

THE UNIVERSAL TROPICALIZATION AND THE BERKOVICH ANALYTIIFICATION

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Given an integral scheme X over a non-archimedean valued field k , we construct a universal closed embedding of X into a k -scheme equipped with a model over the field with one element \mathbb{F}_1 (a generalization of a toric variety). An embedding into such an ambient space determines a tropicalization of X by previous work of the authors, and we show that the set-theoretic tropicalization of X with respect to this universal embedding is the Berkovich analytification X^{an} . Moreover, using the scheme-theoretic tropicalization we previously introduced, we obtain a tropical scheme $\mathcal{Trop}_{\text{univ}}(X)$ whose \mathbb{T} -points give the analytification and that canonically maps to all other scheme-theoretic tropicalizations of X . This makes precise the idea that the Berkovich analytification is the universal tropicalization. When $X = \text{Spec } A$ is affine, we show that $\mathcal{Trop}_{\text{univ}}(X)$ is the limit of the tropicalizations of X with respect to all embeddings in affine space, thus giving a scheme-theoretic enrichment of a well-known result of Payne. Finally, we show that $\mathcal{Trop}_{\text{univ}}(X)$ represents the moduli functor of semivaluations on X , and when $X = \text{Spec } A$ is affine there is a universal semivaluation on A taking values in the idempotent semiring of regular functions on the universal tropicalization.

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1. INTRODUCTION

In recent years, two methods of translating problems from algebraic geometry into other landscapes, both based on non-archimedean valuations, have become increasingly important. The first is *tropicalization* [8, 24, 27], which reduces the complexity of varieties by turning them into finite polyhedral complexes. The second is Berkovich's non-archimedean *analytification* [2], where even the affine line becomes an intricate fractal-like infinitely branching tree.

Analytification is intrinsic, whereas tropicalization depends on the choice of an embedding into a toric variety. Payne showed that these two processes are intimately related: the Berkovich analytification of an affine variety over a complete valued field is the category-theoretic limit of its tropicalizations with respect to embeddings in affine spaces (see [29, Theorem 1.1], as well as [29, Theorem 4.2], [9], and [20, Theorem 6.4]

for global variants). Therefore the analytification is often thought of as an intrinsic and universal tropicalization. This view is also hinted at in the unfinished paper [19], and further justified by the existence of skeletons — polyhedral complexes resembling tropical varieties onto which the analytification admits a strong deformation retraction [2, 3, 4, 12, 13, 18, 28].

A second way to think of the analytification is as a space of semivaluations. Given a non-archimedean valued field k and a k -algebra A , the Berkovich analytification of $\text{Spec } A$ can be defined as the space of all rank-one semivaluations on A compatible with the valuation on k , and this description extends to non-affine schemes with an appropriate notion of (semi)valuation in the non-affine case.

Thus we have two heuristic devices with which to view the analytification: as the universal tropicalization, and as the moduli space of semivaluations. Using the ideas developed in [11] — (1) a generalization of the ambient spaces in which to tropicalize from toric varieties to schemes equipped with models over the field with one element, (2) a refinement of tropicalization from sets to semiring schemes, and (3) a generalization of semivaluations to take values in arbitrary idempotent semirings — we are now able to make both of these ideas into precise statements.

1.1. The universal tropicalization

The embeddings originally used for tropicalization were taken to be in an algebraic torus, as in [8, 32], but Payne and Kajiwara showed that there is a natural extension to arbitrary toric varieties simply by computing the tropicalization separately for each torus-invariant stratum and assembling the results [16, 29]. The underlying reason that toric varieties yield tropicalizations of their subschemes is that they have a distinguished class of monomials in the coordinate ring of each torus-invariant affine patch, and all the gluing of affine patches occurs by localization of monomials.

One is thus led immediately to consider a more general class of ambient spaces in which to tropicalize subschemes: k -schemes Z equipped with a model over \mathbb{F}_1 , the field with one element. This means an \mathbb{F}_1 -scheme Z' (essentially in the sense of [35] or [17, 5, 6]) and an isomorphism $Z \cong Z' \times_{\text{Spec } \mathbb{F}_1} \text{Spec } k$, or in more concrete terms, it means that there is an open affine covering of Z for which the coordinate rings are presented as monoid rings, and the localizations with which these affine patches are glued are induced by localizations of the monoids. See [11, §3] and the references therein for details. This generalization of toric varieties allows, for instance, non-finite type schemes, and more limits exist in this category. In particular, while there is no initial object in the category of embeddings of a scheme X into toric varieties, the category of embeddings into schemes equipped with an \mathbb{F}_1 -model does have an initial object, which we denote $X \hookrightarrow \widehat{X}$. This universal embedding is completely explicit; on each affine patch it is the unit transformation of the base-change adjunction between affine \mathbb{F}_1 -schemes and affine k -schemes. More concretely, if $X = \text{Spec } A$ is affine, \widehat{X} is the spectrum of the monoid ring $k[A]$, and $X \hookrightarrow \widehat{X}$ corresponds to the evaluation map $k[A] \rightarrow A$; so the “monomials” are the elements of A . This construction extends to the case when X is not affine.

In [11] we showed how one can tropicalize a closed subscheme of any integral k -scheme equipped with a model over \mathbb{F}_1 . In this paper we show that one can obtain the Berkovich analytification not merely as a limit of tropicalizations, but as a tropicalization itself.

Theorem A. Let X be an integral scheme over a non-archimedean valued field k . The set-theoretic tropicalization of X with respect to the universal embedding $X \hookrightarrow \widehat{X}$ is canonically identified with the underlying set of the Berkovich analytification X^{an} . Moreover, this identification becomes a homeomorphism when the tropicalization of X is endowed with the “strong Zariski topology” in which the closed subsets are given by closed subschemes.

Remark 1.1.1. It follows immediately from the universal property of the embedding $X \hookrightarrow \widehat{X}$, and functoriality of tropicalization, that the corresponding tropicalization is the limit of *all* tropicalizations (not just those coming from embeddings in toric varieties). The statement that one often obtains the same limit when restricting to a class of embeddings in toric varieties [9, 29] requires an additional argument.

Remark 1.1.2. Given a system of toric embeddings such that the corresponding limit of tropicalizations is the analytification (cf., [9]), one can obtain the analytification as the tropicalization with respect to the (typically infinite) product of these toric embeddings. However, not all varieties can be embedded in a toric variety [38], so for such varieties one must leave the realm of toric tropicalization and embrace more general \mathbb{F}_1 -schemes in order to recover analytification as a tropicalization.

The main construction introduced in [11] is a scheme-theoretic refinement of tropicalization that reduces to the set-theoretic tropicalization upon passing to the set of \mathbb{T} -points, where \mathbb{T} is the tropical semiring $(\mathbb{R} \cup \{\infty\}, \min, +)$. In brief, given a closed embedding locally described as a quotient of a monoid ring $k[M]$, the tropicalization is a quotient of the semiring $\mathbb{T}[M]$, and the equations of the tropicalization are produced by valuating the coefficients of the original equations and then applying the *bend relations* (see §3.1 for a review).

When we apply scheme-theoretic tropicalization to the universal embedding $X \hookrightarrow \widehat{X}$ we obtain a semiring scheme underlying the analytification.

Theorem B. Let X be an integral scheme over a non-archimedean valued ring k . The scheme-theoretic tropicalization associated with the embedding $X \hookrightarrow \widehat{X}$ admits a canonical morphism of \mathbb{T} -schemes to the tropicalization associated with any other closed embedding of X into a locally integral scheme with \mathbb{F}_1 -model. Upon passing to \mathbb{T} -points, these morphisms reduce to the canonical projections from the Berkovich analytification to all set-theoretic tropicalizations.

Accordingly, we call this scheme the *universal tropicalization* of X and denote it by $\mathcal{Trop}_{\text{univ}}(X)$.

Since $\mathcal{Trop}_{\text{univ}}(X)$ is the initial object in the category of tropicalizations (with morphisms induced by morphisms of embeddings), it is trivially the limit (in the category of \mathbb{T} -schemes) of the diagram of *all* scheme-theoretic tropicalizations. We also restrict to the subcategory of tropicalizations from embeddings in affine space and prove a scheme-theoretic refinement of Payne’s affine limit result:

Theorem C. If X is an affine integral scheme of finite type over k , then the limit of its tropicalizations with respect to all closed embeddings in finite-dimensional affine spaces is naturally isomorphic as a \mathbb{T} -scheme to the universal tropicalization $\mathcal{Trop}_{\text{univ}}(X)$.

Remark 1.1.3. For simplicity, we have restricted attention to the affine case of this theorem. A modification of the argument along the lines of [29, Theorem 4.2] should allow one to easily prove the corresponding statement for embeddings of a quasiprojective variety X into toric varieties.

1.2. The moduli space of semivaluations

The category of \mathbb{T} -schemes and the universal tropicalization $\mathit{Trop}_{P_{univ}}(X)$ allow us to make precise the idea that the Berkovich analytification is a moduli space of semivaluations. Let A be a k -algebra, where k is a field equipped with a non-archimedean valuation $\nu : k \rightarrow \mathbb{T}$. Recall that the points of the analytification of $\mathit{Spec} A$ are the rank-one semivaluations on A compatible with ν .

In [11, Definition 2.5.1] and [25, §4.2] a generalization of the notion of semivaluation on a ring was introduced by replacing \mathbb{T} with an arbitrary idempotent semiring S — a semivaluation in this sense is now a multiplicative map $A \rightarrow S$ that satisfies a certain superadditivity condition with respect to the canonical partial order on S . If k is equipped with a semivaluation $\nu : k \rightarrow S$, and T is an S -algebra, then a semivaluation $A \rightarrow T$ is said to be *compatible* with ν if the square

$$\begin{array}{ccc} k & \longrightarrow & S \\ \downarrow & & \downarrow \\ A & \longrightarrow & T \end{array}$$

commutes.

Theorem D. Let k be a field, S an idempotent semiring, and $\nu : k \rightarrow S$ a semivaluation. Given an integral k -algebra A , the S -scheme $\mathit{Trop}_{univ}(\mathit{Spec} A)$ represents the functor on affine S -schemes sending $\mathit{Spec} T$ to the set of semivaluations $A \rightarrow T$ compatible with ν . In particular, there is a universal semivaluation on A compatible with ν and it takes values in the semiring of regular functions on the universal tropicalization.

Remark 1.2.1. This moduli functor of semivaluations was first shown to be representable in semiring schemes by MacPherson in [25, Theorem 6.24]. Thus, the main novelty of this theorem is to identify the representing scheme with the universal tropicalization.

Note that the generalized semivaluations we consider here include higher rank Krull (semi)valuations (where S is a totally ordered idempotent semifield). Thus the universal tropicalization contains information about Huber’s adic space analytification [14] in addition to the rank-one information of the Berkovich analytification. Note also that in order for the universal semivaluation to exist, the total ordering on the value group for semivaluations must indeed be weakened to a partial ordering, as in [11, Definition 2.5.1].

Conventions

Throughout this paper all algebraic objects will be assumed to be commutative. Monoids, rings, and semirings are always assumed to have a multiplicative unit, and semirings are always assumed to have an additive unit as well.

2. THE UNIVERSAL EMBEDDING

Let X be a scheme over a ring R . In this section we will construct an embedding $X \hookrightarrow \widehat{X}$ that is universal among embeddings of X that determine tropicalizations of X , i. e., embeddings over $\text{Spec } R$ into schemes equipped with a locally integral model over the field with one element \mathbb{F}_1 .

2.1. The evaluation map

We shall use the naive version of the field with one element \mathbb{F}_1 as put forward by [17, 5, 35]. Rather than define \mathbb{F}_1 as an object directly, one instead specifies what its categories of modules and algebras should be.

Definition 2.1.1. An \mathbb{F}_1 -module is a set equipped with a distinguished basepoint. An \mathbb{F}_1 -algebra is a monoid-with-zero, i. e., a commutative monoid B (written multiplicatively) with a multiplicative unit 1 and an element 0 such that $b \cdot 0 = 0$ for all $b \in B$. An \mathbb{F}_1 -algebra is said to be *integral* if the set of nonzero elements forms a monoid (i. e., there are no zero-divisors) that is cancellative (i. e., the canonical map from this monoid to its group completion is injective). A *homomorphism* of \mathbb{F}_1 -algebras is a monoid homomorphism sending 0 to 0.

Remark 2.1.2. Note that an \mathbb{F}_1 -algebra with no zero-divisors need not be integral. For example, the \mathbb{F}_1 -algebra $B = \{0, 1, x, y, y^2, y^3, \dots\}$ with the relation $xy = y^2$ has no zero-divisors, but the monoid of non-zero elements fails to be cancellative.

Let R be a (semi)ring. There is a forgetful functor

$$R\text{-mod} \rightarrow \mathbb{F}_1\text{-mod}$$

that sends an R -module to its underlying set with 0 as the distinguished point; we will refer to this as a *scalar restriction* functor. It admits a left adjoint that sends an \mathbb{F}_1 -module to the free R -module generated by the non-basepoint elements; we call this the *scalar extension* functor and denote it by $-\otimes_{\mathbb{F}_1} R$. This adjunction at the level of modules induces an adjoint pair of functors

$$-\otimes_{\mathbb{F}_1} R : \mathbb{F}_1\text{-alg} \rightleftarrows R\text{-alg} : M(-). \quad (1)$$

The scalar restriction functor sends an R -algebra A to its underlying multiplicative monoid $M(A)$; we will write x_a for the element corresponding of $a \in A$. The scalar extension functor sends an \mathbb{F}_1 -algebra B to the R -algebra with one generator x_b for each element $b \in B$ and the relations

$$\begin{aligned} x_a x_b &= x_{ab} \quad \text{for } a, b \in B, \\ x_1 &= 1, \\ x_0 &= 0. \end{aligned}$$

Given an R -algebra A and an \mathbb{F}_1 -algebra B , the adjoint of an R -algebra homomorphism $f : B \otimes_{\mathbb{F}_1} R \rightarrow A$ is the map $B \rightarrow M(A)$ sending b to $x_{f(b)}$.

Lemma 2.1.3. A ring A is an integral domain if, and only if, the \mathbb{F}_1 -algebra $M(A)$ is integral.

Proof. If A is an integral domain, then it has no zero-divisors, so the set of non-zero elements $A \setminus \{0\}$ is a cancellative monoid (a non-cancellative relation $xy = xz$ would imply the existence of a zero divisor: $x(y - z) = 0$). Conversely, if $A \setminus \{0\}$ is a cancellative monoid then $A \setminus \{0\} = M(A) \setminus 0$ injects into its group completion, which is $M(\text{Frac}(A)) \setminus \{0\}$, so A injects into its field of fractions. \square

Given an R -algebra A , we will write \widehat{A} for the R -algebra $M(A) \otimes_{\mathbb{F}_1} R$ (i.e., first apply scalar restriction and then scalar extension). Elements of \widehat{A} are finite formal R -linear combinations of elements x_a for $a \in A \setminus \{0\}$. Note that \widehat{A} is functorial in A ; an R -algebra homomorphism $f : A \rightarrow B$ induces an R -algebra homomorphism $\widehat{f} : \widehat{A} \rightarrow \widehat{B}$ sending $x_a \mapsto x_{f(a)}$. The adjunction (1) comes with a counit natural transformation $\widehat{A} \rightarrow A$ which admits an explicit description. The counit is a surjective R -algebra homomorphism ev defined by

$$ev : x_a \mapsto a.$$

We call this map the *evaluation* because it evaluates a formal R -linear combination of elements of A to an element of A using the arithmetic operations of A :

$$ev \left(\sum_i \lambda_i x_{a_i} \right) = \sum_i \lambda_i a_i,$$

for $\lambda_i \in R$ and $a_i \in A$.

The kernel of ev is

$$\ker(ev) = \left\{ \sum_i \lambda_i x_{a_i} \mid \sum_i \lambda_i a_i = 0 \right\},$$

and it admits a much smaller presentation that will be useful later on.

Proposition 2.1.4. Given an R -algebra A , the kernel of $ev : \widehat{A} \rightarrow A$ is generated as a \mathbb{Z} -module by the following elements:

1. $\lambda x_a - x_{\lambda a}$, for $a \in A$ and $\lambda \in R$;
2. $x_a + x_b + x_c$, for $a, b, c \in A$ with $a + b + c = 0$.

Proof. Given an expression $E = \sum_{i=1}^n \lambda_i x_{a_i}$ in the kernel of ev , we will reduce it to zero by adding an appropriate sequence of elements of the two types above. Adding $x_{\lambda_i a_i} - \lambda_i x_{a_i}$ to E , for each i , reduces it to the expression

$$E' = \sum_{i=1}^n x_{\lambda_i a_i}.$$

This is in the kernel of ev if and only if $\sum_i \lambda_i a_i = 0$. Subtracting first $x_{\lambda_1 a_1} + x_{\lambda_2 a_2} + x_{-\lambda_1 a_1 - \lambda_2 a_2}$ and then $-x_{-\lambda_1 a_1 - \lambda_2 a_2} - x_{\lambda_1 a_1 + \lambda_2 a_2}$ from E' yields an expression of the same form but with one fewer terms. Repeating this inductively eventually yields $x_{\sum_i \lambda_i a_i}$; the subscript is zero by the hypothesis that the original expression E was in the kernel of ev , and $x_a = 0$ in \widehat{A} if and only if $a = 0$. \square

2.2. The universal embedding in the affine case

We now describe how the evaluation map $ev : \widehat{A} \rightarrow A$ gives a universal embedding of $\text{Spec } A$ into an ambient space with the structure required to tropicalize $\text{Spec } A$.

Definition 2.2.1. An \mathbb{F}_1 -framing on an R -scheme X is an \mathbb{F}_1 -scheme Y and a morphism of R -schemes $X \rightarrow Y \times_{\mathbb{F}_1} \text{Spec } R$. We say that a framing is affine if X and Y are both affine. A morphism of \mathbb{F}_1 -framed schemes is a diagram

$$\begin{array}{ccc}
 X & \longrightarrow & Y \times_{\mathbb{F}_1} \text{Spec } R \\
 \downarrow & & \downarrow \\
 X' & \longrightarrow & Y' \times_{\mathbb{F}_1} \text{Spec } R
 \end{array} \tag{2}$$

where the right vertical arrow is induced by a morphism of \mathbb{F}_1 -schemes $Y \rightarrow Y'$. An \mathbb{F}_1 -embedding is an \mathbb{F}_1 -framing that is also a closed immersion.

In other words, the category of \mathbb{F}_1 -framed schemes is simply the comma category of R -schemes over \mathbb{F}_1 -schemes.

For any affine R -scheme $X = \text{Spec } A$, the evaluation map defines an \mathbb{F}_1 -embedding $X \hookrightarrow \widehat{X}$ where the \mathbb{F}_1 -algebra of the ambient space is $M(A)$. We call this the *universal embedding* of X for the following reason. As a purely formal consequence of the adjunction (1), this embedding is the initial object in the category of framings on X , and hence it is initial among \mathbb{F}_1 -embeddings. More precisely,

Proposition 2.2.2. Given an affine framed R -scheme $\alpha : X \rightarrow \text{Spec } B \otimes_{\mathbb{F}_1} R$, there is a unique factorization through the universal embedding

$$X \rightarrow \widehat{X} \xrightarrow{\eta_\alpha} \text{Spec } B \otimes_{\mathbb{F}_1} R.$$

Moreover, this factorization is functorial in the sense that a morphism of affine framed schemes as (2) induces a diagram

$$\begin{array}{ccccc}
 X & \longrightarrow & \widehat{X} & \longrightarrow & Y \times_{\mathbb{F}_1} \text{Spec } R \\
 \downarrow & & \downarrow & & \downarrow \\
 X' & \longrightarrow & \widehat{X}' & \longrightarrow & Y' \times_{\mathbb{F}_1} \text{Spec } R.
 \end{array}$$

Proof. Suppose $X = \text{Spec } A$, so the framing α corresponds to an R -algebra homomorphism

$$B \otimes_{\mathbb{F}_1} R \rightarrow A,$$

which is adjoint to an \mathbb{F}_1 -algebra homomorphism

$$B \rightarrow M(A). \tag{3}$$

The evaluation map $\widehat{A} \rightarrow A$ is adjoint to the identity on $M(A)$. The desired map $\eta_\alpha : \widehat{X} \rightarrow \text{Spec } B \otimes_{\mathbb{F}_1} R$ corresponds to a homomorphism

$$B \otimes_{\mathbb{F}_1} R \rightarrow M(A) \otimes_{\mathbb{F}_1} R$$

that is the functor $- \otimes_{\mathbb{F}_1} R$ applied to a morphism $B \rightarrow M(A)$, but the only such morphism that gives the desired factorization is (3). This shows existence and uniqueness of the factorization η_α . The functoriality claim follows from a similar purely formal manipulation of the adjunction. \square

2.3. The difficulty in globalizing the universal embedding

We would like to construct an extension of the universal embedding functor from affine R -schemes to arbitrary schemes over $\text{Spec } R$, and we will do so in the next section. However, this is not entirely straightforward. The root of the difficulty is that $X \mapsto \widehat{X}$ does not in general send open covers to open covers, as we now explain.

Remark 2.3.1. In the earlier version of this paper, it was incorrectly claimed that the adjunction between scalar restriction and scalar extension for affine schemes directly extends to an adjunction for arbitrary schemes. Although open immersions are sent to open immersions, since the adjunction does not preserve covers, existence of the extension is not immediate. We thank Oliver Lorscheid for pointing this out.

Recall that the category of affine \mathbb{F}_1 -schemes is the opposite of the category of \mathbb{F}_1 -algebras, basic open immersions correspond to localizations, and general \mathbb{F}_1 -schemes can be described by gluing affine patches together along open immersions. See [11, §3] or [35] for further details. The category of schemes over a semiring can also be constructed in essentially the same way.

The scalar restriction and extension functors $R\text{-alg} \rightleftarrows \mathbb{F}_1\text{-alg}$ commute with localizations (see, for example, [7, Paragraph 6.1.13]), so they send open immersions to open immersions. However, the scalar restriction functor M does not send open covers to open covers. The difficulty here is that an affine \mathbb{F}_1 -scheme has very few Zariski open covers.

Proposition 2.3.2. Given an affine \mathbb{F}_1 -scheme Y , a collection of affine open subschemes $\{U_i\}$ is a Zariski cover if and only if $U_i = Y$ for some i .

Proof. Suppose $Y = \text{Spec } B$ for an \mathbb{F}_1 -algebra B . A collection of principal open subschemes $U_i = \text{Spec } B[a_i^{-1}]$ covers $\text{Spec } B$ if and only if the a_i generate the unit ideal in B . The ideal generated by the a_i is simply the set of all elements that are a multiple of one of the a_i ; i.e., it is the multiplicative monoid generated by the a_i . This is the unit ideal if and only if at least one of the a_i is a unit. \square

Example 2.3.3. Let $A = R[z]$ and consider the principal open subschemes of $X = \text{Spec } A$ given by $U = \text{Spec } R[z, z^{-1}]$ and $V = \text{Spec } R[z, (1 - z)^{-1}]$. The pair $\{U, V\}$ is an open cover since z and $1 - z$ generate the unit ideal in the ring A . The scalar restriction functor M sends U and V to principal open subschemes of the affine \mathbb{F}_1 -scheme $\text{Spec } M(A)$. The ideal $(x_z, x_{1-z}) \subset M(A)$ consists of all elements that can be written

in either the form $x_a x_z$ or $x_a x_{1-z}$ for some $a \in A$. Clearly this is not the unit ideal of $M(A)$ since $1 \neq az$ or $a(1-z)$ for any $a \in A$. On $\widehat{X} = \text{Spec } M(A) \otimes_{\mathbb{F}_1} R$, we have open subschemes \widehat{U} and \widehat{V} defined by the ideals (x_z) and (x_{1-z}) respectively, but these ideals do not generate the unit ideal in the R -algebra $M(A) \otimes_{\mathbb{F}_1} R$ so these open subschemes don't cover \widehat{X} .

Thus if X is an affine R -scheme with an affine open cover $\{U_i\}$, the schemes \widehat{U}_i will be affine open subschemes of \widehat{X} , but they will not in general form an open cover. The problem, then, is that if X is not necessarily affine we cannot build \widehat{X} simply by choosing an affine open cover and gluing the universal embeddings of the affine patches in this cover. Different choices of cover will result in different glued objects. In the next subsection, we will see that the way around this obstacle is to use *all* open affine subschemes of X .

2.4. The universal embedding in the non-affine case

We now extend the universal embedding to arbitrary not necessarily affine schemes. The idea is that a scheme X can be written as the colimit of *all* its affine open subschemes U , and so we define \widehat{X} to be the colimit of the corresponding diagram of affine schemes \widehat{U} and open immersions. This colimit exists (one can construct it as a locally ringed space and then easily check that it is a scheme).

Proposition 2.4.1. The universal embeddings $U \rightarrow \widehat{U}$ for affine open subschemes of X glue together to give a embedding $X \rightarrow \widehat{X}$, and on affine schemes this restricts to the previously constructed affine universal embedding. This embedding is initial among \mathbb{F}_1 -framings of X .

Proof. For each affine open U , there is a canonical framing morphism $\alpha_U : U \rightarrow \widehat{U} \rightarrow \widehat{X}$, and when $U \subset V$, the functoriality of Prop. 2.2.2 implies that $\alpha_V|_U = \alpha_U$, so these morphisms glue together to give a closed immersion $X \rightarrow \widehat{X}$.

To show that $X \rightarrow \widehat{X}$ is initial, suppose Y is an \mathbb{F}_1 -scheme and $\alpha : X \rightarrow Y \times_{\mathbb{F}_1} \text{Spec } R$ is a morphism. The target is covered by affine patches of the form $\text{Spec } B \otimes_{\mathbb{F}_1} R$, and for any affine open $U \subset X$ mapping into $\text{Spec } B \otimes_{\mathbb{F}_1} R$, there is a unique factorization

$$U \rightarrow \widehat{U} \rightarrow \text{Spec } B \otimes_{\mathbb{F}_1} R.$$

Once again, by the functoriality of these affine factorization, these glue together to produce the desired factorization $X \rightarrow \widehat{X} \rightarrow Y \times_{\mathbb{F}_1} \text{Spec } R$. Uniqueness of this factorization follows from the fact that it is locally unique. \square

The gluing construction above can also be described in category theoretic terms as a left Kan extension from affine schemes to schemes. First, let us recall the concept of left Kan extension. Consider a pair of functors

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\ G \downarrow & \nearrow & \\ \mathcal{C} & & \end{array}$$

and the question of filling in the diagonal arrow. There may or may not exist a functor $H : \mathcal{C} \rightarrow \mathcal{B}$ making the diagram commute in the sense that there is a natural isomorphism $F \cong H \circ G$. A weaker condition would be to ask for an H and a natural transformation (that is not necessarily invertible) $\omega_H : F \Rightarrow H \circ G$, but now there is a category of such pairs (H, ω_H) . A morphism $(H_1, \omega_1) \rightarrow (H_2, \omega_2)$ is a natural transformation $H_1 \Rightarrow H_2$ that sends ω_1 to ω_2 . The *left Kan extension of F along G* is an initial object in this category; i. e., it is a functor $\text{Lan}_G F : \mathcal{C} \rightarrow \mathcal{B}$ together with a natural transformation $\omega : F \Rightarrow \text{Lan}_G F \circ G$ such that for any other pair (H, ω_H) there is a unique natural transformation $\text{Lan}_G F \Rightarrow H$ sending ω to ω_H . While it may not always exist, when \mathcal{B} admits enough colimits, the left Kan extension can be constructed via a straightforward recipe:

$$\text{Lan}_G F(c) = \text{colim}_{a \in G/c} F(a)$$

where G/c is the comma category of objects $a \in \mathcal{A}$ over c , i. e., pairs $(a \in \mathcal{A}, f : G(a) \rightarrow c)$.

Returning to the case at hand, let \mathcal{O} denote the category of R -schemes and open immersions, and let $\mathcal{O}_{\text{Aff}} \subset \mathcal{O}$ denote the full subcategory of affine schemes. The affine universal embedding $X \mapsto \widehat{X}$ constructed so far in section 2.2 is a functor defined on affine schemes and *all* morphisms, but let us now restrict to open immersions and regard it as a functor

$$E : \mathcal{O}_{\text{Aff}} \rightarrow \text{Sch}/R.$$

Consider the left Kan extension of E along the inclusion j :

$$\begin{array}{ccc} \mathcal{O}_{\text{Aff}} & \xrightarrow{E} & \text{Sch}/R \\ j \downarrow & \nearrow & \\ \mathcal{O} & \text{Lan}_j E & \end{array}$$

Somewhat more concretely, for an R -scheme X , the object $\widehat{X} := \text{Lan}_j E(X)$ can be described as the colimit of \widehat{U} as U runs over the poset $\mathcal{O}_{\text{Aff}}(X)$ of all affine open subschemes of X (this is the category of objects of \mathcal{O}_{aff} over $X \in \mathcal{O}$).

3. TROPICALIZING THE UNIVERSAL EMBEDDING

We now study the tropicalization of an integral scheme X with respect to the universal embedding $X \hookrightarrow \widehat{X}$.

3.1. A brief review of scheme-theoretic tropicalization

In [11] we introduced a generalization and refinement of the Kajiwara-Payne set-theoretic tropicalization of subvarieties of toric varieties over a rank-one valued field. Here we review that construction.

Let \mathbb{T} denote the idempotent semiring $(\mathbb{R} \cup \{\infty\}, \min, +)$ with additive unit $0_{\mathbb{T}} = \infty$ and multiplicative unit $1_{\mathbb{T}} = 0$. Let k be a field equipped with a valuation $\nu : k \rightarrow \mathbb{T}$ (by which we mean a multiplicative and superadditive map preserving the multiplicative and additive unit, respectively). Let B be an integral \mathbb{F}_1 -algebra, and $I \subset B \otimes_{\mathbb{F}_1} k$ an ideal. We can regard I as a k -linear subspace and tropicalize it with respect to ν to

get a tropical linear space $\text{trop}(I) \subset B \otimes_{\mathbb{F}_1} \mathbb{T}$ (or $\text{trop}'(I)$ if we need to emphasize the valuation) which is, by definition, the \mathbb{T} -linear span of the coefficient-wise valuations of the elements of I . The set $\text{trop}(I)$ is automatically a \mathbb{T} -submodule, and moreover it turns out to be an ideal in $B \otimes_{\mathbb{F}_1} \mathbb{T}$ (this requires the assumption that B is integral, see [11, Proposition 6.1.1 and Remark 6.1.2]); i. e., it is a *tropical ideal* as studied by Maclagan and Rinçon in [21, 22, 23]. The congruence $\mathcal{B}\text{trop}(I)$ on $B \otimes_{\mathbb{F}_1} \mathbb{T}$ is generated by the *bend relations*

$$\mathcal{B}(f) : f \sim f_{\widehat{b}}$$

for $f \in \text{trop}(I)$ and b a monomial term in f , where $f_{\widehat{b}}$ denotes the result of deleting b from f .

Now let X be a k -scheme, Y a locally integral \mathbb{F}_1 -scheme (which means Y admits an open affine cover by the spectra of integral \mathbb{F}_1 -algebras), and $\varphi : X \hookrightarrow Y \times_{\mathbb{F}_1} k$ a closed embedding corresponding to a quasi-coherent ideal sheaf \mathcal{I} on $Y \times_{\mathbb{F}_1} k$. For each integral affine patch $U \subset Y$, \mathcal{I} is in particular a k -linear subspace of the space of regular functions on $U \times_{\mathbb{F}_1} k$, and so its tropicalization $\text{trop}(\mathcal{I}(U))$ is a tropical linear space in the space of regular functions on $U \times_{\mathbb{F}_1} \mathbb{T}$. These tropical linear spaces assemble to form a quasi-coherent ideal sheaf $\text{trop}(\mathcal{I})$ on $Y \times_{\mathbb{F}_1} \mathbb{T}$. Applying the bend relations $\mathcal{B}(-)$ on each of the above affine patches then yields a quasi-coherent congruence sheaf $\mathcal{B}\text{trop}(\mathcal{I})$ on $Y \times_{\mathbb{F}_1} \mathbb{T}$; the tropicalization $\text{Trop}_{\varphi}(X)$ (or $\text{Trop}'_{\varphi}(X)$ to emphasize the valuation) of X with respect to the embedding φ is then, by definition, the closed subscheme determined by this congruence sheaf, regarded as a scheme over $\text{Spec } \mathbb{T}$.

A quintessential example of an \mathbb{F}_1 -scheme is a toric variety Y_{Δ} , where the fan Δ provides the model over \mathbb{F}_1 . Set-theoretic tropicalization, as defined by Payne and Kajiwara [16, 29], applies to subvarieties of toric varieties $\varphi : X \hookrightarrow Y_{\Delta}$, and [11, Theorem 6.3.1] shows that the output of that coincides with the \mathbb{T} -points of the scheme-theoretic tropicalization $\text{Trop}_{\varphi}(X)$.

Remark 3.1.1. The idea of generalizing the ambient spaces for tropicalization from toric varieties to \mathbb{F}_1 -schemes first appeared in [31], although they worked only with set-theoretic tropicalization.

Functoriality of tropicalization [11, Proposition 6.4.1] is a scheme-theoretic enrichment of Payne’s observation [30, Corollary 2.6] regarding torus equivariant morphisms of toric varieties. Namely, if $\psi : Y_1 \rightarrow Y_2$ is a map of integral \mathbb{F}_1 -schemes and $\varphi_i : X \hookrightarrow Y_i \times_{\mathbb{F}_1} k$ are closed embeddings forming a commutative triangle

$$\begin{array}{ccc} & X & \\ \varphi_1 \swarrow & & \searrow \varphi_2 \\ Y_1 \times_{\mathbb{F}_1} k & \xrightarrow{\psi \times_{\mathbb{F}_1} k} & Y_2 \times_{\mathbb{F}_1} k, \end{array}$$

then there is an induced morphism of tropicalizations $\text{Trop}_{\varphi_1}(X) \rightarrow \text{Trop}_{\varphi_2}(X)$.

In addition to passing from toric varieties to arbitrary integral \mathbb{F}_1 -schemes as ambient spaces for tropicalization, in [11] we observed that the domain of scheme-theoretic tropicalization, with its functoriality property, naturally admits the following enlargement:

1. The field k can be replaced by an arbitrary ring R .

2. The tropical numbers \mathbb{T} can be replaced by an arbitrary idempotent semiring S .

Item (2) requires a generalization of the definition of (semi)valuation that we discuss in the next section. This generalization includes the case of higher-rank Krull (semi)valuations $k \rightarrow \Gamma \cup \{\infty\}$ simply by giving the totally ordered abelian group Γ the structure of a semiring with ∞ as the additive identity, where multiplication is the group operation in Γ and addition is the minimum with respect to the ordering. Set-theoretic tropicalizations with respect to higher rank Krull valuations were studied in [1]. Higher rank tropicalization has been further studied in [10], where an associated notion of higher rank ‘Hahn’ analytification is introduced.

The output of \mathcal{Trop} is then canonically a scheme over $\text{Spec } S$. Explicitly, when $Y = \text{Spec } B \otimes_{\mathbb{F}_1} R$ and X is defined by an ideal I , we have

$$\mathcal{Trop}(X) = \text{Spec } B \otimes_{\mathbb{F}_1} S / \mathcal{B}trop(I)$$

where $trop(I) \subset B \otimes_{\mathbb{F}_1} S$ is the ideal generated by applying a (generalized) semivaluation $R \rightarrow S$ coefficient-wise to the elements of I .

3.2. Semivaluations

Here we review the definition and properties of the generalized class of semivaluations that were first introduced in [11].

Definition 3.2.1. Given a ring R and an idempotent semiring S , a semivaluation $\nu : R \rightarrow S$ is a map such that

1. $\nu(0) = 0_S$;
2. $\nu(1) = \nu(-1) = 1_S$;
3. (multiplicative): $\nu(ab) = \nu(a)\nu(b)$;
4. (superadditive): $\nu(a + b) + \nu(a) + \nu(b) = \nu(a) + \nu(b)$.

Note that conditions (2) and (3) allow us to rewrite the superadditivity condition (4) in a more symmetric form as

$$\nu(a) + \nu(b) + \nu(c) = \nu(a) + \nu(b) = \nu(a) + \nu(c) = \nu(b) + \nu(c)$$

for any $a, b, c \in R$ satisfying $a + b + c = 0$. More generally, it follows that if $a_1 + a_2 + \dots + a_n = 0$ in R , then the value of the summation $\nu(a_1) + \dots + \nu(a_n)$ in S is unchanged if any one term is omitted; we say that this summation *tropically vanishes*. Also observe that if $R \rightarrow S$ is a semivaluation in this sense, then pre-composition with a ring homomorphism $R' \rightarrow R$ and post-composition with a semiring homomorphism $S \rightarrow S'$ both yield semivaluations.

Note also that when $S = \mathbb{T}$, the above definite reduces to the usual definition of a real (rank 1) semivaluation.

Let $\mathcal{Val}(R, S)$ denote the set of semivaluations from R to S . Fixing S , we can regard $\mathcal{Val}(-, S)$ as a covariant functor $\mathcal{O}_{aff} \rightarrow \text{Sets}$. We can then take the left Kan

extension along the inclusion $\mathcal{O}_{\text{aff}} \hookrightarrow \mathcal{O}$ to extend to arbitrary schemes. Concretely, a point of $\mathcal{V}al(X, S)$ is represented by an affine open subscheme $U = \text{Spec } A$ of X and a semivaluation $\nu : A \rightarrow S$, and two pairs (U_1, ν_1) and (U_2, ν_2) are equivalent if there exists (U_3, ν_3) with $U_3 \subset U_1 \cap U_2$ such that ν_3 maps to ν_1 and ν_2 . We refer to such an object as a semivaluation on X with values in S . This mildly extends the definition of [33, §3.1] to allow semivaluations taking values in more general semirings instead of just \mathbb{T} .

If $\nu : R \rightarrow S$ is a semivaluation, A is an R -algebra, and T is an S -algebra, then we say that a semivaluation $w : A \rightarrow T$ is *compatible* with ν if the diagram

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ A & \longrightarrow & T \end{array}$$

commutes. We write $\mathcal{V}al_\nu(A, T)$ for the set of semivaluations on A taking values in T and compatible with ν ; it is a contravariant functor of R -algebras in the first variable, and a covariant functor of S -algebras in the second variable. As in the paragraph above, left Kan extension along the inclusion $\mathcal{O}_{\text{aff}} \hookrightarrow \mathcal{O}$ extends this to a functor on arbitrary schemes.

When the semivaluation on R takes values in \mathbb{T} and X is an R -scheme, the subset $\mathcal{V}al_\nu(X, \mathbb{T})$ of semivaluations compatible with ν is precisely the underlying set of the Berkovich analytification X^{an} .

3.3. The points of a tropicalization

Let Y be an \mathbb{F}_1 -scheme, and let R be a ring with a semivaluation $\nu : R \rightarrow S$. Given an R -algebra A , an S -algebra T , and a semivaluation $w : A \rightarrow T$ compatible with ν , there is a *tropicalization-of-points* map

$$\text{trop} : (Y \times_{\mathbb{F}_1} \text{Spec } R)(A) \rightarrow (Y \times_{\mathbb{F}_1} \text{Spec } S)(T).$$

This map is defined as follows. Locally $Y \times_{\mathbb{F}_1} \text{Spec } R$ is of the form $\text{Spec } B \otimes_{\mathbb{F}_1} R$ for an \mathbb{F}_1 -algebra B , and an A -valued point is adjoint to a homomorphism $B \rightarrow M(A)$. Since w is multiplicative, the composition

$$B \rightarrow M(A) \xrightarrow{w} M(T)$$

is an \mathbb{F}_1 -algebra homomorphism, and this is adjoint to a S -algebra homomorphism $B \otimes_{\mathbb{F}_1} S \rightarrow T$.

Let X be a locally integral scheme over R with semivaluation $\nu : R \rightarrow S$, and $\varphi : X \hookrightarrow Y \times_{\mathbb{F}_1} \text{Spec } R$ an \mathbb{F}_1 -embedding. These data determine a tropicalization $\text{Trop}_\varphi(X)$, which is a scheme over the idempotent semiring S . Locally, X is cut out by an ideal $I \subset B \otimes_{\mathbb{F}_1} R$, and

$$\text{Trop}_\varphi(X) = \text{Spec } B \otimes_{\mathbb{F}_1} S / \mathcal{B}\text{trop}(I)$$

where $\text{trop}(I) \subset B \otimes_{\mathbb{F}_1} S$ is the ideal generated by applying ν coefficient-wise to the elements of I . Note that $X(A) \subset (Y \times_{\mathbb{F}_1} R)(A)$ and $\text{Trop}_\varphi(X)(T) \subset (Y \times_{\mathbb{F}_1} S)(T)$.

Proposition 3.3.1. The tropicalization-of-points map sends $X(A)$ into $\mathcal{Trop}_\varphi(X)(T)$.

Proof. It suffices to verify this on an affine patch $\text{Spec } B \otimes_{\mathbb{F}_1} R \subset Y \times_{\mathbb{F}_1} R$. If X is defined by an ideal I , then an A -valued point p corresponds to an R -algebra morphism $p : B \otimes_{\mathbb{F}_1} R \rightarrow A$ such that $f(p) = 0$ for all $f \in I$. Thus $w(f(p)) = 0_T$, and then it follows from the symmetric form of the superadditivity condition that if f_i are the monomial terms of f , then the sum $\sum_i w(f_i(p))$ is unchanged if any single term is omitted. It follows that p tropicalizes to a homomorphism $\text{trop}(p) : B \otimes_{\mathbb{F}_1} S \rightarrow T$ that descends to the quotient by the congruence $\mathcal{B}\text{trop}(I)$, and hence $w \circ p$ determines a T -valued point of $\mathcal{Trop}_\varphi(X)$. □

Remark 3.3.2. A T -valued point of $\mathcal{Trop}_\varphi(X)$ can be thought of as a semivaluation on X that is, in an affine patch on which it lives, defined only on the monomials coming from the embedding φ and satisfying superadditivity only for linear combinations of monomials in the defining ideal of X . Moreover, every such partially defined semivaluation comes from a T -point. This bijection appears as Theorem *D* in [15], and it follows directly from the definitions here. This fact provides an illuminating interpretation of our results on the relation between the universal tropicalization or Berkovich analytification and finite tropicalizations: a semivaluation on a scheme X restricts to a partially defined semivaluation on the monomials from any \mathbb{F}_1 -embedding of X , and if the set of monomials is all elements of the coordinate algebra of an affine patch in X then a partially defined semivaluation is the same as a true semivaluation.

Remark 3.3.3. The tropicalization-of-points map is a mild generalization of the Kajiwara-Payne tropicalization map for toric varieties [16, 29]. There are numerous interesting further refinements and generalization of tropicalization, such as the local tropicalization of Popescu-Pampu and Stepanov [31] and the logarithmic tropicalization studied by Ulirsch [36] and its extension to stacks [37]. It would be interesting to combine our construction with these.

3.4. Strong tropical bases

Let B be an integral \mathbb{F}_1 -algebra, $\nu : R \rightarrow S$ a semivaluation from a ring to an idempotent semiring, and $I \subset B \otimes_{\mathbb{F}_1} R$ an ideal. In general, the tropicalized ideal $\text{trop}(I) \subset B \otimes_{\mathbb{F}_1} S$ is not finitely generated, and so the congruence $\mathcal{B}\text{trop}(I)$ is presented by an infinite set of generating relations. There is often a large amount of redundancy in this generating set, and so one can ask about the existence of smaller sets of generating relations. Given a subset $K \subset I$, we can consider the set of tropical polynomials $\{\nu(f)\}_{f \in K}$ and then the congruence generated by their bend relations, $\langle \mathcal{B}(\nu(f)) \rangle_{f \in K}$.

Definition 3.4.1. A *strong tropical basis* for an ideal I is a generating subset $K \subset I$ such that

$$\mathcal{B}\text{trop}(I) = \langle \mathcal{B}(\nu(f)) \rangle_{f \in K}.$$

Remark 3.4.2. In [11, Definition 8.2.1] we proposed a different notion of tropical basis: a scheme-theoretic tropical basis is a collection $\{J_i\}$ of principal ideals that generates I

as an ideal and such that the collection of congruences $\{\mathcal{B}trop(J_i)\}$ generates $\mathcal{B}trop(I)$. If $\{f_i\} \subset I$ is a strong tropical basis then the collection of principal ideals $\{(f_i)\}$ is always a scheme-theoretic tropical basis. However, the converse is not necessarily true: as shown in [11, Example 8.1.1], for the polynomial $f = x^2 + xy + y^2 \in R[x, y]$, the congruence $\mathcal{B}(\nu(f))$ is strictly smaller than the congruence $\mathcal{B}trop(I)$ for $I = (f)$, and so in this case $\{(f)\}$ is trivially a scheme-theoretic tropical basis, while $\{f\}$ is *not* a strong tropical basis.

The following result is the technical heart of this paper.

Proposition 3.4.3. The elements given in Proposition 2.1.4 are a strong tropical basis for the kernel of $ev : \widehat{A} \rightarrow A$, for any semivaluation $\nu : R \rightarrow S$.

Proof. By [11, Lemma 5.1.3(2)] (see also the first paragraph in the proof of [22, Theorem 1.1]), the congruence $\mathcal{B}trop(\ker(ev))$ is spanned as an S -module by the congruences $\mathcal{B}(\nu(f))$ for $f \in \ker(ev)$. Thus it suffices to show that each such congruence $\mathcal{B}(\nu(f))$ is contained in the congruence generated by the bend relations of the semivaluations of the elements of types (1) and (2).

Consider an element $f \in \ker(ev)$; the coefficient-wise semivaluation of f is of the form

$$\nu(f) = \sum_i \nu(\lambda_i)x_{a_i}, \text{ with } \sum_i \lambda_i a_i = 0.$$

The bend relations of the semivaluations of generators of type (1) give the relations

$$x_{\lambda a} \sim \nu(\lambda)x_a, \tag{4}$$

and in particular, $x_{-a} \sim x_a$ since $\nu(-1) = \nu(1) = 1$. Using these relations we see that $\nu(f)$ is equivalent to an element $g \in M(A) \otimes_{\mathbb{F}_1} S$ of the form

$$g = \sum_{i=1}^n x_{b_i}, \text{ with } \sum b_i = 0$$

(here $b_i := \lambda_i a_i$ to keep the notation simpler). We now show that the congruence $\mathcal{B}(g)$ is contained in the congruence

$$J := \langle \mathcal{B}(x_a + x_b + x_{-a-b}) \rangle_{a,b \in A}.$$

The relation $x_{b_1} + x_{b_n} \sim x_{b_2+\dots+b_{n-1}} + x_{b_n}$ from $\mathcal{B}(x_{b_1} + x_{b_2+\dots+b_{n-1}} + x_{b_n})$ gives the relation

$$\begin{aligned} g &= x_{b_1} + x_{b_n} + \sum_{i=2}^{n-1} x_{b_i} \\ &\sim x_{b_2+\dots+b_{n-1}} + x_{b_n} + \sum_{i=2}^{n-1} x_{b_i} \\ &= g_{\widehat{b_1}} + x_{b_2+\dots+b_{n-1}} \end{aligned} \tag{5}$$

in J . Next, consider the relation

$$x_{b_2+\dots+b_\ell} + x_{b_{\ell+1}} \sim x_{b_2+\dots+b_{\ell+1}} + x_{b_{\ell+1}}$$

from $\mathcal{B}(x_{b_2+\dots+b_\ell} + x_{b_{\ell+1}} + x_{-b_2-\dots-b_{\ell+1}})$ and $x_c \sim x_{-c}$; using this repeatedly as ℓ runs from 2 up to $n - 2$ gives

$$\begin{aligned} g_{\widehat{b}_1} &= x_{b_2} + x_{b_3} + \dots + x_{b_n} \\ &\sim x_{b_2+b_3} + x_{b_3} + \dots + x_{b_n} \\ &\quad \vdots \\ &\sim x_{b_2+\dots+b_{n-1}} + x_{b_3} + \dots + x_{b_n} \end{aligned}$$

in J , and hence the idempotency of addition implies that the relation

$$g_{\widehat{b}_1} + x_{b_2+\dots+b_{n-1}} \sim g_{\widehat{b}_1}$$

is in J . Combining this with (5) yields the desired relation $g \sim g_{\widehat{b}_1}$ in J . Since the choice of ordering of the b_i was arbitrary, this shows that the bend relations of g are indeed all contained in J . Using the bend relations of the type (1) elements once again, but this time in the reverse of the direction we used them when passing from $\nu(f)$ to g , we have the relation $g_{\widehat{b}_i} \sim \nu(f)_{\widehat{a}_i}$ in J . Combined with the bend relations of g , this shows that the bend relations $\nu(f)$ are entirely contained in J . \square

3.5. The universal tropicalization

Let S be an idempotent semiring, $\nu : R \rightarrow S$ a semivaluation, and X a scheme over R . Note that the universal embedding $X \hookrightarrow \widehat{X}$ defined in §2.3 yields a tropicalization when the \mathbb{F}_1 -model $M(X)$ of \widehat{X} is locally integral. When X is irreducible, this is equivalent to X being integral; indeed, this can be checked on sufficiently small affine patches by [11, Proposition 3.1.3], and it holds there by Lemma 2.1.3. We therefore assume in this section that X is integral and we will study its tropicalization in \widehat{X} .

Definition 3.5.1. The *universal tropicalization* of X , denoted $\mathcal{Trop}'_{univ}(X)$, is the tropicalization of X with respect to the canonical closed embedding $X \hookrightarrow \widehat{X}$.

Proposition 3.5.2. Let Y be an integral \mathbb{F}_1 -scheme and $\varphi : X \hookrightarrow Y \times_{\mathbb{F}_1} R$ a closed embedding. There is a canonical morphism of S -schemes $\mathcal{Trop}_{univ}(X) \rightarrow \mathcal{Trop}_\varphi(X)$, and it is natural in both X and (Y, φ) .

Proof. This follows immediately from the functoriality of tropicalization [11, Proposition 6.4.1] and the universal property of the embedding $X \hookrightarrow \widehat{X}$ described in Proposition 2.4.1. \square

This proves the first part of Theorem B, and it is because of the above result that the tropicalization of X in \widehat{X} deserves to be called the universal tropicalization of X .

Theorem 3.5.3. The universal tropicalization $\mathcal{Trop}'_{univ}(X)$ represents the contravariant functor on affine S -schemes sending $\text{Spec } T$ to the set $\mathcal{Val}_\nu(X, T)$ of semivaluations $X \rightarrow T$ compatible with ν .

Thus the universal tropicalization is the algebraic moduli space of semivaluations on X . When R is a field k , we obtain Theorem D from the introduction. Moreover, when S is the tropical semiring \mathbb{T} we obtain the set-theoretic bijection part of Theorem A directly from this by passing to the set of \mathbb{T} -points, since on the one hand these are the points of the set-theoretic tropicalization, and on the other hand these are the semivaluations $X \rightarrow \mathbb{T}$, i.e., the points of the Berkovich analytification; the part of the theorem describing the Berkovich topology is explained and proven in §3.7 below.

Proof. [Proof of of Theorem 3.5.3] It suffices to assume that $X = \text{Spec } A$ is affine. In this case the universal tropicalization is

$$\text{Spec } M(A) \otimes_{\mathbb{F}_1} S/\mathcal{B}\text{trop}(\ker(ev)).$$

By Proposition 3.4.3, a T -point of this is a multiplicative map $\alpha : A \rightarrow T$ such that

$$\alpha(\lambda a) = \nu(\lambda)\alpha(a) \text{ for } \lambda \in R \text{ and } a \in A, \tag{6}$$

and (using the fact that $\alpha(-c) = \nu(-1)\alpha(c) = \alpha(c)$ from the equation above),

$$\begin{aligned} \alpha(a) + \alpha(b) + \alpha(a + b) &= \alpha(a) + \alpha(b) \\ &= \alpha(a) + \alpha(a + b) \\ &= \alpha(b) + \alpha(a + b). \end{aligned} \tag{7}$$

The first condition (6) says that α is compatible with the semivaluation on k . In the second condition, (7), the first equality is precisely the superadditivity condition for a semivaluation. We now observe that the remaining two equalities are actually redundant and so impose no additional conditions. We have $\alpha(a) + \alpha(a + b) = \alpha(a) + \alpha(-(a + b))$, and by the first equality of (7) (applied with a and $-(a + b)$ instead of a and b), this is equal to

$$\alpha(a) + \alpha(-(a + b)) + \alpha(a - (a + b)) = \alpha(a) + \alpha(a + b) + \alpha(b).$$

Thus the second equality of (7) follows from the first, and by symmetry between a and b the third one does as well. □

Note that if $X = \text{Spec } A$ is an affine R -scheme, then

$$\mathcal{Trop}'_{univ}(\text{Spec } A) = \text{Spec } M(A) \otimes_{\mathbb{F}_1} S/\mathcal{B}\text{trop}(\ker ev)$$

is an affine S -scheme. Let us abbreviate its algebra of global functions S_A .

Corollary 3.5.4. There is semivaluation $w : A \rightarrow S_A$ that is universal among all semivaluations compatible with ν in the following sense: given any semivaluation $w' : A \rightarrow T$ compatible with ν , there is a unique homomorphism of S -algebras $f : S_A \rightarrow T$ such that $w' = f \circ w$.

The universal semivaluation sends $a \in A$ to $x_a \in M(A) \otimes_{\mathbb{F}_1} S/\mathcal{B}\text{trop}(\ker ev)$.

3.6. An aside on the relation between universal valuations and universal tropicalization

The universal tropicalization discussed above is defined using a fixed semivaluation ν on the base R and the universal embedding of X , in contrast to the tropicalization $\mathcal{Trop}_\varphi^{\nu_{univ}^R}(X)$ from [11, §6.5] which is defined with a fixed embedding and the universal semivaluation ν_{univ}^R on R . The point of Theorem D (Theorem 3.5.3 and Corollary 3.5.4) is that these two objects are closely related, as we now explain.

Given a ring A , one can consider the category of all semivaluations on A , and this category contains an initial object given by the universal semivaluation ν_{univ}^A . If A is an R -algebra, then one can also consider the subcategory of all semivaluations compatible with a given semivaluation ν on R , and this subcategory contains an initial object taking values in the coordinate algebra of the universal tropicalization (i. e., tropicalization of the universal embedding).

Note that every semivaluation on A is compatible with the universal semivaluation ν_{univ}^R on R . Hence, if we consider the universal tropicalization of $\text{Spec } A$ with respect to the universal semivaluation ν_{univ}^R on R , the coordinate algebra of the resulting object will be the semiring in which the universal semivaluation on A takes values.

In geometric terms, the space of all semivaluations on R is (an enrichment of) the Berkovich spectrum $\mathcal{M}(R)$, and the universal tropicalization of $\text{Spec } A$ with respect to the universal semivaluation on R can be viewed as the family of Berkovich analytifications of $\text{Spec } A$ parametrized by $\mathcal{M}(R)$.

3.7. The Berkovich topology as an algebraic topology

We now introduce the *strong Zariski topology* on a semiring scheme, and in the case of a universal tropicalization we show that this topology coincides with the Berkovich topology.

Let X be a scheme over S , viewed as a topological space with a structure sheaf. The Zariski topology on the underlying set of X induces a Zariski topology on the S -points $X(S)$. Explicitly, an open immersion $U \hookrightarrow X$ determines a subset $U(S) \subset X(S)$, and every Zariski open subset is of this form.

Definition 3.7.1. Let X be an S -scheme. The *strong Zariski topology* on $X(S)$ is the topology whose closed subsets are of the form $Z(S)$ for Z a closed subscheme of X .

Note that since the pullback of a closed subscheme is again a closed subscheme, a morphism of schemes $X \rightarrow X'$ induces a map $X(S) \rightarrow X'(S)$ that is continuous with respect to the strong Zariski topology.

Proposition 3.7.2. If S is a ring then the strong Zariski topology and the ordinary Zariski topology coincide, but in general the strong Zariski topology is finer than the Zariski topology.

Proof. The strong Zariski closed subsets are defined by equations of the form $f = g$, whereas Zariski closed subsets are defined by equations of the restricted form $f = 0_S$. I.e.,

Zariski closed subsets are given by ideals, while strong Zariski closed subsets are given by congruences. Over a ring congruences and ideals are in bijection, and so it follows that the strong Zariski topology and the Zariski topology on $X(S)$ coincide. \square

Remark 3.7.3. When S is not a ring then the two topologies can be distinct. For example, over \mathbb{N} the diagonal in $\mathbb{A}^1 \times \mathbb{A}^1$ is strong Zariski closed but not Zariski closed (cf. [7, §6.5.19]), and for $\mathbb{A}^1(\mathbb{N}) = \mathbb{N}$ the strong Zariski topology is the finite complement topology, while the only nontrivial Zariski closed subset is the singleton $\{0\}$.

Note that the structure sheaf \mathcal{O}_X is a sheaf with respect to the Zariski topology, but not with respect to the strong Zariski topology.

The *Euclidean topology* on \mathbb{T} is the topology for which the exponential map $x \mapsto e^{-x}$ gives a homeomorphism with $\mathbb{R}_{\geq 0}$. More generally, if Λ is a (possibly infinite) set then the Euclidean topology on the product space \mathbb{T}^Λ is given by the product topology with the Euclidean topology on each factor.

Lemma 3.7.4. Consider the affine space $\mathbb{A}_{\mathbb{T}}^\Lambda = \text{Spec } \mathbb{T}[x_i \mid i \in \Lambda]$. The strong Zariski topology on $\mathbb{A}^\Lambda(\mathbb{T}) = \mathbb{T}^\Lambda$ is exactly the Euclidean topology.

Proof. We first show that strong Zariski closed sets are also Euclidean closed. A strong Zariski closed set Z is a (possibly infinite) intersection of principal strong Zariski closed subsets $V(f \sim g) = \{x \in \mathbb{T}^\Lambda \mid f(x) = g(x)\}$. Since a polynomial $f \in \mathbb{T}[x_i \mid i \in \Lambda]$ has only finitely many terms corresponding to a finite subset $\text{supp } f \subset \mathbb{N}^\Lambda$, its graph is a finite type polyhedron Γ_f in $\mathbb{T}^{\text{supp } f} \times \mathbb{T}$ crossed with $\mathbb{T}^{\mathbb{N}^\Lambda \setminus \text{supp } f}$. The polyhedra Γ_f and Γ_g are both Euclidean closed, so their intersection is, and hence the set $V(f \sim g)$ is Euclidean closed. Therefore strong Zariski closed sets are Euclidean closed.

We now show that there is a basis for the Euclidean topology consisting of strong Zariski open sets. A basis of open sets for the Euclidean topology is given by Euclidean open boxes: $\prod_{i \in \Lambda} J_i$ with each J_i an open interval in \mathbb{T} , and all but finitely many of them are the whole of \mathbb{T} . Each $J_i = (a_i, b_i)$ or $(b_i, \infty]$ is strong Zariski open, as it is the complement of the strong Zariski closed set $V(x_i + a_i \sim x_i) \cap V(x_i + b_i \sim b_i)$ or $V(x_i + b_i \sim b_i)$, respectively. \square

Remark 3.7.5. As far as we are aware, this observation that closed subschemes generate the Euclidean topology goes back originally to Mikhalkin in [26, Prop. 2.22].

Theorem 3.7.6. The strong Zariski topology on $\mathcal{Trop}_{\mathcal{F}_{univ}}(X)$ coincides with the Berkovich topology.

Proof. Since $\mathcal{Trop}_{\mathcal{F}_{univ}}(X)$ has an open cover by subschemes of the form $\mathcal{Trop}_{\mathcal{F}_{univ}}(U)$ for U an affine open subscheme of X , it suffices to assume that X is affine. The claim follows directly from Lemma 3.7.4 and the fact that the Berkovich topology on X^{an} coincides with the subspace topology it gets from the natural inclusion into $\mathbb{T}^{k[X]}$. \square

Example 3.7.7. Let k be an algebraically closed field that is complete with respect to a non-archimedean absolute value $|\cdot|$. Consider $(\mathbb{A}_k^1)^{an}$; the points of this space are

semivaluations on $k[t]$ that are compatible with the absolute value on k . Given $a \in k$ and $r \in \mathbb{R}_{>0}$, a Berkovich open disk is a subset of the form

$$D_{an}(a, r) = \{\nu \mid e^{-\nu(t-a)} < r\}.$$

A fundamental system of neighborhoods for the Berkovich topology is given by the finite intersections of these Berkovich open disks. The complement of an open disk $D_{an}(a, r)^c$ is the strong Zariski closed set defined by the equation

$$(-\log r)x_1 = (-\log r)x_1 + x_{t-a},$$

where x_1 is the multiplicative unit.

3.8. The map from the analytification to a tropicalization

Let Y be a locally integral \mathbb{F}_1 -scheme and X a k -scheme with a closed embedding $\varphi : X \hookrightarrow Y \times_{\mathbb{F}_1} k$. Given a rank-one valuation $\nu : k \rightarrow \mathbb{T}$, there is a canonical map π from the Berkovich analytification to the set-theoretic tropicalization of X with respect to φ ,

$$\pi : X^{\text{an}} \rightarrow \text{trop}_\varphi(X).$$

This is a slight generalization of the map constructed by Payne in [29]. It can be described on a suitable affine patch as follows. Suppose X is given by $\text{Spec } A$ for some k -algebra A , the \mathbb{F}_1 -scheme Y is $\text{Spec } B$ for some \mathbb{F}_1 -algebra B , and the embedding φ is given by a surjective homomorphism $\varphi^\sharp : B \otimes_{\mathbb{F}_1} k \twoheadrightarrow A$. A point of the analytification is a semivaluation $w : A \rightarrow \mathbb{T}$ compatible with ν , and a point of the set-theoretic tropicalization is a \mathbb{T} -algebra homomorphism $q : B \otimes_{\mathbb{F}_1} \mathbb{T} \rightarrow \mathbb{T}$ such that for each $\sum \lambda_i x_{b_i} \in \ker \varphi^\sharp$, the minimum of the set $\{q(\nu(\lambda_i)x_{b_i})\}$ is either equal to ∞ or is attained at least twice. Given a point $w \in (\text{Spec } A)^{\text{an}}$, the composition

$$B \hookrightarrow B \otimes_{\mathbb{F}_1} k \xrightarrow{\varphi^\sharp} A \xrightarrow{w} \mathbb{T}$$

is multiplicative and so determines a \mathbb{T} -algebra homomorphism $\pi(w) : B \otimes_{\mathbb{F}_1} \mathbb{T} \rightarrow \mathbb{T}$.

Proposition 3.8.1. The homomorphism $\pi(w)$ lies in $\text{trop}_\varphi(X)$.

Proof. Let $f = \sum \lambda_i x_{b_i}$ be an element in the kernel of φ^\sharp , so $w \circ \varphi^\sharp(\sum \lambda_i x_{b_i}) = \infty$. On the other hand, taking the coefficient-wise valuation, $\nu(f) = \sum \nu(\lambda_i)x_{b_i}$, and so

$$\begin{aligned} \pi(w)(\nu(f)) &= \sum \nu(\lambda_i)w \circ \varphi^\sharp(x_{b_i}) \\ &= \sum w \circ \varphi^\sharp(\lambda_i x_{b_i}). \end{aligned}$$

Thus we have

$$\infty = w \circ \varphi^\sharp(\sum \lambda_i x_{b_i}) \geq \sum w \circ \varphi^\sharp(\lambda_i x_{b_i}). \tag{8}$$

Since $w \circ \varphi^\sharp : B \otimes_{\mathbb{F}_1} k \rightarrow \mathbb{T}$ is a semivaluation and \mathbb{T} is totally ordered, a strict inequality

$$w \circ \varphi^\sharp(a + b) > w \circ \varphi^\sharp(a) + w \circ \varphi^\sharp(b)$$

implies $w \circ \varphi^\sharp(a) = w \circ \varphi^\sharp(b)$, and so we can conclude from the above inequality (8) that the minimum of $\{w \circ \varphi^\sharp(\lambda_i x_{b_i})\} \subset \mathbb{T}$ occurs at least twice (or there is only a single term and it is ∞). This shows that $\pi(w)$ is indeed in the set-theoretic tropicalization $\mathit{trop}_\varphi(X)$. \square

Proposition 3.8.2. Upon passing to \mathbb{T} -points, the canonical map of \mathbb{T} -schemes

$$\mathit{Trop}_{univ}(X) \rightarrow \mathit{Trop}_\varphi(X)$$

reduces to the map $\pi : X^{\text{an}} \rightarrow \mathit{trop}_\varphi(X)$.

Proof. This is just a matter of unwinding the definitions. Proposition 2.4.1 provides the canonical map $\widehat{X} \rightarrow Y \times_{\mathbb{F}_1} k$ for which we have a commuting diagram,

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \varphi \\ \widehat{X} & \xrightarrow{\widehat{\pi}} & Y \times_{\mathbb{F}_1} k. \end{array}$$

By restricting attention to suitable affine patches, this diagram is represented at the level of k -algebras by a diagram

$$\begin{array}{ccc} & A & \\ \swarrow ev & & \searrow \varphi^\sharp \\ M(A) \otimes_{\mathbb{F}_1} k & \xleftarrow{\widehat{\pi}^\sharp} & B \otimes_{\mathbb{F}_1} k, \end{array}$$

for a k -algebra A and an \mathbb{F}_1 -algebra B . The bottom arrow $\widehat{\pi}^\sharp$ in this diagram is induced by the morphism of \mathbb{F}_1 -algebras

$$\varphi^\flat : B \rightarrow M(A)$$

that is adjoint to φ^\sharp ; i. e., φ^\flat sends $x_b \in B \otimes_{\mathbb{F}_1} k$ to $\varphi^\sharp(x_b)$ (thought of as an element of $M(A)$). The associated morphism of tropicalizations,

$$B \otimes_{\mathbb{F}_1} \mathbb{T} / \mathcal{B}\mathit{trop}(\ker \varphi^\sharp) \rightarrow M(A) \otimes_{\mathbb{F}_1} \mathbb{T} / \mathcal{B}\mathit{trop}(\ker ev)$$

is also induced by φ^\flat . Consider a \mathbb{T} -point

$$w : M(A) \otimes_{\mathbb{F}_1} \mathbb{T} / \mathcal{B}\mathit{trop}(\ker ev) \rightarrow \mathbb{T}$$

of the universal tropicalization; it is entirely determined by the morphism of \mathbb{F}_1 -algebras $w^\flat : M(A) \rightarrow M(\mathbb{T})$ given by restricting w to monomials. The map $\mathit{Trop}_{univ}(X) \rightarrow \mathit{Trop}_\varphi(X)$ sends w to the \mathbb{T} -point corresponding to the composition of \mathbb{F}_1 -algebra morphisms

$$B \xrightarrow{\varphi^\flat} M(A) \xrightarrow{w^\flat} M(\mathbb{T}),$$

and one easily sees that this agrees with the description of $\pi(w)$ we gave above. \square

By assembling Proposition 3.5.2, Theorem 3.5.3, and the above proposition, we have proven Theorem B from the introduction.

4. LIMITS OF TROPICALIZATIONS

Fix an integral scheme X over a valued ring $\nu : R \rightarrow S$, and let \mathcal{C} denote the category of ‘locally integral \mathbb{F}_1 -embeddings of X ’; that is, an object of \mathcal{C} is a locally integral \mathbb{F}_1 -scheme Y together with a closed embedding $X \hookrightarrow Y \times_{\mathbb{F}_1} R$, and a morphism is a morphism of \mathbb{F}_1 -schemes inducing a commutative triangle of R -schemes. By [11, Proposition 6.4.1], scheme-theoretic tropicalization yields a covariant functor $\mathcal{Trop}_\bullet(X) : \mathcal{C} \rightarrow \text{Sch}_S$.

Because of its universal property (Proposition 2.4.1), the universal embedding $X \hookrightarrow \widehat{X}$ is an initial object in \mathcal{C} . Thus we trivially have that the universal tropicalization is the limit over \mathcal{C} of all tropicalizations of X .

Proposition 4.0.1. There is a canonical isomorphism,

$$\lim_{\varphi \in \mathcal{C}} \mathcal{Trop}_\varphi(X) \cong \mathcal{Trop}_{\text{univ}}(X).$$

It is more interesting to consider the limit over certain subcategories of \mathcal{C} , such as the subcategory $\mathcal{C}_\mathbb{A}$ of all embeddings into affine spaces \mathbb{A}^n (for varying finite n) and torus-equivariant morphisms. Payne showed that if X is an affine variety, then the limit over $\mathcal{C}_\mathbb{A}$ of the set-theoretic tropicalizations of X (considered as topological spaces) is homeomorphic to X^{an} [29, Theorem 1.1]. In the following section we give a scheme-theoretic refinement of this theorem.

4.1. Affine embeddings

Let $X = \text{Spec } A$, for A a finitely generated R -algebra. The category $\mathcal{C}_\mathbb{A}^{\text{op}}$ admits the following explicit algebraic description.

- Objects: Finitely generated free \mathbb{F}_1 -algebras (i. e., finite rank free abelian monoids-with-zero) B equipped with a surjective R -algebra homomorphism $B \otimes_{\mathbb{F}_1} R \twoheadrightarrow A$; equivalently, there is a specified \mathbb{F}_1 -algebra homomorphism $B \rightarrow M(A)$ whose image generates A as an R -algebra.
- Arrows: Homomorphisms of \mathbb{F}_1 -algebras $B_1 \rightarrow B_2$ whose scalar extension commutes with the maps $B_i \otimes_{\mathbb{F}_1} R \twoheadrightarrow A$; equivalently, these are \mathbb{F}_1 -algebra homomorphisms over $M(A)$.

Functoriality of tropicalization therefore gives in this case a functor $\mathcal{C}_\mathbb{A} \rightarrow \text{Sch}_S$, or $\mathcal{C}_\mathbb{A}^{\text{op}} \rightarrow S\text{-alg}$.

Theorem 4.1.1. Let A be a finitely generated integral R -algebra, and suppose R is equipped with a semivaluation $\nu : R \rightarrow S$. The universal tropicalization of $X = \text{Spec } A$ is isomorphic as an S -scheme to the limit of tropicalizations in affine spaces:

$$\mathcal{Trop}'_{\text{univ}}(X) \cong \lim_{\varphi \in \mathcal{C}_\mathbb{A}} \mathcal{Trop}'_\varphi(X).$$

We begin with a lemma, which is used to show that the limit is embedded in the affine S -scheme $\text{Spec } M(A) \otimes_{\mathbb{F}_1} S$. It says that the limit of the affine spaces in which X embeds is \widehat{X} . Let $F : \mathcal{C}_\mathbb{A}^{\text{op}} \rightarrow \mathbb{F}_1\text{-alg}$ be the forgetful functor sending $B \otimes_{\mathbb{F}_1} R \twoheadrightarrow A$ to B .

Lemma 4.1.2. $\text{colim}_{\mathcal{C}_{\mathbb{A}}^{\text{op}}} F \cong M(A)$.

Proof. First, note that the colimit exists because the category of abelian monoids is cocomplete and the colimit of a diagram of monoids-with-zero clearly has a zero element and is the colimit in the subcategory of monoids-with-zero. Let Z denote this colimit. Since arrows in $\mathcal{C}_{\mathbb{A}}^{\text{op}}$ are \mathbb{F}_1 -algebra morphisms over $M(A)$, the universal property of the colimit yields a canonical morphism of \mathbb{F}_1 -algebras $Z \rightarrow M(A)$. We show that this map is surjective and injective.

To show that each $a \in M(A)$ is in the image of $Z \rightarrow M(A)$, it suffices to show that there is an object $B \otimes_{\mathbb{F}_1} R \rightarrow A$ of $\mathcal{C}_{\mathbb{A}}^{\text{op}}$ whose restriction $B \rightarrow M(A)$ contains a in its image. Any finite set $S \subset A$ of R -algebra generators containing a yields a surjection $\mathbb{F}_1[x_1, \dots, x_{|S|}] \otimes_{\mathbb{F}_1} R = R[x_1, \dots, x_{|S|}] \rightarrow A$ with the desired property.

Now we prove injectivity. Suppose $a \in A$ and $a_1, a_2 \in Z$ are two elements that both map to a . Each a_i can be represented by an element a'_i in some finitely generated free \mathbb{F}_1 -algebra $\mathbb{F}_1[S_i]$ over $M(A)$, and without loss of generality we can assume $a'_i \in S_i$. Let $\mathcal{T} := S_1 \cup_{a'_1 \sim a'_2} S_2$. We have a set-map $\mathcal{T} \rightarrow A$ induced by the maps $S_i \rightarrow A$, since a'_1 and a'_2 have the same image in A . These maps $S_i \rightarrow A$ factor through the inclusions $S_i \hookrightarrow \mathcal{T}$, so the image of \mathcal{T} in A generates A as a R -algebra, and hence $\mathbb{F}_1[\mathcal{T}] \rightarrow M(A)$ is an object of $\mathcal{C}_{\mathbb{A}}^{\text{op}}$. Moreover, the inclusions $S_i \hookrightarrow \mathcal{T}$ induce arrows in $\mathcal{C}_{\mathbb{A}}^{\text{op}}$ under which a'_1 and a'_2 are identified. Applying the functor F to these two arrows yields a pair of \mathbb{F}_1 -algebra morphisms $\mathbb{F}_1[S_i] \rightarrow \mathbb{F}_1[\mathcal{T}]$ for which the images of a'_1 with a'_2 coincide, and hence the images of these elements in the colimit Z must be identified as well; i. e., $a_1 = a_2$ in Z . □

We now turn to the proof of Theorem 4.1.1.

Proof. Let $G : \mathcal{C}_{\mathbb{A}}^{\text{op}} \rightarrow S\text{-alg}$ be the functor sending an affine embedding to the algebra of global sections of the structure sheaf of the corresponding tropicalization:

$$(B \otimes_{\mathbb{F}_1} R \xrightarrow{\psi} A) \mapsto B \otimes_{\mathbb{F}_1} S/\mathcal{B}\text{trop}(\ker \psi).$$

We must show that $V := \text{colim}_{\mathcal{C}_{\mathbb{A}}^{\text{op}}} G$ is isomorphic to $W := M(A) \otimes_{\mathbb{F}_1} S/\mathcal{B}\text{trop}(\ker ev)$. Functoriality of tropicalization applied to Proposition 2.4.1 yields, by the universal property of the colimit, a canonical map of S -algebras $V \rightarrow W$. Since colimits commute with tensor products and quotients, it follows from Lemma 4.1.2 that V is a quotient of $M(A) \otimes_{\mathbb{F}_1} S$, say by a congruence J . Then, since the map

$$V = M(A) \otimes_{\mathbb{F}_1} S/J \rightarrow M(A) \otimes_{\mathbb{F}_1} S/\mathcal{B}\text{trop}(\ker ev) = W$$

is induced by the identity on $M(A)$, we see that it is surjective and $J \subset \mathcal{B}\text{trop}(\ker ev)$. We will show that this inclusion on congruences is an equality, i. e., that every relation in the universal tropicalization appears at some stage of the colimit diagram.

By Proposition 3.4.3, it suffices to consider the bend relations coming from the two types of basis elements of the kernel of the evaluation map described in Proposition 2.1.4. Let $a, b, c \in A$ satisfy $a + b + c = 0$, and consider the congruence $\mathcal{B}(x_a + x_b + x_c)$ on $M(A) \otimes_{\mathbb{F}_1} S$. Choose a set $S \subset A$ of R -algebra generators containing a, b, c , corresponding to an object $\psi : \mathbb{F}_1[S] \otimes_{\mathbb{F}_1} R \rightarrow A$ of $\mathcal{C}_{\mathbb{A}}^{\text{op}}$. Then $x_a + x_b + x_c \in \ker \psi$, so the congruence

$\mathcal{B}\text{trop}(\ker \psi)$, which defines the tropicalization $G(\psi)$, contains the bend relations of $x_a + x_b + x_c \in \mathbb{F}_1[\mathcal{S}] \otimes_{\mathbb{F}_1} S$. Since the isomorphism in Lemma 4.1.2 is induced by the structure maps in the objects of $\mathcal{C}_{\mathbb{A}}^{\text{op}}$, the map $G(\psi) \rightarrow V$ sends this polynomial to $x_a + x_b + x_c$ in $M(A) \otimes_{\mathbb{F}_1} S$. Thus the bend relations $\mathcal{B}(x_a + x_b + x_c)$ are contained in the colimit congruence J . Similarly, for $\lambda \in R$ and $a \in A$, an R -algebra generating set $\mathcal{T} \subset A$ containing a and λa yields an object $\gamma : \mathbb{F}_1[\mathcal{T}] \otimes_{\mathbb{F}_1} R \rightarrow A$ of $\mathcal{C}_{\mathbb{A}}^{\text{op}}$ satisfying $\lambda x_a + x_{\lambda a} \in \ker \gamma$. In the congruence defining the tropicalization $G(\gamma)$ we thus have the bend relation $\nu(\lambda)x_a \sim x_{\lambda a}$, and hence this also holds in the colimit congruence J . \square

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REFERENCES

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- [1] S. Banerjee: Tropical geometry over higher dimensional local fields. *J. Reine Angew. Math.* *2015* (2013), 71–87. DOI:10.1515/crelle-2012-0124
 - [2] V. G. Berkovich: Spectral theory and analytic geometry over non-Archimedean fields. *Mathematical Surveys and Monographs*, American Mathematical Society *33*, Providence 1990.
 - [3] V. G. Berkovich: Smooth p -adic analytic spaces are locally contractible. *Invent. Math.* *137* (1999), 1, 1–84. DOI:10.1007/s10107980002a
 - [4] V. G. Berkovich: Smooth p -adic analytic spaces are locally contractible. II. In: *Geometric aspects of Dwork theory*. (Vol. I, II.), Walter de Gruyter, Berlin 2004, pp. 293–370. DOI:10.1515/9783110198133.1.293
 - [5] A. Deitmar: Schemes over \mathbb{F}_1 . In: *Number fields and function fields—two parallel worlds*, *Progr. Math.* *239*, Birkhäuser Boston, Boston, 2005, pp. 87–100.
 - [6] A. Deitmar: \mathbb{F}_1 -schemes and toric varieties. *Beiträge Algebra Geom.* *49* (2008), 2, 517–525.
 - [7] N. Durov: New approach to arakelov geometry. arXiv:0704.2030, 2007.
 - [8] M. Einsiedler, M. Kapranov, and D. Lind: Non-Archimedean amoebas and tropical varieties. *J. Reine Angew. Math.* *601* (2006), 139–157.
 - [9] T. Foster, P. Gross, and S. Payne: Limits of tropicalizations. *Israel J. Math.* *201* (2014), 2, 835–846. DOI:10.1007/s11856-014-1051-x
 - [10] T. Foster and D. Ranganathan: Hahn analytification and connectivity of higher rank tropical varieties. *Manuscr. Math.* *151* (2016), 3–4, 353–374. DOI:10.1007/s00229-016-0841-3
 - [11] J. Giansiracusa and N. Giansiracusa: Equations of tropical varieties. *Duke Math. J.* *165* (2016), 18, 3379–3433.
 - [12] W. Gubler, J. Rabinoff, and A. Werner: Skeletons and tropicalizations. arXiv:1404.7044, 2014.

- [13] E. Hrushovski and F. Loeser: Non-archimedean tame topology and stably dominated types. *Ann. Math. Studies 192*, Princeton University Press, Princeton 2016.
- [14] R. Huber: *Étale cohomology of rigid analytic varieties and adic spaces. Aspects of Mathematics E30*, Friedr. Vieweg and Sohn, Braunschweig 1996.
- [15] J. Jun, K. Mincheva, and J. Tolliver: Vector bundles on tropical schemes. arXiv:2009.03030, 2020.
- [16] T. Kajiwara: Tropical toric geometry. In: *Contemp. Math., Toric topology 460*, Amer. Math. Soc., Providence 2008, pp. 197–207.
- [17] K. Kato: Toric singularities. *Amer. J. Math.* *116* (1994), 5, 1073–1099.
- [18] M. Kontsevich and Y. Soibelman: Affine structures and non-Archimedean analytic spaces. In: *Progr. Math., The Unity Math. 244*, Birkhäuser Boston, Boston 2006. pp. 321–385. DOI:10.1007/0-8176-4467-9_9
- [19] M. Kontsevich and Y. Tschinkel: Non-archimedean kähler geometry. <http://www.ihes.fr/~maxim/TEXTS/Non-archimedean%20Kahler%20geometry.pdf> (unfinished manuscript on the author’s homepage), 2002.
- [20] A. Kuronya, P. Souza, and M. Ulirsch: Tropicalization of toric prevarieties. arXiv:2107.03139, 2021.
- [21] D. Maclagan and F. Rincón: Tropical ideals. *Compos. Math.* *154* (2018), 3, 640–670. DOI:10.1112/S0010437X17008004
- [22] D. Maclagan and F. Rincón: Tropical schemes, tropical cycles, and valuated matroids. *J. Eur. Math. Soc. (JEMS)* *22* (2020), 3, 777–796. DOI:10.4171/jems/932
- [23] D. Maclagan and F. Rincón: Varieties of tropical ideals are balanced. arXiv:2009.14557, 2020.
- [24] D. Maclagan and B. Sturmfels: *Introduction to tropical geometry. American Mathematical Society, Graduate Studies in Mathematics 161*, Providence 2015.
- [25] A. W. Macpherson: Skeleta in non-Archimedean and tropical geometry. arXiv:1311.0502, 2013.
- [26] G. Mikhalkin: *Tropical Geometry. Unfinish draft book.*
- [27] G. Mikhalkin: Tropical geometry and its applications. In: *International Congress of Mathematicians II*, Eur. Math. Soc., Zürich 2006, pp. 827–852.
- [28] M. Mustata and J. Nicaise: Weight functions on non-Archimedean analytic spaces and the Kontsevich–Soibelman skeleton. arXiv:1212.6328, 2013.
- [29] S. Payne: Analytification is the limit of all tropicalizations. *Math. Res. Lett.* *16* (2009), 3, 543–556. DOI:10.4310/MRL.2009.v16.n3.a13
- [30] S. Payne: Fibers of tropicalization. *Math. Z.* *262* (2009), 2, 301–311. DOI:10.1007/s00209-008-0374-x
- [31] P. Popescu-Pampu and D. Stepanov: Local tropicalization. In: *Algebraic and combinatorial aspects of tropical geometry, Contemp. Math. 589*, Amer. Math. Soc., Providence 2013, pp. 253–316. DOI:10.1090/conm/589/11748
- [32] J. Richter-Gebert, B. Sturmfels, and T. Theobald: First steps in tropical geometry. In: *Idempotent mathematics and mathematical physics, Contemp. Math. 377*, Amer. Math. Soc., Providence 2005, pp. 289–317.

- [33] M. Temkin: Relative Riemann–Zariski spaces. *Israel J. Math.* *185* (2011), 1–42. DOI:10.1007/s11856-011-0099-0
- [34] A. Thuillier: Théorie du potentiel sur les courbes en géométrie analytique non archimédienne. applications à la théorie d’arakevov, 2005. <http://tel.ccsd.cnrs.fr/documents/archives0/00/01/09/90/index.html>.
- [35] B. Toën and M. Vaquié: Au-dessous de $\text{Spec } \mathbb{Z}$. *J. K-Theory* *3* (2009), 3, 437–500. DOI:10.1017/is008004027jkt048
- [36] M. Ulirsch: Functorial tropicalization of logarithmic schemes: the case of constant coefficients. *Proc. Lond. Math. Soc.* *3 114* (2017), 6, 1081–1113. DOI:10.1112/plms.12031
- [37] M. Ulirsch: Non-Archimedean geometry of Artin fans. *Adv. Math.* *345* (2019), 346–381. DOI:10.1016/j.aim.2019.01.008
- [38] J. Włodarczyk: Embeddings in toric varieties and prevarieties. *J. Algebraic Geom.* *2* (1993), 4, 705–726.

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