Vector boson pair production at one loop: analytic results for the process q̄qℓℓ̄ℓ'ℓ̄'g

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ABSTRACT: We present compact analytic results for the one-loop amplitude for the process $0 \to q\bar{q}\ell\bar{\ell}\ell'\bar{\ell}'g$, relevant for both the production of a pair of Z and W-bosons in association with a jet. We focus on the gauge-invariant contribution mediated by a loop of quarks. We explicitly include all effects of the loop-quark mass m, appropriate for the production of a pair of Z-bosons. In the limit $m \to 0$, our results are also applicable to the production of W-boson pairs, mediated by a loop of massless quarks. Implemented in a numerical code, the results are fast. The calculation uses novel advancements in spinor-helicity simplification techniques, for the first time applied beyond five-point massless kinematics. We make use of primary decompositions from algebraic-geometry, which now involve non-radical ideals, and p-adic numbers from number theory. We show how to infer whether numerator polynomials belong to symbolic powers of non-radical ideals through numerical evaluations.

KEYWORDS: Scattering Amplitudes, Electroweak Precision Physics, Higher-Order Perturbative Calculations

Contents

1 Introduction

In many respects the numerical calculation of one-loop amplitudes, both in the standard model and in proposed models beyond the standard model, is a solved problem. Following Passarino and Veltman [1] the problem is separated into the calculation of scalar one-loop integrals and the calculation of the coefficients with which these integrals appear in the particular amplitude at hand. The needed finite scalar integrals are provided in refs. [2, 3], whereas the needed singular integrals are provided in ref. [4]. Techniques based on numerical unitarity [5, 6] as well as methods based on iterative calculation of Feynman diagrams [7–10] have been automated into sophisticated tools, which give reliable numerical results for the coefficients of the scalar integrals.

On the other hand, analytic unitarity techniques have also matured so they can give analytic results following an automatic recipe [6, 11–14]. However, the resultant analytical expressions are often complicated so that the singularity structure of the amplitude is hard to divine and numerical evaluations are suboptimal. In this paper we push the analytic techniques a step further by obtaining simpler analytic expressions where the form of the answer, especially with regard to the singularity structure, is manifest. The possible benefits of utilizing the amplitude in such a form are:

- the simpler form may lead to faster numerical evaluation;

- the singularity structure is manifest, with physical poles of as low degree as possible;

- the consequent analytic form leads to improved stability of numerical evaluation;

- the numerical behaviour around singular points can be improved by analytic expansions, if necessary.

The issue of stable evaluation of the amplitude is particular pressing for the case of vector boson pair production. Cuts on the transverse momenta of the decay products of the vector bosons do not exclude the region where the vector sum of the transverse momenta of all the uncoloured particles in the final state is equal to zero. In this kinematic region the amplitudes contain soft and collinear singularities. This issue is especially important in the context of next-to-next-to-leading order calculations since cancellations between real and virtual diagrams occur in the region of zero transverse momentum.

A compact representation of scattering amplitudes is provided in principle by the spinor-helicity formalism [15–17]. These are especially convenient for the case at hand because the factors associated with vector boson decays are simple and the whole family of diboson processes may be described by appropriate dressings of a core set of amplitudes [18]. However, since the spinor products are not all independent, spinor product expressions are not straightforward to simplify. In particular, the application of momentum conservation and Schouten identities leads to many equivalent representations of the same amplitude. The application of systematic reduction techniques (such as momentum twistors or Gröbner basis reduction) does not necessarily result in simpler expressions.

A number of strategies are available to facilitate simplification. The method of momentum twistors [19, 20] allows one to write spinor expressions contributing to an n-point

amplitude in a unique form in terms of $3n - 10$ independent variables. However it is cumbersome to revert to simple spinor expressions as the number of external legs grows. Alternatively, simpler expressions may be obtained by reconstructing multivariate polynomials and rational functions from their evaluation over finite fields [21–25].

Our analytic results will be simplified using large-precision floating point arithmetic and fitting in singular limits [26], as well as using p-adic numbers and technology from algebraic-geometry [27]. The expressions we provide are explicitly rational, i.e. no square roots are present, and contain poles of the lowest degree possible.

1.1 Motivation

Our motivation for this paper is twofold. First, we want to investigate and extend the practical limits on the numerical simplification techniques, alluded to above and to be explained in more detail in section 3. In particular, it is interesting to investigate their feasibility in high multiplicity settings, where new poles appear in the master-integral coefficients. For this purpose, we choose a subset of the diagrams contributing to the one-loop 7-point process $q\bar{q}\ell\bar{\ell}\ell'\bar{\ell}'g$, namely the diagrams including a quark loop. Second, vector boson pair production is an important process, to which these amplitudes contribute. For example, the ZZ final state is one of the decay channels of the Higgs boson, and onshell and offshell calculations are of great interest. The amplitudes we consider contribute to,

$$
\begin{array}{c}
q + \bar{q} \to Z/\gamma^* + Z/\gamma^* + g \\
\Big\downarrow \qquad\quad \Big\downarrow_{\textstyle \to\, \ell'\,\bar{\ell}'} \\
\Big\downarrow_{\textstyle \to\, \ell\,\bar{\ell}}
\end{array}
\tag{1.1}
$$

and to the related processes obtained by crossing the coloured partons. This amplitude receives contributions both at tree level and at one loop. In this paper we report on the one-loop amplitude mediated by quarks with a common mass, m. In the limit $m \to 0$ our results can also be used for the massless quark-loop contributions to the process,

$$
\begin{array}{c}
q + \bar{q} \to W^- + W^+ + g \;. \\
\Big\downarrow \qquad\quad \Big\downarrow_{\textstyle \to\, \nu'\bar{\ell}'} \\
\Big\downarrow_{\textstyle \to\, \ell\,\bar{\nu}}
\end{array}
\tag{1.2}
$$

We note that results for this process, in the limit of massless quarks circulating in the loop, have been calculated in analytic form previously [28], but the resulting amplitudes were only distributed in the form of computer code since they were not sufficiently compact to present otherwise.

In this paper we will present simpler formulae for all the ingredients necessary to assemble the quark-loop amplitude for the process in eq. (1.1). In addition, we have added the simplified results to the code MCFM [29–31], which we will demonstrate leads to improvements in speed.

1.2 Plan of this paper

In section 2 we review the spinor notation and use it to present results for the lowest order amplitude. In section 3 we review and build upon the algebro-geometric methods of

ref. [27], with additional details worked out in appendix C. Section 4 (appendix A) presents the analytic results for the coefficients of box, triangle and bubble scalar integrals coming from quark-loop box (triangle diagrams) contributing to the process in eq. (1.1). Timing results from the numerical implementation of our calculation are described in section 5 and our conclusions are given in section 6.

2 Lowest order amplitude

2.1 Spinor notation

We begin this section by introducing the notation which we shall use to present the tree-graph results for the Z-pair production amplitude, as well as for the one-loop results to be presented in section 4. All results are presented using the standard notation for the kinematic invariants of the process,

$$s_{ab} = (p_a + p_b)^2, \qquad s_{abc} = (p_a + p_b + p_c)^2, \qquad s_{abcd} = (p_a + p_b + p_c + p_d)^2, \qquad (2.1)$$

and the Gram determinant,

$$\Delta_3(a, b, c, d) = (s_{abcd} - s_{ab} - s_{cd})^2 - 4s_{ab}s_{cd}. \qquad (2.2)$$

The Weyl spinor λ_A is a two dimensional complex vector and its complex conjugate is denoted as $\bar{\lambda}_{\dot{A}}$. The spinorial inner product between to Weyl spinors λ_1 and λ_2 is written as,

$$\lambda_{1\,A}\lambda_{2\,B}\varepsilon^{BA} = \lambda_{1\,A}\lambda_2^A = \langle 12 \rangle, \qquad (2.3)$$

$$\bar{\lambda}_{1\,\dot{A}}\bar{\lambda}_{2\,\dot{B}}\varepsilon^{\dot{B}\dot{A}} = \bar{\lambda}_{1\,\dot{A}}\bar{\lambda}_2^{\dot{A}} = [12], \qquad (2.4)$$

where ε is the totally antisymmetric tensor in two dimensions,

$$\varepsilon_{AB} = \varepsilon^{AB} = \varepsilon_{\dot{A}\dot{B}} = \varepsilon^{\dot{A}\dot{B}} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}. \qquad (2.5)$$

For a light-like momentum p, (i.e. $p^2 = (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 = 0$) we have that,

$$\lambda_{a\,A} \equiv \langle a| = \left(\sqrt{p_a^+}\exp(+i\varphi_{p_a}), \sqrt{p_a^-} \right), \quad \lambda_b^A \equiv |b\rangle = \begin{pmatrix} -\sqrt{p_b^-} \\ \sqrt{p_b^+}\exp(+i\varphi_{p_b}) \end{pmatrix}, \qquad (2.6)$$

where

$$e^{\pm i\varphi_p} \equiv \frac{p^1 \pm ip^2}{\sqrt{(p^1)^2 + (p^2)^2}} = \frac{p^1 \pm ip^2}{\sqrt{p^+ p^-}}, \qquad p^\pm = p^0 \pm p^3. \qquad (2.7)$$

For the conjugate spinor we have that,

$$\bar{\lambda}_{a\,\dot{A}} = [a| = \left(\sqrt{p_a^+}\exp(-i\varphi_{p_a}), \sqrt{p_a^-} \right), \quad \bar{\lambda}_b^{\dot{A}} = |b] = \begin{pmatrix} -\sqrt{p_b^-} \\ \sqrt{p_b^+}\exp(-i\varphi_{p_b}) \end{pmatrix}. \qquad (2.8)$$

The corresponding results for the spinor products are

$$\lambda_{a\,A}\lambda_b^A = \langle ab \rangle = \sqrt{p_a^- p_b^+}\exp(+i\varphi_{p_a}) - \sqrt{p_a^+ p_b^-}\exp(+i\varphi_{p_b}), \qquad (2.9)$$

$$\bar{\lambda}_{a\,\dot{A}}\bar{\lambda}_b^{\dot{A}} = [ab] = \sqrt{p_a^+ p_b^-}\exp(-i\varphi_{p_b}) - \sqrt{p_a^- p_b^+}\exp(-i\varphi_{p_a}). \qquad (2.10)$$

The spinor products are antisymmetric, $\langle ba \rangle = -\langle ab \rangle$, $[ba] = -[ab]$ and since the four-vectors are light-like, $s_{ab} = \langle ab \rangle [ba]$. The spinor products also satisfy the Schouten identity,

$$\langle ab \rangle \langle cd \rangle = \langle ad \rangle \langle cb \rangle + \langle ac \rangle \langle bd \rangle, \quad [ab][cd] = [ad][cb] + [ac][bd]. \quad (2.11)$$

In order to write the amplitude in a simple form we introduce some extra notation for more complicated spinor sandwiches. We have for light-like vectors $a \ldots h$,

$$\langle a | \Gamma_{bc} | d] = \langle ab \rangle [bd] + \langle ac \rangle [cd],$$
$$\langle a | \Gamma_{bc|de} | f \rangle = \langle a | \Gamma_{bc} | d] \langle df \rangle + \langle a | \Gamma_{bc} | e] \langle ef \rangle,$$
$$\langle a | \Gamma_{bc|de|fg} | h] = \langle a | \Gamma_{bc|de} | f \rangle [fh] + \langle a | \Gamma_{bc|de} | g \rangle [gh],$$
$$[a | \Gamma_{bc} | d \rangle = [ab] \langle bd \rangle + [ac] \langle cd \rangle,$$
$$[a | \Gamma_{bc|de} | f] = [a | \Gamma_{bc} | d \rangle [df] + [a | \Gamma_{bc} | e \rangle [ef],$$
$$[a | \Gamma_{bc|de|fg} | h \rangle = [a | \Gamma_{bc|de} | f] \langle fh \rangle + [a | \Gamma_{bc|de} | g] \langle gh \rangle. \quad (2.12)$$

We also define $\tilde{\Gamma}$ (note the tilde) to imply anti-symmetrization,

$$\langle a | \tilde{\Gamma}_{bc|de} | f \rangle = \langle a | \Gamma_{bc|de} | f \rangle - \langle a | \Gamma_{de|bc} | f \rangle,$$
$$[a | \tilde{\Gamma}_{bc|de} | f] = [a | \Gamma_{bc|de} | f] - [a | \Gamma_{de|bc} | f],$$
$$[a | \tilde{\Gamma}_{bc|de|fg} | h \rangle = [a | \Gamma_{bc|de|fg} | h \rangle - [a | \Gamma_{fg|de|bc} | h \rangle. \quad (2.13)$$

Lastly, we point out that eq. (2.12) and eq. (2.13) imply definitions for non-fully contracted spinor strings. These un-contracted spinor strings are useful to express generators with open indices for some ideals in section 3.3.

The basic amplitude which we calculate is the one for the process involving four leptons, two quarks and one gluon,

$$A_7(1_q^-, 2_{\bar{q}}^+, 3_\ell^-, 4_{\bar{\ell}}, 5_{\ell'}^-, 6_{\bar{\ell}'}^+, 7_g^+), \quad (2.14)$$

with all particles outgoing; the superscript denotes the helicity and the subscript shows the type of particle. The couplings required to reconstruct the physical amplitude will be given later.

2.2 Tree graphs

We will first setup the notation for the reduced amplitudes removing the colour matrix (for emission of a gluon with colour index B), and powers of the coupling constants,

$$\mathcal{A}_7^{\text{tree},B}(1,2,3,4,5,6,7) = 4ig_s e^4 (t^B)_{i_1 \bar{\imath}_2} A_7^{\text{tree}}(1,2,3,4,5,6,7). \quad (2.15)$$

The colour matrix t^B is normalized such that $\text{tr}(t^A t^B) = \delta^{AB}$. The indices i_1 and $\bar{\imath}_2$ denote the colours of the quark and anti-quark line, $i_1, \bar{\imath}_2 = \{1,2,3\}$. The strong and electromagnetic couplings are denoted by g_s and e. Written in this form $\mathcal{A}_7^{\text{tree},B}$ is exactly the amplitude where the production of both pairs of leptons (off a unit electric charge quark

line) is mediated by virtual photons. The appropriate coupling factors and propagators for Z boson production will be added below. The result for the reduced tree amplitude is,

$$A_7^{\text{tree}}(1^-, 2^+, 3^-, 4^+, 5^-, 6^+, 7^+) = -\frac{\langle 13 \rangle}{\langle 17 \rangle \, s_{34} \, s_{56} \, s_{134}} \tag{2.16}$$

$$\times \left[\frac{\langle 13 \rangle \, [34] \, [26] \, \langle 5|\Gamma_{26}|7]}{s_{256}} + \frac{\langle 5|\Gamma_{13}|4] \langle 1|\Gamma_{27}|6]}{\langle 72 \rangle} \right].$$

This amplitude was first presented in ref. [18] although our labelling of the lepton momenta, see eq. (2.14), differs from the notation in that paper. The remaining tree amplitudes are obtained by symmetry operations. These correspond to flipping the helicities of the leptons, e.g.

$$A_7^{\text{tree}}(1^-, 2^+, 3^+, 4^-, 5^-, 6^+, 7^+) = A_7^{\text{tree}}(1^-, 2^+, 4^-, 3^+, 5^-, 6^+, 7^+), \tag{2.17}$$

flipping the helicities of the quarks, e.g.,

$$A_7^{\text{tree}}(1^+, 2^-, 3^-, 4^+, 5^-, 6^+, 7^+) = A_7^{\text{tree}}(2^-, 1^+, 3^-, 4^+, 5^-, 6^+, 7^+), \tag{2.18}$$

and reversing the helicity of the gluon, e.g.,

$$A_7^{\text{tree}}(1^+, 2^-, 3^-, 4^+, 5^-, 6^+, 7^-) = -A_{\text{sr}}(1^-, 2^+, 4^-, 3^+, 6^-, 5^+, 7^+)\Big|_{\langle . \rangle \leftrightarrow [.]}. \tag{2.19}$$

2.2.1 Restoring the couplings

We define the left- and right-handed couplings of a Z boson to quarks and leptons by,

$$v_{L,q} = \frac{\tau_q - 2Q_q \sin^2 \theta_W}{2 \sin \theta_W \cos \theta_W}, \qquad v_{R,q} = -\frac{Q_q \sin \theta_W}{\cos \theta_W}, \tag{2.20}$$

$$v_{L,e} = \frac{-1 + 2 \sin^2 \theta_W}{2 \sin \theta_W \cos \theta_W}, \qquad v_{R,e} = \frac{\sin \theta_W}{\cos \theta_W}, \tag{2.21}$$

$$v_{L,n} = \frac{1}{2 \sin \theta_W \cos \theta_W}, \qquad v_{R,n} = 0, \tag{2.22}$$

where Q_q is the charge of the quark (in units of the positron charge) and $\tau_q = +1$ for up-type quarks and $\tau_q = -1$ for down-type quarks. The full tree amplitude for the ZZ case is then,

$$\mathcal{A}_7^{\text{tree},B}(1, 2, 3, 4, 5, 6, 7) = 4ie^4 g_s (t^B)_{i_1 \bar{i}_2}$$

$$\times (q_{34}Q_q + v_{34}v_q P(s_{34}, M_Z)) \, (q_{56}Q_q + v_{56}v_q P(s_{56}, M_Z))$$

$$\times \left(A_7^{\text{tree}}(1, 2, 3, 4, 5, 6, 7) + A_7^{\text{tree}}(1, 2, 5, 6, 3, 4, 7) \right), \tag{2.23}$$

where q_{34}, v_{34} (q_{56}, v_{56}) label the charge and coupling factors for the leptons appearing in the decay of $Z(p_{34})$ ($Z(p_{56})$) as given in eqs. (2.21) and (2.22). The Z-boson propagator factor is given by,

$$P(s, M) = \frac{s}{s - M^2}, \tag{2.24}$$

where M is the (complex) mass of the vector boson. Dressed this way, these amplitudes account for the effect of virtual photons as well as Z bosons.

For completeness, we also present the tree graph results for the contribution of singly-resonant diagrams $Z(p_{3456}) \to \ell_3 \bar{\ell}_4 Z(p_{56})$. The colour stripped amplitudes, reduced as in eq. (2.15), for this contribution are,

$$
A_7^{\text{sr}}(1^-, 2^+, 3^-, 4^+, 5^-, 6^+, 7^-) =
$$
$$
\left(\frac{\langle 35 \rangle [24] [2|\Gamma_{17|35}|6]}{s_{356}} + \frac{[64] \langle 3|\Gamma_{17}|2] \langle 5|\Gamma_{46}|2]}{s_{456}} \right) \frac{1}{[27][71]s_{3456}s_{56}}, \tag{2.25}
$$

and,

$$
A_7^{\text{sr}}(1^-, 2^+, 3^-, 4^+, 5^-, 6^+, 7^+) =
$$
$$
\left(\frac{\langle 13 \rangle [64] \left\langle 5|\Gamma_{46|27}|1\right\rangle}{s_{456}} + \frac{\langle 53 \rangle \langle 1|\Gamma_{35}|6] \langle 1|\Gamma_{27}|4]}{s_{356}} \right) \frac{1}{\langle 27 \rangle \langle 17 \rangle s_{3456}s_{56}}. \tag{2.26}
$$

The quark helicities can be flipped by, for example,

$$
A_7^{\text{sr}}(1^+, 2^-, 3^-, 4^+, 5^-, 6^+, 7^+) = -A_7^{\text{sr}}(2^-, 1^+, 3^-, 4^+, 5^-, 6^+, 7^+), \tag{2.27}
$$

where we note that there is an additional sign-flip compared to the corresponding relation for the double-resonant contribution in eq. (2.18). Amplitudes with lepton helicities $(5^+, 6^-)$ are obtained by the interchange $5 \leftrightarrow 6$. For lepton helicities $(3^+, 4^-)$ we have, for example,

$$
A_7^{\text{sr}}(1^+, 2^-, 3^+, 4^-, 5^+, 6^-, 7^+) = -A_7^{\text{sr}}(1^-, 2^+, 3^-, 4^+, 5^-, 6^+, 7^-)\Big|_{\langle . \rangle \leftrightarrow [.]}. \tag{2.28}
$$

The contribution of the singly resonant diagrams to the full amplitude, dressed with couplings and adding also the term with the role of the Z bosons interchanged, is given by,

$$
\mathcal{A}_7^{\text{tree,sr},B}(1, 2, 3, 4, 5, 6, 7) = 4ie^4 g_s (t^B)_{i_1 \bar{i}_2} \tag{2.29}
$$
$$
\times \Big[\left(q_{34}q_{56} + v_{34}v_{56}P(s_{56}, M_Z) \right) \left(q_{34}Q_q + v_{34}v_q P(s_{3456}, M_Z) \right) A_7^{\text{sr}}(1, 2, 3, 4, 5, 6, 7)
$$
$$
+ \left(q_{34}q_{56} + v_{34}v_{56}P(s_{34}, M_Z) \right) \left(q_{56}Q_q + v_{56}v_q P(s_{3456}, M_Z) \right) A_7^{\text{sr}}(1, 2, 5, 6, 3, 4, 7) \Big].
$$

3 Geometric reconstruction

Any one-loop amplitude can be written as a sum of scalar master integrals with definite coefficients and rational terms [1]. Therefore, we write,

$$
A_7^{\text{1-loop}} \sim \sum_{i,j,k} d_{\{i \times j \times k\}} D_0(p_i, p_j, p_k; m) + \sum_{i,j} c_{\{i \times j\}} C_0(p_i, p_j; m)
$$
$$
+ \sum_i b_{\{i\}} B_0(p_i; m) + r. \tag{3.1}
$$

The exact definitions of the scalar integrals D_0, C_0, and B_0 are given in appendix B. For the case at hand, the rational term is fully determined by the mass dependence of the

coefficients [13]. In this section, we deal with the problem of simplifying the coefficients of the master integrals: $d_{\{i \times j \times k\}}$, $c_{\{i \times j\}}$ and $b_{\{i\}}$. Even though the coefficients are obtained through analytic unitarity methods, we phrase the simplification problem as a reconstruction problem from numerical samples. In this context, we treat all three types of coefficients on the same footing, thus we give them a generic name $\mathcal{C}_i$. These coefficients $\mathcal{C}_i$ are rational functions of the external kinematics. They can be written in common denominator form as

$$\mathcal{C}_i(\lambda, \tilde{\lambda}) = \frac{\mathcal{N}_i(\lambda, \tilde{\lambda})}{\prod_j \mathcal{D}_j(\lambda, \tilde{\lambda})^{q_{ij}}}, \tag{3.2}$$

where $(\lambda, \tilde{\lambda})$ denote the set of right- and left-handed Weyl spinors, which we treat as independent. That is, we consider the $\mathcal{C}_i$ in the analytical continuation to complex momenta. The exponents q_{ij} are integers and they are allowed to take negative values, thus denoting common factors in the numerator. When the exponents q_{ij} are as low as possible we refer to the denominator of eq. (3.2) as the least common denominator. By definition, the least common denominator form is free of spurious poles and of poles of higher order than necessary. As a first step, we bring the expressions obtained by analytic unitarity techniques, which may contain spurious poles or poles of higher order than necessary, to least common denominator form (see section 3.2). However, at this stage the numerator is not obtained explicitly, as it would be impractically complicated.[1]

To better comprehend and more efficiently manipulate the analytical expressions, we take a geometric perspective, following ref. [27]. In section 3.1 we review the most important algebro-geometric concepts that are needed, namely ideals and varieties in spinor space. Qualitatively, since the $\mathcal{C}_i$ are functions of many complex variables, they are evaluated in a multi-dimensional space analogous to the complex plane. Their poles and zeros are surfaces (a.k.a. *varieties*) of one less dimension than the full space. What the least common numerator achieves is to manifest the behaviour of the rational functions on these varieties, by removing indeterminate forms such as $\infty - \infty$ or $0/0$. In fact, one may think of spurious poles as removable discontinuities on these varieties. Moreover, since in a multivariate setting the (complex) dimension of the poles is not zero, as in the univariate case, the rational functions have further non-trivial behaviour at the intersection of the poles, that is on varieties of lower dimensions, where multiple denominator factors vanish simultaneously. For example, the form of the tree amplitude in eq. (2.16),

$$A_7^{\text{tree}} = -\frac{\langle 13 \rangle}{\langle 17 \rangle \, s_{34} \, s_{56} \, s_{134}} \left[\frac{\langle 13 \rangle \, [34] \, [26] \, \langle 5|\Gamma_{26}|7]}{s_{256}} + \frac{\langle 5|\Gamma_{13}|4] \langle 1|\Gamma_{27}|6]}{\langle 72 \rangle} \right], \tag{3.3}$$

manifestly has a simple pole on the variety where both s_{256} and $\langle 72 \rangle$ vanish, that is at the intersection of the varieties where s_{256} and $\langle 72 \rangle$ vanish individually, as they appear in different denominators. Our approach to simplifying the expressions is to manifest as many of these cancellations as we can, either by partial fraction decompositions or by inserting appropriate structures in the numerators (see section 3.3). This often implies a

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[1]The most complicated coefficient in common denominator form would be $b_{12}^{(0)}$. To parametrize its numerator one would need an homogeneous polynomial of degree 30 in the spinors, with millions of terms.

choice as it may be hard, if not altogether impossible, to manifest all such cancellations simultaneously. For example, by using repeated applications of the Schouten identity and momentum conservation, the tree amplitude of eq. (3.3) can also be written as,

$$A_7^{\text{tree}} = \frac{\langle 13\rangle}{s_{34}\,s_{56}\,s_{256}\,\langle 72\rangle}\left[\frac{\langle 5|\Gamma_{26}|4]\,\langle 12\rangle\,[26]}{\langle 17\rangle} + \frac{\langle 5|\Gamma_{13}|4]\langle 5|\Gamma_{26}|7]\,[65]}{s_{134}} + \langle 52\rangle\,[26]\,[74]\right].$$
(3.4)

The form of eq. (3.4) makes clear that the amplitude only has a simple pole also on the variety where both s_{134} and $\langle 17\rangle$ vanish, which is not apparent from eq. (3.3).

In our theoretical approach we make use of elements of both the algebro-geometric reconstruction procedure of ref. [27] and of the iterated, in-limit reconstruction strategy of ref. [26]. On a more practical level, we rely on the computer algebra system `Singular` [32], through the `Python` interface `syngular` [33], on the multi-precision floating-point arithmetic package `mpmath` [34] and on an in-house `Python` implementation of p-adic numbers $(\mathbb{Q}_p)$ with variable size mantissa and explicit precision tracking.

3.1 Ideals and varieties in spinor space

For the purposes of algebro-geometric computations, let us begin by considering polynomials in the components of the Weyl spinors of the massless external legs. Mathematically, we say that these polynomials belong to the polynomial ring defined as,

$$S_n = \mathbb{F}\big[|1\rangle, [1|, \ldots, |n\rangle, [n|\big],$$
(3.5)

where the spinors are understood to be taken component-wise and we employ standard spinor-helicity notation. We stress that, in the analytic continuation to complex momenta, the spinors are independent. The field $\mathbb{F}$ is taken to be either the complex numbers $(\mathbb{C})$ or the p-adic numbers $(\mathbb{Q}_p)$. In practice, we usually work over $\mathbb{C}$ and all geometric considerations are always understood to be over the complex numbers, as it is often required for the field to be algebraically closed. However, in some circumstances evaluations over $\mathbb{Q}_p$ can be very useful since the scale hierarchy is easier to control.

In order to discuss the geometric properties of the amplitude coefficients, we introduce the algebraic concept of an ideal, which is denoted by a pair of angle brackets.[2] An ideal is defined to be the set of all polynomial linear combinations of an initial set of polynomials which are called generators. As algebraic objects, ideals have well defined algebraic operations, addition, multiplication etc. In particular, a physically important ideal of S_n is that of momentum conservation,

$$J_{\Lambda_n} = \left\langle \sum_{i=1}^{n} |i\rangle[i|\right\rangle_{S_n},$$
(3.6)

where the subscript denotes the ring to which the ideal belongs. Note that, we can write this ideal as a single tensor generator, or as four generators taking the tensor component by component. If instead, we were to define the momentum conservation ideal in the subring

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[2]Unfortunately, this common notation from algebraic geometry clashes with standard spinor-helicity notation. Nevertheless, all expressions are unambiguous: the brackets denoting ideals must be balanced.

of spinor products (see eqs. (2.9) and (2.10) and ref. [27, section 2.2]) it would require n^2 contractions of momentum conservation from eq. (3.6), plus an additional $2\binom{n}{4}$ Schouten identities from eq. (2.11).

For the spinors to describe a physically meaningful phase space, they must satisfy momentum conservation. That is, any polynomial belonging to the momentum conservation ideal J_{Λ_n} has to be considered a rewriting of zero. For this reason, it is convenient to introduce the quotient ring,

$$R_n = S_n / J_{\Lambda_n}. \tag{3.7}$$

This is the set of all polynomials in spinor components where any pair of polynomials differing by a member of J_{Λ_n} is considered to be equivalent. That is, the elements of R_n are not polynomials, but equivalence classes of polynomials. For example, the following are six different ways to write the same element of R_7,

$$\Big\{\, [7|\Gamma_{12|34}|7], [7|\Gamma_{34|56}|7], [7|\Gamma_{56|12}|7], -[7|\Gamma_{34|12}|7], -[7|\Gamma_{56|34}|7], -[7|\Gamma_{12|56}|7] \,\Big\}. \tag{3.8}$$

Just like we can define ideals of S_n, we can also define ideals of R_n. In fact, there exists a one-to-one map between ideals of R_n and ideals of S_n that contain J_{Λ_n}, given by,

$$\Big\langle\, p_1, \ldots, p_k \,\Big\rangle_{R_n} \sim \Big\langle\, p_1, \ldots, p_k, \sum_{i=1}^{n} |i\rangle[i| \,\Big\rangle_{S_n}, \tag{3.9}$$

for an arbitrary set of generators $p_1, \ldots, p_k$. The numerators $\mathcal{N}_i$ and the denominators $\prod_j \mathcal{D}_j^{q_{ij}}$ of the coefficients $\mathcal{C}_i$ belong to R_n. Therefore, the coefficients $\mathcal{C}_i$ belong to the field of fractions of R_n, denoted as $FF(R_n)$. However, note that it is not entirely trivial to define a field of fractions over a quotient ring (see ref. [35, chapter 5]). In particular, R_n needs to be a so-called integral domain, and R_3 is not.[3] For $n \geq 4$, R_n is an integral domain and $FF(R_n)$ is well defined. We will work in R_7.

The key geometric concept that we consider is that of a variety. We denote a variety U associated to an ideal J as $U = V(J)$. For an ideal $J = \langle p_1, \ldots, p_k \rangle_{S_n}$, $V(J)$ is defined as the set of points $(\lambda, \tilde{\lambda}) \in \mathbb{F}^{4n}$ such that the generators of J evaluate to zero,

$$V(\langle p_1, \ldots, p_k \rangle_{S_n}) = \Big\{ (\lambda, \tilde{\lambda}) \in \mathbb{F}^{4n} \,:\, p_i(\lambda, \tilde{\lambda}) = 0 \text{ for } 1 \leq i \leq k \Big\}. \tag{3.10}$$

Through the correspondence of eq. (3.9), the same definition applies to varieties associated to ideals of R_n. Note that in the latter case, all varieties will be sub-varieties of $V(J_{\Lambda_n})$. Similarly, to every variety $U \subseteq \mathbb{F}^{4n}$ we can associate an ideal J of S_n, denoted as $J = I(U)$, and defined as the set of polynomials which vanish on U,

$$I(U) = \Big\{ p \in S_n \,:\, p(\lambda, \tilde{\lambda}) = 0 \text{ for all } (\lambda, \tilde{\lambda}) \in U \Big\}. \tag{3.11}$$

Varieties and ideals have well defined dimensions. For example, $\dim(V(J_{\Lambda_n})) = (4n - 4)$, as 4 constraints are imposed in $\mathbb{F}^{4n}$. We also define codimension as the complement of dimension w.r.t. the dimension of the full space, i.e. $\mathrm{codim}(V(J_{\Lambda_n})) = 4$. Note that the

---

[3]This follows from the fact that $\big\langle 0 \big\rangle_{R_3}$ is not a prime ideal (see below).

codimension need not always match the number of generators. If it is possible to find a set of generators with as few elements as the codimension, then we describe the ideal as of maximal codimension. As the quotient of a polynomial ring by a maximal codimension ideal, R_n is a Cohen-Macaulay ring [27]. This property of R_n has a number of useful implications which we will review shortly. In the context of R_n, as $V(\langle 0 \rangle_{R_n}) \sim V(J_{\Lambda_n})$, a codimension-one variety will be a $[(4n-4)-1]$-dimensional variety contained in $V(J_{\Lambda_n})$ and a codimension-two variety will have dimension $[(4n-4)-2]$. The poles of the rational coefficients are codimension-one varieties.

It turns out that the order of a pole is not necessarily well defined on all varieties. If a variety is comprised of multiple irreducible components, which we call branches, then the degree of divergence need not be the same for all of them. Therefore, it is important to be able to identify the branches of reducible varieties. Any variety admits a unique minimal decomposition,

$$U = \bigcup_{k=1}^{n_B(U)} U_k \,, \tag{3.12}$$

where $n_B(U)$ denotes the number of branches U_k, which are themselves irreducible, i.e. $n_B(U_k) = 1$, and where no branch is contained in another $U_i \not\subseteq U_j$ for all $i \neq j$. If U itself is irreducible then we simply have $n_B(U) = 1$.

To compute a minimal decompositions of a variety, one can rely on the corresponding concept for ideals, that is on a so-called minimal primary decomposition. This requires the introduction of a few more concepts from algebraic geometry. An ideal Q is said to be primary if

$$ab \in Q \;\Rightarrow\; \text{either } a \in Q \text{ or } b^k \in Q \,, \tag{3.13}$$

where k is a positive integer, and a and b are elements of the ring over which Q is defined. That is, an ideal is primary if whenever a product belongs to the ideal then either factor, possibly to some power, does as well. A useful operation on ideals is the radical operation,

$$\sqrt{J} = \{a : a^k \in J \text{ for some } k \in \mathbb{Z}_{>0}\} \,, \tag{3.14}$$

which is itself an ideal. Following this definition, an ideal J is said to be radical if $J = \sqrt{J}$. By combining these two concepts we obtain a third one: an ideal is said to be prime if it is both primary and radical. We will generally denote a prime ideal as P. Given a primary ideal Q, by definition, $\sqrt{Q}$ is a prime ideal P. One than says that Q is P-primary.

We are now in a position to discuss minimal primary decompositions. Given any ideal J, in analogy to eq. (3.12), we can write,

$$J = \bigcap_{l=1}^{n_Q(J)} Q_l \,, \tag{3.15}$$

where $n_Q(J)$ denotes the number of primary ideals Q_l in the primary decomposition. For the decomposition to be minimal, all associated prime ideals $P_l = \sqrt{Q_l}$ must be distinct. Furthermore, it must not be possible to remove any primary ideal from the decomposition without changing the result, i.e. $Q_m \not\supseteq \bigcap_{l \neq m} Q_l$. Now let $U = V(J)$. The subset of prime

ideals P_l such that $P_l = I(U_k)$ for some k is called the set of minimal associated primes. It has $n_B(U)$ elements and we denote it as $\mathrm{minAssoc}(J)$. The complement of this subset is the set of so-called embedded components. An embedded prime P_l is such that $V(P_l)$ is redundant (and thus absent) in eq. (3.12), but such that Q_l is not redundant in eq. (3.15). The number of embedded components is $n_Q(J) - n_B(U)$. The only non-unique components in a primary decomposition are those primary ideals Q_l associated to embedded primes P_l. A first useful consequence of R_n being a Cohen-Macaulay ring is that maximal codimension ideals in R_n are equi-dimensional, i.e. they are free of embedded components [27],

$$J = \langle p_1, \ldots, p_m \rangle_{R_n} \quad \text{s.t.} \quad \mathrm{codim}(J) = m \quad \implies \quad n_Q(J) = n_B(V(J)) \,. \tag{3.16}$$

All minimal primary decompositions that we present are for maximal codimension ideals in R_7, and thus are unambiguously defined. It is also useful to note that S_n, J_n and R_n are symmetric under permutations of the external legs and a swap of left- and right-handed spinors. We can make use of these properties to aid the computation of primary decompositions, and only present those not related by a symmetry.

Minimal (primary) decompositions allow us to systematically discuss the location of poles and zeros in multivariate contexts. However, poles and zeros are also characterized by an order (or degree), which we shall now briefly review. We also refer to the order of a pole as the degree of divergence of a given function on a variety, and to the order of a zero as its degree of vanishing. Note that the two concepts are closely related, as if a function f has a pole of order κ, then $1/f$ has a zero of the same order. Therefore, it suffices to discuss functions vanishing to a given order on a given variety (see ref. [27], section 2.4). In the univariate case, a function $f(z)$ is said to have a zero of order κ at z_0 if its Taylor series around z_0 starts at $(z - z_0)^\kappa$. Note that evaluating $f(z)$ at $z_0 + \epsilon$, $|\epsilon| \ll 1$, yields ϵ^κ, which gives access numerically to the order κ. In the language of algebraic-geometry one would say that $f(z)$ belongs to the ideal $\langle z - z_0 \rangle^\kappa_{\mathbb{C}[z]}$, where the power J^κ of an ideal J can be defined recursively in terms of ideal products[4]

$$J^0 = \langle 1 \rangle \quad \text{together with} \quad J^\kappa = J \cdot J^{\kappa-1} \,. \tag{3.17}$$

Analogously, in the context of R_n and referring back to the rational function $\mathcal{C}_i$, we say that a numerator $\mathcal{N}_i$ vanishes to order κ at a point $(\eta, \tilde\eta)$ of $\mathbb{F}^{4n}$ if it belongs to the ideal

$$\mathfrak{m}^\kappa_{(\eta,\tilde\eta)} = \langle \lambda - \eta, \tilde\lambda - \tilde\eta \rangle^\kappa_{R_n} \,. \tag{3.18}$$

Generally, we are interested in situations where $\mathcal{N}_i$ does not vanish just at an isolated point $(\eta, \tilde\eta)$ but at all points $(\eta, \tilde\eta) \in U$, for some variety U. In this case, the membership statement must hold for all points on U

$$\mathcal{N}_i \text{ vanishes at order } \kappa \text{ on } U \text{ if } \mathcal{N}_i \in \bigcap_{(\eta,\tilde\eta) \in U} \mathfrak{m}^\kappa_{(\eta,\tilde\eta)} \,. \tag{3.19}$$

---

[4]The product of two ideals $J = \langle p_1, \ldots, p_a \rangle$ and $K = \langle q_1, \ldots, q_b \rangle$ is the ideal generated by the outer product of the generators of J and K, i.e. $J \cdot K = \langle p_i q_j, 1 \le i \le a, 1 \le j \le b \rangle$.

Let $U = V(Q_l)$ for some primary ideal Q_l, whose associated prime is $P_l = \sqrt{Q_l}$. The Zariski-Nagata theorem [36–38] states that

$$\bigcap_{(\eta,\tilde{\eta})\in V(Q_l)} \mathfrak{m}_{(\eta,\tilde{\eta})}^{\kappa} \subseteq P_l^{\langle\kappa\rangle} . \tag{3.20}$$

The quantity $P_l^{\langle\kappa\rangle}$ is the so-called κ^{th} symbolic power of the ideal P_l. It is closely related to the standard power P_l^{κ} defined in eq. (3.17). To see why a refined notion of power is needed, let us consider the primary decomposition of P_l^{κ},[5]

$$P_l^{\kappa} = P_l^{\langle\kappa\rangle} \cap Q_1^{\text{em.}} \cap \cdots \cap Q_m^{\text{em.}} , \tag{3.21}$$

where we have made manifest that $P_l^{\langle\kappa\rangle}$ is the only non-embedded component in the decomposition of P_l^{κ}. In other words, $P_l^{\langle\kappa\rangle}$ is the unique P_l-primary ideal in the decomposition of P_l^{κ}. Membership to $P_l^{\langle\kappa\rangle}$ is less restrictive than membership to P_l^{κ}, as the embedded components $Q^{\text{em.}}$ would imply additional non-trivial vanishing properties on sub-varieties of $V(Q_l)$. In ref. [27, section 3] it was shown that the power κ can be reliably identified numerically by evaluating $\mathcal{N}_i$ in proximity of $V(Q_l)$. Furthermore, it was pointed out that another useful consequence of R_n being a Cohen-Macaulay ring is that symbolic powers of maximal codimension ideals in R_n coincide with the standard powers,

$$J = \langle p_1, \ldots, p_m \rangle_{R_n} \quad \text{s.t.} \quad \text{codim}(J) = m \quad \Longrightarrow \quad J^{\langle\kappa\rangle} = J^{\kappa} . \tag{3.22}$$

Finally, let us point out that for an irreducible variety U with a codimension higher than one it is useful to not only consider membership of $\mathcal{N}_i$ to symbolic powers of the associated prime ideal $P = I(U)$, but also to more generic primary ideals whose radical is P. This could be interpreted as a more generic notion of order of a pole or zero when the dimension of the cotangent space to the variety is greater than one, that is when there is freedom as to the choice of direction in approaching U. These concepts are explored further in appendix C and are further motivated by the appearance of non-radical ideals beyond 5-point massless kinematics.

3.2 Poles and zeros as codimension-one varieties

We begin by determining the denominators of the coefficients $\mathcal{C}_i$, which belong to $FF(R_7)$. Poles and zeros of these rational functions are irreducible varieties of codimension one, with an associated degree of vanishing or divergence. We observe that the poles needed for the one-loop amplitudes under our consideration in this paper are of the form,

$$\mathcal{D} = \{\langle ab\rangle, \langle a|\Gamma_{bc}|a], \langle a|\Gamma_{bc|de}|a\rangle, \Delta_3(a,b,c,d)\} , \tag{3.23}$$

where the indices a, b, c, d, e are assumed to be distinct and in the set $\{1, \ldots, 7\}$. All associated codimension-one ideals, that is ideals of the form $\langle D_j\rangle_{R_7}$, are prime, meaning that the corresponding varieties $V(\langle D_j\rangle_{R_7})$ are irreducible. Then, it follows that the order of the poles and zeros can be determined by either a pair of high-precision floating point

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[5]Note that even if P_l is prime, there is no guaranteed that P_l^{κ} is primary.

evaluations in the limit approaching the variety, as by ref. [26], or by a single p-adic evaluation at a point close to the variety, as by ref. [27]. With this procedure we obtain the least common denominators (LCD),

$$\mathcal{D}_{\mathrm{LCD},\,i} = \prod_j \mathcal{D}_j(\lambda, \tilde{\lambda})^{q_{ij}}\,, \text{ with } q_{ij} > 0\,. \tag{3.24}$$

We stress that since the pole orders are determined numerically, it follows that all spurious poles are automatically removed and the physical ones are of as low degree as possible. Common factors in the numerator ($q_{ij} < 0$) are also obtained in this way.

3.3   Partial fractions and numerators from codimension-two varieties

To proceed in the simplification we aim to constrain the numerators $\mathcal{N}_i$. For this purpose we study of their behaviour on codimension-two varieties. In particular, it is convenient to consider the behaviour of the numerators on those codimension-two varieties that originate from the intersection of varieties of codimension one corresponding to poles of the coefficients. We can now assume we have access to numerical evaluations of the numerators $\mathcal{N}_i$, as the denominators have been determined in section 3.2 from the study of varieties of codimension one. The following discussion is applicable to any quotient ring R_n with a multiplicity n bigger than four,[6] thus we will omit the ideal subscripts.

Given a pair of distinct poles $(\mathcal{D}_\alpha, \mathcal{D}_\beta)$, we consider the intersection of the associated varieties, which is equivalent to the variety associated to the sum of the ideals,

$$V(\langle \mathcal{D}_\alpha \rangle) \cap V(\langle \mathcal{D}_\beta \rangle) = V(\langle \mathcal{D}_\alpha \rangle + \langle \mathcal{D}_\beta \rangle) = V(\langle \mathcal{D}_\alpha, \mathcal{D}_\beta \rangle)\,. \tag{3.25}$$

Contrary to the ideals of codimension one introduced in the previous section, these ideals $\langle \mathcal{D}_\alpha, \mathcal{D}_\beta \rangle$ of codimension two do not always correspond to irreducible varieties. As the numerator may vanish to different orders on the different branches of $V(\langle \mathcal{D}_\alpha, \mathcal{D}_\beta \rangle)$, we compute the primary decompositions, as by eq. (3.15). They read,

$$\langle \mathcal{D}_\alpha, \mathcal{D}_\beta \rangle = \bigcap_{l=1}^{n_Q(\langle \mathcal{D}_\alpha, \mathcal{D}_\beta \rangle)} Q_l\,, \tag{3.26}$$

where, by eq. (3.16), we have

$$n_Q(\langle \mathcal{D}_\alpha, \mathcal{D}_\beta \rangle) = n_U\big(V(\langle \mathcal{D}_\alpha, \mathcal{D}_\beta \rangle)\big)\,. \tag{3.27}$$

That is, we are in a special situation where the minimal decomposition of the associated variety reads

$$V(\langle \mathcal{D}_\alpha, \mathcal{D}_\beta \rangle) = \bigcup_{l=1}^{n_U(V(\langle \mathcal{D}_\alpha, \mathcal{D}_\beta \rangle))} V(Q_l)\,. \tag{3.28}$$

For each irreducible variety $V(Q_l)$ we then generate either a single or a pair of nearby phase-space points, depending on whether the field $\mathbb{F}$ is taken to be $\mathbb{Q}_p$ or $\mathbb{C}$ respectively,

---

[6]While R_4 is an integral domain, it is not a unique factorization domain, as in R_4 we have $\langle 12 \rangle [12] = \langle 34 \rangle [34]$. Therefore, codimension-one ideals $\langle \mathcal{D}_\alpha \rangle_{R_4}$ may be reducible and the common denominator is not unique. Hence, the discussion in section 3.2, and in the current section, cannot be directly applied in R_4.

and thus obtain the degree of vanishing of the numerators $\mathcal{N}_i$. Given a prime ideal $P_l = \langle p_1, \ldots, p_r, q_1, \ldots, q_4 \rangle$, where $J_{\Lambda_7} = \langle q_1, \ldots, q_4 \rangle$, the phase-space point we require is a set $(\eta^{(\epsilon)}, \tilde{\eta}^{(\epsilon)}) \in \mathbb{F}^{28}$ such that,

$$p_i(\eta^{(\epsilon)}, \tilde{\eta}^{(\epsilon)}) = \epsilon^{\kappa_i} \quad \text{and} \quad q_j(\eta^{(\epsilon)}, \tilde{\eta}^{(\epsilon)}) = \epsilon^{\kappa_q}, \tag{3.29}$$

where κ_q is the working precision ($\kappa_q \gg \kappa_i$), and κ_i is the largest integer such that $p_i \in P_l^{\langle \kappa_i \rangle}$. Usually, but not always, one has $\kappa_i = 1$. Such a phase-space point can be built with the method described in ref. [27, section 3], which is also easily adapted to $\mathbb{F} = \mathbb{C}$. The parameter ϵ denotes a small quantity with respect to the absolute value of the chosen field. For each variety $V(Q_l)$ this results in a constraint on the numerator of the form

$$\mathcal{N}_i(\eta^{(\epsilon)}, \tilde{\eta}^{(\epsilon)}) \sim \epsilon^{\kappa_l} \implies \mathcal{N}_i \in P_l^{\langle \kappa_l \rangle}. \tag{3.30}$$

To achieve compact representations of the coefficients, as well as to be able to reconstruct one pole residue at a time, we wish to interpret as many as possible of these constraints on the numerators in terms of a partial fraction decomposition of $\mathcal{C}_i$. To illustrate this let us now restrict our discussion to those codimension-two ideals $\langle \mathcal{D}_\alpha, \mathcal{D}_\beta \rangle$ which are radical, that is to those ideals such that $Q_l = P_l$ for all l. The extension of the following reasoning to the generic case is addressed in appendix C. Let κ be the largest power such that $\mathcal{N}_i$ vanishes to order κ on all branches $V(Q_l)$ of $V(\langle \mathcal{D}_\alpha, \mathcal{D}_\beta \rangle)$, we have,

$$\mathcal{N}_i \in \bigcap_l P_l^{\langle \kappa \rangle} \implies \mathcal{N}_i \in \langle \mathcal{D}_\alpha, \mathcal{D}_\beta \rangle^{\langle \kappa \rangle}, \quad \text{if } P_l = Q_l. \tag{3.31}$$

By eq. (3.22) we can then explicitly expand the symbolic power as,

$$\langle \mathcal{D}_\alpha, \mathcal{D}_\beta \rangle^{\langle \kappa \rangle} = \langle \mathcal{D}_\alpha, \mathcal{D}_\beta \rangle^\kappa = \langle \mathcal{D}_\alpha^\kappa, \, \mathcal{D}_\alpha^{\kappa-1} \mathcal{D}_\beta, \, \ldots, \, \mathcal{D}_\alpha \mathcal{D}_\beta^{\kappa-1}, \, \mathcal{D}_\beta^\kappa \rangle. \tag{3.32}$$

This can be interpreted in terms of a partial-fraction decomposition of the rational function,

$$\mathcal{C}_i(\lambda, \tilde{\lambda}) = \frac{1}{\prod_{j \neq \alpha, \beta} \mathcal{D}_j(\lambda, \tilde{\lambda})^{q_{ij}}} \sum_{k=0}^\kappa \frac{\mathcal{N}_{ik}(\lambda, \tilde{\lambda})}{\mathcal{D}_\alpha^{q_\alpha - \kappa + k} \mathcal{D}_\beta^{q_\beta - k}}, \tag{3.33}$$

which is manifestly free from spurious poles. This decomposition is maximal in the sense that it is not possible to replace κ with $\kappa + 1$. However, it may still be possible to further refine the decomposition as some $\mathcal{N}_{ik}$ may be proportional to $\mathcal{D}_\alpha$ or $\mathcal{D}_\beta$. How to obtain and interpret these additional constraints is addressed in appendix C. We perform this analysis for the vast majority of pairs of poles, $(\mathcal{D}_\alpha, \mathcal{D}_\beta)$. The required primary decompositions are presented in section 3.3.1. In some rare cases, the partial fraction decomposition is expected from the structure of unitarity cuts, and this analysis is thus not needed. For example, one of the two bubble coefficients in section 4.3 involves two three-mass Gram-determinant poles, but these are clearly associated to different triple cuts and thus separable.

Since the degree of vanishing of $\mathcal{N}_i$ need not be uniform on all branches of the ideal, it follows that not all constraints can be interpreted this way. The extra degree of vanishing of the numerator beyond κ on a particular branch is then purely a statement about

the structure of the numerators $\mathcal{N}_{ik}$. For example, we observe that the integral coefficients presented in section 4 often diverge less strongly on one of the two branches of $V\left(\left\langle \langle 7|\Gamma_{34|56}|7\rangle, [7|\Gamma_{34|56}|7]\right\rangle\right)$, which are given in eq. (3.41). Therefore, besides a partial-fraction decomposition, we obtain further information on the numerators. For instance, if the degree of divergence is lower on the second branch than on the first one, the numerators corresponding to the leading poles of $\langle 7|\Gamma_{34|56}|7\rangle$ and $[7|\Gamma_{34|56}|7]$ may contain contractions of the third generator $\tilde{\Gamma}_{12|34|56}$ given in the second primary ideal of eq. (3.41). Thus, the primary decompositions allow one to uncover numerator structures with specific vanishing properties in certain regions of phase space.

So far we have achieved a systematic partial fraction decomposition between pairs of poles, and a way to identify new spinor structures for the numerators. The next step is to combine all the decompositions and numerator constraints together, so that the resulting expression is simple. However, note that in general not all ideal-membership constraints to which the common numerator $\mathcal{N}_i$ is subject carry over to the individual $\mathcal{N}_{ik}$. For example, if $\mathcal{N}_i \in J_1 \cap J_2$, with $J_1 = \langle \mathcal{D}_\alpha, \mathcal{D}_\beta\rangle^{\langle \kappa_1\rangle}$ and $J_2 = \langle \mathcal{D}_\gamma, \mathcal{D}_\delta\rangle^{\langle \kappa_2\rangle}$, assuming $(\alpha, \beta, \gamma, \delta)$ to be distinct, then we could attempt a partial fraction decomposition of the form

$$\mathcal{C}_i(\lambda, \tilde{\lambda}) = \frac{1}{\prod_{j\neq\alpha,\beta,\gamma,\delta} \mathcal{D}_j(\lambda, \tilde{\lambda})^{q_{ij}}} \sum_{k_1=0}^{\kappa_1} \sum_{k_2=0}^{\kappa_2} \frac{\mathcal{N}_{ik_1k_2}(\lambda, \tilde{\lambda})}{\mathcal{D}_\alpha^{q_\alpha-\kappa_1+k_1} \mathcal{D}_\beta^{q_\beta-k_1} \mathcal{D}_\gamma^{q_\gamma-\kappa_2+k_2} \mathcal{D}_\delta^{q_\delta-k_2}} . \tag{3.34}$$

However, this may or may not be a valid decomposition of the coefficient $\mathcal{C}_i$. To see this, let us consider the simplified case $\kappa_1 = \kappa_2 = 1$. We can refer back to eq. (3.33) and use the first J_1 constraint, that is, we can write $\mathcal{N}_i = \mathcal{N}_{i0}\mathcal{D}_\alpha + \mathcal{N}_{i1}\mathcal{D}_\beta$. To then achieve the decomposition of eq. (3.34) we need $\mathcal{N}_{i0} \in J_2$ and $\mathcal{N}_{i1} \in J_2$. Yet, we are guaranteed this to be true only if $J_1 \cap J_2 = J_1 \cdot J_2$, which is not the case in general. On the other hand, failure of the intersection to equal the product does not automatically imply that an ansatz of the form of eq. (3.34) has to fail. This makes it highly non-trivial to make use of multiple constraints while performing a partial-fraction decomposition. In fact, it is not hard to find cases where, given a certain partial-fraction decomposition, it becomes impossible to make all other numerator constraints manifest. This can lead to spurious singular behaviour even in the absence of spurious poles. That is, spurious divergences on codimension-two varieties are possible even in the absence of spurious divergences on codimension-one varieties. A well known example of this type of spurious singularities is the appearance of s_{ij} poles in lieu of $\langle ij\rangle$ and/or $[ij]$ poles in gauge-theory amplitudes. An example of a partial fraction decomposition which makes it impossible to manifest all numerator constraints is given in section 3.3.2. Because of this subtlety, which warrants further investigation in the future, for the time being we take an heuristic guess-and-check approach when combining multiple constraints. That is, we assume that multiple constraints can be naively combined, for example as in eq. (3.34), and attempt to fit the free coefficients in the ansatz for the numerators with the in-limit reconstruction strategy of ref. [26]. If this fails, we relax one or more constraints until the reconstruction succeeds. When a choice has to be made as to which constraints to manifest, on grounds of numerical accuracy, we tend to favor inserting appropriate spinor expressions in the numerators, as opposed to performing further partial fraction decompositions, because the former lead to indeterminates of the form $0/0$ while

the latter to $\infty - \infty$. In fact, barring overflows in the exponent, multiplication preserves the relative accuracy of the mantissa, whereas the addition of two large numbers which results in a small number may cause a loss of accuracy.

3.3.1 Codimension-two primary decompositions

In this sub-section we present several primary decompositions which were used in the present computation. Unless otherwise stated, all ideals are understood to be taken in the quotient ring R_7. The following is not meant to be a complete list of primary decompositions for codimension-two ideals at seven point. In fact, including spinor strings of the form $\langle a|\Gamma_{bc}|d]$ and three-particle Mandelstam invariants s_{abc} one obtains hundreds of distinct codimension-two varieties, which is beyond the current scope. The notation employed is as in eq. (3.26), that is, the left-hand side consists of a reducible ideal and the right-hand side gives its decomposition as in intersection of primary ideals. Whenever a primary ideal Q_l does not correspond to its associated prime P_l, i.e. if $Q_l \neq \sqrt{Q_l}$, then we also write an equation of the form $\sqrt{Q_l} = P_l$, with P_l explicitly given by a set of generators.

There are five independent ideals generated by pairs of two-particle invariants. Only one is not prime, and it splits into a "collinear" branch and a "soft" branch,

$$
\begin{aligned}
&\langle\langle 12\rangle, \langle 13\rangle\rangle = \langle\langle 12\rangle, \langle 13\rangle, \langle 23\rangle\rangle \cap \langle|1\rangle\rangle\,, \\
&\text{while} \quad \langle\langle 12\rangle, \langle 34\rangle\rangle\,, \ \langle\langle 12\rangle, [12]\rangle\,, \ \langle\langle 12\rangle, [13]\rangle\,, \ \langle\langle 12\rangle, [34]\rangle \quad \text{are prime}\,.
\end{aligned}
\tag{3.35}
$$

We note that compared to the codimension-two five-point primary decompositions presented in ref. [27], the ideal generated by all angle brackets is absent. This is explained by the following observation,

$$
\mathrm{codim}\left(\langle\langle ij\rangle \ \forall \ i \neq j \in (1,\ldots,n)\,\rangle_{R_n}\right) = n - 3\,,
\tag{3.36}
$$

which can be easily checked up to very high multiplicity.

There are also five independent ideals generated by a two-particle invariant and a parity-invariant spinor string,

$$
\begin{aligned}
\langle\langle 12\rangle, \langle 1|\Gamma_{23}|1]\rangle &= \langle\langle 12\rangle, \langle 13\rangle, \langle 23\rangle\rangle \cap \langle|1\rangle\rangle \cap \langle\langle 12\rangle, [13]\rangle\,, \\
\langle\langle 12\rangle, \langle 1|\Gamma_{34}|1]\rangle &= \langle|1\rangle\rangle \cap \langle\langle 12\rangle, \langle 1|\Gamma_{34}|1], \langle 2|\Gamma_{34}|1]\rangle\,, \\
\langle\langle 12\rangle, \langle 3|\Gamma_{12}|3]\rangle &= \langle\langle 12\rangle, \langle 13\rangle, \langle 23\rangle\rangle \cap \langle\langle 12\rangle, \Gamma_{12}|3]\rangle\,, \\
\text{while} \quad \langle\langle 12\rangle, \langle 3|\Gamma_{14}|3]\rangle&\,, \ \langle\langle 12\rangle, \langle 3|\Gamma_{45}|3]\rangle \quad \text{are prime}\,.
\end{aligned}
\tag{3.37}
$$

As the primary decompositions of eq. (3.35) and eq. (3.37) are the same as those in R_6, it seems natural to conjecture that they should in fact hold for all $R_{n\geq 6}$.

Proceeding to ideals involving a two-particle invariant and a longer spinor chain we identify six ideals, of which three are not primary,

$$
\begin{aligned}
&\langle\langle 12\rangle, \langle 7|\Gamma_{34|56}|7\rangle\rangle = \langle\langle 12\rangle, \langle 17\rangle, \langle 27\rangle\rangle \cap \langle\langle 12\rangle, \Gamma_{12|34}|7\rangle\rangle\,, \\
&\langle[12], \langle 7|\Gamma_{34|56}|7\rangle\rangle = \langle[12], \Gamma_{12}|7\rangle\rangle \cap \langle[12], \langle 7|\Gamma_{56}|1], \langle 7|\Gamma_{56}|2]\rangle\,, \\
&\langle\langle 17\rangle, \langle 7|\Gamma_{34|56}|7\rangle\rangle = \langle\langle 17\rangle, |7\rangle\langle 7|\rangle \cap \langle\langle 12\rangle, \langle 17\rangle, \langle 27\rangle\rangle \cap \langle\langle 17\rangle, \langle 7|\Gamma_{56}|2], \langle 1|\Gamma_{56}|2]\rangle\,, \\
&\text{while} \quad \langle[17], \langle 7|\Gamma_{34|56}|7\rangle\rangle\,, \ \langle\langle 13\rangle, \langle 7|\Gamma_{34|56}|7\rangle\rangle\,, \ \langle[13], \langle 7|\Gamma_{34|56}|7\rangle\rangle \quad \text{are prime}\,.
\end{aligned}
\tag{3.38}
$$

In the above $\langle\langle 17\rangle, |7\rangle\langle 7|\rangle$ is clearly not radical, as it contains the outer product of $|7\rangle$ with itself, i.e. it is primary but not prime. The associated prime is simply the soft ideal,

$$\sqrt{\langle\langle 17\rangle, |7\rangle\langle 7|\rangle} = \langle |7\rangle\rangle. \qquad (3.39)$$

We also make extensive use of two primary decompositions of codimension-two ideals generated by pairs of spinor chains,

$$\langle\langle 7|\Gamma_{12}|7], \langle 7|\Gamma_{34|56}|7]\rangle = \langle\langle 12\rangle, \langle 17\rangle, \langle 27\rangle\rangle \cap \langle[12], \Gamma_{12}|7]\rangle \cap \langle\langle 7|\Gamma_{12}|7], |7\rangle\langle 7|\rangle$$
$$\cap \langle\langle 7|\Gamma_{34}|7], \langle 7|\Gamma_{56}|7], \langle 7|\Gamma_{34|56}|7], [7|\Gamma_{34|56}|7]\rangle, \qquad (3.40)$$
$$\langle\langle 7|\Gamma_{34|56}|7], [7|\Gamma_{34|56}|7]\rangle = \langle\langle 7|\Gamma_{34}|7], \langle 7|\Gamma_{56}|7], \langle 7|\Gamma_{34|56}|7], [7|\Gamma_{34|56}|7]\rangle$$
$$\cap \langle\langle 7|\Gamma_{34|56}|7], [7|\Gamma_{34|56}|7], \tilde{\Gamma}_{12|34|56}\rangle. \qquad (3.41)$$

In particular, the latter provides significant constraints, both in terms of partial fractions and in terms of numerator spinor structures, since the involved polynomials are of high degree. We recall that the tilde denotes anti-symmetrization, as defined in eq. (2.13). One of the primaries is again not radical, and it has the same associated prime as eq. (3.39),

$$\sqrt{\langle\langle 7|\Gamma_{12}|7], |7\rangle\langle 7|\rangle} = \langle |7\rangle\rangle. \qquad (3.42)$$

Lastly, we consider a few codimension two ideals involving three-mass Gram determinants. These are often not radical, but generally they are primary,

$$\begin{aligned}
\langle\langle 12\rangle, \Delta_3(1,2,3,4)\rangle &= \langle\langle 12\rangle, (s_{567} - s_{34})^2\rangle, \\
\langle\langle 1|\Gamma_{34}|2], \Delta_3(1,2,3,4)\rangle &= \langle\langle 1|\Gamma_{34}|2], (s_{134} - s_{234})^2\rangle,
\end{aligned} \qquad (3.43)$$

where we have made explicit the existence of a perfect-square polynomial in the ideals. Then, the radicals can be shown to be,

$$\begin{aligned}
\sqrt{\langle\langle 12\rangle, \Delta_3(1,2,3,4)\rangle} &= \langle\langle 12\rangle, (s_{567} - s_{34})\rangle, \\
\sqrt{\langle\langle 1|\Gamma_{34}|2], \Delta_3(1,2,3,4)\rangle} &= \langle\langle 1|\Gamma_{34}|2], (s_{134} - s_{234})\rangle.
\end{aligned} \qquad (3.44)$$

The attentive reader may recognize that $\langle 1|\Gamma_{34}|2]$ is not one of the poles listed in eq. (3.23), however it is often a zero of the residue of the three-mass Gram pole, hence we choose to include it here. Finally, there is also strong evidence for the following primary decomposition

$$\begin{aligned}
\langle\langle 7|\Gamma_{34|56}|7], \Delta_3(3,4,5,6)\rangle &= \langle\langle 7|\Gamma_{34|56}|7], \Delta_3(3,4,5,6), \tilde{\Gamma}_{34|56}|7]\langle 7|\tilde{\Gamma}_{34|56}\rangle, \\
\sqrt{\langle\langle 7|\Gamma_{34|56}|7], \Delta_3(3,4,5,6)\rangle} &= \langle\Delta_3(3,4,5,6), \tilde{\Gamma}_{34|56}|7]\rangle.
\end{aligned} \qquad (3.45)$$

The primality of all prime ideals can be proven via the test presented in appendix B.3 of ref. [27], except the very last one of eq. (3.45), which remains to be proven. It is also possible to explicitly check that the intersection of the primaries equals the reducible ideal, and whenever a primary is not radical one can check that $\sqrt{Q_l} = P_l$ by verifying $\dim(Q_l) = \dim(P_l)$ and that $Q_l/P_l^\infty = \langle 1\rangle$, where the latter operation denotes ideal saturation. In the supplementary material we provide a `Python` script that performs these checks.

3.3.2 Example of simultaneous constraints and spurious singularities

Let us now consider an example to illustrate the subtlety with combining multiple numerator constraints when interpreting some in terms of partial-fraction decompositions. We take a part of the integral coefficient $\tilde{c}^{(2)}_{12 \times 56}$ from eq. (4.28), namely,

$$\mathcal{C} = \frac{[2|\tilde{\Gamma}_{12|34|56}|1\rangle}{s_{12}\langle 7|\Gamma_{34|56}|7\rangle[7|\Gamma_{34|56}|7\rangle}\,. \tag{3.46}$$

The numerator belongs to the following two ideals: $J_1 = \left\langle \langle 7|\Gamma_{34|56}|7\rangle, [7|\Gamma_{34|56}|7\rangle, \tilde{\Gamma}_{12|34|56} \right\rangle$ and $J_2 = \left\langle \langle 12\rangle, [12] \right\rangle$. That is, the numerator $\mathcal{N}$ belongs to $J_1 \cap J_2$. One can recognize J_1 as the second primary ideal in the primary decomposition of eq. (3.41). At the same time, $\mathcal{N}$ does not belong to the first primary ideal in that same decomposition. Therefore, the two poles $\langle 7|\Gamma_{34|56}|7\rangle$ and $[7|\Gamma_{34|56}|7\rangle$ cannot be separated without introducing spurious poles.[7] However, the numerator does belong to the ideal $\langle \langle 12\rangle, [12] \rangle$, even if this is perhaps not entirely manifest. It becomes apparent by expanding the above as,

$$[2|\tilde{\Gamma}_{12|34|56}|1\rangle = \langle 12\rangle[2|\Gamma_{56|34}|2] + [12]\langle 1|\Gamma_{56|34}|1\rangle\,, \tag{3.47}$$

which can then be written as a partial fraction decomposition of the form,

$$\mathcal{C} = \frac{[2|\Gamma_{34|56}|2]}{[12]\langle 7|\Gamma_{34|56}|7\rangle[7|\Gamma_{34|56}|7\rangle} + \frac{\langle 1|\Gamma_{34|56}|1\rangle}{\langle 12\rangle\langle 7|\Gamma_{34|56}|7\rangle[7|\Gamma_{34|56}|7\rangle}\,. \tag{3.48}$$

Note that now it has become impossible to have a $\tilde{\Gamma}_{12|34|56}$ factor in either numerator, as their mass dimensions are not sufficient. We can see this might happen as product and intersection are not equal for the ideals J_1 and J_2. These kinds of consideration have implications on the stability of the expressions. For instance, the latter partial-fraction form of eq. (3.48) is potentially unstable near the variety associated to the second primary ideal in the primary decomposition of eq. (3.41) (double-pole cancelling to give a simple pole in the sum of the fractions).

4 Integral coefficients

In this section we present results for the one-loop master-integral coefficients for the process under consideration, starting for the six one-loop diagrams shown in figure 1 which contribute to doubly-resonant Z-boson pair production in association with a jet. These results have been calculated using standard analytic techniques [6, 11–14] and subsequently simplified using the methods of section 3. The one-loop colour amplitudes have the decomposition,

$$\mathcal{A}^{1\text{-loop},B}_7(1,2,3,4,5,6,7) = 4ig_se^4\frac{g_s^2}{16\pi^2}(t^B)_{i_1\,\bar{\imath}_2}A^{1\text{-loop}}_7(1,2,3,4,5,6,7), \tag{4.1}$$

---

[7]Note that if one were to introduce a spurious pole of, say, the form $\langle 7|\Gamma_{34}|7\rangle$, then the numerator would indeed vanish on both branches and a partial-fraction decomposition would be possible.
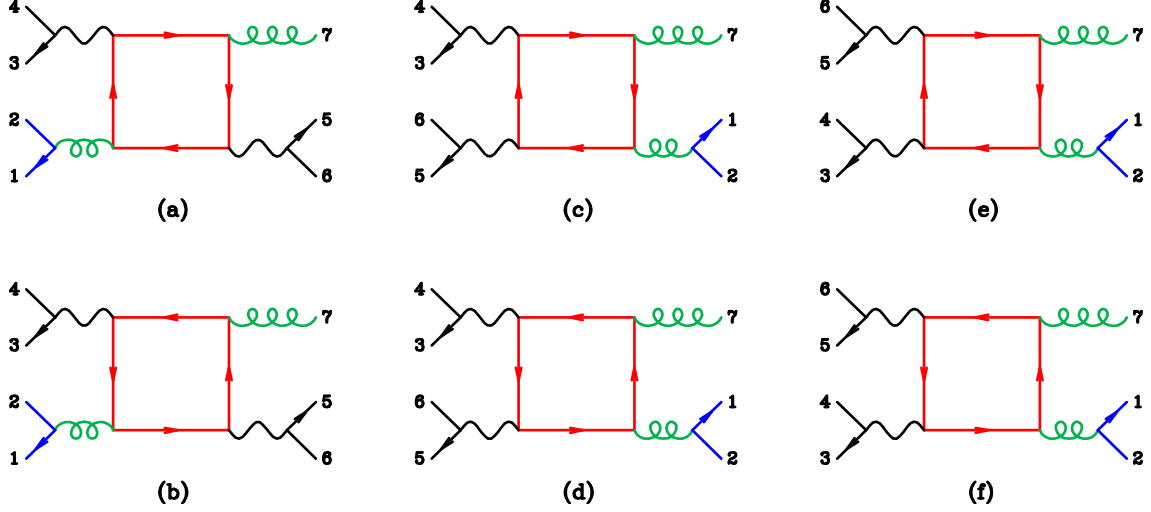
Figure 1. Box diagrams containing a quark loop.

where the SU(3) colour matrix in the fundamental representation, t^B, is normalized such that tr $t^A t^B = \delta^{AB}$. The quark and antiquark colour indices are i_1 and $\bar{\imath}_2$. The colour stripped amplitude can be expressed in terms of scalar integrals and rational terms r.

$$
\begin{aligned}
A_7^{1\text{-loop}}(1^{h_1}, 2^{h_2}, 3^{h_3}, 4^{h_4}, 5^{h_5}, 6^{h_6}, 7^{h_7}) &= \frac{\bar{\mu}^{4-n}}{r_\Gamma} \frac{1}{i\pi^{n/2}} \int \mathrm{d}^n\ell \, \frac{\mathrm{Num}(\ell)}{\prod_i d_i(\ell)} \\
&= \sum_{i,j,k} d_{\{i\times j\times k\}}(1^{h_1}, 2^{h_2}, 3^{h_3}, 4^{h_4}, 5^{h_5}, 6^{h_6}, 7^{h_7}) \, D_0(p_i, p_j, p_k; m) \\
&\quad + \sum_{i,j} c_{\{i\times j\}}(1^{h_1}, 2^{h_2}, 3^{h_3}, 4^{h_4}, 5^{h_5}, 6^{h_6}, 7^{h_7}) \, C_0(p_i, p_j; m) \\
&\quad + \sum_i b_{\{i\}}(1^{h_1}, 2^{h_2}, 3^{h_3}, 4^{h_4}, 5^{h_5}, 6^{h_6}, 7^{h_7}) \, B_0(p_i; m) \\
&\quad + r(1^{h_1}, 2^{h_2}, 3^{h_3}, 4^{h_4}, 5^{h_5}, 6^{h_6}, 7^{h_7}) \,.
\end{aligned}
\tag{4.2}
$$

The definitions of the scalar integrals are given in appendix B. As in eq. (2.14) we present results for one specific helicity choice,

$$
A_7^{1\text{-loop}}(1^-, 2^+, 3^-, 4^+, 5^-, 6^+, 7^+) \,.
\tag{4.3}
$$

Other helicities are obtained by permutation of the arguments.

The tree amplitude that we presented in eqs. (2.15), (2.16) was exactly the amplitude for production of a gluon and two lepton pairs off a massless, unit-charge, quark line. The production of the two lepton pairs was mediated by a virtual photon, and the propagators and couplings appropriate for Z-boson exchange were added subsequently as detailed in eq. (2.23). For the quark loop amplitudes, because of the mass running in the quark loop, we need to maintain helicity information. Therefore the amplitudes that we present below generalize the coupling of the "photon" to the massive quark line as follows,

$$
-ie\gamma^\mu \to -ie(v_R \gamma^\mu \gamma_R + v_L \gamma^\mu \gamma_L)\,, \quad \text{where } \gamma_{R/L} = \frac{1}{2}(1 \pm \gamma_5)\,.
\tag{4.4}
$$

The amplitude for true photon exchange is recovered by setting $v_L = v_R = 1$. Given this helicity information it is straightforward to add the couplings and propagators appropriate for ZZ production.

In several cases we observe that, while overall a given coefficient $\mathcal{C}$ may be of mixed symmetry under a certain swap operation $\mathcal{S}$, some of its poles are actually either fully symmetric or anti-symmetric. In these cases, we find it convenient to split the coefficients in their symmetric and anti-symmetric parts with respect to $\mathcal{S}$ as,

$$\mathcal{C}^{(S)} = \frac{1}{2}\big[\mathcal{C} + \mathcal{C}\big|_{\mathcal{S}}\big], \quad \mathcal{C}^{(A)} = \frac{1}{2}\big[\mathcal{C} - \mathcal{C}\big|_{\mathcal{S}}\big]. \tag{4.5}$$

4.1 Results for box coefficients

We begin with a decomposition of the box coefficients in terms of the mass m and the left-handed and right-handed couplings of the vector bosons to the quark loop

$$\begin{aligned}
d_{\{i \times j \times k\}} =\ & (v_L^2 + v_R^2) \ \left[m^0 d^{(0)}_{\{i \times j \times k\}} + m^2 d^{(2)}_{\{i \times j \times k\}} + m^4 d^{(4)}_{\{i \times j \times k\}} \right] \\
& + v_L v_R \quad \left[m^2 \tilde{d}^{(2)}_{\{i \times j \times k\}} + m^4 \tilde{d}^{(4)}_{\{i \times j \times k\}} \right] .
\end{aligned} \tag{4.6}$$

i, j and k thus represent the outgoing momenta at three of the four corners of the box. Analogous expansions will follow for triangle and bubble coefficients.

4.1.1 Results for $\{12 \times 34 \times 56\}$ box

The result for the quark mass-independent piece of this box is,

$$\begin{aligned}
d^{(0)}_{\{12 \times 34 \times 56\}} = \Bigg\{ & -\frac{\langle 57 \rangle^2 \langle 7|\Gamma_{12}|4]^2 \left\langle 7|\Gamma_{56|34}|1 \right\rangle^2 (s_{127}s_{567} - s_{12}s_{56})}{4 \langle 56 \rangle \langle 12 \rangle [34] \left\langle 7|\Gamma_{56|12}|7 \right\rangle^4} \Bigg\} \\
& + \Bigg\{ \quad \Bigg\}_{1 \leftrightarrow 5, 2 \leftrightarrow 6} .
\end{aligned} \tag{4.7}$$

It is manifestly symmetric under a swap of the opposite corners of the box diagram, that is $(1,2) \leftrightarrow (5,6)$. To present the result for the m^2 coefficient we employ the decomposition of eq. (4.5), as the $\langle 7|\Gamma_{34|56}|7 \rangle^3$ and the $[7|\Gamma_{34|56}|7]^2$ poles have simple properties under the interchange. We also remind the reader that there exist identities such as

$$\left\langle 7|\Gamma_{34|56}|7 \right\rangle = -\left\langle 7|\Gamma_{12|56}|7 \right\rangle . \tag{4.8}$$

The symmetric part reads

$$\begin{aligned}
d^{(2,S)}_{\{12 \times 34 \times 56\}} = \Bigg\{ & \frac{[67]^2 \langle 3|\Gamma_{12}|7][2|\Gamma_{34|56}|7][2|\tilde{\Gamma}_{12|34|56}|3 \rangle}{4 [12] \langle 34 \rangle [56] \left\langle 7|\Gamma_{34|56}|7 \right\rangle [7|\Gamma_{34|56}|7]^2} + \\
& -\frac{\langle 57 \rangle^2 \langle 7|\Gamma_{12}|4] \left\langle 1|\Gamma_{34|56}|7 \right] [4|\tilde{\Gamma}_{12|34|56}|1 \rangle}{\langle 12 \rangle [34] \langle 56 \rangle \left\langle 7|\Gamma_{34|56}|7 \right\rangle^3} - \frac{\langle 17 \rangle [67]^2 \langle 7|\Gamma_{56}|4][2|\tilde{\Gamma}_{12|34|56}|3 \rangle}{4 [56] \left\langle 7|\Gamma_{34|56}|7 \right\rangle^2 [7|\Gamma_{34|56}|7]}
\end{aligned}$$

$$+\frac{\langle 13\rangle [27][67]^2\langle 3|\Gamma_{12}|7]}{8s_{12}\langle 34\rangle [56][7|\Gamma_{34|56}|7]}+\frac{\langle 13\rangle [46]\langle 57\rangle\left\langle 1|\Gamma_{34|56}|7\right]\left\langle 7|\Gamma_{12|34|56}|7\right]}{4\langle 12\rangle s_{34}s_{56}\left\langle 7|\Gamma_{34|56}|7\right\rangle^2}$$

$$+\frac{\langle 15\rangle [46]\langle 7|\Gamma_{56}|4]\langle 7|\Gamma_{12}|7]\left\langle 1|\Gamma_{34|56}|7\right]}{4\langle 12\rangle [34]s_{56}\left\langle 7|\Gamma_{34|56}|7\right\rangle^2}-\frac{\langle 13\rangle\langle 35\rangle [67]\left\langle 1|\Gamma_{34|56}|7\right]}{8\langle 12\rangle\langle 34\rangle\langle 56\rangle [56]\left\langle 7|\Gamma_{34|56}|7\right\rangle}+$$

$$-\frac{\langle 15\rangle [46][47]\left\langle 1|\Gamma_{34|56}|7\right]}{8\langle 12\rangle [34]\langle 56\rangle [56]\left\langle 7|\Gamma_{34|56}|7\right\rangle}+\frac{\langle 15\rangle [27][46]\langle 7|\Gamma_{56}|4]}{8[34]\langle 56\rangle [56]\left\langle 7|\Gamma_{34|56}|7\right\rangle}+$$

$$-\frac{[26][27]\langle 37\rangle^2[67]}{8[12]\langle 34\rangle [56]\left\langle 7|\Gamma_{34|56}|7\right\rangle}-\frac{\langle 35\rangle^2\left\langle 1|\Gamma_{34|56}|7\right]^2}{4\langle 12\rangle\langle 34\rangle\langle 56\rangle\left\langle 7|\Gamma_{34|56}|7\right\rangle^2}+$$

$$-\frac{[46]^2\left\langle 1|\Gamma_{34|56}|7\right]^2}{2\langle 12\rangle [34][56]\left\langle 7|\Gamma_{34|56}|7\right\rangle^2}+\frac{[24]\langle 57\rangle\langle 5|\Gamma_{36}|4]\left\langle 1|\Gamma_{34|56}|7\right]}{2[34]\langle 56\rangle\left\langle 7|\Gamma_{34|56}|7\right\rangle^2}+$$

$$-\frac{\langle 35\rangle\langle 13\rangle\langle 7|\Gamma_{12}|6]\left\langle 1|\Gamma_{34|56}|7\right]}{2\langle 12\rangle\langle 34\rangle\left\langle 7|\Gamma_{34|56}|7\right\rangle^2}-\frac{[26]\langle 37\rangle^2[67]\left\langle 1|\Gamma_{34|56}|7\right]}{4\langle 34\rangle [56]\left\langle 7|\Gamma_{34|56}|7\right\rangle^2}$$

$$+\langle 35\rangle\left\langle 1|\Gamma_{34|56}|7\right]\frac{\langle 17\rangle [34]\langle 35\rangle-\langle 15\rangle\langle 7|\Gamma_{12}|4]}{4\langle 12\rangle\langle 56\rangle\left\langle 7|\Gamma_{34|56}|7\right\rangle^2}+\frac{\langle 17\rangle\langle 34\rangle [46]^2\left\langle 1|\Gamma_{34|56}|7\right]}{2\langle 12\rangle [56]\left\langle 7|\Gamma_{34|56}|7\right\rangle^2}$$

$$+\frac{\langle 15\rangle [46]\langle 7|\Gamma_{12}|4]\left\langle 1|\Gamma_{34|56}|7\right]}{2\langle 12\rangle [34]\left\langle 7|\Gamma_{34|56}|7\right\rangle^2}-\frac{[12]\langle 17\rangle [46]^2\left\langle 1|\Gamma_{34|56}|7\right]}{2[34][56]\left\langle 7|\Gamma_{34|56}|7\right\rangle^2}-\frac{\langle 13\rangle\langle 15\rangle [46]}{\langle 12\rangle\left\langle 7|\Gamma_{34|56}|7\right\rangle}+$$

$$-3\frac{\langle 13\rangle [26]\langle 35\rangle}{4\langle 34\rangle\left\langle 7|\Gamma_{34|56}|7\right\rangle}-\frac{\langle 13\rangle\left(\langle 13\rangle\langle 5|\Gamma_{12}|6]-\langle 13\rangle\langle 5|\Gamma_{34}|6]+\langle 15\rangle\langle 35\rangle [56]\right)}{2\langle 12\rangle\langle 34\rangle\left\langle 7|\Gamma_{34|56}|7\right\rangle}+$$

$$-3\frac{\langle 15\rangle\langle 13\rangle\left(2\langle 5|\Gamma_{12}|4]-[34]\langle 35\rangle\right)}{16\langle 12\rangle\langle 56\rangle\left\langle 7|\Gamma_{34|56}|7\right\rangle}-\frac{\langle 13\rangle\langle 1|\Gamma_{57}|6]\langle 3|\Gamma_{57}|6]}{2\langle 12\rangle\langle 34\rangle [56]\left\langle 7|\Gamma_{34|56}|7\right\rangle}+$$

$$-\langle 15\rangle\frac{8\langle 12\rangle [24][46]\langle 56\rangle+6[14]\langle 15\rangle\langle 16\rangle [46]+3[14]\langle 15\rangle^2[45]+3\langle 15\rangle [24]\langle 26\rangle [46]}{16\langle 12\rangle [34]\langle 56\rangle\left\langle 7|\Gamma_{34|56}|7\right\rangle}$$

$$+\frac{[46]^2\langle 1|\Gamma_{34}|2]}{2[34][56]\left\langle 7|\Gamma_{34|56}|7\right\rangle}\Bigg\}+\left\{}\right\}_{1\leftrightarrow 5,2\leftrightarrow 6}. \tag{4.9}$$

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The anti-symmetric part reads

$$d^{(2,A)}_{\{12\times34\times56\}} = \left\{ - \frac{\langle17\rangle\langle35\rangle[46]\left\langle1|\Gamma_{34|56}|7\right\rangle\left\langle7|\Gamma_{12|34|56}|7\right]}{4\langle12\rangle s_{34}s_{56}\left\langle7|\Gamma_{12|56}|7\right\rangle^2} \right.$$

$$+\frac{\left\langle5|\Gamma_{34|12}|7\right\rangle(-2\langle13\rangle[27]\langle5|\Gamma_{12}|4]+\langle13\rangle[27][34]\langle35\rangle+\langle15\rangle[47]\langle3|\Gamma_{14}|2])}{8s_{12}s_{34}\langle56\rangle\left\langle7|\Gamma_{12|56}|7\right\rangle}$$

$$\left.+\frac{\langle15\rangle[27][46]\langle7|\Gamma_{56}|4]}{8[34]s_{56}\left\langle7|\Gamma_{12|56}|7\right\rangle}+\frac{\langle13\rangle[27][67]^2\langle3|\Gamma_{12}|7]}{8s_{12}\langle34\rangle[56][7|\Gamma_{12|56}|7\rangle}\right\} - \left\{\quad\right\}_{1\leftrightarrow5,2\leftrightarrow6}. \tag{4.10}$$

The m^4 piece reads

$$d^{(4)}_{\{12\times34\times56\}} = \frac{[2|\tilde{\Gamma}_{34|12|56}|1\rangle}{s_{12}s_{34}s_{56}\left\langle7|\Gamma_{34|56}|7\right\rangle}\left(\frac{[4|\tilde{\Gamma}_{34|12|56}|3\rangle[6|\tilde{\Gamma}_{34|12|56}|5\rangle}{\left\langle7|\Gamma_{34|56}|7\right\rangle[7|\Gamma_{34|56}|7\rangle} - \langle35\rangle[46]\right). \tag{4.11}$$

The helicity flip pieces of this box amplitude (cf. eq. (4.6)) are given by,

$$\tilde{d}^{(2)}_{\{12\times34\times56\}} = \frac{\langle17\rangle\langle35\rangle[46]\left\langle1|\Gamma_{34|56}|7\right\rangle\left\langle7|\Gamma_{12|34|56}|7\right]}{\langle12\rangle s_{34}s_{56}\left\langle7|\Gamma_{34|56}|7\right\rangle^2} + \frac{\langle35\rangle[27]^2[46]}{2[12]s_{34}s_{56}} +$$

$$-\frac{\langle35\rangle[26][27]\left\langle7|\Gamma_{12|34|56}|4\right]}{2[12]s_{34}s_{56}\left\langle7|\Gamma_{34|56}|7\right\rangle}+\frac{\langle13\rangle\langle15\rangle[46]\left\langle7|\Gamma_{12|34|56}|7\right]}{2\langle12\rangle s_{34}s_{56}\left\langle7|\Gamma_{34|56}|7\right\rangle}+$$

$$-\frac{\langle35\rangle[47]\left\langle1|\Gamma_{27}|6\right]\left\langle1|\Gamma_{34|56}|7\right\rangle}{2\langle12\rangle s_{34}s_{56}\left\langle7|\Gamma_{34|56}|7\right\rangle}-\frac{\langle35\rangle[27]^2[67]\langle3|\Gamma_{56}|7]}{2[12]\langle34\rangle s_{56}[7|\Gamma_{34|56}|7\rangle} \tag{4.12}$$

and

$$\tilde{d}^{(4)}_{\{12\times34\times56\}} = \frac{2\langle35\rangle[46][2|\tilde{\Gamma}_{34|12|56}|1\rangle}{s_{12}s_{34}s_{56}\left\langle7|\Gamma_{34|56}|7\right\rangle}. \tag{4.13}$$

Note that eqs. (4.11) and (4.13) are symmetric under the exchanges $3\leftrightarrow5, 4\leftrightarrow6$. This will be important in the following because these functions also supply the m^4 pieces of the box symmetric under this exchange, i.e. $d^{(4)}_{\{56\times12\times34\}}$ and $\tilde{d}^{(4)}_{\{56\times12\times34\}}$.

4.1.2 Results for $\{12\times56\times34\}$ box

The box coefficients for $d_{\{12\times56\times34\}}$ are all obtained from the above results by exchange,

$$d^{(i)}_{\{12\times56\times34\}} = d^{(i)}_{\{12\times34\times56\}}\Big|_{3\leftrightarrow5,4\leftrightarrow6}, \qquad \tilde{d}^{(i)}_{\{12\times56\times34\}} = \tilde{d}^{(i)}_{\{12\times34\times56\}}\Big|_{3\leftrightarrow5,4\leftrightarrow6}. \tag{4.14}$$

4.1.3 Results for $\{56\times12\times34\}$ box

All the coefficients for this box are fully symmetric under the exchange $3\leftrightarrow5, 4\leftrightarrow6$, which is a symmetry of the relevant diagrams, figures 1(a) and 1(b). The mass-independent piece is determined by eq. (4.7),

$$d^{(0)}_{\{56\times12\times34\}} = d^{(0)}_{\{12\times34\times56\}}\Big|_{1\to5,2\to6,3\to1,4\to2,5\to3,6\to4}. \tag{4.15}$$

The coefficient proportional to m^2 has certain elements in common with the suitably permuted $d^{(2,S)}_{\{12\times34\times56\}}$ from eq. (4.9) so we write,

$$
d^{(2)}_{\{56\times12\times34\}} - d^{(2,S)}_{\{12\times34\times56\}}\big|_{\{1\to5,2\to6,3\to1,4\to2,5\to3,6\to4\}}
$$

$$
= \Bigg\{ \frac{[24]\,\langle 57\rangle\,\big\langle 7|\Gamma_{34|12|56}|7\big]\,(\langle 13\rangle\,\big\langle 5|\Gamma_{12|34}|7\big\rangle + 2\,\langle 12\rangle\,\langle 35\rangle\,\langle 7|\Gamma_{34}|2])}{4\,s_{12}s_{34}\,\langle 56\rangle\,\big\langle 7|\Gamma_{34|56}|7\big\rangle^2} +
$$

$$
- \frac{\langle 35\rangle\,[47]\,\langle 1|\Gamma_{34}|2]\,\big\langle 5|\Gamma_{12|34}|7\big\rangle}{4\,s_{12}s_{34}\,\langle 56\rangle\,\big\langle 7|\Gamma_{34|56}|7\big\rangle} - 3\frac{\langle 13\rangle\,[27]\,\langle 35\rangle\,\big\langle 5|\Gamma_{12|34}|7\big\rangle}{8\,s_{12}\,\langle 34\rangle\,\langle 56\rangle\,\big\langle 7|\Gamma_{34|56}|7\big\rangle} +
$$

$$
- \frac{\langle 13\rangle\,\langle 15\rangle\,[47]\,\big\langle 5|\Gamma_{12|34}|7\big\rangle}{8\,\langle 12\rangle\,s_{34}\,\langle 56\rangle\,\big\langle 7|\Gamma_{34|56}|7\big\rangle} - 3\frac{[24]\,[27]\,\langle 35\rangle\,\big\langle 5|\Gamma_{12|34}|7\big\rangle}{8\,[12]\,s_{34}\,\langle 56\rangle\,\big\langle 7|\Gamma_{34|56}|7\big\rangle} +
$$

$$
-3\frac{[24]\,\langle 35\rangle\,[67]\,\langle 7|\Gamma_{34}|2]}{8\,[12]\,s_{34}\,\big\langle 7|\Gamma_{34|56}|7\big\rangle} + \frac{\langle 13\rangle\,\langle 15\rangle\,[47]\,[67]}{16\,\langle 12\rangle\,s_{34}s_{56}}
$$

$$
+ \frac{\langle 35\rangle\,[47]\,[67]\,\langle 1|\Gamma_{34}|7]\langle 1|\Gamma_{56}|7]}{16\,\langle 12\rangle\,s_{34}s_{56}[7|\Gamma_{34|56}|7]} \Bigg\} + \Bigg\{ \Bigg\}_{3\leftrightarrow5,4\leftrightarrow6} . \tag{4.16}
$$

The m^4 term is supplied by eq. (4.11),

$$
d^{(4)}_{\{56\times12\times34\}} = d^{(4)}_{\{12\times34\times56\}} . \tag{4.17}
$$

For the box coefficients for terms with a helicity flip on the massive quark line we have,

$$
\tilde{d}^{(2)}_{\{56\times12\times34\}} = \Bigg\{ \frac{1}{4s_{34}s_{56}}\Bigg[\langle 35\rangle\,[46]\,\big\langle 7|\Gamma_{56|12|34}|7\big]\,\frac{(2\langle 7|\Gamma_{34}|2]^2 - [12]^2\,\langle 17\rangle^2)}{[12]\,\big\langle 7|\Gamma_{34|56}|7\big\rangle^2}
$$

$$
+2\frac{\langle 13\rangle^2\,[34]\,[67]\,\big\langle 7|\Gamma_{34|12}|5\big\rangle}{\langle 12\rangle\,\big\langle 7|\Gamma_{34|56}|7\big\rangle} - 2\frac{\langle 13\rangle\,\langle 56\rangle\,[47]\,[67]^2\,\langle 1|\Gamma_{34}|7]}{\langle 12\rangle\,[7|\Gamma_{12|56}|7]}
$$

$$
+ \frac{\langle 37\rangle\,[47]\,[67]\,(\langle 13\rangle\,[23]\,\langle 57\rangle - \langle 15\rangle\,\langle 7|\Gamma_{15}|2])}{\big\langle 7|\Gamma_{34|56}|7\big\rangle} +
$$

$$
-2\frac{\langle 13\rangle\,[67]\,([26]\,[34]\,\langle 37\rangle\,\langle 56\rangle - \langle 57\rangle\,[24]\,(s_{34}+s_{37}))}{\big\langle 7|\Gamma_{34|56}|7\big\rangle} +
$$

$$
- \frac{\langle 35\rangle\,([24]\,[26]\,\big\langle 7|\Gamma_{56|12|34}|7\big] + [47]\,[67]\,\langle 7|\Gamma_{34}|2]^2)}{[12]\,\big\langle 7|\Gamma_{34|56}|7\big\rangle}\Bigg]\Bigg\} + \Bigg\{ \Bigg\}_{3\leftrightarrow5,4\leftrightarrow6} . \tag{4.18}
$$

The m^4 term is supplied by eq. (4.13),

$$
\tilde{d}^{(4)}_{\{56\times12\times34\}} = \tilde{d}^{(4)}_{\{12\times34\times56\}} . \tag{4.19}
$$

Because a rank two box integral is cut constructible, the vanishing of the rational piece requires the following relationship,

$$\tilde{d}^{(4)}_{\{56\times12\times34\}} = \tilde{c}^{(2)}_{\{7\times12\}} + \tilde{c}^{(2)}_{\{7\times34\}} + \tilde{c}^{(2)}_{\{7\times56\}} + \tilde{c}^{(2)}_{\{34\times56\}} + \tilde{c}^{(2)}_{\{12\times56\}} + \tilde{c}^{(2)}_{\{12\times34\}}. \qquad (4.20)$$

The c coefficients in this eq. (4.20) are presented in subsection 4.2.

4.2 Results for triangle coefficients

The triangle coefficients have no terms which are quartic in the mass of the quark,

$$c_{\{i\times j\}} = (v_L^2 + v_R^2)\left[m^0 c^{(0)}_{\{i\times j\}} + m^2 c^{(2)}_{\{i\times j\}}\right] + v_L v_R\left[m^2 \tilde{c}^{(2)}_{\{i\times j\}}\right]. \qquad (4.21)$$

4.2.1 Results for {34 × 56} triangle

$$
\begin{aligned}
c^{(0)}_{\{34\times56\}} = \Bigg\{ & \frac{\langle17\rangle\langle37\rangle\langle7|\Gamma_{12}|6](s_{35}+s_{36}+s_{45}+s_{46})(-2\langle15\rangle\langle7|\Gamma_{12}|4]-\langle17\rangle\langle5|\Gamma_{36}|4])}{2\langle12\rangle\langle7|\Gamma_{12|56}|7\rangle^3} \\
& +\frac{\langle17\rangle\langle37\rangle^2\langle7|\Gamma_{12}|6]^2\Delta_3(3,4,5,6)(2\langle17\rangle(s_{35}+s_{36}+s_{45}+s_{46})-\langle1|\Gamma_{34|56}|7\rangle)}{4\langle12\rangle\langle34\rangle[56]\langle7|\Gamma_{12|56}|7\rangle^4} \\
& +\frac{\langle17\rangle\langle37\rangle\langle7|\Gamma_{12}|6](s_{35}+s_{36}+s_{45}+s_{46})^2(2\langle17\rangle\langle35\rangle[56]-3\langle17\rangle\langle34\rangle[46]-\langle13\rangle\langle7|\Gamma_{12}|6])}{4\langle12\rangle\langle34\rangle[56]\langle7|\Gamma_{12|56}|7\rangle^3} \\
& +\frac{\langle17\rangle\langle37\rangle\langle7|\Gamma_{12}|6]\langle1|\Gamma_{34|56}|7\rangle(\langle56\rangle[64]-\langle53\rangle[34])}{\langle12\rangle\langle7|\Gamma_{12|56}|7\rangle^3} \\
& +\frac{\langle17\rangle^2(s_{35}+s_{36}+s_{45}+s_{46})(\langle34\rangle^2[46]^2+\langle35\rangle^2[56]^2)}{4\langle12\rangle\langle34\rangle[56]\langle7|\Gamma_{12|56}|7\rangle^2} \\
& +\frac{(s_{35}+s_{36}+s_{45}+s_{46})(\langle13\rangle\langle7|\Gamma_{12}|6]+\langle17\rangle\langle34\rangle[46])^2}{4\langle12\rangle\langle34\rangle[56]\langle7|\Gamma_{12|56}|7\rangle^2} + \\
& -3\frac{\langle3|\Gamma_{56}|4]\langle5|\Gamma_{34}|6]\langle1|\Gamma_{34|56}|1\rangle(s_{127}-s_{34}+s_{56})(s_{127}+s_{34}-s_{56})}{4\langle12\rangle\langle7|\Gamma_{12|56}|7\rangle\Delta_3(3,4,5,6)^2} + \\
& -5\frac{\langle17\rangle^2\langle3|\Gamma_{56}|4]\langle5|\Gamma_{34}|6](s_{127}+s_{34}-s_{56})(s_{127}-s_{34}+s_{56})}{8\langle12\rangle\langle7|\Gamma_{12|56}|7\rangle^2\Delta_3(3,4,5,6)} + \\
& -\frac{\langle3|\Gamma_{56}|4]\langle7|\Gamma_{12}|6]\langle5|\Gamma_{34|12}|7\rangle\langle1|\Gamma_{34|56}|1\rangle}{\langle12\rangle\langle7|\Gamma_{12|56}|7\rangle^2\Delta_3(3,4,5,6)} \\
& +\frac{\langle13\rangle\langle57\rangle s_{127}\langle1|\tilde{\Gamma}_{34|56}|7\rangle(2\langle5|\Gamma_{34}|6][45]-(s_{345}-s_{346})[46])}{4\langle12\rangle\langle7|\Gamma_{12|56}|7\rangle^2\Delta_3(3,4,5,6)} \\
& +\frac{[34]\langle1|\tilde{\Gamma}_{34|56}|7\rangle\langle45\rangle(2\langle3|\Gamma_{56}|4]\langle45\rangle-(s_{456}-s_{356})\langle35\rangle)(6\langle13\rangle\langle7|\Gamma_{12}|6]+3\langle34\rangle[46]\langle17\rangle)}{4\langle12\rangle\langle7|\Gamma_{12|56}|7\rangle^2\Delta_3(3,4,5,6)} \\
& +\frac{\langle13\rangle(2\langle5|\Gamma_{34}|6][45]-(s_{345}-s_{346})[46])(2\langle56\rangle\langle1|\Gamma_{34}|6]-11\langle15\rangle s_{127}+14s_{34}\langle15\rangle)}{8\langle12\rangle\langle7|\Gamma_{12|56}|7\rangle\Delta_3(3,4,5,6)} \\
& +\frac{(2\langle3|\Gamma_{56}|4]\langle45\rangle-(s_{456}-s_{356})\langle35\rangle)(5[46]\langle1|\Gamma_{34|56}|1\rangle-16\langle13\rangle[34]\langle1|\Gamma_{34}|6])}{8\langle12\rangle\langle7|\Gamma_{12|56}|7\rangle\Delta_3(3,4,5,6)}
\end{aligned}
$$

$$+(-12\langle 35\rangle[46]\langle 1|\Gamma_{34|56}|7\rangle + 4\langle 15\rangle\langle 7|\Gamma_{12}|6]\langle 3|\Gamma_{27}|4] +$$

$$-2\langle 13\rangle[34]\langle 35\rangle\langle 7|\Gamma_{34}|6] + 3\langle 17\rangle\langle 34\rangle[46]^2\langle 56\rangle)\frac{\langle 17\rangle}{8\langle 12\rangle\langle 7|\Gamma_{12|56}|7\rangle^2}$$

$$+\frac{\langle 13\rangle\langle 7|\Gamma_{12}|6](-4[14]\langle 15\rangle\langle 17\rangle - 2\langle 15\rangle[24]\langle 27\rangle - 9\langle 17\rangle[34]\langle 35\rangle)}{4\langle 12\rangle\langle 7|\Gamma_{12|56}|7\rangle^2}$$

$$+\frac{\langle 13\rangle\langle 15\rangle[46]}{\langle 12\rangle\langle 7|\Gamma_{12|56}|7\rangle}\bigg\} + \bigg\{\quad\bigg\}_{3\leftrightarrow 5,4\leftrightarrow 6} \tag{4.22}$$

$$c_{\{34\times 56\}}^{(2)} = \bigg\{\frac{1}{4s_{34}s_{56}}\bigg[\Delta_3(3,4,5,6)[6|\tilde{\Gamma}_{12|34|56}|3\rangle$$

$$\times\frac{(-[27][34](\langle 31\rangle\langle 57\rangle - \langle 37\rangle\langle 15\rangle) - [12]\langle 15\rangle\langle 1|\Gamma_{67}|4] + [12]\langle 15\rangle\langle 17\rangle[47])}{s_{12}[7|\Gamma_{56|34}|7]\langle 7|\Gamma_{56|34}|7\rangle^2} +$$

$$-\langle 16\rangle\langle 5|\Gamma_{34}|6][6|\tilde{\Gamma}_{12|34|56}|3\rangle$$

$$\times\frac{(-2(s_{356}-s_{456})(\langle 15\rangle[54]-\langle 17\rangle[74]) + 4\langle 3|\Gamma_{56}|4](\langle 15\rangle[53]-\langle 17\rangle[73]))}{\langle 12\rangle[7|\Gamma_{56|34}|7]\langle 7|\Gamma_{56|34}|7\rangle^2} +$$

$$-\langle 15\rangle(s_{345}-s_{346})[6|\tilde{\Gamma}_{12|34|56}|3\rangle\frac{(\langle 1|\Gamma_{67}|4](s_{356}-s_{456}) - 2\langle 1|\Gamma_{67}|3]\langle 3|\Gamma_{56}|4])}{\langle 12\rangle[7|\Gamma_{56|34}|7]\langle 7|\Gamma_{56|34}|7\rangle^2}$$

$$+\frac{\langle 15\rangle[24]\langle 37\rangle\langle 7|\Gamma_{12}|6]\Delta_3(3,4,5,6)}{s_{12}\langle 7|\Gamma_{56|34}|7\rangle^2} - \frac{[12]\langle 13\rangle\langle 15\rangle[47][67](s_{345}-s_{346})(s_{356}-s_{456})}{2s_{12}\langle 7|\Gamma_{56|34}|7\rangle[7|\Gamma_{56|34}|7]} +$$

$$-\frac{\langle 13\rangle[47]\Delta_3(3,4,5,6)(2\langle 5|\Gamma_{12}|7][26]+[12]\langle 15\rangle[67])}{2s_{12}\langle 7|\Gamma_{56|34}|7\rangle[7|\Gamma_{56|34}|7]} - 2\frac{\langle 13\rangle\langle 1|\Gamma_{56}|4]\langle 5|\Gamma_{34}|6]}{\langle 12\rangle\langle 7|\Gamma_{56|34}|7\rangle}$$

$$+\langle 13\rangle[47]\langle 5|\Gamma_{34}|6](s_{356}-s_{456})\frac{([27](s_{45}+s_{46}-s_{35}-s_{36})+2[12]\langle 15\rangle[57])}{s_{12}\langle 7|\Gamma_{56|34}|7\rangle[7|\Gamma_{56|34}|7]}$$

$$+\frac{\langle 12\rangle[27]^2\langle 3|\Gamma_{56}|4]\langle 5|\Gamma_{34}|6](s_{35}+s_{45}+s_{36}+s_{46})}{s_{12}\langle 7|\Gamma_{56|34}|7\rangle[7|\Gamma_{56|34}|7]} +$$

$$-2\langle 13\rangle\langle 3|\Gamma_{56}|4]\langle 5|\Gamma_{34}|6]\frac{([12]\langle 15\rangle[37][57]+2[27][47]\langle 4|\Gamma_{56}|3])}{s_{12}\langle 7|\Gamma_{56|34}|7\rangle[7|\Gamma_{56|34}|7]}$$

$$+\frac{\Delta_3(3,4,5,6)[6|\tilde{\Gamma}_{12|34|56}|3\rangle[4|\tilde{\Gamma}_{12|34|56}|5\rangle}{[7|\Gamma_{56|34}|7]\langle 7|\Gamma_{56|34}|7\rangle^2}\left(\frac{2\langle 17\rangle^2}{\langle 12\rangle\langle 7|\Gamma_{56|34}|7\rangle} + \frac{[27]^2}{[12][7|\Gamma_{56|34}|7]}\right) +$$

$$-\frac{\langle 1|\Gamma_{34|56}|1\rangle(2\langle 3|\Gamma_{56}|4]\langle 5|\Gamma_{34}|6](s_{35}+s_{45}+s_{36}+s_{46})+\langle 35\rangle[46]\Delta_3(3,4,5,6))}{\langle 12\rangle\langle 7|\Gamma_{56|34}|7\rangle\Delta_3(3,4,5,6)}\bigg]\bigg\}$$

$$+\bigg\{\quad\bigg\}_{3\leftrightarrow 5,4\leftrightarrow 6}. \tag{4.23}$$

The result for the helicity flip part of this triangle is,

$$\tilde{c}_{\{34\times 56\}}^{(2)} = \langle 35\rangle[46]\frac{\langle 17\rangle^2\Delta_3(3,4,5,6) - 2\langle 1|\Gamma_{56|34}|1\rangle\langle 7|\Gamma_{34|56}|7\rangle}{\langle 12\rangle s_{34}s_{56}\langle 7|\Gamma_{34|56}|7\rangle^2}. \tag{4.24}$$

4.2.2 Results for $\{12 \times 56\}$ triangle

The mass-independent term is obtained by exchange,

$$c_{\{12\times56\}}^{(0)} = c_{\{34\times56\}}^{(0)}\Big|_{1\leftrightarrow3,2\leftrightarrow4} \,. \tag{4.25}$$

The m^2 piece contains both symmetric and anti-symmetric parts,

$$
\begin{aligned}
c_{\{12\times56\}}^{(2,S)} = \Bigg\{ &\frac{1}{16 s_{12} s_{34} s_{56}} \Bigg[-4\Delta_3(1,2,5,6)[6|\tilde{\Gamma}_{12|34|56}|1\rangle \\
&\times \frac{([12][47](\langle13\rangle\langle57\rangle - \langle17\rangle\langle35\rangle) + [34]\langle35\rangle(\langle3|\Gamma_{67}|2] + \langle37\rangle[72]))}{[7|\Gamma_{12|56}|7]\langle7|\Gamma_{12|56}|7\rangle^2} + \\
&-8[34]\langle36\rangle\langle5|\Gamma_{12}|6][6|\tilde{\Gamma}_{12|34|56}|1\rangle \\
&\times \frac{(s_{156} - s_{256})(\langle35\rangle[52] - \langle37\rangle[72]) - 2\langle1|\Gamma_{56}|2](\langle35\rangle[51] - \langle37\rangle[71])}{[7|\Gamma_{12|56}|7]\langle7|\Gamma_{12|56}|7\rangle^2} + \\
&-4[34]\langle35\rangle(s_{125} - s_{126})[6|\tilde{\Gamma}_{12|34|56}|1\rangle\frac{(2\langle3|\Gamma_{67}|1]\langle1|\Gamma_{56}|2] - \langle3|\Gamma_{67}|2](s_{156} - s_{256}))}{[7|\Gamma_{12|56}|7]\langle7|\Gamma_{12|56}|7\rangle^2} + \\
&-4\frac{[34]\langle13\rangle[26]\langle37\rangle\langle57\rangle\Delta_3(1,2,5,6)}{\langle7|\Gamma_{12|56}|7\rangle^2} + \\
&-12\frac{[47]\langle3|\Gamma_{12}|7]\langle1|\Gamma_{56}|2]\langle5|\Gamma_{12}|6](s_{15} + s_{16} + s_{25} + s_{26})}{[7|\Gamma_{12|56}|7]\langle7|\Gamma_{12|56}|7\rangle} + \\
&-4\frac{\langle13\rangle[27][47]\langle5|\Gamma_{12}|6](s_{156} - s_{256})(s_{12} + s_{13} + s_{14} + s_{23} + s_{24} + s_{35} + s_{36} + s_{45} + s_{46})}{[7|\Gamma_{12|56}|7]\langle7|\Gamma_{12|56}|7\rangle} \\
&+8\frac{[47]\langle1|\Gamma_{56}|2]\langle5|\Gamma_{12}|6](-\langle3|\Gamma_{12}|7]s_{12} + \langle12\rangle[17][27]\langle37\rangle - \langle13\rangle[17](s_{37} + s_{47}))}{[7|\Gamma_{12|56}|7]\langle7|\Gamma_{12|56}|7\rangle} \\
&+2\frac{\langle13\rangle[34][27]\langle35\rangle[67](\Delta_3(1,2,5,6) - (s_{125} - s_{126})(s_{156} - s_{256}))}{[7|\Gamma_{12|56}|7]\langle7|\Gamma_{12|56}|7\rangle} \\
&+4([27](s_{156} - s_{256})(\langle35\rangle[57] - \langle3|\Gamma_{46}|7]) + 2[17]\langle1|\Gamma_{56}|2]\langle3|\Gamma_{46}|7])\frac{\langle13\rangle[34]\langle5|\Gamma_{12}|6]}{[7|\Gamma_{12|56}|7]\langle7|\Gamma_{12|56}|7\rangle} \\
&+4[27][47]\langle5|\Gamma_{12}|6]\frac{(2\langle12\rangle[14]\langle34\rangle\langle1|\Gamma_{56}|2] - (2s_{34}\langle13\rangle + \langle12\rangle\langle3|\Gamma_{56}|2])(s_{156} - s_{256}))}{[7|\Gamma_{12|56}|7]\langle7|\Gamma_{12|56}|7\rangle} + \\
&-4\frac{\Delta_3(1,2,5,6)[6|\tilde{\Gamma}_{12|34|56}|1\rangle[2|\tilde{\Gamma}_{12|34|56}|5\rangle}{[7|\Gamma_{12|56}|7]\langle7|\Gamma_{12|56}|7\rangle^2}\left(\frac{2\langle37\rangle^2[34]}{\langle7|\Gamma_{12|56}|7\rangle} + \frac{[47]^2\langle34\rangle}{[7|\Gamma_{12|56}|7]}\right) \\
&+4\frac{[34]\langle3|\Gamma_{56|12}|3\rangle}{\langle7|\Gamma_{12|56}|7\rangle}\left(\frac{2\langle1|\Gamma_{56}|2]\langle5|\Gamma_{12}|6](s_{15} + s_{16} + s_{25} + s_{26})}{\Delta_3(1,2,5,6)} + \langle15\rangle[26]\right)\Bigg]\Bigg\} \\
&+\Bigg\{\quad\Bigg\}_{1\leftrightarrow5,2\leftrightarrow6} \tag{4.26}
\end{aligned}
$$

$$
\begin{aligned}
c_{\{12\times56\}}^{(2,A)} = \Bigg\{ &\frac{1}{16 s_{12} s_{34} s_{56}} \\
&\times \Bigg[\frac{\langle37\rangle\Delta_3(1,2,5,6)(-2\langle15\rangle[26]\langle7|\Gamma_{12}|4] - 2\langle13\rangle[26][34]\langle57\rangle - 4[12]\langle15\rangle\langle17\rangle[46])}{\langle7|\Gamma_{12|56}|7\rangle^2}
\end{aligned}
$$

$$+\frac{\langle 13\rangle\,[24]\,\langle 5|\Gamma_{12}|6]\,(-4(s_{15}+s_{16}+s_{25}+s_{26})+8(s_{347}-s_{12}))}{\langle 7|\Gamma_{12|56}|7\rangle}+$$

$$-\frac{\langle 15\rangle\,[27]\,[67]\,([7|\tilde{\Gamma}_{12|56}|4\rangle\langle 7|\tilde{\Gamma}_{12|56}|3\rangle-2\langle 3|\Gamma_{12}|4]\Delta_3(1,2,5,6))}{\langle 7|\Gamma_{12|56}|7\rangle\,[7|\Gamma_{12|56}|7]}$$

$$+4\frac{[24]\,\langle 35\rangle\,[67]\,\langle 7|\tilde{\Gamma}_{12|56}|1\rangle}{\langle 7|\Gamma_{12|56}|7\rangle}\Big]\Big\}-\Big\{\ \Big\}_{1\leftrightarrow 5,2\leftrightarrow 6}. \tag{4.27}$$

The result for the helicity flip part of this triangle is,

$$\tilde{c}^{(2)}_{\{12\times 56\}}=\frac{\langle 53\rangle}{s_{34}s_{56}}\Bigg[-\frac{[46]\,\langle 17\rangle\,\langle 7|\Gamma_{34}|2]\Delta_3(1,2,5,6)}{s_{12}\,\langle 7|\Gamma_{56|34}|7\rangle^2}+\frac{[46]\,\langle 1|\Gamma_{56}|2]\,(s_{15}+s_{16}+s_{25}+s_{26})}{s_{12}\,\langle 7|\Gamma_{56|34}|7\rangle}+$$

$$-\frac{[47]\,[2|\tilde{\Gamma}_{12|34|56}|1\rangle[6|\tilde{\Gamma}_{12|56}|7]}{s_{12}\,\langle 7|\Gamma_{56|34}|7\rangle\,[7|\Gamma_{56|34}|7]}-2\,[26]\,\frac{([24]\,(s_{15}+s_{16}+s_{25}+s_{26}+s_{56})+[34]\,\langle 3|\Gamma_{56}|2])}{[12]\,\langle 7|\Gamma_{56|34}|7\rangle}$$

$$+\frac{\langle 15\rangle\,[56]\,(2\langle 1|\Gamma_{27}|4]-4\,\langle 12\rangle\,[24])}{\langle 12\rangle\,\langle 7|\Gamma_{56|34}|7\rangle}\Bigg]. \tag{4.28}$$

4.2.3 Results for $\{12\times 34\}$ triangle

The results for this triangle are obtained by exchange,

$$c^{(i)}_{\{12\times 34\}}=c^{(i)}_{\{12\times 56\}}\Big|_{3\leftrightarrow 5,4\leftrightarrow 6}\,, \tag{4.29}$$

$$\tilde{c}^{(2)}_{\{12\times 34\}}=\tilde{c}^{(2)}_{\{12\times 56\}}\Big|_{3\leftrightarrow 5,4\leftrightarrow 6}\,. \tag{4.30}$$

4.2.4 Results for $\{7\times 12\}$ triangle

The mass-independent part of the coefficient is determined by infrared relations that ensure the cancellation of $1/\epsilon$ poles in the massless case. In terms of the box integral coefficients defined above we have,

$$\frac{c^{(0)}_{\{7\times 12\}}}{s_{12}-s_{127}}=\frac{d^{(0)}_{\{12\times 56\times 34\}}}{s_{127}s_{347}-s_{12}s_{34}}+\frac{d^{(0)}_{\{12\times 34\times 56\}}}{s_{127}s_{567}-s_{12}s_{56}}\,. \tag{4.31}$$

$$c^{(2)}_{\{7\times 12\}}=\Bigg\{\frac{1}{4s_{34}s_{56}}\Bigg[4\frac{\langle 17\rangle\,\langle 34\rangle\,\langle 57\rangle\,[56]\,[27]\,[47]^2\,\langle 5|\Gamma_{12}|7]}{\langle 7|\Gamma_{12}|7]\,\langle 7|\Gamma_{56|34}|7\rangle\,[7|\Gamma_{56|34}|7]}$$

$$+4\frac{\langle 17\rangle^2\,\langle 37\rangle\,\langle 57\rangle\,[43]\,([65]\,\langle 5|\Gamma_{34}|7]\,\langle 73\rangle-[67]\,\langle 7|\Gamma_{34|56}|3\rangle)(s_{35}+s_{36}+s_{45}+s_{46})}{\langle 12\rangle\,\langle 7|\Gamma_{56|34}|7\rangle^3}$$

$$+\frac{\langle 17\rangle\,\langle 57\rangle\,[27]\,[47]\,([65]\,\langle 5|\Gamma_{34}|7]\,\langle 73\rangle-[67]\,\langle 7|\Gamma_{34|56}|3\rangle)(s_{35}+s_{36}+s_{45}+s_{46})}{\langle 7|\Gamma_{56|34}|7\rangle^2\,[7|\Gamma_{56|34}|7]}$$

$$+2\frac{\langle 17\rangle^2\,\langle 37\rangle\,\langle 5|\Gamma_{34}|6]([47]\,(s_{35}+s_{36}+s_{45}+s_{46})+[4|\Gamma_{12|34}|7])}{\langle 12\rangle\,\langle 7|\Gamma_{56|34}|7\rangle^2}$$

$$+ \frac{[26]\,[27]\,\langle 37\rangle\,\langle 57\rangle\,\langle 7|\Gamma_{12}|4]\,(s_{35}+s_{36}+s_{45}+s_{46})}{[12]\,\big\langle 7|\Gamma_{56|34}|7\big\rangle^2} +$$

$$-\frac{\langle 17\rangle\,\langle 37\rangle\,(2\,[24]\,[67]\,\big\langle 5|\Gamma_{12|34}|7\big\rangle + [12]\,[46]\,\big\langle 1|\Gamma_{56|12}|5\big\rangle)}{\big\langle 7|\Gamma_{56|34}|7\big\rangle^2}$$

$$+\frac{\langle 17\rangle\,\langle 37\rangle\,[12]}{\big\langle 7|\Gamma_{56|34}|7\big\rangle^2}$$

$$\times \Big(\langle 1|\Gamma_{37}|4] \langle 5|\Gamma_{34}|6] - 2\,\langle 13\rangle\,[34]\,\langle 5|\Gamma_{34}|6] + \langle 12\rangle\,[24]\,\langle 5|\Gamma_{12}|6] - \langle 15\rangle\,[56]\,\langle 5|\Gamma_{36}|4] \Big)$$

$$+2\,\frac{[27]^2\,\langle 56\rangle\,[67]\,\langle 3|\Gamma_{12}|7]\,[43]\,([65]\,\langle 5|\Gamma_{34}|7]\,\langle 73\rangle - [67]\,\big\langle 7|\Gamma_{34|56}|3\big\rangle)}{[12]\,\big\langle 7|\Gamma_{56|34}|7\big\rangle\,[7|\Gamma_{56|34}|7]^2}$$

$$+\frac{[27]\,[47]\,\langle 37\rangle\,[65]\,(-2\langle 5|\Gamma_{34}|2]\langle 5|\Gamma_{12}|7] + 3\,\langle 15\rangle\,[12]\,\langle 5|\Gamma_{12}|7] - \langle 15\rangle^2\,[12]\,[17])}{[12]\,\big\langle 7|\Gamma_{56|34}|7\big\rangle\,[7|\Gamma_{56|34}|7]}$$

$$+\big[\,[27]\,[47]\,\langle 37\rangle\,(3\,\langle 15\rangle\,[67]\,(s_{356}+s_{456}) + 2\,[67]\,\big\langle 1|\Gamma_{56|12}|5\big\rangle +$$

$$+ \langle 15\rangle\,\langle 25\rangle\,[27]\,[56] - 2\,\langle 57\rangle\,[67]\,\langle 1|\Gamma_{56}|7])\big]\,\frac{1}{\big\langle 7|\Gamma_{56|34}|7\big\rangle\,[7|\Gamma_{56|34}|7]} +$$

$$-\frac{[27]\,\langle 37\rangle\,[46]\,(\langle 53\rangle\,[32] + \langle 54\rangle\,[42] - 3\,\langle 56\rangle\,[62])}{[12]\,\big\langle 7|\Gamma_{56|34}|7\big\rangle}$$

$$+\frac{\langle 13\rangle\,\langle 57\rangle\,([24]\,[67] + 3\,[26]\,[47])}{\big\langle 7|\Gamma_{56|34}|7\big\rangle} \Big] \Big\} + \Big\{ \quad \Big\}_{3\leftrightarrow 5, 4\leftrightarrow 6} \tag{4.32}$$

$$\tilde{c}^{(2)}_{\{7\times 12\}} = \frac{1}{s_{34}s_{56}}\Bigg[\frac{\langle 17\rangle^2\,\langle 35\rangle\,[46]\,\langle 7|\Gamma_{12}|7]\,(s_{35}+s_{36}+s_{45}+s_{46})}{\langle 12\rangle\,\big\langle 7|\Gamma_{34|56}|7\big\rangle^2}$$

$$+\frac{\langle 7|\Gamma_{12}|7]}{\big\langle 7|\Gamma_{34|56}|7\big\rangle}\Big(\frac{\langle 13\rangle\,\langle 15\rangle\,[46]}{\langle 12\rangle} - \frac{[24]\,[26]\,\langle 35\rangle}{[12]}\Big)$$

$$+ \langle 35\rangle\,[47]\,[67]\,\Big(\frac{\langle 17\rangle^2}{\langle 12\rangle\,\big\langle 7|\Gamma_{34|56}|7\big\rangle} - \frac{[27]^2}{[12]\,[7|\Gamma_{34|56}|7]}\Big)\Bigg]. \tag{4.33}$$

4.2.5 Results for $\{7\times 34\}$ triangle

The infrared condition here is,

$$\frac{c^{(0)}_{\{7\times 34\}}}{s_{34}-s_{347}} = \frac{d^{(0)}_{\{12\times 56\times 34\}}}{s_{127}s_{347}-s_{12}s_{34}} + \frac{d^{(0)}_{\{56\times 12\times 34\}}}{s_{347}s_{567}-s_{34}s_{56}}. \tag{4.34}$$

$$c^{(2,S)}_{\{7\times 34\}} = \frac{1}{4s_{12}s_{34}s_{56}}\Bigg\{4\frac{s_{34}\,\langle 12\rangle\,[56]\,[27]^2\,\langle 37\rangle\,[47]\,\langle 57\rangle\,\langle 5|\Gamma_{34}|7]}{\big\langle 7|\Gamma_{34}|7]\,\big\langle 7|\Gamma_{12|56}|7\big\rangle\,[7|\Gamma_{12|56}|7]} +$$

$$-4\frac{\langle 17\rangle\,\langle 37\rangle^2\,\langle 57\rangle\,[12]\,[34]\,([65]\,\langle 5|\Gamma_{12}|7]\,\langle 71\rangle - [67]\,\big\langle 7|\Gamma_{12|56}|1\big\rangle)(s_{15}+s_{16}+s_{25}+s_{26})}{\big\langle 7|\Gamma_{12|56}|7\big\rangle^3} +$$

$$-2\frac{[47]^2\,[67]\,\langle 1|\Gamma_{34}|7]\,[12]\,\langle 34\rangle\,\langle 56\rangle\,([65]\,\langle 5|\Gamma_{12}|7]\,\langle 71\rangle - [67]\,\langle 7|\Gamma_{12|56}|1\rangle)}{\langle 7|\Gamma_{12|56}|7]\,[7|\Gamma_{12|56}|7]^2}+$$

$$-\langle 37\rangle\,[47]\,([65]\,\langle 5|\Gamma_{12}|7]\,\langle 71\rangle - [67]\,\langle 7|\Gamma_{12|56}|1\rangle)(s_{15}+s_{16}+s_{25}+s_{26})$$

$$\times\frac{(2\,\langle 1|\Gamma_{34|26}|5\rangle + \langle 15\rangle\,(s_{347}+s_{134}+s_{13}+s_{14}))\,[12]}{\langle 7|\Gamma_{12|56}|7\rangle^2\,[7|\Gamma_{12|56}|7]}+$$

$$-\langle 37\rangle\,\langle 57\rangle\,(s_{15}+s_{16}+s_{25}+s_{26})$$

$$\times\frac{(\langle 13\rangle\,[26]\,(s_{347}+s_{23}+s_{24})+\langle 13\rangle\,[2|\Gamma_{34|15}|6]-2\,[27]\,\langle 37\rangle\,\langle 1|\Gamma_{25}|6])\,[43]}{\langle 7|\Gamma_{12|56}|7\rangle^2}+$$

$$-2\frac{\langle 37\rangle^2\,([65]\,\langle 5|\Gamma_{12}|7]\,\langle 71\rangle - [67]\,\langle 7|\Gamma_{12|56}|1\rangle)([12]\,\langle 15\rangle - [26]\,\langle 56\rangle)\,[43]}{\langle 7|\Gamma_{12|56}|7\rangle^2}$$

$$+\frac{[27]\,\langle 37\rangle\,[47]\,\langle 5|\Gamma_{34}|7](-\,\langle 15\rangle\,[56]\,(s_{15}+s_{16}+s_{25}+s_{26})+2\,\langle 12\rangle\,[26]\,s_{567})}{\langle 7|\Gamma_{12|56}|7\rangle\,[7|\Gamma_{12|56}|7]}$$

$$+\frac{[27]\,[47]\,\langle 5|\Gamma_{34}|7](2\,[23]\,\langle 35\rangle\,\langle 37\rangle + 4\,[24]\,\langle 37\rangle\,\langle 45\rangle - 2\,[24]\,\langle 35\rangle\,\langle 47\rangle)\,\langle 12\rangle\,[56]}{\langle 7|\Gamma_{12|56}|7\rangle\,[7|\Gamma_{12|56}|7]}+$$

$$-\frac{\langle 13\rangle\,\langle 37\rangle\,([26]\,\langle 5|\Gamma_{34}|7] + 2\,[27]\,\langle 5|\Gamma_{12}|6])\,[43]}{\langle 7|\Gamma_{12|56}|7\rangle}$$

$$+\frac{\langle 12\rangle\,[26]\,\langle 37\rangle\,(2\,[24]\,\langle 5|\Gamma_{34}|7] + [12]\,\langle 15\rangle\,[47])}{\langle 7|\Gamma_{12|56}|7\rangle}\bigg\}+\bigg\{\quad\bigg\}_{1\leftrightarrow5,2\leftrightarrow6}\tag{4.35}$$

$$c^{(2,A)}_{\{7\times34\}}=\frac{1}{4s_{12}s_{34}s_{56}}\bigg\{\frac{\langle 37\rangle\,\langle 57\rangle\,\langle 7|\Gamma_{34}|7]\,[34]\,(\langle 13\rangle\,[26]\,s_{127}+[24]\,\langle 34\rangle\,\langle 1|\Gamma_{34}|6])}{\langle 7|\Gamma_{12|56}|7\rangle^2}+$$

$$-\frac{\langle 15\rangle\,[24]\,\langle 37\rangle\,\langle 7|\Gamma_{34}|7](\langle 7|\Gamma_{34}|6]s_{567}-[67]\,\langle 7|\Gamma_{12|56}|7\rangle)}{\langle 7|\Gamma_{12|56}|7\rangle^2}\bigg\}-\bigg\{\quad\bigg\}_{1\leftrightarrow5,2\leftrightarrow6}.\tag{4.36}$$

The result for the helicity flip part of this triangle is,

$$\tilde{c}^{(2)}_{\{7\times34\}}=-\,\langle 35\rangle\,[46]\,\langle 7|\Gamma_{34}|7]\frac{(-\,[12]\,\langle 17\rangle^2\,s_{347}-2\,\langle 12\rangle\,[12]\,\langle 17\rangle\,\langle 7|\Gamma_{34}|2]+\langle 12\rangle\,\langle 7|\Gamma_{34}|2]^2)}{s_{12}s_{34}s_{56}\,\langle 7|\Gamma_{12|56}|7\rangle^2}+$$

$$-\langle 7|\Gamma_{34}|7]\frac{(-\,[12]\,\langle 13\rangle\,\langle 15\rangle\,[46]+\langle 12\rangle\,[24]\,[26]\,\langle 35\rangle)}{s_{12}s_{34}s_{56}\,\langle 7|\Gamma_{12|56}|7\rangle}.\tag{4.37}$$

4.2.6 Results for $\{7\times56\}$ triangle

The results for this triangle are obtained by exchange,

$$c^{(i)}_{\{7\times56\}}=c^{(i)}_{\{7\times34\}}\Big|_{3\leftrightarrow5,4\leftrightarrow6}\,,\tag{4.38}$$

$$\tilde{c}^{(2)}_{\{7\times56\}}=\tilde{c}^{(2)}_{\{7\times34\}}\Big|_{3\leftrightarrow5,4\leftrightarrow6}\,.\tag{4.39}$$

4.3 Results for bubble coefficients

The bubble coefficients are all independent of the mass. Consequently the bubble integrals do not contribute to the LR structure. In addition since the full result for the amplitude

is UV finite we have that

$$b^{(0)}_{\{12\}} + b^{(0)}_{\{34\}} + b^{(0)}_{\{56\}} + b^{(0)}_{\{127\}} + b^{(0)}_{\{347\}} + b^{(0)}_{\{567\}} = 0 \,. \tag{4.40}$$

4.3.1 Results for {127} bubble

This function is symmetric under the exchange $(3 \leftrightarrow 5, 4 \leftrightarrow 6)$.

$$
\begin{aligned}
b^{(0,S)}_{\{127\}} = \Bigg\{ & \frac{\left\langle 1|\Gamma_{56|27}|1\right\rangle \langle 5|\Gamma_{36}|4]^2}{2\Delta_3(3,4,5,6)\,\langle 12\rangle\,[34]\,\langle 56\rangle\,\langle 7|\Gamma_{34|56}|7\rangle} + \\
& -\frac{s_{127}[7|\Gamma_{34|56}|7](\langle 17\rangle\,\langle 57\rangle\,\langle 7|\Gamma_{12}|4])^2}{\langle 12\rangle\,[34]\,\langle 56\rangle\,\langle 7|\Gamma_{12}|7]^2\,\langle 7|\Gamma_{34|56}|7\rangle^3} - \frac{\langle 13\rangle\,\langle 17\rangle\,(s_{456}-s_{356})\langle 5|\Gamma_{34}|6]\langle 7|\Gamma_{12}|4]}{2\Delta_3(3,4,5,6)\,\langle 12\rangle\,\langle 7|\Gamma_{34|56}|7\rangle^2} \\
& +\frac{\langle 1|\Gamma_{27}|4]\,\langle 17\rangle\,(s_{127}-s_{56})\langle 5|\Gamma_{34}|6]\langle 7|\Gamma_{12}|4]}{2\Delta_3(3,4,5,6)\,\langle 12\rangle\,[34]\,\langle 7|\Gamma_{34|56}|7\rangle^2} - \frac{\left\langle 1|\Gamma_{35|12}|7\right\rangle\,\langle 17\rangle\,\langle 3|\Gamma_{56}|4]\langle 5|\Gamma_{34}|6]}{2\Delta_3(3,4,5,6)\,\langle 12\rangle\,\langle 7|\Gamma_{34|56}|7\rangle^2} + \\
& -\frac{\langle 15\rangle\,\langle 17\rangle\,\langle 57\rangle\,s_{127}(s_{127}-s_{34})\langle 3|\Gamma_{56}|4]}{2\Delta_3(3,4,5,6)\,\langle 12\rangle\,\langle 56\rangle\,\langle 7|\Gamma_{34|56}|7\rangle^2} + \frac{\langle 15\rangle\,\langle 17\rangle\,[47]\,\langle 57\rangle\,\langle 7|\Gamma_{12}|4]}{2\,\langle 12\rangle\,[34]\,\langle 56\rangle\,\langle 7|\Gamma_{34|56}|7\rangle^2} \\
& +\frac{\langle 17\rangle\,\langle 57\rangle\,\langle 7|\Gamma_{12}|4]([27]\,[46]\,\langle 56\rangle - [12]\,\langle 15\rangle\,[47]+[27]\,[34]\,\langle 35\rangle)}{2\,[34]\,\langle 56\rangle\,\langle 7|\Gamma_{12}|7]\,\langle 7|\Gamma_{34|56}|7\rangle^2} + \\
& -\frac{3(\langle 17\rangle^2\,\Delta_3(3,4,5,6)-4\left\langle 1|\Gamma_{56|27}|1\right\rangle\left\langle 7|\Gamma_{34|56}|7\right\rangle)\langle 3|\Gamma_{56}|4]\langle 5|\Gamma_{34}|6]s_{127}}{4\Delta_3(3,4,5,6)^2\,\langle 12\rangle\,\langle 7|\Gamma_{34|56}|7\rangle^2} \\
& +\frac{3\,\langle 17\rangle\,[27]\,[47]\,\langle 57\rangle^2\,s_{127}\langle 7|\Gamma_{12}|4]}{2\,[34]\,\langle 56\rangle\,\langle 7|\Gamma_{12}|7]^2\,\langle 7|\Gamma_{34|56}|7\rangle^2}\Bigg\} + \Bigg\{\quad\Bigg\}_{3\leftrightarrow 5,4\leftrightarrow 6}\,. \tag{4.41}
\end{aligned}
$$

4.3.2 Results for {12} bubble

This function is symmetric under the exchange $(3 \leftrightarrow 5, 4 \leftrightarrow 6)$.

$$
\begin{aligned}
b^{(0)}_{\{12\}} = \Bigg\{ & \frac{\langle 57\rangle^2\,\langle 7|\Gamma_{12}|4]^2[2|\tilde{\Gamma}_{12|34|56}|1\rangle}{[34]\,\langle 56\rangle\,\langle 7|\Gamma_{12}|7]\,\langle 7|\Gamma_{34|56}|7\rangle^3} - \frac{s_{12}\,\langle 17\rangle\,[27]\,[47]\,\langle 57\rangle^2\,\langle 7|\Gamma_{12}|4]}{2\,[34]\,\langle 56\rangle\,\langle 7|\Gamma_{12}|7]^2\,\langle 7|\Gamma_{34|56}|7\rangle^2} + \\
& -\frac{[12]\,\langle 17\rangle^2\,\langle 37\rangle^2\,\langle 57\rangle^2\,[7|\Gamma_{34|56}|7]}{2\,\langle 34\rangle\,\langle 56\rangle\,\langle 7|\Gamma_{12}|7]\,\langle 7|\Gamma_{34|56}|7\rangle^3} - \frac{\langle 12\rangle\,[27]^2\,\langle 37\rangle^2\,\langle 57\rangle^2}{2\,\langle 34\rangle\,\langle 56\rangle\,\langle 7|\Gamma_{12}|7]\,\langle 7|\Gamma_{34|56}|7\rangle^2} + \\
& -\langle 17\rangle\,\langle 57\rangle\,[12]\,[47]\frac{(-3\,\langle 15\rangle\,\langle 7|\Gamma_{12}|4]+\langle 17\rangle\,\langle 5|\Gamma_{12}|4])}{2\,[34]\,\langle 56\rangle\,\langle 7|\Gamma_{12}|7]\,\langle 7|\Gamma_{34|56}|7\rangle^2} + \frac{\langle 12\rangle\,\langle 57\rangle^2\,[24]^2}{[34]\,\langle 56\rangle\,\langle 7|\Gamma_{34|56}|7\rangle^2} \\
& +\frac{\langle 12\rangle\,[24]^2\,\langle 5|\Gamma_{34|12}|5\rangle}{2\,[34]\,\langle 56\rangle\,\langle 7|\Gamma_{34|56}|7\rangle\,\Delta_3(1,2,3,4)} + \frac{\langle 12\rangle\,[24]\,\langle 57\rangle\,\langle 5|\Gamma_{12}|4]\langle 7|\Gamma_{34}|2](s_{12}-s_{567})}{2\,[34]\,\langle 56\rangle\,\langle 7|\Gamma_{34|56}|7\rangle^2\,\Delta_3(1,2,3,4)}
\end{aligned}
$$

$$+\frac{s_{12}\langle 35\rangle^2 \langle 1|\Gamma_{34}|2]}{2\langle 34\rangle\langle 56\rangle\langle 7|\Gamma_{34|56}|7\rangle\Delta_3(1,2,3,4)}+\frac{s_{12}(s_{12}-s_{567})\langle 17\rangle\langle 35\rangle\langle 37\rangle\langle 5|\Gamma_{34}|2]}{2\langle 34\rangle\langle 56\rangle\langle 7|\Gamma_{34|56}|7\rangle^2\Delta_3(1,2,3,4)}$$

$$+\frac{\langle 15\rangle\langle 37\rangle([12]\langle 15\rangle\langle 37\rangle-2[12]\langle 17\rangle\langle 35\rangle+2[24]\langle 34\rangle\langle 57\rangle)}{2\langle 34\rangle\langle 56\rangle\langle 7|\Gamma_{34|56}|7\rangle^2}+$$

$$-\frac{3\langle 1|\Gamma_{34}|2]\langle 3|\Gamma_{12}|4]\langle 5|\Gamma_{34|12}|5\rangle(s_{34}-s_{567}-s_{12})}{\langle 56\rangle\langle 7|\Gamma_{34|56}|7\rangle\Delta_3^2(1,2,3,4)}$$

$$+\frac{\langle 17\rangle\langle 5|\Gamma_{34|12}|5\rangle(\langle 37\rangle[12]\langle 1|\Gamma_{23}|4]-\langle 13\rangle\langle 7|\Gamma_{12}|4][12]+7\langle 3|\Gamma_{12}|4]\langle 7|\Gamma_{56}|2])}{2\langle 56\rangle\langle 7|\Gamma_{34|56}|7\rangle^2\Delta_3(1,2,3,4)}$$

$$+\{-2[12]\langle 17\rangle[34]\langle 35\rangle-4\langle 15\rangle[24]\langle 7|\Gamma_{34}|1]-4[24]\langle 25\rangle\langle 7|\Gamma_{34}|2]$$

$$+\langle 35\rangle[34]\langle 7|\Gamma_{34}|2]-2\langle 34\rangle\langle 57\rangle[24][34]\}$$

$$\times\frac{[12]\langle 12\rangle\langle 17\rangle\langle 35\rangle}{2\langle 56\rangle\langle 7|\Gamma_{34|56}|7\rangle^2\Delta_3(1,2,3,4)}$$

$$+\langle 37\rangle\langle 1|\tilde{\Gamma}_{12|34}|7\rangle\frac{\left(-[12]\langle 15\rangle\langle 5|\Gamma_{12}|4]-2[24]\langle 5|\Gamma_{34|67}|5\rangle\right)}{2\langle 56\rangle\langle 7|\Gamma_{34|56}|7\rangle^2\Delta_3(1,2,3,4)}$$

$$+\left(2\langle 13\rangle\langle 17\rangle[12]\langle 5|\Gamma_{12}|4]+2\langle 15\rangle\langle 3|\Gamma_{12}|4]\langle 7|\Gamma_{56}|2]+\langle 12\rangle\langle 35\rangle[24]\langle 7|\Gamma_{56}|2]\right)$$

$$\times\frac{\langle 5|\tilde{\Gamma}_{12|34}|7\rangle}{2\langle 56\rangle\langle 7|\Gamma_{34|56}|7\rangle^2\Delta_3(1,2,3,4)}$$

$$+\left\{-7[24]\langle 25\rangle\langle 35\rangle\langle 1|\Gamma_{34}|2]+[24]^2\langle 12\rangle\langle 45\rangle\langle 35\rangle+\langle 35\rangle^2[24]\langle 12\rangle[23]\right.$$

$$+\langle 15\rangle\langle 23\rangle[24][12]\langle 15\rangle+[12]\langle 13\rangle\langle 15\rangle([14]\langle 15\rangle-2[34]\langle 35\rangle)+$$

$$\left.-7[24]\langle 35\rangle\langle 15\rangle(s_{13}+s_{14})+5\langle 15\rangle[24]\langle 34\rangle\langle 5|\Gamma_{12}|4]\right\}$$

$$\left.\times\frac{1}{2\langle 56\rangle\langle 7|\Gamma_{34|56}|7\rangle\Delta_3(1,2,3,4)}\right\}$$

$$+\left\{\quad\right\}_{3\leftrightarrow 5,4\leftrightarrow 6}.\tag{4.42}$$

4.3.3 Results for remaining bubble coefficients

The remaining bubble coefficients are all obtained by exchange,

$$b^{(0)}_{\{347\}}=b^{(0)}_{\{127\}}\Big|_{1\leftrightarrow 3,2\leftrightarrow 4},\qquad b^{(0)}_{\{567\}}=b^{(0)}_{\{127\}}\Big|_{1\leftrightarrow 5,2\leftrightarrow 6},\tag{4.43}$$

$$b^{(0)}_{\{34\}}=b^{(0)}_{\{12\}}\Big|_{1\leftrightarrow 3,2\leftrightarrow 4},\qquad b^{(0)}_{\{56\}}=b^{(0)}_{\{12\}}\Big|_{1\leftrightarrow 5,2\leftrightarrow 6}.\tag{4.44}$$
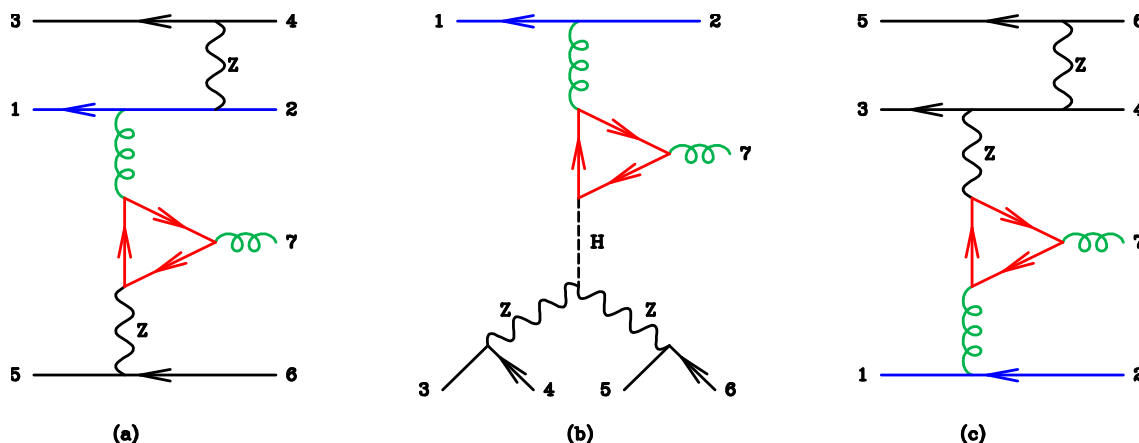
Figure 2. Representative diagrams for additional processes (a) doubly resonant anomaly diagram; (b) Higgs boson contribution; (c) singly resonant anomaly diagram.

4.3.4 Results for rational term

The rational piece is determined by the triangle coefficients proportional to m^2 and the unique box coefficient proportional to m^4, (see eqs. (4.11), (4.19), (4.14))

$$r = \frac{1}{2}(c^{(2)}_{\{7 \times 12\}} + c^{(2)}_{\{7 \times 34\}} + c^{(2)}_{\{7 \times 56\}} + c^{(2)}_{\{34 \times 56\}} + c^{(2)}_{\{12 \times 56\}} + c^{(2)}_{\{12 \times 34\}} - d^{(4)}_{\{12 \times 34 \times 56\}}). \quad (4.45)$$

5 Numerical implementation

We can directly compare our analytic results against those obtained using Recola2 [10], with the model file 'SM_FERM' that only computes the effects of fermion loops. In order to perform a direct comparison we must also account for additional contributions to the amplitude from triangle diagrams in which the Z bosons are not both attached to the quark loop. Representative diagrams for these three processes are shown in figure 2 and explicit results for the contributions are given in appendix A.

We have performed a comparison for the $ZZ+$jet process, obtaining perfect agreement. For the purposes of this comparison we have used a physical value of the top-quark mass and set the bottom quark mass to zero. We observe that the analytic amplitudes presented here are more than an order of magnitude faster than their Recola2 counterparts.

6 Conclusions

In this paper, we have presented compact analytical expressions for the rational coefficients of the master integrals of the one-loop QCD helicity amplitudes for the production of a pair of vector bosons in association with a jet. We have focused on the contribution mediated by a closed quark loop, and retained full dependence on the quark mass. The results are expressed in spinor-helicity variables by factoring out propagators involving masses and by expressing the results in terms of the massless decay products of the vector bosons, which are therefore considered fully off mass shell.

Due to the large number of scales involved in 7-point phase space and the fact that the rational coefficients are ratios of polynomials subject to constraints, namely momentum-conservation and Schouten identities, simplifying the analytical expressions is a complex task. To tackle it, we rely and expand upon recent advances in spinor-helicity simplification techniques based on algebraic geometry and numerical sampling in singular limits. In particular, for the first time beyond five-point amplitudes, we systematically identify irreducible varieties in spinor space, thus quantitatively identifying the pole structure of the amplitude in the analytical continuation to complex momenta. We also observe that some of the involved ideals are not radical, meaning special care is required in making the connection between numerical evaluations and membership to symbolic powers. We employ floating-point and p-adic evaluations close to these irreducible varieties to infer membership of numerator polynomials to symbolic powers of prime ideals. Subsequently, this allows to identify possible partial-fraction decompositions as well as new numerator structures. We fit the numerators by sampling near singular varieties.

The usefulness of compact analytical expressions, and related simplification techniques, goes beyond that of mere theoretical understanding. As the computational load on the Worldwide LHC Computing Grid is predicted to fall short of the demands in the near future [39], speeding up matrix-element providers by using simplified analytic expressions [40] would improve the performance of event generators and aid LHC data analysis. Furthermore, in light of phenomenological applications, and in particular for the numerical stability in singular regions, it would be interesting to understand the interplay between the primary decompositions in complexified momentum space and the real subset of the latter needed for physical kinematics.

Acknowledgments

A Triangle contributions

This section presents results for the triangle contributions illustrated in figure 2. There are three types of triangle contributions to enumerate:

1. Axial anomaly diagrams with one Z boson coupling to the external quark line and the other to the fermion loop, figure 2(a).

2. Higgs-mediated contributions with the Higgs boson coupling to the fermion loop, figure 2(b).

3. Single-resonant axial anomaly diagrams, with one Z boson coupled to the fermion loop whose decays products subsequently radiate the second Z boson, figure 2(c).

These contributions all take a very simple form since they can be obtained by contracting suitable currents with known results for triangle loops containing two off-shell gluons and either a Z or a Higgs boson.

A.1 Double resonant axial anomaly

We first consider contributions such as those depicted in figure 2(a). The amplitude for the production of a Z boson by two offshell gluons has been given for example in ref. [41] and, more specifically for the case at hand, in appendix A of ref. [42]. The amplitude for this contribution is given by contracting this result with the appropriate currents. The basic amplitude is,

$$
A_{ax,56}(1^-, 2^+, 3^-, 4^+, 5^-, 6^+, 7^+) = \frac{\mathcal{G}\,[76]}{\sin\theta_W \cos\theta_W s_{34} s_{56}}
$$
$$
\times \left(\frac{\langle 5|\Gamma_{13}|4]\,[27]\,\langle 13\rangle}{s_{134}} - \frac{\langle 51\rangle\,\langle 3|\Gamma_{24}|7]\,[42]}{s_{234}} \right),
\tag{A.1}
$$

with other helicity amplitudes obtained trivially by symmetries. For example,

$$
A_{ax,56}(1^-, 2^+, 3^-, 4^+, 5^-, 6^+, 7^-) = -A_{ax,56}(2^-, 1^+, 4^-, 3^+, 6^-, 5^+, 7^+)\Big|_{\langle.\rangle \leftrightarrow [.]}, \tag{A.2}
$$

$$
A_{ax,56}(1^-, 2^+, 3^-, 4^+, 5^+, 6^-, 7^+) = A_{ax,56}(1^-, 2^+, 3^-, 4^+, 6^-, 5^+, 7^+), \tag{A.3}
$$

$$
A_{ax,56}(1^-, 2^+, 3^+, 4^-, 5^-, 6^+, 7^+) = A_{ax,56}(1^-, 2^+, 4^-, 3^+, 5^-, 6^+, 7^+), \tag{A.4}
$$

$$
A_{ax,56}(1^+, 2^-, 3^+, 4^-, 5^-, 6^+, 7^+) = -A_{ax,56}(2^-, 1^+, 3^-, 4^+, 5^-, 6^+, 7^+). \tag{A.5}
$$

The loop integral factor $\mathcal{G}$ is defined by $\mathcal{G} = F_1(p_{1234}, p_{56}; m_t) - F_1(p_{1234}, p_{56}; m_b)$ where,

$$
F_1(p_1, p_2; m) = \frac{1}{2(p_2^2 - p_1^2)} \left[1 + 2m^2 C_0(p_1, p_2; m) + \frac{p_2^2}{p_2^2 - p_1^2} (B_0(p_2^2; m) - B_0(p_1^2; m)) \right].
\tag{A.6}
$$

Including the overall factors and also accounting for the additional contribution where the vector boson decay products $((3,4)$ and $(5,6))$ are interchanged we have,

$$
\mathcal{A}_7^{ax,B} = 4ie^4 \frac{g_s}{16\pi^2} t^B \left[P_Z(e_q, q_{34}, v_q, v_{34}) P(s_{56}, M_Z) v_{56} A_{ax,56}(1,2,3,4,5,6,7) \right.
$$
$$
\left. + P_Z(e_q, q_{56}, v_q, v_{56}) P(s_{34}, M_Z) v_{34} A_{ax,56}(1,2,5,6,3,4,7) \right].
\tag{A.7}
$$

A.2 Higgs contribution

We now address another component of the vector boson pair production amplitude, which is the piece containing an intermediate Higgs boson as shown in figure 2(b). This result has been known for almost 35 years [43, 44]. The relevant amplitude is,

$$
0 \to q(p_1) + \bar{q}(p_2) + H(VV) + g(p_7).
\tag{A.8}
$$

This process is of interest since it is one of the simplest processes to illustrate the fundamental role of the Higgs boson in cancelling bad high energy behaviour [45]. The amplitude for the production of a Higgs boson by two offshell gluons has been given for example in refs. [46, 47]. Using that result and attaching the decays of the Z bosons we obtain,

$$\mathcal{A}_7^{h,B}(1^-, 2^+, 3^-, 4^+, 5^-, 6^+, 7^+) = 4ie^4 \frac{g_s^2}{16\pi^2} t^B \tag{A.9}$$

$$\times \frac{F_T \, l_{34} l_{56}}{4 \sin^2 \theta_W \cos^2 \theta_W} \frac{P(s_{127}, M_H)}{s_{127}} \frac{\langle 12 \rangle [27]^2}{(s_{127} - s_{12})} \frac{\langle 35 \rangle [64]}{s_{12}} \frac{P(s_{34}, M_Z)}{s_{34}} \frac{P(s_{56}, M_Z)}{s_{56}},$$

where

$$F_T = \frac{4m^2}{s_{127} - s_{12}} \left[1 + \frac{s_{12}}{s_{127} - s_{12}} \big(B_0(s_{127}; m) - B_0(s_{12}; m) \big) \right.$$

$$\left. + \left(2m^2 - \frac{(s_{127} - s_{12})}{2} \right) C_0(p_{12}, p_7; m) \right], \tag{A.10}$$

and the coupling factors l_{34} and l_{56} are the left-handed couplings of the Z boson decay products (equal to either $v_{L,e}$ or $v_{L,n}$ in eqs. (2.21) and (2.22)). Amplitudes for quarks and leptons of opposite helicity are trivially obtained by interchanging labels ($1 \leftrightarrow 2$, $3 \leftrightarrow 4$ or $5 \leftrightarrow 6$) and coupling factors. The amplitude for a negative helicity gluon is obtained by making the replacement $\langle 12 \rangle [27]^2 \to [12] \langle 17 \rangle^2$ in eq. (A.9).

A.3 Single resonant axial anomaly

This contribution corresponds to diagrams such as the one in figure 2(c). As in the double-resonant case, the result for this contribution can be obtained by contracting the appropriate currents with the known result from refs. [41, 42]. In this case there are two essential amplitudes,

$$A_{\rm sr}(1^-, 2^+, 3^-, 4^+, 5^-, 6^+, 7^+) = \frac{\mathcal{F}}{\sin \theta_W \cos \theta_W} \frac{P(s_{127}, M_Z)}{s_{127}} [27]$$

$$\times \left[l_{34} \frac{P_Z(q_{34}, q_{56}, l_{34}, l_{56}, s_{56})}{s_{56}} \left(\frac{\langle 13 \rangle [46] \langle 5 | \Gamma_{4+6} | 7]}{s_{456}} + \frac{\langle 35 \rangle [47] \langle 1 | \Gamma_{3+5} | 6]}{s_{356}} \right) + \right.$$

$$\left. - l_{56} \frac{P_Z(q_{34}, q_{56}, l_{34}, l_{56}, s_{34})}{s_{34}} \left(\frac{\langle 15 \rangle [46] \langle 3 | \Gamma_{6+4} | 7]}{s_{346}} + \frac{\langle 35 \rangle [67] \langle 1 | \Gamma_{5+3} | 4]}{s_{345}} \right) \right], \tag{A.11}$$

and,

$$A_{\rm sr}(1^-, 2^+, 3^-, 4^+, 5^+, 6^-, 7^+) = \frac{\mathcal{F}}{\sin \theta_W \cos \theta_W} \frac{P(s_{127}, M_Z)}{s_{127}} [27]$$

$$\times \left[l_{34} \frac{P_Z(q_{34}, q_{56}, l_{34}, r_{56}, s_{56})}{s_{56}} \left(\frac{\langle 13 \rangle [45] \langle 6 | \Gamma_{4+5} | 7]}{s_{456}} + \frac{\langle 36 \rangle [47] \langle 1 | \Gamma_{36} | 5]}{s_{356}} \right) \right.$$

$$\left. + r_{56} \frac{P_Z(q_{34}, q_{56}, l_{34}, r_{56}, s_{34})}{s_{34}} \left(\frac{\langle 16 \rangle [45] \langle 3 | \Gamma_{45} | 7]}{s_{345}} + \frac{\langle 36 \rangle [57] \langle 1 | \Gamma_{36} | 4]}{s_{346}} \right) \right]. \tag{A.12}$$

We note that, even accounting for coupling changes, the two are not related by a $5 \leftrightarrow 6$ interchange because one of the terms flips sign. These amplitudes depend explicitly on the

charges of the decay products (q_{34}, q_{56}) and their left- (l_{34}, l_{56}) and right-handed (r_{34}, r_{56}) couplings to Z bosons. The combination of couplings and propagator factors is,

$$P_Z(q_1, q_2, v_1, v_2, s) = q_1 q_2 + v_1 v_2 \, P(s, M_Z) \,. \tag{A.13}$$

The loop factor is $\mathcal{F} = F_1(p_{12}, p_{3456}, m_t) - F_1(p_{12}, p_{3456}, m_b)$ where F_1 has already been specified in eq. (A.6). Remaining amplitudes are obtained by symmetry operations. These correspond to flipping the helicities of the quarks, e.g.,

$$A_{\rm sr}(1^+, 2^-, 3^-, 4^+, 5^+, 6^-, 7^+) = A_{\rm sr}(2^-, 1^+, 3^-, 4^+, 5^+, 6^-, 7^+) \,, \tag{A.14}$$

flipping the helicities of the leptons, e.g.

$$A_{\rm sr}(1^-, 2^+, 3^+, 4^-, 5^-, 6^+, 7^+) = -A_{\rm sr}(1^-, 2^+, 4^-, 3^+, 6^+, 5^-, 7^+) \Big|_{l_{34} \to r_{34}, r_{56} \to l_{56}} \,, \tag{A.15}$$

and flipping the gluon helicity, e.g.

$$A_{\rm sr}(1^-, 2^+, 3^-, 4^+, 5^-, 6^+, 7^+) = -A_{\rm sr}(2^-, 1^+, 4^-, 3^+, 6^+, 5^-, 7^+) \Big|_{\langle . \rangle \leftrightarrow [.]} \,. \tag{A.16}$$

Including overall factors we then have,

$$\mathcal{A}_7^{{\rm sr}, B} = 4ie^4 \frac{g_s^3}{16\pi^2} t^B A_{\rm sr} \,. \tag{A.17}$$

B Integrals

This section gives the precise definition of the scalar integrals. We define the denominators of the integrals as follows,

$$D(\ell) = \ell^2 - m^2 + i\varepsilon \,. \tag{B.1}$$

Either the denominators all have a common non-zero mass, for the case of ZZ production, or the mass can be taken to be zero, $m = 0$ for the case of WW production. In the latter case we ignore the contributions of top and bottom loops. The momenta running through the propagators are,

$$\begin{aligned}
\ell_1 &= \ell + p_1 = \ell + q_1 \\
\ell_{12} &= \ell + p_1 + p_2 = \ell + q_2 \\
\ell_{123} &= \ell + p_1 + p_2 + p_3 = \ell + q_3 \\
\ell_{1234} &= \ell + p_1 + p_2 + p_3 + p_4 = \ell + q_4 \,.
\end{aligned} \tag{B.2}$$

The p_i are the external momenta, whereas the q_i are the off-set momenta in the propagators. In terms of these denominators the integrals are,

$$\begin{aligned}
B_0(p_1; m) &= \frac{\bar{\mu}^{4-n}}{r_\Gamma} \frac{1}{i\pi^{n/2}} \int {\rm d}^n \ell \, \frac{1}{D(\ell)\, D(\ell_1)} \,, \\
C_0(p_1, p_2; m) &= \frac{1}{i\pi^2} \int {\rm d}^4 \ell \, \frac{1}{D(\ell)\, D(\ell_1)\, D(\ell_{12})} \,, \\
D_0(p_1, p_2, p_3; m) &= \frac{1}{i\pi^2} \int {\rm d}^4 \ell \, \frac{1}{D(\ell)\, D(\ell_1)\, D(\ell_{12})\, D(\ell_{123})} \,,
\end{aligned} \tag{B.3}$$

where $r_\Gamma = 1/\Gamma(1 - \epsilon) + O(\epsilon^3)$ and $\bar{\mu}$ is an arbitrary mass scale.

C Asymmetric approaches and ring extensions

In section 3.3.1 we presented several primary decompositions, some of which involve ideals which are not radical, while eq. (3.31), which is the basis for the partial-fraction decomposition of eq. (3.33), requires the involved ideal $\langle \mathcal{D}_\alpha, \mathcal{D}_\beta \rangle$ to be radical. A natural question which arises is then how to generalize this to non-radical ideals. Furthermore, even when the condition on eq. (3.31) is satisfied, i.e. when the ideal is radical, the phase-space point chosen in eq. (3.29) is a very specific one, which satisfies $\mathcal{D}_\alpha \sim \mathcal{D}_\beta \sim \epsilon$. In this case $\mathcal{D}_\alpha, \mathcal{D}_\beta$ vanish symmetrically. It was shown in ref. [26] that by allowing unequal (i.e. asymmetric) degrees of vanishing it is possible glean more information on the numerators. In this appendix, we address both points in unified way with algebraic geometry. In particular, we are going to reduce the asymmetric case to the symmetric one and, in doing so, provide an algebraic interpretation to evaluations in asymmetric approaches.

First of all, let us generalize eq. (3.31) to a special class of non-radical ideals $\langle \mathcal{D}_\alpha, \mathcal{D}_\beta \rangle$, specifically to those whose primary components are (symbolic) powers of their associated primes.[8] To achieve this, let us recall that the κth symbolic power $Q^{\langle \kappa \rangle}$ of any P-primary ideal Q can be defined as the P-primary component of Q^κ, that is,

$$Q^\kappa = Q^{\langle \kappa \rangle} \cap Q_1^{\text{em.}} \cap \cdots \cap Q_m^{\text{em.}}, \tag{C.1}$$

where $Q^{\langle \kappa \rangle}$ is P-primary and the $Q_{1 \leq i \leq m}^{\text{em.}}$ are embedded. See, for instance, ref. [48, lemma 1.18] for why this is a valid definition. It follows that,

$$Q^{\langle \kappa \rangle} = P^{\langle s\kappa \rangle} \quad \text{if} \quad Q = P^s. \tag{C.2}$$

We denote the exponent as s because it is the saturation index of P in Q. Now, let the primary decomposition of $\langle \mathcal{D}_\alpha, \mathcal{D}_\beta \rangle$ read,

$$\langle \mathcal{D}_\alpha, \mathcal{D}_\beta \rangle = \bigcap_l Q_l = \bigcap_l P_l^{s_l}, \tag{C.3}$$

and define κ such that,

$$\kappa = \min\big(\kappa_l \,:\, \mathcal{N}_i \text{ vanishes to order } s_l \cdot \kappa_l \text{ on } V(Q_l)\big). \tag{C.4}$$

We have then achieved a generalization of eq. (3.31), it now reads,

$$\mathcal{N}_i \in \bigcap_l P_l^{\langle s_l \cdot \kappa \rangle} = \bigcap_l Q_l^{\langle \kappa \rangle} \;\Rightarrow\; \mathcal{N}_i \in \langle \mathcal{D}_\alpha, \mathcal{D}_\beta \rangle^{\langle \kappa \rangle}, \;\text{ if } Q_l = P_l^{s_l}. \tag{C.5}$$

This generalization is, however, insufficient: not a single non-radical primary ideal obtained in section 3.3.1 is a power of the associated prime. Since the prime ideal $P_l = \sqrt{Q_l}$ is unique, if no positive integer s_l exists such that $Q_l = P_l^{s_l}$, then it is not possible to find the desired prime ideal P_l in R_n such that the condition on eq. (C.5) is satisfied. Nevertheless, even if such a P_l does not exist in R_n, we can achieve this by extending the ring in which the ideal is defined by allowing it to include roots of polynomials.

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[8]By definition of primary, if a primary ideal is a power of the associated prime ideal, then this power is also a symbolic power.

For our purposes, it suffices to consider a ring extension involving a single s^{th}-root. This can be achieved by extending the quotient ring R_n of eq. (3.7) by a single variable x, and by taking the quotient with respect to an ideal defining x^s as a member q of R_n. That is, we define the extended ring $R_n^\sharp$ as,

$$R_n^\sharp = R_n[x] \,/\, \langle x^s - q \rangle_{R_n[x]} \,, \tag{C.6}$$

where $R_n[x]$ denotes the ring of polynomials in x with coefficients in R_n. Given this definition, $R_n^\sharp$ is a quotient ring of (an extension of) a quotient ring. Nevertheless, we can also regard $R_n^\sharp$ as a simple quotient ring, just like R_n. In fact, by the *Third Isomorphism Theorem* [35], we have,

$$R_n^\sharp \cong S_n[x] \,\Big/\, \Big\langle \sum_{i=1}^n |i\rangle[i], x^s - q \Big\rangle_{S_n[x]} \,. \tag{C.7}$$

That is, $R_n^\sharp$ is isomorphic to the quotient of the (extended) polynomial ring $S_n[x]$, by an ideal whose generators define both momentum conservation and x^s as a polynomial in S_n. Since this ideal is of maximal codimension, by the same reasoning which makes R_n a Cohen-Macaulay ring [27], $R_n^\sharp$ is also Cohen-Macaulay. Effectively, we have extended R_n by $\sqrt[s]{q}$. Therefore, with a slight abuse of notation, let us simply denote x as $\sqrt[s]{q}$. In the following, $R_n^\sharp$ will denote different extensions of R_n, but it will always be clear which one is being considered at any one time depending on the polynomial appearing under the root. We are now in a position to build an ideal $P^\sharp$ of $R_n^\sharp$ such that,

$$(P^\sharp)^s \cap R_n = Q \,, \tag{C.8}$$

even if no prime ideal P of R_n exists such that $P^s = Q$. Note that using the symbolic power instead of the standard power in eq. (C.8) would again be redundant: either the symbolic power coincides with the standard power, or the embedded components in the primary decomposition of the standard power must become redundant in the intersection with R_n.[9] By the same reasoning, given eq. (C.8), it can be shown that,

$$(P^\sharp)^{\langle s\kappa \rangle} \cap R_n = Q^{\langle \kappa \rangle} \,. \tag{C.9}$$

As the numerators $\mathcal{N}_i$ belong to the quotient ring R_n, we are always free to add the intersection with R_n to a membership statement of the form $\mathcal{N}_i \in P^{\sharp\langle s\kappa \rangle}$. Thus, assuming we can find an appropriate $P^\sharp$ and up to a suitable re-scaling of the symbolic power, we are now in a position to numerically obtain information about membership to $Q^{\langle \kappa \rangle}$, independently of whether Q is the power of a prime ideal in R_n. That is, we have completely generalized eq. (C.5). Furthermore, by extending the reasoning to $R_n^\sharp$, we have provided an interpretation to the κth symbolic power of a non-radical ideal $I = \bigcap_l Q_l$ free of embedded components as the set of polynomials vanishing to degree $s_l\kappa$ on the varieties $V(P_l^\sharp)$ in $R_n^\sharp$, where the relation between Q_l and $P_l^\sharp$ is given by eq. (C.8).

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[9]To show this, one has to remember that Q is a primary ideal, and that inclusion is preserverd in the intersection with a subring, i.e. $\sqrt{(P^\sharp)^{\langle s \rangle}} \subset \sqrt{Q_i^{\sharp,\text{em.}}} \Rightarrow \sqrt{(P^\sharp)^{\langle s \rangle}} \cap R_n \subseteq \sqrt{Q_i^{\sharp,\text{em.}}} \cap R_n$.

Let us now consider applications to the problem at hand. Starting from the first non-radical ideal that we encountered in eqs. (3.38) and (3.39), for the "sharp" prime ideal in the ring extension we can write,

$$\left\langle \sqrt{\langle 17 \rangle}, |7\rangle \right\rangle_{R_7^\sharp},$$ (C.10)

such that taking the second power we obtain,

$$\left\langle \sqrt{\langle 17 \rangle}, |7\rangle \right\rangle_{R_7^\sharp}^2 = \left\langle \langle 17 \rangle, |7\rangle \sqrt{\langle 17 \rangle}, |7\rangle \langle 7| \right\rangle_{R_7^\sharp}.$$ (C.11)

Then, intersecting with R_n, we have,

$$\left\langle \sqrt{\langle 17 \rangle}, |7\rangle \right\rangle_{R_7^\sharp}^2 \cap R_7 = \left\langle \langle 17 \rangle, |7\rangle \langle 7| \right\rangle_{R_7},$$ (C.12)

that is, we have effectively removed the generator involving the radical. This is an explicit example of the form of eq. (C.8). From this construction we observe that the phase-space point required to infer membership to symbolic powers of the ideal of eq. (C.10) is such that,

$$\sqrt{\langle 17 \rangle} \sim \epsilon, \ |7\rangle \sim \epsilon \Rightarrow \langle 17 \rangle \sim \epsilon^2, \ |7\rangle \sim \epsilon,$$ (C.13)

while a standard "symmetric" approach to $V\left(\langle |7\rangle \rangle\right)$ reads,

$$\langle 17 \rangle \sim \epsilon, \ |7\rangle \sim \epsilon.$$ (C.14)

An analogous construction can be followed for the ideals in eqs. (3.43) and (3.44), for example we can write,

$$\left\langle \sqrt{\langle 12 \rangle}, (s_{567} - s_{34}) \right\rangle_{R_7^\sharp},$$ (C.15)

where we stress that the different meaning of $R_7^\sharp$ between eq. (C.15) and eq. (C.10). In this case, we also have that the ideal is of maximal codimension, thus its symbolic powers must coincide with normal powers. Once again, we have constructed the desired ideal,

$$\left\langle \sqrt{\langle 12 \rangle}, (s_{567} - s_{34}) \right\rangle_{R_7^\sharp}^2 \cap R_7 = \left\langle \langle 12 \rangle, \Delta_3(1, 2, 3, 4) \right\rangle_{R_7}.$$ (C.16)

Finally, as promised, the same approach can also be employed independently of whether an ideal is radical or not, in order to obtain further data regarding the pole structure of the integral coefficients. For instance, let us reconsider the primary decomposition of eq. (3.41). Within a suitable ring extension $R_7^\sharp$, we can write,

$$\left\langle \langle 7|\Gamma_{34|56}|7\rangle, \sqrt{[7|\Gamma_{34|56}|7]} \right\rangle_{R_7^\sharp} = \left\langle \langle 7|\Gamma_{34}|7], \langle 7|\Gamma_{56}|7], \langle 7|\Gamma_{34|56}|7\rangle, \sqrt{[7|\Gamma_{34|56}|7]} \right\rangle_{R_7^\sharp}$$
$$\cap \left\langle \langle 7|\Gamma_{34|56}|7\rangle, \sqrt{[7|\Gamma_{34|56}|7]}, \tilde{\Gamma}_{12|34|56} \right\rangle_{R_7^\sharp},$$ (C.17)

where we verified again in $R_7^\sharp$ the equality as well as the primality of the ideals in the r.h.s. We can than combine constraints from the simultaneous membership to a symbolic power of $\left\langle \langle 7|\Gamma_{34|56}|7\rangle, [7|\Gamma_{34|56}|7] \right\rangle_{R_7}$ and $\left\langle \langle 7|\Gamma_{34|56}|7\rangle, \sqrt{[7|\Gamma_{34|56}|7]} \right\rangle_{R_7^\sharp}$ to obtain refined partial fraction decompositions. To see this in practice, let us refer back to section 4. It can be seen that a number of coefficients, such as $d^{(2,A)}_{\{12\times34\times56\}}$ in eq. (4.10), $\tilde{d}^{(2)}_{\{12\times34\times56\}}$ in eq. (4.12)

and $\tilde{c}^{(2)}_{\{7\times 12\}}$ in eq. (4.33), have a double pole on $V\left(\left\langle\langle 7|\Gamma_{34|56}|7\rangle\right\rangle\right)$ and a simple pole on $V\left(\left\langle[7|\Gamma_{34|56}|7]\right\rangle\right)$, i.e. they read,

$$\mathcal{C}_i \propto \frac{\mathcal{N}_i}{\langle 7|\Gamma_{34|56}|7\rangle^2[7|\Gamma_{34|56}|7]}\,. \tag{C.18}$$

Their numerator in least common denominator form belongs to $\left\langle\langle 7|\Gamma_{34|56}|7\rangle, [7|\Gamma_{34|56}|7]\right\rangle_{R_7}$, therefore we can write them as,

$$\mathcal{C}_i \propto \frac{\mathcal{N}_{i1}}{\langle 7|\Gamma_{34|56}|7\rangle^2} + \frac{\mathcal{N}_{i2}}{\langle 7|\Gamma_{34|56}|7\rangle[7|\Gamma_{34|56}|7]}\,. \tag{C.19}$$

However, by probing them in the asymmetric approach we also obtain the constraint,

$$\mathcal{N}_i \in \left\langle \langle 7|\Gamma_{34|56}|7\rangle, \sqrt{[7|\Gamma_{34|56}|7]} \right\rangle^{\langle 2\rangle}_{R_7^\sharp} \cap R_7\,. \tag{C.20}$$

As this is a maximal codimension ideal the symbolic power coincides with the normal power. Thus, we conclude that a more accurate representation is,

$$\mathcal{C}_i \propto \frac{\mathcal{N}_{i1}}{\langle 7|\Gamma_{34|56}|7\rangle^2} + \frac{\mathcal{N}_{i2}}{[7|\Gamma_{34|56}|7]}\,. \tag{C.21}$$

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