. Here we rely on the combination of the estimates derived in the previous sections and suitably tailored averaging lemmas from kinetic theory, applied in our context for distributional subsolutions to kinetic Hamilton-Jacobi equations. In Section 7 we show that optimizers of the variational problems in duality provide weak solutions to the MFG system and, conversely, weak solutions are also optimizers of the variational problems. Furthermore, strong convexity yields (partial) uniqueness of these solutions. Section 8 is devoted to the derivation of higher order Sobolev estimates for the weak solutions. These require further assumptions on the data.

We end the paper with two appendix sections. In Appendix A we discuss the time regularity of distributional subsolutions to kinetic Hamilton-Jacobi equations which allow us to construct suitable notions of time traces. Finally, in Appendix B we show that truncations and maxima of distributional subsolutions to kinetic Hamilton-Jacobi equations remain distributional subsolutions to suitably modified equations.

2. Standing assumptions and main results

In this section we state our main results on the existence, uniqueness and Sobolev regularity of solutions to the MFG system.

We define \mathcal{F} and \mathcal{G} to be the anti-derivatives of the coupling functions f and g with respect to m:

$$\mathcal{F}(x,v,m) = \int_0^m f(x,v,m') \,\mathrm{d}m' \qquad \mathcal{G}(x,v,m) = \int_0^m g(x,v,m') \,\mathrm{d}m'.$$

Throughout, we make the following assumptions on the Hamiltonian and coupling functions.

Assumption 1.

(H1) The Hamiltonian H is continuous in all variables, and convex and differentiable with respect to p. Furthermore, for some r > 1, H satisfies bounds of the form

$$\frac{1}{cr}|p|^{r} - C_{H} \le H(x, v, p) \le \frac{c}{r}|p|^{r} + C_{H},$$
(2.1)

for all $(x, v, p) \in \mathbb{M} \times \mathbb{R}^d \times \mathbb{R}^d$ and some constants c > 0 and $C_H \ge 0$. Finally, the function $H_0(x, v) := H(x, v, 0)$ has positive part $(H_0)_+ \in C_0(\mathbb{M} \times \mathbb{R}^d)$, where $C_0(\mathbb{M} \times \mathbb{R}^d)$ denotes the closure of the space $C_c(\mathbb{M} \times \mathbb{R}^d)$ with respect to the uniform norm.

(H2) \mathcal{F} is continuous in all variables and strictly convex and differentiable with respect to *m* for m > 0. Moreover, it satisfies the growth condition

$$\frac{1}{cq}m^q - C_F(x,v) \le \mathcal{F}(x,v,m) \le \frac{c}{q}m^q + C_F(x,v), \quad m \ge 0$$
(2.2)

where q > 1 and the function $C_F \in L^1(\mathbb{M} \times \mathbb{R}^d)$. For m < 0, we set $\mathcal{F}(x, v, m) = +\infty$.

(H3) G is continuous and strictly convex. Moreover, it satisfies the growth condition

$$\frac{1}{c}m^s - C_G(x,v) \le \mathcal{G}(x,v,m) \le cm^s + C_G(x,v), \quad m \ge 0,$$
(2.3)

for some $C_G \in L^1(\mathbb{M} \times \mathbb{R}^d)$ and $1 < s \leq q$. For m < 0, we set $\mathcal{G}(x, v, m) = +\infty$.

(H4) The initial datum $m_0 \in C_b(\mathbb{M} \times \mathbb{R}^d)$ is a probability density.

We note that since m_0 is imposed to be a bounded probability density, by interpolation, it is uniformly bounded in $L^{\alpha}(\mathbb{M} \times \mathbb{R}^d)$, for any $\alpha \in [1, +\infty]$. We emphasize that here we impose growth conditions on \mathcal{F}, \mathcal{G} rather than on f, g.

Example 2.1. For any q > 1 and continuous bounded function c such that $c \ge c_0$ with $c_0 > 0$ a strictly positive constant, the function

$$\mathcal{F}(x, v, m) = \begin{cases} c(x, v)m^q & m \ge 0 \\ +\infty & m < 0, \end{cases}$$

satisfies the given assumptions.

Definition 2.2 (Reachable set). It will be useful to define the set $U_{m_0} \subseteq [0, T] \times \mathbb{M} \times \mathbb{R}^d$ to be the set of points potentially reachable by a collection of agents initially distributed according to m_0 and evolving according to the control system

$$\dot{x} = v, \quad \dot{v} = a,$$

for some control $a \in C([0, T]; \mathbb{R}^d)$. Observe that the previous control system satisfies the

classical Kalman rank condition, and so we have

$$\mathcal{U}_{m_0} = \{0\} \times \{m_0 > 0\} \cup (0, T] \times \mathbb{M} \times \mathbb{R}^d.$$

Under these standing assumptions, we define the following notion of weak solution to the MFG system.

Definition 2.3. We say that (u, m) is a weak solution to (1.1), if the following are fulfilled:

- (i) $u \in L^1_{loc}(\mathcal{U}_{m_0}), D_{\nu}u \in L^r_{loc}(\mathcal{U}_{m_0}) \text{ and } m|D_{\nu}u|^r \in L^1((0,T) \times \mathbb{M} \times \mathbb{R}^d);$
- (ii) $m \in L^q((0, T) \times \mathbb{M} \times \mathbb{R}^d)$ and $m_T \in L^s(\mathbb{M} \times \mathbb{R}^d)$;
- (iii) $(u_0)_+ \in (L^{\infty} + L^{q'})(\mathbb{M} \times \mathbb{R}^d)$ and $(u_0)_-$ is a locally finite Radon measure supported in $\{m_0 > 0\}$.

(iv)
$$\begin{cases} -\partial_t u - v \cdot D_x u + H(x, v, D_v u) \leq f(x, v, m), \text{ in } \mathscr{D}'((0, T) \times \mathbb{M} \times \mathbb{R}^d) \\ u_T \leq g(\cdot, \cdot, m_T), \text{ in } \mathscr{D}'(\mathbb{M} \times \mathbb{R}^d). \end{cases}$$

- (v) The continuity equation from (1.1) holds in $\mathscr{D}'((0,T) \times \mathbb{M} \times \mathbb{R}^d)$.
- (vi) $\int_{\mathbb{M}\times\mathbb{R}^d} m_0 u_0(\,\mathrm{d} x\,\mathrm{d} v)$ is finite.
- (vii) The following energy equality holds:

$$\int_{\mathbb{M}\times\mathbb{R}^{d}} m_{0}u_{0}(\,\mathrm{d}x\,\mathrm{d}v) - \int_{\mathbb{M}\times\mathbb{R}^{d}} g(x,v,m_{T})m_{T}\,\mathrm{d}x\,\mathrm{d}v$$

$$= \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} f(x,v,m)m\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t$$

$$+ \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \left[D_{p_{v}}H(x,v,D_{v}u) \cdot D_{v}u - H(x,v,D_{v}u) \right]m\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t.$$
(2.4)

2.1. Existence and uniqueness

The first of our main results is the existence and uniqueness of these weak solutions.

Theorem 2.4. Let Assumption 1 hold. Then there exists a weak solution (u, m) of the mean field game system (1.1) in the sense of Definition 2.3. This solution is unique, in the sense that if (u_1, m_1) and (u_2, m_2) are both weak solutions in the sense of Definition 2.3, then $m_1 = m_2$ almost everywhere and $u_1 = u_2$ almost everywhere on the set $\{m_1 > 0\}$.

2.2. Regularity

Our second main result is Sobolev regularity for weak solutions of the mean field games system (1.1). For this result we assume quadratic growth of the Hamiltonian (r=2) and stronger convexity and regularity hypotheses on the data, as follows.

Assumption 2.

(H5) (Conditions on the coupling functions) There exists C > 0 such that the functions f, g satisfy

$$\begin{aligned} |f(x_1, v_1, m) - f(x_2, v_2, m)| &\leq C(m^{q-1} + 1)(|x_1 - x_2| + |v_1 - v_2|) \\ &\forall (x_1, v_1), (x_2, v_2) \in \mathbb{M} \times \mathbb{R}^d, \ m \geq 0. \end{aligned}$$
(2.5)

and

$$\begin{aligned} |g(x_1, v_1, m) - g(x_2, v_2, m)| &\leq C(m^{s-1} + 1)(|x_1 - x_2| + |v_1 - v_2|) \\ &\forall (x_1, v_1), (x_2, v_2) \in \mathbb{M} \times \mathbb{R}^d, \ m \geq 0. \end{aligned}$$
(2.6)

Moreover, there exists $c_f, c_g > 0$ such that

$$(f(x, v, \tilde{m}) - f(x, v, m))(\tilde{m} - m) \ge c_f \min\{\tilde{m}^{q-2}, m^{q-2}\} |\tilde{m} - m|^2 \,\forall \tilde{m}, m \ge 0, \, \tilde{m} \ne m.$$
(2.7)

$$(g(x, v, \tilde{m}) - g(x, v, m))(\tilde{m} - m) \ge c_g \min\{\tilde{m}^{s-2}, m^{s-2}\}|\tilde{m} - m|^2 \,\forall \tilde{m}, m \ge 0, \, \tilde{m} \ne m.$$
(2.8)

In the above assumptions, if q < 2 or s < 2 one should interpret 0^{q-2} and 0^{s-2} as $+\infty$. In this way, when $\tilde{m} = 0$, for instance, (2.7) reduces to $f(x, v, m)m \ge c_f m^q$, as in the more regular case $q \ge 2$. Similar comments can be made for (2.8).

(H6) (Quadratic growth and strong coercivity assumption on H) Suppose that r = 2and there exist $j_1, j_2 : \mathbb{R}^d \to \mathbb{R}^d$ and $c_H > 0$ such that

$$H(x, v, P) + L(x, v, W) - P \cdot W \ge c_H |j_1(P) - j_2(W)|^2.$$
(2.10)

In particular, and in light of our restriction (2.1), we assume that j_1 and j_2 have linear growth.

(H7)
$$L(\cdot, \cdot, W) \in C^2(\mathbb{M} \times \mathbb{R}^d)$$
 and $|D^2_{xx}L(x, v, W)|, |D^2_{xv}L(x, v, W)|, |D^2_{vv}L(x, v, W)| \leq C_0 |W|^2 + C_0, \forall (x, v, W) \in \mathbb{M} \times \mathbb{R}^d \times \mathbb{R}^d.$

Under these additional assumptions, we prove the following result. The proof is carried out in Section 8.

Theorem 2.5. Suppose that (u, m) is a weak solution to (1.1) in the sense of Definition 2.3 and that (H5), (H6), (H7) hold.

Then, there exists $\overline{C} > 0$ such that

$$\|m^{\frac{q}{2}-1}D_{x,\nu}m\|_{L^{2}_{loc}((0,T]\times\mathbb{M}\times\mathbb{R}^{d})}\leq \bar{C}, \|m^{1/2}D_{x,\nu}D_{\nu}u\|_{L^{2}_{loc}((0,T]\times\mathbb{M}\times\mathbb{R}^{d})}\leq \bar{C}$$

and

$$\|m_T^{\frac{s}{2}-1}D_{x,\nu}m_T\|_{L^2(\mathbb{M}\times\mathbb{R}^d)}\leq \bar{C}.$$

Remark 2.6. The estimates appearing in this statement are informal; we in fact obtain uniform L^2 -type summability of differential quotients (see estimate (8.8) below). The corresponding Sobolev estimates, however, are more delicate to obtain, because these would need to be understood in the sense of weighted Sobolev spaces or more generally in the sense of Sobolev spaces with respect to measures. Their precise versions would need to involve tangent spaces with respect to the measure *m*, but these are beyond the scope of the current article. We refer to [55] on this topic.

3. Variational problems in duality

We will prove existence of a solution to the MFG system (1.1) through a variational characterization. In this section we set up the variational problems used to obtain solutions. We recall that here and throughout the rest of the article, we will work under Assumption 1.

3.1. Optimal control of the Hamilton-Jacobi equation: smooth setting

We define the Fenchel conjugates of \mathcal{F} and \mathcal{G} respectively by

$$\mathcal{F}^*(x,\nu,\beta) := \sup_{m \ge 0} \left\{ \beta m - \mathcal{F}(x,\nu,m) \right\} \qquad \mathcal{G}^*(x,\nu,u) := \sup_{m \ge 0} \left\{ um - \mathcal{G}(x,\nu,m) \right\}.$$

Under our assumptions on \mathcal{F} , we have the bounds

$$\begin{cases} c|\beta|^{q'} - C_F(x,\nu) \le \mathcal{F}^*(x,\nu,\beta) \le c^{-1}|\beta|^{q'} + C_F(x,\nu) & \beta > 0, \\ -\mathcal{F}(x,\nu,0) = 0 \le \mathcal{F}^*(x,\nu,\beta) \le -\inf_{m \ge 0} \mathcal{F}(x,\nu,m) \le C_F(x,\nu) & \beta \le 0, \end{cases}$$
(3.1)

where q' = q/(q-1) denotes the Hölder conjugate exponent of q. Note also that \mathcal{F}^* is non-decreasing. Similar observations hold for \mathcal{G}^* .

Using this, we define the following functional: for $u \in C_b^1([0, T] \times \mathbb{M} \times \mathbb{R}^d)$, let

$$\mathcal{A}(u) := \int_0^I \int_{\mathbb{M}\times\mathbb{R}^d} \mathcal{F}^*(x, v, -\partial_t u - v \cdot D_x u + H(x, v, D_v u)) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t$$
$$- \int_{\mathbb{M}\times\mathbb{R}^d} u(0, x, v) m_0(x, v) \, \mathrm{d}x \, \mathrm{d}v + \int_{\mathbb{M}\times\mathbb{R}^d} \mathcal{G}^*(x, v, u(T, x, v)) \, \mathrm{d}x \, \mathrm{d}v,$$

whenever the integrals are meaningful, and set $A(u) = +\infty$ otherwise. We define a first variational problem associated to this problem.

Problem 3.1. Minimize $\mathcal{A}(u)$ over $u \in E_0$, where E_0 denotes the space

$$E_0 := \{ u \in C_b^1([0,T] \times \mathbb{M} \times \mathbb{R}^d) : |\nu| | D_x u | \in L^\infty([0,T] \times \mathbb{M} \times \mathbb{R}^d) \}.$$
(3.2)

Remark 3.2. E_0 is a Banach space when equipped with the norm

$$\|u\|_{E_0} := \|u\|_{L^{\infty}} + \|D_{\nu}u\|_{L^{\infty}} + \|(1+|\nu|)D_xu\|_{L^{\infty}}$$

3.2. Optimal control of the continuity equation

To state the dual problem we define the Lagrangian $L: \mathbb{M} \times \mathbb{R}^{2d} \to \mathbb{R}$, which is the Fenchel conjugate of the Hamiltonian *H* in the last variable. In other words, for any $(x, v, \alpha) \in \mathbb{M} \times \mathbb{R}^{2d}$, we define

$$L(x, \nu, \alpha) := \sup_{p \in \mathbb{R}^d} \big\{ \alpha \cdot p - H(x, \nu, p) \big\}.$$

Note that L then satisfies upper and lower bounds of the form

$$\frac{1}{c_L}|\alpha|^{r'}-C_L\leq L(x,v,\alpha)\leq c_L|\alpha|^{r'}+C_L,$$

where r' = r/(r-1) denotes the Hölder conjugate exponent of *r*.

For pairs $(m, w) \in L^1([0, T] \times \mathbb{M} \times \mathbb{R}^d) \times L^1([0, T] \times \mathbb{M} \times \mathbb{R}^d)$, we define the functional

$$\mathcal{B}(m,w) := \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} \mathcal{F}(x,v,m) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t + \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} L\left(x,v,-\frac{w}{m}\right) m \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \\ + \int_{\mathbb{M}\times\mathbb{R}^d} \mathcal{G}(x,v,m_T(x,v)) \, \mathrm{d}x \, \mathrm{d}v,$$

with the convention that

$$L\left(x, v, -\frac{w}{m}\right)m = \begin{cases} 0 & m = w = 0, \\ +\infty & m = 0, w \neq 0. \end{cases}$$

We then define a second variational problem, (formally) dual to the first.

Problem 3.3. Minimize $\mathcal{B}(m, w)$ over the set $\mathcal{K}_{\mathcal{B}}$ of pairs $(m, w) \in L^1([0, T] \times \mathbb{M} \times \mathbb{R}^d) \times (L^1([0, T] \times \mathbb{M} \times \mathbb{R}^d))^d$ with $m \ge 0$, subject to (m, w) satisfying the following continuity equation:

$$\partial_t m + v \cdot D_x m + \operatorname{div}_v w = 0, \text{ in } \mathscr{D}'((0, T) \times \mathbb{M} \times \mathbb{R}^d)$$
(3.3)

and $m|_{t=0} = m_0$ in the sense of a weak trace.

Remark 3.4. Let us comment on the weak trace of m with respect to the time variable. Since we are interested in competitors (m, w) for which $\mathcal{B}(m, w)$ is finite, there must exist a vector-valued measurable function $V \in L^{r'}(m \operatorname{dx} \operatorname{dv} \operatorname{dt})$, that is, for which

$$\int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} |V|^{r'} m\,\mathrm{d} x\,\mathrm{d} v\,\mathrm{d} t < +\infty,$$

such that w = Vm (i.e. V is the density of w with respect to m). So, we notice that the previous equation can be written as

$$\partial_t m + \operatorname{div}_x(vm) + \operatorname{div}_v(Vm) = 0.$$

Since $Vm = w \in L^1([0, T] \times \mathbb{M} \times \mathbb{R}^d)$, we have $|V|m \in L^1([0, T] \times \mathbb{M} \times \mathbb{R}^d)$. We are then able to prove that *m* has a narrowly continuous representative $[0, T] \ni t \mapsto m_t \in \mathscr{P}(\mathbb{M} \times \mathbb{R}^d)$, so that in particular $m|_{t=0}$ and $m|_{t=T}$ are meaningful. This is essentially a consequence of [56, Lemma 8.1.2], with minor modifications to account for the fact that vm is only locally integrable; we sketch this in the appendix in Lemma A.12.

3.3. Duality

Lemma 3.5. We have the following duality:

$$\inf_{u\in E_0}\mathcal{A}(u)=-\min_{(m,w)\in \mathcal{K}_B}\mathcal{B}(m,w)$$

Proof. This is an application of the classical Fenchel–Rockafeller duality theorem. Recall that we defined the Banach space E_0 above in (3.2). Then let E_1 be defined by

$$E_1 := C_b^0([0,T] \times \mathbb{M} \times \mathbb{R}^d; \mathbb{R}) \times C_b^0([0,T] \times \mathbb{M} \times \mathbb{R}^d; \mathbb{R}^d);$$

we will express elements of E_1 as pairs (ϕ, ψ) of continuous bounded functions, where ϕ is real-valued and ψ is vector-valued. E_1 is a Banach space with respect to the uniform norm. On these spaces we define the respective functionals

$$\mathcal{A}_0(u) := -\int_{\mathbb{M}\times\mathbb{R}^d} u(0,x,v) m_0(x,v) \,\mathrm{d}x \,\mathrm{d}v + \int_{\mathbb{M}\times\mathbb{R}^d} \mathcal{G}^*(x,v,u(T,x,v)) \,\mathrm{d}x \,\mathrm{d}v$$

and

$$\mathcal{A}_1(\phi,\psi) := \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} \mathcal{F}^*(x,\nu,-\phi+H(x,\nu,\psi)) \,\mathrm{d}x \,\mathrm{d}\nu \,\mathrm{d}t$$

Note that these functionals are convex. We also define the bounded linear map Λ : $E_0 \rightarrow E_1$ by

$$\Lambda u := (\partial_t u + v \cdot D_x u, D_v u).$$

Then

$$\mathcal{A}(u) = \mathcal{A}_0(u) + \mathcal{A}_1(\Lambda u).$$

We wish to apply Fenchel-Rockafeller duality. In order to do this we must verify the existence of $u \in E_0$ such that $\mathcal{A}_0(u), \mathcal{A}_1(\Lambda u) < +\infty$ and \mathcal{A}_1 is continuous at Λu . For example, we may take u to be of the form

$$u(t, x, v) := \zeta(x - vt, v) + 2C_H(t - T),$$

where C_H denotes the constant from the bounds on the Hamiltonian (2.1). We then take $\zeta \in C_b^1(\mathbb{M} \times \mathbb{R}^d)$ non-negative to have sufficiently strong decay at infinity so that

$$\zeta \in L^{s'}(\mathbb{M} \times \mathbb{R}^d), \quad D_{x,v}\zeta \in L^{rq'}(\mathbb{M} \times \mathbb{R}^d).$$
(3.4)

Explicitly, for the case $\mathbb{M} = \mathbb{T}^d$ we may take for example

$$\zeta(x,v) = \zeta(v) = (1 + |v|^2)^{-k/2}; \qquad k > \max\left\{\frac{d}{s'}, \frac{d}{rq'}\right\},$$

in which case $|v||D_x u| = 0$ and therefore $u \in E_0$.

For the case $\mathbb{M} = \mathbb{R}^d$ we may take

$$\zeta(x,v) = \zeta(v) = (1+|x|^2+|v|^2)^{-k/2}; \qquad k > \max\left\{\frac{2d}{s'}, \frac{2d}{rq'}\right\}.$$

In this case,

$$D_x u = D_x \zeta(x - vt, v) = \frac{-k(x - vt)}{(1 + |x - vt|^2 + |v|^2)^{1 + k/2}},$$

and so

$$|v||D_{x}u| = \frac{k|v||x-vt|}{(1+|x-vt|^{2}+|v|^{2})^{1+k/2}} \leq \frac{k}{2}(1+|x-vt|^{2}+|v|^{2})^{-k/2} \leq \frac{k}{2},$$

which implies that $u \in E_0$.

Then, in either case,

$$-\partial_t u - v \cdot D_x u + H(x, v, D_v u) = -2C_H + H(x, v, D_v u) \le c |D_v u|^r - C_H$$

It follows that the positive part satisfies

$$[-\partial_t u - v \cdot D_x u + H(x, v, D_v u)]_+ \leq c |D_v u|^r \mathbb{1}_{\{c|D_v u|^r > C_H\}} \in L^{q'}([0, T] \times \mathbb{M} \times \mathbb{R}^d)$$

and thus by the bounds on \mathcal{F}^* (3.1) we obtain

$$\int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} \mathcal{F}^*(x,v,-\partial_t u-v\cdot D_x u+H(x,v,D_v u))\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t<+\infty.$$

That is, $\mathcal{A}_1(\Lambda u)$ is finite.

Moreover, $u_T \in L^{s'}(\mathbb{M} \times \mathbb{R}^d)$ and thus

$$\int_{\mathbb{M}\times\mathbb{R}^d} \mathcal{G}^*(x,\nu,u(T,x,\nu)) \,\mathrm{d}x \,\mathrm{d}\nu < +\infty.$$

Finally, since $u_0 = \zeta - 2C_H T$ and m_0 is a probability density,

$$-\int_{\mathbb{M}\times\mathbb{R}^d} u(0,x,\nu)m_0(x,\nu)\,\mathrm{d} x\,\mathrm{d} \nu<+\infty.$$

Thus $\mathcal{A}_0(u)$ is finite.

Now we verify that A_1 is continuous at Λu with respect to convergence in E_1 . Consider the sequence of pairs $(\phi_n, \psi_n) \in E_1$, $n \in \mathbb{N}$, such that

$$\phi_n = \partial_t u + v \cdot D_x u + \delta_n, \quad \psi_n = D_v u + \varepsilon_n,$$

where $(\delta_n, \varepsilon_n) \in E_1$ satisfy $\|(\delta_n, \varepsilon_n)\|_{L^{\infty}} \leq 2^{-n}$. Then

$$-\phi_n + H(x, v, \psi_n) = -2C_H - \delta_n + H(x, v, D_v u + \varepsilon_n).$$

Using the bounds (2.1) on the Hamiltonian, we obtain

$$\begin{aligned} -\phi_n + H(x, \nu, \psi_n) &\leq -2C_H - \delta_n + C_H + \frac{c}{r} |D_\nu u + \varepsilon_n|^r \\ &\leq -C_H - \delta_n + 2^{r-1} \frac{c}{r} |D_\nu u|^r + 2^{r-1} \frac{c}{r} |\varepsilon_n|^r \\ &\leq -C_H + C(2^{-n} + 2^{-rn}) + C|D_\nu u|^r, \end{aligned}$$

for some constant C > 0. Therefore, for all *n* large enough that $C(2^{-n} + 2^{-m}) < C_H$, for the positive part we have

$$\left[-\phi_n + H(x, v, \psi_n)\right]_+ \le C |D_v u|^r.$$

Then the bounds (3.1) on \mathcal{F}^* imply that

$$\mathcal{F}^*(-\phi_n + H(x, \nu, \psi_n)) \leq 2C_F + C|D_\nu u|^{rq'} \in L^1([0, T] \times \mathbb{M} \times \mathbb{R}^d),$$

(where the constant C > 0 has changed line to line). The right hand side is in L^1 because we constructed ζ to satisfy (3.4). We may therefore use it as a dominating function: since $(\delta_n, \varepsilon_n)$ certainly converges to zero pointwise (in fact in uniform norm), and \mathcal{F}^* is continuous with respect to the variable β , by dominated convergence we may conclude that

$$\mathcal{A}_1[(\phi_n, \psi_n)] = \lim_{n \to +\infty} \int_0^T \int_{\mathbb{M} \times \mathbb{R}^d} \mathcal{F}^*(x, \nu, -\phi_n + H(x, \nu, \psi_n)) \, \mathrm{d}x \, \mathrm{d}\nu \, \mathrm{d}t$$
$$= \int_0^T \int_{\mathbb{M} \times \mathbb{R}^d} \mathcal{F}^*(x, \nu, -\partial_t u - \nu \cdot D_x u + H(x, \nu, D_\nu u)) \, \mathrm{d}x \, \mathrm{d}\nu \, \mathrm{d}t$$
$$= \mathcal{A}_1(\Lambda u).$$

Thus A_1 is indeed continuous at Λu .

It remains to check that \mathcal{A} is bounded below on E_0 . Let $u \in E_0$ and set $\beta := -\partial_t u - v \cdot D_x u + H(x, v, D_v u)$. Then, using the growth assumptions on \mathcal{F}^* and \mathcal{G}^* , similarly to the inequality (4.2) below, we have

$$egin{aligned} \mathcal{A}(u) &\geq \|eta_+\|_{L^{q'}}^{q'} + \|u(T,\,\cdot\,,\cdot)_+\|_{L^{s'}}^{s'} - \int_{\mathbb{M} imes\mathbb{R}^d} [TC_F(x,
u) + C_G(x,
u)] \,\mathrm{d}x \,\mathrm{d}
u \ &- \int_{\mathbb{M} imes\mathbb{R}^d} u(0,x,
u)_+ m_0(x,
u) \,\mathrm{d}x \,\mathrm{d}
u \ &\geq \|eta_+\|_{L^{q'}}^{q'} + \|u(T,\,\cdot\,,\cdot)_+\|_{L^{s'}}^{s'} - C - \Big(\|(u_T)_+\|_{L^{s'}} + C_HT + T^{1/q}\|eta_+\|_{L^{q'}}\Big) (\|m_0\|_{L^1} + \|m_0\|_{L^q}) \ &\geq \inf_{a,b\geq 0} \left\{a^{q'} - c_0a + b^{s'} - c_0b - c_0 \right\} > -\infty, \end{aligned}$$

where c_0 was set to be a large positive constant depending only on m_0, T, C_H, C_F, C_G .

Therefore, we are in position to apply the Fenchel–Rockafeller duality theorem (cf. [57, Chapter 3, Theorem 4.1]), to conclude

$$\inf_{u\in E_0} \mathcal{A}(u) = \max_{(m,w)\in E_1'} \{-\mathcal{A}_0^*(\Lambda^*(m,w)) - \mathcal{A}_1^*(-(m,w))\}.$$

Here E'_1 denotes the dual space of E_1 . By [58, IV.6] the dual space of C_b^0 may be identified with the space of bounded, regular, finitely additive set functions. Thus E'_1 is the space of pairs (m, w), where m is a real-valued regular finitely additive set function, and w is a \mathbb{R}^d -valued regular finitely additive set function.

It remains to identify

$$\max_{(m,w)\in E'_1} \left\{ -\mathcal{A}^*_0(\Lambda^*(m,w)) - \mathcal{A}^*_1(-(m,w)) \right\}$$

In what follows, we are going to show that the above maximization problem actually admits solutions in a better space than E'_1 . So, we have

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$$\max_{(m,w)\in E_1'} \left\{ -\mathcal{A}_0^*(\Lambda^*(m,w)) - \mathcal{A}_1^*(-(m,w)) \right\} = \max_{(m,w)\in \tilde{E}_1'} \left\{ -\mathcal{A}_0^*(\Lambda^*(m,w)) - \mathcal{A}_1^*(-(m,w)) \right\},$$

where the set \tilde{E}'_1 stands for pairs (m, w) such that m is a finite Radon measure on $[0, T] \times \mathbb{M} \times \mathbb{R}^d$ and w is a finite vector-valued Radon measure on $[0, T] \times \mathbb{M} \times \mathbb{R}^d$ taking values in \mathbb{R}^d . The proof of this is postponed to Lemma 3.6 below.

Then, by arguing as in [19, Section 3.3], we may identify that

$$\max_{(m,w)\in E'_1} \{-\mathcal{A}^*_0(\Lambda^*(m,w)) - \mathcal{A}^*_1(-(m,w))\} = \max_{(m,w)\in E'_1} -\mathcal{B}(m,w)$$

where the maximum is taken over $(m, w) \in E'_1$ such that $(m, w) \in L^1([0, T] \times \mathbb{M} \times \mathbb{R}^d) \times L^1([0, T] \times \mathbb{M} \times \mathbb{R}^d; \mathbb{R}^d)$ and $m \ge 0$ almost everywhere, such that

$$\partial_t m + v \cdot D_x m + \operatorname{div}_v w = 0$$
 in $\mathscr{D}'((0,T) \times \mathbb{M} \times \mathbb{R}^d)$, $m|_{t=0} = m_0$.

Thus

$$\inf_{u\in E_0}\mathcal{A}(u)=-\min_{(m,w)\in \mathcal{K}_B}\mathcal{B}(m,w).$$

Lemma 3.6. Using the notations and assumptions from Lemma 3.5, we have

$$\max_{(m,w)\in E'_1} \left\{ -\mathcal{A}^*_0(\Lambda^*(m,w)) - \mathcal{A}^*_1(-(m,w)) \right\} = \max_{(m,w)\in \tilde{E}'_1} \left\{ -\mathcal{A}^*_0(\Lambda^*(m,w)) - \mathcal{A}^*_1(-(m,w)) \right\},$$

Proof. Observe that any pair $(m, w) \in E'_1$ induces functionals on C_c^0 and $(C_c^0)^d$. Therefore, there exist a signed Radon measure \tilde{m} with finite total variation and a finite vector-valued measure \tilde{w} which coincide with, respectively, m and w on (the closure with respect to the uniform norm of) C_c^0 and $(C_c^0)^d$. Then

$$\mathcal{A}_{1}^{*}(-(m,w)) = \sup_{(\phi,\psi)\in E_{1}} \left\{ \langle -m,\phi\rangle + \langle -w,\psi\rangle - \int_{0}^{t} \int_{\mathbb{M}\times\mathbb{R}^{d}} \mathcal{F}^{*}(x,\nu,-\phi+H(x,\nu,\psi)) \, \mathrm{d}x \, \mathrm{d}\nu \right\}$$

By considering functions of the form $\phi = l\chi + H_0$, for $H_0(x, v) := H(x, v, 0)$ (note that our assumptions on H imply in particular that $H_0 \in C_b$) and any non-negative $\chi \in C_b^0$ and l > 0, and $\psi = 0$, we find that $\mathcal{A}_1^*(-(m, w)) = +\infty$ unless m is a positive functional. Indeed, note that

$$\int_0^t \int_{\mathbb{M}\times\mathbb{R}^d} \mathcal{F}^*(x,\nu,\ -l\chi) \, \mathrm{d}x \, \mathrm{d}\nu \leq \int_0^t \int_{\mathbb{M}\times\mathbb{R}^d} \sup_{\beta\leq 0} \mathcal{F}^*(x,\nu,\beta) \, \mathrm{d}x \, \mathrm{d}\nu < +\infty,$$

and $\sup_{l>0}\langle -m, l\chi \rangle = +\infty$ if $\langle m, \chi \rangle < 0$.

Next, by taking the supremum over the smaller set $(\phi, \psi) \in C_c^0 \times (C_c^0)^d$ we have

$$\begin{split} \mathcal{A}_{1}^{*}(-(m,w)) &\geq \sup_{(\phi,\psi)\in C_{c}^{0}\times(C_{c}^{0})^{d}} \left\{ \langle -m,\phi+H_{0}\rangle + \langle -w,\psi\rangle \\ &-\int_{0}^{t}\int_{\mathbb{M}\times\mathbb{R}^{d}}\mathcal{F}^{*}(x,v,-\phi-H_{0}+H(x,v,\psi))\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \right\} \\ &= \sup_{(\phi,\psi)\in C_{c}^{0}\times(C_{c}^{0})^{d}} \left\{ \langle -\tilde{m},\phi+H_{0}\rangle + \langle -\tilde{w},\psi\rangle \\ &-\int_{0}^{t}\int_{\mathbb{M}\times\mathbb{R}^{d}}\mathcal{F}^{*}(x,v,-\phi-H_{0}+H(x,v,\psi))\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \right\} \\ &- \langle m-\tilde{m},H_{0}\rangle. \end{split}$$

Let us underline that the assumption on H_0 plays a crucial role, otherwise the integral of \mathcal{F}^* might not be finite for compactly supported test functions.

Since *H* is convex, for any $\chi_R \in C_b^0$ such that $0 \le \chi_R \le 1$,

$$H(x, \nu, \chi_r \psi) \leq \chi_R H(x, \nu, \psi) + (1 - \chi_R) H_0(x, \nu).$$

Thus

$$-\phi\chi_R - H_0 + H(x, v, \chi_R\psi) \leq \chi_R(-\phi - H_0 + H(x, v, \psi)),$$

and in particular we can compare the positive parts:

$$(-\phi\chi_R - H_0 + H(x,\nu,\chi_R\psi))_+ \le (\chi_R(-\phi - H_0 + H(x,\nu,\psi)))_+.$$

Since \mathcal{F}^* is non-decreasing,

$$\mathcal{F}^*(x,\nu,-\phi\chi_R-H_0+H(x,\nu,\chi_R\psi))\leq \sup_{\beta<0}\mathcal{F}^*(x,\nu,\beta)+\mathcal{F}^*(x,\nu,-\phi-H_0+H(x,\nu,\psi))\in L^1.$$

Hence, for all $\phi \in C^0_b, \, \psi \in (C^0_b)^d$ such that

$$\int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} \mathcal{F}^*(x,\nu,-\phi-H_0+H(x,\nu,\psi))\,\mathrm{d}x\,\mathrm{d}\nu\,\mathrm{d}t<+\infty,$$

by dominated convergence we have

$$\int_0^t \int_{\mathbb{M}\times\mathbb{R}^d} \mathcal{F}^*(x, \nu, -\phi - H_0 + H(x, \nu, \psi)) \, \mathrm{d}x \, \mathrm{d}\nu \, \mathrm{d}t$$
$$= \lim_{R \to +\infty} \int_0^t \int_{\mathbb{M}\times\mathbb{R}^d} \mathcal{F}^*(x, \nu, -\phi_R - H_0 + H(x, \nu, \psi_R)) \, \mathrm{d}x \, \mathrm{d}\nu \, \mathrm{d}t$$

where $\phi_R = \phi \chi_R$, $\psi_R = \psi \chi_R$ for some continuous $0 \le \chi_R \le 1$ converging pointwise to the constant function 1 as *R* tends to positive infinity. We conclude that, for any \tilde{m} , \tilde{w} (respectively signed, vector-valued) Radon measures with finite total variation,

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$$\begin{split} \sup_{(\phi,\psi)\in C^0_c\times (C^0_c)^d} & \left\{ \langle -\tilde{m},\phi+H_0\rangle + \langle -\tilde{w},\psi\rangle - \int_0^t \int_{\mathbb{M}\times\mathbb{R}^d} \mathcal{F}^*(x,v,-\phi-H_0+H(x,v,\psi)) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \right\} \\ &= \sup_{(\phi,\psi)\in C^0_b\times (C^0_b)^d} \left\{ \langle -\tilde{m},\phi\rangle + \langle -\tilde{w},\psi\rangle - \int_0^t \int_{\mathbb{M}\times\mathbb{R}^d} \mathcal{F}^*(x,v,-\phi+H(x,v,\psi)) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \right\}, \end{split}$$

where we have used that H_0 is also a C_b^0 function in order to relabel ϕ . We have thus proved that

$$\mathcal{A}_1^*(-(m,w)) \geq \mathcal{A}_1^*(-(\tilde{m},\tilde{w})) - \langle m - \tilde{m}, H_0 \rangle.$$

Next, note that if $m \in (C_b^0)'$ is a positive functional with Radon measure part \tilde{m} , then $m - \tilde{m}$ is also a positive functional: given $0 \le \phi \in C_b^0$, let $0 \le \chi_R \le 1$ be a sequence of continuous functions, non-decreasing with R and converging pointwise to the constant function 1 as R tends to positive infinity. Then, since $0 \le \phi \chi_R \le \phi$, by dominated convergence and the positivity of m,

$$\langle \tilde{m}, \phi \rangle = \lim_{R \to 0} \langle \tilde{m}, \phi \chi_R \rangle = \lim_{R \to 0} \langle m, \phi \chi_R \rangle \le \langle m, \phi \rangle$$

Since $(H_0)_+ \in C_0$, $\langle m - \tilde{m}, (H_0)_+ \rangle = 0$ and thus $\langle m - \tilde{m}, H_0 \rangle \leq 0$ for all m such that $\mathcal{A}_1^*(-(m, w,))$ is finite. Then

$$-\mathcal{A}_1^*(-(m,w)) \leq -\mathcal{A}_1^*(-(\tilde{m},\tilde{w})).$$

We now consider \mathcal{A}_0^* . We assume from now on that $m \in (C_b^0)'$ is a *positive* functional, since we only wish to consider (m, w) for which $\mathcal{A}_1^*(-(m, w)) < +\infty$.

$$\mathcal{A}_{0}^{*}(\Lambda^{*}(m,w)) = \sup_{u \in E_{0}} \left\{ \langle \Lambda^{*}(m,w), u \rangle - \int_{\mathbb{M} \times \mathbb{R}^{d}} u(0,x,v) m_{0}(x,v) \, \mathrm{d}x \, \mathrm{d}v - \int_{\mathbb{M} \times \mathbb{R}^{d}} \mathcal{G}^{*}(x,v,u(T,x,v)) \, \mathrm{d}x \, \mathrm{d}v \right\}$$
$$= \sup_{u \in E_{0}} \left\{ \langle (m,w), \Lambda u \rangle - \int_{\mathbb{M} \times \mathbb{R}^{d}} u(0,x,v) m_{0}(x,v) \, \mathrm{d}x \, \mathrm{d}v - \int_{\mathbb{M} \times \mathbb{R}^{d}} \mathcal{G}^{*}(x,v,u(T,x,v)) \, \mathrm{d}x \, \mathrm{d}v \right\}.$$

Then, taking supremum over the smaller set $u \in C_c^1$, we have

$$\mathcal{A}_0^*(\Lambda^*(m,w)) \ge \sup_{u \in C_c^1} \left\{ \langle (m,w), \Lambda u \rangle - \int_{\mathbb{M} \times \mathbb{R}^d} u(0,x,v) m_0(x,v) \, \mathrm{d}x \, \mathrm{d}v - \int_{\mathbb{M} \times \mathbb{R}^d} \mathcal{G}^*(x,v,u(T,x,v)) \, \mathrm{d}x \, \mathrm{d}v \right\}.$$

If $u \in C_c^1$, then $\Lambda u \in C_c^0$. Thus

$$\mathcal{A}_0^*(\Lambda^*(m,w)) \geq \sup_{u \in C_c^1} \bigg\{ \langle (\tilde{m}, \tilde{w}), \Lambda u \rangle - \int_{\mathbb{M} \times \mathbb{R}^d} u(0, x, v) m_0(x, v) \, \mathrm{d}x \, \mathrm{d}v - \int_{\mathbb{M} \times \mathbb{R}^d} \mathcal{G}^*(x, v, u(T, x, v)) \, \mathrm{d}x \, \mathrm{d}v \bigg\}.$$

We show that the right hand side is in fact equal to $\mathcal{A}_0^*(\Lambda^*(\tilde{m},\tilde{w}))$: given $u \in E_0$, let $\chi_R \in C_c^1(\mathbb{M} \times \mathbb{R}^d)$ be a sequence of cutoff functions such that $0 \leq \chi_R \leq 1$. We construct χ_R such that their support is contained in $\bar{B}_{2R}(0) \subset \mathbb{M} \times \mathbb{R}^d$, the closed ball of radius 2R, $\chi_R = 1$ on $\bar{B}_R(0)$, the closed ball of radius R, and $\|\nabla\chi_R\| \leq \frac{C}{R}$ for some constant C > 0 independent of R. Thus note in particular that $\chi_R \to 1$ and $\nabla\chi_R \to 0$ pointwise as $R \to +\infty$. Let $u_R := u\chi_R$. Then

$$|\Lambda u_R| = |\partial_t u_R + v \cdot D_x u_R| \le C ||u||_{E_0}, \quad |u_R(0, x, v)| \le |u(0, x, v)|, \quad [u_R(T, x, v)]_+ \le [u(T, x, v)]_+.$$

Since $m \in (C_b^0)'$, we have $\langle \tilde{m}, C \| u \|_{E_0} \rangle = C \| u \|_{E_0} \langle \tilde{m}, 1 \rangle < +\infty$. Moreover $u(0, \cdot)$ is bounded and therefore integrable with respect to m_0 . Finally, note that

$$\mathcal{G}^*(x,\nu,u_R(T,x,\nu)) \leq \mathcal{G}^*(x,\nu,u(T,x,\nu)) + \sup_{\beta_T < 0} \mathcal{G}^*(x,\nu,\beta_T).$$

Hence, if

$$\int_{\mathbb{M} imes\mathbb{R}^d} \mathcal{G}^*(x,
u,u(T,x,
u)) \,\mathrm{d}x \,\mathrm{d}
u < +\infty,$$

we may apply the dominated convergence theorem to find that

$$\langle (\tilde{m}, \tilde{w}), \Lambda u \rangle - \int_{\mathbb{M} \times \mathbb{R}^d} u(0, x, v) m_0(x, v) \, dx \, dv - \int_{\mathbb{M} \times \mathbb{R}^d} \mathcal{G}^*(x, v, u(T, x, v)) \, dx \, dv$$

=
$$\lim_{R \to +\infty} \langle (\tilde{m}, \tilde{w}), \Lambda u_R \rangle - \int_{\mathbb{M} \times \mathbb{R}^d} u_R(0, x, v) m_0(x, v) \, dx \, dv - \int_{\mathbb{M} \times \mathbb{R}^d} \mathcal{G}^*(x, v, u_R(T, x, v)) \, dx \, dv.$$

This completes the proof that the suprema over E_0 and C_c^1 are equal for the Radon measure parts. We conclude that

$$-\mathcal{A}_0^*(\Lambda^*(m,w)) \leq -\mathcal{A}_0^*(\Lambda^*(\tilde{m},\tilde{w})).$$

Now observe that, since the set \tilde{E}'_1 is contained in E'_1 (it is precisely the set of Radon measure parts of elements of E'_1),

$$\begin{split} \max_{(m,w)\in E_1'} \{-\mathcal{A}_0^*(\Lambda^*(m,w)) - \mathcal{A}_1^*(-(m,w))\} &\leq \sup_{(m,w)\in E_1'} \{-\mathcal{A}_0^*(\Lambda^*(\tilde{m},\tilde{w})) - \mathcal{A}_1^*(-(\tilde{m},\tilde{w}))\} \\ &\leq \sup_{(\tilde{m},\tilde{w})\in \tilde{E}_1'} \{-\mathcal{A}_0^*(\Lambda^*(\tilde{m},\tilde{w})) - \mathcal{A}_1^*(-(\tilde{m},\tilde{w}))\} \\ &\leq \sup_{(m,w)\in E_1'} \{-\mathcal{A}_0^*(\Lambda^*(m,w)) - \mathcal{A}_1^*(-(m,w))\}. \end{split}$$

All of the above inequalities are therefore equalities. Moreover, since

$$-\mathcal{A}_0^*(\Lambda^*(m,w)) - \mathcal{A}_1^*(-(m,w)) \leq -\mathcal{A}_0^*(\Lambda^*(\tilde{m},\tilde{w})) - \mathcal{A}_1^*(-(\tilde{m},\tilde{w})),$$

if (m, w) attains the supremum then the same is true of the Radon measure part (\tilde{m}, \tilde{w}) . Thus, without loss of generality, the optimizer is given by some $(m, w) \in \tilde{E}'_1$, i.e. a finite measure and a finite \mathbb{R}^d -valued measure.

Remark 3.7. Let us notice that the minimizer of $\mathcal{B}(m, w)$ is unique (by the convexity of \mathcal{F}, \mathcal{G} and L in their last variables). Moreover, the growth conditions on \mathcal{F}, \mathcal{G} and L imply that $m \in L^q((0, T) \times \mathbb{M} \times \mathbb{R}^d)$, $m_T \in L^s(\mathbb{M} \times \mathbb{R}^d)$ and $\frac{|w|''}{m''-1} \in L^1((0, T) \times \mathbb{M} \times \mathbb{R}^d)$. Furthermore, by Hölder's inequality, $w \in L^p((0, T) \times \mathbb{M} \times \mathbb{R}^d)$), with $p := \frac{r'q}{r'+q-1}$. These arguments are similar to the ones in [20, Theorem 2.1] and [19, Lemma 2]. Furthermore, the equation satisfied by m conserves mass, so that $m \in L^\infty_t L^1_{x,v}$, and in fact $||m_t||_{L^1_{t,v}} = ||m_0||_{L^1_{t,v}}$ for all $t \in [0, T]$.

3.4. The relaxed problem

The third problem we define is a relaxation of Problem 3.1. Consider the functional

$$\widetilde{\mathcal{A}}(u,\beta,\beta_T) := \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} \mathcal{F}^*(x,v,\beta) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t - \int_{\mathbb{M}\times\mathbb{R}^d} m_0(x,v) u_0(\,\mathrm{d}x \,\mathrm{d}v) \\ + \int_{\mathbb{M}\times\mathbb{R}^d} \mathcal{G}^*(\beta_T) \, \mathrm{d}x \, \mathrm{d}v.$$

Problem 3.8. Minimize $\tilde{\mathcal{A}}(u, \beta, \beta_T)$ over the set $\mathcal{K}_{\mathcal{A}}$ of triples $(u, \beta, \beta_T) \in L^1_{loc}(\mathcal{U}_{m_0}) \times L^1_{loc}(\mathcal{U}_{m_0}) \times L^1(\mathbb{M} \times \mathbb{R}^d)$ satisfying

- The positive part of *u* satisfies $u_+ \in L^1_{loc}([0, T] \times \mathbb{M} \times \mathbb{R}^d)$;
- The positive part of β satisfies $\beta_+ \in L^{q'}([0, T] \times \mathbb{M} \times \mathbb{R}^d)$.
- The positive part of β_T satisfies $(\beta_T)_+ \in L^{s'}(\mathbb{M} \times \mathbb{R}^d)$;
- $D_{\nu}u \in L^r_{\text{loc}}(\mathcal{U}_{m_0}),$

and subject to (3.5), understood in the sense of Definition 3.9.

Definition 3.9. We say that a triple (u, β, β_T) that belongs to the spaces from Problem 3.8 is a weak distributional solution to

$$\begin{cases} -\partial_t u - v \cdot D_x u + H(x, v, D_v u) \le \beta, & \text{in } (0, T) \times \mathbb{M} \times \mathbb{R}^d, \\ u_T \le \beta_T, & \text{in } \mathbb{M} \times \mathbb{R}^d, \end{cases}$$
(3.5)

if

$$\int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} u[\partial_{t}\phi + \operatorname{div}(v\phi)] + \phi H(x, v, D_{v}u) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t$$

$$\leq \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \beta \phi \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t + \int_{\mathbb{M}\times\mathbb{R}^{d}} \beta_{T}\phi_{T} \, \mathrm{d}x \, \mathrm{d}v,$$
(3.6)

for any $\phi \in C^1_c((0,T] \times \mathbb{M} \times \mathbb{R}^d)$ nonnegative.

Remark 3.10.

- (i) Let us emphasize that the weak form (3.6) encodes both inequalities from (3.5), as we show this in Lemma A.6.
- (ii) u_0 is similarly understood as a certain notion of a trace at t=0 in a weak sense. In particular, the term

$$-\int_{\mathbb{M}\times\mathbb{R}^d}m_0(x,\nu)u_0(\,\mathrm{d} x\,\mathrm{d} \nu)=-\langle u_0,m_0\rangle,$$

which appears in the definition of $\tilde{\mathcal{A}}$ is to be understood as in Definition A.10. Moreover, we underline that this quantity is set to be $+\infty$, if there exist $\phi \in C_c^1(\{m_0 > 0\})$ nonnegative such that $\langle u_0, \phi \rangle = -\infty$.

4. The Hamilton–Jacobi equation

In this section, we analyze the equation (3.5). We take the assumptions appropriate to the minimization problem we will consider. Therefore, we suppose throughout that $(u, \beta, \beta_T) \in \mathcal{K}_A$ is such that $\tilde{\mathcal{A}}(u, \beta, \beta_T) < +\infty$. From the finiteness of the energy we deduce in particular that

$$-\int_{\mathbb{M}\times\mathbb{R}^d}u_0m_0<+\infty.$$

4.1. Upper bounds

We prove upper bounds on u. First, we observe that for any constant $l \in \mathbb{R}$ the function $(u-l)_+ := \max\{u-l, 0\}$ satisfies (see Lemma B.1)

$$\begin{cases} -(\partial_t + v \cdot D_x)(u-l)_+ + H(x,v,D_vu)\mathbb{1}_{\{u>l\}} \leq \beta\mathbb{1}_{\{u>l\}}, & \text{in } \mathscr{D}'((0,T) \times \mathbb{M} \times \mathbb{R}^d), \\ (u_T - l)_+ \leq (\beta_T - l)_+, & \text{in } \mathscr{D}'(\mathbb{M} \times \mathbb{R}^d). \end{cases}$$

$$(4.1)$$

We use the notation $L^{\infty} + L^{q'}$ to denote the set of functions

$$\{h=h_1+h_2:h_1\in L^\infty,h_2\in L^{q'}\},$$

which becomes a Banach space when equipped with the norm

$$\|h\|_{L^{\infty}+L^{q'}} := \inf\{\|h_1\|_{L^{\infty}} + \|h_2\|_{L^{q'}} : h = h_1 + h_2\}.$$

We also use the notation $L^1 \cap L^q$ to denote the intersection of L^1 and L^q made into a Banach space under the norm

$$\|h\|_{L^1\cap L^q} := \max\{\|h\|_{L^1}, \|h\|_{L^q}\}.$$

Note that the dual space is given by $(L^1 \cap L^q)^* = L^{\infty} + L^{q'}$. Lemma 4.1. Let $l \in \mathbb{R}$ be given and let $(u, \beta, \beta_T) \in \mathcal{K}_A$ satisfy (4.1).

(i) Then
$$(u-l)_{+} \in L^{\infty}_{t}(L^{\infty}+L^{q'})_{x,v}$$
, with the a priori estimate
 $\|(u-l)_{+}\|_{L^{\infty}_{t}(L^{\infty}+L^{q'})_{x,v}} \leq \|(\beta_{T}-l)_{+}\|_{(L^{\infty}+L^{q'})(\mathbb{M}\times\mathbb{R}^{d})} + C_{H}T + T^{1/q}\|\beta_{+}\|_{L^{q'}}$
 $\leq \|(\beta_{T}-l)_{+}\|_{L^{t'}(\mathbb{M}\times\mathbb{R}^{d})} + C_{H}T + T^{1/q}\|\beta_{+}\|_{L^{q'}}.$

(ii) Suppose in addition that $\tilde{\mathcal{A}}(u,\beta,\beta_T) < C_{\tilde{A}}$. Then, there exists

$$C = C ig(C_{ ilde{\mathcal{A}}}, \|m_0\|_{L^1 \cap L^q}, T, C_F, C_G ig) > 0$$

such that

$$\|\beta_+\|_{L^{q'}((0,T)\times\mathbb{M}\times\mathbb{R}^d)} + \|(\beta_T)_+\|_{L^{s'}(\mathbb{M}\times\mathbb{R}^d)} \le C$$

Proof. First, let us note that, since $(\beta_T)_+ \in L^{s'}(\mathbb{M} \times \mathbb{R}^d)$ and $s' \ge q'$ (by Assumption 1), $(\beta_T)_+ \in (L^{\infty} + L^{q'})(\mathbb{M} \times \mathbb{R}^d)$ and thus also $(\beta_T - l)_+ \in (L^{\infty} + L^{q'})(\mathbb{M} \times \mathbb{R}^d)$.

(i) Let $t \in [0, T)$ be fixed. Let $0 \le \psi \in C_c^{\infty}(\mathbb{M} \times \mathbb{R}^d)$ and consider $\zeta(\tau, x, \nu) := \psi(x + (t - \tau)\nu, \nu).$

Then ζ is smooth and compactly supported and satisfies

$$\partial_{\tau}\zeta + \nu \cdot D_x\zeta = 0$$

By using ζ as a test function for $(u - l)_+$ over $\tau \in [t, T]$, we obtain

$$\begin{split} \int_{\mathbb{M}\times\mathbb{R}^d} (u-l)_+(t,x,v)\psi(x,v)\,\mathrm{d}x\,\mathrm{d}v &\leq \int_{\mathbb{M}\times\mathbb{R}^d} (\beta_T-l)_+\zeta_T\,\mathrm{d}x\,\mathrm{d}v \\ &+ \int_t^T \int_{\mathbb{M}\times\mathbb{R}^d} \zeta[\beta-H(x,v,D_vu)]\mathbbm{1}_{\{u>l\}}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau. \end{split}$$

Recall that when we write $(u - l)_+(t, \cdot, \cdot)$, we are always referring to the version of u that is weakly right continuous with respect to time (cf. Appendix A, Lemma A.1).

Since $H \ge -C_H$, we have

$$\int_{\mathbb{M}\times\mathbb{R}^d} (u-l)_+(t,x,v)\psi(x,v)\,\mathrm{d}x\,\mathrm{d}v \leq \int_{\mathbb{M}\times\mathbb{R}^d} (\beta_T-l)_+\zeta_T\,\mathrm{d}x\,\mathrm{d}v + \int_t^T \int_{\mathbb{M}\times\mathbb{R}^d} \zeta[\beta_++C_H]\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau.$$

Then

$$\begin{split} \int_{\mathbb{M}\times\mathbb{R}^d} (u_t - l)_+ \psi \, \mathrm{d}x \, \mathrm{d}\nu &\leq \|(\beta_T - l)_+\|_{L^\infty + L^{q'}} \|\zeta_T\|_{L^1 \cap L^q} \\ &+ \|\zeta\|_{L^q([t, T]\times\mathbb{M}\times\mathbb{R}^d)} \|\beta_+\|_{L^{q'}([t, T]\times\mathbb{M}\times\mathbb{R}^d)} + C_H \|\zeta\|_{L^1([t, T]\times\mathbb{M}\times\mathbb{R}^d)}. \end{split}$$

We compute

$$\begin{aligned} \|\zeta\|_{L^{1}([t,T]\times\mathbb{M}\times\mathbb{R}^{d})} &= \int_{t}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \psi(x+(t-\tau)\nu,\nu) \, \mathrm{d}x \, \mathrm{d}\nu \, \mathrm{d}\tau \\ &= \int_{t}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \psi(x,\nu) \, \mathrm{d}x \, \mathrm{d}\nu \, \mathrm{d}\tau = (T-t) \|\psi\|_{L^{1}(\mathbb{M}\times\mathbb{R}^{d})}. \end{aligned}$$

Similarly

$$\|\zeta_T\|_{L^1(\mathbb{M}\times\mathbb{R}^d)} = \|\psi\|_{L^1(\mathbb{M}\times\mathbb{R}^d)}$$

and

$$\|\zeta\|_{L^q([t,T]\times\mathbb{M}\times\mathbb{R}^d)}=(T-t)^{1/q}\|\psi\|_{L^q(\mathbb{M}\times\mathbb{R}^d)}.$$

Thus

$$\int_{\mathbb{M}\times\mathbb{R}^d} (u_t - l)_+ \psi \, \mathrm{d}x \, \mathrm{d}\nu \leq \left(\|(\beta_T - l)_+\|_{L^\infty + L^{q'}} + C_H T + T^{1/q} \|\beta_+\|_{L^{q'}} \right) \|\psi\|_{L^1 \cap L^q}.$$

This extends by density to all non-negative $\psi \in (L^1 \cap L^q)(\mathbb{M} \times \mathbb{R}^d)$, and general $\psi \in (L^1 \cap L^q)(\mathbb{M} \times \mathbb{R}^d)$ by non-negativity of $(u - l)_+$. We conclude by the fact that $\|(\beta_T - l)_+\|_{L^{\infty} + L^{q'}} \le \|(\beta_T - l)_+\|_{L^{q'}}$. The result follows.

(i) By the definition of $\tilde{\mathcal{A}}$ and the assumptions on the data one has

$$\begin{split} \tilde{\mathcal{A}}(u,\beta,\beta_{T}) &\geq \|\beta_{+}\|_{L^{q'}}^{q'} + \|(\beta_{T})_{+}\|_{L^{s'}}^{s'} - \int_{\mathbb{M}\times\mathbb{R}^{d}} [TC_{F}(x,\nu) + C_{G}(x,\nu)] \, \mathrm{d}x \, \mathrm{d}\nu \\ &- \int_{\mathbb{M}\times\mathbb{R}^{d}} u(0,x,\nu)_{+} m_{0}(x,\nu) \, \mathrm{d}x \, \mathrm{d}\nu \\ &\geq \|\beta_{+}\|_{L^{q'}}^{q'} + \|(\beta_{T})_{+}\|_{L^{s'}}^{s'} - C - \left(\|(\beta_{T})_{+}\|_{L^{s'}} + C_{H}T + T^{1/q}\|\beta_{+}\|_{L^{q'}}\right) \|m_{0}\|_{L^{1}\cap L^{q}}, \end{split}$$

$$(4.2)$$

where in the last inequality we used the estimate from (i). This further yields the claim in (ii). $\hfill \square$

Corollary 4.2. Let (u, β, β_T) be as in the statement of Lemma 4.1 such that there exists $C_{\tilde{A}} > 0$ with $\tilde{A}(u, \beta, \beta_T) < C_{\tilde{A}}$. Then, there exists $C = C(C_{\tilde{A}}, ||m_0||_{L^1 \cap L^q}, T, C_F, C_G) > 0$ such that the following hold.

- (i) $||(u_0)_+||_{L^{\infty}+L^{q'}(\mathbb{M}\times\mathbb{R}^d)} \leq C;$
- (ii) $\int_{\mathbb{M}\times\mathbb{R}^d} m_0(u_0)_{-}(\,\mathrm{d} x\,\mathrm{d} v) \leq C.$

Proof. We notice that (i) is a simple consequence of Lemma 4.1(i)-(ii), by setting l=0 and t=0 (in the sense of weak trace, given in Definition A.5).

For (ii), we observe

$$\begin{split} \int_{\mathbb{M}\times\mathbb{R}^d} m_0(u_0)_{-}(\,\mathrm{d}x\,\mathrm{d}v) &= -\int_{\mathbb{M}\times\mathbb{R}^d} m_0 u_0(\,\mathrm{d}x\,\mathrm{d}v) + \int_{\mathbb{M}\times\mathbb{R}^d} (u_0)_+ m_0\,\mathrm{d}x\,\mathrm{d}v\\ &\leq \tilde{\mathcal{A}}(u,\beta,\beta_T) - \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} \mathcal{F}^*(x,v,\beta)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t\\ &- \int_{\mathbb{M}\times\mathbb{R}^d} \mathcal{G}^*(\beta_T)\,\mathrm{d}x\,\mathrm{d}v + \|(u_0)_+\|_{L^\infty + L^{q'}}\|m_0\|_{L^1\cap L^q}. \end{split}$$

By the bounds (3.1) on \mathcal{F}^* and the corresponding estimates for \mathcal{G}^* ,

$$\sup_{\beta} \{-\mathcal{F}^*(x,\nu,\beta)\} \leq C_F(x,\nu), \quad \sup_{\beta_T} \{-\mathcal{G}^*(x,\nu,\beta_T)\} \leq C_G(x,\nu).$$

Hence, using the above bounds and (i), we obtain

$$\int_{\mathbb{M}\times\mathbb{R}^d} m_0(u_0)_{-}(\,\mathrm{d} x\,\mathrm{d} \nu) \leq C_{\tilde{\mathcal{A}}} + \|C_F\|_{L^1} + \|C_G\|_{L^1} + C\|m_0\|_{L^1\cap L^q},$$

which completes the proof.

4.2. Local L^1 bounds

Next, we prove bounds on the negative parts of u and β . We will obtain $L^1_{loc}(\mathcal{U}_{m_0})$ bounds, by use of a duality argument involving a certain class of test functions which satisfy the continuity equation associated to the control system.

Lemma 4.3. Let $a \in C_b^1([0,T] \times \mathbb{M} \times \mathbb{R}^d; \mathbb{R}^d)$ be a bounded control. Let $\phi_0 \in C_c^1(\mathbb{M} \times \mathbb{R}^d)$ satisfy $0 \le \phi_0 \le m_0$. Let $\phi \in C_c^1(\mathcal{U}_{m_0})$ be the solution of the continuity equation

$$\partial_t \phi + v \cdot D_x \phi + \operatorname{div}_v(a\phi) = 0, \qquad \phi|_{t=0} = \phi_0.$$

Then, for any $(u, \beta, \beta_T) \in \mathcal{K}_A$ such that $\tilde{\mathcal{A}}(u, \beta, \beta_T) < +\infty$, the following hold:

• $u \in L^{\infty}_t L^1_{x,v}(\phi)$, that is,

$$\operatorname{ess\,sup}_{t\in[0,\,T]}\int_{\mathbb{M}\times\mathbb{R}^d}|u_t|\phi_t\,\mathrm{d} x\,\mathrm{d} \nu<+\infty.$$

- The negative part of β satisfies $\beta_{-} \in L^{1}_{t,x,v}(\phi)$.
- The negative part of β_T satisfies $(\beta_T)_- \in L^1_{x,v}(\phi_T)$.
- The following estimate holds:

$$\begin{aligned} \|u\|_{L^{\infty}_{t}L^{1}_{x,\nu}(\phi)} + \|\beta_{-}\|_{L^{1}_{t,x,\nu}(\phi)} + \|(\beta_{T})_{-}\|_{L^{1}_{x,\nu}(\phi_{T})} + \|D_{\nu}u\|_{L^{r}_{t,x,\nu}(\phi)}^{r} \\ &\leq C(a,\phi,m_{0},H,T)(1+\|\beta_{+}\|_{L^{q'}} + \|(\beta_{T})_{+}\|_{L^{\infty}+L^{q'}}) - \int_{\mathbb{M}\times\mathbb{R}^{d}} u_{0}m_{0} \,\mathrm{d}x \,\mathrm{d}\nu. \end{aligned}$$

Proof. Note the following properties of ϕ :

- ϕ is non-negative,
- ϕ has compact support contained in \mathcal{U}_{m_0} .
- $\phi \in (L^1 \cap L^\infty)_{t,x,\nu}.$

In particular, since $\phi_t \in (L^1 \cap L^q)(\mathbb{M} \times \mathbb{R}^d)$ for any $t \in [0, T]$, then

$$\int_{\mathbb{M}\times\mathbb{R}^d} (u_t)_+ \phi_t \, \mathrm{d}x \, \mathrm{d}\nu \le \|\phi_t\|_{(L^1 \cap L^q)_{x,\nu}} \|u_+\|_{L^{\infty}_t(L^{\infty} + L^{q'})_{x,\nu}}$$

By Lemma 4.1,

$$\|u_{+}\|_{L^{\infty}_{t}(L^{\infty}+L^{q'})_{x,v}} \leq C(T,H) \left(1 + \|(\beta_{T})_{+}\|_{(L^{\infty}+L^{q'})(\mathbb{M}\times\mathbb{R}^{d})} + \|\beta_{+}\|_{L^{q'}_{t,x,v}}\right)$$
(4.3)

and thus for $t \in [0, T]$, we have

$$\int_{\mathbb{M}\times\mathbb{R}^{d}} (u_{t})_{+} \phi_{t} \, \mathrm{d}x \, \mathrm{d}v \leq C(T, H, \phi) \Big(1 + \|(\beta_{T})_{+}\|_{(L^{\infty} + L^{q'})(\mathbb{M}\times\mathbb{R}^{d})} + \|\beta_{+}\|_{L^{q'}_{t, x, v}} \Big),$$

where $(u_t)_+$ is understood in the sense of weak trace (cf. Lemma A.1, Definition A.5).

For the negative part we make use of the equation. A density argument shows that ϕ is admissible as a test function in the weak form of the Hamilton-Jacobi inequality satisfied by u. Thus for $0 \le s < t \le T$,

$$\int_{\mathbb{M}\times\mathbb{R}^d} u_s \phi_s \, \mathrm{d}x \, \mathrm{d}v - \int_{\mathbb{M}\times\mathbb{R}^d} u_t \phi_t \, \mathrm{d}x \, \mathrm{d}v + \int_s^t \int_{\mathbb{M}\times\mathbb{R}^d} u(\partial_t \phi + v \cdot D_x \phi) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}\tau \\ + \int_s^t \int_{\mathbb{M}\times\mathbb{R}^d} \phi H(x, v, D_v u) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}\tau \le \int_s^t \int_{\mathbb{M}\times\mathbb{R}^d} \beta \phi \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}\tau.$$

We apply this in the case s = 0, $t \in (0, T]$. Using the fact that ϕ satisfies the continuity equation in a pointwise sense,

$$\int_{\mathbb{M}\times\mathbb{R}^d} u_0\phi_0 \,\mathrm{d}x \,\mathrm{d}v - \int_{\mathbb{M}\times\mathbb{R}^d} u(t,x,v)\phi(t,x,v) \,\mathrm{d}x \,\mathrm{d}v \\ - \int_0^t \int_{\mathbb{M}\times\mathbb{R}^d} u \mathrm{d}iv_v(a\phi) \,\mathrm{d}x \,\mathrm{d}v \,\mathrm{d}\tau + \int_0^t \int_{\mathbb{M}\times\mathbb{R}^d} \phi H(x,v,D_v u) \,\mathrm{d}x \,\mathrm{d}v \,\mathrm{d}\tau \le \int_0^t \int_{\mathbb{M}\times\mathbb{R}^d} \beta\phi \,\mathrm{d}x \,\mathrm{d}v \,\mathrm{d}\tau.$$

Here, let us notice that we have used the existence of weak traces in the sense of Lemma A.1. In particular the integral $\int_{\mathbb{M}\times\mathbb{R}^d} u_0\phi_0 \,dx \,d\nu$ is meaningful and finite, since $\operatorname{spt}(\phi_0) \subseteq \operatorname{spt}(m_0)$ (Definition A.10).

Since $D_{\nu}u \in L^{r}_{loc}(\mathcal{U}_{m_{0}})$ and $a\phi \in C^{1}$ has compact support contained in $\mathcal{U}_{m_{0}}$, we may integrate by parts to obtain

$$-\int_0^t\int_{\mathbb{M}\times\mathbb{R}^d} u\,\operatorname{div}_\nu(a\phi)\,\mathrm{d}x\,\mathrm{d}\nu\,\mathrm{d}\tau=\int_0^t\int_{\mathbb{M}\times\mathbb{R}^d} a\cdot D_\nu u\,\phi\,\mathrm{d}x\,\mathrm{d}\nu\,\mathrm{d}\tau.$$

Then estimate

$$\left|\int_0^t\int_{\mathbb{M}\times\mathbb{R}^d}a\cdot D_\nu u\ \phi\,\mathrm{d}x\,\mathrm{d}\nu\,\mathrm{d}\tau\right|\leq \|a\|_{L^{r'}(\phi)}\|D_\nu u\|_{L^r(\phi)},$$

where, in order to lighten the notation, we have used the shorthand

$$\|h\|_{L^p(\phi)} := \left(\int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} |h|^p \phi \,\mathrm{d}x \,\mathrm{d}v \,\mathrm{d}t\right)^{1/p}, \quad p \in [1, +\infty)$$

to denote L^p norms with respect to the measure on $[0, T] \times \mathbb{M} \times \mathbb{R}^d$ with density ϕ with respect to Lebesgue measure. Thus

$$-\int_{\mathbb{M}\times\mathbb{R}^d} u_t \phi_t \, \mathrm{d}x \, \mathrm{d}v + \int_0^t \int_{\mathbb{M}\times\mathbb{R}^d} \phi H(x, v, D_v u) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}\tau \\ \leq \|a\|_{L^{r'}(\phi)} \|D_v u\|_{L^r(\phi)} - \int_{\mathbb{M}\times\mathbb{R}^d} u_0 \phi_0 \, \mathrm{d}x \, \mathrm{d}v + \int_0^t \int_{\mathbb{M}\times\mathbb{R}^d} \beta \phi \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}\tau.$$

Using the lower bounds on the Hamiltonian H, rearranging terms and using Young's inequality for products (with a small parameter), we obtain

$$- \int_{\mathbb{M}\times\mathbb{R}^d} u_t \phi_t \, \mathrm{d}x \, \mathrm{d}v + \int_0^t \int_{\mathbb{M}\times\mathbb{R}^d} \beta_- \phi \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}\tau + c \int_0^t \int_{\mathbb{M}\times\mathbb{R}^d} \phi |D_v u|^r \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}\tau \\ \leq C ||a||_{L^{r'}(\phi)}^{r'} + C_H \int_0^t \int_{\mathbb{M}\times\mathbb{R}^d} \phi \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}\tau - \int_{\mathbb{M}\times\mathbb{R}^d} u_0 \phi_0 \, \mathrm{d}x \, \mathrm{d}v + \int_0^t \int_{\mathbb{M}\times\mathbb{R}^d} \beta_+ \phi \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}\tau.$$

Then

$$-\int_{\mathbb{M}\times\mathbb{R}^{d}}u_{t}\phi_{t}\,\mathrm{d}x\,\mathrm{d}v+\int_{0}^{t}\int_{\mathbb{M}\times\mathbb{R}^{d}}\beta_{-}\phi\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau+c\int_{0}^{t}\int_{\mathbb{M}\times\mathbb{R}^{d}}\phi|D_{\nu}u|^{r}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}\tau$$

$$\leq C(a,\phi)\big(C_{H}+\|\beta_{+}\|_{L^{q'}}\big)-\int_{\mathbb{M}\times\mathbb{R}^{d}}u_{0}m_{0}\,\mathrm{d}x\,\mathrm{d}v+\int_{\mathbb{M}\times\mathbb{R}^{d}}(u_{0})_{+}(m_{0}-\phi_{0})\,\mathrm{d}x\,\mathrm{d}v.$$

$$(4.4)$$

Finally, since $0 \le m_0 - \phi_0 \le m_0 \in L^1 \cap L^q$, we use the $(L^{\infty} + L^{q'})_{x,v}$ bounds on the positive part $(u_0)_+$ (Equation (4.3)) to conclude that

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$$\int_{\mathbb{M}\times\mathbb{R}^d} |u_t|\phi_t \,\mathrm{d}x \,\mathrm{d}v + \int_0^t \int_{\mathbb{M}\times\mathbb{R}^d} \beta_-\phi \,\mathrm{d}x \,\mathrm{d}v \,\mathrm{d}\tau + c \int_0^t \int_{\mathbb{M}\times\mathbb{R}^d} \phi |D_v u|^r \,\mathrm{d}x \,\mathrm{d}v \,\mathrm{d}\tau$$

$$\leq C(a,\phi,m_0,H,T) (1 + \|\beta_+\|_{L^{q'}} + \|(\beta_T)_+\|_{L^{\infty}+L^{q'}}) - \int_{\mathbb{M}\times\mathbb{R}^d} u_0 m_0 \,\mathrm{d}x \,\mathrm{d}v.$$

Notice that by setting t = T, (4.4) and the fact that $u_T \leq \beta_T$ (together with the bounds that we already have on $(\beta_T)_+$) readily yield also that $(\beta_T)_- \in L^1(\phi_T)$.

This completes the proof.

Corollary 4.4. Let $(u, \beta, \beta_T) \in \mathcal{K}_A$ such that $\tilde{\mathcal{A}}(u, \beta, \beta_T) < +\infty$. Then $u \in L^1_{\text{loc}}(\mathcal{U}_{m_0}), \beta_- \in L^1_{\text{loc}}(\mathcal{U}_{m_0}), (\beta_T)_- \in L^1_{\text{loc}}(\mathbb{M} \times \mathbb{R}^d)$ and $D_v u \in L^r_{\text{loc}}(\mathcal{U}_{m_0})$.

Proof. First, consider a compact set

$$K \subset \bigcup_{a \in C^1_b, \, 0 \le \phi_0 \in C^1_c(\{m_0 > 0\})} \{\phi > 0\}$$

where $\phi \in C^1$ denotes the solution of the continuity equation

$$\partial_t \phi + v \cdot D_x \phi + \operatorname{div}_v(a\phi) = 0, \qquad \phi|_{t=0} = \phi_0.$$
 (4.5)

By compactness of K, there exist finitely many ϕ_i , i = 1, ..., k such that

$$K \subset \bigcup_{i=1}^{k} \{\phi_i > 0\}.$$

The function $\max_i \phi_i$ is continuous and so

$$0 < \delta_K := \inf_K \max_i \phi_i.$$

Then

$$\|u\|_{L^{1}(K)}+\|eta_{-}\|_{L^{1}(K)}+\|D_{
u}u\|_{L^{r}(K)}\leq \delta_{K}^{-1}\sum_{i=1}^{k}\|u\|_{L^{1}(\phi_{i})}+\|eta_{-}\|_{L^{1}(\phi_{i})}+\|D_{
u}u\|_{L^{r}(\phi_{i})}.$$

By Lemma 4.3, this leads to the estimate

$$\|u\|_{L^{1}(K)} + \|\beta_{-}\|_{L^{1}(K)} + \|D_{\nu}u\|_{L^{r}(K)} \leq C,$$

where $C = C(K, \tilde{\mathcal{A}}(u, \beta, \beta_T))$. We now claim that

$${\mathcal U}_{m_0} \subset \mathop{\cup}\limits_{a \in C^1_b, \, 0 \leq \phi_0 \in C^1_c(\{m_0 > 0\})} \{ \phi > 0 \}.$$

This follows from the controllability of the ODE system

$$\dot{x} = v, \quad \dot{v} = a \tag{4.6}$$

on $\mathbb{M} \times \mathbb{R}^d$. That is, for any initial datum $(x_0, v_0) \in \mathbb{M} \times \mathbb{R}^d$ and target $(t_*, x_*, v_*) \in (0, T] \times \mathbb{M} \times \mathbb{R}^d$, there exists a control function *a* such that the solution $(x(\tau), v(\tau))$ of the ODE (4.5) with $(x(0), v(0)) = (x_0, v_0)$ satisfies $(x(t_*), v(t_*)) = (x_*, v_*)$.

Next, note that (since m_0 is continuous) $\{m_0 > 0\}$ contains a closed ball $\overline{B}_r(x_0, v_0)$ for some point (x_0, v_0) and some r > 0. Thus there exists $0 \le \phi_0 \in C_c^1(\{m_0 > 0\})$ such

that $\phi_0 > 0$ on $\overline{B}_r(x_0, v_0)$. Consider the solution ϕ of (4.4) for the control *a* found above and with this choice of ϕ_0 . It follows that $(t_*, x_*, v_*) \in \{\phi > 0\}$.

Finally, we notice that by the structure of the set \mathcal{U}_{m_0} , we have the bound $(\beta_T)_- \in L^1_{loc}(\mathbb{M} \times \mathbb{R}^d)$.

5. Duality for the relaxed problem

Theorem 5.1. Problems 3.3 and 3.8 are in duality:

$$\inf_{(u,\beta,\beta_T)\in\mathcal{K}_{\mathcal{A}}}\mathcal{A}(u,\beta,\beta_T)=-\min_{(m,w)\in\mathcal{K}_{\mathcal{B}}}\mathcal{B}(m,w)$$

Proof. For $u \in C_b^1([0,T] \times \mathbb{M} \times \mathbb{R}^d)$ such that $\mathcal{A}(u) < +\infty$, the triple $(u, -\partial_t u - v \cdot D_x u + H(x,v,D_v u), u_T)$ lies in $\mathcal{K}_{\mathcal{A}}$. Thus

$$\inf_{(u,\beta)\in\mathcal{K}_{\mathcal{A}}}\tilde{\mathcal{A}}(u,\beta,\beta_T)\leq \inf_{u\in C_b^1}\mathcal{A}(u).$$

By the duality result of Lemma 3.5,

$$\inf_{(u,\beta,\beta_T)\in\mathcal{K}_{\mathcal{A}}}\tilde{\mathcal{A}}(u,\beta,\beta_T)\leq \inf_{u\in C_b^1}\mathcal{A}(u)=-\min_{(m,w)\in\mathcal{K}_{\mathcal{B}}}\mathcal{B}(m,w).$$

It therefore remains only to prove the reverse inequality. This follows from Lemma 5.2 below, which states that for all $(u, \beta, \beta_T) \in \mathcal{K}_A$ and $(m, w) \in \mathcal{K}_B$,

$$\mathcal{A}(u,\beta,\beta_T) \geq -\mathcal{B}(m,w).$$

Taking the infimum over $(u, \beta, \beta_T) \in \mathcal{K}_A$ and supremum over $(m, w) \in \mathcal{K}_B$ gives

$$\inf_{(u,\beta,\beta_T)\in\mathcal{K}_{\mathcal{A}}}\tilde{\mathcal{A}}(u,\beta,\beta_T)\geq-\min_{(m,w)\in\mathcal{K}_{\mathcal{B}}}\mathcal{B}(m,w)$$

as required.

Lemma 5.2. Let $(u, \beta, \beta_T) \in \mathcal{K}_A$ and $(m, w) \in \mathcal{K}_B$ such that $\tilde{\mathcal{A}}(u, \beta, \beta_T), \mathcal{B}(m, w) < +\infty$. Then

$$\hat{\mathcal{A}}(u,\beta,\beta_T) + \mathcal{B}(m,w) \geq 0.$$

In the proof of this lemma we require the following observation regarding the commutator between the operator $v \cdot D_x$ and the operator given by convolution with a fixed function.

Lemma 5.3. Let $\chi : \mathbb{R}^d \to [0, +\infty)$ be a function such that $(1 + |\nu|)\chi \in L^1_{\nu}(\mathbb{R}^d)$. Let $h \in L^p_{\nu}(\mathbb{R}^d)$ for $p \in [1, +\infty]$. Then

$$\nu\chi * h - \chi * (\nu h) = (\nu\chi) * h.$$

Proof. By direct computation, for all $v \in \mathbb{R}^d$,

$$\begin{aligned} v\chi * h(v) - \chi * (vh) &= v \int_{\mathbb{R}^d} h(v-z)\chi(z) \, \mathrm{d}z - \int_{\mathbb{R}^d} (v-z)h(v-z)\chi(z) \, \mathrm{d}z \\ &= \int_{\mathbb{R}^d} z\chi(z)h(v-z) \, \mathrm{d}z \\ &= (v\chi) * h. \end{aligned}$$

Proof of Lemma 5.2. The overall idea of the proof is to use m as a test function in the weak form of the inequality

$$-\partial_t u - v \cdot D_x u + H(x, v, D_v u) \le \beta$$

and its terminal condition

$$u_T \leq \beta_T$$
.

To make this valid, we must first introduce an approximation procedure.

First, we introduce a lower cutoff on u and β . Let $l \leq 0$ and define $u_l := \max\{u, l\}$. Similarly, for $k \leq 0$, let $\beta_k := \max\{\beta, k\}$. Then by Lemma B.1 we obtain

$$\begin{cases} -\partial_t u_l - v \cdot D_x u_l + H(x, v, D_v u) \mathbb{1}_{\{u > l\}} &\leq \beta_k \mathbb{1}_{\{u > l\}}, \quad (0, T) \times \mathbb{M} \times \mathbb{R}^d \\ (u_T)_l &\leq (\beta_T)_l, \quad \mathbb{M} \times \mathbb{R}^d, \end{cases}$$
(5.1)

in the sense of distributions. By Lemma 4.1, $u_l \in L_t^{\infty}(L^{\infty} + L^{q'})_{x,v}$. We emphasize that k and l are taken to be possibly independent at this point.

Next, we approximate *m* by a function in C_c^1 , which is then an admissible test function for the Hamilton-Jacobi Equation (5.1). We regularize *m* by convolution with a mollifier. For ease of presentation, it will be convenient to work with the time, space and velocity variables separately. Fix $\chi \in C_c^{\infty}(\mathbb{R}^d)$ and define χ_{ε} for $\varepsilon > 0$ by

$$\chi_{\varepsilon}(v) := \varepsilon^{-d} \chi\left(\frac{v}{\varepsilon}\right).$$

For the space variable, consider $\psi \in C_c^{\infty}(\mathbb{R}^d)$ and for $\delta > 0$ let

$$\psi_{\delta}(x) := \delta^{-d} \psi\left(\frac{x}{\delta}\right).$$

For the time variable, fix $\theta \in C_c^{\infty}(\mathbb{R})$ and for $\eta > 0$ let

$$\theta_{\eta}(t) := \eta^{-1} \theta\left(\frac{t}{\eta}\right).$$

We then define the full mollifier φ by

$$\varphi(t, x, v) = \theta_{\eta}(t)\psi_{\delta}(x)\chi_{\varepsilon}(v)$$

Then define the smooth functions

$$\tilde{m} := \varphi *_{t,x,v} m$$
 and $\tilde{w} = \varphi *_{t,x,v} w$.

Notice that for the convolution in time, (m, w) needs to be extended. We choose the following extensions. We set $w(t, \cdot, \cdot) = 0$ to for t < 0 and t > T. Then, if t < 0, we set $m(t, \cdot, \cdot)$ to be the solution to the problem

$$\left\{ egin{array}{ll} \partial_t m + v \cdot D_x m = 0, & ext{ in } (-\eta, 0) imes \mathbb{M} imes \mathbb{R}^d, \ m(0,\,\cdot\,,\cdot) = m_0, & ext{ in } \mathbb{M} imes \mathbb{R}^d. \end{array}
ight.$$

Similarly, for t > T we set $m(t, \cdot, \cdot)$ to be the solution to

$$\left\{ egin{array}{ll} \partial_t m + v \cdot D_x m = 0, & ext{ in } (T, T + \eta) imes \mathbb{M} imes \mathbb{R}^d, \ m(T, \cdot, \cdot) = m_T, & ext{ in } \mathbb{M} imes \mathbb{R}^d, \end{array}
ight.$$

where m_T is the trace of *m* in time at t = T.

As the final step in the approximation, we localize \tilde{m} . As localizers we consider smooth functions $\zeta_R \in C_c^{\infty}(\mathbb{M} \times \mathbb{R}^d)$ such that

$$\zeta_R(x,\nu) = \begin{cases} 1 & |x| \le R^2, |\nu| \le R\\ 0 & |x| > 2R^2, |\nu| > 2R. \end{cases} \quad |D_x \zeta_R| \le \frac{C}{R^2}, |D_\nu \zeta_R| \le \frac{C}{R} \quad . \tag{5.2}$$

We then define $\tilde{m}^{(R)} := \zeta_R \tilde{m}$ and $\tilde{w}^{(R)} := \zeta_R \tilde{w}$.

Then $\tilde{m}^{(R)}$ satisfies the equation

$$\partial_t \tilde{m}^{(R)} + v \cdot D_x \tilde{m}^{(R)} + \operatorname{div}_v \tilde{w}^{(R)} = \mathcal{E}_{\eta, \, \delta, \, \varepsilon, \, R},$$

where the error term is given by

$$\mathcal{E}_{\eta,\delta,\varepsilon,R} := \zeta_R \ [\nu \cdot D_x, \chi_{\varepsilon} *_{\nu}] \theta_{\eta} *_t \psi_{\delta} *_x m + (\nu \cdot D_x \zeta_R) \tilde{m} + (D_{\nu} \zeta_R) \cdot \tilde{w}.$$
(5.3)

Here, we use the standard commutator notation $[\Lambda_1, \Lambda_2]f := \Lambda_1(\Lambda_2 f) - \Lambda_2(\Lambda_1 f)$, where Λ_1, Λ_2 are some operators acting on the function *f*.

5.1. Convergence of the error term

We show that the error term $\mathcal{E}_{\eta,\delta,\varepsilon,R}$ defined by (5.3) converges to zero in the space $L^1_t(L^1 + L^q)_{x,\nu}$, as $R \to +\infty$ and $\eta, \varepsilon, \delta \to 0$, under a certain relationship between these parameters.

For the first term, either for p = 1 or p = q, using the explicit formula for the commutator we estimate

$$\begin{split} \|\zeta_{R} \ [v \cdot D_{x}, \chi_{\varepsilon} *_{v}] \theta_{\eta} *_{t} \psi_{\delta} *_{x} m \|_{L_{t}^{1} L_{x,v}^{p}} &\leq \| [v \cdot D_{x}, \chi_{\varepsilon} *_{v}] (\theta_{\eta} \psi_{\delta}) *_{t,x} m \|_{L_{t}^{1} L_{x,v}^{p}} \\ &\leq \| [v, \chi_{\varepsilon} *_{v}] \theta_{\eta} *_{t} (D_{x} \psi_{\delta}) *_{x} m \|_{L_{t}^{1} L_{x,v}^{p}} \\ &\leq \| (v\chi_{\varepsilon}) *_{v} \theta_{\eta} *_{t} (D_{x} \psi_{\delta}) *_{x} m \|_{L_{t}^{1} L_{x,v}^{p}} \\ &\leq \| v\chi_{\varepsilon} \|_{L^{1}} \| \theta_{\eta} \|_{L^{1}} \| D_{x} \psi_{\delta} \|_{L^{1}} \| m \|_{L_{t}^{1} L_{x,v}^{p}} \\ &\leq \varepsilon \delta^{-1} \cdot CT^{1/p'} \| m \|_{L_{t,x,v}^{p}}, \end{split}$$

where we have used Lemma 5.3 in the third inequality. Thus by choosing $\varepsilon = \varepsilon(\delta)$ sufficiently small with respect to δ , we may ensure that

$$\lim_{\delta\to 0} \sup_{\eta,R} \|\zeta_R \ [\nu \cdot D_x, \chi_{\varepsilon(\delta)} *_\nu] \theta_\eta *_t \psi_\delta *_x m \|_{L^1_t L^p_{x,\nu}} = 0.$$

For the second term, observe that for all R > 0, $|v \cdot D_x \zeta_R| \leq CR^{-1}$ and thus

$$\begin{split} \| (v \cdot D_x \zeta_R) \tilde{m} \|_{L^1_t (L^1 \cap L^q)_{x,v}} &\leq C R^{-1} \| \tilde{m} \|_{L^1_t (L^1 \cap L^q)_{x,v}} \\ &\leq C R^{-1} \| m \|_{L^1_t (L^1 \cap L^q)_{x,v}} \\ &\leq C_T R^{-1} \| m \|_{(L^1 \cap L^q)_{t,v,v}}. \end{split}$$

It follows that

$$\lim_{R\to+\infty}\sup_{\eta,\,\delta,\,\varepsilon}\|(\nu\cdot D_x\zeta_R)\tilde{m}\|_{L^1_t(L^1\cap L^q)_{x,\,\nu}}=0$$

For the third term, for either p = 1 or p = q we have

$$\begin{split} \| (D_{\nu}\zeta_{R}) \cdot \tilde{w} \|_{L^{1}_{t}L^{p}_{x,\nu}} &\leq \frac{C}{R} \| \tilde{w} \|_{L^{1}_{t}L^{p}_{x,\nu}} \\ &\leq \frac{C}{R} \| \varphi \|_{L^{1}_{t}L^{p}_{x,\nu}} \| w \|_{L^{1}_{t,x,\nu}} \\ &\leq \frac{C}{R} \delta^{-d/p'} \varepsilon^{-d/p'} \| w \|_{L^{1}_{t,x,\nu}} \end{split}$$

Taking $\varepsilon = \varepsilon(\delta)$ as above, we can then ensure this term converges to zero by choosing $R = R(\delta)$ sufficiently large with respect to δ and $\varepsilon(\delta)$. Thus, for this choice of $\varepsilon(\delta), R(\delta)$, we have

$$\lim_{\delta\to 0}\sup_{\eta}\|(D_{\nu}\zeta_{R})\cdot \tilde{w}\|_{L^{1}_{t}L^{p}_{x,\nu}}=0$$

Altogether, we have found that there exists a regime $R = R(\delta)$ and $\varepsilon = \varepsilon(\delta)$ such that

$$\lim_{\delta\to 0}\sup_{\eta}\|\mathcal{E}_{\eta,\,\delta,\,\varepsilon,\,R}\|_{L^1_tL^p_{x,\,\nu}}=0$$

5.2. Testing the equation

Using $\tilde{m}^{(R)}$ as a test function in the weak form of the equation for u_b one obtains

$$\begin{split} \int_{\mathbb{M}\times\mathbb{R}^d} & u_l(0,x,v)\tilde{m}^{(R)}(0,x,v)\,\mathrm{d}x\,\mathrm{d}v - \int_{\mathbb{M}\times\mathbb{R}^d} (\beta_T)_l \tilde{m}^{(R)}(T,x,v)\,\mathrm{d}x\,\mathrm{d}v \\ &+ \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} u_l(\partial_t \tilde{m}^{(R)} + v \cdot D_x \tilde{m}^{(R)})\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t + \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} \tilde{m}^{(R)} H(x,v,D_v u)\,\mathbb{1}_{\{u>l\}}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \\ &\leq \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} \beta_k \tilde{m}^{(R)}\,\mathbb{1}_{\{u>l\}}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t. \end{split}$$

Using the equation satisfied by $\tilde{m}^{(R)}$, we have

$$\int_{\mathbb{M}\times\mathbb{R}^{d}} u_{l}(0,x,v)\tilde{m}^{(R)}(0,x,v) \, \mathrm{d}x \, \mathrm{d}v - \int_{\mathbb{M}\times\mathbb{R}^{d}} (\beta_{T})_{l} \tilde{m}^{(R)}(T,x,v) \, \mathrm{d}x \, \mathrm{d}v \\
- \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} u_{l} \mathrm{d}v_{v} \tilde{w}^{(R)} \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t + \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \tilde{m}^{(R)} H(x,v,D_{v}u) \mathbb{1}_{\{u>l\}} \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \qquad (5.4)$$

$$\leq \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \beta_{k} \tilde{m}^{(R)} \mathbb{1}_{\{u>l\}} \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t - \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} u_{l} \, \mathcal{E}_{\eta,\delta,\varepsilon,R} \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t.$$

Next, note that $D_{\nu}u \in L^r_{loc}(\mathcal{U}_{m_0})$. By the chain rule for Lipschitz functions composed with Sobolev-regular functions,

$$D_{\nu}u_l=D_{\nu}u\mathbb{1}_{\{u>l\}}.$$

Thus, using the definition of distributional derivative we may integrate by parts to obtain

$$-\int_0^T\int_{\mathbb{M}\times\mathbb{R}^d}u_l\,\operatorname{div}_{\nu}\tilde{w}^{(R)}\,\mathrm{d}x\,\mathrm{d}\nu\,\mathrm{d}t=\int_0^T\int_{\mathbb{M}\times\mathbb{R}^d}\tilde{w}^{(R)}\cdot D_{\nu}u\mathbbm{1}_{\{u>l\}}\,\mathrm{d}x\,\mathrm{d}\nu\,\mathrm{d}t.$$

Since $\mathcal{B}(m, w)$ is finite, w is absolutely continuous with respect to m. It follows that there exists $\tilde{\alpha} \in L^{r'}([0, T] \times \mathbb{M} \times \mathbb{R}^d; \tilde{m})$ such that

$$\tilde{w} = \tilde{\alpha} \tilde{m}$$

Thus

$$\int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} \left(\tilde{w}^{(R)} \cdot D_v u + \tilde{m}^{(R)} H(x, v, D_v u) \right) \mathbb{1}_{\{u>l\}} dx dv dt$$

=
$$\int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} (\tilde{\alpha} \cdot D_v u + H(x, v, D_v u)) \tilde{m}^{(R)} \mathbb{1}_{\{u>l\}} dx dv dt$$

$$\geq -\int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} L(x, v, -\tilde{\alpha}) \tilde{m}^{(R)} \mathbb{1}_{\{u>l\}} dx dv dt.$$

Substituting this, we obtain

$$\int_{\mathbb{M}\times\mathbb{R}^{d}} u_{l}(0,x,v)\tilde{m}^{(R)}(0,x,v) \, dx \, dv - \int_{\mathbb{M}\times\mathbb{R}^{d}} (\beta_{T})_{l} \tilde{m}^{(R)}(T,x,v) \, dx \, dv
- \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} L(x,v,-\tilde{\alpha})\tilde{m}^{(R)} \mathbb{1}_{\{u>l\}} \, dx \, dv \, dt$$

$$\leq \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \beta_{k} \tilde{m}^{(R)} \mathbb{1}_{\{u>l\}} \, dx \, dv \, dt - \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} u_{l} \, \mathcal{E}_{\eta,\delta,\varepsilon,R} \, dx \, dv \, dt.$$
(5.5)

We have shown above that there exists a regime $R = R(\delta)$ and $\varepsilon = \varepsilon(\delta)$ such that the final term converges to zero uniformly in η as δ tends to zero, since $u_l \in L_t^{\infty}(L^{\infty} + L^{q'})_{x,v}$. We now discuss the convergence of the other terms.

5.3. Boundary terms

We consider the boundary terms at t = 0, T. Note that Lemma 4.1 and Corollary 4.2 yield $u_l(0, \cdot) \in L^{q'} + L^{\infty}$ (since $(u_0)_+ \in L^{\infty} + L^{q'}$ and $u_l(0, \cdot)$ is bounded below), while by Lemma 4.1 and Corollary 4.4 we have $(\beta_T)_l \in L^{s'} + L^{\infty}$.

We first show that $\tilde{m}_0^{(R)}$ converges to m_0 in $L^1 \cap L^q$, and $\tilde{m}_T^{(R)}$ converges to m_T in $L^1 \cap L^q$, in the limit as δ tends to zero for a certain regime $\eta = \eta(\delta)$ and $R = R(\delta)$, $\varepsilon = \varepsilon(\delta)$ according to the regime already found above.

For t = 0, T, we write

$$\tilde{m}_t^{(R)} - m_t = (\tilde{m}_t^{(R)} - \zeta_R \psi_\delta *_x \chi_\varepsilon *_\nu m_t) + \zeta_R (\psi_\delta *_x \chi_\varepsilon *_\nu m_t - m_t) + m_t (\zeta_R - 1).$$
(5.6)

We first note that $m_0 \in (L^1 \cap L^q)_{x,v}$ by the assumption that it is a bounded probability density, while $m_T \in L^s_{x,v}$ since the energy $\mathcal{B}(m,w)$ is finite, and $m_T \in L^1_{x,v}$ since the continuity equation conserves mass.

Then, since $|m_t(\zeta_R - 1)| \le m_t$, if $m_t \in L^p$ (where $p \in \{1, q\}$ or $p \in \{1, s\}$) then by dominated convergence

$$\lim_{R\to+\infty}\|m_t(\zeta_R-1)\|_{L^p_{x,\nu}}=0.$$

Moreover, by continuity of translations in L^p ,

$$\lim_{\delta\to 0} \|\psi_{\delta} *_x \chi_{\varepsilon(\delta)} *_v m_t - m_t\|_{L^p_{x,v}} = 0.$$

Since for all R > 0

$$\|\zeta_R(\psi_{\delta}*_x\chi_{\varepsilon}*_vm_t-m_t)\|_{L^p_{x,v}}\leq \|\psi_{\delta}*_x\chi_{\varepsilon}*_vm_t-m_t\|_{L^p_{x,v}},$$

it follows that

$$\lim_{\delta\to 0}\sup_{R}\left\|\zeta_{R}(\psi_{\delta}*_{x}\chi_{\varepsilon(\delta)}*_{\nu}m_{t}-m_{t})\right\|_{L^{p}_{x,\nu}}=0.$$

Therefore, the latter two terms of (5.6) converge to zero as δ tends to zero with $\varepsilon = \varepsilon(\delta)$ and $R = R(\delta)$ as already specified above, in $(L^1 \cap L^q)_{x,v}$ for t = 0 and in $(L^1 \cap L^s)_{x,v}$ for t = T.

It remains to estimate the difference $\psi_{\delta} *_x \chi_{\varepsilon} *_v m_t - \tilde{m}_t$. For any function $f \in L^p_{x,v}$ $(p \in [1, +\infty]$ to be specified),

$$\int_{\mathbb{M}\times\mathbb{R}^d} (\psi_{\delta} *_x \chi_{\varepsilon} *_v m_t - \tilde{m}_t) \zeta_R f \, \mathrm{d}x \, \mathrm{d}v = \int_{\mathbb{M}\times\mathbb{R}^d} (m_t - (\theta_{\eta} *_t m)_t) \psi_{\delta} *_x \chi_{\varepsilon} *_v (\zeta_R f) \, \mathrm{d}x \, \mathrm{d}v.$$

We use the notation $\tilde{f} := \psi_{\delta} *_x \chi_{\varepsilon} *_{\nu}(\zeta_R f)$. Writing the time convolution explicitly, we obtain

$$\int_{\mathbb{M}\times\mathbb{R}^d} (m_t - (\theta_\eta *_t m)_t) \tilde{f} \, \mathrm{d}x \, \mathrm{d}v = \int_{-\infty}^{\infty} \int_{\mathbb{M}\times\mathbb{R}^d} \theta_\eta(\tau) (m_t - m_{t-\tau}) \tilde{f} \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}\tau.$$

Next, we use estimates on $\partial_t m$:

$$\langle m_t - m_{t-\tau}, \tilde{f} \rangle = \int_{t-\tau}^t \langle \partial_t m, \tilde{f} \rangle_s \, \mathrm{d}s = \int_{t-\tau}^t \langle m, v \cdot \nabla_x \tilde{f} \rangle_s + \langle w, \nabla_v \tilde{f} \rangle_s \, \mathrm{d}s.$$

Then, since $m, w \in L^1$,

$$|\langle m_t - m_{t-\tau}, \tilde{f} \rangle| \leq \left(\int_{t-\tau}^t \|m_s\|_{L^1_{x,v}} + \|w_s\|_{L^1_{x,v}} \, \mathrm{d}s \right) \left(\|v \cdot \nabla_x \tilde{f}\|_{L^{\infty}} + \|\nabla_v \tilde{f}\|_{L^{\infty}} \right).$$

We estimate \tilde{f} :

$$\begin{aligned} \| \boldsymbol{\nu} \cdot \nabla_x \, \tilde{f} \|_{L^{\infty}} &= \| \boldsymbol{\nu} \cdot \nabla_x \psi_{\delta} *_x \chi_{\varepsilon} *_{\boldsymbol{\nu}} (\zeta_R f) \|_{L^{\infty}} \\ &\leq (R + C\varepsilon) \| \nabla_x \psi_{\delta} \|_{L^{p'}} \| \chi_{\varepsilon} \|_{L^{p'}} \| f \|_{L^p} \\ &\lesssim (R + C\varepsilon) \delta^{-(1+d/p)} \varepsilon^{-d/p} \| f \|_{L^p} \end{aligned}$$

and similarly

$$\begin{split} \|\nabla_{\nu} f\|_{L^{\infty}} &= \|\psi_{\delta} *_{x} \nabla_{\nu} \chi_{\varepsilon} *_{\nu} (\zeta_{R} f)\|_{L^{\infty}} \\ &\leq \|\psi_{\delta}\|_{L^{p'}} \|\nabla_{\nu} \chi_{\varepsilon}\|_{L^{p'}} \|f\|_{L^{p}} \\ &\lesssim \delta^{-d/p} \varepsilon^{-(1+d/p)} \|f\|_{L^{p}}. \end{split}$$

Thus

$$|\langle m_t - m_{t-\tau}, \tilde{f} \rangle| \leq C(\delta, \varepsilon, R) \left(\int_{t-\tau}^t \|m_s\|_{L^1_{x,v}} + \|w_s\|_{L^1_{x,v}} \,\mathrm{d}s \right) \|f\|_{L^p_{x,v}}.$$

Finally

$$\begin{split} \left| \int_{\mathbb{M}\times\mathbb{R}^d} \left(m_t - (\theta_\eta *_t m)_t \right) \tilde{f} \, \mathrm{d}x \, \mathrm{d}\nu \right| &\leq C(\delta, \varepsilon, R) \left(\int_{t-\eta}^{t+\eta} \|m_s\|_{L^1_{x,v}} + \|w_s\|_{L^1_{x,v}} \, \mathrm{d}s \right) \|f\|_{L^p_{x,v}} \\ &\leq \omega(\eta) \ C(\delta, \varepsilon, R) \ \|f\|_{L^p_{x,v}}, \end{split}$$

where $\lim_{\eta\to 0} \omega(\eta) = 0$. Thus it is possible to choose $\eta = \eta(\delta)$ depending on $\delta, \varepsilon(\delta), R(\delta)$ in such a way that

$$\lim_{\delta \to 0} \omega(\eta) \ C(\delta, \varepsilon, R) = 0, \quad \lim_{\delta \to 0} \eta(\delta) = 0.$$

We apply this in the case $f = f_i$ for i = 1, 2, where

$$(\beta_T)_l = f_1 + f_2 \text{ or } u_l(0) = f_1 + f_2, \quad f_1 \in \begin{cases} L_{x,v}^{s'} \text{ if } t = T \\ L_{x,v}^{q'} \text{ if } t = 0 \end{cases} \quad f_2 \in L_{x,v}^{\infty}.$$

Consequently, for t = 0, T,

$$\int_{\mathbb{M}\times\mathbb{R}^d} \tilde{m}_0^{(R)} u_l(0) \,\mathrm{d} x \,\mathrm{d} \nu \to \int_{\mathbb{M}\times\mathbb{R}^d} m_0 u_l(0) \,\mathrm{d} x \,\mathrm{d} \nu,$$

and

$$\int_{\mathbb{M}\times\mathbb{R}^d} \tilde{m}_T^{(R)}(\beta_T)_l \,\mathrm{d} x \,\mathrm{d} \nu \to \int_{\mathbb{M}\times\mathbb{R}^d} m_T(\beta_T)_l \,\mathrm{d} x \,\mathrm{d} \nu,$$

as $\delta \to 0$ with η , *R*, ε chosen to depend on δ in the manner specified.

Finally, we take the limit $l \rightarrow -\infty$. For the term, t = 0, convergence holds by monotonicity, and the limit is finite since

$$-\int_{\mathbb{M}\times\mathbb{R}^d}u_0m_0\,\mathrm{d}x\,\mathrm{d}\nu<+\infty$$

by finiteness of $\tilde{\mathcal{A}}(u, \beta, \beta_T)$. For the term t = T, we first note that

$$\int_{\mathbb{M}\times\mathbb{R}^d} (\beta_T)_l m_T(x,\nu) \, \mathrm{d}x \, \mathrm{d}\nu \leq \int_{\mathbb{M}\times\mathbb{R}^d} \mathcal{G}(m_T) \, \mathrm{d}x \, \mathrm{d}\nu + \int_{\mathbb{M}\times\mathbb{R}^d} \mathcal{G}^*((\beta_T)_l) \, \mathrm{d}x \, \mathrm{d}\nu.$$

The second term on the right hand side converges due to the assumption on \mathcal{G} (2.3), since the integrand is dominated by

$$\sup_{u<0}\mathcal{G}^*(u)\leq C_G\in L^1(\mathbb{M}\times\mathbb{R}^d).$$

5.4. Term involving βm

Since $m \in L^1 \cap L^q$, by standard results on approximation by mollification in L^p spaces we have

$$\lim_{(\eta,\,\delta,\,\varepsilon)\to 0,\,R\to+\infty}\|\tilde{m}^{(R)}-m\|_{L^1\cap L^q}=0,$$

and thus the same limit holds with η, ε, R chosen to depend on δ as described above. Then, since $\beta_k \in L^{\infty} + L^{q'}$, we deduce that

$$\lim_{(\delta,\varepsilon)\to 0, R\to +\infty} \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} \beta_k \tilde{m}^{(R)} \mathbb{1}_{\{u>l\}} \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t = \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} \beta_k m \mathbb{1}_{\{u>l\}} \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t$$

By the definition of Fenchel conjugate,

$$\int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} \beta_k m \mathbb{1}_{\{u>l\}} \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \leq \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} (\mathcal{F}^*(x,v,\beta_k) + \mathcal{F}(x,v,m)) \mathbb{1}_{\{u>l\}} \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t.$$

We then take the limit $l \to -\infty$. Note \mathcal{F} and \mathcal{F}^* are both lower bounded by integrable functions (conditions (2.2) and (3.1)). Then, by monotonicity,

$$\lim_{l \to -\infty} \int_0^1 \int_{\mathbb{M} \times \mathbb{R}^d} \left(\mathcal{F}(x, v, m) - \inf_m \mathcal{F}(x, v, m) \right) \mathbb{1}_{\{u > l\}} \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t$$
$$= \int_0^T \int_{\mathbb{M} \times \mathbb{R}^d} \left(\mathcal{F}(x, v, m) - \inf_m \mathcal{F}(x, v, m) \right) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t.$$

Moreover

$$\lim_{l\to\infty}\int_0^T\int_{\mathbb{M}\times\mathbb{R}^d}\inf_m \mathcal{F}(x,v,m)\mathbb{1}_{\{u>l\}}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t = \int_0^T\int_{\mathbb{M}\times\mathbb{R}^d}\inf_m \mathcal{F}(x,v,m)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t$$

Since the lower bound is integrable and $\mathcal{F}(\cdot, \cdot, m)$ has finite integral by finiteness of the energy, both of these limits are finite. Thus

$$\lim_{l\to\infty}\int_0^T\int_{\mathbb{M}\times\mathbb{R}^d}\mathcal{F}(x,v,m)\mathbb{1}_{\{u>l\}}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t=\int_0^T\int_{\mathbb{M}\times\mathbb{R}^d}\mathcal{F}(x,v,m)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t.$$

A similar argument shows that

$$\lim_{l\to -\infty} \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} \mathcal{F}^*(x,\nu,\beta_k) \mathbb{1}_{\{u>l\}} \,\mathrm{d}x \,\mathrm{d}\nu \,\mathrm{d}t = \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} \mathcal{F}^*(x,\nu,\beta_k) \,\mathrm{d}x \,\mathrm{d}\nu \,\mathrm{d}t$$

where the right hand side is finite.

Finally, we consider $k \to -\infty$. Note that $\sup_{\beta \le 0} \mathcal{F}^*(x, v, \beta) \le C_F(x, v) \in L^1$ by assumption (see (3.1)). Since $\mathcal{F}^*(\beta)$ is a continuous non-decreasing function of β , as k decreases to negative infinity $\mathcal{F}^*(\cdot, \cdot, \beta_k)$ is decreasing and converges almost everywhere to $\mathcal{F}^*(\cdot, \cdot, \beta)$. Thus we deduce the convergence

$$\lim_{k\to-\infty}\int_{\{\beta\leq 0\}}\mathcal{F}^*(x,\nu,\beta_k)\,\mathrm{d}x\,\mathrm{d}\nu\,\mathrm{d}t=\int_{\{\beta\leq 0\}}\mathcal{F}^*(x,\nu,\beta)\,\mathrm{d}x\,\mathrm{d}\nu\,\mathrm{d}t.$$

By the bounds (3.1), the right hand side is finite. Moreover, for any $k \leq 0$,

$$\int_{\{\beta>0\}} \mathcal{F}^*(x,\nu,\beta_k) \,\mathrm{d}x \,\mathrm{d}\nu \,\mathrm{d}t = \int_{\{\beta>0\}} \mathcal{F}^*(x,\nu,\beta) \,\mathrm{d}x \,\mathrm{d}\nu \,\mathrm{d}t.$$

Thus we conclude that

$$\lim_{k\to\infty}\int_0^T\int_{\mathbb{M}\times\mathbb{R}^d}\mathcal{F}^*(x,\nu,\beta_k)\,\mathrm{d}x\,\mathrm{d}\nu\,\mathrm{d}t=\int_0^T\int_{\mathbb{M}\times\mathbb{R}^d}\mathcal{F}^*(x,\nu,\beta)\,\mathrm{d}x\,\mathrm{d}\nu\,\mathrm{d}t$$

5.5. Lagrangian term

For the term involving the Lagrangian, we use a similar argument as was used in [20]. This argument is based on the joint convexity of L(x, v, -w/m)m as a function of (m, w). In our case we must additionally account for the convergence of the localizer ζ_R . By convexity, for all (t, x, v), the integrand satisfies the inequality

$$\begin{split} & L\left(x,v,-\frac{\tilde{w}}{\tilde{m}}\right)\tilde{m}\zeta_{R}\mathbb{1}_{\{u>l\}} \\ & \leq \int_{0}^{T}\int_{\mathbb{M}\times\mathbb{R}^{d}}\theta_{\eta}(t-t')\psi_{\delta}(x-x')\chi_{\varepsilon}(v-v')L\left(x,v,-\frac{w(t',x',v')}{m(t',x',v')}\right)m(t',x',v')\,\mathrm{d}t'\,\mathrm{d}x'\,\mathrm{d}v'\,\,\zeta_{R}\mathbb{1}_{\{u>l\}} \\ & = \theta_{\eta}*_{t}\psi_{\delta}*_{x}\chi_{\varepsilon}*_{v}\left[L\left(x,v,-\frac{w}{m}\right)m\right]\zeta_{R}\mathbb{1}_{\{u>l\}}. \end{split}$$

Then, note that

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$$\begin{aligned} \theta_{\eta} *_{t} \psi_{\delta} *_{x} \chi_{\varepsilon} *_{v} \bigg[L \bigg(x, v, -\frac{w}{m} \bigg) m \bigg] (t, x, v) &= \theta_{\eta} *_{t} \psi_{\delta} *_{x} \chi_{\varepsilon} *_{v} \bigg[L \bigg(\cdot, -\frac{w}{m} \bigg) m \bigg] (t, x, v) \\ &+ \theta_{\eta} *_{t} \psi_{\delta} *_{x} \chi_{\varepsilon} *_{v} \bigg[L \bigg(x, v, -\frac{w}{m} \bigg) m - L \bigg(\cdot, \cdot, -\frac{w}{m} \bigg) m \bigg] (t, x, v). \end{aligned}$$

Then, since $L(\cdot, \cdot, -\frac{w}{m})m \in L^1((0,T) \times \mathbb{M} \times \mathbb{R}^d)$, $\theta_{\eta} *_t \psi_{\delta} *_x \chi_{\varepsilon} *_v [L(\cdot, \cdot, \frac{w}{m})m]$ converges to $L(\cdot, \cdot, -\frac{w}{m})m$ in $L^1((0,T) \times \mathbb{M} \times \mathbb{R}^d)$ as $\eta, \delta, \varepsilon$ tend to zero. Since $0 \le \zeta_R \le 1$,

$$\lim_{\eta,\,\delta,\,\varepsilon\to 0} \sup_{R} \left\| \left\{ \theta_{\eta} *_{t} \psi_{\delta} *_{x} \chi_{\varepsilon} *_{v} \left[L\left(\cdot,\,\cdot\,,\,-\frac{w}{m}\right) m \right] - L\left(\cdot,\,\cdot\,,\,-\frac{w}{m}\right) m \right\} \zeta_{R} \mathbb{1}_{\{u>l\}} \right\|_{L^{1}} = 0.$$

It follows that if we take the regime $R = R(\delta), \varepsilon = \varepsilon(\delta), \eta = \eta(\delta)$ established above, then

$$\lim_{\delta \to 0} \left\| \left\{ \theta_{\eta} *_{t} \psi_{\delta} *_{x} \chi_{\varepsilon} *_{v} \left[L\left(\cdot, \cdot, -\frac{w}{m}\right) m \right] - L\left(\cdot, \cdot, -\frac{w}{m}\right) m \right\} \zeta_{R} \mathbb{1}_{\{u > l\}} \right\|_{L^{1}} = 0.$$

We stay with this regime and consider the remaining term

$$\int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \theta_{\eta} *_{t} \psi_{\delta} *_{x} \chi_{\varepsilon} *_{v} \left\{ L\left(x, v, -\frac{w}{m}\right)m - L\left(\cdot, \cdot, -\frac{w}{m}\right)m \right\}(t, x, v) \zeta_{R}(x, v) \mathbb{1}_{\{u>l\}} \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t.$$

The integrand converges to zero almost everywhere: note then that

$$\left|\theta_{\eta}*_{t}\psi_{\delta}*_{x}\chi_{\varepsilon}*_{v}\left\{L\left(x,v,-\frac{w}{m}\right)m-L\left(\cdot,\cdot,-\frac{w}{m}\right)m\right\}\zeta_{R}\mathbb{1}_{\{u>l\}}\right|\leq\theta_{\eta}*_{t}\psi_{\delta}*_{x}\chi_{\varepsilon}*_{v}\left[\left(C+\frac{|w|^{\prime\prime}}{m^{\prime\prime}}\right)m\right].$$

The right hand side converges in L^1 to $Cm + |w|^{r'}m^{1-r'}$ as δ tends to zero. Thus by dominated convergence we conclude that

$$\lim_{\delta\to 0}\int_0^T\int_{\mathbb{M}\times\mathbb{R}^d}L\left(x,\nu,-\frac{\tilde{w}}{\tilde{m}}\right)\tilde{m}\zeta_R\mathbb{1}_{\{u>l\}}\,\mathrm{d}x\,\mathrm{d}\nu\,\mathrm{d}t\leq \int_0^T\int_{\mathbb{M}\times\mathbb{R}^d}L\left(x,\nu,-\frac{w}{m}\right)m\mathbb{1}_{\{u>l\}}\,\mathrm{d}x\,\mathrm{d}\nu\,\mathrm{d}t.$$

The reverse inequality follows from Fatou's lemma. Thus

$$\lim_{\delta\to 0}\int_0^T\int_{\mathbb{M}\times\mathbb{R}^d}L\left(x,\nu,-\frac{\tilde{w}}{\tilde{m}}\right)\tilde{m}\zeta_R\mathbb{1}_{\{u>l\}}\,\mathrm{d}x\,\mathrm{d}\nu\,\mathrm{d}t=\int_0^T\int_{\mathbb{M}\times\mathbb{R}^d}L\left(x,\nu,-\frac{w}{m}\right)m\mathbb{1}_{\{u>l\}}\,\mathrm{d}x\,\mathrm{d}\nu\,\mathrm{d}t.$$

Finally, we take the limit $l \to -\infty$. Since $L(x, v, -\frac{w}{m})m \in L^1$, in the limit we obtain

$$\int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} L\left(x, v, -\frac{w}{m}\right) m \,\mathrm{d}x \,\mathrm{d}v \,\mathrm{d}t.$$

5.6. Conclusion

From the discussion above, we have obtained

$$\begin{split} &\int_{\mathbb{M}\times\mathbb{R}^d} u_0 m_0 \, \mathrm{d}x \, \mathrm{d}v - \int_{\mathbb{M}\times\mathbb{R}^d} \mathcal{G}(x, v, m_T) \, \mathrm{d}x \, \mathrm{d}v - \int_{\mathbb{M}\times\mathbb{R}^d} \mathcal{G}^*(x, v, u_T) \, \mathrm{d}x \, \mathrm{d}v - \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} L\left(x, v, -\frac{w}{m}\right) m \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \\ &\leq \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} (\mathcal{F}^*(x, v, \beta) + \mathcal{F}(x, v, m)) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t, \end{split}$$
where all terms are finite. Rearranging this inequality, we obtain the statement $\tilde{\mathcal{A}}(u,\beta,\beta_T) + \mathcal{B}(m,w) \ge 0.$

Corollary 5.4. Let $(u, \beta, \beta_T) \in \mathcal{K}_A$ and $(m, w) \in \mathcal{K}_B$ be such that $\tilde{\mathcal{A}}(u, \beta, \beta_T) < +\infty$ and $\mathcal{B}(m, w) < +\infty$.

Then

$$\beta_{-}m \in L^{1}((0,T) \times \mathbb{M} \times \mathbb{R}^{d}) \text{ and } (\beta_{T})_{-}m_{T} \in L^{1}(\mathbb{M} \times \mathbb{R}^{d}).$$

Moreover, for almost all $t \in [0, T]$ *,*

$$\int_{\mathbb{M}\times\mathbb{R}^d} [u_t m_t - \beta_T m_T] \, \mathrm{d}x \, \mathrm{d}v - \int_t^T \int_{\mathbb{M}\times\mathbb{R}^d} L\left(x, v, -\frac{w}{m}\right) m \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \leq \int_t^T \int_{\mathbb{M}\times\mathbb{R}^d} \beta m \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t,$$

and

$$\int_{\mathbb{M}\times\mathbb{R}^d} [u_0 m_0 - u_t m_t] \, \mathrm{d}x \, \mathrm{d}v - \int_0^t \int_{\mathbb{M}\times\mathbb{R}^d} L\left(x, v, -\frac{w}{m}\right) m \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \le \int_0^t \int_{\mathbb{M}\times\mathbb{R}^d} \beta m \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t.$$
(5.7)

In particular,

$$\int_{\mathbb{M}\times\mathbb{R}^d} [u_0 m_0 - \beta_T m_T] \, \mathrm{d}x \, \mathrm{d}v - \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} L\left(x, v, -\frac{w}{m}\right) m \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \le \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} \beta m \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t.$$
(5.8)

Proof. This is a consequence of the proof of Lemma 5.2 and in particular the inequality (5.5). First, let us show the first part of the statement, i.e. that $\beta_{-}m \in L^{1}((0,T) \times \mathbb{M} \times \mathbb{R}^{d})$ and $(\beta_{T})_{-}m_{T} \in L^{1}(\mathbb{M} \times \mathbb{R}^{d})$.

As in the mentioned proof, let us first pass to the limit with η , δ , ε , R in the inequality (5.5). Then, we pass to the limit as $l \to -\infty$ and $k \to -\infty$ the remaining terms.

All the terms, except the ones involving $((\beta_T)_{-})_l m_T$ and $(\beta_{-})_k m \mathbb{1}_{\{u>l\}}$ pass to the limit, as in the proof. After rearranging, we also find that both $\int_{\mathbb{M}\times\mathbb{R}^d} ((\beta_T)_{-})_l m_T \, dx \, dv$ and $\int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} (\beta_{-})_k m \mathbb{1}_{\{u>l\}} \, dx \, dv \, dt$ are uniformly bounded, independently of l and k. Therefore, the monotone convergence theorem yields

$$\lim_{l\to\infty}\int_{\mathbb{M}\times\mathbb{R}^d}((\beta_T)_-)_l m_T\,\mathrm{d}x\,\mathrm{d}\nu=\int_{\mathbb{M}\times\mathbb{R}^d}(\beta_T)_-m_T\,\mathrm{d}x\,\mathrm{d}\nu$$

and

$$\lim_{l\to\infty}\lim_{k\to\infty}\int_0^T\int_{\mathbb{M}\times\mathbb{R}^d}(\beta_-)_k m\mathbb{1}_{\{u>l\}}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t=\int_0^T\int_{\mathbb{M}\times\mathbb{R}^d}\beta_-m\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t.$$

The summability results follow, and so does (5.8).

For the case of general $t \in [0, T]$, we begin by testing the equation to find that, for example,

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$$\begin{split} \int_{\mathbb{M}\times\mathbb{R}^d} & u_l(t,x,v)\tilde{m}^{(R)}(t,x,v)\,\mathrm{d}x\,\mathrm{d}v - \int_{\mathbb{M}\times\mathbb{R}^d} (\beta_T)_l \tilde{m}^{(R)}(T,x,v)\,\mathrm{d}x\,\mathrm{d}v \\ & - \int_t^T \int_{\mathbb{M}\times\mathbb{R}^d} L(x,v,\,-\tilde{\alpha})\tilde{m}^{(R)}\,\mathbb{1}_{\{u>l\}}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \\ & \leq \int_t^T \int_{\mathbb{M}\times\mathbb{R}^d} \beta_k \tilde{m}^{(R)}\,\mathbb{1}_{\{u>l\}}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t - \int_t^T \int_{\mathbb{M}\times\mathbb{R}^d} u_l\,\,\mathcal{E}_{\eta,\delta,\varepsilon,R}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t. \end{split}$$

We note that we are referring to the version of u_l that is weakly right continuous in time (see Appendix A). The only term that requires attention is

$$\int_{\mathbb{M}\times\mathbb{R}^d} u_l(t,x,\nu) \tilde{m}^{(R)}(t,x,\nu) \,\mathrm{d}x \,\mathrm{d}\nu.$$

For almost all $t \in [0, T]$, $u_l(t) \in (L^{q'} + L^{\infty})(\mathbb{M} \times \mathbb{R}^d)$ and $m_t \in L^1 \cap L^q$. Thus the arguments for the boundary terms in Lemma 5.2 show that

$$\lim_{\delta \to 0} \int_{\mathbb{M} \times \mathbb{R}^d} u_l(t, x, v) \tilde{m}^{(R)}(t, x, v) \, \mathrm{d}x \, \mathrm{d}v = \int_{\mathbb{M} \times \mathbb{R}^d} u_l(t, x, v) m(t, x, v) \, \mathrm{d}x \, \mathrm{d}v.$$

The limit $l \to -\infty$ is then taken by monotone convergence, noting that $(u_t)_+ m_t \in L^1(\mathbb{M} \times \mathbb{R}^d)$. Note that the argument for the case (5.7) shows that limit is not negative infinity, since all other terms have finite limits.

Corollary 5.5. Let $(u, \beta, \beta_T) \in \mathcal{K}_A$ and $(m, w) \in \mathcal{K}_B$ be such that $\tilde{\mathcal{A}}(u, \beta, \beta_T) < +\infty$ and $\mathcal{B}(m, w) < +\infty$. Then

- (1) $m|D_{\nu}u|^r$ is uniformly bounded in $L^1((0,T) \times \mathbb{M} \times \mathbb{R}^d)$, by a constant that depends only on the data and $\tilde{\mathcal{A}}(u,\beta,\beta_T)$ and $\mathcal{B}(m,w)$.
- (2) The following estimate holds:

$$\int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} mH(x,v,D_vu) + mL\left(x,v,-\frac{w}{m}\right) + D_vu \cdot w \, dx \, dv \, dt$$

$$\leq \int_{\mathbb{M}\times\mathbb{R}^d} \beta_T m_T \, dx \, dv - \int_{\mathbb{M}\times\mathbb{R}^d} u_0 m_0 \, dx \, dv + \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} \beta m \, dx \, dv \, dt$$

$$+ \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} mL\left(x,v,-\frac{w}{m}\right) \, dx \, dv \, dt.$$

Proof. This is a consequence of (5.4). Using the same notation as in the proof of Lemma 5.2, we rewrite (5.4) as

$$\begin{split} \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \tilde{m}^{(R)} H(x, v, D_{v}u) \mathbb{1}_{\{u>l\}} dx dv dt \\ &\leq \int_{\mathbb{M}\times\mathbb{R}^{d}} (\beta_{T})_{l} \tilde{m}^{(R)}(T, x, v) dx dv - \int_{\mathbb{M}\times\mathbb{R}^{d}} u_{l}(0, x, v) \tilde{m}^{(R)}(0, x, v) dx dv \\ &+ \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \beta_{k} \tilde{m}^{(R)} \mathbb{1}_{\{u>l\}} dx dv dt - \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} D_{v}u_{l} \cdot \tilde{w}^{(R)} \mathbb{1}_{\{u>l\}} dx dv dt \\ &- \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} u_{l} \mathcal{E}_{\eta, \delta, \varepsilon, R} dx dv dt \\ &\leq \int_{\mathbb{M}\times\mathbb{R}^{d}} (\beta_{T})_{+} \tilde{m}^{(R)}(T, x, v) dx dv + \int_{\mathbb{M}\times\mathbb{R}^{d}} [u_{l}(0, x, v)]_{-} \tilde{m}^{(R)}(0, x, v) dx dv \\ &+ \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \beta_{+} \tilde{m}^{(R)} dx dv dt - \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} D_{v}u_{l} \cdot \tilde{w}^{(R)} \mathbb{1}_{\{u>l\}} dx dv dt \\ &- \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} u_{l} \mathcal{E}_{\eta, \delta, \varepsilon, R} dx dv dt. \end{split}$$

Let us observe that for some $\theta > 0$ parameter that we choose later, Young's inequality yields

$$\begin{split} &- \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} D_{\nu} u_{l} \cdot \tilde{w}^{(R)} \mathbb{1}_{\{u>l\}} \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \\ &= - \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \theta(\tilde{m}^{(R)})^{\frac{1}{r}} D_{\nu} u_{l} \cdot \tilde{w}^{(R)} \theta^{-1} (\tilde{m}^{(R)})^{-\frac{1}{r}} \mathbb{1}_{\{u>l\}} \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \\ &\leq \frac{1}{r} \theta^{r} \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \tilde{m}^{(R)} |D_{\nu} u_{l}|^{r} \mathbb{1}_{\{u>l\}} \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t + \frac{1}{r'} \theta^{-r'} \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} |\tilde{w}^{(R)}|^{r'} (\tilde{m}^{(R)})^{-\frac{r'}{r}} \mathbb{1}_{\{u>l\}} \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t. \end{split}$$

We notice that $-\frac{r'}{r} = 1 - r'$. Thus, by using the growth condition (2.1) on the Hamiltonian and choosing θ appropriately, we can conclude that there exists a constant C > 0 (independent of the parameters $l, k, R, \eta, \varepsilon, \delta$), such that after passing to the limit with $R, \eta, \varepsilon, \delta$, as in the proof of Lemma 5.2, we obtain

$$\begin{split} \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} m |D_v u_l|^r \, \mathbb{1}_{\{u>l\}} \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t &\leq C + \int_{\mathbb{M}\times\mathbb{R}^d} (\beta_T)_+ m(T, x, v) \, \mathrm{d}x \, \mathrm{d}v \\ &+ \int_{\mathbb{M}\times\mathbb{R}^d} [u_l(0, x, v)]_- m_0(x, v) \, \mathrm{d}x \, \mathrm{d}v \\ &+ \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} \beta_+ m \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t + C \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} |w|^{r'} m^{1-r'} \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t. \end{split}$$

Since the right hand side of this inequality is uniformly bounded, independently of l (by Lemma 4.1(ii), Corollary 4.2(ii) and Remark 3.7), the result follows by Fatou's lemma by sending $l \rightarrow -\infty$.

Using (5.4), we have

$$\begin{split} \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \tilde{m}^{(R)} H(x,v,D_{v}u) \mathbb{1}_{\{u>l\}} \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t + \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \tilde{m}^{(R)} L\left(x,v,-\frac{\tilde{w}}{\tilde{m}}\right) \mathbb{1}_{\{u>l\}} \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \\ &+ \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} D_{v}u \cdot \tilde{w}^{(R)} \mathbb{1}_{\{u>l\}} \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \\ &\leq \int_{\mathbb{M}\times\mathbb{R}^{d}} (\beta_{T})_{l} \tilde{m}^{(R)}(T,x,v) \, \mathrm{d}x \, \mathrm{d}v - \int_{\mathbb{M}\times\mathbb{R}^{d}} u_{l}(0,x,v) \tilde{m}^{(R)}(0,x,v) \, \mathrm{d}x \, \mathrm{d}v \\ &+ \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \beta_{k} \tilde{m}^{(R)} \mathbb{1}_{\{u>l\}} \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t + \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \tilde{m}^{(R)} L\left(x,v,-\frac{\tilde{w}}{\tilde{m}}\right) \mathbb{1}_{\{u>l\}} \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \\ &- \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} u_{l} \, \mathcal{E}_{\eta,\delta,\varepsilon,R} \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t. \end{split}$$

Passing to the limit with $R, \eta, \varepsilon, \delta$, as in the proof of Lemma 5.2, by Fatou's lemma we obtain

$$\int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} mH(x,v,D_{v}u) \mathbb{1}_{\{u>l\}} dx dv dt + \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} mL\left(x,v,-\frac{w}{m}\right) \mathbb{1}_{\{u>l\}} dx dv dt + \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} D_{v}u \cdot w \mathbb{1}_{\{u>l\}} dx dv dt \leq \int_{\mathbb{M}\times\mathbb{R}^{d}} (\beta_{T})_{l}m(T,x,v) dx dv - \int_{\mathbb{M}\times\mathbb{R}^{d}} u_{l}(0,x,v)m(0,x,v) dx dv + \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \beta_{k}m \mathbb{1}_{\{u>l\}} dx dv dt + \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} mL\left(x,v,-\frac{w}{m}\right) \mathbb{1}_{\{u>l\}} dx dv dt.$$

Finally, taking the limit $l, k \rightarrow -\infty$ as in the proofs of Lemma 5.2 and Corollary 5.4, we obtain

$$\int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} mH(x,v,D_{v}u) + mL\left(x,v,-\frac{w}{m}\right) + D_{v}u \cdot w \, dx \, dv \, dt$$

$$\leq \int_{\mathbb{M}\times\mathbb{R}^{d}} \beta_{T}m_{T} \, dx \, dv - \int_{\mathbb{M}\times\mathbb{R}^{d}} u_{0}m_{0} \, dx \, dv + \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \beta m \, dx \, dv \, dt$$

$$+ \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} mL\left(x,v,-\frac{w}{m}\right) \, dx \, dv \, dt.$$

6. Existence of a solution of the relaxed problem

In this section we prove the existence of a solution for the relaxed problem. Consider a minimizing sequence $(u_n, \beta_n, (\beta_T)_n)_n$. We will extract a convergent subsequence, and show that the limit constitutes a minimizer of the objective functional.

6.1. Compactness of the minimizing sequence

The following proposition enables the extraction of a convergent subsequence.

Proposition 6.1. Let $(u_n)_n$ be a sequence of solutions to the Hamilton-Jacobi equations

$$-\partial_t u_n - v \cdot D_x u_n + H(x, v, D_v u_n) = \beta_n,$$

Assume that:

- The family $(u_n)_n$ is uniformly bounded in $L^{\infty}_t L^1_{x,v, \text{loc}}(\mathcal{U}_{m_0})$.
- The family $(D_v u_n)_n$ is uniformly bounded in $L^r_{loc}(\mathcal{U}_{m_0})$.
- The family $(\beta_n)_n$ is uniformly bounded in $L^1_{loc}(\mathcal{U}_{m_0})$.

Then there exists a subsequence $(u_{n_j})_j$ that is strongly convergent in $L^1_{loc}(\mathcal{U}_{m_0})$.

The proof of this result will be a consequence of some intermediate results that we detail below.

Remark 6.2. Let us notice that the assumptions of Proposition 6.1 hold true by Corollary 4.4 and Lemma 4.1.

Proposition 6.1 is proved by treating the Hamilton-Jacobi equation as a kinetic transport equation with right hand side bounded in $L^1_{loc}(\mathcal{U}_{m_0})$:

$$-\partial_t u_n - v \cdot D_x u_n = \beta_n - H(x, v, D_v u_n).$$

A form of compactness for the solutions can be obtained by using an *averaging lemma*. Averaging lemmas are results in kinetic theory showing that, for L^p -bounded families of solutions to the kinetic transport equation, with L^p -bounded source terms, the velocity averages

$$\rho_{\phi}[u](t,x) := \int_{\mathbb{R}^d} u(t,x,v)\phi(v) \, \mathrm{d}v \qquad \phi \in C^{\infty}_c(\mathbb{R}^d),$$

enjoy additional fractional Sobolev regularity and/or strong L^p -compactness. In our case we are in the setting p = 1, and we use an L^1 averaging result from [49]. It is necessary to assume a certain equi-integrability condition on the solution u_n . This condition is defined below.

Definition 6.3 (Equi-integrability in velocity). Let $(u_{\lambda})_{\lambda \in \Lambda}$ be a bounded family in $L^{1}_{loc}([0, T] \times \mathbb{M} \times \mathbb{R}^{d})$. The family is locally equi-integrable in ν if, for all $\varepsilon > 0$ and all compact sets $K \subset [0, T] \times \mathbb{M} \times \mathbb{R}^{d}$, there exists $\eta > 0$ such that for all measurable families $(A_{t,x})_{(t,x)\in[0,T]\times\mathbb{M}}$ of measurable subsets of \mathbb{R}^{d} for which $\sup_{(t,x)\in[0,T]\times\mathbb{M}} |A_{t,x}| < \eta$,

$$\int_0^T \int_{\mathbb{M}} \int_{A_{t,x}} \mathbb{1}_K |u_{\lambda}(t,x,v)| \, \mathrm{d} v \, \mathrm{d} x \, \mathrm{d} t < \varepsilon \qquad \text{for all } \lambda \in \Lambda.$$

The required averaging lemma is quoted below. This result was proved in [49] for the stationary case, i.e. the equation $v \cdot D_x u = \beta$. The result can be adapted to the time dependent equation by standard techniques; see [50] or [51] for statements in the time dependent setting.

Theorem 6.4. Let $(u_{\lambda})_{\lambda \in \Lambda}$ be a bounded family in $L^{\infty}([0, T]; L^{1}_{loc}(\mathbb{M} \times \mathbb{R}^{d}))$ satisfying

$$\partial_t u_{\lambda} + v \cdot D_x u_{\lambda} = \beta_{\lambda}, \tag{6.1}$$

where $(\beta_{\lambda})_{\lambda \in \Lambda}$ is a bounded family in $L^{1}_{loc}([0,T] \times \mathbb{M} \times \mathbb{R}^{d})$. Assume that $(u_{\lambda})_{\lambda \in \Lambda}$ is equi-integrable in v. Then

- The family $(u_{\lambda})_{\lambda \in \Lambda}$ is locally equi-integrable in all variables in $[0, T] \times \mathbb{M} \times \mathbb{R}^{d}$.
- For each φ ∈ C_c(ℝ^d), the family of averages (ρ_φ[u_λ])_{λ∈Λ} is relatively (strongly) compact in L¹_{loc}([0, T] × M).

In our setting we expect to have local summability estimates in \mathcal{U}_{m_0} rather than $[0, T] \times \mathbb{M} \times \mathbb{R}^d$. To deal with this technicality we make use of a localization procedure: given a compact set K, consider a smooth bump function ζ supported in K. If u_{λ} satisfies the kinetic transport Equation (6.2), then $u_{\lambda}\zeta$ satisfies

$$\partial_t(u_\lambda\zeta) + v \cdot D_x(u_\lambda\zeta) = \beta_\lambda\zeta + (\partial_t\zeta + v \cdot \nabla_x\zeta)u_\lambda.$$
(6.2)

The right hand side of the above equation is bounded in $L^1([0, T] \times \mathbb{M} \times \mathbb{R}^d)$, uniformly in λ , as long as u_{λ} and β_{λ} are bounded in $L^1_{loc}(\mathcal{U}_{m_0})$.

We wish to apply this to the solutions of the Hamilton-Jacobi equation. To do this, we verify the equi-integrability condition. To prove equi-integrability, we make use of the L^r estimates available for the *v*-derivative $D_v u$.

Lemma 6.5. Let $(u_{\lambda})_{\lambda \in \Lambda}$ be a bounded family in $L_t^{\infty} L_{x,v,\text{loc}}^1(\mathcal{U}_{m_0})$. Assume that $(D_v u_{\lambda})_{\lambda \in \Lambda}$ is a bounded family in $L_{\text{loc}}^r(\mathcal{U}_{m_0})$. Then:

- $(u_{\lambda})_{\lambda \in \Lambda}$ is bounded in $L^{1}_{t,x}L^{\alpha}_{v, \text{loc}}(\mathcal{U}_{m_{0}})$, where $1/\alpha = 1/r 1/d$ if r < d, or any $\alpha < +\infty$ if $r \geq d$.
- $(u_{\lambda})_{\lambda \in \Lambda}$ is equi-integrable in v, locally in \mathcal{U}_{m_0} .

Proof. We obtain higher integrability in the velocity variable by using Sobolev embedding. We first apply a localization procedure. Given a compact set $K \subset U_{m_0}$, let ζ_K denote a smooth bump function with compact support contained in U_{m_0} , such that ζ_K takes values contained in [0, 1] and $\zeta_K \equiv 1$ on K. Then

$$D_{\nu}(u_{\lambda}\zeta_{K})=D_{\nu}u_{\lambda}\ \zeta_{K}+u_{\lambda}\ D_{\nu}\zeta_{K}.$$

Let *K'* denote the support of ζ_K . Then

$$\|D_{\nu}(u_{\lambda}\zeta_{K})\|_{L^{1}} \leq C(\zeta_{K})\Big(\|D_{\nu}u_{\lambda}\|_{L^{r}(K')}+\|u_{\lambda}\|_{L^{\infty}_{t}L^{1}_{x,\nu}(K')}\Big),$$

thus $D_{\nu}(u_{\lambda}\zeta_{K})$ is bounded in L^{1} , uniformly in λ . Moreover it is compactly supported.

We then apply Sobolev embedding in the v variable. Letting $1^* := \frac{d}{d-1}$, we have

$$\|u_{\lambda}\|_{L^{1}_{t,x}L^{1^{*}}_{\nu}(K)} \leq \|u_{\lambda}\zeta_{K}\|_{L^{1}_{t,x}L^{1^{*}}_{\nu}} \leq C_{d}\|D_{\nu}(u_{\lambda}\zeta_{K})\|_{L^{1}} \leq C(d,\zeta_{K})\Big(\|D_{\nu}u_{\lambda}\|_{L^{r}(K')} + \|u_{\lambda}\|_{L^{\infty}_{t}L^{1}_{x,\nu}(K')}\Big)$$

Thus $(u_{\lambda})_{\lambda}$ is uniformly bounded in $L^{1}_{t,x}L^{1^{*}}_{\nu,\text{loc}}$.

We now apply a bootstrap argument: it follows from the above that $D_{\nu}(u_{\lambda}\zeta_{K})$ is bounded in $L_{t,x}^{1}L_{\nu}^{1*}$, and thus $(u_{\lambda})_{\lambda}$ is bounded in $L_{t,x}^{1}L_{\nu,\text{loc}}^{\alpha}$ for $\alpha = (1 - 2/d)^{-1}$. This process can be repeated until we obtain that $(u_{\lambda})_{\lambda}$ is bounded in $L_{t,x}^{1}L_{\nu,\text{loc}}^{r*}$, for $1/r_{*} = 1/r - 1/d$, if r < d; otherwise we may obtain $L_{t,x}^{1}L_{\nu,\text{loc}}^{\alpha}$ for any $\alpha < +\infty$.

We now prove local equi-integrability. Let $(A_{t,x})_{(t,x)\in[0,T]\times\mathbb{M}}$ be a measurable family of measurable subsets of \mathbb{R}^d , such that $|A_{t,x}| < \eta$ for all t, x. Then

$$\int_{0}^{T} \int_{\mathbb{M}} \int_{A_{t,x}} \mathbb{1}_{K} |u_{\lambda}| \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t \leq \| \mathbb{1}_{A_{t,x}} \mathbb{1}_{K} \|_{L^{\infty}_{t,x} L^{d'}_{v}} \| u_{\lambda} \mathbb{1}_{K} \|_{L^{1}_{t,x} L^{d'}_{v}}$$

where ' denotes a Hölder conjugate exponent. From the condition on the measure of $A_{t,x}$, we have

$$\int_{0}^{T} \int_{\mathbb{M}} \int_{A_{t,x}} \mathbb{1}_{K} |u_{\lambda}(t,x,\nu)| \, \mathrm{d}\nu \, \mathrm{d}x \, \mathrm{d}t \leq C(K) \sup_{t,x} |A_{t,x}|^{1/\alpha'} ||u_{\lambda}||_{L^{1}_{t,x}L^{2}_{\nu}(K)} \leq C(K) ||u_{\lambda}||_{L^{1}_{t,x}L^{2}_{\nu}(K)} \eta^{1/\alpha'} ||u_{\lambda}||_{L^{1}_{t,x}L^{2}_{\nu}(K)} \leq C(K) ||u_{\lambda}||_{L^{1}_{t,x}L^{2}_{\nu}(K)} ||$$

which proves equi-integrability.

It follows that, under the assumptions of Proposition 6.1, Theorem 6.4 can be applied to $(\zeta_K u_n)_n$ for any $\zeta_K \in C_c^{\infty}(\mathcal{U}_{m_0})$. We deduce strong L^1_{loc} -compactness for the averages $(\rho_{\phi}[u_n\zeta_K])_n$. We now use this to prove strong compactness for the full solutions u_n .

Lemma 6.6. Assume that the family $(u_n)_n$ satisfies the following:

- $(u_n)_n$ is uniformly bounded in $L_t^{\infty} L_{x,v}^1$.
- $(u_n)_n$ is equi-integrable in all variables.
- $(u_n)_n$ share the same compact support K.
- $(D_v u_n)_n$ is uniformly bounded in $L^1_{t,x} L^r_{v}$.
- For each $\phi \in C_c^{\infty}(\mathbb{R}^d)$, the family of averages $(\rho_{\phi}[u_n])_n$ is relatively (strongly) compact in L^1_{loc} .

Then the family $(u_n)_n$ is relatively (strongly) compact in L^1 .

Proof. First, note that the first two assumptions imply the weak L^1 sequential compactness of $(u_n)_n$. We pass (without relabelling) to a weakly convergent subsequence u_n , and let u denote the weak limit. In the remainder of the proof we improve the mode of convergence of u_n to u to strong convergence in L^1 .

Step 1: Approximation by smoothing in ν . We approximate u_n by a function that is smooth with respect to the ν variable. Fix $\phi \in C_c^{\infty}(\mathbb{R}^d)$ and define, for $\varepsilon > 0$,

$$\phi_{\varepsilon}(v) := \varepsilon^{-d} \phi\left(\frac{v}{\varepsilon}\right).$$

Let

$$u_{n,\varepsilon}(t,x,\nu) := \int_{\mathbb{R}^d} u_n(t,x,\nu') \phi_{\varepsilon}(\nu-\nu') \,\mathrm{d}\nu'.$$

Step 2: Compactness for the approximations. Fix $v_* \in \mathbb{R}^d$. For any $\psi \in L^{\infty}_{t,x}$ with compact support we consider testing the sequence $(u_n)_n$ against the test function

$$\psi(t,x)\phi_{\varepsilon}(v_*-v).$$

Since

$$\langle u_n, \psi \phi_{\varepsilon}(v_* - \cdot) \rangle \rightarrow \langle u, \psi \phi_{\varepsilon}(v_* - \cdot) \rangle, \ n \rightarrow +\infty,$$

we deduce that $u_{n,\varepsilon}(\cdot, \cdot, v_*)$ converges weakly in $L^1_{t,x,\text{loc}}$ to $u*_v\phi_{\varepsilon}(\cdot, \cdot, v_*)$ as $n \to +\infty$.

Note moreover for each fixed $v \in \mathbb{R}^d$, $u_{n,\varepsilon}(t,x,v)$ is a velocity average with respect to the test function $\phi_{\varepsilon}(v-\cdot)$. Therefore, by Theorem 6.4 the convergence in fact holds strongly in $L^1_{t,x,\text{loc}}$ for each $v \in \mathbb{R}^d$.

Furthermore, for fixed $\varepsilon > 0$, the family $(u_{n,\varepsilon})_n$ is equi-continuous in ν into $L_t^{\infty} L_x^1$: indeed

$$\begin{aligned} |u_{n,\varepsilon}(t,x,\nu+h) - u_{n,\varepsilon}(t,x,\nu)| &= \left| \int_{\mathbb{R}^d} u_n(t,x,\nu') \left(\phi_{\varepsilon}(\nu-\nu'+h) - \phi_{\varepsilon}(\nu-\nu') \right) \mathrm{d}\nu' \right. \\ &\leq |h| \|\nabla \phi_{\varepsilon}\|_{L^{\infty}} \int_{\mathbb{R}^d} |u_n(t,x,\nu')| \, \mathrm{d}\nu'. \end{aligned}$$

Thus

$$\|u_{n,\varepsilon}(\cdot,\cdot,\nu+h)-u_{n,\varepsilon}(\cdot,\cdot,\nu)\|_{L^{\infty}_{t}L^{1}_{x}}\leq C_{\varepsilon}\sup_{m}\|u_{m}\|_{L^{\infty}_{t}L^{1}_{x,\nu}}\|h\|_{L^{\infty}_{t}L^{1}_{x,\nu}}$$

By an Arzelà-Ascoli argument the convergence therefore holds locally uniformly in v, with respect to the strong topology on $L^1_{t,x,\text{loc}}$: that is, for all compact sets $K_v \subset \mathbb{R}^d$ and $K_{t,x} \subset [0,T] \times \mathbb{R}^d$,

$$\lim_{n\to\infty}\sup_{\nu\in K_{\nu}}\|u_{n,\varepsilon}(\nu)-u*\phi_{\varepsilon}(\nu)\|_{L^{1}_{t,x}(K_{t,x})}=0.$$

Consequently, the convergence holds in L^1_{loc} ; in fact, since $u_{n,\varepsilon} - u * \phi_{\varepsilon}$ is supported for all n in $K + B_{C\varepsilon}(0)$, the convergence holds in L^1 .

Step 3: Removing the approximation.

The bound on $D_{\nu}u_n$ implies that, for any $h \in \mathbb{R}^d$,

$$\|u_n(t, x, \cdot +h) - u_n(t, x, \cdot)\|_{L^r} \le \|h\| \|D_v u_n(t, x, \cdot)\|_{L^r}$$

It follows that

$$\|u_{n,\varepsilon}-u_n\|_{L^1_{t,x}L^r_{\nu}} \lesssim \varepsilon \sup_m \|D_{\nu}u_m\|_{L^1_{t,x}L^r_{\nu}}$$

Indeed, by definition of $u_{n,\varepsilon}$,

$$[u_{n,\varepsilon}-u_n](t,x,\nu)=\int_{\mathbb{R}^d}[u_n(t,x,\nu-h)-u_n(t,x,\nu)]\phi_{\varepsilon}(h)\,\mathrm{d}h.$$

Thus, for any $g \in L_{\nu}^{r'}(\mathbb{R}^d)$,

$$\begin{split} \int_{\mathbb{R}^d} [u_{n,\varepsilon} - u_n](t,x,v)g(v) \, \mathrm{d}v &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} [u_n(t,x,v-h) - u_n(t,x,v)] \phi_{\varepsilon}(h)g(v) \, \mathrm{d}h \, \mathrm{d}v \\ &\leq \|g\|_{L_v^{\nu'}} \int_{\mathbb{R}^d} \|u_n(t,x,\cdot+h) - u_n(t,x,\cdot)\|_{L_v^{\nu}} |\phi_{\varepsilon}(h)| \, \mathrm{d}h \\ &\leq \|g\|_{L_v^{\nu'}} \|D_v u_n(t,x,\cdot)\|_{L_v^{\nu}} \int_{\mathbb{R}^d} |h| |\phi_1\left(\frac{h}{\varepsilon}\right) |\varepsilon^{-d} \, \mathrm{d}h \\ &\leq C_\phi \varepsilon \|D_v u_n(t,x,\cdot)\|_{L_v^{\nu}} \|g\|_{L_v^{\nu'}}. \end{split}$$

That is,

$$\|u_{n,\varepsilon}(t,x,\cdot)-u_n(t,x,\cdot)\|_{L^r_{\nu}} \leq \varepsilon \|D_{\nu}u_n(t,x,\cdot)\|_{L^r_{\nu}},$$

then, one integrates in t, x and takes supremum.

Finally, estimate

$$\begin{split} \|u_{n}-u\|_{L^{1}_{t,x,v}} &\leq \|u_{n}-u_{n,\varepsilon}\|_{L^{1}_{t,x,v}} + \|u_{n,\varepsilon}-u*_{v}\phi_{\varepsilon}\|_{L^{1}_{t,x,v}} + \|u*_{v}\phi_{\varepsilon}-u\|_{L^{1}_{t,x,v}} \\ &\leq C_{K}\|u_{n}-u_{n,\varepsilon}\|_{L^{1}_{t,x}L^{r}_{v}} + \|u_{n,\varepsilon}-u*_{v}\phi_{\varepsilon}\|_{L^{1}_{t,x,v}} + C_{K}\|u*_{v}\phi_{\varepsilon}-u\|_{L^{1}_{t,x}L^{r}_{v}} \\ &\lesssim C_{K}\varepsilon\sup_{m}\|D_{v}u_{m}\|_{L^{1}_{t,x}L^{r}_{v}} + \|u_{n,\varepsilon}-u*_{v}\phi_{\varepsilon}\|_{L^{1}_{t,x,v}} + \|u*_{v}\phi_{\varepsilon}-u\|_{L^{1}_{t,x}L^{r}_{v}}. \end{split}$$

Thus

$$\limsup_{n\to\infty} \|u_n-u\|_{L^1_{t,x,\nu}} \lesssim_{K} \varepsilon \sup_{m} \|D_{\nu}u_m\|_{L^1_{t,x}L^r_{\nu}} + \|u*_{\nu}\phi_{\varepsilon}-u\|_{L^1_{t,x}L^r_{\nu}} \to 0$$

as $\varepsilon \to 0$, which completes the proof.

Proof of Proposition 6.1. The proof of this proposition follows by applying Lemma 6.6 to $(\zeta_K u_n)_n$.

It remains to obtain the necessary convergence of $(\beta_n)_n$ and $(\beta_{T,n})_n$.

Lemma 6.7. Let $(u_n, \beta_n, \beta_{T,n})$ be a minimizing sequence for Problem 3.8. There exists a modification $(u_n, \tilde{\beta}_n, \tilde{\beta}_{T,n})$ of this sequence that is also minimizing such that $(\tilde{\beta}_n)_n$ is weakly precompact in $L^1_{loc}(\mathcal{U}_{m_0})$ and $(\tilde{\beta}_{T,n})_n$ is weakly precompact in $L^1_{loc}(\mathbb{M} \times \mathbb{R}^d)$.

Proof. We replace β_n with some $\tilde{\beta}_n \geq \beta_n$ and $(\beta_{T,n})_n$ with $\tilde{\beta}_{T,n} \geq \beta_{T,n}$ such that $(\tilde{\beta}_n, \tilde{\beta}_{T,n})$ is uniformly integrable, and $(u_n, \tilde{\beta}_n, \tilde{\beta}_{T,n})$ is still a minimizing sequence. We do this in a similar manner to [20]: since $(\beta_n)_-$ is bounded in $L^1_{\text{loc}}(\mathcal{U}_{m_0})$, using a compact exhaustion of \mathcal{U}_{m_0} and a diagonal argument, by [59] it is possible to pass to a

subsequence such that the following holds for some $J_n \in \mathbb{R}$. We define $\tilde{\beta}_n$ by

$$(\beta_n)_- := (\beta_n)_- \mathbb{1}_{\{(\beta_n)_- \leq J_n\}}, \quad (\beta_n)_+ = (\beta_n)_+.$$

Then it is possible to choose J_n in such a way that:

- For each compact set $K \subset \mathcal{U}_{m_0}$, the sequence $(\tilde{\beta}_n)_{-1} \mathbb{1}_K$ is uniformly integrable.
- The measure of the set $\{(\beta_n)_- > J_n\} \cap K$ converges to zero as *n* tends to infinity.

We use the exact same construction for $\tilde{\beta}_{T,n}$, and we can get the same properties (now taking $K \subset \mathbb{M} \times \mathbb{R}^d$).

We notice, that by construction the constraints

$$-\partial_t u_n - v \cdot D_x u_n + H(x, v, D_v u_n) \le \beta_n$$

and

$$u_{T,n} \leq \tilde{\beta}_{T,n}$$

are still satisfied. Finally,

$$\left| \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} \mathcal{F}^*(x,\nu,\tilde{\beta}_n) \, \mathrm{d}x \, \mathrm{d}\nu \, \mathrm{d}t - \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} \mathcal{F}^*(x,\nu,\beta_n) \, \mathrm{d}x \, \mathrm{d}\nu \, \mathrm{d}t \right|$$

$$\leq \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} |\mathcal{F}^*(x,\nu,0) - \mathcal{F}^*(x,\nu,\beta_n)| \mathbb{1}_{\{\beta_n \leq -J_n\}} \, \mathrm{d}x \, \mathrm{d}\nu \, \mathrm{d}t.$$

By the estimate (3.1), the integrand on the right hand side is dominated by $2C_F \in L^1$, and thus the right hand side converges to zero as *n* tends to infinity.

The exact same arguments apply to \mathcal{G}^* and $\tilde{\beta}_{T,n}$ too. Thus $(u_n, \tilde{\beta}_n, \tilde{\beta}_{T,n})$ is a minimizing sequence. Moreover, there exists (u, β, β_T) such that up to passing to a subsequence, $(u_n)_n$ converges to u strongly in $L^1_{loc}(\mathcal{U}_{m_0})$, $(\tilde{\beta}_n)_n$ converges weakly to β in $L^1_{loc}(\mathcal{U}_{m_0})$ and $(\tilde{\beta}_{T,n})_n$ converges to β_T weakly in $L^1_{loc}(\mathbb{M} \times \mathbb{R}^d)$.

6.2. Existence of a minimizer of $\tilde{\mathcal{A}}$ over $\mathcal{K}_{\mathcal{A}}$

In this subsection, we prove that there exists a minimizer (u, β, β_T) by passing to the limit in the functional

$$\begin{split} \tilde{\mathcal{A}}(u_n, \tilde{\beta}_n, \tilde{\beta}_{T,n}) &:= \int_0^T \int_{\mathbb{M} \times \mathbb{R}^d} \mathcal{F}^*(x, v, \tilde{\beta}_n) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t - \int_{\mathbb{M} \times \mathbb{R}^d} u_n(0, x, v) m_0(x, v) \, \mathrm{d}x \, \mathrm{d}v \\ &+ \int_{\mathbb{M} \times \mathbb{R}^d} \mathcal{G}^*(\tilde{\beta}_{T, n}(x, v)) \, \mathrm{d}x \, \mathrm{d}v. \end{split}$$

Theorem 6.8. Under our standing assumptions, the functional \tilde{A} admits a minimizer over $\mathcal{K}_{\mathcal{A}}$.

Proof. Let $(u_n, \beta_n, \beta_{T,n})_{n \in \mathbb{N}}$ be a minimizing sequence. Without loss of generality, for example by considering $u_n \in C^1$, we may assume equality in the Hamilton-Jacobi equation:

$$-\partial_t u_n - v \cdot D_x u_n + H(x, v, D_v u_n) = \beta_n, \quad u_n(T, x, v) = \beta_{T, n}(x, v).$$

For this minimizing sequence we have, for some constant C > 0,

$$\sup_{n} \mathcal{A}(u_n, \beta_n, \beta_{T,n}) \leq C.$$

We have discussed that this implies uniform in n bounds on the following quantities:

$$\begin{aligned} \|(\beta_{T,n})_{+}\|_{L^{g'}(\mathbb{M}\times\mathbb{R}^{d})}, \int_{\mathbb{M}\times\mathbb{R}^{d}} (u_{0,n})_{-}m_{0} \, dx \, dv, \\ \|(\beta_{T,n})_{-}\|_{L^{1}_{x,v,\text{loc}}(\mathbb{M}\times\mathbb{R}^{d})}, \|(u_{0,n})_{+}\|_{(L^{\infty}+L^{q'})_{x,v}}, \\ \|D_{v}u_{n}\|_{L^{r}_{\text{loc}}(\mathcal{U}_{m_{0}})}, \|(\beta_{n})_{+}\|_{L^{q'}_{t,x,v}}, \|(\beta_{n})_{-}\|_{L^{1}_{\text{loc}}(\mathcal{U}_{m_{0}})}. \end{aligned}$$

To get the uniform integrability on $(\beta_n)_-$ and $(\beta_{T,n})_-$, we perform the surgery argument as in Lemma 6.7. So, let $(u_n, \tilde{\beta}_n, \tilde{\beta}_{T,n})_{n \in \mathbb{N}}$ be the modification of the minimizing sequence (which will still have uniformly bounded energy). By Proposition 6.1 we know that $(u_n)_{n \in \mathbb{N}}$ is strongly precompact in $L^1_{loc}(\mathcal{U}_{m_0})$, while Lemma 6.7 yields that $(\tilde{\beta}_n)_{n \in \mathbb{N}}$ and $(\tilde{\beta}_{T,n})_{n \in \mathbb{N}}$ are weakly precompact in $L^1_{loc}(\mathcal{U}_{m_0})$ and $L^1_{loc}(\mathbb{M} \times \mathbb{R}^d)$, respectively. In particular, after passing to a subsequence let us denote by u the strong $L^1_{loc}(\mathcal{U}_{m_0})$ limit of $(u_n)_n$. In what follows, to ease the notation, we drop the tilde symbol, but whenever we write β_n and $\beta_{T,n}$, we mean the corresponding modified versions.

Passing to further subsequences (that we do not relabel), there exist limit functions so that we may also assume the following weak convergences:

- $(\beta_n)_+ \longrightarrow \beta_+$, weakly in $L^{q'}_{t,x,\nu}([0,T] \times \mathbb{M} \times \mathbb{R}^d)$, as $n \to +\infty$.
- $\beta_n \rightharpoonup \beta$, weakly in $L^1_{loc}(\mathcal{U}_{m_0})$, as $n \to +\infty$.
- $(\beta_{T,n})_+ \rightarrow (\beta_T)_+$, weakly in $L^{s'}(\mathbb{M} \times \mathbb{R}^d)$, as $n \to +\infty$.
- $\beta_{n,T} \rightarrow \beta_T$, weakly in $L^1_{loc}(\mathbb{M} \times \mathbb{R}^d)$, as $n \to +\infty$.
- $D_{\nu}u_n \rightarrow D_{\nu}u$, weakly in $L^r_{loc}(\mathcal{U}_{m_0})$, as $n \rightarrow +\infty$.

With these convergences in hand, we are ready to pass to the limit in the Hamilton-Jacobi inequality constraint and the functional. Note that the weak form of the inequality (Definition 3.9) implies that, for all *n* and all test functions $\phi \in C_c^{\infty}((0, T] \times \mathbb{M} \times \mathbb{R}^d)$ such that $\phi \geq 0$,

$$\int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} (\partial_{t}\phi + v \cdot D_{x}\phi)u_{n} + \phi H(x, v, D_{v}u_{n}) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \leq \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \phi\beta_{n} \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t + \int_{\mathbb{M}\times\mathbb{R}^{d}} \phi(T, x, v)\beta_{T, n} \, \mathrm{d}x \, \mathrm{d}v.$$
(6.3)

Note again that ϕ is compactly supported in \mathcal{U}_{m_0} . By the weak convergence of $D_v u_n$ in $L_{loc}^r(\mathcal{U}_0)$ and the convexity of H it follows that

$$\int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} \phi H(x,v,D_v u) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t = \int_{\mathcal{U}_{m_0}} \phi H(x,v,D_v u) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t$$

$$\leq \liminf_n \int_{\mathcal{U}_{m_0}} \phi H(x,v,D_v u_n) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t = \liminf_n \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} \phi H(x,v,D_v u_n) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t.$$

All the other convergences stated above are sufficient to guarantee convergence against ϕ . So, we obtain that the limit (u, β, β_T) satisfies (3.6).

Next, we consider the convergence in the functional. In addition to the previous convergences, along the previously chosen subsequence, we have

• $(u_{0,n})_+ \stackrel{\star}{\rightharpoonup} (\bar{u}_0)_+$, weakly-* in $(L^{\infty} + L^{q'})(\mathbb{M} \times \mathbb{R}^d)$, as $n \to +\infty$.

The convergence of the sequence $((u_{0,n})_{-}m_0)_n$ requires special attention. The boundedness of this sequence in $L^1(\mathbb{M} \times \mathbb{R}^d)$ lets us conclude that there exists a nonnegative Radon measure ν such that after passing to a subsequence (that we do not relabel)

$$(u_{0,n})_{-}m_0 \stackrel{\star}{\rightharpoonup} \nu$$
, as $n \to +\infty$.

This means in particular that for all $\phi \in C_c(\mathbb{M} \times \mathbb{R}^d)$, we have

$$\int_{\mathbb{M}\times\mathbb{R}^d} \phi(u_{0,n})_{-} m_0 \,\mathrm{d}x \,\mathrm{d}\nu \to \int_{\mathbb{M}\times\mathbb{R}^d} \phi\nu(\,\mathrm{d}x \,\mathrm{d}\nu), \text{ as } n \to +\infty.$$

Since the sequence $((u_{0,n})_{-}m_{0})_{n\in\mathbb{N}}$ is supported in the open set $\{m_{0} > 0\}$, we get that $\operatorname{spt}(\nu) \subseteq \operatorname{spt}(m_{0})$. Now, let us take $\phi \in C_{c}(\{m_{0} > 0\})$ arbitrary and define $\psi := \phi/m_{0}$. Since $m_{0} \in C(\mathbb{M} \times \mathbb{R}^{d})$, by assumption, we have that $\psi \in C_{c}(\{m_{0} > 0\})$ and so

$$\int_{\mathbb{M}\times\mathbb{R}^d} \psi(u_{0,n})_{-} m_0 \, \mathrm{d}x \, \mathrm{d}\nu = \int_{\mathbb{M}\times\mathbb{R}^d} (\phi/m_0)(u_{0,n})_{-} m_0 \, \mathrm{d}x \, \mathrm{d}\nu$$
$$= \int_{\mathbb{M}\times\mathbb{R}^d} \phi(u_{0,n})_{-} \, \mathrm{d}x \, \mathrm{d}\nu \to \int_{\mathbb{M}\times\mathbb{R}^d} (\phi/m_0)\nu(\,\mathrm{d}x \,\mathrm{d}\nu), \text{as } n \to +\infty.$$

Thus, this means that as $n \to +\infty$, $(u_{n,0})_-$ converges weakly-* to the nonnegative Radon measure $(\bar{u}_0)_- := \frac{1}{m_0} \cdot \nu$, i.e. $(u_0)_-$ has density $\frac{1}{m_0}$ with respect to ν . We notice that this means that $(\bar{u}_0)_-$ is absolutely continuous with respect to ν . In fact, we also have that ν is absolutely continuous with respect to $(\bar{u}_0)_-$, and so we can write $\nu = m_0 \cdot (\bar{u}_0)_-$.

Let us take now $\phi \in C_c^{\infty}(\mathcal{U}_{m_0})$, and test the inequalities satisfied by $(u_n, \beta_n, \beta_{T,n})$, similarly to (6.3), to obtain

$$\begin{split} \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} (\partial_t \phi + v \cdot D_x \phi) u_n + \phi H(x, v, D_v u_n) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t &\leq \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} \phi \beta_n \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \\ + \int_{\mathbb{M}\times\mathbb{R}^d} \phi(T, x, v) \beta_{T, n} \, \mathrm{d}x \, \mathrm{d}v - \int_{\mathbb{M}\times\mathbb{R}^d} \phi(0, x, v) u_{0, n} \, \mathrm{d}x \, \mathrm{d}v. \end{split}$$

Incorporating also the previously described convergence of $(u_{0,n})_n$, we can pass to the limit along the chosen subsequence and obtain

$$\int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} (\partial_t \phi + v \cdot D_x \phi) u + \phi H(x, v, D_v u) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t$$
$$\leq \int_{\mathbb{M}\times\mathbb{R}^d} \beta_T \phi_T \, \mathrm{d}x \, \mathrm{d}v - \int_{\mathbb{M}\times\mathbb{R}^d} \phi_0 \bar{u}_0(\, \mathrm{d}x \, \mathrm{d}v) + \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} \phi \beta \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t,$$

where $\bar{u}_0 := (\bar{u}_0)_+ - (\bar{u}_0)_-$. We notice that \bar{u}_0 is a signed Radon measure, supported in spt (m_0) .

Having in hand this last inequality, we readily check that the assumptions of Lemma A.11 are fulfilled with the choice of $\beta_0 = \bar{u}_0$ and β_T as before. This means in particular that u satisfies

$$\begin{cases} -\partial_t u - v \cdot D_x u + H(x, v, D_v u) \le \beta, \text{ in } \mathscr{D}'((0, T) \times \mathbb{M} \times \mathbb{R}^d) \\ u_0 \ge \bar{u}_0 \text{ in } \mathscr{D}'(\{m_0 > 0\}); u_T \le \beta_T \text{ in } \mathscr{D}'(\mathbb{M} \times \mathbb{R}^d), \end{cases}$$

where when writing the traces u_0 and u_T , we are referring to the right continuous version of u in time. Since by construction, $\langle \bar{u}_0, m_0 \rangle = \int_{\mathbb{M} \times \mathbb{R}^d} (\bar{u}_0)_+ m_0 \, dx \, dv - \langle (\bar{u}_0)_-, m_0 \rangle$ is finite, we have that $\langle u_0, m_0 \rangle$ is meaningful and finite, with

$$-\langle u_0, m_0 \rangle \leq -\langle \bar{u}_0, m_0 \rangle$$

Lower semicontinuity of the term involving $-\int_{\mathbb{M}\times\mathbb{R}^d} m_0 u_0(\mathrm{d} x \mathrm{d} v)$.

Claim.

$$\int_{\{m_0>0\}} m_0(u_0)_-(\operatorname{d} x \operatorname{d} \nu) \leq \int_{\mathbb{M}\times\mathbb{R}^d} m_0(\bar{u}_0)_-(\operatorname{d} x \operatorname{d} \nu) \leq \liminf_{n\to\infty} \int_{\mathbb{M}\times\mathbb{R}^d} (u_{0,n})_- m_0 \operatorname{d} x \operatorname{d} \nu.$$

Proof of Claim. First, notice that since $u_0 - \bar{u}_0$ is a positive distribution, it can be represented by a Radon measure. We may therefore write, for some $\nu_0 \in \mathcal{M}_+(\mathbb{M} \times \mathbb{R}^d)$, such that $\operatorname{spt}(\nu_0) \subseteq \operatorname{spt}(m_0)$ and

$$u_0 = \bar{u}_0 + \nu_0 = \left[(\bar{u}_0)_+ + \nu_0 \right] - (\bar{u}_0)_-$$

It follows that the Hahn-Jordan decomposition of u_0 satisfies

$$(u_0)_+ \leq (\bar{u}_0)_+ + \nu_0, (u_0)_- \leq (\bar{u}_0)_-.$$

Now consider any compactly supported function $\zeta \in C_c(\{m_0 > 0\})$, such that $0 \leq \zeta \leq 1$. Then

$$\int_{\mathbb{M}\times\mathbb{R}^d} \zeta m_0(\bar{u}_0)_-(\,\mathrm{d} x\,\mathrm{d} v) = \lim_{n\to\infty} \int_{\mathbb{M}\times\mathbb{R}^d} \zeta(u_{0,n})_- m_0\,\mathrm{d} x\,\mathrm{d} v \leq \liminf_{n\to\infty} \int_{\mathbb{M}\times\mathbb{R}^d} (u_{0,n})_- m_0\,\mathrm{d} x\,\mathrm{d} v.$$

Since $(u_0)_- \leq (\bar{u}_0)_-$ as measures,

$$\lim_{\mathbb{M}\times\mathbb{R}^d} \zeta m_0(u_0)_{-}(\,\mathrm{d} x\,\mathrm{d} v) \leq \liminf_{n\to\infty} \int_{\mathbb{M}\times\mathbb{R}^d} (u_{0,n})_{-} m_0\,\mathrm{d} x\,\mathrm{d} v$$

Then take a non-decreasing sequence of functions ζ_k such that ζ_k converges pointwise to the indicator function of the set $\{m_0 > 0\}$ as k tends to infinity: consider for example functions such that

$$\zeta_k(x,
u) = egin{cases} 1 & ext{if } m_0(x,
u) > 2^{-k} \ 0 & ext{if } m_0(x,
u) \leq 2^{-(k+1)}. \end{cases}$$

This is always possible since m_0 is continuous. Then, by monotone convergence, we indeed have

$$\int_{\mathbb{M}\times\mathbb{R}^d} m_0(u_0)_-(\,\mathrm{d} x\,\mathrm{d} v) = \lim_{k\to+\infty} \int_{\mathbb{M}\times\mathbb{R}^d} \zeta_k m_0(u_0)_-(\,\mathrm{d} x\,\mathrm{d} v) \leq \liminf_{n\to\infty} \int_{\mathbb{M}\times\mathbb{R}^d} (u_{0,n})_- m_0\,\mathrm{d} x\,\mathrm{d} v,$$

as desired and the claim follows.

By the weak star convergence of $(u_{0,n})_+$ to $(\bar{u}_0)_+$ in $(L^{\infty} + L^{q'})(\mathbb{M} \times \mathbb{R}^d)$, we also have that, for \bar{u}_0 , the positive part $(u_{0,n})_+m_0$ converges to $(\bar{u}_0)_+m_0$ strongly in $L^1(\mathbb{M} \times \mathbb{R}^d)$. Since $-u_0 \leq -\bar{u}_0$ as signed measures, we deduce that

$$-\int_{\mathbb{M}\times\mathbb{R}^d} m_0 u_0(\,\mathrm{d} x\,\mathrm{d} \nu) \leq -\int_{\mathbb{M}\times\mathbb{R}^d} m_0 \bar{u}_0(\,\mathrm{d} x\,\mathrm{d} \nu) \leq \liminf_n -\int_{\mathbb{M}\times\mathbb{R}^d} u_{0,n} m_0\,\mathrm{d} x\,\mathrm{d} \nu \,\,, \qquad (6.4)$$

as required.

The term involving \mathcal{G}^* .

For the term involving \mathcal{G}^* , we notice that by convexity

$$\int_{\mathbb{M}\times\mathbb{R}^d} \mathcal{G}^*(x,\nu,\beta_T) \, \mathrm{d}x \, \mathrm{d}\nu \leq \liminf_{n \to +\infty} \int_{\mathbb{M}\times\mathbb{R}^d} \mathcal{G}^*(x,\nu,\beta_{T,n}) \, \mathrm{d}x \, \mathrm{d}\nu.$$
(6.5)

Indeed, by classical results (cf. [57, Proposition I.2.3, Corollary I.2.2]), this is a consequence of the convexity of the integrand in the last variable and Fatou's lemma that yields the lower semi-continuity with respect to the strong topology on $L^1_{loc}(\mathbb{M} \times \mathbb{R}^d)$.

The term involving \mathcal{F}^* .

First note that, for functions β such that $\beta_+ \in L^{q'}([0,T] \times \mathbb{M} \times \mathbb{R}^d)$ and $\beta_- \in L^1_{loc}([0,T] \times \mathbb{M} \times \mathbb{R}^d)$, by (3.1) the following inequality holds:

$$|\mathcal{F}^*(x,\nu,\beta)| \leq c^{-1}|\beta_+|^{q'} + C_F(x,\nu) \in L^1([0,T] \times \mathbb{M} \times \mathbb{R}^d).$$

Thus

$$\int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \mathcal{F}^{*}(x,v,\beta) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t = \int_{\mathcal{U}_{m_{0}}} \mathcal{F}^{*}(x,v,\beta) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t.$$
(6.6)

Indeed, since $\mathcal{U}_{m_0} = \{0\} \times \{m_0 > 0\} \cup (0, T) \times \mathbb{M} \times \mathbb{R}^d$, for all $\delta > 0$ we have

$$\left|\int_{0}^{T}\int_{\mathbb{M}\times\mathbb{R}^{d}}\mathcal{F}^{*}(x,\nu,\beta)\,\mathrm{d}x\,\mathrm{d}\nu\,\mathrm{d}t-\int_{\mathcal{U}_{m_{0}}}\mathcal{F}^{*}(x,\nu,\beta)\,\mathrm{d}x\,\mathrm{d}\nu\,\mathrm{d}t\right|\leq\int_{0}^{\delta}\int_{\{m_{0}>0\}}|\mathcal{F}^{*}(x,\nu,\beta)|\,\mathrm{d}x\,\mathrm{d}\nu\,\mathrm{d}t.$$

The integrand is bounded by the L^1 function $|\mathcal{F}^*(x, v, \beta)|$ and converges to zero almost everywhere as δ tends to zero. Thus, taking $\delta \to 0$ we obtain (6.6). A similar equality holds for all β_n .

Therefore, by the convexity of \mathcal{F}^* (and by arguments similar to the one for \mathcal{G}^*), we conclude that

$$\int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \mathcal{F}^{*}(x,v,\beta) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t = \int_{\mathcal{U}_{m_{0}}} \mathcal{F}^{*}(x,v,\beta) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \leq \liminf_{n \to +\infty} \int_{\mathcal{U}_{m_{0}}} \mathcal{F}^{*}(x,v,\beta_{n}) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \\ = \liminf_{n \to +\infty} \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \mathcal{F}^{*}(x,v,\beta_{n}) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t.$$

$$(6.7)$$

Thus, collecting all the previous arguments, one deduces that

$$\mathcal{A}(u,\beta,\beta_T) \leq \liminf_{n\to+\infty} \mathcal{A}(u_n,\beta_n,w_n).$$

The thesis of the theorem follows.

Corollary 6.9. In the setting and notation of the previous theorem, in fact $u_0 = \bar{u}_0$ on $\{m_0 > 0\}$.

Proof. Since (u, β, β_T) is a minimizer,

$$\begin{split} \tilde{\mathcal{A}}(u,\beta,\beta_{T}) &\leq -\int_{\mathbb{M}\times\mathbb{R}^{d}} m_{0}\bar{u}_{0}(\,\mathrm{d}x\,\mathrm{d}v) + \liminf_{n \to +\infty} \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \mathcal{F}^{*}(x,v,\beta_{n})\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \\ &+ \liminf_{n \to \infty} \int_{\mathbb{M}\times\mathbb{R}^{d}} \mathcal{G}^{*}(x,v,\beta_{T,n})\,\mathrm{d}x\,\mathrm{d}v \\ &\leq \lim_{n \to +\infty} \left(-\int_{\mathbb{M}\times\mathbb{R}^{d}} m_{0}u_{0,n}(\,\mathrm{d}x\,\mathrm{d}v) + \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \mathcal{F}^{*}(x,v,\beta_{n})\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t + \int_{\mathbb{M}\times\mathbb{R}^{d}} \mathcal{G}^{*}(\beta_{T,n})\,\mathrm{d}x\,\mathrm{d}v \right) \\ &= \tilde{\mathcal{A}}(u,\beta,\beta_{T}), \end{split}$$

where in the last equality we have used that $(u_n, \beta_n, \beta_{T,n})$ is a minimizing sequence.

All the above inequalities are therefore equalities. From the inequalities (6.4), (6.5) and (6.7) for each of the terms, we deduce that

$$-\int_{\mathbb{M}\times\mathbb{R}^d} m_0 u_0(\,\mathrm{d} x\,\mathrm{d} \nu) = -\int_{\mathbb{M}\times\mathbb{R}^d} m_0 \bar{u}_0(\,\mathrm{d} x\,\mathrm{d} \nu).$$

It follows that $u_0 = \bar{u}_0$ as signed measures on $\{m_0 > 0\}$. Indeed, first note that $u_0 \ge \bar{u}_0$ as signed measures, or in other words $u_0 - \bar{u}_0$ is a nonnegative measure. For any nonnegative test function $\phi_0 \in C_c(\{m_0 > 0\})$ we have $m_0 \ge \varepsilon > 0$ on the support of ϕ_0 , for some $\varepsilon > 0$. Thus there exists a constant C such that $\phi_0 \leq Cm_0$. Thus

$$0 \leq \int_{\mathbb{M}\times\mathbb{R}^d} \phi_0(u_0 - \bar{u}_0)(\,\mathrm{d} x\,\mathrm{d} \nu) \leq C \!\!\int_{\mathbb{M}\times\mathbb{R}^d} m_0(u_0 - \bar{u}_0)(\,\mathrm{d} x\,\mathrm{d} \nu) = 0.$$

Thus $u_0 = \bar{u}_0$ as signed measures on $\{m_0 > 0\}$.

7. Existence and uniqueness of a solution to the MFG system

In this section we prove Theorem 2.4. First, we show that the minimizers of Problems 3.3 and 3.8 that we have obtained in the previous sections provide weak solutions (u, m) of the MFG.

Theorem 7.1. Let (u, β, β_T) be a minimizer of \tilde{A} over \mathcal{K}_A and let (m, w) be a minimizer of \mathcal{B} over \mathcal{K}_B . Then

(i)
$$\beta(t, x, v,) = f(x, v, m(t, x, v))$$
 for a.e. $(t, x, v) \in (0, T) \times \mathbb{M} \times \mathbb{R}^d$,
 $\beta_T(x, v) = g(x, v, m_T(x, v))$ for a.e. $(x, v) \in \mathbb{M} \times \mathbb{R}^d$;

(ii)
$$w(t,x,v) = -m(t,x,v)D_{p_v}H(x,v,D_vu(t,x,v))$$
 for a.e. $(t,x,v) \in (0,T) \times \mathbb{M} \times \mathbb{R}^d$.

As a consequence, (u, m) is a weak solution to (1.1) in the sense of Definition 2.3. *Proof.* By Theorem 5.1,

$$\mathcal{A}(u,\beta,\beta_T) + \mathcal{B}(m,w) = 0.$$

Substituting the definitions of the functionals, we obtain

$$\int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \mathcal{F}(x,v,m) + \mathcal{F}^{*}(x,v,\beta) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t - \int_{\mathbb{M}\times\mathbb{R}^{d}} m_{0}u_{0}(\,\mathrm{d}x \,\mathrm{d}v) \\ + \int_{\mathbb{M}\times\mathbb{R}^{d}} \mathcal{G}(x,v,m_{T}) + \mathcal{G}^{*}(x,v,\beta_{T}) \, \mathrm{d}x \, \mathrm{d}v + \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} L\left(x,v,-\frac{w}{m}\right) m \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t = 0.$$

Fenchel's inequality then implies that

$$\int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} \beta m \,\mathrm{d}x \,\mathrm{d}v \,\mathrm{d}t + \int_{\mathbb{M}\times\mathbb{R}^d} \beta_T m_T - m_0 u_0 \,\mathrm{d}x \,\mathrm{d}v + \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} L\left(x, v, -\frac{w}{m}\right) m \,\mathrm{d}x \,\mathrm{d}v \,\mathrm{d}t \leq 0.$$

By Corollary 5.4, the left hand side is non-negative, and therefore equality holds:

$$\int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \beta m \,\mathrm{d}x \,\mathrm{d}v \,\mathrm{d}t + \int_{\mathbb{M}\times\mathbb{R}^{d}} \beta_{T} m_{T} - m_{0} u_{0} \,\mathrm{d}x \,\mathrm{d}v + \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} L\left(x, v, -\frac{w}{m}\right) m \,\mathrm{d}x \,\mathrm{d}v \,\mathrm{d}t = 0$$
(7.1)

Moreover, equality also holds almost everywhere in the applications of Fenchel's inequality. Thus the following hold almost everywhere in $[0, T] \times \mathbb{M} \times \mathbb{R}^d$:

$$\beta = f(x, v, m(t, x, v)), \qquad \beta_T = g(x, v, m(T, x, v)).$$
 (7.2)

By (7.1) and Corollary 5.5,

$$\int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} mH(x,v,D_vu) + mL\left(x,v,-\frac{w}{m}\right) + D_vu\cdot w \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \leq 0.$$

By Fenchel's inequality, the integrand on the right hand side is non-negative; we deduce that equality holds in the above estimate and thus the integrand is equal almost everywhere to zero. It follows that

$$\frac{w}{m} = -D_{p_v}H(x, v, D_v u)$$

almost everywhere on the support of *m*. Moreover,

$$mL\left(x,v,-\frac{w}{m}\right) = m\left(D_{v}u \cdot D_{p_{v}}H(x,v,D_{v}u) - H(x,v,D_{v}u)\right).$$
(7.3)

The energy equality then follows from substituting (7.2) and (7.3) into (7.1). \Box

We show now, conversely, that weak solutions to the MFG system are in fact minimizers in the corresponding variational problems. The proof of this result follows similar ideas as the corresponding ones from [20,21].

Theorem 7.2. Let (u, m) be a weak solution to (1.1) in the sense of Definition 2.3. Then by setting $\beta := f(\cdot, \cdot, m)$, $\beta_T := g(\cdot, \cdot, m_T)$ and $w := -mD_{p_v}H(\cdot, \cdot, D_vu)$, we find that (m, w) is a solution of Problem 3.3, while (u, β, β_T) is a solution of Problem 3.8.

Proof. First let us notice that by Fenchel's equality one has

$$\mathcal{F}^*(\cdot, \cdot, f(\cdot, \cdot, m)) = \mathcal{F}(\cdot, \cdot, m) - mf(\cdot, \cdot, m).$$

We define the Borel set $B := \{(t, x, v) \in [0, T] \times \mathbb{M} \times \mathbb{R}^d : f(x, v, m(t, x, v)) \ge 0\}$. Restricted to this set, we find

$$-C_F \leq \mathcal{F}^*(\cdot, \cdot, f(\cdot, \cdot, m)) \leq \mathcal{F}(\cdot, \cdot, m)$$
, a.e. in B ,

where in the first inequality we have used our assumptions (3.1). Since, C_F , $\mathcal{F}^*(\cdot, \cdot, 0)$ and $\mathcal{F}(\cdot, \cdot, m)$ are summable, this implies in particular that $\mathcal{F}^*(\cdot, \cdot, f(\cdot, \cdot, m)_+) \in L^1([0, T] \times \mathbb{M} \times \mathbb{R}^d)$. Using the growth condition on \mathcal{F}^* we find furthermore that $f(\cdot, \cdot, m)_+ = \beta_+ \in L^{q'}([0, T] \times \mathbb{M} \times \mathbb{R}^d)$.

Now, on B^c , i.e. when $f(\cdot, \cdot, m) \leq 0$, we find

$$0 \le mf(\cdot, \cdot, m)_{-} = -mf(\cdot, \cdot, m) = \mathcal{F}^{*}(\cdot, \cdot, f(\cdot, \cdot, m)) - \mathcal{F}(\cdot, \cdot, m)$$
$$\le \sup_{\beta < 0} \mathcal{F}^{*}(\cdot, \cdot, \beta) - \mathcal{F}(\cdot, \cdot, m).$$

Again, the summability of the right hand side, we find that $mf(\cdot, \cdot, m)_{-} \in L^{1}([0, T] \times \mathbb{M} \times \mathbb{R}^{d})$. Using the exact same arguments for \mathcal{G}^{*} , we find similarly that $(\beta_{T})_{+} \in L^{s'}(\mathbb{M} \times \mathbb{R}^{d})$ and $m_{T}(\beta_{T})_{-} \in L^{1}(\mathbb{M} \times \mathbb{R}^{d})$.

Moreover, we have that $D_v u \in L^r_{loc}(\mathcal{U}_{m_0})$, $m \in L^1([0,T] \times \mathbb{M} \times \mathbb{R}^d)$ and $w \in L^1([0,T] \times \mathbb{M} \times \mathbb{R}^d; \mathbb{R}^d)$, so (m, w) and (u, β, β_T) are admissible competitors for the two optimization problems.

Now, take $(\bar{u}, \bar{\beta}, \bar{\beta}_T)$ as an admissible competitor for the problem involving the functional $\tilde{\mathcal{A}}$. By the convexity and differentiability of \mathcal{F}^* and \mathcal{G}^* in their last variable we have

$$\begin{split} \tilde{\mathcal{A}}(\bar{u},\bar{\beta},\bar{\beta}_{T}) &= \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \mathcal{F}^{*}(x,v,\bar{\beta}) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t - \int_{\mathbb{M}\times\mathbb{R}^{d}} m_{0}\bar{u}_{0}(\,\mathrm{d}x \,\mathrm{d}v) + \int_{\mathbb{M}\times\mathbb{R}^{d}} \mathcal{G}^{*}(\bar{\beta}_{T}) \, \mathrm{d}x \, \mathrm{d}v \\ &\geq \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \mathcal{F}^{*}(x,v,\beta) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t + \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \partial_{\beta} \mathcal{F}^{*}(x,v,\beta)(\bar{\beta}-\beta) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \\ &- \int_{\mathbb{M}\times\mathbb{R}^{d}} m_{0}u_{0}(\,\mathrm{d}x \, \mathrm{d}v) + \int_{\mathbb{M}\times\mathbb{R}^{d}} m_{0}(u_{0}-\bar{u}_{0})(\,\mathrm{d}x \, \mathrm{d}v) \\ &+ \int_{\mathbb{M}\times\mathbb{R}^{d}} \mathcal{G}^{*}(\beta_{T}) \, \mathrm{d}x \, \mathrm{d}v + \int_{\mathbb{M}\times\mathbb{R}^{d}} \partial_{\beta_{T}} \mathcal{G}^{*}(\beta_{T})(\bar{\beta}_{T}-\beta_{T}) \, \mathrm{d}x \, \mathrm{d}v \\ &= \tilde{\mathcal{A}}(u,\beta,\beta_{T}) + \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} m(\bar{\beta}-f(\cdot,\,\cdot\,,m)) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t + \int_{\mathbb{M}\times\mathbb{R}^{d}} m_{0}(u_{0}-\bar{u}_{0})(\,\mathrm{d}x \, \mathrm{d}v) \\ &+ \int_{\mathbb{M}\times\mathbb{R}^{d}} m_{T}(\bar{\beta}_{T}-g(\cdot,\,\cdot\,,m_{T})) \, \mathrm{d}x \, \mathrm{d}v \end{split}$$

where we have used the fact that $mf(\cdot, \cdot, m) \in L^1([0, T] \times \mathbb{M} \times \mathbb{R}^d)$ and $m_Tg(\cdot, \cdot, m_T) \in L^1(\mathbb{M} \times \mathbb{R}^d)$ (by the arguments at the beginning of this proof). Moreover, $m\bar{\beta} \in L^1([0, T] \times \mathbb{M} \times \mathbb{R}^d)$ and $m_T\bar{\beta}_T \in L^1(\mathbb{M} \times \mathbb{R}^d)$ (cf. Corollary 5.4) and

$$\partial_{\beta}\mathcal{F}^{*}(x,v,\beta) = \partial_{\beta}\mathcal{F}^{*}(x,v,f(x,v,m)) = \partial_{\beta}\mathcal{F}^{*}(x,v,\partial_{m}\mathcal{F}(x,v,m)) = m,$$

$$\partial_{\beta_{T}}\mathcal{G}^{*}(x,v,\beta_{T}) = \partial_{\beta}\mathcal{G}^{*}(x,v,g(x,v,m_{T})) = \partial_{\beta_{T}}\mathcal{G}^{*}(x,v,\partial_{m_{T}}\mathcal{G}(x,v,m_{T})) = m_{T}.$$

Now, using (2.4), one obtains

$$\begin{split} \tilde{\mathcal{A}}(\bar{u},\bar{\beta},\bar{\beta}_T) &\geq \tilde{\mathcal{A}}(u,\beta,\beta_T) + \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} m\bar{\beta} \,\mathrm{d}x \,\mathrm{d}v \,\mathrm{d}t + \int_{\mathbb{M}\times\mathbb{R}^d} m_T \bar{\beta}_T \,\mathrm{d}x \,\mathrm{d}v - \int_{\mathbb{M}\times\mathbb{R}^d} m_0 \bar{u}_0(\,\mathrm{d}x \,\mathrm{d}v) \\ &+ \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} L(\cdot,\,\cdot\,,\,-w/m) m \,\mathrm{d}x \,\mathrm{d}v \,\mathrm{d}t, \end{split}$$

where in the last line we have used

$$D_{p_{\nu}}H(\cdot, \cdot, D_{\nu}u) \cdot D_{\nu}u - H(\cdot, \cdot, D_{\nu}u) = L(\cdot, \cdot, D_{p_{\nu}}H(\cdot, \cdot, D_{\nu}u)).$$

By Corollary 5.4 we conclude that $\tilde{\mathcal{A}}(\bar{u}, \bar{\beta}, \bar{\beta}_T) \geq \tilde{\mathcal{A}}(u, \beta, \beta_T)$, as desired.

Using the very same ideas and the convexity of \mathcal{F} and \mathcal{G} , we can conclude similarly that (m, w) must be a minimizer in Problem 3.3.

Finally, we show that solutions in the sense of Definition 2.3 are unique, again following similar ideas as the corresponding ones from [21]. One major difference, however, is that we develop a suitable comparison principle for the distributional solutions to the corresponding Hamilton-Jacobi inequalities. This completes the proof of Theorem 2.4.

Proof of Theorem 2.4. The existence of a weak solution (u, m) follows from combining Theorem 6.8 (existence of a minimizer for \tilde{A}), Theorem 5.1 (duality, and the fact that the infimum for \tilde{B} is attained) and Theorem 7.1 (minimizers are weak solutions in the sense of Definition 2.3).

For the uniqueness, we first apply Theorem 7.2 to obtain that for $i = 1, 2, (u_i, f(\cdot, \cdot, m_i), g(\cdot, \cdot, m_i(T)))$ are minimizers of $\tilde{\mathcal{A}}$ over $\mathcal{K}_{\tilde{\mathcal{A}}}$ and $(m_i, -m_iD_pH(\cdot, \cdot, D_vu_i))$ are minimizers

of $\tilde{\mathcal{B}}$ over $\mathcal{K}_{\tilde{\mathcal{B}}}$. Since the minimizer of $\tilde{\mathcal{B}}$ is unique by strict convexity, $m_1 = m_2 =: m$ almost everywhere and $-m_1 D_p H(\cdot, \cdot, D_v u_1) = -m_2 D_p H(\cdot, \cdot, D_v u_2) =: w$ almost everywhere.

To show that $u_1 = u_2$ almost everywhere on the set $\{m > 0\}$, we first define $u = \max\{u_1, u_2\}$. By Lemma B.2, u also satisfies the Hamilton-Jacobi inequality, with $\beta = f(\cdot, \cdot, m)$ and $\beta_T = g(\cdot, \cdot, m_T)$. Since $u_i \le u$ for i = 1, 2, we have

$$-\int_{\mathbb{M}\times\mathbb{R}^d} m_0 u_0(\,\mathrm{d} x\,\mathrm{d} \nu) \leq -\int_{\mathbb{M}\times\mathbb{R}^d} m_0[u_i]_0(\,\mathrm{d} x\,\mathrm{d} \nu),$$

and thus $\tilde{\mathcal{A}}(u, \beta, \beta_T) \leq \tilde{\mathcal{A}}(u_i, \beta, \beta_T)$. Since u_i is a minimizer, equality holds. By duality, equality then holds in the energy inequalities of Corollary 5.4 for u and m, with β, β_T, w as defined previously. Thus, for almost all $t \in [0, T]$,

$$\int_{\mathbb{M}\times\mathbb{R}^d} u_t m_t dx dv = \int_t^T \int_{\mathbb{M}\times\mathbb{R}^d} L\left(x, v, -\frac{w}{m}\right) m dx dv dt + \int_t^T \int_{\mathbb{M}\times\mathbb{R}^d} \beta m dx dv dt + \int_{\mathbb{M}\times\mathbb{R}^d} \beta_T m_T dx dv.$$

The same is true replacing u by u_i , and so

$$\int_{\mathbb{M}\times\mathbb{R}^d} u_t m_t \,\mathrm{d}x \,\mathrm{d}v = \int_{\mathbb{M}\times\mathbb{R}^d} (u_i)_t m_t \,\mathrm{d}x \,\mathrm{d}v, \qquad i=1,2.$$

Thus, since also $u_i \leq u$, we deduce that $u_i = u$ almost everywhere on the set $\{m > 0\}$.

8. Sobolev estimates on the solutions

In this section, we obtain Sobolev estimates on the optimizers of the variational problems, and hence on weak solutions for the MFG system (1.1). The general idea is to "compare" the optimility of the optimizers in the variational problems with their carefully chosen translates. Then using strong convexity of the data one can deduce differential quotient estimates.

These results are inspired by [23,24]. However, because of the kinetic nature of the model we need completely new ideas when we consider perturbations. So, the estimates that we obtain are on suitable kinetic differential operators applied to the solutions. Another crucial difference between our results and the ones in [23,24] is that our Sobolev estimates in the x and v variables are local in time on (0, T]. The main reason behind this is that we have a weaker notion of trace for u_0 , that we cannot ensure to be stable under perturbations. This imposed further technical complications that require us to work in the case of r=2.

We emphasize that these estimates are consequences of the *stronger* convexity and regularity assumptions on the data stated in Assumption 2.

8.1. Local in time Sobolev estimates

Let $\zeta : [0, T] \to \mathbb{R}$ be a smooth cutoff function such that $\zeta(0) = 0$ and $\zeta(t) \ge 0$ for all t > 0. We define $\eta : [0, T] \to \mathbb{R}$ as $\eta(t) := \int_0^t \zeta(s) \, ds$.

For competitors (m, w) in Problem 3.3, without loss of generality one might assume the representation w = Vm, for a suitable vector field V. Let $\delta \in \mathbb{R}^d$ with $|\delta| \ll 1$ and define

$$m^{\delta}(t,x,v) := m(t,x+\eta(t)\delta,v+\zeta(t)\delta) \text{ and } V^{\delta}(t,x,v) := V(t,x+\eta(t)\delta,v+\zeta(t)\delta) - \zeta'(t)\delta.$$

We use the notation $w^{\delta} := V^{\delta} m^{\delta}$.

We notice that by construction, if (m, w) = (m, Vm) is a distributional solution to (3.3), so is $(m^{\delta}, w^{\delta}) = (m^{\delta}, V^{\delta}m^{\delta})$ and $m^{\delta}(0, \cdot, \cdot) = m_0$.

Similarly, for competitors (u, β, β_T) in Problem 3.8 we define

$$u^{\delta}(t, x, v) := u(t, x + \eta(t)\delta, v + \zeta(t)\delta), \ \beta^{\delta}(t, x, v) := \beta(t, x + \eta(t)\delta, v + \zeta(t)\delta),$$

and $\beta^{\delta}_{T}(x, v) := \beta_{T}(x + \eta(t)\delta, v + \zeta(t)\delta).$

Furthermore, we define

$$\begin{split} H^{\delta}(x,v,\xi) &:= H(x+\eta(t)\delta,v+\zeta(t)\delta,\xi) + \zeta'(t)\delta\cdot\xi,\\ \mathcal{F}^{\delta}(x,v,\theta) &:= \mathcal{F}(x+\eta(t)\delta,v+\zeta(t)\delta,\theta),\\ \mathcal{G}^{\delta}(x,v,\theta) &:= \mathcal{G}(x+\eta(t)\delta,v+\zeta(t)\delta,\theta). \end{split}$$

When computing the Legendre transforms of these functions in their last variables we obtain

$$\begin{aligned} (H^{\delta})^*(x,v,\xi) &:= H^*(x+\eta(t)\delta,v+\zeta(t)\delta,\xi-\zeta'(t)\delta),\\ (\mathcal{F}^{\delta})^*(x,v,\theta) &:= \mathcal{F}^*(x+\eta(t)\delta,v+\zeta(t)\delta,\theta),\\ (\mathcal{G}^{\delta})^*(x,v,\theta) &:= \mathcal{G}^*(x+\eta(t)\delta,v+\zeta(t)\delta,\theta). \end{aligned}$$

Let us notice that H^{δ} satisfies in particular the hypotheses imposed in Assumptions 1. Correspondingly, we define the functionals $\tilde{\mathcal{A}}^{\delta}$ and \mathcal{B}^{δ} and the constraint sets $\mathcal{K}^{\delta}_{\mathcal{A}}$ and $\mathcal{K}^{\delta}_{\mathcal{B}}$, using the shifted versions of the data functions. By construction, as a consequence of a change of variable formula, the proof of the following lemma is immediate.

Lemma 8.1. (m, w) is an optimizer of \mathcal{B} over $\mathcal{K}_{\mathcal{B}}$ if and only if (m^{δ}, w^{δ}) is an optimizer of \mathcal{B}^{δ} over $\mathcal{K}_{\mathcal{B}}^{\delta}$. Similarly, (u, β, β_T) is an optimizer of $\tilde{\mathcal{A}}$ over $\mathcal{K}_{\mathcal{A}}$ if and only if $(u^{\delta}, \beta^{\delta}, \beta_T^{\delta})$ is an optimizer of $\tilde{\mathcal{A}}^{\delta}$ over $\mathcal{K}_{\mathcal{A}}^{\delta}$.

Proof. We provide the proof of one of the statements only, the other ones follow similar steps. Suppose that (m^{δ}, w^{δ}) is an optimizer of \mathcal{B}^{δ} over $\mathcal{K}^{\delta}_{\mathcal{B}}$. This means in particular the minimality of the quantity

$$\begin{split} &\int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \mathcal{F}^{\delta}(x,v,m^{\delta}) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t + \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} (H^{\delta})^{*} \left(x,v,-\frac{w^{\delta}}{m^{\delta}}\right) m \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \\ &+ \int_{\mathbb{M}\times\mathbb{R}^{d}} \mathcal{G}^{\delta}(x,v,m_{T}^{\delta}(x,v)) \, \mathrm{d}x \, \mathrm{d}v \\ &= \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \mathcal{F}(x+\eta(t)\delta,v+\zeta(t)\delta,m(t,x+\eta(t)\delta,v+\zeta(t)\delta)) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \\ &+ \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \mathcal{H}^{*} \left(x+\eta(t)\delta,v+\zeta(t)\delta,-\frac{w(t,x+\eta(t)\delta,v+\zeta(t)\delta)}{m(t,x+\eta(t)\delta,v+\zeta(t)\delta)}\right) m \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \\ &+ \int_{\mathbb{M}\times\mathbb{R}^{d}} \mathcal{G}(x+\eta(t)\delta,v+\zeta(t)\delta,m_{T}(x+\eta(t)\delta,v+\zeta(t)\delta)) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \\ &= \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \mathcal{F}(x,v,m) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t + \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} H^{*} \left(x,v,-\frac{w}{m}\right) m \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \\ &+ \int_{\mathbb{M}\times\mathbb{R}^{d}} \mathcal{G}(x,v,m_{T}(x,v)) \, \mathrm{d}x \, \mathrm{d}v, \end{split}$$

where in the last equality we have used the change of variables $(x, v) \mapsto (x - \eta(t)\delta, v - \zeta(t)\delta)$. So, this means that the minimality of (m^{δ}, w^{δ}) , after a change of variables, yields the minimality of (m, w).

Now we are ready to state the main result of this subsection.

Theorem 8.2. Suppose that (u, m) is a weak solution to (1.1) in the sense of Definition 2.3 and that (H5), (H6), (H7) hold.

Then, there exists $\overline{C} > 0$ such that

$$\|m^{\frac{q}{2}-1}(\eta D_x + \zeta D_\nu)m\|_{L^2((0,\,T]\times\mathbb{M}\times\mathbb{R}^d)} \le \bar{C}, \|m^{1/2}(\eta D_x + \zeta D_\nu)D_\nu u\|_{L^2((0,\,T]\times\mathbb{M}\times\mathbb{R}^d)} \le \bar{C}$$

and

$$\|m_T^{\frac{k}{2}-1}(\eta(T)D_x+\zeta(T)D_\nu)m_T\|_{L^2(\mathbb{M}\times\mathbb{R}^d)}\leq \bar{C}.$$

Remark 8.3. As for Theorem 2.5, this is an informal statement: the result we obtain is on suitable difference quotients as in estimate (8.8) below.

Proof of Theorem 8.2. Let $(u_n, \beta_n, \beta_{T,n})_{n \in \mathbb{N}}$ be a minimizing sequence for Problem 3.8 such that $u_n \in C_c^1([0, T] \times \mathbb{M} \times \mathbb{R}^d)$,

$$\beta_n = -\partial_t u_n - v \cdot D_x u_n + H(x, v, D_v u_n), \ \beta_{T, n} = u(T, \cdot, \cdot).$$

Let us recall that after passing to a subsequence, that we do not relabel, as a consequence of Proposition 6.1, Lemma 6.7 and by Claim 2 in the proof of Theorem 7.1, we have that

- $(\beta_n)_+ \longrightarrow \beta_+$, weakly in $L^{q'}([0,T] \times \mathbb{M} \times \mathbb{R}^d)$, as $n \to +\infty$.
- $(\beta_n)_- \rightarrow \beta_-$, weakly in $L^1_{\text{loc}}(\mathcal{U}_{m_0})$, as $n \to +\infty$.
- $(\beta_{T,n})_+ \rightarrow (\beta_T)_+$, weakly in $L^{s'}(\mathbb{M} \times \mathbb{R}^d)$, as $n \to +\infty$.

- $(\beta_{T,n})_{-} \rightarrow (\beta_{T})_{-}$, weakly in $L^{1}_{loc}(\mathbb{M} \times \mathbb{R}^{d})$, as $n \rightarrow +\infty$.
- $(u_{0,n})_+ \stackrel{\star}{\rightharpoonup} (\tilde{u}_0)_+$, weakly-* in $(L^{\infty} + L^{q'})(\mathbb{M} \times \mathbb{R}^d)$, as $n \to +\infty$.
- $D_{\nu}u_n \rightarrow D_{\nu}u$, weakly in $L_m^r([0,T] \times \mathbb{M} \times \mathbb{R}^d)$, as $n \rightarrow +\infty$.

Notice that the previous arguments imply also that the subsequence can be chosen such that for all M < 0

$$\beta_n \mathbb{1}_{\{\beta_n \ge M\}} \stackrel{\star}{\rightharpoonup} \beta \mathbb{1}_{\{\beta \ge M\}}, \text{ weakly} - \ast \text{ in } (L^{\infty} + L^{q'})([0, T] \times \mathbb{M} \times \mathbb{R}^d) \text{ as } n \to +\infty$$
(8.1)

and

$$\beta_{T,n} \mathbb{1}_{\{\beta_{T,n} \ge M\}} \stackrel{\star}{\rightharpoonup} \beta_T \mathbb{1}_{\{\beta_T \ge M\}}, \text{weakly} - * \text{ in } (L^{\infty} + L^{s'})(\mathbb{M} \times \mathbb{R}^d) \text{ as } n \to +\infty.$$

Furthermore, by Theorem 7.1, we have that $\beta = f(\cdot, \cdot, m)$ and $\beta_T = g(\cdot, \cdot, m_T)$. Let $w = -mD_{p_v}H(\cdot, \cdot, D_vu)$.

Fix $\delta \in \mathbb{R}^d$ such that $|\delta| \ll 1$ and $\zeta : [0, T] \to \mathbb{R}$ as described at the beginning of this subsection.

Now, the main idea is to use u_n^{δ} as a test function in the weak formulation of the equation satisfied by (m, w) and u_n as test function in the weak form of the equation satisfied by (m^{δ}, w^{δ}) . Then we combine these inequalities with the energy equality (2.4) written for (m, w) and (m^{δ}, w^{δ}) , respectively, and rely on the strong convexity and regularity properties of the data to deduce a differential quotient estimate.

Following these steps, we obtain

$$\int_{\mathbb{M}\times\mathbb{R}^d} \left[\beta_{T,n}^{\delta} m_T - u_{0,n}^{\delta} m_0\right] \mathrm{d}x \, \mathrm{d}v \geq \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} (H^{\delta}(x,v,D_v u_n^{\delta}) - \beta_n^{\delta}) m + D_v u_n^{\delta} \cdot w \, \mathrm{d}x \, \mathrm{d}t.$$

We combine this with the energy equality (2.4) for (m, w) to get

$$\int_{\mathbb{M}\times\mathbb{R}^{d}} (\beta_{T,n}^{\delta} - g(\cdot, \cdot, m_{T}))m_{T} - (u_{0,n}^{\delta} - u_{0})m_{0} \,\mathrm{d}x \,\mathrm{d}\nu$$

$$\geq \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} (H^{\delta}(x, \nu, D_{\nu}u_{n}^{\delta}) + H^{*}(x, \nu, -w/m) + D_{\nu}u_{n}^{\delta} \cdot w/m - \beta_{n}^{\delta} + f(\cdot, \cdot, m))m \,\mathrm{d}x \,\mathrm{d}\nu \,\mathrm{d}t.$$

$$(8.2)$$

Similarly, using u_n as a test function in the weak form of the equation for (m^{δ}, w^{δ}) and combining with (2.4) for (m^{δ}, w^{δ}) ,

$$\int_{\mathbb{M}\times\mathbb{R}^{d}} \left[(\beta_{T,n} - g^{\delta}(x,v,m_{T}^{\delta}))m_{T}^{\delta} - (u_{0,n} - u_{0}^{\delta})m_{0}^{\delta} \right] dx dv$$

$$\geq \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} (H(x,v,D_{v}u_{n}) + (H^{\delta})^{*}(x,v,-w^{\delta}/m^{\delta})$$

$$+ D_{v}u_{n} \cdot w^{\delta}/m^{\delta} - \beta_{n} + f^{\delta}(m^{\delta}))m^{\delta} dx dt$$
(8.3)

Adding (8.2) and (8.3), after some changes of variables (translations) and a Taylor expansion of *L*, we deduce

$$\begin{split} \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} (H(x+\eta(t)\delta, v+\zeta(t)\delta, D_{v}u_{n}^{\delta}) + H^{*}(x+\eta(t)\delta, v+\zeta(t)\delta, -w/m) + D_{v}u_{n}^{\delta} \cdot w/m)m \, dx \, dv \, dt \\ &+ \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} (H(x-\eta(t)\delta, v-\zeta(t)\delta, D_{v}u_{n}^{-\delta}) + H^{*}(x-\eta(t)\delta, v-\zeta(t)\delta, -w/m) + D_{v}u_{n}^{-\delta} \cdot w/m)m \, dx \, dv \, dt \\ &\leq \int_{\mathbb{M}\times\mathbb{R}^{d}} \left[\beta_{T,n}(m_{T}^{\delta} + m_{T}^{-\delta}) - 2g(x, v, m_{T})m_{T} \right] \, dx \, dv - \int_{\mathbb{M}\times\mathbb{R}^{d}} 2(u_{0,n} - u_{0})m_{0} \, dx \, dv \\ &+ \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \left[\beta_{n}^{\delta} + \beta_{n}^{-\delta} - 2f(m) \right] m \, dx \, dv \, dt, \\ &+ \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \left[-H^{*}(x, v, -w/m) + H^{*}(x+\eta(t)\delta, v+\zeta(t)\delta, -w/m) - \zeta'(t)\delta \cdot D_{v}u_{n}^{\delta} \right] m \, dx \, dv \, dt \\ &+ \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \left[-H^{*}(x, v, -w/m) + H^{*}(x-\eta(t)\delta, v-\zeta(t)\delta, -w/m) + \zeta'(t)\delta \cdot D_{v}u_{n}^{-\delta} \right] m \, dx \, dv \, dt \\ &= \int_{\mathbb{M}\times\mathbb{R}^{d}} \left[\beta_{T,n}(m_{T}^{\delta} + m_{T}^{-\delta}) - 2g(x, v, m_{T})m_{T} \right] \, dx \, dv - \int_{\mathbb{M}\times\mathbb{R}^{d}} 2(u_{0,n} - u_{0})m_{0} \, dx \, dv \\ &+ \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \left[\beta_{n}^{\delta} + \beta_{n}^{-\delta} - 2f(m) \right] m \, dx \, dv \, dt, \\ &+ \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \left[\beta_{n}^{\delta} + \beta_{n}^{-\delta} - 2f(m) \right] m \, dx \, dv \, dt, \\ &+ \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \left[\beta_{n}^{\delta} + \beta_{n}^{-\delta} - 2f(m) \right] m \, dx \, dv \, dt \\ &+ \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \left[\beta_{n}^{\delta} + \beta_{n}^{-\delta} - 2f(m) \right] m \, dx \, dv \, dt \\ &+ \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \left[\beta_{n}^{\delta} + \beta_{n}^{-\delta} - 2f(m) \right] m \, dx \, dv \, dt \\ &+ \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \left[\beta_{n}^{\delta} + \beta_{n}^{-\delta} - 2f(m) \right] m \, dx \, dv \, dt \\ &+ \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \left[\beta_{n}^{\delta} + \beta_{n}^{-\delta} - 2f(m) \right] m \, dx \, dv \, dt \\ &+ \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \left[\beta_{n}^{\delta} + \beta_{n}^{-\delta} - 2f(m) \right] m \, dx \, dv \, dt \\ &+ \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \left[\beta_{n}^{\delta} + \beta_{n}^{-\delta} - 2f(m) \right] m \, dx \, dv \, dt \\ &+ \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \left[\beta_{n}^{\delta} + \beta_{n}^{-\delta} - 2f(m) \right] m \, dx \, dv \, dt \\ &+ \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \left[\beta_{n}^{\delta} + \beta_{n}^{-\delta} - 2f(m) \right] m \, dx \, dv \, dt \\ &+ \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \left[\beta_{n}^{\delta} + \beta_{n}^{-\delta} - 2f(m) \right] m \, dx \, dv \, dt \\ &+ \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \left[\beta_{n}^{\delta} + \beta_{n}^{-\delta} - 2f($$

where we have also used the facts that by the choice of η and ζ , we have $u_{0,n}^{\delta} = u_{0,n}$, $u_0^{\delta} = u_0$ and $m_0^{\delta} = m_0$.

Our aim now is to pass to the limit $n \to +\infty$ in (8.4) and derive a differential quotient estimate. For this, we consider each of the terms separately.

Step 1. First, we notice that by (H7) and by the fact that $\frac{|w|'}{m'^{-1}} \in L^1([0, T] \times \mathbb{M} \times \mathbb{R}^d)$, there exists C > 0 such that

$$\begin{split} &\int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} \int_0^1 \int_s^{-s} \left\langle \left[\eta^2(t) D_{xx}^2 H^* + \zeta^2(t) D_{vv}^2 H^* \right] (x + \tau \eta(t)\delta, v + \tau \zeta(t)\delta, - w/m)\delta, \delta \right\rangle m \, \mathrm{d}\tau \, \mathrm{d}s \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \\ &+ \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} \int_0^1 \int_s^{-s} \left\langle 2\eta(t)\zeta(t) D_{xv}^2 H^*(x + \tau \eta(t)\delta, v + \tau \zeta(t)\delta, - w/m)\delta, \delta \right\rangle m \, \mathrm{d}\tau \, \mathrm{d}s \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \\ &\leq C_0 |\delta|^2 \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} \left(\frac{|w|^{r\prime}}{m^{r\prime-1}} + m \right) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \leq C |\delta|^2. \end{split}$$

Step 2. Second, let us notice that by the fact that $m \in (L^1 \cap L^q)([0, T] \times \mathbb{M} \times \mathbb{R}^d)$ and by (8.1), for any M < 0 we have

$$\lim_{n \to +\infty} \int_0^T \int_{\mathbb{M} \times \mathbb{R}^d} \beta_n^{\pm \delta} \mathbb{1}_{\{\beta_n^{\pm \delta} \ge M\}} m \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t = \int_0^T \int_{\mathbb{M} \times \mathbb{R}^d} f^{\pm \delta}(m^{\pm \delta}) \mathbb{1}_{\{f^{\pm \delta}(m^{\pm \delta}) \ge M\}} m \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t.$$

Therefore,

$$\begin{split} \limsup_{n \to +\infty} \int_0^T \int_{\mathbb{M} \times \mathbb{R}^d} \left[\beta_n^{\delta} + \beta_n^{-\delta} - 2f(m) \right] m \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \\ &\leq \lim_{n \to +\infty} \int_0^T \int_{\mathbb{M} \times \mathbb{R}^d} \left[\beta_n^{\delta} \mathbb{1}_{\{\beta_n^{\delta} \ge M\}} + \beta_n^{-\delta} \mathbb{1}_{\{\beta_n^{-\delta} \ge M\}} - 2f(m) \right] m \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \\ &= \int_0^T \int_{\mathbb{M} \times \mathbb{R}^d} \left[f^{\delta}(m^{\delta}) \mathbb{1}_{\{f^{\delta}(m^{\delta}) \ge M\}} + f^{-\delta}(m^{-\delta}) \mathbb{1}_{\{f^{-\delta}(m^{-\delta}) \ge M\}} - 2f(m) \right] m \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t. \end{split}$$

Now, sending $M \to -\infty$, we conclude that

$$\limsup_{n \to +\infty} \int_{0}^{T} \int_{\mathbb{M} \times \mathbb{R}^{d}} \left[\beta_{n}^{\delta} + \beta_{n}^{-\delta} - 2f(m) \right] m \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \leq \int_{0}^{T} \int_{\mathbb{M} \times \mathbb{R}^{d}} \left[f^{\delta}(m^{\delta}) + f^{-\delta}(m^{-\delta}) - 2f(m) \right] m \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t,$$
(8.5)

where we have used the fact that $f(m)m, (f^{\delta}(m^{\delta}))_+, (f^{-\delta}(m^{-\delta}))_+ \in L^1$ so that the integrand is upper bounded by an L^1 function to allow us to apply the monotone convergence theorem. Since the left hand side of inequality (8.4) is bounded from below by zero, it follows the right hand side of (8.5) is not negative infinity.

By the very same arguments one can conclude that

$$\begin{split} \limsup_{n \to +\infty} \int_{\mathbb{M} \times \mathbb{R}^d} \left[\beta_{T,n} (m_T^{\delta} + m_T^{-\delta}) - 2g(x, v, m_T) m_T \right] \mathrm{d}x \, \mathrm{d}v \\ & \leq \int_{\mathbb{M} \times \mathbb{R}^d} \left[g^{\delta}(m_T^{\delta}) + g^{-\delta}(m_T^{-\delta}) - 2g(m_T) \right] m_T \, \mathrm{d}x \, \mathrm{d}v. \end{split}$$

Step 3. By Young's inequality, we have

$$\int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} \left[\zeta'(t)\delta \cdot D_v u_n^{-\delta} - \zeta'(t)\delta \cdot D_v u_n^{\delta} \right] m \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t + c \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} |D_v u_n^{-\delta} - D_v u_n^{\delta}|^2 m \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t,$$

where c > 0 is an arbitrary constant, and $C = C(c, T, \zeta') > 0$.

Step 4. By the previous steps we can conclude that

$$\int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} (H(x\pm\eta(t)\delta,\nu\pm\zeta(t)\delta,D_\nu u_n^{\pm\delta}) + H^*(x\pm\eta(t)\delta,\nu\pm\zeta(t)\delta,-w/m) + D_\nu u_n^{\pm\delta}\cdot w/m)m\,\mathrm{d}x\,\mathrm{d}\nu\,\mathrm{d}t - c\int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} |D_\nu u_n^{-\delta} - D_\nu u_n^{\delta}|^2 m\,\mathrm{d}x\,\mathrm{d}\nu\,\mathrm{d}t$$

is uniformly bounded above, independently of $n \in \mathbb{N}$. Let us recall that $\frac{|w|^2}{m} \in L^1((0,T) \times \mathbb{M} \times \mathbb{R}^d)$, and so is $H^*(x \pm \eta(t)\delta, v \pm \zeta(t)\delta, -w/m)m \in L^1((0,T) \times \mathbb{M} \times \mathbb{R}^d)$. Using the growth condition on H, by choosing c > 0 small enough in our application of Young's inequality we deduce that $D_v u_n^{\pm \delta}$ is uniformly bounded in $L^2_m((0,T) \times \mathbb{M} \times \mathbb{R}^d; \mathbb{R}^d)$. By a change of variable, one can similarly deduce that $D_v u_n$ is uniformly bounded in $L^2_{m\pm\delta}((0,T) \times \mathbb{M} \times \mathbb{R}^d; \mathbb{R}^d)$.

Claim. After passing to a subsequence that we do not relabel, we have $D_{\nu}u_n^{\pm\delta} \rightarrow D_{\nu}u^{\pm\delta}$ weakly in $L^2_m((0,T) \times \mathbb{M} \times \mathbb{R}^d; \mathbb{R}^d)$, as $n \rightarrow +\infty$.

Proof of Claim. By the uniform boundedness of the sequence, we know that there exists a subsequence of it (that we do not relabel) and $\xi \in L^2_m((0, T) \times \mathbb{M} \times \mathbb{R}^d; \mathbb{R}^d)$, as weak limit, i.e.

$$\int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} D_{v} u_{n}^{\pm\delta} \cdot \phi m \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \to \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \xi \cdot \phi m \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t,$$
$$\forall \phi \in L_{m}^{2}((0,T) \times \mathbb{M} \times \mathbb{R}^{d}; \mathbb{R}^{d}), \text{as } n \to +\infty.$$

Thus, we aim to show that $\xi = D_{\nu}u^{\pm\delta}$. As $D_{\nu}u_n^{\pm\delta} \rightarrow D_{\nu}u^{\pm\delta}$, weakly in $L^2_{loc}(\mathcal{U}_{m_0})$, as $n \rightarrow +\infty$, we can argue similarly as in the proof of Claim 2, in the proof of Theorem 7.1 to deduce the claim.

Step 5. By summarizing, (8.4) implies that

$$\begin{split} &\int_{0}^{T}\int_{\mathbb{M}\times\mathbb{R}^{d}}(H(x+\eta(t)\delta,\nu+\zeta(t)\delta,D_{\nu}u_{n}^{\delta})+H^{*}(x+\eta(t)\delta,\nu+\zeta(t)\delta,-w/m)+D_{\nu}u_{n}^{\delta}\cdot w/m)m\,\mathrm{d}x\,\mathrm{d}\nu\,\mathrm{d}t\\ &+\int_{0}^{T}\int_{\mathbb{M}\times\mathbb{R}^{d}}(H(x-\eta(t)\delta,\nu-\zeta(t)\delta,D_{\nu}u_{n}^{-\delta})+H^{*}(x-\eta(t)\delta,\nu-\zeta(t)\delta,-w/m)+D_{\nu}u_{n}^{-\delta}\cdot w/m)m\,\mathrm{d}x\,\mathrm{d}\nu\,\mathrm{d}t\\ &-c\int_{0}^{T}\int_{\mathbb{M}\times\mathbb{R}^{d}}|D_{\nu}u_{n}^{-\delta}-D_{\nu}u_{n}^{\delta}|^{2}m\,\mathrm{d}x\,\mathrm{d}\nu\,\mathrm{d}t\\ &\leq\int_{\mathbb{M}\times\mathbb{R}^{d}}\left[\beta_{T,n}(m_{T}^{\delta}+m_{T}^{-\delta})-2g(x,\nu,m_{T})m_{T}\right]\mathrm{d}x\,\mathrm{d}\nu-\int_{\mathbb{M}\times\mathbb{R}^{d}}2(u_{0,n}-u_{0})m_{0}\,\mathrm{d}x\,\mathrm{d}\nu\\ &+\int_{0}^{T}\int_{\mathbb{M}\times\mathbb{R}^{d}}\left[\beta_{n}^{\delta}+\beta_{n}^{-\delta}-2f(m)\right]m\,\mathrm{d}x\,\mathrm{d}\nu\,\mathrm{d}t\\ &+C|\delta|^{2}. \end{split}$$

Using the additional assumption (2.10) and the inequality $|a + b|^2 \le 2(a^2 + b^2)$, for c > 0 sufficiently small, one can conclude that there exists $c_0 > 0$ depending only on the data, such that

$$c_{0} \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} |D_{\nu}u_{n}^{-\delta} - D_{\nu}u_{n}^{\delta}|^{2}m \, dx \, d\nu \, dt$$

$$\leq \int_{\mathbb{M}\times\mathbb{R}^{d}} \left[\beta_{T,n}(m_{T}^{\delta} + m_{T}^{-\delta}) - 2g(x,\nu,m_{T})m_{T}\right] \, dx \, d\nu - \int_{\mathbb{M}\times\mathbb{R}^{d}} 2(u_{0,n} - u_{0})m_{0} \, dx \, d\nu$$

$$+ \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \left[\beta_{n}^{\delta} + \beta_{n}^{-\delta} - 2f(m)\right] m \, dx \, d\nu \, dt$$

$$+ C|\delta|^{2}.$$
(8.6)

Now, our aim is to pass to the limit with $n \to +\infty$ first in (8.6). For this we take $\liminf_{n\to+\infty}$ of the left hand side and $\limsup_{n\to+\infty}$ of the right hand side. We notice that the term $-\int_{\mathbb{M}\times\mathbb{R}^d} 2u_{0,n}m_0 \, dx \, dv$ needs special attention, since we do not have upper semicontinuity of it. Because of this, we add to both sides of (8.6) the quantity

$$2\int_0^T\int_{\mathbb{M}\times\mathbb{R}^d}\mathcal{F}^*(x,v,\beta_n)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t+2\int_{\mathbb{M}\times\mathbb{R}^d}\mathcal{G}^*(\beta_{T,n})\,\mathrm{d}x\,\mathrm{d}v$$

before passing to the limit. Thus we obtain

$$\begin{split} &\lim_{n \to +\infty} c_0 \int_0^T \int_{\mathbb{M} \times \mathbb{R}^d} |D_v u_n^{-\delta} - D_v u_n^{\delta}|^2 m \, dx \, dv \, dt \\ &+ \lim_{n \to +\infty} 2 \int_0^T \int_{\mathbb{M} \times \mathbb{R}^d} \mathcal{F}^*(x, v, \beta_n) \, dx \, dv \, dt + \lim_{n \to +\infty} 2 \int_{\mathbb{M} \times \mathbb{R}^d} \mathcal{G}^*(\beta_{T, n}) \, dx \, dv \\ &\leq \lim_{n \to +\infty} \int_{\mathbb{M} \times \mathbb{R}^d} \left[\beta_{T, n}(m_T^{\delta} + m_T^{-\delta}) - 2g(x, v, m_T) m_T \right] \, dx \, dv + 2 \int_{\mathbb{M} \times \mathbb{R}^d} u_0 m_0 \, dx \, dv \\ &+ \lim_{n \to +\infty} \int_0^T \int_{\mathbb{M} \times \mathbb{R}^d} \left[\beta_n^{\delta} + \beta_n^{-\delta} - 2f(m) \right] m \, dx \, dv \, dt \\ &+ \lim_{n \to +\infty} \left(-2 \int_{\mathbb{M} \times \mathbb{R}^d} u_{0, n} m_0 \, dx \, dv + 2 \int_0^T \int_{\mathbb{M} \times \mathbb{R}^d} \mathcal{F}^*(x, v, \beta_n) \, dx \, dv \, dt + 2 \int_{\mathbb{M} \times \mathbb{R}^d} \mathcal{G}^*(\beta_{T, n}) \, dx \, dv \right) \\ &+ C |\delta|^2. \end{split}$$

All the arguments in the previous steps allow us to pass to the limit. By the fact that $(u_n, \beta_n, \beta_{T,n})$ is a minimizing sequence, we get that

$$\begin{split} & \limsup_{n \to +\infty} \left(-2 \int_{\mathbb{M} \times \mathbb{R}^d} u_{0,n} m_0 \, \mathrm{d}x \, \mathrm{d}v + 2 \int_0^T \int_{\mathbb{M} \times \mathbb{R}^d} \mathcal{F}^*(x,v,\beta_n) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t + 2 \int_{\mathbb{M} \times \mathbb{R}^d} \mathcal{G}^*(\beta_{T,n}) \, \mathrm{d}x \, \mathrm{d}v \right) \\ & \lim_{n \to +\infty} \left(-2 \int_{\mathbb{M} \times \mathbb{R}^d} u_{0,n} m_0 \, \mathrm{d}x \, \mathrm{d}v + 2 \int_0^T \int_{\mathbb{M} \times \mathbb{R}^d} \mathcal{F}^*(x,v,\beta_n) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t + 2 \int_{\mathbb{M} \times \mathbb{R}^d} \mathcal{G}^*(\beta_{T,n}) \, \mathrm{d}x \, \mathrm{d}v \right) \\ &= 2 \tilde{\mathcal{A}}(u,\beta,\beta_t) \\ &= 2 \left(- \int_{\mathbb{M} \times \mathbb{R}^d} u_0 m_0 \, \mathrm{d}x \, \mathrm{d}v + \int_0^T \int_{\mathbb{M} \times \mathbb{R}^d} \mathcal{F}^*(x,v,\beta) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t + 2 \int_{\mathbb{M} \times \mathbb{R}^d} \mathcal{G}^*(\beta_T) \, \mathrm{d}x \, \mathrm{d}v \right). \end{split}$$

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So, after simplification, one obtains

$$c_{0} \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} |D_{\nu}u^{-\delta} - D_{\nu}u^{\delta}|^{2} m \, \mathrm{d}x \, \mathrm{d}\nu \, \mathrm{d}t$$

$$\leq \int_{\mathbb{M}\times\mathbb{R}^{d}} \left[g^{\delta}(m_{T}^{\delta}) + g^{-\delta}(m_{T}^{-\delta}) - 2g(x, \nu, m_{T}] m_{T} \, \mathrm{d}x \, \mathrm{d}\nu \right.$$

$$\left. + \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \left[f^{\delta}(m^{\delta}) + f^{-\delta}(m^{-\delta}) - 2f(m) \right] m \, \mathrm{d}x \, \mathrm{d}\nu \, \mathrm{d}t$$

$$\left. + C |\delta|^{2}.$$

$$(8.7)$$

Now, using (2.5) and (2.7) the very same arguments as in [23, computation (4.25)] yield

$$\int_{\mathbb{M}\times\mathbb{R}^d} (f^{\delta}(m^{\delta}) + f^{-\delta}(m^{-\delta}) - 2f(m))m \,\mathrm{d}x \,\mathrm{d}v$$

$$\leq C|\delta|^2 \left(1 + \int_{\mathbb{M}\times\mathbb{R}^d} \min\{m^{\delta}, m\}^q \,\mathrm{d}x \,\mathrm{d}v\right) - \frac{c_f}{2} \int_{\mathbb{M}\times\mathbb{R}^d} \min\{(m^{\delta})^{q-2}, m^{q-2}\} |m^{\delta} - m|^2 \,\mathrm{d}x \,\mathrm{d}v.$$

Similarly, (2.6) and (2.8) yield

$$\begin{split} &\int_{\mathbb{M}\times\mathbb{R}^d} \left(g^{\delta}(m_T^{\delta}) + g^{-\delta}(m_T^{-\delta}) - 2g(m_T) \right) m_T \, \mathrm{d}x \, \mathrm{d}v \\ &\leq C |\delta|^2 \left(1 + \int_{\mathbb{M}\times\mathbb{R}^d} \min\{m_T^{\delta}, m_T\}^s \, \mathrm{d}x \, \mathrm{d}v \right) - \frac{c_g}{2} \int_{\mathbb{M}\times\mathbb{R}^d} \min\{(m_T^{\delta})^{s-2}, m_T^{s-2}\} |m_T^{\delta} - m_T|^2 \, \mathrm{d}x \, \mathrm{d}v. \end{split}$$

Combining these estimates with (8.7), we get

$$c_{0} \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} |D_{\nu}u^{-\delta} - D_{\nu}u^{\delta}|^{2} m \, dx \, d\nu \, dt + \frac{c_{f}}{2} \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \min\{(m^{\delta})^{q-2}, m^{q-2}\} |m^{\delta} - m|^{2} \, dx \, d\nu \, dt + \frac{c_{g}}{2} \int_{\mathbb{M}\times\mathbb{R}^{d}} \min\{(m_{T}^{\delta})^{s-2}, m_{T}^{s-2}\} |m_{T}^{\delta} - m_{T}|^{2} \, dx \, d\nu \leq C |\delta|^{2}.$$
(8.8)

Dividing by $\left|\delta\right|^2$ and letting $\delta \to 0$, we easily obtain the result.

8.1.1. Recovering estimates on the operator $(tD_x + D_v)$ applied to solutions

By choosing a specific structure for the cutoff function ζ , we can recover estimates on more particular differential operators. Suppose that $\zeta(t) = 0$ for $t \in [0, t_0/2]$, and $\zeta(t) =$ 1, for $t \in (t_0, T]$ for some $t_0 \in (0, T)$ (to be chosen to be arbitrary), in such a way that also $\eta(t_0) = t_0$. Then in Theorem 8.2, the operator $(\eta D_x + \zeta D_v)$, for $t > t_0$ becomes $(tD_x + D_v)$. So, one can state the following local in time corollary.

Corollary 8.4. Suppose that the assumptions of Theorem 8.2 take place. Then, there exists $\bar{C} > 0$ such that

$$\|m^{\frac{q}{2}-1}(tD_{x}+D_{\nu})m\|_{L^{2}_{loc}((0,T]\times\mathbb{M}\times\mathbb{R}^{d})}\leq \bar{C}, \|m^{1/2}(tD_{x}+D_{\nu})D_{\nu}u\|_{L^{2}_{loc}((0,T]\times\mathbb{M}\times\mathbb{R}^{d})}\leq \bar{C}$$

and

$$\|m_T^{\frac{s}{2}-1}(TD_x+D_v)m_T\|_{L^2(\mathbb{M}\times\mathbb{R}^d)}\leq \bar{C}.$$

8.1.2. Recovering estimates on the operator D_x applied to solutions

Now suppose that $\eta(t) = 0$ for $t \in [0, t_0/2]$ and $\eta(t) = 1$ for $t \in (t_0, T]$ (where $t_0 \in (0, T)$ can be chosen arbitrarily). We still require that $\zeta := \eta'$. With this choice of cutoff functions η, ζ , we can formulate the following result as a corollary of Theorem 8.2.

Corollary 8.5. Suppose that the assumptions of Theorem 8.2 take place. Then, there exists $\overline{C} > 0$ such that

$$\|m^{\frac{q}{2}-1}D_xm\|_{L^2_{\mathrm{loc}}((0,\,T] imes \mathbb{M} imes \mathbb{R}^d)} \leq ar{C}, \|m^{1/2}D_xD_
uu\|_{L^2_{\mathrm{loc}}((0,\,T] imes \mathbb{M} imes \mathbb{R}^d)} \leq ar{C}$$

and

$$\|m_T^{\frac{s}{2}-1}D_xm_T\|_{L^2(\mathbb{M}\times\mathbb{R}^d)}\leq \bar{C}.$$

8.1.3. Proof of Theorem 2.5

Finally, the proof of Theorem 2.5 follows from the previous two corollaries and the inequality

$$|D_{\nu}h| \leq |(tD_x + D_{\nu})h| + T|D_xh| \quad \text{for all } t \in [0, T],$$

(8.1.3) for any Sobolev function h.

Acknowledgements

We thank Mikaela Iacobelli for useful discussions regarding Lemma A.4. We thank the two anonymous referees for carefully reading our manuscript and for their valuable comments.

Disclosure statement

The authors report that there are no competing interests to declare.

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Appendix A. Time regularity

In this appendix, we collect some facts about the regularity with respect to time of solutions u of

$$-\partial_t u - v \cdot D_x u + H(x, v, D_v u) \le \beta, \quad \text{in } \mathscr{D}'((0, T) \times \mathbb{M} \times \mathbb{R}^d).$$
(A.1)

By this we mean that, for any non-negative test function $0 \le \phi \in C_c^{\infty}((0, T) \times \mathbb{M} \times \mathbb{R}^d)$,

$$\int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} (\partial_{t}\phi + v \cdot D_{x}\phi)u + \phi \ H(x, v, D_{v}u) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \le \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \beta\phi \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t.$$
(A.2)

What we discuss is close to the standard theory of distributional solutions. However, in our case technical difficulties arise since, firstly, (A.1) is an inequality and, secondly, we wish to work on the atypical domain U_{m_0} . We therefore found it useful to clarify several points. Our main goal is to give a precise sense to the specification of boundary data for this problem at time t = T, and to give a meaning to u_0 (the 'value of u at time t = 0'), which appears in the functional \tilde{A} defining the variational problem.

Throughout this appendix we impose the following summability conditions on the pair $(u,\beta) \in L^1_{loc}(\mathcal{U}_{m_0}) \times L^1_{loc}(\mathcal{U}_{m_0})$ and that *H* satisfies (2.1).

Assumption 3. The pair $(u, \beta) \in L^1_{loc}(\mathcal{U}_{m_0}) \times L^1_{loc}(\mathcal{U}_{m_0})$ satisfies the following assumptions:

- The positive part of β satisfies $\beta_+ \in L^{q'}([0, T] \times \mathbb{M} \times \mathbb{R}^d)$;
- $D_v u \in L^r_{\operatorname{loc}}(\mathcal{U}_{m_0});$

Under Assumption 3, by a density argument the weak form (A.2) extends additionally to test functions in $C_c^1((0,T) \times \mathbb{M} \times \mathbb{R}^d)$.

Lemma A.1. Let $(u, \beta) \in L^1_{loc}(\mathcal{U}_{m_0}) \times L^1_{loc}(\mathcal{U}_{m_0})$ be a distributional solution to (A.1) satisfying Assumption 3. Then

(1) for any $\phi \in C^1_c((0,T) \times \mathbb{M} \times \mathbb{R}^d)$ the function

$$(0,T) \ni t \mapsto \langle \phi(t), u(t) \rangle := \int_{\mathbb{M} \times \mathbb{R}^d} \phi(t,x,v) u(t,x,v) \, \mathrm{d}x \, \mathrm{d}v$$

is of locally bounded variation and therefore has a right continuous representative with a countable number of jump discontinuities.

(2) There exists a path $(0, T) \ni t \mapsto \tilde{u}_t \in C_c^1(\mathbb{M} \times \mathbb{R}^d)'$ which is right continuous with respect to the weak-star topology on $C_c^1(\mathbb{M} \times \mathbb{R}^d)'$ and such that $\tilde{u}_t = u_t$ as elements of $(C_c^1)'$ for almost every $t \in (0, T)$.

Proof. Since $\beta + \partial_t u + v \cdot D_x u - H(x, v, D_v u)$ is a positive distribution, it is given by some Radon measure ν on $(0, T) \times \mathbb{M} \times \mathbb{R}^d$. We have

$$\partial_t u = -v \cdot D_x u + H(x, v, D_v u) - \beta + \nu =: \mu, \tag{A.3}$$

which we will use to deduce weak time regularity for u.

Consider a test function $\phi \in C^1_c((0,T) \times \mathbb{M} \times \mathbb{R}^d)$. The function

$$f_{\phi}: (0,T) \to \mathbb{R}$$

$$t \mapsto \langle \phi, u(t) \rangle := \int_{\mathbb{M} \times \mathbb{R}^d} \phi(t,x,v) u(t,x,v) \, \mathrm{d}x \, \mathrm{d}v$$
(A.4)

has distributional derivative

$$f'_{\phi} = \frac{\mathrm{d}}{\mathrm{d}t} \langle \phi, u \rangle = \langle \partial_t \phi + v \cdot D_x \phi, u \rangle + \langle \phi, H(x, v, D_v u) + \nu - \beta \rangle. \tag{A.5}$$

By Assumption 3, u, β and $H(x, v, D_v u)$ are all locally integrable functions on \mathcal{U}_{m_0} and so in particular on $(0, T) \times \mathbb{M} \times \mathbb{R}^d$. Thus the distributional derivative f'_{ϕ} defined in (A.5) is a Radon measure on (0, T), and so the path (A.4) is of locally bounded variation.

It follows that f_{ϕ} has a unique right-continuous version. That is, there exists a set $E_{\phi} \subset [0, T]$ of full measure and a right continuous function \tilde{f}_{ϕ} such that $\tilde{f}_{\phi}(t) = f_{\phi}(t)$ for all $t \in E_{\phi}$. The function \tilde{f}_{ϕ} satisfies

$$\tilde{f}_{\phi}(t) = \tilde{f}_{\phi}(s) + f'_{\phi}((s,t])$$

for all 0 < s < t < T.

Now consider $\psi \in C_c^1(\mathbb{M} \times \mathbb{R}^d)$ (independent of time). The path $t \mapsto \langle u(t), \psi \rangle$ has time derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi,u(t)\rangle=\langle v\cdot\nabla_x\psi,u\rangle+\langle\psi,H(x,v,D_vu)+\nu-\beta\rangle.$$

For each compact subset $K \subset \mathbb{M} \times \mathbb{R}^d$, we define the following Radon measure on (0, T): for $A \subset (0, T)$ Borel,

$$\mu_{K}(A) := \sup_{\nu: \exists (x,\nu) \in K} |\nu| \|u\|_{L^{1}(A \times K)} + \|H(\cdot, \cdot, D_{\nu}u) - \beta\|_{L^{1}(A \times K)} + \nu(A \times K).$$

For the right-continuous versions we have, for all 0 < s < t < T and all $\psi \in C_c^1(\mathbb{M} \times \mathbb{R}^d)$ with support contained in K,

$$|\tilde{f}_{\psi}(t) - \tilde{f}_{\psi}(s)| \le \|\psi\|_{C^{1}} \mu_{K}((s, t]).$$
(A.6)

Using these estimates, it is possible to construct a right continuous version of u: that is, a path $t \mapsto \tilde{u}_t \in C_c^1(\mathbb{M} \times \mathbb{R}^d)'$ that is right continuous with respect to the weak-star topology.

Such a construction is classical, but because of the lack of a precise reference in our context, we sketch the main ideas here. Take a countable dense set $Z \subset C_c^1(\mathbb{M} \times \mathbb{R}^d)$; there is a full measure set $E \subset (0,T)$ such that $\langle \psi, u(t) \rangle = \tilde{f}_{\psi}(t)$ for all $t \in E$ and all $\psi \in Z$, and moreover $u(t) \in L^1_{loc}(\mathbb{M} \times \mathbb{R}^d)$ (the latter is true for almost all t since u is L^1_{loc}). Then

$$\langle \tilde{u}(t), \psi \rangle := \tilde{f}_{\psi}(t)$$

defines a bounded linear functional on Z, for all $t \in E$. The estimate (A.6) can be used to show that this is in fact true for all $t \in (0, T)$. The resulting functional $\tilde{u}(t)$ extends by density to a continuous linear functional on $C_c^1(\mathbb{M} \times \mathbb{R}^d)$. Then the estimate (A.6) can be used to prove that $\langle \tilde{u}(t), \psi \rangle$ is right continuous for all $\psi \in C_c^1(\mathbb{M} \times \mathbb{R}^d)$, not just on Z.

Next, we construct the extension of \tilde{u} to the boundaries t = 0, T.

Definition A.2 (Transport shift). Let $t \in \mathbb{R}$. The operator $\mathcal{T}_t : C_c(\mathbb{M} \times \mathbb{R}^d) \to C_c(\mathbb{M} \times \mathbb{R}^d)$ is defined by

$$\mathcal{T}_t \phi(x, v) = \phi(x - tv, v)$$

Remark A.3 (Group property). For any $s, t \in \mathbb{R}$, $\mathcal{T}_s \mathcal{T}_t = \mathcal{T}_{s+t}$.

Lemma A.4. Let u be a solution to (A.1) and let \tilde{u} be its right continuous representative, obtained in Lemma A.1. Let $\psi \in C_c^1(\mathbb{M} \times \mathbb{R}^d)$ be non-negative. Consider the function $(0,T) \ni t \mapsto \langle T_t \psi, \tilde{u}(t) \rangle$. Then

- (1) As t tends to $T \langle T_t \psi, \tilde{u}(t) \rangle$ either tends to a finite limit or to positive infinity.
- (2) As t tends to $0 + \langle T_t \psi, \tilde{u}(t) \rangle$ either tends to a finite limit or to negative infinity.

Proof. Observe that

$$(\partial_t + \mathbf{v} \cdot \nabla_x) \mathcal{T}_t \boldsymbol{\psi} = \mathbf{0}.$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathcal{T}_t\psi,\tilde{u}(t)\rangle=\langle \mathcal{T}_t\psi,H(x,\nu,D_\nu u)+\nu-\beta\rangle.$$

Then the negative part of the time derivative satisfies

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathcal{T}_t\psi,\tilde{u}(t)\rangle\right]_{-}\leq \langle \mathcal{T}_t\psi,C_H+\beta_+\rangle\in L^1(0,T).$$

Thus $\langle \mathcal{T}_t \psi, \tilde{u}(t) \rangle$ can be written as the difference of monotone functions, where the decreasing part is absolutely continuous on (0, T) and can be extended to finite limits at the endpoints. By monotonicity, the increasing part either has a finite limit at t = T, or tends to positive infinity; similarly, at t = 0 it either has a finite limit or tends to negative infinity.

Definition A.5 (Weak traces). For any $\psi_T, \psi_0 \in C^1_c(\mathbb{M} \times \mathbb{R}^d)$, let

$$\langle \psi_T, u_T \rangle := \lim_{t \to T_-} \langle \mathcal{T}_{t-T} \psi_T, \tilde{u}(t) \rangle, \qquad \langle \psi_0, u_0 \rangle := \lim_{t \to 0+} \langle \mathcal{T}_t \psi_0, \tilde{u}(t) \rangle.$$
(A.7)

These define linear maps from $C_c^1(\mathbb{M} \times \mathbb{R}^d)$ to $\mathbb{R} \cup \{-\infty\}$ in the case of u_0 , and $\mathbb{R} \cup \{+\infty\}$ in the case of u_T .

We now suppose that, in addition to the weak Hamilton-Jacobi inequality in the interior (A.2), u satisfies the following: for all non-negative test functions $\phi \in C_c^1((0, T] \times \mathbb{M} \times \mathbb{R}^d)$

$$\int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} u[\partial_{t}\phi + \operatorname{div}_{x}(v\phi)] + \phi H(x, v, D_{v}u) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \leq \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \beta \phi \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t + \int_{\mathbb{M}\times\mathbb{R}^{d}} \beta_{T}\phi_{T} \, \mathrm{d}x \, \mathrm{d}v.$$
(A.8)

In our setting, we will have that $\beta_T \in L^1_{loc}(\mathbb{M} \times \mathbb{R}^d)$ is a given function whose positive part satisfies $(\beta_T)_+ \in L^{s'}(\mathbb{M} \times \mathbb{R}^d)$. In this case, we show below that the time trace u_T , enjoys some more properties.

Lemma A.6. If u satisfies (A.8) with $\beta_T \in \mathcal{M}(\mathbb{M} \times \mathbb{R}^d)$, then u_T as defined in (A.7) is a bounded linear functional on $C_c^1(\mathbb{M} \times \mathbb{R}^d)$ and $u_T \leq \beta_T$ in the sense of distributions: that is, for all $\psi_T \in C_c^1(\mathbb{M} \times \mathbb{R}^d)$ non-negative,

$$\langle \psi_T, u_T \rangle \leq \langle \psi_T, \beta_T \rangle.$$

In particular, we have that $\langle \psi, u_T \rangle = +\infty$ does not occur for any $\psi \in C_c^1(\mathbb{M} \times \mathbb{R}^d)$.

Remark A.7. Since then $\beta_T - u_T$ is a positive distribution, if $\beta_T \in L^1_{loc}(\mathbb{M} \times \mathbb{R}^d)$ then we in fact have that u_T is represented by a signed Radon measure with absolutely continuous positive part.

Proof of Lemma A.6. In what follows, we will use the right continuous representative of u constructed in Lemma A.1. By the abuse of notation, we write simply u for \tilde{u} . Fix $\psi_T \in C_c^1(\mathbb{M} \times \mathbb{R}^d)$ non-negative. For each $\varepsilon > 0$ small consider a smooth, non-negative test function $\eta_{\varepsilon} : [0, T] \to \mathbb{R}$, chosen such that $\eta(t) = 0$ for all $0 \le t \le T - \varepsilon$ and the derivative η'_{ε} satisfies

$$0 \leq \eta'_{arepsilon} \leq arepsilon^{-1}, \qquad \eta'_{arepsilon}(t) = egin{cases} 0 & t \in [0, T-arepsilon] \ arepsilon^{-1} & t \in [T-arepsilon+arepsilon^2, T-arepsilon^2] \end{cases}$$

Note that as a consequence of the fundamental theorem of calculus, one has $\lim_{\epsilon \to 0} \eta_{\epsilon}(T) = 1$.

We define the following non-negative test function $\phi_{\varepsilon} \in C^{1}_{c}((0,T] \times \mathbb{M} \times \mathbb{R}^{d})$:

$$\phi_{\varepsilon}(t, x, v) = \eta_{\varepsilon}(t)\psi_{T}(x + (T - t)v, v).$$

Substituting this choice of ϕ into (A.8), we obtain

$$\int_{T-\varepsilon}^{T} \eta_{\varepsilon}'(t) \int_{\mathbb{M} \times \mathbb{R}^{d}} u(t) \psi_{T}(x + (T-t)v, v) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \le \eta_{\varepsilon}(T) \int_{\mathbb{M} \times \mathbb{R}^{d}} \beta_{T} \psi_{T} \, \mathrm{d}x \, \mathrm{d}v + \alpha_{1}(\varepsilon).$$
(A.9)

where

$$\alpha_1(\varepsilon) = \int_{T-\varepsilon}^T \int_{\mathbb{M}\times\mathbb{R}^d} \beta \phi_\varepsilon \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t - \int_{T-\varepsilon}^T \int_{\mathbb{M}\times\mathbb{R}^d} \phi_\varepsilon H(x,v,D_v u) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t.$$

Note that $\lim_{\varepsilon \to 0} \alpha_1(\varepsilon) = 0$, since β and $H(x, \nu, D_{\nu}u)$ are locally integrable and ϕ_{ε} are bounded in L^{∞} , uniformly in ε .

We exclude the possibility that $\langle \psi_T, u_T \rangle = +\infty$. Indeed, if this occurs, then for any M > 0, there exists ε' such that for any $t \in [T - \varepsilon', T]$,

$$M < \int_{\mathbb{M}\times\mathbb{R}^d} u(t)\psi_T(x+(T-t)v,v)\,\mathrm{d}x\,\mathrm{d}v.$$

Then by bounding the left hand side of inequality (A.9) from below we obtain that for any $\varepsilon < \varepsilon'$,

$$M(1-2\varepsilon) \leq \eta_{\varepsilon}(T) \int_{\mathbb{M} \times \mathbb{R}^d} \beta_T \psi_T \, \mathrm{d}x \, \mathrm{d}\nu + \alpha_1(\varepsilon).$$

Taking the limit $\varepsilon \to 0$ gives

$$M \leq \int_{\mathbb{M}\times\mathbb{R}^d} \beta_T \psi_T \,\mathrm{d}x \,\mathrm{d}\nu.$$

Since this holds for any M > 0, we derive a contradiction. Thus—using also Lemma A.4— u_T is in fact a linear map from $C_c^1(\mathbb{M} \times \mathbb{R}^d)$ to \mathbb{R} . We note also that the map

$$t \mapsto \int_{\mathbb{M}\times\mathbb{R}^d} u(t)\psi_T(x+(T-t)v,v)\,\mathrm{d}x\,\mathrm{d}v$$

extends to a function that is bounded and continuous (from the left) at t = T.

Next, we show that $u_T \leq \beta_T$ as functionals on $C^1_c(\mathbb{M} \times \mathbb{R}^d)$. We have

$$\frac{1}{\varepsilon} \int_{T-\varepsilon}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} u\psi_{T}(x+(T-t)\nu,\nu) \,\mathrm{d}x \,\mathrm{d}\nu \,\mathrm{d}t \le \eta_{\varepsilon}(T) \int_{\mathbb{M}\times\mathbb{R}^{d}} \beta_{T}\psi_{T} \,\mathrm{d}x \,\mathrm{d}\nu + \alpha_{2}(\varepsilon), \tag{A.10}$$

where

$$\alpha_2(\varepsilon) := \alpha_1(\varepsilon) + \int_{T-\varepsilon}^T (\varepsilon^{-1} - \eta_{\varepsilon}') \int_{\mathbb{M} \times \mathbb{R}^d} u \psi_T(x + (T-t)v, v) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t.$$

For the second term here we have

$$\begin{split} &\int_{T-\varepsilon}^{T} (\varepsilon^{-1} - \eta_{\varepsilon}') \int_{\mathbb{M} \times \mathbb{R}^{d}} u \psi_{T}(x + (T-t)v, v) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \bigg| \\ &\leq \bigg| \int_{T-\varepsilon}^{T-\varepsilon+\varepsilon^{2}} |\varepsilon^{-1} - \eta_{\varepsilon}'|| \langle u(t), \psi_{T}(x + (T-t)v, v) \rangle| \, \mathrm{d}t \bigg| \\ &+ \bigg| \int_{T-\varepsilon^{2}}^{T} |\varepsilon^{-1} - \eta_{\varepsilon}'|| \langle u(t), \psi_{T}(x + (T-t)v, v) \rangle| \, \mathrm{d}t \bigg| \\ &\leq 2\varepsilon \| \langle u(t), \psi_{T}(x + (T-t)v, v) \rangle \|_{L^{\infty}[T-\varepsilon, T]}, \end{split}$$

which converges to zero as $\varepsilon \to 0$ since the trajectory $\langle u(t), \psi_T(x + (T - t)\nu, \nu) \rangle$ is bounded near t = T. Thus $\lim_{\varepsilon \to 0} \alpha_2(\varepsilon) = 0$.

Taking the limit $\varepsilon \to 0$ in inequality (A.10), we conclude that

$$\langle \psi_T, u_T \rangle \leq \langle \psi_T, \beta_T \rangle.$$

Since $\beta_T - u_T$ is a positive linear functional on $C_c^1(\mathbb{M} \times \mathbb{R}^d)$, it is bounded, and therefore u_T is also a bounded linear functional on $C_c^1(\mathbb{M} \times \mathbb{R}^d)$.

Corollary A.8. If u satisfies (A.8) with $\beta_T \in M(\mathbb{M} \times \mathbb{R}^d)$ then, in the notation of equation (A.3), the measure ν extends to a finite Radon measure on $(0, T] \times \mathbb{M} \times \mathbb{R}^d$ given by $\nu(A) = \nu(A \mathbb{1}_{(0,T)})$.

Proof. We show that, for any non-negative test function $\phi \in C_c^1((0, T] \times \mathbb{M} \times \mathbb{R}^d)$,

$$\langle \nu, \phi \mathbb{1}_{(0,T)} \rangle < +\infty. \tag{A.11}$$

It suffices to prove (A.11) for test functions of the form $\phi(t, x, v) = \theta(t)T_t\psi(x, v)$, where $\theta \in C_c^1(0, T]$ and $\psi \in C_c^1(\mathbb{M} \times \mathbb{R}^d)$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\phi,u(t)\rangle=\theta'(t)\langle\mathcal{T}_t\psi,u\rangle+\langle\phi,H(x,\nu,D_\nu u)+\nu-\beta\rangle\geq\theta'(t)\langle\mathcal{T}_t\psi,u\rangle+\langle\phi,\nu-C_H-\beta_+\rangle.$$

It follows that (once again using the right continuous version of u), for t < T,

$$\langle \nu, \phi \mathbb{1}_{(0,t]} \rangle \leq \langle \phi, u(t) \rangle - \int_0^t \int_{\mathbb{M} \times \mathbb{R}^d} \theta'(s) \langle \mathcal{T}_s \psi, u \rangle + \phi(C_H + \beta_+) \, \mathrm{d}x \, \mathrm{d}\nu \, \mathrm{d}s$$

Taking the limit $t \to T$, by definition of u_T ,

$$\langle
u, \phi \mathbb{1}_{(0,T)}
angle \leq \langle \phi, u_T
angle - \int_0^T \int_{\mathbb{M} imes \mathbb{R}^d} heta'(s) \langle \mathcal{T}_s \psi, u
angle + \phi(C_H + eta_+) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}s < +\infty.$$

We now discuss the trace of u at t=0: u_0 as defined in Definition A.7. $\langle \psi, u_0 \rangle$ is defined for all $\psi \in C_c^1(\mathbb{M} \times \mathbb{R}^d)$. Our aim is to give a meaning to the quantity $\langle m_0, u_0 \rangle$, which appears in the definition of the functional $\tilde{\mathcal{A}}$. In the case where $m_0 \in C_c^1(\mathbb{M} \times \mathbb{R}^d)$ this is straightforward, noting that we allow the possible value $-\infty$. We now consider the more general case where $m_0 \in C(\mathbb{M} \times \mathbb{R}^d)$.

Lemma A.9. Assume that, for all $\phi \in C_c^1(\{m_0 > 0\})$, $\langle \phi, u_0 \rangle \neq -\infty$. Then u_0 is represented by a Radon measure on $\{m_0 > 0\}$. Furthermore, the positive part $(u_0)_+$ has the property that

$$\int_{\{m_0>0\}} m_0 \, \mathrm{d}(u_0)_+(x,v) < +\infty.$$

Proof. Let $\phi \in C^1_c(\mathbb{M} \times \mathbb{R}^d)$ be non-negative. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathcal{T}_t\phi,\tilde{u}(t)\rangle\geq-\langle \mathcal{T}_t\phi,C_H+\beta_+\rangle,$$

we have

$$\langle \phi, u_0 \rangle \leq \int_0^T \langle \mathcal{T}_t \phi, C_H + \beta_+ \rangle \, \mathrm{d}t + \langle \mathcal{T}_T \phi, \beta_T \rangle =: S\phi$$

The right hand side is linear in ϕ and satisfies

$$\left|\int_0^T \langle \mathcal{T}_t \phi, C_H + \beta_+ \rangle \, \mathrm{d}t + \langle \mathcal{T}_T \phi, \beta_T \rangle \right| \le \|\phi\|_{L^\infty} \Big(\|C_H + \beta_+\|_{L^1(K_{[0,T]})} + \|\beta_T\|_{L^1(K_T)} \Big);$$

here K_T denotes the set

$$K_T := \{ (x + vT, v) : (x, v) \in K \},\$$

where K is the support of ϕ , and $K_{[0,T]}$ is the set

$$K_{[0,T]} := \{ (x + vt, v) : (x, v) \in K, t \in [0,T] \}$$

Thus S defines a bounded linear functional on $C_c(\mathbb{M} \times \mathbb{R}^d)$. In particular it is a distribution; moreover, it is represented by a signed Radon measure.

Observe next that $S - u_0$ is a positive linear functional on $C_c^1(\{m_0 > 0\})$, and thus bounded and a distribution. By positivity it is given by a Radon measure ν_0 on $\{m_0 > 0\}$. We deduce that

$$u_0 = S - \nu_0$$

That is, u_0 is a signed Radon measure.

Moreover, from the definition of the Hahn-Jordan decomposition we have the following estimate for the positive part:

$$\langle \phi, (u_0)_+ \rangle \leq \int_0^T \langle \mathcal{T}_t \phi, C_H + \beta_+ \rangle \, \mathrm{d}t + \langle \mathcal{T}_T \phi, (\beta_T)_+ \rangle.$$
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Let $\phi_n \in C_c(\mathbb{M} \times \mathbb{R}^d)$ be an increasing sequence of functions such that ϕ_n converges to m_0 as $n \to \infty$. Since

$$\sup_{n} \langle \phi_{n}, (u_{0})_{+}
angle \leq \int_{0}^{T} \langle {\mathcal{T}}_{t} m_{0}, C_{H} + eta_{+}
angle \, \mathrm{d}t + \langle {\mathcal{T}}_{T} m_{0}, (eta_{T})_{+}
angle \leq C_{H} T \| m_{0} \|_{L^{2}} + T^{1/q} \| eta_{+} \|_{L^{q'}} \| m_{0} \|_{L^{q}} + \| (eta_{T})_{+} \|_{L^{q'}} \| m_{0} \|_{L^{s}},$$

we conclude that $\langle m_0, (u_0)_+ \rangle$ is finite.

Based on the previous lemma, we make the following definition.

Definition A.10. We define $-\langle m_0, u_0 \rangle$ as follows:

(1) If there exists $\phi \in C_c^1(\{m_0 > 0\})$ such that $\langle \phi, u_0 \rangle = -\infty$, then we define

$$-\langle m_0, u_0 \rangle = +\infty$$

(2) Otherwise, let $\phi_n \in C_c(\{m_0 > 0\})$ be an increasing sequence of functions such that ϕ_n converges to m_0 as $n \to \infty$, and define

$$-\langle m_0, u_0 \rangle = -\lim_{n \to \infty} \langle \phi_n, u_0 \rangle$$

This is well-defined (allowing for the possible value $+\infty$) by Lemma A.9.

Lemma A.11. Suppose that the assumptions of Lemma A.6 hold and suppose in addition that

$$\int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} u[\partial_{t}\phi + \operatorname{div}_{x}(v\phi)] + \phi H(x, v, D_{v}u) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \leq \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \beta\phi \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t + \int_{\mathbb{M}\times\mathbb{R}^{d}} \beta_{T}\phi_{T} \, \mathrm{d}x \, \mathrm{d}v - \int_{\mathbb{M}\times\mathbb{R}^{d}} \beta_{0}\phi_{0} \, \mathrm{d}x \, \mathrm{d}v,$$
(A.12)

holds for all $\phi \in C_c^1(\mathcal{U}_{m_0})$, where $\beta_0 \in \mathcal{M}(\{m_0 > 0\})$ is also given. Then for the trace u_0 of the right continuous version of u we have

$$\beta_0 \leq u_0$$
, in $\mathscr{D}'(\{m_0 > 0\})$

and in particular $\langle u_0, \psi \rangle \neq -\infty$ for any $\psi \in C^1_c(\{m_0 > 0\})$.

If in addition we suppose that β_0 is such that $\langle \beta_0, m_0 \rangle$ is meaningful and finite, then $\langle u_0, m_0 \rangle$ is finite and

$$\langle \beta_0, m_0 \rangle \leq \langle u_0, m_0 \rangle.$$

Proof. The proof of this result follows the same lines as the proof of Lemma A.6, so we point out only the main differences. Let *u* stand for the right continuous representative constructed in Lemma A.1. Fix $\psi_0 \in C_c^1(\{m_0 > 0\})$ non-negative. For each $\varepsilon > 0$ small consider a smooth, non-negative test function $\eta_{\varepsilon} : [0, T] \to \mathbb{R}$, chosen such that $\eta(t) = 0$ for all $\varepsilon \le t \le T$ and the derivative η'_{ε} satisfies

$$-arepsilon^{-1} \leq \eta_arepsilon^\prime \leq 0, \qquad \eta_arepsilon^\prime(t) = egin{cases} -arepsilon^{-1} & t\in [arepsilon^2,arepsilon-arepsilon^2], \ 0 & t\in [arepsilon,T]. \end{cases}$$

Note that as a consequence of the fundamental theorem of calculus, one has $\lim_{\varepsilon \to 0} \eta_{\varepsilon}(0) = 1$. We define the following non-negative test function $\phi_{\varepsilon} \in C_c^1(\mathcal{U}_{m_0})$:

$$\phi_{\varepsilon}(t, x, v) = \eta_{\varepsilon}(t)\psi_{T}(x - tv, v)$$

Substituting this choice of ϕ into (A.12), we obtain

$$\int_0^\varepsilon \eta_\varepsilon'(t) \int_{\mathbb{M}\times\mathbb{R}^d} u(t) \psi_T(x-t\nu,\nu) \, \mathrm{d}x \, \mathrm{d}\nu \, \mathrm{d}t \leq -\eta_\varepsilon(0) \langle \beta_0,\psi_0\rangle + \alpha_1(\varepsilon).$$

where

$$\alpha_1(\varepsilon) = \int_0^\varepsilon \int_{\mathbb{M}\times\mathbb{R}^d} \beta \phi_\varepsilon \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t - \int_0^\varepsilon \int_{\mathbb{M}\times\mathbb{R}^d} \phi_\varepsilon H(x, v, D_v u) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t.$$

As before, we note that $\lim_{\epsilon \to 0} \alpha_1(\epsilon) = 0$. We exclude the possibility that $\langle u_0, \psi_0 \rangle = -\infty$. For this, we rewrite the previous inequality as

$$\eta_{\varepsilon}(0)\langle\beta_{0},\psi_{0}\rangle-\alpha_{1}(\varepsilon)\leq-\int_{0}^{\varepsilon}\eta_{\varepsilon}'(t)\int_{\mathbb{M}\times\mathbb{R}^{d}}u(t)\psi_{T}(x-t\nu,\nu)\,\mathrm{d}x\,\mathrm{d}\nu\,\mathrm{d}t,$$

and use the same arguments as when proving $\langle u_T, \psi_T \rangle \neq +\infty$ in the proof of Lemma A.6.

Therefore, u_0 defines a linear map on $C_c^1(\{m_0 > 0\})$. Having this, we can show the inequality $\langle \beta_0, \psi_0 \rangle \leq \langle u_0, \psi_0 \rangle$ in the same way as corresponding inequality in Lemma A.6.

Now, using the Definition A.10, $\langle u_0, m_0 \rangle$ is meaningful, having also the possibility that it is $-\infty$. However, if the additional assumption that $\langle \beta_0, m_0 \rangle$ is finite takes place, taking a an increasing sequence of test functions, we find that $\langle \beta_0, m_0 \rangle \leq \langle u_0, m_0 \rangle$, so clearly, the latter term cannot be $-\infty$.

Through similar arguments it is possible to justify the existence of weak time traces for competitors m in Problem 3.3, thereby giving meaning to the initial value problem

$$\begin{cases} \partial_t m + v \cdot D_x m + \operatorname{div}_v w = 0, \text{ in } \mathscr{D}'((0, T) \times \mathbb{M} \times \mathbb{R}^d) \\ m|_{t=0} = m_0. \end{cases}$$

Recall that in Remark 3.4 we established that, in the cases of interest to us, there exists a function $V \in L''(m \operatorname{dx} \operatorname{dv} \operatorname{dt})$ such that w = Vm, and so we may assume that *m* is a distributional solution of the following equation:

$$\partial_t m + \operatorname{div}_x(vm) + \operatorname{div}_v(Vm) = 0,$$

with $|V|m \in L^1([0, T] \times \mathbb{M} \times \mathbb{R}^d)$.

This setting is much more standard since here the time derivatives $\frac{d}{dt}\langle\phi,m_t\rangle$ will be in $L^1[0,T]$ for any $\phi \in C_c^1(\mathbb{M} \times \mathbb{R}^d)$ rather than measures, that is, we expect absolutely continuous rather than right continuous trajectories. Moreover we can work on the whole space $[0,T] \times \mathbb{M} \times \mathbb{R}^d$ rather than only the reachable set \mathcal{U}_{mo} .

Deducing that m has a narrowly continuous representative is essentially an application of [56, Lemma 8.1.2]. However, since we do not necessarily have

$$\int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} |v| m \,\mathrm{d} x \,\mathrm{d} v \,\mathrm{d} t < +\infty,$$

due to the unbounded drift v, we cannot immediately apply this lemma. Below we briefly sketch the adaptation to our case.

Lemma A.12. (See [56, Lemma 8.1.2]). Let $0 \le m \in (L^1 \cap L^q)([0, T] \times \mathbb{M} \times \mathbb{R}^d)$ be a non-negative function satisfying

$$\partial_t m + \operatorname{div}_x(vm) + \operatorname{div}_v(Vm) = 0$$

in the sense of distributions on $(0,T) \times \mathbb{M} \times \mathbb{R}^d$, where V is given, such that $|V|m \in L^1([0,T] \times \mathbb{M} \times \mathbb{R}^d)$.

Then there exists a continuous curve $\tilde{m}_{\bullet} : [0, T] \to (C_c^1(\mathbb{M} \times \mathbb{R}^d))'$ such that $\tilde{m}_t = m_t$ for almost all $t \in [0, T]$. Thus \tilde{m}_0 is well-defined as an element of $(C_c^1(\mathbb{M} \times \mathbb{R}^d))'$ (or in fact, by positivity, a Radon measure).

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Furthermore, if \tilde{m}_0 is a probability measure, then \tilde{m}_{\bullet} extends uniquely to a narrowly continuous curve in the space of probability measures, i.e. $\tilde{m}_{\bullet} \in C([0, T]; \mathcal{P}(\mathbb{M} \times \mathbb{R}^d))$.

Proof. Since $m, |V|m \in L^1([0, T] \times \mathbb{M} \times \mathbb{R}^d)$, for any compact set $K \subset \mathbb{M} \times \mathbb{R}^d$ we have

$$\int_0^1 \int_K (|v|+|V|) m \,\mathrm{d}x \,\mathrm{d}v \,\mathrm{d}t < +\infty.$$

It follows that, as in the proof of [56, Lemma 8.1.2], we may select a dense subset $\{\phi_n\}_{n\in\mathbb{N}}$ of $C_c^1(\mathbb{M}\times\mathbb{R}^d)$ and take a version \tilde{m}_t of m_t such that $t\mapsto\langle\phi_n,\tilde{m}_t\rangle$ is continuous with respect to t for all n and $\tilde{m}_t = m_t$ for almost all $t \in [0, T]$, and a define a unique weak-* continuous extension of \tilde{m}_{\bullet} to $(C_c^1(\mathbb{M}\times\mathbb{R}^d))'$. Thus \tilde{m}_0 is well-defined as the unique element of $(C_c^1(\mathbb{M}\times\mathbb{R}^d))'$ satisfying

$$\langle \phi, ilde{m}_0
angle := \lim_{t o 0} \langle \phi, ilde{m}_t
angle \quad ext{for all } \phi \in C^1_c(\mathbb{M} imes \mathbb{R}^d).$$

Moreover, since \tilde{m} is non-negative, in fact \tilde{m}_t is a Radon measure on $\mathbb{M} \times \mathbb{R}^d$ for all $t \in [0, T]$.

Furthermore, it follows from the continuity of \tilde{m}_{\bullet} in the weak-* sense of $(C_c^1(\mathbb{M} \times \mathbb{R}^d))'$, and the fact that \tilde{m} is locally finite, that, for any (N.B. now time-dependent) $\phi \in C_c^1([0, T] \times \mathbb{M} \times \mathbb{R}^d)$, the path $t \mapsto \langle \phi(t, \cdot), \tilde{m}_t \rangle$ is continuous. Thus we may also use the final argument from [56, Lemma 8.1.2] (similar to our argument for the time traces at the boundary in Lemma A.6) to prove the following equality (c.f. [56, Equation (8.1.4)]): for any $\phi \in C_c^1([0, T] \times \mathbb{M} \times \mathbb{R}^d)$ and any $0 \le t_1 \le t_2 \le T$,

$$\langle \phi(t_2, \cdot), \tilde{m}_{t_2} \rangle - \langle \phi(t_1, \cdot), \tilde{m}_{t_1} \rangle = \int_{t_1}^{t_2} \langle \partial_t \phi + \nu \cdot D_x \phi + V \cdot D_\nu \phi, \tilde{m}_t \rangle \, \mathrm{d}t. \tag{A.13}$$

Next we wish to show that, if \tilde{m}_0 is a probability measure, then \tilde{m}_t is a probability measure for all $t \in [0, T]$. If this is the case, then we may apply [56, Remark 5.1.6]—if $\tilde{m}_{t_n} \rightarrow \tilde{m}_t$ in the sense of distributions as *n* tends to infinity, then this convergence also holds in the narrow sense—to deduce that \tilde{m}_t is a narrowly continuous path in the space of probability measures, as desired.

To do this we use the argument of Lemma A.4 to avoid the need for *vm* to be integrable. First, fix a sequence $\zeta_R \in C_c^{\infty}(\mathbb{M} \times \mathbb{R}^d)$ of smooth, compactly supported functions, approximating the constant function 1 in a monotone limit as *R* tends to infinity—that is, let ζ_R satisfy the assumptions given in Equation (5.2). We note in particular that $\|D\zeta_R\|_{L^{\infty}} \leq C/R$ for some constant C > 0 independent of *R*. Then, for each t_* , consider the test function $\mathcal{T}_{t-t_*}\zeta_R$ (recall \mathcal{T} from Definition A.2), which satisfies

$$(\partial_t + v \cdot D_x) \mathcal{T}_{t-t_*} \zeta_R = 0, \quad \mathcal{T}_{t-t_*} \zeta_R|_{t=t_*} = \zeta_R.$$

Using (A.13) with $t_1 = 0$, $t_2 = t_*$ and $\phi = \mathcal{T}_{t-t_*}\zeta_R$, we find that

$$\langle \zeta_R, \tilde{m}_{t_*} \rangle = \langle \mathcal{T}_{-t_*} \zeta_R, \tilde{m}_0 \rangle + \int_0^{t_*} \langle D_\nu(\mathcal{T}_{t-t_*} \zeta_R), V \tilde{m}_t \rangle \,\mathrm{d}t.$$

We observe that, for all $t \in [0, t_*]$,

$$|D_{\nu}(\mathcal{T}_{t-t_{*}}\zeta_{R})| = |[(t_{*}-t)D_{x}+D_{\nu}]\zeta_{R}| \le (1+t_{*})\|D\zeta_{R}\|_{L^{\infty}} \le \frac{C(1+t_{*})}{R}$$

Thus (since $\tilde{m}_t = m_t$ for almost all t)

$$\left|\int_0^{t_*} \langle D_{\nu}(\mathcal{T}_{t-t_*}\zeta_R), V\tilde{m}_t \rangle \, \mathrm{d}t \right| \leq \frac{C(1+t_*)}{R} \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} |V| m_t \, \mathrm{d}x \, \mathrm{d}\nu \, \mathrm{d}t.$$

The right hand side tends to zero as R tends to infinity, since $|V|m \in L^1([0, T] \times \mathbb{M} \times \mathbb{R}^d)$. Moreover, since \tilde{m}_0 is a probability measure and $\mathcal{T}_{-t_*}\zeta_R$ increases monotonically to 1 pointwise as R tends to infinity, it follows that

$$\lim_{R\to 0} \langle \mathcal{T}_{-t_*}\zeta_R, \tilde{m}_0 \rangle = 1.$$

Finally, since \tilde{m}_{t_*} is a Radon measure and $\lim_{R\to\infty} \langle \zeta_R, \tilde{m}_{t_*} \rangle$ is a monotone limit,

$$\tilde{m}_{t_*}(\mathbb{M}\times\mathbb{R}^d) = \lim_{R\to\infty} \langle \zeta_R, \tilde{m}_{t_*} \rangle = 1.$$

That is, \tilde{m}_{t_*} is a probability measure for all $t_* \in [0, T]$. This completes the proof.

Appendix B. Truncations and maxima

Given a distributional solution to the Hamilton-Jacobi inequality

$$\begin{aligned} &-\partial_t u - v \cdot D_x u + H(x, v, D_v u) \le \beta, \quad \text{in } \mathscr{D}'((0, T) \times \mathbb{M} \times \mathbb{R}^d) \\ &u_T \le \beta_T, \quad \text{in } \mathscr{D}'(\mathbb{M} \times \mathbb{R}^d) \end{aligned} \tag{B.1}$$

in the sense of Definition 3.9, we show that the truncations of u from below, that is, the functions max $\{u, l\}$ for some l < 0, satisfy a similar inequality. In a similar vein, we also show that given u^1 and u^2 both satisfying (B.1), their maximum satisfies the same inequality (B.1).

Lemma B.1. Let $u \in L^1_{loc}((0, T) \times \mathbb{M} \times \mathbb{R}^d)$ satisfy (B.1) in the sense of distributions. Assume that $\beta \in L^1_{loc}((0, T) \times \mathbb{M} \times \mathbb{R}^d)$ and $D_v u \in L^r_{loc}((0, T) \times \mathbb{M} \times \mathbb{R}^d)$. Then $u_l := (u - l)_+$ satisfies

$$\begin{aligned} &-\partial_t u_l - v \cdot D_x u_l + H(x, v, D_v u_l) \mathbb{1}_{\{u > l\}} \le \beta \mathbb{1}_{\{u > l\}}, \quad \text{in } \mathscr{D}'((0, T) \times \mathbb{M} \times \mathbb{R}^d) \\ &\left[(u - l)_+ \right]_T \le (\beta_T - l)_+ \quad \text{in } \mathscr{D}'(\mathbb{M} \times \mathbb{R}^d). \end{aligned}$$

A similar result holds for the truncation $u \lor l = (u - l)_+ + l$. Moreover it suffices to consider the case l = 0.

Lemma B.2. Let u_1 , u_2 satisfy Assumption 3 and (B.1). Then $u = \max\{u_1, u_2\}$ also satisfies (B.1).

The result here is in the spirit of renormalization [60]. Bouchut [61, Theorem 1.1] proved a chain rule for the kinetic transport operator, i.e. the identity

$$(\partial_t + v \cdot D_x)h(u) = h'(u)(\partial_t + v \cdot D_x)u$$
(B.2)

that applies when *h* is a Lipschitz function and $\partial_t u + v \cdot D_x u \in L^1_{loc}$. However, since in our case $\partial_t u + v \cdot D_x u$ may only be a measure, we are not able to use this result directly, or indeed prove a chain rule with equality as in Equation (B.2). Nevertheless, the ideas of the proofs in [24, 61] can be used to obtain the inequality that is sufficient for our case.

The argument proceeds in several steps.

B.1. Extension

We define the following time shift and extension of *u* on the time interval $(-2\eta, T + 2\eta)$ for $\eta > 0$:

$$\tilde{u}(t, x, v) = \begin{cases} u(t, x, v), & t \in (0, T) \\ 0, & t \in [T, T + 2\eta]. \end{cases}$$

Then, defining

$$\tilde{\beta}(t, x, v) = \begin{cases} \beta(t, x, v), & t \in (0, T) \\ H(x, v, 0), & t \in [T, T + 2\eta], \end{cases}$$

 \tilde{u} satisfies the following inequality in the sense of distributions on $(0, T + 2\eta) \times \mathbb{M} \times \mathbb{R}^d$ (see the similar construction in [21, Section 6.3]):

$$-\partial_t \tilde{u} - v \cdot D_x \tilde{u} + H(x, v, D_v \tilde{u}) \le \tilde{\beta} + \beta_T \ \partial_t \mathbf{1}_{[T, T+2\eta]}, \quad \text{in } \mathscr{D}'((0, T+2\eta) \times \mathbb{M} \times \mathbb{R}^d).$$

B.2. Fenchel's inequality

Since L is the Fenchel conjugate of H, for any continuously differentiable vector field $a \in C_b^1$ we have

$$-\partial_t \tilde{u} - v \cdot D_x \tilde{u} - a \cdot D_v \tilde{u} \le L(x, v, -a) + \beta + \beta_T \ \partial_t \mathbf{1}_{[T, T+2\eta]}, \quad \text{in } \mathscr{D}'((0, T+2\eta) \times \mathbb{M} \times \mathbb{R}^d).$$

B.3. Regularisation

Fix non-negative, symmetric, unit mass mollifiers $\chi, \psi \in C_c^{\infty}(\mathbb{R}^d)$ and $\theta \in C_c^{\infty}(\mathbb{R})$. Assume that θ is supported on the set [-1, 1]. Then define, for $\varepsilon, \delta, \eta > 0$,

$$\chi_{\varepsilon}(v) := \varepsilon^{-d} \chi\left(\frac{v}{\varepsilon}\right), \quad \psi_{\delta}(x) := \delta^{-d} \psi\left(\frac{x}{\delta}\right), \quad \theta_{\eta}(t) := \eta^{-1} \theta\left(\frac{t}{\eta}\right).$$

Then define the full mollifier φ by

$$\varphi(t, x, v) = \theta_{\eta}(t)\psi_{\delta}(x)\chi_{\varepsilon}(v).$$

Then the regularization $u_{\eta, \varepsilon, \delta} := \tilde{u} * \varphi$ satisfies the following inequality, in a pointwise sense on $(\eta, T + \eta] \times \mathbb{M} \times \mathbb{R}^d$:

$$-(\partial_t + v \cdot D_x + a \cdot D_v)u_{\eta,\varepsilon,\delta} \leq L_{\eta,\varepsilon,\delta} + \beta_{\eta,\varepsilon,\delta} + \varphi(t-T,\cdot)*_{x,v}\beta_T + \mathcal{E}_{\eta,\varepsilon,\delta},$$

where $\beta_{\eta,\varepsilon,\delta} := \tilde{\beta} * \varphi$ and $L_{\eta,\varepsilon,\delta} := L(x, v, -a) * \varphi$, while $\mathcal{E}_{\eta,\varepsilon,\delta}$ denotes the commutator

$$\mathcal{E}_{\eta,\varepsilon,\delta} := \theta_{\eta} *_t [\chi_{\varepsilon} *_v \psi_{\delta} *_x (v \cdot D_x + a \cdot D_v) u - (v \cdot D_x + a \cdot D_v) \chi_{\varepsilon} *_v \psi_{\delta} *_x u].$$

With the choice $\varepsilon = \delta^2$ this error converges to zero in L^1_{loc} by Lemma 5.3 and [60].

B.4. The maximum function

We fix a smooth approximation of the functions $x_+ = \max\{x, 0\}$ and $\max\{x_1, x_2\}$. First, for each $\alpha > 0$ we fix a smooth function $\gamma_{\alpha}(x)$ approximating x_+ in such a way that $0 \le \gamma_{\alpha}(x) \le x_+$ and $0 \le \gamma'_{\alpha}(x) \le 1$ for all $\alpha > 0$ and $\lim_{\alpha \to 0} \gamma_{\alpha}(x) = x_+$ for all x and $\lim_{\alpha \to 0} \gamma'_{\alpha}(x) = 1$ for x > 0. Note in particular that then $\gamma'_{\alpha}(x) = 0$ for all $\alpha > 0$ and $x \le 0$, so that γ'_{α} converges pointwise to the function $\mathbb{1}_{\{x>0\}}$ as α tends to zero.

We similarly define an approximation h_{α} of the maximum function by

$$h_{\alpha}(x_1, x_2) := x_2 + \gamma_{\alpha}(x_1 - x_2).$$

Observe that h_{α} satisfies $0 \leq \partial_{x_i} h_{\alpha} \leq 1$ for i = 1, 2, and

$$\partial_{x_1}h_{\alpha}(x_1,x_2) + \partial_{x_2}h_{\alpha}(x_1,x_2) = 1.$$

B.5. Inequality for the truncations and maximum

Since γ'_{α} is non-negative,

$$-(\partial_{t}+\nu\cdot D_{x}+a\cdot D_{\nu})u_{\eta,\varepsilon,\delta} \ \gamma_{\alpha}'(u_{\eta,\varepsilon,\delta}) \leq \left[\beta_{\eta,\varepsilon,\delta}+L_{\eta,\varepsilon,\delta}+\varphi(t-T,\cdot)*_{x,\nu}\beta_{T}+\mathcal{E}_{\eta,\varepsilon,\delta}\right]\gamma_{\alpha}'(u_{\eta,\varepsilon,\delta}).$$
(B.3)

Thus, since $\gamma_{\alpha} \in C^{1}$ and $u_{\eta,\varepsilon,\delta}$ is smooth in all variables, applying the usual chain rule

$$-(\partial_t + v \cdot D_x + a \cdot D_v)\gamma_{\alpha}(u_{\eta,\varepsilon,\delta}) \leq \left[\beta_{\eta,\varepsilon,\delta} + L_{\eta,\varepsilon,\delta} + \varphi(t-T,\cdot)*_{x,v}\beta_T + \mathcal{E}_{\eta,\varepsilon,\delta}\right]\gamma_{\alpha}'(u_{\eta,\varepsilon,\delta})$$

Similarly, given two subsolutions u^1 and u^2 ,

$$\begin{aligned} &-(\partial_t + v \cdot D_x + a \cdot D_v)h_{\alpha}(u^1_{\eta,\varepsilon,\delta}, u^2_{\eta,\varepsilon,\delta}) \\ &\leq \left[\beta_{\eta,\varepsilon,\delta} + L_{\eta,\varepsilon,\delta} + \varphi(t-T, \cdot)*_{x,v}\beta_T + \mathcal{E}^1_{\eta,\varepsilon,\delta}\right]\partial_1h_{\alpha}(u^1_{\eta,\varepsilon,\delta}, u^2_{\eta,\varepsilon,\delta}) \\ &+ \left[\beta_{\eta,\varepsilon,\delta} + L_{\eta,\varepsilon,\delta} + \varphi(t-T, \cdot)*_{x,v}\beta_T + \mathcal{E}^2_{\eta,\varepsilon,\delta}\right]\partial_2h_{\alpha}(u^1_{\eta,\varepsilon,\delta}, u^2_{\eta,\varepsilon,\delta}) \end{aligned}$$

Thus

$$\begin{aligned} &-(\partial_t + v \cdot D_x + a \cdot D_v)h_{\alpha}(u^1_{\eta, \varepsilon, \delta}, u^2_{\eta, \varepsilon, \delta}) \leq \beta_{\eta, \varepsilon, \delta} + L_{\eta, \varepsilon, \delta} + \varphi(t - T, \cdot) *_{x, v}\beta_T \\ &+ \mathcal{E}^1_{\eta, \varepsilon, \delta}\partial_1 h_{\alpha}(u^1_{\eta, \varepsilon, \delta}, u^2_{\eta, \varepsilon, \delta}) + \mathcal{E}^2_{\eta, \varepsilon, \delta}\partial_2 h_{\alpha}(u^1_{\eta, \varepsilon, \delta}, u^2_{\eta, \varepsilon, \delta}). \end{aligned}$$

B.6. Limits

T + m = -

We now take the limit as the smoothing parameters tend to zero in the previous inequalities. This procedure yields the proofs of the Lemmas B.1 and B.2. We continue to choose $\varepsilon = \delta^2$ to ensure convergence of the commutator. We detail the procedure in the case of the truncation (B.3); the case of the maximum function is similar.

Proof of Lemma B.1. We first test the inequality (B.3) with an arbitrary non-negative smooth function $\zeta \in C_c^{\infty}((0, T] \times \mathbb{M} \times \mathbb{R}^d)$. We fix an extension of ζ to a function $\zeta \in C_c^{\infty}((0, T+1] \times \mathbb{M} \times \mathbb{R}^d)$ and consider integrating over $[0, T+\eta] \times \mathbb{M} \times \mathbb{R}^d$. For all $\eta > 0$ small enough that the support of ζ is contained in $(\eta, T+1] \times \mathbb{M} \times \mathbb{R}^d$,

$$\begin{split} \int_{0}^{t+\eta} \int_{\mathbb{M}\times\mathbb{R}^{d}} &\gamma_{\alpha}(u_{\eta,\varepsilon,\delta})(\partial_{t}\zeta + v \cdot D_{x}\zeta + \operatorname{div}_{\nu}(a\zeta)) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \\ &\leq \int_{0}^{T+\eta} \int_{\mathbb{M}\times\mathbb{R}^{d}} \left[\beta_{\eta,\varepsilon,\delta} + L_{\eta,\varepsilon,\delta} + \mathcal{E}_{\eta,\varepsilon,\delta}\right] \gamma_{\alpha}'(u_{\eta,\varepsilon,\delta}) \zeta \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \\ &+ \int_{T-\eta}^{T+\eta} \int_{\mathbb{M}\times\mathbb{R}^{d}} \gamma_{\alpha}'(u_{\eta,\varepsilon,\delta}(t,x,v)) \theta_{\eta}(T-t) \chi_{\varepsilon} *_{\nu} \psi_{\delta} *_{x} \beta_{T} \zeta(t,x,v) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \end{split}$$

We have used that $u_{\eta,\varepsilon,\delta}(T+\eta, x, v) = 0$. Since $0 \le \gamma'_{\alpha} \le 1$, we may estimate the boundary term from above to obtain

$$\begin{split} \int_{0}^{T+\eta} \int_{\mathbb{M}\times\mathbb{R}^{d}} \gamma_{\alpha}(u_{\eta,\varepsilon,\delta})(\partial_{t}\zeta + v \cdot D_{x}\zeta + \operatorname{div}_{\nu}(a\zeta)) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \\ & \leq \int_{0}^{T+\eta} \int_{\mathbb{M}\times\mathbb{R}^{d}} \left[\beta_{\eta,\varepsilon,\delta} + L_{\eta,\varepsilon,\delta} + \mathcal{E}_{\eta,\varepsilon,\delta}\right] \gamma_{\alpha'}(u_{\eta,\varepsilon,\delta}) \zeta \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \\ & + \int_{\mathbb{M}\times\mathbb{R}^{d}} \left[\beta_{T}(x,v)\right]_{+} \varphi * \zeta(T,x,v) \, \mathrm{d}x \, \mathrm{d}v. \end{split}$$

Since $\tilde{u}, \tilde{\beta}, L(x, v, -a) \in L^1_{loc}$, $u_{\eta, \varepsilon, \delta}, \beta_{\eta, \varepsilon, \delta}, L_{\eta, \varepsilon, \delta}$ converge respectively to these strongly in L^1_{loc} by standard results on convolutions. We have already noted that $\mathcal{E}_{\eta, \varepsilon, \delta}$ converges to zero in L^1_{loc} . We therefore also obtain pointwise convergence along a subsequence. Similarly, $\varphi * \zeta$ converges to ζ pointwise since ζ is smooth. From this we obtain convergence of all terms, by continuity of γ_{α} and γ'_{α} and applying dominated convergence. Hence we obtain the following inequality:

$$\int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} \gamma_{\alpha}(u) (\partial_{t}\zeta + v \cdot D_{x}\zeta + \operatorname{div}_{\nu}(a\zeta)) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \leq \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} [\beta + L(x, v, -a)] \gamma_{\alpha'}(u) \zeta \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \\ + \int_{\mathbb{M}\times\mathbb{R}^{d}} [\beta_{T}(x, v)]_{+} \zeta(T, x, v) \, \mathrm{d}x \, \mathrm{d}v.$$

The convergences as $\alpha \to 0$ all follow by dominated convergence: for example, since $u \in L^1_{loc}((0,T] \times \mathbb{M} \times \mathbb{R}^d)$ and ζ has support contained in $(0,T] \times \mathbb{M} \times \mathbb{R}^d$, we have

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$$\gamma_{\alpha}(u)(\partial_{t}\zeta + v \cdot D_{x}\zeta + \operatorname{div}_{\nu}(a\zeta)) \leq |u||(\partial_{t} + v \cdot D_{x})\zeta| \in L^{1}.$$

A similar argument is used for the term involving β_T .

For the remaining term, use that $|\gamma'_{\alpha}| \leq 1$ (the bound being uniform in $\alpha > 0$), and both β and $H(x, v, D_v u)$ are in $L^1_{loc}((0, T] \times \mathbb{M} \times \mathbb{R}^d)$ by assumption. Then

$$\int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} u_+(\partial_t \zeta + v \cdot D_x \zeta + \operatorname{div}_{\nu}(a\zeta)) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \le \int_0^T \int_{\mathbb{M}\times\mathbb{R}^d} [\beta + L(x, v, -a)] 1_{\{u>0\}} \zeta \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t + \int_{\mathbb{M}\times\mathbb{R}^d} [\beta_T]_+ \zeta_T \, \mathrm{d}x \, \mathrm{d}v.$$

Finally, taking a sequence of vector fields *a* converging in $L^{r'}((0,T) \times \mathbb{M} \times \mathbb{R}^d)$ to $-D_pH(x,v,D_vu_+)$, we conclude that

$$\int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} u_{+}(\partial_{t}\zeta + v \cdot D_{x}\zeta) \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \leq \int_{0}^{T} \int_{\mathbb{M}\times\mathbb{R}^{d}} [\beta - H(x, v, D_{v}u_{+})] \mathbf{1}_{\{u>0\}} \zeta \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}t \\ + \int_{\mathbb{M}\times\mathbb{R}^{d}} [\beta_{T}]_{+} \zeta_{T} \, \mathrm{d}x \, \mathrm{d}v,$$

that is, the following holds in the sense of distributions:

$$\begin{aligned} &-\partial_t u_+ - \nu \cdot D_x u_+ + H(x,\nu,D_\nu u_+) \mathbf{1}_{\{u>0\}} \leq \beta \mathbb{1}_{\{u>0\}}, \quad \text{in } \mathscr{D}'((0,T) \times \mathbb{M} \times \mathbb{R}^d) \\ & [u_+]_T \leq (\beta_T)_+ \quad \text{in } \mathscr{D}'(\mathbb{M} \times \mathbb{R}^d). \end{aligned}$$

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