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Revisiting inclusive decay widths of charmed mesons

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Abstract: Determining for the first time the Darwin operator contribution for the non-leptonic charm-quark decays and using new non-perturbative results for the matrix elements of $\Delta C = 0$ four-quark operators, including eye-contractions, we present a comprehensive study of the lifetimes of charmed mesons and inclusive semileptonic decay rates as well as the ratios, within the framework of the Heavy Quark Expansion (HQE). We find good agreement with experiment for the ratio $\tau(D^+)/\tau(D^0)$, for the total D_s^+-meson decay rate, for the semileptonic rates of all three mesons D^0, D^+ and D_s^+, and for the semileptonic ratio $\Gamma_{sl}^{D^+}/\Gamma_{sl}^{D^0}$. The total decay rates of the D^0 and D^+ mesons are underestimated in our HQE approach and we suspect that this is due to missing higher-order QCD corrections to the free charm quark decay and the Pauli interference contribution. For the $\mathrm{SU(3)}_F$ breaking ratios $\tau(D_s^+)/\tau(D^0)$ and $\Gamma_{sl}^{D_s^+}/\Gamma_{sl}^{D^0}$ our predictions lie closer to one than experiment. This might originate from the poor knowledge of the non-perturbative parameters μ_G^2, μ_π^2 and ρ_D^3 in the D^0 and D_s^+ systems. These parameters could be determined by experimental studies of the moments of inclusive semileptonic D meson decays.

Keywords: Flavour Symmetries, Semi-Leptonic Decays

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Contents

1 Introduction

Lifetimes of charm mesons are determined experimentally very precisely [1][1] and show a pattern which is clearly less monotonous than in the b-sector, with values spreading over a rather large range. Moreover, also inclusive semileptonic branching fractions have been measured [1], and recently an update for the D_s^+-meson has been released by the BESIII Collaboration [3]. A summary of the current experimental status is presented in table 1. While in the bottom sector, the approximation that the meson decay can be described

[1]New results from Belle II have recently been made public [2]: $\tau(D^0) = 410.5 \pm 1.1 \pm 0.8$ fs, $\tau(D^+) = 1030.4 \pm 4.7 \pm 3.1$ fs.

	D^0	D^+	D_s^+
τ [ps]	0.4101(15)	1.040(7)	0.504(4)
Γ [ps^{-1}]	2.44(1)	0.96(1)	1.98(2)
$\tau(D_q)/\tau(D^0)$	1	2.54(2)	1.20(1)
$\mathrm{Br}(D_q \to X e^+ \nu_e)[\%]$	6.49(11)	16.07(30)	6.30(16)
$\dfrac{\Gamma(D_q \to X e^+ \nu_e)}{\Gamma(D^0 \to X e^+ \nu_e)}$	1	0.977(26)	0.790(26)

Table 1. Status of the experimental determinations of the lifetime and the semileptonic branching fractions of the lightest charmed mesons ($D_q \in \{D^0, D^+, D_s^+\}$). All values are taken from the PDG [1] apart from the semileptonic D_s^+-meson decays which were recently measured by the BESIII Collaboration [3].

in terms of the free b-quark decay is experimentally well accommodated, for the charm system this is poorly justified. A systematic way to study this assumption is provided by the heavy quark expansion (HQE) — see refs. [4, 5] for early references or ref. [6] for a recent review, according to which the inclusive decay width of a meson containing a heavy charm quark can be written as

$$\Gamma(D) = \Gamma_3 + \Gamma_5 \frac{\langle \mathcal{O}_5 \rangle}{m_c^2} + \Gamma_6 \frac{\langle \mathcal{O}_6 \rangle}{m_c^3} + \ldots + 16\pi^2 \left(\tilde{\Gamma}_6 \frac{\langle \tilde{\mathcal{O}}_6 \rangle}{m_c^3} + \tilde{\Gamma}_7 \frac{\langle \tilde{\mathcal{O}}_7 \rangle}{m_c^4} + \ldots \right), \tag{1.1}$$

with the matrix element of the $\Delta C = 0$ operators given by $\langle \mathcal{O}_Y \rangle = \langle D | \mathcal{O}_Y | D \rangle / (2 m_D)$. Their numerical size is expected to be of the order of the hadronic scale $\Lambda_{\mathrm{QCD}} \leq 1\,\mathrm{GeV}$, but the actual value must be determined with a non-perturbative calculation. Note that in eq. (1.1) quantities labelled by a tilde refer to the contribution of four-quark operators, while those without a tilde correspond to two-quark operators, cf. figure 1. The Wilson coefficients Γ_i in eq. (1.1) can be computed perturbatively and admit the following expansion in the strong coupling α_s, i.e.

$$\Gamma_i = \Gamma_i^{(0)} + \frac{\alpha_s(m_c)}{4\pi} \Gamma_i^{(1)} + \left[\frac{\alpha_s(m_c)}{4\pi} \right]^2 \Gamma_i^{(2)} + \ldots . \tag{1.2}$$

In the present work we will try to shed further light into the question, whether the expansion parameters $\alpha_s(m_c)$ and $\Lambda_{\mathrm{QCD}}/m_c$ are small enough in order to ensure a meaningful convergence of the HQE. The Particle Data Group [1] quotes, for the pole and $\overline{\mathrm{MS}}$ mass of the charm quark, the values

$$m_c^{\mathrm{Pole}} = (1.67 \pm 0.07)\ \mathrm{GeV}\,, \qquad \overline{m}_c(\overline{m}_c) = (1.27 \pm 0.02)\ \mathrm{GeV}, \tag{1.3}$$

while the dependence of the strong coupling on both the charm scale and the loop order (obtained using the RunDec package [7]) is shown in table 2. In our numerical analysis we

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$\alpha_s(m_c)$	$m_c = 1.67\,\text{GeV}$	$m_c = 1.48\,\text{GeV}$	$m_c = 1.27\,\text{GeV}$
2-loop	0.322	0.346	0.373
5-loop	0.329	0.356	0.387

Table 2. Numerical values of the strong coupling α_s evaluated at different scales and loop order, obtained using the RunDec package [7].

use the 5-loop running of the strong coupling. While the determination of the $\overline{\text{MS}}$ mass is theoretically well founded, that of the pole mass seems to be affected by a potential breakdown of perturbation theory. On the other side, the pole mass is the natural expansion parameter of the HQE. The relation between the two mass schemes, up to third order in the strong coupling, reads [8–10]

$$\begin{aligned} m_c^{\text{Pole}} &= \overline{m}_c(\overline{m}_c)\left[1+\frac{4}{3}\frac{\alpha_s(\overline{m}_c)}{\pi}+10.43\left(\frac{\alpha_s(\overline{m}_c)}{\pi}\right)^2+116.5\left(\frac{\alpha_s(\overline{m}_c)}{\pi}\right)^3\right] \\ &= \overline{m}_c(\overline{m}_c)\left[1+0.1642+0.1582+0.2176\right], \end{aligned} \tag{1.4}$$

where we have used the 5-loop result for the strong coupling at the scale 1.27 GeV. Due to the fact that Γ_3 depends on the fifth power of the charm pole mass, see section 2.2, one obtains quite different results according to how higher orders in eq. (1.4) are treated. Specifically, by truncating the expansion in eq. (1.4) at first order in α_s, from $\overline{m}_c(\overline{m}_c) = 1.27\,\text{GeV}$, we obtain for the pole mass the value $m_c^{\text{Pole}} = 1.479\,\text{GeV}$, which leads respectively to

$$\left(m_c^{\text{Pole}}\right)^5 = \overline{m}_c(\overline{m}_c)^5\left[1+0.1642\right]^5 = 2.14\,\overline{m}_c(\overline{m}_c)^5, \tag{1.5}$$

taking the fifth power of m_c^{Pole}, and

$$\left(m_c^{\text{Pole}}\right)^5 \approx \overline{m}_c(\overline{m}_c)^5\left[1+5\cdot 0.1642\right] = 1.82\,\overline{m}_c(\overline{m}_c)^5, \tag{1.6}$$

further expanding up to the first order in α_s. The result in eq. (1.6) is about 15% smaller than the one in eq. (1.5). Instead, by including also all the higher order terms given in eq. (1.4), we get

$$\left(m_c^{\text{Pole}}\right)^5 = \overline{m}_c(\overline{m}_c)^5\left[1+0.1642+0.1582+0.2176\right]^5 = 8.66\,\overline{m}_c(\overline{m}_c)^5, \tag{1.7}$$

which is roughly a factor four larger than the value in eq. (1.5).

In the following, we will thus consider four different quark mass schemes:

1. Use eq. (1.4) to first order in α_s, since this is the order to which most of the Wilson coefficients are known. In this case we fix $m_c^{\text{Pole}} = 1.48\,\text{GeV}$ and express everything in terms of the pole mass. A further possibility would be to consider the expansion in eq. (1.4) to be an asymptotic one, whose smallest correction appears at order α_s^2, which is where we stop the expansion. In this case we get the pole mass value from PDG, $m_c^{\text{Pole}} = 1.67\,\text{GeV}$. We did numerical tests for this large value of the charm quark mass and the results for decay rates are roughly 70–90% larger than

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the values obtained using $m_c^{\rm Pole} = 1.48\,{\rm GeV}$. Since we expect this enhancement to be compensated by missing NNLO corrections to the non-leptonic decay rates, we will not separately present results for $m_c^{\rm Pole} = 1.67\,{\rm GeV}$.

2. Express the c-quark mass in terms of the $\overline{\rm MS}$ mass [11],

$$m_c^{\rm Pole} = \overline{m}_c(\overline{m}_c)\left[1 + \frac{4}{3}\frac{\alpha_s(\overline{m}_c)}{\pi}\right], \tag{1.8}$$

taking $\overline{m}_c(\overline{m}_c) = 1.27\,{\rm GeV}$ [1], and expand consistently up to order α_s. Because of the dependence on the fifth power of the charm-quark mass, in this case Γ_3 is affected by a large correction $5 \times (4/3)(\alpha_s/\pi)$.

3. Express the c-quark mass in terms of the kinetic mass [12, 13]. The kinetic scheme has been introduced in order to obtain a short distance definition of the heavy quark mass which allows a faster convergence of the perturbative series and is still valid at small scales $\mu \sim 1\,{\rm GeV}$. The relation between the kinetic scheme and the $\overline{\rm MS}$ and Pole schemes can be found, up to N^{3}LO corrections, in ref. [14]. At order α_s one has

$$m_c^{\rm Pole} = m_c^{\rm Kin}\left[1 + \frac{4\alpha_s}{3\pi}\left(\frac{4}{3}\frac{\mu^{\rm cut}}{m_c^{\rm Kin}} + \frac{1}{2}\left(\frac{\mu^{\rm cut}}{m_c^{\rm Kin}}\right)^2\right)\right], \tag{1.9}$$

where $\mu^{\rm cut}$ is the Wilsonian cutoff separating the perturbative and non-perturbative regimes. Using $\overline{m}_c(\overline{m}_c)$ as an input, the authors of ref. [14] obtain

$$m_c^{\rm Kin}(1{\rm GeV}) = 1.128\,{\rm GeV} \quad ({\rm N^3LO}), \tag{1.10}$$

$$m_c^{\rm Kin}(1{\rm GeV}) = 1.206\,{\rm GeV} \quad ({\rm NLO}). \tag{1.11}$$

Comparing with eq. (1.8) it follows that the kinetic scheme might be preferred to the $\overline{\rm MS}$ scheme if the term in the round brackets of eq. (1.9) would give a suppression factor. For $\mu^{\rm cut} = 1\,{\rm GeV}$ and $m_c^{\rm Kin} = 1.2\,{\rm GeV}$, this is not the case, while using lower values i.e. $\mu^{\rm cut} < 1\,{\rm GeV}$, the convergence of the series could be improved, however this would bring in an additional uncertainty due to the closeness to the non-perturbative scale $\Lambda_{\rm QCD}$. In our numerical analysis we will investigate the kinetic scheme with $\mu^{\rm cut} = 0.5\,{\rm GeV}$. From ref. [14] we take the following value

$$m_c^{\rm kin}(0.5\,{\rm GeV}) = 1.363\,{\rm GeV}, \tag{1.12}$$

obtained for consistency at NLO in α_s and using as an input $\overline{m}_c(\overline{m}_c)$.

4. In addition, we will consider the $1S$-mass scheme defined as [15–17]:

$$m_c^{\rm Pole} = m_c^{1S}\left(1 + \frac{(C_F\,\alpha_s)^2}{8}\right), \tag{1.13}$$

where $C_F = 4/3$, and the $1S$ mass $m_c^{1S} \approx 1.44\,{\rm GeV}$ is obtained using the conversion from the $\overline{\rm MS}$-scheme (implemented in the RunDec package [7]) at one-loop level. Note that the correction within the $1S$ scheme in fact starts at order α_s^2 which however is still considered to be a NLO (not NNLO) effect [15].[2]

[2] Similarly, another possibility would be to study the potential subtracted mass [18].

The above arguments clearly indicate the importance of including higher order perturbative QCD corrections to the decay rates.

With this work we present a study of the total decay rate of the D^0, D^+ and D_s^+ mesons, of their lifetime ratios $\tau(D^+)/\tau(D^0)$ and $\tau(D_s^+)/\tau(D^0)$ and of the semileptonic branching fractions $\mathrm{Br}(D_q \to X e^+ \nu_e)$ using state-of-the-art expressions for the $\Delta C = 0$ Wilson coefficients and for the non-perturbative parameters. Γ_3 is known at NLO-QCD [19–26] for non-leptonic decays. NNLO-QCD [27–36] and NNNLO-QCD [37, 38] corrections have been computed for semileptonic decays, while for non-leptonic decays NNLO corrections have been determined in the massless case and in full QCD (i.e. no effective Hamiltonian was used) in ref. [39]. Γ_5 was determined at LO-QCD for both semileptonic and non-leptonic decays [40–43]. For the semileptonic modes even NLO-QCD corrections are available [44–46]. In the b-system, Γ_6 was first computed at LO-QCD in ref. [47] and recently the NLO-QCD corrections were determined in ref. [48], both for the semileptonic case only. Very recently Γ_6 has been determined also for non-leptonic decays [49–51] and the coefficient was found to be large. For semileptonic D-meson decays, Γ_6 was determined in ref. [52], see also the recent ref. [53], while the corresponding results for the non-leptonic charm modes are presented for the first time in this work. $\tilde{\Gamma}_6$ is known at NLO-QCD for lifetimes of B-meson [54, 55] and of D-meson [56], while $\tilde{\Gamma}_7$ and $\tilde{\Gamma}_8$ have been estimated in LO-QCD in refs. [57, 58].

On the non-perturbative side, at dimension-five, the matrix element of the chromo-magnetic operator can be determined from spectroscopy, while for the kinetic operator there exist several heavy quark effective theory (HQET) determinations with lattice simulations [59–63] and using sum rules [12, 64, 65]. The matrix elements of the four-quark operators $\langle \tilde{\mathcal{O}}_6 \rangle$ have been computed using HQET sum rules [66]. Violations of $\mathrm{SU}(3)_F$ and so far undetermined eye-contractions could yield visible effects and a calculation of these corrections with HQET sum rules — following ref. [67] — has been performed in ref. [68]. Corresponding lattice results for the matrix elements of the four-quark operators would be highly desirable. We emphasise that the matrix element of the dimension-six Darwin operator, $\langle \mathcal{O}_6 \rangle$, can be expressed in terms of the above Bag parameters by taking into account the equation of motion for the gluon field strength tensor.

The paper is organised as follows. In section 2, after briefly introducing the effective Hamiltonian describing the c-quark decays, we analyse in detail the structure of the HQE, discussing each of the short-distance contributions in eq. (1.1). In section 3, we describe how the corresponding non-perturbative parameters are determined. Numerical results for the total D-meson decay widths, their ratios, as well as for the semileptonic branching fractions, are presented in section 4. Finally, we conclude in section 5 with an outlook on how to further improve the theoretical predictions in the charm sector. The numerical input used in the analysis are collected in appendix A, the complete expressions for the coefficients of the Darwin operator for non-leptonic c-quark decays are presented in appendix B, while in appendix C we show the parametrisation of the matrix elements of the four-quark operators.

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μ_1[GeV]	1	1.27	1.36	1.44	1.48	3
$C_1(\mu_1)$	1.25 (1.34)	1.20 (1.27)	1.19 (1.26)	1.18 (1.25)	1.18 (1.24)	1.10 (1.15)
$C_2(\mu_1)$	−0.48 (−0.62)	−0.39 (−0.50)	−0.40 (−0.53)	−0.37 (−0.49)	−0.37 (−0.48)	−0.24 (−0.32)
$C_3(\mu_1)$	0.03 (0.02)	0.02 (0.01)	0.02 (0.01)	0.01 (0.01)	0.01 (0.01)	0.00 (0.00)
$C_4(\mu_1)$	−0.06 (−0.04)	−0.05 (−0.03)	−0.04 (−0.03)	−0.04 (−0.02)	−0.04 (−0.02)	−0.01 (−0.01)
$C_5(\mu_1)$	0.01 (0.01)	0.01 (0.01)	0.01 (0.01)	0.01 (0.01)	0.01 (0.01)	0.00 (0.00)
$C_6(\mu_1)$	−0.08 (−0.05)	−0.05 (−0.03)	−0.05 (−0.03)	−0.04 (−0.03)	−0.04 (−0.03)	−0.01 (−0.01)

Table 3. Comparison of the Wilson coefficients at NLO-QCD (LO-QCD) for different values of μ_1.

2 The total decay rate

2.1 Effective Hamiltonian and HQE

The non-leptonic decay of a charm quark $c \to q_1 \bar{q}_2 u$ ($q_i = u, d, s$) is governed by the effective $\Delta C = 1$ Hamiltonian (see e.g. ref. [69])

$$\mathcal{H}_{\rm eff}^{\rm NL} = \frac{G_F}{\sqrt{2}} \left[\sum_{q_{1,2}=d,s} \lambda_{q_1 q_2} \left[C_1(\mu_1)\, Q_1^{q_1 q_2} + C_2(\mu_1)\, Q_2^{q_1 q_2} \right] - \lambda_b \sum_{j=3}^{6} C_j(\mu_1) Q_j \right] + \text{h.c.}, \tag{2.1}$$

where $\lambda_{q_1 q_2} = V_{cq_1}^* V_{uq_2}$ and $\lambda_b = V_{cb}^* V_{ub}$ are the CKM factors, $C_i(\mu_1)$ denote the Wilson coefficients at the renormalisation scale $\mu_1 \sim m_c$, $Q_{1,2}^{q_1 q_2}$ are tree-level $\Delta C = 1$ operators[3]

$$Q_1^{q_1 q_2} = \left(\bar{q}_1^i \gamma_\rho (1 - \gamma_5) c^i \right) \left(\bar{u}^j \gamma^\rho (1 - \gamma_5) q_2^j \right), \tag{2.2}$$

$$Q_2^{q_1 q_2} = \left(\bar{q}_1^i \gamma_\rho (1 - \gamma_5) c^j \right) \left(\bar{u}^j \gamma^\rho (1 - \gamma_5) q_2^i \right), \tag{2.3}$$

while Q_j, $j = 3 \dots 6$ are penguin operators, which can only arise in the singly Cabibbo suppressed decays $c \to s\bar{s}u$ and $c \to d\bar{d}u$ or in further suppressed pure penguin decays like $c \to u\bar{u}u$. Values of the Wilson coefficients at different scales are shown in table 3 both at NLO-QCD and LO-QCD.

We see that the Wilson coefficients of the penguin operators are very small, additionally their contributions are also strongly suppressed by the CKM factor $\lambda_b \ll \lambda_{q_1 q_2}$. Therefore, in our analysis, we neglect the effect of the penguin operators, given the current limited theoretical accuracy in the charm sector.

[3] In our notation, $Q_1^{q_1 q_2}$ is the colour-singlet operator.

The complete effective Hamiltonian describing all possible c-quark decays is a sum of non-leptonic, semileptonic and radiative contributions, namely

$$\mathcal{H}_{\rm eff} = \mathcal{H}_{\rm eff}^{\rm NL} + \mathcal{H}_{\rm eff}^{\rm SL} + \mathcal{H}_{\rm eff}^{\rm rare}\,, \tag{2.4}$$

where $\mathcal{H}_{\rm eff}^{\rm NL}$ is given in eq. (2.1),

$$\mathcal{H}_{\rm eff}^{\rm SL} = \frac{G_F}{\sqrt{2}} \sum_{q=d,s} \sum_{\ell=e,\mu} V_{cq}^* \, Q^{q\ell} + \text{h.c.}\,, \tag{2.5}$$

with the semileptonic operator $Q^{q\ell} = (\bar{q}\gamma^\mu(1-\gamma_5)c)\,(\bar{\nu}_\ell\gamma_\mu(1-\gamma_5)\ell)$, while $\mathcal{H}_{\rm eff}^{\rm rare}$ describes decays like $D \to \pi\ell^+\ell^-$, whose branching fraction is much smaller than those corresponding to tree-level transitions. Hence, in the following we neglect rare decays and we do not show an explicit expression for $\mathcal{H}_{\rm eff}^{\rm rare}$.

The total decay width of a heavy D meson with mass m_D and four-momentum p_D^μ can be written as

$$\Gamma(D) = \frac{1}{2m_D} \sum_X \int_{\rm PS} (2\pi)^4 \delta^{(4)}(p_D - p_X)\; |\langle X(p_X)|\mathcal{H}_{\rm eff}|D(p_D)\rangle|^2, \tag{2.6}$$

where PS denotes the phase space integration and we have summed over all possible final states X into which the D meson can decay. Eq. (2.6) can be related, via the optical theorem, to the discontinuity of the forward scattering matrix element of the time ordered product of the double insertion of the effective Hamiltonian, i.e.

$$\Gamma(D) = \frac{1}{2m_D} {\rm Im}\langle D|\mathcal{T}|D\rangle\,, \tag{2.7}$$

with the transition operator

$$\mathcal{T} = i \int d^4x \, T\left\{\mathcal{H}_{\rm eff}(x)\,,\mathcal{H}_{\rm eff}(0)\right\}\,. \tag{2.8}$$

In the framework of the HQE, the four-momentum of the decaying c-quark is parametrised in terms of "large" and "small" components as

$$p_c^\mu = m_c\, v^\mu + k^\mu, \tag{2.9}$$

where $v^\mu = p^\mu/m_D$ denotes the four-velocity of the D-meson and $k^\mu \to iD^\mu$, with D^μ being the covariant derivative with respect to the background gluon field, accounts for the residual interaction of the c-quark with the light degrees of freedom inside the hadron, i.e. soft gluons and quarks. At the same time, the heavy charm quark field is redefined as

$$c(x) = e^{-im_c v\cdot x} c_v(x)\,, \tag{2.10}$$

to remove the large fraction of the c-quark momentum. Using eqs. (2.9) and (2.10), $\Gamma(D)$ in eq. (2.7) can be expanded in the small quantity $D^\mu/m_c \sim \Lambda_{\rm QCD}/m_c$, leading to the series in eq. (1.1) — for more details see e.g. ref. [70], or ref. [49] for a more recent treatment. The result is schematically shown in figure 1. The first diagram on the top line of figure 1,

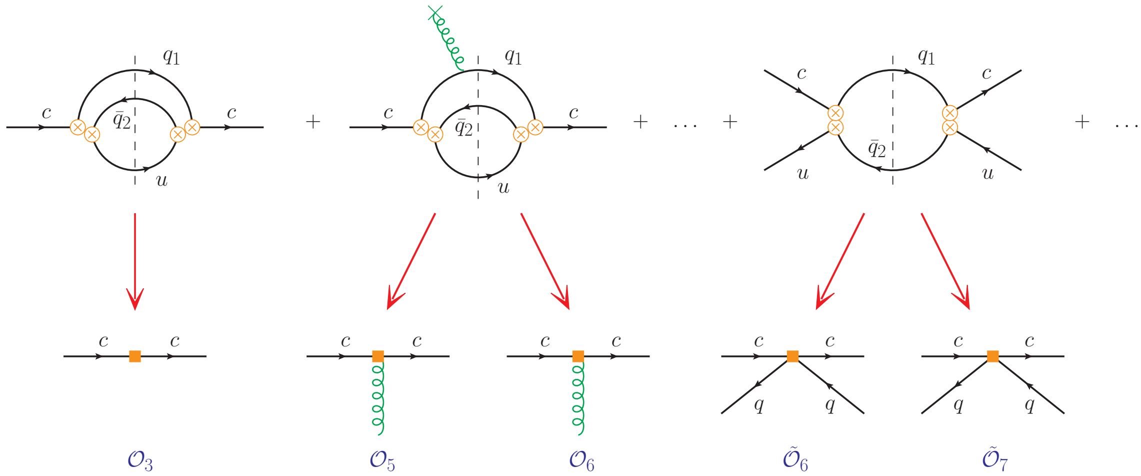

Figure 1. The diagrams describing contributions to the HQE in eq. (1.1). The crossed circles denote the $\Delta C = 1$ operators Q_i of the effective Hamiltonian while the squares denote the local $\Delta C = 0$ operators $\mathcal{O}_i$ and $\tilde{\mathcal{O}}_i$. The two-loop and the phase space enhanced one-loop diagrams correspond respectively to the two-quark operators $\mathcal{O}_i$ and to the four-quark operators $\tilde{\mathcal{O}}_i$ in the HQE.

corresponding to the limit $m_c \to \infty$, represents the decay of a free c-quark, while power corrections due to the interaction of the heavy quark with soft gluons and quarks are described respectively by the second and third diagrams on the top line of figure 1. Finally, before discussing the individual terms in eq. (1.1) separately, it is worth emphasizing that the field c_v is related to the effective heavy quark field h_v, introduced in the framework of the HQET (see e.g. ref. [71]), by

$$c_v(x) = h_v(x) + \frac{i \not{D}_\perp}{2m_c} h_v(x) + \mathcal{O}\left(\frac{1}{m_c^2}\right), \tag{2.11}$$

where $D_\perp^\mu = D^\mu - (v \cdot D)\, v^\mu$.

2.2 Dimension-three contribution

The leading term in eq. (1.1), $\Gamma_3^{(0)}$, can be schematically written as

$$\Gamma_3^{(0)} = \Gamma_0\, c_3 = \Gamma_0 \left[f\left(z_s, z_e, z_{\nu_e}\right) + f\left(z_s, z_\mu, z_{\nu_\mu}\right) + |V_{ud}|^2 \mathcal{N}_a\, f\left(z_s, z_u, z_d\right) + \ldots \right], \tag{2.12}$$

where we define

$$\Gamma_0 = \frac{G_F^2 m_c^5}{192\pi^3} |V_{cs}|^2, \tag{2.13}$$

and we introduce the dimensionless mass parameter $z_q = m_q^2/m_c^2$. Note than we neglect the neutrino as well as the electron, the up and down quarks masses, i.e. $z_\nu = z_e = z_u = z_d = 0$, while $z_s \neq 0 \neq z_\mu$. The first two terms in c_3 in eq. (2.12) correspond to the semileptonic modes $c \to s e^+ \nu_e$ and $c \to s \mu^+ \nu_\mu$, while the third term to the Cabibbo favoured decay $c \to s u \bar{d}$. The ellipsis stand for CKM suppressed contributions. The dependence on the $\Delta C = 1$ Wilson coefficients is absorbed in the combination $\mathcal{N}_a = 3\, C_1^2 + 3\, C_2^2 + 2\, C_1 C_2$. The behaviour of $\mathcal{N}_a$ as function of the renormalisation scale, both at LO- and NLO-QCD, is

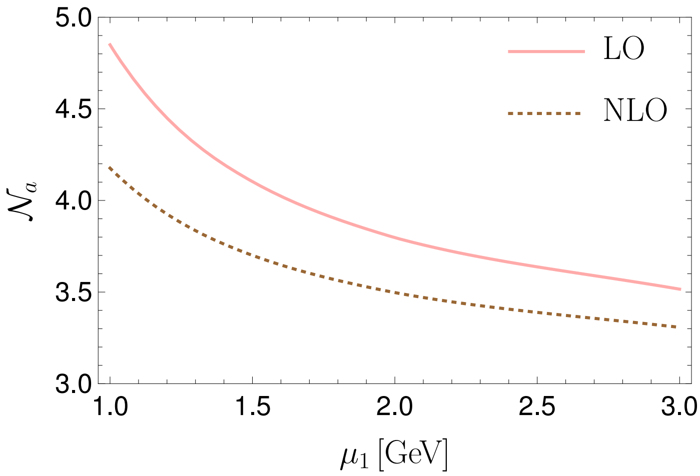

Figure 2. Scale dependence of the Wilson coefficient combination $\mathcal{N}_a = 3\,C_1^2 + 3\,C_2^2 + 2\,C_1 C_2$.

μ_1 [GeV]	1	1.27	1.36	1.44	1.48	3
$\mathcal{N}_a$(LO)	4.85	4.35	4.23	4.15	4.12	3.52
$\mathcal{N}_a$(NLO)	4.18	3.86	3.79	3.74	3.71	3.31

Table 4. Comparison of $\mathcal{N}_a$ at LO- and NLO-QCD, for different values of the renormalisation scale μ_1.

shown in table 4 and in figure 2, indicating a visible shift from LO to NLO and a moderate reduction of the scale uncertainty in the NLO result. Else there are no cancellations in $\mathcal{N}_a$ that might lead to numerical instability. The phase-space function $f(a,b,c)$ in eq. (2.12) describes the effect of the final state masses. In the case of one massive particle, it reduces to the well-known expression

$$f(z,0,0) = 1 - 8z + 8z^3 - z^4 - 12z^2 \ln z\,, \qquad f\left(z_s,0,0\right) \approx 1 - 0.03\,, \tag{2.14}$$

which shows that the contribution due to the finite s-quark mass is small. The analytic expression of $f(a,b,c)$ for two different masses in the final state, can be found e.g. in the appendix of ref. [72].

By including also NLO-QCD corrections, Γ_3 can be schematically presented as

$$\Gamma_3 = \Gamma_0 \left[3\,C_1^2\,\mathcal{C}_{3,11} + 2\,C_1 C_2\,\mathcal{C}_{3,12} + 3\,C_2^2\,\mathcal{C}_{3,22} + \mathcal{C}_{3,\mathrm{SL}}\right]\,, \tag{2.15}$$

where a summation over all the modes is implicitly assumed. At NLO, the expressions for $\mathcal{C}_{3,11}$, $\mathcal{C}_{3,22}$ and $\mathcal{C}_{3,\mathrm{SL}}$ are taken from ref. [19], where the computation was done for three different final state masses, hence we can easily use these results for all c-quark decay modes. For the coefficient $\mathcal{C}_{3,12}$ we use ref. [22] for the $c \to s\bar{d}u$, $c \to d\bar{s}u$ and $c \to d\bar{d}u$ decay channels, while the result of ref. [26] is used in the case of final state with two massive s-quarks, $c \to s\bar{s}u$.

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Mass scheme	$\Gamma_3^{\rm LO}$ [ps^{-1}]	$\Gamma_3^{\rm NLO}$ [ps^{-1}]
Pole ($m_c = 1.48\,\mathrm{GeV}$)	$1.45^{+0.17}_{-0.14}$	$1.52^{+0.20}_{-0.16}$
$\overline{\rm MS}$ (Eq. (1.8))	$0.69^{+0.06}_{-0.09}$	$1.32^{+0.06}_{-0.03}$
Kinetic (Eq. (1.9))	$0.97^{+0.10}_{-0.11}$	$1.47^{+0.27}_{-0.30}$
$1S$ (Eq. (1.13))	$1.25^{+0.14}_{-0.13}$	$1.50^{+0.31}_{-0.25}$

Table 5. Numerical values of $\Gamma_3^{\rm LO} = \Gamma_3^{(0)}$ and $\Gamma_3^{\rm NLO} = \Gamma_3^{(0)} + \alpha_s(m_c)/(4\pi)\,\Gamma_3^{(1)}$ using different schemes for the c-quark mass. The uncertainties are obtained by varying the renormalisation scale μ_1 between 1 GeV and 3 GeV.

Neglecting final state masses and approximating $|V_{ud}|^2 \approx 1$ the following expression was determined in 1991 [20], i.e.

$$c_3^{\rm NLO} - c_3^{\rm LO} = 8\,\frac{\alpha_s}{4\pi}\left[\underbrace{\left(\frac{25}{4} - \pi^2\right)}_{<0} \underbrace{+(C_1^2 + C_2^2)\left(\frac{31}{4} - \pi^2\right)}_{<0} \underbrace{-\frac{2}{3}C_1 C_2\left(\frac{7}{4} + \pi^2\right)}_{\geq 0}\right]. \tag{2.16}$$

The first term on the r.h.s. of eq. (2.16) stems from semileptonic decays and the next two terms from non-leptonic channels. For non-leptonic b-quark decays the NLO corrections are negative, while for charm quarks decays the third term will dominate over the second one and the correction becomes positive. Moreover, there is a sizable enhancement of the α_s-corrections in the non-leptonic b-quark decays due to finite charm quark mass effects [21–23, 26] — the corresponding increase in charm quark decays is much less pronounced as $m_c^2/m_b^2 \approx 0.1 \gg m_s^2/m_c^2 \approx 0.005$.

The numerical values for Γ_3 both in LO- and NLO-QCD, for different c-quark mass schemes are shown in table 5. The range of NLO-QCD values from $1.3\,\mathrm{ps}^{-1}$ to $1.5\,\mathrm{ps}^{-1}$ for the free charm-quark decay at NLO-QCD, is in good agreement with the experimental determinations in table 1. Moreover we observe small ($< 5\%$) corrections due to a non-vanishing strange quark mass. Interestingly the NLO-QCD result is affected by strong cancellations. We in fact observe a suppression of the non-leptonic contribution because of the opposite sign between the NLO corrections to the diagrams describing QCD corrections to the upper left diagram of figure 1 and the QCD corrections intrinsic to the $\Delta C = 1$ Wilson coefficients. A further cancellation is then present between the semileptonic and the non-leptonic modes. This behaviour can be nicely read of the result in the Pole scheme:

$$\Gamma_3 = \Gamma_3^{\rm LO}\left[1 + \left(\overbrace{\underbrace{1.84}_{\rm diag.} - \underbrace{0.74}_{\rm WC}}^{\rm NL} - \overbrace{0.67}^{\rm SL}\right)\frac{\alpha_s}{\pi} + \mathcal{O}\left(\frac{\alpha_s}{\pi}\right)^2\right]. \tag{2.17}$$

Expressing the pole mass in terms of a short distance mass like the $\overline{\rm MS}$ scheme, an additional large NLO correction arises from the conversion factor of m_c^5, which is the origin of the large shift between the LO and the NLO values in the $\overline{\rm MS}$-, the kinetic and the

1S-schemes, see table 5. We find e.g. in the $\overline{\rm MS}$ scheme

$$\Gamma_3 = \Gamma_3^{\rm LO}\left[1+\left(\overbrace{\underbrace{2.10}_{\rm diag.}-\underbrace{0.70}_{\rm WC}}^{\rm NL}-\overbrace{0.71}^{\rm SL}+\overbrace{6.66}^{\rm conv.fac.}\right)\frac{\alpha_s}{\pi}+\mathcal{O}\left(\frac{\alpha_s}{\pi}\right)^2\right]. \tag{2.18}$$

The corrections due to the mass conversion also make the overall semileptonic NLO term in the $\overline{\rm MS}$ scheme positive.

To get a first indication of the behaviour of the QCD series for the decay rate at higher orders, we briefly discuss here the NNLO [35] and NNNLO [37] corrections for the semileptonic b-quark decay and the preliminary NNLO-QCD corrections for the non-leptonic b-quark decay [39]. In the Pole mass scheme [37]

$$\frac{\Gamma_3(B\to X_c\ell\bar\nu_\ell)}{\Gamma_3^{\rm LO}(B\to X_c\ell\bar\nu_\ell)} = 1-1.72\frac{\alpha_s(\mu)}{\pi}-13.09\left(\frac{\alpha_s(\mu)}{\pi}\right)^2-163.3\left(\frac{\alpha_s(\mu)}{\pi}\right)^3 = 1-0.12-0.06-0.05, \tag{2.19}$$

the semileptonic decay rate gets large negative corrections, and in the $\overline{\rm MS}$-scheme

$$\frac{\Gamma_3(B\to X_c\ell\bar\nu_\ell)}{\Gamma_3^{\rm LO}(B\to X_c\ell\bar\nu_\ell)} = 1+3.07\frac{\alpha_s(\mu)}{\pi}+13.36\left(\frac{\alpha_s(\mu)}{\pi}\right)^2+62.7\left(\frac{\alpha_s(\mu)}{\pi}\right)^3 = 1+0.21+0.06+0.02, \tag{2.20}$$

one finds [37] sizable positive corrections — driven by the conversion of the quark mass from the Pole scheme to the $\overline{\rm MS}$-scheme and indicating again the importance of higher order perturbative corrections. For the semileptonic charm quark decay one finds even larger corrections,[4] e.g. in the Pole mass scheme

$$\frac{\Gamma_3(D\to X\ell^+\nu_\ell)}{\Gamma_3^{\rm LO}(D\to X\ell^+\nu_\ell)} = 1-2.41\frac{\alpha_s(\mu)}{\pi}-23.4\left(\frac{\alpha_s(\mu)}{\pi}\right)^2-321.5\left(\frac{\alpha_s(\mu)}{\pi}\right)^3 = 1-0.25-0.26-0.37, \tag{2.21}$$

which clearly spoils the perturbative approach and makes the use of different quark mass schemes mandatory.

Regarding the NNLO-QCD corrections to the non-leptonic decay rates, ref. [39] presents a partial result (not resumming the large logarithms, neglecting effects of the operator $Q_2^{q_1q_2}$ and assuming a vanishing charm quark mass) for the b-quark. In the pole mass scheme the authors obtain

$$\frac{\Gamma(b\to c\bar ud)}{3\Gamma(b\to ce\bar\nu)} = 1+1\,\frac{\alpha_s(\mu)}{\pi}+67.1\left(\frac{\alpha_s(\mu)}{\pi}\right)^2. \tag{2.22}$$

It is interesting to note that from the coefficient of the α_s^2 term, 67.1, a contribution of 54.7 stems from not summing large logarithms of the form $\ln(M_W/m_b)$ and $\ln^2(M_W/m_b)$. Using eq. (2.19) and the fact that, in the approximations of ref. [39] the ratio between non-leptonic and semileptonic rate is equal to 3 at LO-QCD, yields

$$\Gamma(b\to c\bar ud) = \Gamma^{\rm LO}(b\to c\bar ud)\left[1-0.7\,\frac{\alpha_s(\mu)}{\pi}+52.3\left(\frac{\alpha_s(\mu)}{\pi}\right)^2\right]. \tag{2.23}$$

[4]Results presented by Matteo Fael at CHARM 2020:
https://indico.nucleares.unam.mx/event/1488/session/12/contribution/56/material/slides/0.pdf.

For non-leptonic charm-quark decays the logarithms become even larger and we find that the coefficient of the α_s^2 term increases from 52.3 to 91.2, which clearly indicates the necessity of summing the large logarithms. Neglecting final state masses seems to be well justified in the charm system. In order to further estimate the effect of neglecting the operator $Q_2^{q_1 q_2}$, we set in our code $C_1 = 1$ and $C_2 = 0$ and we get in the Pole scheme

$$\frac{\Gamma_3^{\rm NL}}{\Gamma_3^{\rm NL,LO}} = 1 - 1.4\,\frac{\alpha_s(\mu)}{\pi}, \tag{2.24}$$

while the result with the full inclusion of the effective Hamiltonian yields a very different value of the QCD corrections

$$\frac{\Gamma_3^{\rm NL}}{\Gamma_3^{\rm NL,LO}} = 1 + 1.6\,\frac{\alpha_s(\mu)}{\pi}. \tag{2.25}$$

All in all we conclude that, higher order corrections seem to be crucial for a reliable determination of Γ_3. Despite being conceptually very interesting, the result of ref. [39] is not useful for phenomenological applications and a full NNLO determination of the non-leptonic decay rate using the effective Hamiltonian would be highly desirable.

2.3 Dimension-five contribution

The first corrections to the free charm-quark decay arise at order $1/m_c^2$ and describe the effect of the kinetic and the chromomagnetic operators. Their matrix elements are parametrised by the two non-perturbative inputs μ_π^2 and μ_G^2, i.e.

$$2m_D\,\mu_\pi^2(D) = -\langle D(p)|\bar{c}_v(iD_\mu)(iD^\mu)c_v|D(p)\rangle\,, \tag{2.26}$$
$$2m_D\,\mu_G^2(D) = \langle D(p)|\bar{c}_v(iD_\mu)(iD_\nu)(-i\sigma^{\mu\nu})c_v|D(p)\rangle\,, \tag{2.27}$$

with $\sigma^{\mu\nu} = (i/2)[\gamma^\mu, \gamma^\nu]$. Both the operators receive a contribution from the expansion of the dimension-three matrix element $\langle D(p)|\bar{c}_v c_v|D(p)\rangle$ [70]. However, the chromomagnetic operator receives further contributions due to the expansion of the short distance coefficient c_3 and of the quark-propagator in the external gluon field [41, 42, 49] — see the second diagram on the top line of figure 1. Hence, at order $1/m_c^2$, we can schematically write

$$\Gamma_5 \frac{\langle \mathcal{O}_5\rangle}{m_c^2} = \Gamma_0 \left[c_{\mu_\pi} \frac{\mu_\pi^2}{m_c^2} + c_G\,\frac{\mu_G^2}{m_c^2}\right]. \tag{2.28}$$

The coefficient of the kinetic operator is related to the dimension-three contribution,[5] $c_{\mu_\pi} = -c_3^{(0)}/2$, and the chromomagnetic coefficient c_G can be decomposed as

$$c_G = 3\,C_1^2\,\mathcal{C}_{G,11} + 2\,C_1 C_2\,\mathcal{C}_{G,12} + 3\,C_2^2\,\mathcal{C}_{G,22} + \mathcal{C}_{G,SL}\,, \tag{2.29}$$

where again a summation over all the modes is assumed. The individual contributions $\mathcal{C}_{G,nm}$ for non-leptonic modes can be found e.g. in the appendix of ref. [49]. In the latter

[5]Since the dimension-5 contribution for non-leptonic modes is known only at LO in QCD, we use the dimension-three coefficient c_3 just at LO-QCD for c_{μ_π}.

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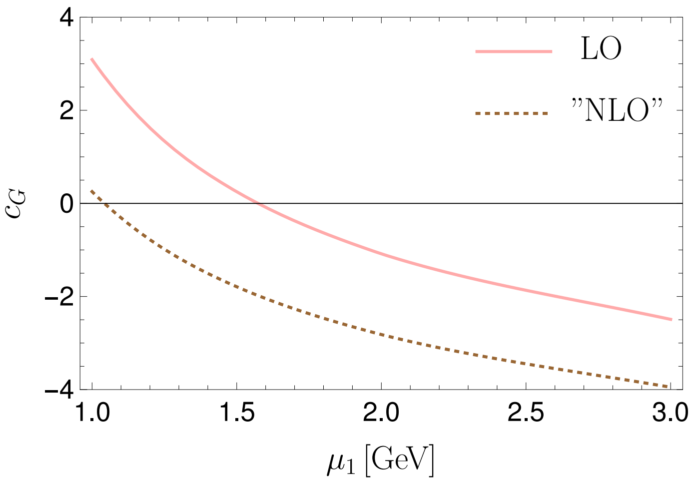

Figure 3. Scale dependence of the coefficient of the chromomagnetic operator.

reference, the coefficients of the chromomagnetic operator were determined for the non-leptonic B-meson decays, however, since there are no IR-divergences at this order, it is straightforward to obtain the corresponding results for the charm-sector, namely by replacing $m_b \to m_c$, $m_c \to m_s$, etc. For the semileptonic decay $c \to s\mu^+\nu_\mu$, the expression for two different mass parameters $z_s \neq 0 \neq z_\mu$ can be found in the appendix of ref. [72].[6] By neglecting the strange and muon masses and by considering only the dominant CKM modes, the result for c_G becomes very compact, i.e.

$$c_G \approx -|V_{ud}|^2 \left[\frac{9}{2}\left(C_1^2 + C_2^2\right) + 19\, C_1 C_2\right] - 3. \tag{2.30}$$

Because of the large coefficient in front of C_1C_2 and of its negative value, eq. (2.30) can be affected by cancellations. In figure 3 we plot c_G in eq. (2.29), as a function of the renormalisation scale μ_1 while in table 6 we list the numerical result for some reference values of μ_1. For c_G a change of sign occurs in the region between 1 and 2 GeV — leading to a large uncertainty due to scale variation. Note, that the "NLO" result shown in figure 3 and table 6 only includes QCD corrections due to the $\Delta C = 1$ Wilson coefficients. A complete calculation of the NLO-QCD corrections to c_G is still missing (these corrections are only known for the semileptonic case) and would be very desirable in order to reduce the huge scale dependence. The numerical values of the non-perturbative parameters μ_π^2 and μ_G^2 will be discussed in sections 3.2 and 3.1.

2.4 Dimension-six two-quark operator contribution

By determining higher order $1/m_c$ corrections in the expansion, respectively, of the quark-propagator, of the matrix elements of mass dimension-three and mass dimension-five, and of the corresponding short-distance coefficients, see e.g. refs. [41, 42, 49, 73] for details, one

[6] Since $m_s \approx m_\mu \approx 100\,\text{MeV} \ll m_c$, in principle one can safely set $z_s = z_\mu$ and use the non-leptonic expressions for the semileptonic modes, e.g. $c \to s\bar{s}u$ for $c \to s\mu\bar{\nu}_\mu$ by setting $N_c = 1$, $C_1 = 1$, $C_2 = 0$.

μ_1 [GeV]	1	1.27	1.36	1.44	1.48	3
$c_G^{\rm NL}$(LO)	6.20	4.34	3.91	3.58	3.43	0.62
$c_G^{\rm SL}$(LO)	-3.11	-3.11	-3.11	-3.11	-3.11	-3.11
c_G(LO)	3.09	1.23	0.80	0.47	0.32	-2.49
c_G("NLO$''$)	0.25	-1.06	-1.37	-1.62	-1.74	-3.95

Table 6. Comparison of the coefficients $c_G^{\rm SL}$, $c_G^{\rm NL}$, and $c_G = c_G^{\rm SL} + c_G^{\rm NL}$ for different values of the renormalisation scale μ_1 at LO and "NLO", setting for reference $m_c = 1.5\,\text{GeV}$.

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obtains the dimension-six contribution to $\Gamma(D)$, which can be compactly written as

$$\Gamma_6 \frac{\langle \mathcal{O}_6 \rangle}{m_c^3} = \Gamma_0 \, c_{\rho_D} \frac{\rho_D^3}{m_c^3} \,, \tag{2.31}$$

with the matrix element of the Darwin operator given by[7]

$$2 m_D \, \rho_D^3(D) = \langle D(p) | \bar{c}_v (i D_\mu)(i v \cdot D)(i D^\mu) c_v | D(p) \rangle \,. \tag{2.32}$$

The coefficient c_{ρ_D} can be decomposed into

$$c_{\rho_D} = 3\, C_1^2 \, \mathcal{C}_{\rho_D,11} + 2\, C_1 C_2 \, \mathcal{C}_{\rho_D,12} + 3\, C_2^2 \, \mathcal{C}_{\rho_D,22} + \mathcal{C}_{\rho_D,SL} \,, \tag{2.33}$$

including both non-leptonic and semileptonic contributions. For B-mesons decays, the non-leptonic coefficients were computed recently in refs. [49–51]. In order to determine the corresponding expressions for the charm system, some subtleties have to be considered. In b-quark decays, one assumes $m_b \sim m_c \gg \Lambda_{\rm QCD}$, and the coefficient of the Darwin operator for the semileptonic $b \to c \ell \bar{\nu}_\ell$ decays is a finite function of $\rho = m_c^2/m_b^2$, which however diverges in the limit $\rho \to 0$, i.e. in correspondence of the $b \to u \ell \bar{\nu}_\ell$ transitions. This is due to the fact that, the radiation of a soft gluon off a massless quark-propagator leads to IR singularities at dimension-six. In non-leptonic b-quark decays, one has to further deal with the emission of a soft gluon from the internal light u-, d-, and s-quark lines. The corresponding IR divergences are of the form $\log(m_q/m_b)$, for $q = u, d, s$, and are removed by taking into account the mixing between the four-quark operators with external q quarks and the Darwin operator under renormalisation, for details see e.g. refs. [49, 50, 74, 75]. Because of $m_c \gg m_s \sim \Lambda_{\rm QCD}$, it follows that one cannot trivially generalise the results from the b- to the c-sector, i.e. by only replacing $m_b \to m_c$, $m_c \to m_s$, etc., since there are further contributions due to the mixing of four-quark operators with external s-quarks which must be additionally included. Specifically, this leads to a modification of the coefficients proportional to C_1^2 and $C_1 C_2$. Using the same procedure as discussed in ref. [49], we have recomputed the coefficients of the Darwin operator required for the study of D-meson decays. The analytic expressions for $\mathcal{C}_{\rho_D,nm}$, including the full s-quark mass dependence, however finite in the limit $m_s \to 0$, are presented in appendix B for

[7]Note that with the given definition for the dimension-six two-quark operators, in terms of full covariant derivatives, the contribution of the spin-orbit operator to the decay width vanishes.

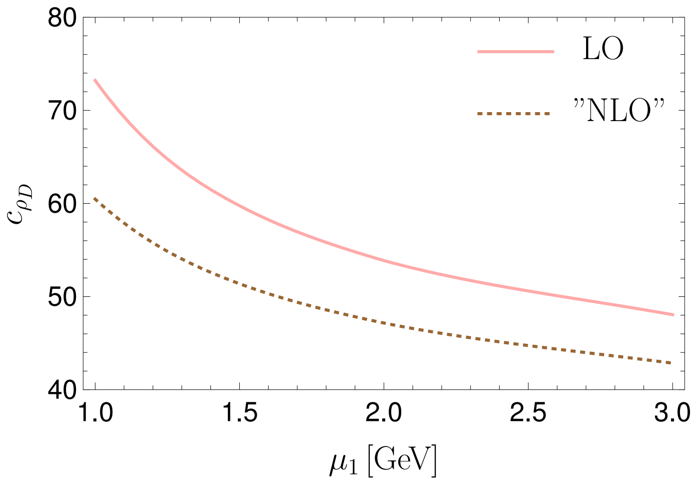

Figure 4. Scale dependence of the coefficient of the Darwin operator.

μ_1 [GeV]	1	1.27	1.36	1.44	1.48	3
$c_{\rho_D}^{\rm NL}$(LO)	60.6	51.7	49.6	48.1	47.5	35.4
$c_{\rho_D}^{\rm SL}$(LO)	12.6	12.6	12.6	12.6	12.6	12.6
c_{ρ_D}(LO)	73.2	64.3	62.3	60.8	60.1	48.1
c_{ρ_D}(“NLO$''$)	60.5	54.5	53.1	52.1	51.6	42.8

Table 7. Numerical values of $c_{\rho_D}^{\rm SL}$, $c_{\rho_D}^{\rm NL}$, and $c_{\rho_D} = c_{\rho_D}^{\rm SL} + c_{\rho_D}^{\rm NL}$ for different values of the renormalisation scale μ_1 at LO and “NLO” with $\mu_0 = m_c = 1.5\,$GeV.

all non-leptonic modes. To obtain the corresponding expression for $\mathcal{C}_{\rho_D,SL}$, it is sufficient to set in the results for the non-leptonic decays $N_c = 1$, $C_1 = 1$, $C_2 = 0$ and $z_s = z_\mu$ for the $c \to s\mu^+\nu_\mu$ mode. In particular, we confirm the results in ref. [53].

Again, by neglecting the strange and muon masses and by considering only the dominant CKM modes, one finds

$$c_{\rho_D} \approx |V_{ud}|^2 \left(18\, C_1^2 - \frac{68}{3}\, C_1 C_2 + 18\, C_2^2\right) + 12\,. \tag{2.34}$$

It is interesting to note that in this combination all terms have the same sign and no cancellations arise. In figure 4 we show the dependence of the function c_{ρ_D} in eq. (2.33) on the renormalisation scale μ_1 and in table 7 we quote the numerical result for some reference values of μ_1. The determination of the matrix element of the Darwin operator will be discussed in section 3.3.

2.5 Dimension-six four-quark operator contribution

The perturbative coefficients in eq. (1.1) considered so far are independent of the spectator quark in the D meson, in fact its effect appears only in the corresponding matrix elements of the dimension-five and dimension-six operators. Starting at order $1/m_c^3$, there are also

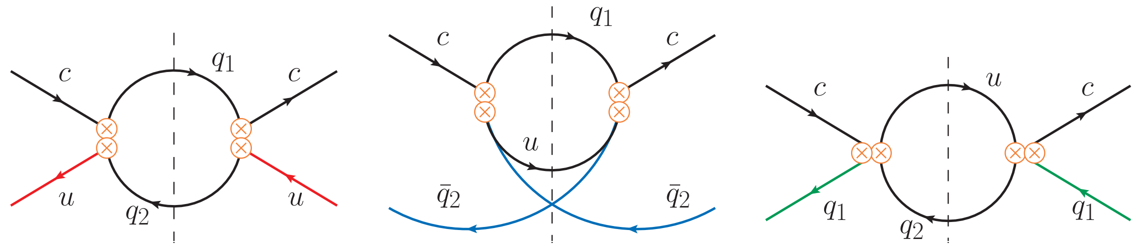

Figure 5. Spectator quark effects in the HQE expansion: WE (left), PI (middle) and WA (right).

one-loop contributions, cf. $\tilde{\Gamma}_6$ in eq. (1.1), in which the spectator quark is directly involved. These correspond respectively to the weak exchange (WE), Pauli interference (PI) and weak annihilation (WA) diagrams, depicted in figure 5.[8] Note that compared to the terms discussed above, these contributions imply a phase space enhancement factor of $16\pi^2$. The corresponding $\Delta C = 0$ four quark operators of dimension-six are:[9]

$$O_1^q = (\bar{c}\,\gamma_\mu(1-\gamma_5)q)\,(\bar{q}\,\gamma^\mu(1-\gamma_5)c), \tag{2.35}$$
$$O_2^q = (\bar{c}(1-\gamma_5)q)\,(\bar{q}(1+\gamma_5)c), \tag{2.36}$$
$$T_1^q = (\bar{c}\,\gamma_\mu(1-\gamma_5)T^A q)\,(\bar{q}\,\gamma^\mu(1-\gamma_5)T^A c), \tag{2.37}$$
$$T_2^q = (\bar{c}(1-\gamma_5)T^A q)\,(\bar{q}(1+\gamma_5)T^A c), \tag{2.38}$$

where T^A is a colour matrix and a summation over colour indices is implied. The parameterisation of the matrix elements of the operators in eqs. (2.35)–(2.38) in QCD is given in appendix C. However, by evaluating the matrix elements in the framework of the HQET, one obtains the following set of operators, i.e.[10]

$$\tilde{O}_1^q = (\bar{h}_v\,\gamma_\mu(1-\gamma_5)q)\,(\bar{q}\,\gamma^\mu(1-\gamma_5)h_v), \tag{2.39}$$
$$\tilde{O}_2^q = (\bar{h}_v(1-\gamma_5)q)\,(\bar{q}(1+\gamma_5)h_v), \tag{2.40}$$
$$\tilde{T}_1^q = (\bar{h}_v\,\gamma_\mu(1-\gamma_5)T^A q)\,(\bar{q}\,\gamma^\mu(1-\gamma_5)T^A h_v), \tag{2.41}$$
$$\tilde{T}_2^q = (\bar{h}_v(1-\gamma_5)T^A q)\,(\bar{q}(1+\gamma_5)T^A h_v), \tag{2.42}$$

here h_v denotes the HQET field defined by eqs. (2.10), (2.11). The matrix elements of these operators are parameterised as

$$\langle D_q|\tilde{O}_i^q\,|D_q\rangle = F^2(\mu)\,m_{D_q}\,\tilde{B}_i^q, \tag{2.43}$$
$$\langle D_q|\tilde{O}_i^{q'}\,|D_q\rangle = F^2(\mu)\,m_{D_q}\,\tilde{\delta}_i^{q'q}, \qquad q \neq q'\,, \tag{2.44}$$

where $q, q' = u, d, s$, $\tilde{B}_i^q$ denote the Bag parameters in HQET, with $\tilde{B}_{1,2}^q$ corresponding to the colour-singlet operators, and $\tilde{B}_{3,4}^q \equiv \tilde{\epsilon}_{1,2}^q$ to the colour-octet ones, and $F(\mu)$ is the HQET decay constant, defined as

$$\langle 0|\bar{q}\gamma^\mu\gamma_5 h_v|D_q(v)\rangle_{\rm HQET} = i\,F(\mu)\,\sqrt{m_{D_q}}\,v^\mu. \tag{2.45}$$

[8]In the case of semileptonic decays, only the WA topology can contribute.
[9]Sometimes, we will use the short-hand notation O_i^q, $i = 1, 2, 3, 4$ assuming $O_3^q \equiv T_1^q$, $O_4^q \equiv T_2^q$.
[10]Note that all quantities defined in HQET are labelled by a tilde, contrary to those in QCD.

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Similarly, the QCD decay constant f_{D_q} is given by[11]

$$\langle 0|\bar{q}\gamma^\mu\gamma_5 c|D_q(p)\rangle_{\rm QCD} = i f_{D_q}\, p^\mu, \tag{2.46}$$

with $p = m_{D_q} v$. The relation between f_{D_q} and $F(\mu)$ up to α_s and $1/m_c$ corrections can be found e.g. in refs. [76, 77]. At the scale $\mu = m_c$, it reads

$$f_{D_q} = \frac{F(m_c)}{\sqrt{m_{D_q}}}\left(1 - \frac{2}{3}\frac{\alpha_s(m_c)}{\pi} + \frac{G_1(m_c)}{m_c} + 6\,\frac{G_2(m_c)}{m_c} - \frac{1}{2}\frac{\bar{\Lambda} - m_q}{m_c}\right), \tag{2.47}$$

where $\bar{\Lambda} = m_{D_q} - m_c$, and the parameters G_1 and G_2 characterise matrix elements of non-local operators. Note that in our analysis we express the parameter $F(m_c)$ in eqs. (2.43), (2.44), in terms of f_{D_q} using eq. (2.47). This brings additional α_s corrections — which become part of NLO dimension-six contribution — as well as $1/m_c$ ones. The latter, as it will be explained in detail in section 2.6, can be partially absorbed in the contribution of some of the dimension-seven operators in HQET.

In vacuum insertion approximation (VIA), the Bag parameters of the colour-singlet operators are equal to one, $\tilde{B}^q_{1,2} = 1$, and the Bag parameters of the colour-octet operators vanish, $\tilde{\epsilon}^q_{1,2} = 0$. Note that throughout this work we assume isospin symmetry, i.e.

$$\tilde{B}^u_i = \tilde{B}^d_i. \tag{2.48}$$

The quantities $\tilde{\delta}^{q'q}_i$ in eq. (2.44) describe the so-called eye-contractions, see figure 6, and characterize "subleading" (compared to the large Bag parameters) effects in the non-perturbative matrix elements — in VIA all eye-contractions vanish i.e. $\tilde{\delta}^{q'q}_i = 0$. However, beyond VIA, the matrix elements of the four-quark operators with external q' quark differ from zero even when the spectator quark q in the D_q meson does not coincide with the quark q', as reflected by $\tilde{\delta}^{q'q}_i$ in eq. (2.44). Note that in our notation the eye-contractions with $q = q'$, are in fact included in the Bag parameters $\tilde{B}^q_i$. And again, due to isospin symmetry we will use:

$$\tilde{\delta}^{uq'}_i = \tilde{\delta}^{dq'}_i, \qquad \tilde{\delta}^{q'u}_i = \tilde{\delta}^{q'd}_i, \qquad q' = u, d, s\,.$$

The Bag parameters $\tilde{B}^q_i$ and $\tilde{\delta}^{qq'}_i$ have been determined using HQET sum rules, specifically the Bag parameters $\tilde{B}^q_i$ for the $D^{+,0}$ mesons were calculated in ref. [66], while corrections due to the strange quark mass as well as the contribution of the eye-contractions, see figure 6, have been computed for the first time in ref. [68]. The numerical values of the Bag parameters will be briefly discussed in section 3.4 and they are summarised in appendix A.

By considering only the dominant CKM modes and by neglecting the effect of the eye-contractions, at LO-QCD and at dimension-six, the contribution of four-quark operators

[11]The subscript 'QCD' or 'HQET' on the states is usually omitted, however for clarity it is specified in the definition of the decay constant.

to the D-mesons decay rate reads

$$
\begin{aligned}
16\pi^2\,\tilde{\Gamma}_6^{D^0}\,\frac{\langle\tilde{O}_6\rangle^{D^0}}{m_c^3} &= \Gamma_0|V_{ud}^*|^2\,16\pi^2\frac{M_{D^0}f_{D^0}^2}{m_c^3}\,(1-z_s)^2 \\
&\quad\left\{\left(\frac{1}{3}C_1^2+2\,C_1C_2+3\,C_2^2\right)\left[(\tilde{B}_2^u-\tilde{B}_1^u)+z_s\left(2\tilde{B}_2^u-\frac{\tilde{B}_1^u}{2}\right)\right]\right. \\
&\qquad\left.+2\,C_1^2\left[(\tilde{\epsilon}_2^u-\tilde{\epsilon}_1^u)+z_s\left(2\,\tilde{\epsilon}_2^u-\frac{\tilde{\epsilon}_1^u}{2}\right)\right]\right\}, \qquad (2.49)
\end{aligned}
$$

$$
\begin{aligned}
16\pi^2\,\tilde{\Gamma}_6^{D^+}\,\frac{\langle\tilde{O}_6\rangle^{D^+}}{m_c^3} &= \Gamma_0|V_{ud}^*|^2\,16\pi^2\frac{M_{D^+}f_{D^+}^2}{m_c^3}(1-z_s)^2 \\
&\quad\left\{\left(C_1^2+6\,C_1C_2+C_2^2\right)\tilde{B}_1^d+6\left(C_1^2+C_2^2\right)\tilde{\epsilon}_1^d\right\}, \qquad (2.50)
\end{aligned}
$$

$$
\begin{aligned}
16\pi^2\,\tilde{\Gamma}_6^{D_s^+}\,\frac{\langle\tilde{O}_6\rangle^{D_s^+}}{m_c^3} &= \Gamma_0|V_{ud}^*|^2\,16\pi^2\frac{M_{D_s^+}f_{D_s^+}^2}{m_c^3} \\
&\quad\left\{\left(3\,C_1^2+2\,C_1C_2+\frac{1}{3}C_2^2+\frac{2}{|V_{ud}^*|^2}\right)\left(\tilde{B}_2^s-\tilde{B}_1^s\right)+2\,C_2^2\,(\tilde{\epsilon}_2^s-\tilde{\epsilon}_1^s)\right\}, \qquad (2.51)
\end{aligned}
$$

respectively, for the WE, PI and WA topologies. Note that in the latter we have neglected the muon mass in the semileptonic decay $c\to s\mu^+\nu_\mu$. In eqs. (2.49)–(2.51) some interesting numerical effects are arising. First, in the charm system, one expects that the contribution due to the spectator quark is of similar size compared to the leading term Γ_3 in the HQE, unless some additional cancellations are present. Using the pole mass $m_c^{\rm Pole}=1.48\,{\rm GeV}$ and Lattice QCD values for the decay constants [78] we roughly obtain that

$$16\pi^2\frac{M_{D^0}f_{D^0}^2}{m_c^3} = 4.1 \approx \mathcal{O}(c_3)\,, \qquad (2.52)$$

$$16\pi^2\frac{M_{D_s^+}f_{D_s^+}^2}{m_c^3} = 6.0 \approx \mathcal{O}(c_3)\,. \qquad (2.53)$$

This result has led the authors of ref. [79] to propose a different ordering for the HQE series in the charm sector. However, to investigate further the size of four-quark contributions, we consider the combinations of Wilson coefficients that appear in eqs. (2.49)–(2.51), i.e.

$$C_{\rm WE}^S = \frac{1}{3}C_1^2+2\,C_1C_2+3\,C_2^2, \qquad C_{\rm WE}^O = 2\,C_1^2, \qquad (2.54)$$

$$C_{\rm PI}^S = C_1^2+6\,C_1C_2+C_2^2, \qquad C_{\rm PI}^O = 6\,(C_1^2+C_2^2)\,, \qquad (2.55)$$

$$C_{\rm WA}^S = 3\,C_1^2+2\,C_1C_2+\frac{1}{3}C_2^2, \qquad C_{\rm WA}^O = 2\,C_2^2\,, \qquad (2.56)$$

where the superscripts S and O refer to coefficient in front of the colour-singlet and colour-octet Bag parameters, respectively. A comparison of these combinations for different values of the renormalisation scale μ_1 is shown in table 8.

As one can see, the combination of Wilson coefficients multiplying the colour-singlet Bag parameters of WE is strongly suppressed. Note that, depending on whether we disregard α_s^2 corrections in these combinations of $\Delta C=1$ Wilson coefficients — as we do —

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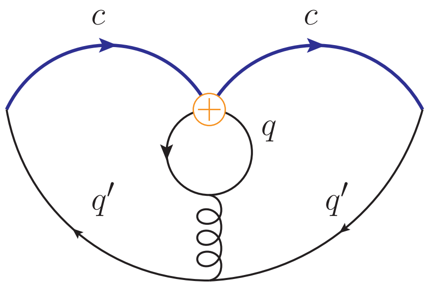

Figure 6. Diagram describing the eye-contractions.

μ_1 [GeV]	1	1.27	1.36	1.44	1.48	3
$C^S_{\rm WE}$(LO)	0.09	0.03	0.02	0.02	0.01	0.01
$C^S_{\rm WE}$(NLO)	-0.03	-0.03	-0.03	-0.02	-0.02	0.04
$C^O_{\rm WE}$(LO)	3.57	3.24	3.16	3.11	3.08	2.63
$C^O_{\rm WE}$(NLO)	3.11	2.89	2.83	2.79	2.77	2.44
$C^S_{\rm PI}$(LO)	-2.80	-2.12	-1.96	-1.85	-1.79	-0.79
$C^S_{\rm PI}$(NLO)	-1.74	-1.28	-1.16	-1.08	-1.04	-0.27
$C^O_{\rm PI}$(LO)	13.0	11.4	11.0	10.7	10.6	8.50
$C^O_{\rm PI}$(NLO)	10.6	9.55	9.31	9.13	9.05	7.60
$C^S_{\rm WA}$(LO)	3.82	3.61	3.56	3.53	3.51	3.24
$C^S_{\rm WA}$(NLO)	3.57	3.42	3.38	3.36	3.35	3.16
$C^O_{\rm WA}$(LO)	0.77	0.55	0.51	0.47	0.46	0.21
$C^O_{\rm WA}$(NLO)	0.41	0.30	0.27	0.25	0.24	0.10

Table 8. Comparison of the combinations $C^{S,O}_{\rm WE,PI,WA}$, respectively at LO- and NLO-QCD, for different values of the renormalisation scale μ_1.

or not, we can get even different signs for $C^{\rm S}_{\rm WE}$ at NLO. Moreover, in eq. (2.49) the Bag parameters of the colour singlet operators exactly cancel in VIA at leading order in $1/m_c$. The coefficient of the colour-octet operator is, on the other hand, not suppressed for weak exchange, indicating that both singlet and octet operators might be equally important in this case. For Pauli interference, the combinations of Wilson coefficients multiplying the colour-singlet operators are significantly enhanced compared to those in WE, the same holds for the colour-octet operators. Note that $C^O_{\rm PI}$ and $C^S_{\rm PI}$ get large modifications (and even a flip of sign) compared to the case $C_1 = 1$ and $C_2 = 0$ revealing the importance of gluon radiative corrections. Moreover $C^O_{\rm PI}$ is enhanced compared to $C^S_{\rm PI}$, indicating that both singlet and octet operators might be equally important for Pauli interference. For weak annihilation, the corresponding combination in front of the colour-singlet operators is large. On the other hand, the Bag parameters of the colour singlet operators exactly cancel in VIA at leading order in $1/m_c$.

The above arguments show that, by neglecting the effect of the colour-octet operators in VIA, one might be led to misleading conclusions, and therefore an accurate determination of the deviation of the Bag parameters from their VIA values, using non-perturbative methods like HQET sum rules or lattice simulations, is necessary.

Finally, by including all CKM modes as well as NLO-QCD corrections, the contribution of four-quark operators to the total decay width at order $1/m_c^3$ schematically reads

$$16\pi^2\,\tilde{\Gamma}_6^{D_q}\frac{\langle\tilde{\mathcal{O}}_6\rangle^{D_q}}{m_c^3} = \frac{\Gamma_0}{|V_{cs}|^2}\sum_{i=1}^{4}\Bigg\{\sum_{q_1,q_2=d,s}|\lambda_{q_1q_2}|^2\Bigg[A^{\rm WE}_{i,q_1q_2}\frac{\langle D_q|\tilde{O}_i^{u}|D_q\rangle}{m_c^3}+A^{\rm PI}_{i,q_1q_2}\frac{\langle D_q|\tilde{O}_i^{q_2}|D_q\rangle}{m_c^3}$$
$$+A^{\rm WA}_{i,q_1q_2}\frac{\langle D_q|\tilde{O}_i^{q_1}|D_q\rangle}{m_c^3}\Bigg]+\sum_{q_1=d,s}|V_{cq_1}|^2\sum_{\ell=e,\mu}\Bigg[A^{\rm WA}_{i,q_1\ell}\frac{\langle D_q|\tilde{O}_i^{q_1}|D_q\rangle}{m_c^3}\Bigg]\Bigg\}, \tag{2.57}$$

where the matrix elements of the four-quark operators are given in eqs. (2.43), (2.44), and the short-distance coefficients for the WE, PI and WA topologies, cf. figure 5 are denoted by $A^{\rm WE}_{i,q_1q_2}$, $A^{\rm PI}_{i,q_1q_2}$ and $A^{\rm WA}_{i,q_1q_2}$, $A^{\rm WA}_{i,q_1\ell}$, respectively. NLO corrections to $A^{\rm WE}_{i,q_1q_2}$ and $A^{\rm PI}_{i,q_1q_2}$ have been computed for HQET operators in ref. [55]. The corresponding results for $A^{\rm WA}_{i,q_1q_2}$ can be obtained by Fierz transforming the $\Delta C = 1$ operators given in eqs. (2.2), (2.3). Since the Fierz symmetry is respected also at one-loop level, the functions $A^{\rm WA}_{i,q_1q_2}$ are derived from $A^{\rm WE}_{i,q_1q_2}$ by replacing $C_1 \leftrightarrow C_2$. For the semileptonic modes, the coefficients $A^{\rm WA}_{i,q_1\ell}$ have been determined in ref. [56]. Note that in our analysis, we treat the contribution of the $\tilde{\delta}_i^{q'q}$ parameters as a subleading "NLO" effect, therefore their coefficients are included only at LO-QCD. To demonstrate the importance of the NLO-QCD corrections to the spectator effects, we show in table 9 the dimension-six contributions to the D-meson decay widths (see eq. (2.57)) splitting the LO and NLO parts, both in VIA and using HQET SR results for the Bag parameters. NLO-QCD corrections turn out to have an essential numerical effect for the four-quark contributions. In the case of the D^0 and D_s^+ mesons these corrections lift the helicity suppression of weak exchange and weak annihilation being present in LO-QCD when using VIA. For the D_s^+ meson, in addition to the CKM dominant WA contribution, there is a correction due to CKM suppressed but nevertheless large PI topology. In the case of the D^+ meson the overall contribution from Pauli interference turns out to be huge, of the order of -2.5 ps^{-1}. In addition, the NLO correction to Pauli interference turn also out to be very large, 50%–100% of the LO term depending on the mass scheme. Already in the B system this NLO-QCD corrections were found to be of the order of 30% for the ratio $\tau(B^+)/\tau(B_d)$, see e.g. ref. [54] in the Pole scheme. Thus, neglecting these contributions for charm lifetime studies, as done in ref. [80], is clearly not justified and a knowledge of NNLO-QCD corrections to the four-quark contributions would be highly desirable.

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Mass scheme	D^0	D^+	D_s^+
		VIA	
Pole	$\underbrace{-0.014}_{\text{NLO}} = \underbrace{0.000}_{\text{LO}} \underbrace{-0.014}_{\Delta\text{NLO}}$	$\underbrace{-2.64}_{\text{NLO}} = \underbrace{-1.68}_{\text{LO}} \underbrace{-0.97}_{\Delta\text{NLO}}$	$\underbrace{-0.20}_{\text{NLO}} = \underbrace{-0.12}_{\text{LO}} \underbrace{-0.08}_{\Delta\text{NLO}}$
$\overline{\text{MS}}$	$\underbrace{-0.010}_{\text{NLO}} = \underbrace{0.000}_{\text{LO}} \underbrace{-0.010}_{\Delta\text{NLO}}$	$\underbrace{-2.49}_{\text{NLO}} = \underbrace{-1.23}_{\text{LO}} \underbrace{-1.25}_{\Delta\text{NLO}}$	$\underbrace{-0.18}_{\text{NLO}} = \underbrace{-0.08}_{\text{LO}} \underbrace{-0.10}_{\Delta\text{NLO}}$
Kinetic	$\underbrace{-0.012}_{\text{NLO}} = \underbrace{0.000}_{\text{LO}} \underbrace{-0.012}_{\Delta\text{NLO}}$	$\underbrace{-2.53}_{\text{NLO}} = \underbrace{-1.42}_{\text{LO}} \underbrace{-1.11}_{\Delta\text{NLO}}$	$\underbrace{-0.19}_{\text{NLO}} = \underbrace{-0.10}_{\text{LO}} \underbrace{-0.09}_{\Delta\text{NLO}}$
$1S$	$\underbrace{-0.013}_{\text{NLO}} = \underbrace{0.000}_{\text{LO}} \underbrace{-0.013}_{\Delta\text{NLO}}$	$\underbrace{-2.60}_{\text{NLO}} = \underbrace{-1.58}_{\text{LO}} \underbrace{-1.02}_{\Delta\text{NLO}}$	$\underbrace{-0.19}_{\text{NLO}} = \underbrace{-0.11}_{\text{LO}} \underbrace{-0.08}_{\Delta\text{NLO}}$
		HQET SR	
Pole	$\underbrace{0.007}_{\text{NLO}} = \underbrace{0.019}_{\text{LO}} \underbrace{-0.012}_{\Delta\text{NLO}}$	$\underbrace{-2.89}_{\text{NLO}} = \underbrace{-1.87}_{\text{LO}} \underbrace{-1.02}_{\Delta\text{NLO}}$	$\underbrace{-0.21}_{\text{NLO}} = \underbrace{-0.16}_{\text{LO}} \underbrace{-0.05}_{\Delta\text{NLO}}$
$\overline{\text{MS}}$	$\underbrace{0.020}_{\text{NLO}} = \underbrace{0.014}_{\text{LO}} \underbrace{+0.006}_{\Delta\text{NLO}}$	$\underbrace{-2.72}_{\text{NLO}} = \underbrace{-1.37}_{\text{LO}} \underbrace{-1.35}_{\Delta\text{NLO}}$	$\underbrace{-0.20}_{\text{NLO}} = \underbrace{-0.12}_{\text{LO}} \underbrace{-0.08}_{\Delta\text{NLO}}$
Kinetic	$\underbrace{0.014}_{\text{NLO}} = \underbrace{0.016}_{\text{LO}} \underbrace{-0.002}_{\Delta\text{NLO}}$	$\underbrace{-2.76}_{\text{NLO}} = \underbrace{-1.58}_{\text{LO}} \underbrace{-1.18}_{\Delta\text{NLO}}$	$\underbrace{-0.20}_{\text{NLO}} = \underbrace{-0.13}_{\text{LO}} \underbrace{-0.07}_{\Delta\text{NLO}}$
$1S$	$\underbrace{0.009}_{\text{NLO}} = \underbrace{0.018}_{\text{LO}} \underbrace{-0.008}_{\Delta\text{NLO}}$	$\underbrace{-2.84}_{\text{NLO}} = \underbrace{-1.76}_{\text{LO}} \underbrace{-1.08}_{\Delta\text{NLO}}$	$\underbrace{-0.21}_{\text{NLO}} = \underbrace{-0.15}_{\text{LO}} \underbrace{-0.06}_{\Delta\text{NLO}}$

Table 9. Dimension-six contributions to D-meson decay widths (see eq. (2.57)) (in ps^{-1}) and split up into LO-QCD and NLO-QCD corrections within different mass schemes and both in VIA and using the HQET SR for Bag parameters.

2.6 Dimension-seven four-quark operator contribution

The dimension-six four-quark operator contribution discussed in the previous section, is obtained by neglecting in the expression of the incoming momentum $p^\mu = p_c^\mu + p_q^\mu$ the effect due to the small momentum of the light spectator quark $p_q \sim \Lambda_{\text{QCD}}$. Including also corrections linear in the quantity p_q/m_c, leads to the contribution of order $1/m_c^4$ to $\Gamma(D)$, which can be described in terms of the following basis of dimension-seven operators,

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defined in full QCD, i.e.[12]

$$P_1^q = m_q\,(\bar{c}(1-\gamma_5)q)(\bar{q}(1-\gamma_5)c)\,, \tag{2.58}$$

$$P_2^q = \frac{1}{m_c}(\bar{c}\overleftarrow{D}_\nu\gamma_\mu(1-\gamma_5)D^\nu q)(\bar{q}\gamma^\mu(1-\gamma_5)c)\,, \tag{2.59}$$

$$P_3^q = \frac{1}{m_c}(\bar{c}\overleftarrow{D}_\nu(1-\gamma_5)D^\nu q)(\bar{q}(1+\gamma_5)c)\,, \tag{2.60}$$

together with the corresponding colour-octet operators S_1^q, S_2^q, S_3^q, containing the generators T^A, and again a summation over colour indices is implied. Due to the presence in eqs. (2.59), (2.60) of a covariant derivative acting on the charm quark field, which scales as m_c at this order, there is no immediate power counting for these operators, cf. the HQET operators in eqs. (2.62), (2.63). Moreover, note that this basis differs from the one used in ref. [58] for the computation of dimension-seven and dimension-eight contributions.

In order to evaluate the matrix element of the dimension-seven four-quark operators using the framework of the HQET, one has to further expand the charm quark momentum, according to $p^\mu = m_c v^\mu + k^\mu + p_q^\mu$, see eq. (2.9), as well as to include $1/m_c$ corrections to the effective heavy quark field and to the HQET Lagrangian, retaining only terms linear in k/m_c and p_q/m_c. The small residual momentum of the charm quark k^μ will result in a covariant derivative acting on h_v and the small momentum of the spectator quark p_q^μ will result in a covariant derivative acting on the light quark field q. In this case, one obtains the following basis, which includes the local operators

$$\tilde{P}_1^q = m_q\,(\bar{h}_v(1-\gamma_5)q)(\bar{q}(1-\gamma_5)h_v)\,, \tag{2.61}$$

$$\tilde{P}_2^q = (\bar{h}_v\gamma_\mu(1-\gamma_5)(iv\cdot D)q)(\bar{q}\gamma^\mu(1-\gamma_5)h_v)\,, \tag{2.62}$$

$$\tilde{P}_3^q = (\bar{h}_v(1-\gamma_5)(iv\cdot D)q)(\bar{q}(1+\gamma_5)h_v)\,, \tag{2.63}$$

and

$$\tilde{R}_1^q = (\bar{h}_v\gamma_\mu(1-\gamma_5)q)(\bar{q}\gamma^\mu(1-\gamma_5)(i\not{D})h_v)\,, \tag{2.64}$$

$$\tilde{R}_2^q = (\bar{h}_v(1-\gamma_5)q)(\bar{q}(1+\gamma_5)(i\not{D})h_v)\,, \tag{2.65}$$

supplemented by the corresponding colour-octet operators $\tilde{S}_{1,2,3}^q$ and $\tilde{U}_{1,2}^q$, and the non-local operators

$$\tilde{M}_{1,\pi}^q = i\int d^4y\, T\left[\tilde{O}_1^q(0), (\bar{h}_v(iD)^2h_v)(y)\right], \tag{2.66}$$

$$\tilde{M}_{2,\pi}^q = i\int d^4y\, T\left[\tilde{O}_2^q(0), (\bar{h}_v(iD)^2h_v)(y)\right], \tag{2.67}$$

$$\tilde{M}_{1,G}^q = i\int d^4y\, T\left[\tilde{O}_1^q(0), \frac{1}{2}g_s\left(\bar{h}_v\sigma_{\alpha\beta}G^{\alpha\beta}h_v\right)(y)\right], \tag{2.68}$$

$$\tilde{M}_{2,G}^q = i\int d^4y\, T\left[\tilde{O}_2^q(0), \frac{1}{2}g_s\left(\bar{h}_v\sigma_{\alpha\beta}G^{\alpha\beta}h_v\right)(y)\right], \tag{2.69}$$

[12] Notice that in e.g. ref. [56] it is used a redundant basis in which the operator denoted by P_2^q is related to P_1^q by hermitean conjugation, namely $P_2^q = m_q\,(\bar{c}(1+\gamma_5)q)(\bar{q}(1+\gamma_5)c) = (P_1^q)^\dagger$. Since these two operators lead to the same matrix element, we only include P_1^q in our basis. It then follows that the operators here denoted by P_2^q and P_3^q, correspond respectively to P_3^q and P_4^q in the notation of ref. [56]. Furthermore, we point out that the matrix element of the operator P_4^q in eqs. (C5) and (C6) of ref. [56] should have the opposite sign.

also supplemented by the corresponding colour-octet operators.[13] We see that, compared to the QCD basis, there are in addition the two local operators $\tilde{R}_1^q$ and $\tilde{R}_2^q$ (and also the corresponding colour-octet ones), which emerge from the expansion in eq. (2.11), and the four non-local operators $\tilde{M}_{1,\pi}^q$, $\tilde{M}_{2,\pi}^q$, $\tilde{M}_{1,G}^q$ and $\tilde{M}_{2,G}^q$ (and the corresponding colour-octet ones) which are obtained by taking the time-ordered product of the dimension-six operators with the $1/m_c$ correction to the HQET Lagrangian, see e.g. ref. [71] for details.

We parametrise the matrix elements of the operators in eqs. (2.61)–(2.69) using VIA and account for deviations from it by including the corresponding Bag parameters, as it is explicitly shown in appendix C. However, since for these matrix elements there is no non-perturbative evaluation available yet, in our analysis we have to rely only on VIA. It follows that, at LO-QCD the matrix element of the dimension-seven operators listed above, can be expressed in terms of the HQET non-perturbative parameters $F(\mu)$, $G_1(\mu)$, $G_2(\mu)$, and $\bar{\Lambda}$, so far determined only with large uncertainties. For this reason, we prefer to use as an input the QCD decay constant f_D, which is computed very precisely using Lattice QCD [78]. In doing so, we obtain that in VIA and at the matching scale $\mu = m_c$, the contribution of the local operators $\tilde{R}_{1,2}^q$ as well as that of the non-local ones $\tilde{M}_{1,\pi}^q$, $\tilde{M}_{2,\pi}^q$, $\tilde{M}_{1,G}^q$ and $\tilde{M}_{2,G}^q$ can be entirely absorbed in the QCD decay constant f_D, cf. eq. (2.47) (more precisely, in the matrix element of the dimension-six QCD operators in eqs. (2.35), (2.36), which are proportional to f_D), so that we are left only with the $1/m_c$ contribution due to the operators $\tilde{P}_{1,2,3}^q$, analogously to the QCD case.[14]

To make this point more clear, we consider as an example the contribution due to Pauli interference at LO-QCD and up to order $1/m_c^4$, in the case of $c \to s\bar{d}u$ transition, which constitutes the dominant correction to $\Gamma(D^+)$,

$$\operatorname{Im} \mathcal{T}^{\rm PI} = \Gamma_0 \, |V_{ud}^*|^2 \, \frac{32\pi^2}{m_c^3} (1 - z_s)^2 \Bigg[C_{\rm PI}^S \left(\tilde{O}_1^d + \frac{\tilde{R}_1^d}{m_c} + \frac{\tilde{M}_{1,\pi}^d}{m_c} + \frac{\tilde{M}_{1,G}^d}{m_c} + 2 \frac{1+z_s}{1-z_s} \frac{\tilde{P}_3^q}{m_c} \right) + (\text{colour-octet part}) \Bigg], \tag{2.70}$$

with $C_{\rm PI}^S$ defined in eq. (2.55). By evaluating the matrix element of $\operatorname{Im}\mathcal{T}^{\rm PI}$ in VIA, the contribution due to the colour-octet operators vanishes. Moreover, using the parametrisation for the matrix elements of the four-quark operators given in eq. (2.43) and in appendix C, we obtain in VIA and setting $\mu = m_c$, that

$$\begin{aligned} \langle D^+ | \tilde{O}_1^d + \frac{\tilde{R}_1^d}{m_c} + \frac{\tilde{M}_{1,\pi}^d}{m_c} + \frac{\tilde{M}_{1,G}^d}{m_c} | D^+ \rangle_{\rm HQET} &= F^2(m_c) \, m_{D^+} \left[1 - \frac{\bar{\Lambda}}{m_c} + \frac{2\,G_1(m_c)}{m_c} + \frac{12\,G_2(m_c)}{m_c} \right] \\ &= f_D^2 \, m_{D^+}^2 = \langle D^+ | O_1^d | D^+ \rangle_{\rm QCD}, \end{aligned} \tag{2.71}$$

where in the second line we have used the conversion between the QCD and HQET decay constants given in eq. (2.47), showing that the contribution of the local operators $\tilde{R}_i^q$ and

[13] Operators which vanish due to the equation of motion $(iv \cdot D) h_v = 0$ are not shown.

[14] In the matrix element of $\tilde{P}_{1,2,3}^q$ one can replace the HQET decay constant with the QCD one, up to higher order corrections.

non-local operators $\tilde{M}^q_{i,\pi}$ and $\tilde{M}^q_{i,G}$ is entirely absorbed in the QCD decay constant. Note that, by neglecting the effect due to the strange quark mass and using VIA we reproduce the approximate result of eq. (19) in ref. [79].

The same argument applies also to the remaining topologies i.e. WE and WA. However, it is worth mentioning that in VIA and neglecting the strange quark mass, the contribution of WE and WA exactly vanishes at LO-QCD, due to the helicity suppression. This suppression is lifted once the s-quark mass or perturbative gluon corrections are included, and in this case it becomes again manifest that the contributions of $\tilde{R}^q_i$, $\tilde{\mathcal{M}}^q_{i,\pi}$ and $\tilde{\mathcal{M}}^q_{i,G}$ in HQET can be completely absorbed in f_D by evaluating the matrix elements in VIA.[15] We note that a detailed analysis of the dimension-seven contributions within the HQET has been performed in ref. [77] for the case of $B-\bar{B}$-mixing. Specifically, it was found that in VIA, subleading power corrections due to non-local operators can be entirely absorbed in the definition of the QCD decay constant, and that the residual $1/m_b$ corrections, due to the running of the local dimension-seven operators from the scale m_b to $\mu \sim 1\,\text{GeV}$, is numerically small ($\sim 5\%$ for ref. [77]).[16]

Finally, by summing over all the CKM modes, at LO-QCD, the dimension-seven contribution can therefore be presented as (with $q = u, d, s$)

$$16\pi^2\,\tilde{\Gamma}_7^{D_q}\,\frac{\langle\tilde{\mathcal{O}}_7\rangle^{D_q}}{m_c^4} = \frac{\Gamma_0}{|V_{cs}|^2}\sum_{i=1}^{3}\Bigg\{\sum_{q_1,q_2=d,s}|\lambda_{q_1q_2}|^2\Bigg[G^{\rm WE}_{i,q_1q_2}\frac{\langle D_q|\tilde{P}^u_i|D_q\rangle}{m_c^4}+G^{\rm PI}_{i,q_1q_2}\frac{\langle D_q|\tilde{P}^{q_2}_i|D_q\rangle}{m_c^4}$$
$$+G^{\rm WA}_{i,q_1q_2}\frac{\langle D_q|\tilde{P}^{q_1}_i|D_q\rangle}{m_c^4}\Bigg]+\sum_{q_1=d,s}|V_{cq_1}|^2\sum_{\ell=e,\mu}\Bigg[G^{\rm WA}_{i,q_1\ell}\frac{\langle D_q|\tilde{P}^{q_1}_i|D_q\rangle}{m_c^4}\Bigg]\Bigg\}$$
$$+(\text{colour-octet part}), \tag{2.72}$$

where the matrix elements of the dimension-seven operators are presented in appendix C. We confirm the results for the short-distance coefficients $G^{\rm WE}_{i,q_1q_2}$, $G^{\rm PI}_{i,q_1q_2}$ and $G^{\rm WA}_{i,q_1q_2}$, $G^{\rm WA}_{i,q_1\ell}$ presented in ref. [56]. Note that, due to the current accuracy of the analysis, at dimension-seven we include only the contribution of the valence-quark, therefore e.g. $\langle D^0|P^s_i|D^0\rangle = 0$. Numerical values of the dimension-seven contributions to the decay rates and the ratios will be presented in section 4. In table 10 we show the central values of the dimension-seven contributions in ps^{-1} in the kinetic mass scheme and we find for the D^+ meson a correction that is almost as large as the leading dimension three term, see table 5.

3 Determination of the non-perturbative parameters

In the present section, we discuss the numerical determination for the matrix elements of the operators introduced in sections 2.2–2.6. We start with the operators of the lowest mass dimension.

[15] Note, that for the operator O^q_2 the contribution of R^q_2 is absorbed by the combination $(m_D\, f_D/m_c)^2 \approx (1 + 2\,\bar{\Lambda}/m_c)\, f_D^2$.

[16] By neglecting the effect of running down to a lower scale, from ref. [77] one can see that in VIA the QCD decay constant entirely absorbs all the $1/m_b$ contributions.

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	D^0	D^+	D_s^+
$16\pi^2\, \tilde{\Gamma}_7^{D_q} \dfrac{\langle \tilde{\mathcal{O}}_7 \rangle^{D_q}}{m_c^4}$ $[\mathrm{ps}^{-1}]$	4.6×10^{-7}	1.05	0.10

Table 10. Dimension-seven contributions to D-meson decay widths (see eq. (2.72)) in ps^{-1} within VIA in the kinetic mass scheme.

3.1 Parameters of the chromomagnetic operator

For the B system many of non-perturbative parameters have been determined by performing fits to the experimental data for inclusive semileptonic decays [81, 82]. In the case of the chromomagnetic operator, one finds [82]

$$\mu_G^2(B) = (0.306 \pm 0.050)\,\mathrm{GeV}^2\,. \tag{3.1}$$

Assuming heavy quark symmetry we expect the corresponding parameter in the D system to have a similar size. Another way of estimating the value of μ_G^2 is to use the well-known spectroscopy relation [83]

$$\mu_G^2(D_{(s)}) = \frac{3}{2} m_c \,(M_{D^*_{(s)}} - M_{D_{(s)}})\,, \tag{3.2}$$

which holds up to power corrections. Using the value for the meson masses given in the PDG [1] and setting $m_c = 1.27\,\mathrm{GeV}$, we obtain the following estimates:

$$\mu_G^2(D) = (0.268 \pm 0.107)\,\mathrm{GeV}^2, \qquad \mu_G^2(D_s) = (0.274 \pm 0.110)\,\mathrm{GeV}^2, \tag{3.3}$$

where we have conservatively added an uncertainty of 40% due to unknown power corrections of order $1/m_c$. The values in eq. (3.3) are roughly 19% smaller than those obtained from experimental fits for semileptonic B-meson decays, see eq. (3.1). Moreover, eq. (3.2) leads to a tiny amount of $\mathrm{SU}(3)_F$-symmetry breaking of $\approx 2\%$, which might, however, be enhanced by the neglected power corrections. In the literature many times instead of eq. (3.2) the relation [71, 84]

$$\mu_G^2(D_{(s)}) = \frac{3}{4}\left(M^2_{D^*_{(s)}} - M^2_{D_{(s)}}\right) \tag{3.4}$$

is adopted, which coincides with eq. (3.2) up to corrections of order $1/m_c$. Numerically we find that eq. (3.4) yields

$$\mu_G^2(D) = 0.41\,\mathrm{GeV}^2\,, \qquad \mu_G^2(D_s^+) = 0.44\,\mathrm{GeV}^2\,, \tag{3.5}$$

which are roughly 23% higher than that in eq. (3.1). In our numerical analysis, we will use the average value of the two determinations in eq. (3.3) and eq. (3.5). This gives

$$\mu_G^2(D) = (0.34 \pm 0.10)\,\mathrm{GeV}^2, \qquad \mu_G^2(D_s^+) = (0.36 \pm 0.10)\,\mathrm{GeV}^2\,, \tag{3.6}$$

which agrees well with the one in eq. (3.1).

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Source	LQCD [59]	LQCD [60]	Exp. fit [81]	Exp. fit [82]	QCD SR [65]	QCD SR [64]
$\mu_\pi^2[\mathrm{GeV}^2]$	0.05(22)	0.314(15)	0.465(68)	0.477(56)	0.10(5)	0.6(1)

Table 11. Different determinations of $\mu_\pi^2(B)$ available in the literature.

Thus, from eq. (2.28), we expect corrections to the total decay rate due to the chromomagnetic operator, $c_G\,\mu_G^2/(c_3\,m_c^2)$ ranging between -6% and $+8\%$ with respect to the leading free-quark decay contribution. A large part of the sizable uncertainty derives from the cancellations in the coefficient c_G, shown in table 6 and figure 3, which could be reduced with a complete determination of the NLO-QCD corrections to c_G. For semileptonic rates the contribution of the chromomagnetic operator can be even of the order of 20%, see section 4.3.

An experimental determination of $\mu_G^2(D)$ from inclusive semileptonic D-meson decays could further reduce the uncertainties and could in particular give some insight into the numerical size of $\mathrm{SU}(3)_F$ breaking.

3.2 Parameters of the kinetic operator

For the matrix element of the kinetic operator no precise determination is available so far in the charm sector. Several predictions of μ_π^2 available in the literature for the B-meson cover a large range of values, see table 11. Assuming heavy quark symmetry one can use the value obtained from the recent fit of the semileptonic B-meson decays [82]:

$$\mu_\pi^2(B) \,=\, (0.477 \pm 0.056)\,\mathrm{GeV}^2\,, \tag{3.7}$$

to get the following estimate for the D-meson

$$\mu_\pi^2(D) \,=\, (0.48 \pm 0.20)\,\mathrm{GeV}^2. \tag{3.8}$$

In the above, we have again added a conservative uncertainty of 40% to account for the breaking of the heavy quark symmetry. This value clearly fulfills the theoretical bound $\mu_\pi^2 \geq \mu_G^2$, see e.g. the review [85]. Thus we expect from eq. (2.28) corrections due to the kinetic operator of the order of -10%, which is also found in section 4.3 — both for the total decay rate and the semileptonic one.

The $\mathrm{SU}(3)_F$ breaking effects for the kinetic operator have been estimated in refs. [56, 86]

$$\mu_\pi^2(D_s^+) - \mu_\pi^2(D^0) \,\approx\, 0.09\,\mathrm{GeV}^2\,, \tag{3.9}$$

leading to the following estimate we use for the D_s meson:

$$\mu_\pi^2(D_s^+) = (0.57 \pm 0.23)\,\mathrm{GeV}^2. \tag{3.10}$$

Again a more precise experimental determination of μ_π^2 from fits to semileptonic D^+, D^0 and D_s^+ meson decays — as it was done for the B^+ and B^0 decays — would be very desirable.

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3.3 Parameters of the Darwin operator

For the matrix element of the Darwin operator no theoretical determination for the charm sector is available. We again could assume heavy quark symmetry and use the corresponding value in the B-system, obtained from recent fits of the semileptonic decays [82]:

$$\rho_D^3(B) = (0.185 \pm 0.031)\,\text{GeV}^3\,, \tag{3.11}$$

and add quadratically an uncertainty of 40% for the transition from the B to the D system, leading to a first estimate of

$$\rho_D^3(D)^I = (0.185 \pm 0.080)\,\text{GeV}^3\,. \tag{3.12}$$

Alternatively the Darwin parameter can be related to the matrix elements of the dimension-six four-quark operators through the equation of motion for the gluon field. At leading order in $1/m_Q$ one obtains:

$$\rho_D^3(H) = \frac{g_s^2}{18} f_H^2\, m_H \left[2\,\tilde{B}_2^{q'} - \tilde{B}_1^{q'} + \frac{3}{4}\tilde{\epsilon}_1^{q'} - \frac{3}{2}\tilde{\epsilon}_2^{q'} + \sum_{q=u,d,s}\left(2\tilde{\delta}_2^{qq'} - \tilde{\delta}_1^{qq'} + \frac{3}{4}\tilde{\delta}_3^{qq'} - \frac{3}{2}\tilde{\delta}_4^{qq'}\right)\right] + \mathcal{O}\left(\frac{1}{m_Q}\right), \tag{3.13}$$

where H is a heavy hadron with the mass m_H and the decay constant f_H, $q' = u, d, s$ is the light valence quark in the H-hadron, and the Bag parameters $\tilde{B}_1^q$, $\tilde{B}_2^q$, $\tilde{\epsilon}_1^q$, $\tilde{\epsilon}_2^q$, $\tilde{\delta}_1^{qq'}$ $\tilde{\delta}_2^{qq'}$ $\tilde{\delta}_3^{qq'}$ and $\tilde{\delta}_4^{qq'}$ were introduced in section 2.5. Their numerical values are summarised in table 17 in appendix C. The strong coupling g_s has its origin in the non-perturbative regimes — e.g. ref. [87] suggests to set $\alpha_s = 1$.

Using the input from the appendix A and applying eq. (3.13) we derive estimates of ρ_D^3 for B- and D-mesons both in VIA and using the HQET SR results for the Bag parameters. The values are summarised in table 12 for the three different choices, $\alpha_s(\mu = 1.5\,\text{GeV})$, $\alpha_s(\mu = 1\,\text{GeV})$ and $\alpha_s = 1$.

Setting $\alpha_s = 1$ in eq. (3.13) yields values for ρ_D^3 that are close to the one determined from the fit of semileptonic B meson decays, eq. (3.11), indicating $1/m_b$-corrections in eq. (3.13) of the order of +30%. Moreover, we find that VIA gives in eq. (3.13) values which are very close to the HQET sum rule ones. We emphasise that due to the sizeable $\text{SU}(3)_F$ breaking in the decay constants, eq. (3.13) leads also to a sizable $\text{SU}(3)_F$ breaking for the non-perturbative parameters $\rho_D^3(D)$, $\rho_D^3(D_s^+)$. Taking the values corresponding to $\alpha_s = 1$ and using HQET SR results for the bag parameters we get the second estimate (last column in table 12)

$$\rho_D^3(D)^{II} = (0.056 \pm 0.022)\,\text{GeV}^3\,, \qquad \rho_D^3(D_s^+)^{II} = (0.082 \pm 0.033)\,\text{GeV}^3\,, \tag{3.14}$$

where we have again added a 40% uncertainty. Finally, another possibility to extract $\rho_D^3(D)$ is to substitute in eq. (3.13) the values of the Bag parameters in VIA, which gives

$$\rho_D^3(H) \approx \frac{g_s^2}{18} f_H^2\, m_H. \tag{3.15}$$

$\rho_D^3[\mathrm{GeV}^3]$	$\mu = 1.5\,\mathrm{GeV}$ VIA	$\mu = 1.5\,\mathrm{GeV}$ HQET	$\mu = 1.0\,\mathrm{GeV}$ VIA	$\mu = 1.0\,\mathrm{GeV}$ HQET	$\alpha_s = 1$ VIA	$\alpha_s = 1$ HQET
B^+, B_d	0.048	0.047	0.066	0.064	0.133	0.129
B_s	0.072	0.070	0.098	0.095	0.199	0.193
D^+, D^0	0.021	0.020	0.027	0.026	0.059	0.056
D_s^+	0.030	0.029	0.040	0.038	0.086	0.082

Table 12. Values of $\rho_D^3(H)$ for B- and D-mesons in VIA and using HQET SR for Bag parameters for three different choices of α_s in eq. (3.13).

Assuming a similar size for the strong coupling in both the B- and D-meson matrix elements, from eq. (3.15) one obtains:

$$\rho_D^3(D) \approx \frac{f_D^2\, m_D}{f_B^2\, m_B}\, \rho_D^3(B)\,, \qquad \rho_D^3(D_s) \approx \frac{f_{D_s}^2 m_{D_s}}{f_B^2\, m_B}\, \rho_D^3(B)\,. \tag{3.16}$$

Using the most precise determination of the decay constants from Lattice QCD [78], and of the meson masses from PDG [1] and taking into account the value of $\rho_D^3(B)$ in eq. (3.11), leads to the following estimates:

$$\rho_D^3(D)^{III} = (0.082 \pm 0.035)\,\mathrm{GeV}^3\,, \qquad \rho_D^3(D_s)^{III} = (0.119 \pm 0.052)\,\mathrm{GeV}^3\,, \tag{3.17}$$

where we again assign in addition a conservative 40% uncertainty due to missing power corrections. These values are consistent with the numbers shown in table 12 for $\alpha_s = 1$. Contrary to the case of the dimension-five non-perturbative parameters, in eq. (3.17) one observes a large $\mathrm{SU}(3)_f$ -symmetry breaking of $\approx 46\%$, similar to the $\approx 49\%$ that one obtains for the $B_{(s)}$-mesons, mostly stemming from the ratios f_{B_s}/f_{B_d} and $f_{D_s^+}/f_{D^0}$. In our numerical analysis we use the values shown in eq. (3.17), which lies between the estimates obtained in eq. (3.12) and eq. (3.14).

Again, here a more precise experimental determination of ρ_D^3 from fits to semileptonic D^+, D^0 and D_s^+ meson decays — as it was done for the B^+ and B^0 decays — would be very desirable and could have a significant effect on the phenomenology of inclusive charm decays.

3.4 Bag parameters of dimension-six and dimension-seven

The dimension-six Bag parameters of the D^+ and D^0 mesons have been determined using HQET Sum Rules [66]; strange quark mass corrections, relevant for the Bag parameter of the D_s^+ meson, as well as eye-contractions have been computed for the first time in ref. [68]. The results are collected in table 17 and the HQET sum rules suggest values for the Bag parameter that are very close to VIA.

For the dimension-seven Bag parameters (defined in HQET), we apply VIA. As one can see from appendix C, the matrix elements of dimension-seven operators in HQET depend

VIA					
Observable	Pole	$\overline{\text{MS}}$	Kinetic	$1S$	Exp. value
$\Gamma(D^0)[\text{ps}^{-1}]$	1.71	1.49	1.58	1.66	2.44
$\Gamma(D^+)[\text{ps}^{-1}]$	0.22	-0.01	0.11	0.18	0.96
$\bar{\Gamma}(D_s^+)[\text{ps}^{-1}]$	1.76	1.51	1.61	1.71	1.88
$\tau(D^+)/\tau(D^0)$	2.55	2.56	2.53	2.54	2.54
$\bar{\tau}(D_s^+)/\tau(D^0)$	0.97	0.99	0.98	0.98	1.30
$B_{sl}^{D^0}[\%]$	5.43	6.55	6.14	5.75	6.49
$B_{sl}^{D^+}[\%]$	13.8	16.6	15.6	14.6	16.07
$B_{sl}^{D_s^+}[\%]$	7.12	8.42	7.95	7.50	6.30
$\Gamma_{sl}^{D^+}/\Gamma_{sl}^{D^0}$	1.00	1.00	1.00	1.00	0.985
$\Gamma_{sl}^{D_s^+}/\Gamma_{sl}^{D^0}$	1.06	1.05	1.05	1.05	0.790

Table 13. Central values of the charm observables in different quark mass schemes using VIA for the matrix elements of the 4-quark operators compared to the corresponding experimental values (last column).

also on the parameters $\bar{\Lambda}_{(s)} = m_{D_{(s)}} - m_c$, for which we use the following ranges [68]

$$\begin{aligned} \bar{\Lambda} &= (0.5 \pm 0.1)\,\text{GeV}, \\ \bar{\Lambda}_s &= (0.6 \pm 0.1)\,\text{GeV}. \end{aligned} \tag{3.18}$$

4 Numerical results

In this section, using all the ingredients described above, we present the theoretical prediction for the total and semileptonic decay rates, and for their ratios. All the input used in our numerical analysis are collected in appendix A. For each observable, we investigate several quark mass schemes (with the kinetic scheme as default) and compare the corresponding results using both VIA and HQET SR values for the Bag parameters. The uncertainties quoted below are obtained by varying all the input parameters within their intervals. For the renormalisation scales, we fix the central values to $\mu_1 = \mu_0 = 1.5\,\text{GeV}$ and vary both of them independently between 1 and 3 GeV. Moreover we add an estimated uncertainty due to missing higher power corrections. The results are discussed in the following subsections and they are summarised in tables 13, 14, 15 and in figure 7.

4.1 The total decay rates

We start by investigating the theory prediction of the total decay rates, which are expected to have sizable uncertainties due to the dependence of the free quark decay on the fifth

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Observable	Pole	$\overline{\text{MS}}$	Kinetic	$1S$	Exp. value
	HQET SR				
$\Gamma(D^0)[\text{ps}^{-1}]$	1.73	1.52	1.61	1.68	2.44
$\Gamma(D^+)[\text{ps}^{-1}]$	-0.03	-0.24	-0.12	-0.06	0.96
$\bar\Gamma(D_s^+)[\text{ps}^{-1}]$	1.75	1.50	1.60	1.69	1.88
$\tau(D^+)/\tau(D^0)$	2.83	2.83	2.80	2.82	2.54
$\bar\tau(D_s^+)/\tau(D^0)$	0.99	1.01	1.00	1.00	1.30
$B_{sl}^{D^0}\,[\%]$	5.26	6.42	6.00	5.59	6.49
$B_{sl}^{D^+}\,[\%]$	13.4	16.3	15.2	14.2	16.07
$B_{sl}^{D_s^+}\,[\%]$	7.10	8.36	7.91	7.48	6.30
$\Gamma_{sl}^{D^+}/\Gamma_{sl}^{D^0}$	1.002	1.001	1.001	1.002	0.985
$\Gamma_{sl}^{D_s^+}/\Gamma_{sl}^{D^0}$	1.08	1.06	1.07	1.08	0.790

Table 14. Central values of the charm observables in different quark mass schemes using HQET sum rule results [66, 68] for the matrix elements of the 4-quark operators compared to the corresponding experimental values (last column).

Observable	HQE prediction	Exp. value
$\Gamma(D^0)[\text{ps}^{-1}]$	$1.61 \pm 0.37^{+0.46}_{-0.37}{}^{+0.01}_{-0.01}$	2.44 ± 0.01
$\Gamma(D^+)[\text{ps}^{-1}]$	$-0.12 \pm 0.77^{+0.59}_{-0.28}{}^{+0.25}_{-0.10}$	0.96 ± 0.01
$\bar\Gamma(D_s^+)[\text{ps}^{-1}]$	$1.60 \pm 0.44^{+0.52}_{-0.41}{}^{+0.02}_{-0.01}$	1.88 ± 0.02
$\tau(D^+)/\tau(D^0)$	$2.80 \pm 0.85^{+0.01}_{-0.14}{}^{+0.11}_{-0.26}$	2.54 ± 0.02
$\bar\tau(D_s^+)/\tau(D^0)$	$1.00 \pm 0.16^{+0.02}_{-0.03}{}^{+0.01}_{-0.01}$	1.30 ± 0.01
$B_{sl}^{D^0}\,[\%]$	$6.00 \pm 1.57^{+0.33}_{-0.28}$	6.49 ± 0.11
$B_{sl}^{D^+}\,[\%]$	$15.23 \pm 4.07^{+0.83}_{-0.72}$	16.07 ± 0.30
$B_{sl}^{D_s^+}\,[\%]$	$7.91 \pm 2.64^{+0.43}_{-0.38}$	6.30 ± 0.16
$\Gamma_{sl}^{D^+}/\Gamma_{sl}^{D^0}$	$1.001 \pm 0.008 \pm 0.001$	0.985 ± 0.028
$\Gamma_{sl}^{D_s^+}/\Gamma_{sl}^{D^0}$	$1.07 \pm 0.24 \pm 0.01$	0.790 ± 0.026

Table 15. HQE predictions for all the ten observables in the kinetic scheme (second column), using HQET SR results for the Bag parameters. The first uncertainty is parametric, the second and third uncertainties are due to μ_1- and μ_0-scales variation, respectively. The results are compared with the corresponding experimental measurements (third column).

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power of the charm quark mass and due to large perturbative and power corrections. The central values for the HQE prediction of the decay widths in different mass schemes, are shown in the three first rows of table 13, using VIA for the Bag parameters and of table 14 using the HQET sum rules results. In table 15 we show the theoretical prediction including the corresponding uncertainties within the kinetic mass scheme and using the HQET SR values for the dimension-six Bag parameters — the same result is visualised in figure 7. In each table, the corresponding experimental determinations are listed in the last column. For the D_s^+ meson an additional subtlety is arising due to the large branching fraction of the leptonic decay $D_s^+ \to \tau^+\nu_\tau$, which is not included in the HQE, since the tau lepton is more massive than the charm quark. Using the experimental value of the leptonic branching ratio [1] (online update)

$$\mathrm{Br}(D_s^+ \to \tau^+\nu_\tau) = (5.48 \pm 0.23)\%\,, \tag{4.1}$$

we therefore define a reduced decay rate $\bar{\Gamma}(D_s^+)$:

$$\bar{\Gamma}(D_s^+) \equiv \Gamma(D_s^+) - \Gamma(D_s^+ \to \tau^+\nu_\tau) = (1.88 \pm 0.02)\,\mathrm{ps}^{-1}\,, \tag{4.2}$$

leading also to a reduced lifetime ratio

$$\frac{\bar{\tau}(D_s^+)}{\tau(D^0)} = 1.30 \pm 0.01\,. \tag{4.3}$$

The first and main result we deduce from table 15 and figure 7, is that the HQE gives values of $\Gamma(D^0)$, $\Gamma(D^+)$ and $\Gamma(D_s^+)$ which lie in the ballpark of the experimental numbers. Looking closer we find that our prediction for $\Gamma(D_s)$ is in good agreement with experiment (within large uncertainties), while the total decay rates of the D^0 and D^+ mesons are underestimated. As a reason for that we suspect missing NNLO-QCD corrections to the free charm quark decay. Second, using different mass schemes yields similar results, and further higher order correction will reduce the differences between these schemes. Due to the fact that the values of the HQET Bag parameters [66, 68] listed in table 17 are close to the corresponding ones in VIA, the predictions shown in table 13 and in table 14 do not differ much. A peculiar role is played by the D^+ meson, where we get a huge theoretical uncertainty stemming from the large negative value of the Pauli interference contribution at dimension-six. This term actually dominates the total decay rate. Moreover, the large negative value is further enhanced by NLO-QCD corrections, but partly compensated by the dimension-seven contribution. Here further studies of the Bag parameters, e.g. via an independent confirmation of the HQET sum rule results with lattice QCD, as well as calculation of higher order QCD corrections to dimension-six and dimension-seven might yield deeper insights.

In order to further analyse the size of the individual contributions to the total decay rate, we show below the numerical coefficients of each non-perturbative parameter in the HQE, using the central values for the input in appendix A and (as an example) the kinetic scheme for the charm mass with $\mu^{\rm cut} = 0.5\,$GeV, namely[17]

$$
\begin{aligned}
\Gamma(D^0) = \Gamma_0 \Big[& \underbrace{6.15}_{c_3^{\rm LO}} + \underbrace{2.95}_{\Delta c_3^{\rm NLO}} - 1.66\, \frac{\mu_\pi^2(D)}{\rm GeV^2} + 0.13\, \frac{\mu_G^2(D)}{\rm GeV^2} + 23.6\, \frac{\rho_D^3(D)}{\rm GeV^3} \\
& -1.60\, \tilde{B}_1^q + 1.53\, \tilde{B}_2^q - 21.0\, \tilde{\epsilon}_1^q + 19.2\, \tilde{\epsilon}_2^q + \underbrace{0.00}_{\rm dim-7, VIA} \\
& -10.7\, \tilde{\delta}_1^{qq} + 1.53\, \tilde{\delta}_2^{qq} + 54.6\, \tilde{\delta}_3^{qq} + 0.13\, \tilde{\delta}_4^{qq} - 29.2\, \tilde{\delta}_1^{sq} + 28.8\, \tilde{\delta}_2^{sq} + 0.56\, \tilde{\delta}_3^{sq} + 2.36\, \tilde{\delta}_4^{sq} \Big] \\
= 6.15\, \Gamma_0 \Big[& 1 + 0.48 - 0.13\, \frac{\mu_\pi^2(D)}{0.48\,\rm GeV^2} + 0.01\, \frac{\mu_G^2(D)}{0.34\,\rm GeV^2} + 0.31\, \frac{\rho_D^3(D)}{0.082\,\rm GeV^3} \\
& - \underbrace{0.01}_{\rm dim-6, VIA} - 0.005\, \frac{\delta \tilde{B}_1^q}{0.02} + 0.005\, \frac{\delta \tilde{B}_2^q}{0.02} + 0.137\, \frac{\tilde{\epsilon}_1^q}{-0.04} - 0.125\, \frac{\tilde{\epsilon}_2^q}{-0.04} + \underbrace{0.00}_{\rm dim-7, VIA} \\
& - 0.0045\, r_1^{qq} - 0.0004\, r_2^{qq} - 0.0035\, r_3^{qq} + 0.0000\, r_4^{qq} \\
& - 0.0109\, r_1^{sq} - 0.0079\, r_2^{sq} - 0.0000\, r_3^{sq} + 0.0001\, r_4^{sq} \Big] \, .
\end{aligned}
\tag{4.4}
$$

In the second equality in eq. (4.4) we have normalised the HQE parameters $\mu_\pi^2(D)$, $\mu_G^2(D)$ and $\rho_D^3(D)$ to their central values. Moreover, we introduce

$$
\tilde{B}_i^q = 1 + \delta \tilde{B}_i^q \, , \tag{4.5}
$$

to indicate deviations from VIA and we conservatively normalise $\delta \tilde{B}_i^q$ to 0.02. The matrix elements of the colour-octet operators are normalised to -0.04 — here using the central value of the HQET determination for $\tilde{\epsilon}_2^q$ might underestimate its effect due to the quoted HQET uncertainties. Furthermore, we introduce also the ratios $r_i^{qq'} \equiv \tilde{\delta}_i^{qq'} / \langle \tilde{\delta}_i^{qq'} \rangle$, with $\langle \tilde{\delta}_i^{qq'} \rangle$ being the central values listed in table 17.

For the neutral D meson we find a convergent series, with the largest correction due to the QCD corrections to the free quark decay and the contribution of the Darwin operator. Here a calculation of the NNLO-QCD corrections to the free-quark decay would be very desirable, as well as a more profound determination of the value of the matrix element of the Darwin operator. Note that since we take as a central value $\mu_1 = 1.5\,$GeV, the coefficient of the chromomagnetic operator in eq. (4.4) turns out accidentally to be very small, see figure 3. In fact, varying the renormalisation scale μ_1 between 1 and 3 GeV one finds quite sizable contribution of $\sim$ 5–10% due to $\mu_G^2(D)$. Because of the helicity suppression, we get only small contributions from the weak exchange diagrams. In LO-QCD and VIA these corrections actually vanish, the small value ≈ -0.01 stems from NLO-QCD corrections, which break the helicity suppression. Nevertheless, depending on the size of the $\tilde{\epsilon}_i^q$, the colour-octet operator could give contributions of a similar size as the kinetic operator. Finally, according to the HQET SR determination, the numerical effect of the eye-contractions does not seem to be pronounced.

[17]Here and hereafter, in the Bag parameters we use the same label q both for u or d-quarks, reflecting the isospin symmetry, namely $\tilde{B}_i^u = \tilde{B}_i^d \equiv \tilde{B}_i^q$ and $\tilde{\delta}_i^{ud} = \tilde{\delta}_i^{du} \equiv \tilde{\delta}_i^{qq}$, $\tilde{\delta}_i^{us} = \tilde{\delta}_i^{ds} \equiv \tilde{\delta}_i^{qs}$, $\tilde{\delta}_i^{su} = \tilde{\delta}_i^{sd} \equiv \tilde{\delta}_i^{sq}$.

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Similarly, we get for the D^+-meson decay width:

$$\begin{aligned}
\Gamma(D^+) = \Gamma_0 \Big[& \underbrace{6.15}_{c_3^{\rm LO}} + \underbrace{2.95}_{\Delta c_3^{\rm NLO}} - 1.66 \frac{\mu_\pi^2(D)}{\mathrm{GeV}^2} + 0.13 \frac{\mu_G^2(D)}{\mathrm{GeV}^2} + 23.6 \frac{\rho_D^3(D)}{\mathrm{GeV}^3} \\
& - 16.9\, \tilde{B}_1^q + 0.56\, \tilde{B}_2^q + 84.0\, \tilde{\epsilon}_1^q - 1.34\, \tilde{\epsilon}_2^q + \underbrace{6.76}_{\rm dim-7} \\
& - 0.06\, \tilde{\delta}_1^{qq} + 0.06\, \tilde{\delta}_2^{qq} - 16.8\, \tilde{\delta}_3^{qq} + 16.9\, \tilde{\delta}_4^{qq} - 29.3\, \tilde{\delta}_1^{sq} + 28.8\, \tilde{\delta}_2^{sq} + 0.56\, \tilde{\delta}_3^{sq} + 2.36\, \tilde{\delta}_4^{sq} \Big] \\
= 6.15\, \Gamma_0 \Big[& 1 + 0.48 - 0.13 \frac{\mu_\pi^2(D)}{0.48\, \mathrm{GeV}^2} + 0.01 \frac{\mu_G^2(D)}{0.34\, \mathrm{GeV}^2} + 0.31 \frac{\rho_D^3(D)}{0.082\, \mathrm{GeV}^3} \\
& - \underbrace{2.66}_{\rm dim-6,VIA} - 0.055 \frac{\delta \tilde{B}_1^q}{0.02} + 0.002 \frac{\delta \tilde{B}_2^q}{0.02} - 0.546 \frac{\tilde{\epsilon}_1^q}{-0.04} + 0.009 \frac{\tilde{\epsilon}_2^q}{-0.04} + \underbrace{1.10}_{\rm dim-7,VIA} \\
& - 0.0000\, r_1^{qq} - 0.0000\, r_2^{qq} + 0.0011\, r_3^{qq} + 0.0008\, r_4^{qq} \\
& - 0.0109\, r_1^{sq} - 0.0080\, r_2^{sq} - 0.0000\, r_3^{sq} + 0.0001\, r_4^{sq} \Big] , \qquad (4.6)
\end{aligned}$$

where we observe huge negative corrections due to Pauli interference. In VIA we get from dimension-six (summing LO and NLO-QCD) a $\approx -270\%$ correction to the LO-free-quark decay. Dimension-seven yields a large positive correction of $+110\%$. Because of the almost perfect cancellation between the three dominant terms, $16\pi^2 \left(\tilde{\Gamma}_6^{(0)} + \alpha_s/\pi \tilde{\Gamma}_6^{(1)}\right) \langle \tilde{\mathcal{O}}_6 \rangle^{\rm VIA}/m_c^3$, Γ_3 and $16\pi^2 \tilde{\Gamma}_7^{(0)} \langle \tilde{\mathcal{O}}_7 \rangle^{\rm VIA}/m_c^4$, the HQE series for $\Gamma(D^+)$ becomes very sensitive to sub-dominant terms, e.g. higher order QCD corrections to $\tilde{\Gamma}_6$, $\tilde{\Gamma}_7$, Γ_3, Γ_5 and Γ_6, and to deviations of the Bag parameter from VIA. In this case it might also be interesting to further study estimates of higher orders in the HQE, see e.g. refs. [57, 58]. Else, we get for the two-quark $\Delta C = 0$ contributions the same (due to isospin) size of corrections as in the D^0 case and we find, based on the HQET sum rules estimates, again that the eye-contractions give only tiny corrections.

Finally, we have for the D_s^+-meson decay width:

$$\begin{aligned}
\Gamma(D_s^+) = \Gamma_0 \Big[& \underbrace{6.15}_{c_3^{\rm LO}} + \underbrace{2.95}_{\Delta c_3^{\rm NLO}} - 1.66 \frac{\mu_\pi^2(D_s)}{\mathrm{GeV}^2} + 0.13 \frac{\mu_G^2(D_s)}{\mathrm{GeV}^2} + 23.6 \frac{\rho_D^3(D_s)}{\mathrm{GeV}^3} \\
& - 49.6\, \tilde{B}_1^s + 48.4\, \tilde{B}_2^s - 13.7\, \tilde{\epsilon}_1^s + 18.8\, \tilde{\epsilon}_2^s + \underbrace{0.63}_{\rm dim-7} \\
& - 15.8\, \tilde{\delta}_1^{qs} + 2.34\, \tilde{\delta}_2^{qs} + 55.4\, \tilde{\delta}_3^{qs} + 25.0\, \tilde{\delta}_4^{qs} \Big] \\
= 6.15\, \Gamma_0 \Big[& 1 + 0.48 - 0.15 \frac{\mu_\pi^2(D_s)}{0.57\, \mathrm{GeV}^2} + 0.01 \frac{\mu_G^2(D_s)}{0.36\, \mathrm{GeV}^2} + 0.46 \frac{\rho_D^3(D_s)}{0.119\, \mathrm{GeV}^3} \\
& - \underbrace{0.20}_{\rm dim-6,VIA} - 0.161 \frac{\delta \tilde{B}_1^s}{0.02} + 0.157 \frac{\tilde{B}_2^s}{0.02} + 0.089 \frac{\tilde{\epsilon}_1^s}{-0.04} + 0.122 \frac{\tilde{\epsilon}_2^s}{0.04} + \underbrace{0.10}_{\rm dim-7,VIA} \\
& - 0.0064\, r_1^{qs} - 0.0007\, r_2^{qs} - 0.0036\, r_3^{qs} + 0.0012\, r_4^{qs} \Big] , \qquad (4.7)
\end{aligned}$$

where we find again a converging series with the dominant contribution coming from the NLO-QCD corrections to the free quark decay and the Darwin term. For the latter a more reliable determination of the corresponding non-perturbative matrix elements would be highly desirable. In VIA, the four-quark operators show again a pronounced cancellation between dimension-six and dimension-seven contributions.

4.2 The lifetime ratios

In order to eliminate the contribution of the free-quark decay, we calculate the lifetime ratios as

$$\frac{\tau(D^+_{(s)})}{\tau(D^0)} = 1 + \left[\Gamma^{\rm HQE}(D^0) - \Gamma^{\rm HQE}(D^+_{(s)})\right] \tau^{\rm exp}(D^+_{(s)})\,, \tag{4.8}$$

where $\Gamma^{\rm HQE}(D^0)$ and $\Gamma^{\rm HQE}(D^+_{(s)})$ are given in eqs. (4.4) and (4.6), (4.7), respectively. In these ratios, Γ_3 cancels exactly and Γ_5 and Γ_6 cancel up to isospin or $SU(3)_F$ breaking corrections in the corresponding non-perturbative matrix elements. The lifetime ratios should then be dominated by the contribution of four-quark operators.

The central values for the HQE prediction of the lifetime ratios in several mass schemes are shown in the fourth and fifth rows of table 13, table 14, table 15 and in figure 7 and it turns out that the large lifetime ratio $\tau(D^+)/\tau(D^0)$ is well reproduced in all schemes, while in the case of $\tau(D^+_s)/\tau(D^0)$ the HQE predictions lie closer to one compared to the experimental values. The latter theory result is dominated by $SU(3)_F$ breaking differences of the non-perturbative matrix elements μ_π^2, μ_G^2 and ρ_D^3, which are only very roughly known, see section 3. With future, more precise determinations of these parameters our conclusion might significantly change for this lifetime ratio.

The large lifetime ratio $\tau(D^+)/\tau(D^0)$ can be expressed as

$$\begin{aligned}
\frac{\tau(D^+)}{\tau(D^0)} &= 1+2.46\,\tilde{B}_1^q+0.16\,\tilde{B}_2^q-16.9\,\tilde{\epsilon}_1^q+3.31\,\tilde{\epsilon}_2^q \underbrace{-1.09}_{\rm dim-7,VIA} \\
&\quad -1.71\,\tilde{\delta}_1^{qq}+0.24\,\tilde{\delta}_2^{qq}+1.15\,\tilde{\delta}_3^{qq}-2.71\,\tilde{\delta}_4^{qq}+0.01\,\tilde{\delta}_1^{sq}-0.01\,\tilde{\delta}_2^{sq}+0.00\,\tilde{\delta}_3^{sq}+0.00\,\tilde{\delta}_4^{sq} \\
&= 1 \underbrace{+\,2.62}_{\rm dim-6,VIA} \underbrace{-\,1.09}_{\rm dim-7,VIA} +0.049\,\frac{\delta\tilde{B}_1^q}{0.02}+0.003\,\frac{\delta\tilde{B}_2^q}{0.02}+0.676\,\frac{\tilde{\epsilon}_1^q}{-0.04}-0.132\,\frac{\tilde{\epsilon}_2^q}{-0.04} \\
&\quad -0.004\,r_1^{qq}-0.000\,r_2^{qq}-0.005\,r_3^{qq}-0.001\,r_4^{qq}\,.
\end{aligned} \tag{4.9}$$

In VIA, we predict a lifetime ratio of 2.5, which is already quite close to the experimental value. Again, we observe here a sizable cancellation between dimension-six and dimension-seven contributions. In order to improve the theoretical prediction, a more precise determination of the Bag parameters of the colour-octet operators is mandatory, as well as of the perturbative higher order QCD corrections in $\tilde{\Gamma}_6$ and $\tilde{\Gamma}_7$.

And finally we get for the lifetime ratio $\tau(D_s^+)/\tau(D^0)$:

$$\begin{aligned}
\frac{\tau(D_s^+)}{\tau(D^0)} &= 1+0.14\,\frac{\mu_\pi^2(D_s)-\mu_\pi^2(D)}{\mathrm{GeV}^2}-0.01\,\frac{\mu_G^2(D_s)-\mu_G^2(D)}{\mathrm{GeV}^2}-1.93\,\frac{\rho_D^3(D_s)-\rho_D^3(D)}{\mathrm{GeV}^3}\underbrace{-0.05}_{\mathrm{dim-7,VIA}}\\
&\quad -0.13\,\tilde{B}_1^q+0.13\,\tilde{B}_2^q+4.06\,\tilde{B}_1^s-3.96\,\tilde{B}_2^s-1.72\,\tilde{\epsilon}_1^q+1.57\,\tilde{\epsilon}_2^q+1.12\,\tilde{\epsilon}_1^s-1.54\,\tilde{\epsilon}_2^s\\
&\quad -0.88\,\tilde{\delta}_1^{qq}+0.13\,\tilde{\delta}_2^{qq}+4.47\,\tilde{\delta}_3^{qq}+0.01\,\tilde{\delta}_4^{qq}-2.39\,\tilde{\delta}_1^{qs}+2.36\,\tilde{\delta}_2^{qs}+0.05\,\tilde{\delta}_3^{qs}+0.19\,\tilde{\delta}_4^{qs}\\
&\quad +1.29\,\tilde{\delta}_1^{sq}-0.19\,\tilde{\delta}_2^{sq}-4.54\,\tilde{\delta}_3^{sq}-2.04\,\tilde{\delta}_4^{sq}\\
&= 1+0.012\,\frac{\mu_\pi^2(D_s)-\mu_\pi^2(D)}{0.09\,\mathrm{GeV}^2}-0.0002\,\frac{\mu_G^2(D_s)-\mu_G^2(D)}{0.02\,\mathrm{GeV}^2}-0.071\,\frac{\rho_D^3(D_s)-\rho_D^3(D)}{0.037\,\mathrm{GeV}^3}\\
&\quad \underbrace{+0.10}_{\mathrm{dim-6,VIA}}\ \underbrace{-0.05}_{\mathrm{dim-7,VIA}}\ -0.003\,\frac{\delta\tilde{B}_1^q}{0.02}+0.003\,\frac{\delta\tilde{B}_2^q}{0.02}+0.081\,\frac{\delta\tilde{B}_1^s}{0.02}-0.079\,\frac{\delta\tilde{B}_2^s}{0.02}\\
&\quad +0.069\,\frac{\tilde{\epsilon}_1^q}{-0.04}-0.063\,\frac{\tilde{\epsilon}_2^q}{-0.04}-0.045\,\frac{\tilde{\epsilon}_1^s}{-0.04}-0.062\,\frac{\tilde{\epsilon}_2^s}{0.04}\\
&\quad -0.0023\,r_1^{qq}-0.0002\,r_2^{qq}-0.0018\,r_3^{qq}+0.0000\,r_4^{qq}\\
&\quad -0.0055\,r_1^{qs}-0.0040\,r_2^{qs}-0.0000\,r_3^{qs}+0.0001\,r_4^{qs}\\
&\quad +0.0032\,r_1^{sq}+0.0003\,r_2^{sq}+0.0018\,r_3^{sq}-0.0006\,r_4^{sq}\,. \qquad (4.10)
\end{aligned}$$

With the estimates of μ_π^2, μ_G^2 and ρ_D^3 from section 3 we find that the largest individual $\mathrm{SU(3)}_F$ breaking effect ($\approx -7\%$) comes from the Darwin term. Using VIA we obtain a correction of $+5\%$ due to the four-quark contributions of dimension-six and dimension-seven — finite values of the matrix elements of the colour-octet operators as well as of $\delta\tilde{B}_{1,2}^s$ might lead to numerically similar effects. Else we have a large number of smaller $\mathrm{SU(3)}_F$ breaking effects, which can be both positive and negative.

4.3 The semileptonic decay widths and their ratios

For discussing the inclusive semileptonic decays of D mesons, we introduce the short-hand notations $\Gamma_{sl}^D \equiv \Gamma(D \to Xe^+\nu_e)$ and $B_{sl}^D \equiv \mathrm{Br}(D \to Xe^+\nu_e)$. We determine the theory value of the semileptonic branching ratio as

$$B_{sl}^{D,\mathrm{HQE}} = \Gamma_{sl}^{D,\mathrm{HQE}} \cdot \tau(D)^{\mathrm{exp}}\,. \qquad (4.11)$$

The central values for the HQE prediction of the lifetime ratios in several mass schemes are shown in the sixth, seventh and eighth row of table 13, table 14 and table 15 and in figure 7.

The semileptonic decay rate of the D^0 meson can be written (in the kinetic scheme) as

$$\begin{aligned}
\Gamma_{sl}^{D^0} &= \Gamma_0\Big[\underbrace{1.02}_{c_3^{\mathrm{LO}}}+\underbrace{0.16}_{\Delta c_3^{\mathrm{NLO}}}-0.27\,\frac{\mu_\pi^2(D)}{\mathrm{GeV}^2}-0.84\,\frac{\mu_G^2(D)}{\mathrm{GeV}^2}+2.48\,\frac{\rho_D^3(D)}{\mathrm{GeV}^3}\\
&\qquad -0.28\,\tilde{\delta}_1^{qq}+0.28\,\tilde{\delta}_2^{qq}-5.23\,\tilde{\delta}_1^{sq}+5.23\,\tilde{\delta}_2^{sq}\Big]\\
&= 1.02\,\Gamma_0\Big[1+0.16-0.13\,\frac{\mu_\pi^2(D)}{0.48\,\mathrm{GeV}^2}-0.28\,\frac{\mu_G^2(D)}{0.34\,\mathrm{GeV}^2}+0.20\,\frac{\rho_D^3(D)}{0.082\,\mathrm{GeV}^3}\\
&\qquad -0.0007\,r_1^{qq}-0.0005\,r_2^{qq}-0.0118\,r_1^{sq}-0.0087\,r_2^{sq}\Big], \qquad (4.12)
\end{aligned}$$

where as for the total D^0-meson decay width we find a converging series, with the largest correction due to the dimension-five operators, followed by the Darwin operator contribution and the NLO-QCD corrections to the free quark decay. Note that only the non-valence four-quark operator contributions (eye-contractions) are present here.

For the semileptonic D^+-meson decay we obtain

$$\begin{aligned}
\Gamma_{sl}^{D^+} &= \Gamma_0 \Big[\underbrace{1.02}_{c_3^{\rm LO}} + \underbrace{0.16}_{\Delta c_3^{\rm NLO}} - 0.27\,\frac{\mu_\pi^2(D)}{\rm GeV^2} - 0.84\,\frac{\mu_G^2(D)}{\rm GeV^2} + 2.48\,\frac{\rho_D^3(D)}{\rm GeV^3} + \underbrace{0.00}_{\rm dim-7,VIA} \\
&\quad - 0.28\,\tilde{B}_1^q + 0.28\,\tilde{B}_2^q - 0.09\,\tilde{\epsilon}_1^q + 0.09\,\tilde{\epsilon}_2^q - 5.24\,\tilde{\delta}_1^{sq} + 5.24\,\tilde{\delta}_2^{sq}\Big] \\
&= 1.02\,\Gamma_0 \Big[1 + 0.16 - 0.13\,\frac{\mu_\pi^2(D)}{0.48\,\rm GeV^2} - 0.28\,\frac{\mu_G^2(D)}{0.34\,\rm GeV^2} + 0.20\,\frac{\rho_D^3(D)}{0.082\,\rm GeV^3} \\
&\quad - \underbrace{0.00}_{\rm dim-6,7,VIA} - 0.005\,\frac{\delta\tilde{B}_1^q}{0.02} + 0.005\,\frac{\delta\tilde{B}_2^q}{0.02} + 0.004\,\frac{\tilde{\epsilon}_1^q}{-0.04} - 0.004\,\frac{\tilde{\epsilon}_2^q}{-0.04} \\
&\quad - 0.0118\,r_1^{sq} - 0.0088\,r_2^{sq}\Big]\,,
\end{aligned} \tag{4.13}$$

where we find the same series as for the neutral D-meson supplemented by contributions from CKM suppressed weak annihilation, which vanish in VIA both at dimension-six and dimension-seven. Deviations from VIA give very small corrections.

For the D_s^+-meson we obtain

$$\begin{aligned}
\Gamma_{sl}^{D_s^+} &= \Gamma_0 \Big[\underbrace{1.02}_{c_3^{\rm LO}} + \underbrace{0.16}_{\Delta c_3^{\rm NLO}} - 0.27\,\frac{\mu_\pi^2(D_s)}{\rm GeV^2} - 0.84\,\frac{\mu_G^2(D_s)}{\rm GeV^2} + 2.48\,\frac{\rho_D^3(D_s)}{\rm GeV^3} + \underbrace{0.00}_{\rm dim-7,VIA} \\
&\quad - 7.63\,\tilde{B}_1^s + 7.63\,\tilde{B}_2^s - 2.55\,\tilde{\epsilon}_1^s + 2.37\,\tilde{\epsilon}_2^s - 0.41\,\tilde{\delta}_1^{qs} + 0.41\,\tilde{\delta}_2^{qs}\Big] \\
&= 1.02\,\Gamma_0 \Big[1 + 0.16 - 0.15\,\frac{\mu_\pi^2(D_s)}{0.57\,\rm GeV^2} - 0.30\,\frac{\mu_G^2(D_s)}{0.36\,\rm GeV^2} + 0.29\,\frac{\rho_D^3(D_s)}{0.119\,\rm GeV^3} \\
&\quad - \underbrace{0.00}_{\rm dim-6,VIA} - 0.15\,\frac{\delta\tilde{B}_1^s}{0.02} + 0.15\,\frac{\delta\tilde{B}_2^s}{0.02} + 0.10\,\frac{\tilde{\epsilon}_1^s}{-0.04} + 0.09\,\frac{\tilde{\epsilon}_2^s}{0.04} \\
&\quad - 0.0010\,r_1^{qs} - 0.0007\,r_2^{qs}\Big]\,,
\end{aligned} \tag{4.14}$$

where we have a larger contribution due to CKM dominant weak annihilation as well as $SU(3)_F$ breaking corrections. Again, in VIA the four-quark contributions vanish both at dimension-six and dimension-seven, but now deviations from VIA might give sizable corrections.

Using the experimental values for the D^0 lifetime and semileptonic branching fraction, we determine the semileptonic ratios in the following way

$$\frac{\Gamma_{sl}^{D^+}}{\Gamma_{sl}^{D^0}} = 1 + \left[\Gamma_{sl}^{D^+} - \Gamma_{sl}^{D^0}\right]^{\rm HQE} \left[\frac{\tau(D^0)}{B_{sl}^{D^0}}\right]^{\rm exp}\,, \tag{4.15}$$

$$\frac{\Gamma_{sl}^{D_s^+}}{\Gamma_{sl}^{D^0}} = 1 + \left[\Gamma_{sl}^{D_s^+} - \Gamma_{sl}^{D^0}\right]^{\rm HQE} \left[\frac{\tau(D^0)}{B_{sl}^{D^0}}\right]^{\rm exp}\,, \tag{4.16}$$

where $\left[\Gamma_{sl}^{D^0}\right]^{\rm HQE}$, $\left[\Gamma_{sl}^{D^+}\right]^{\rm HQE}$ and $\left[\Gamma_{sl}^{D_s^+}\right]^{\rm HQE}$ are given in eqs. (4.12), (4.13) and (4.14), respectively.

The HQE values of these ratios are shown in the ninth and tenth rows of tables 13, 14 and 15 and in figure 7. In agreement with experiment HQE predicts values for $\Gamma_{sl}^{D^+}/\Gamma_{sl}^{D^0}$ very close to one. Using the inputs from appendix A the HQE prefers also for $\Gamma_{sl}^{D_s^+}/\Gamma_{sl}^{D^0}$ values close to one, while experiment find a value as low as 0.79 — again a more profound determination of μ_G^2, μ_π^2 and ρ_D^3 as well as an inclusion of dimension-seven contributions with two-quarks operators for D mesons might change this conclusion.

We expand $\Gamma_{sl}^{D^+}/\Gamma_{sl}^{D^0}$ as

$$\begin{aligned}
\frac{\Gamma_{sl}^{D^+}}{\Gamma_{sl}^{D^0}} &= 1 - 0.27\,\tilde{B}_1^q + 0.27\,\tilde{B}_2^q - 0.09\,\tilde{\epsilon}_1^q + 0.08\,\tilde{\epsilon}_2^q \underbrace{+\,0.00}_{\rm dim-7,VIA} \\
&\quad + 0.27\,\tilde{\delta}_1^{qq} - 0.27\,\tilde{\delta}_2^{qq} - 0.01\,\tilde{\delta}_1^{sq} + 0.01\,\tilde{\delta}_2^{sq} \\
&= 1 + \underbrace{0.00}_{\rm dim-6,7,VIA} - 0.005\,\frac{\delta\tilde{B}_1^q}{0.02} + 0.005\,\frac{\delta\tilde{B}_2^q}{0.02} + 0.004\,\frac{\tilde{\epsilon}_1^q}{-0.04} - 0.003\,\frac{\tilde{\epsilon}_2^q}{-0.04}\,.
\end{aligned} \tag{4.17}$$

Due to isospin symmetry, in eq. (4.17) the contributions of the kinetic, chromomagnetic and the Darwin operators vanish. Moreover, in VIA there is also no correction due to the spectator quark effects. Thus this ratio, within the framework of the HQE, is predicted to be very close to one.

Finally, we obtain for the ratio $\Gamma_{sl}^{D_s^+}/\Gamma_{sl}^{D^0}$[18]

$$\begin{aligned}
\frac{\Gamma_{sl}^{D_s^+}}{\Gamma_{sl}^{D^0}} &= 1 - 0.27\,\frac{\mu_\pi^2(D_s) - \mu_\pi^2(D)}{\rm GeV^2} - 0.82\,\frac{\mu_G^2(D_s) - \mu_G^2(D)}{\rm GeV^2} + 2.42\,\frac{\rho_D^3(D_s) - \rho_D^3(D)}{\rm GeV^3} \\
&\quad - 7.47\,\tilde{B}_1^s + 7.47\,\tilde{B}_2^s - 2.50\,\tilde{\epsilon}_1^s + 2.32\,\tilde{\epsilon}_2^s \underbrace{+\,0.00}_{\rm dim-7,VIA} \\
&\quad + 0.27\,\tilde{\delta}_1^{qq} - 0.27\,\tilde{\delta}_2^{qq} + 5.11\,\tilde{\delta}_1^{sq} - 5.11\,\tilde{\delta}_2^{sq} - 0.40\,\tilde{\delta}_1^{qs} + 0.40\,\tilde{\delta}_2^{qs} \\
&= 1 - 0.024\,\frac{\mu_\pi^2(D_s) - \mu_\pi^2(D)}{0.09\,\rm GeV^2} - 0.016\,\frac{\mu_G^2(D_s) - \mu_G^2(D)}{0.02\,\rm GeV^2} + 0.090\,\frac{\rho_D^3(D_s) - \rho_D^3(D)}{0.037\,\rm GeV^3} \\
&\quad + \underbrace{0.00}_{\rm dim-6,7,VIA} - 0.15\,\frac{\delta\tilde{B}_1^s}{0.02} + 0.15\,\frac{\delta\tilde{B}_2^s}{0.02} + 0.10\,\frac{\tilde{\epsilon}_1^s}{-0.04} + 0.09\,\frac{\tilde{\epsilon}_2^s}{0.04} \\
&\quad + 0.0007\,r_1^{qq} + 0.0005\,r_2^{qq} + 0.0118\,r_1^{sq} + 0.0087\,r_2^{sq} - 0.0001\,r_1^{qs} - 0.0007\,r_2^{qs}\,,
\end{aligned} \tag{4.18}$$

which is dominated by $SU(3)_F$-symmetry breaking corrections. The Darwin operator gives a sizable positive contribution to the ratio, which is partly compensated by the kinetic and the chromomagnetic terms. The spectator effects give in VIA a vanishing contribution, but deviations from VIA could sizably affect the ratio and also eye-contractions could yield a visible effect — here again a more precise determination of the non-perturbative parameters is necessary in order to make more profound statements.

[18] We note here a typo in the corresponding expression of this ratio in ref. [56]. In eq. (40) of ref. [56], the sign in front of the contribution of the kinetic operator has to be changed.

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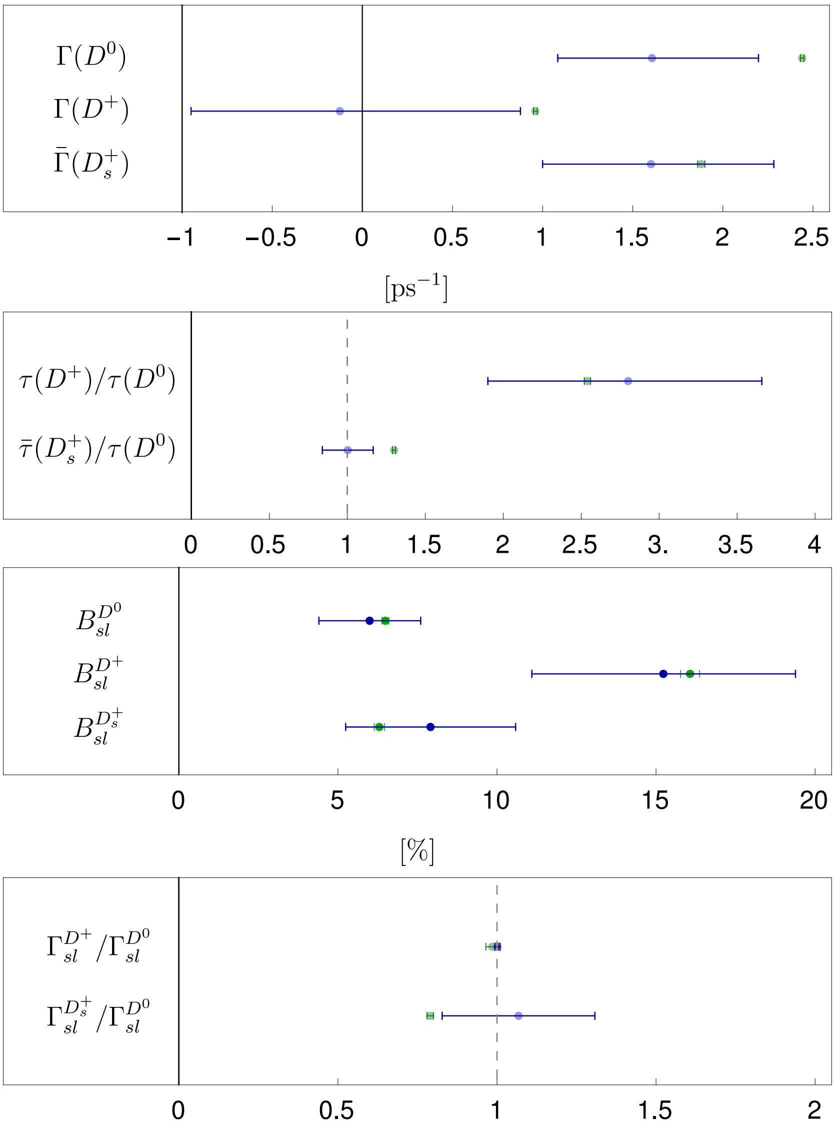

Figure 7. A comparison of the HQE prediction for the charm observables in the kinetic scheme (blue) with the corresponding experimental data (green).

5 Conclusions and outlook

We have performed a comprehensive study of charmed mesons lifetimes, of their ratios and of the inclusive semileptonic decay rates. Compared to previous studies we have included for the first time the sizeable contribution due to the Darwin term in the charm sector (with new expressions shown in appendix B), non-perturbative estimates of the eye-contractions [68] and strange quark mass corrections to the Bag parameters of the D_s^+ meson [68]. Moreover we have studied different mass schemes for the charm quark.

In particular our new study supersedes the one done by some of us in ref. [66] and we could clarify in the present work that the dimension-seven operators $\tilde{R}_{1,2}^q$ (introduced in ref. [66] as $P_{5,6}^q$) can be absorbed in the definition of the QCD decay constant. In contrast

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to the present work, ref. [66] could describe the experimental number for $\tau(D_s^+)/\tau(D^0)$ by fitting the Bag parameters in order to accommodate the experimental value of $\Gamma_{sl}^{D_s^+}/\Gamma_{sl}^{D^0}$ — this can be achieved by demanding e.g. for the difference $\tilde{B}_1^s - \tilde{B}_2^s \approx 0.032$, which is in slight tension with the HQET sum rule result $\tilde{B}_1^s - \tilde{B}_2^s = 0.004^{+0.019}_{-0.012}$ we are using here.

Ref. [80] also studies charm mesons, albeit restricting exclusively to a LO-QCD analysis. Different quark mass schemes can only be distinguished starting from NLO-QCD onwards — working at leading order in QCD only, the different quark mass schemes used in our work would induce a relative uncertainty to the free-quark decay of $(1.48/1.27)^5 \approx 2.15$, which is clearly not acceptable. Moreover, as can be nicely read from table 9, NLO-QCD corrections to the four-quark operators can dominate over the LO contribution. Using exclusively LO-QCD expressions is thus a far too crude and unnecessary assumption in the charm sector. Furthermore, ref. [80] considers only the $\overline{\text{MS}}$ scheme for the charm quark mass and obviously the recently determined Darwin term and the eye-contractions could not have been included, since they were not known at that point of time.

Finally, there is also some overlap with two recent studies of the B_c lifetime [88, 89]. The first paper [88] considers also the free charm quark decay Γ_3 and the second one [89] the total D-meson decay rate without the free charm quark decay, i.e. $\Gamma(D) - \Gamma_3$. For Γ_3 the authors of ref. [88] consider three quark mass schemes: $\overline{\text{MS}}$, $1S$ and the meson mass scheme. They find in table 3 and 4 of their paper values in the $\overline{\text{MS}}/1S$ scheme which are slightly smaller/slightly larger than our values in table 5: $1.0\,\text{ps}^{-1}$ vs $1.3\,\text{ps}^{-1}$ and $1.7\,\text{ps}^{-1}$ vs $1.5\,\text{ps}^{-1}$. Since they in principle use the same NLO-QCD expressions as we do, we expect the slight difference to root in a different treatment of higher orders in α_s and some differences in the values of the input parameters. As in our study, they also find a relatively small effect due to a non-vanishing strange quark mass. In ref. [89] the authors determine the D-meson decay rate without the free charm quark decay. In that respect they consider all the corrections we also take into account, except contributions of dimension-seven and eye-contractions. In the end, when considering the D^+ meson they obtain values for the B_c-meson decay rate of around $3.3\,\text{ps}^{-1}$ (see table III of ref. [89]), compared to the experimental value of $1.961(35)\,\text{ps}^{-1}$. We naively estimate that an inclusion of the dimension-seven contribution to the D^+ meson decay rate would decrease their result by about $1.1\,\text{ps}^{-1}$, see table 10, and bring it in nice agreement with the measurement. On the other hand, these missing dimension-seven contributions might be partially compensated by the corresponding contributions to the B_c-meson decay rate. Here a further investigation might be necessary to clarify this point.

Our main numerical results are presented in tables 13, 14 and 15 and in figure 7. At a first glance all considered observables lie in the ballpark of the experimental results. In particular, we find good agreement with experiment for the ratio $\tau(D^+)/\tau(D^0)$, for the total D_s^+-meson decay rate, for the semileptonic rates of all three mesons D^0, D^+ and D_s^+, and for the semileptonic ratio $\Gamma_{sl}^{D^+}/\Gamma_{sl}^{D^0}$. The values obtained with different mass schemes for the charm quark overlap and the exclusive use of only one scheme might underestimate the uncertainties. Including higher orders in the perturbative QCD expansion will further alleviate the differences among the mass schemes. Looking, as a starting point, at the structure of the contributions to the total decay rates and neglecting spectator effects, we find that the NLO-QCD corrections to the free quark decay give the dominant correction (of the order of 50% of LO-QCD free quark decay), followed by the Darwin term (of the order of 30% of LO-QCD free quark decay). In the case of semileptonic decay rates the chromomagnetic term provides the dominant contribution (of the order of 30%), followed by the Darwin term and NLO-QCD corrections to the free quark decay. Turning now to the spectator effects, we find them to be tiny for $\Gamma(D^0)$, $\Gamma_{sl}^{D^0}$ and $\Gamma_{sl}^{D^+}$, but they provide visible corrections to $\Gamma_{sl}^{D_s^+}$ and $\Gamma(D_s^+)$ — in the latter case we find also sizable cancellations between dimension-six and dimension-seven contributions. For the D^+ meson we find, however, a huge negative Pauli interference contribution — with a substantial part stemming from the NLO-QCD corrections. Moreover, one observes here a significant cancellation between dimension-six and dimension-seven terms related to Pauli interference. The values of the HQET Bag parameters entering the spectator effects are close to the VIA values, deviations from the latter can, however, lead to sizable effects in $\Gamma(D^+)$ and to visible effects in $\Gamma(D^0)$, $\Gamma(D_s^+)$ and $\Gamma_{sl}^{D_s^+}$. Based on the HQET sum rule results [68] we find that eye-contractions constitute only subleading corrections, they might, however, turn out to be relevant for $\Gamma_{sl}^{D_s^+}/\Gamma_{sl}^{D^0}$ and $\tau(D_s^+)/\tau(D^0)$, when more precise non-perturbative estimates will become available. In the end, the total decay rates of the D^0 and D^+ mesons stay underestimated in our HQE approach and we suspect that this is due to missing higher-order QCD corrections to the free charm quark decay and the Pauli interference contribution. For the $SU(3)_F$ breaking ratios $\tau(D_s^+)/\tau(D^0)$ and $\Gamma_{sl}^{D_s^+}/\Gamma_{sl}^{D^0}$ our predictions lie closer to one than experiment. This might originate from the poor knowledge of the non-perturbative parameters μ_G^2, μ_π^2 and ρ_D^3 in the D^0 and D_s^+ systems, as discussed in section 3.

Our numerical analysis shows that there are many possibilities for future improvements of the HQE predictions in the charm sector:

- $\Gamma_3^{(2)}$: NNLO-QCD [27–38] contributions to the semileptonic decays have been found to be large and NLO-QCD corrections to the non-leptonic decay rates represent one of the dominant corrections. Moreover we observe that at NLO-QCD there is pronounced cancellation — see eq. (2.17) and eq. (2.18) — which might not be necessarily present at NNLO-QCD. Thus a first determination of the NNLO-QCD corrections to the non-leptonic decays might have some sizable impact on the numerical studies of the total decay rates.

- $\Gamma_5^{(1)}$: cancellations in the coefficient c_G for the total decay rate, shown in eq. (6) and figure 3 lead to large uncertainties, even the sign of these corrections is ambiguous. Here a determination of the QCD-corrections to the coefficient c_G for the non-leptonic case might considerably improve the situation.

- $\Gamma_6^{(1)}$: the Wilson oefficients of the Darwin operator are large, therefore QCD corrections for the non-leptonic case might be important.

- $\Gamma_{7,8}^{(0)}$: since the dimension-six contribution is sizable, the LO-QCD determination of the dimension-seven and dimension-eight contributions with two-quark operators for the non-leptonic case might bring some additional insights on the convergence of the HQE in the charm sector.

- $\tilde{\Gamma}_6^{(2)}$, $\tilde{\Gamma}_7^{(1)}$: Pauli interference dominates the total decay rate of the D^+ meson. Currently $\tilde{\Gamma}_6^{(0)}$, $\tilde{\Gamma}_6^{(1)}$ and $\tilde{\Gamma}_7^{(0)}$ are known and their numerical values were found to be huge, see e.g. table 9. Thus further QCD corrections will turn out to be very important.

- $\tilde{\Gamma}_8^{(0)}$: since the four-quark dimension-six contribution can dominate the total decay rate and $\tilde{\Gamma}_7^{(0)}$ is also very sizable, a further study of the dimension-eight contributions might bring further insights on the convergence of the HQE in the charm sector, see refs. [57, 58].

- More precise determinations for the parameters μ_G^2, μ_π^2 and ρ_D^3 — both for the D^0 and the D_s^+ mesons: the Darwin term and the chromomagnetic term provide large corrections to the decay rates and they are poorly known — in particular the size of $\mathrm{SU}(3)_F$ breaking effects is largely unknown. An experimental determination of μ_G^2, μ_π^2 and ρ_D^3 from fits to semileptonic D^+-, D^0- and D_s^+-meson decays — as done in the B system, see e.g. ref. [81] — would be very desirable. This might be doable at BESIII, Belle II and a future tau-charm factory. Moreover, new theoretical determinations, e.g. via lattice simulations or sum rules could be undertaken.

- Independent lattice determination of the matrix elements of the four-quark operators of dimension-six: here we have currently only HQET sum rule determinations [66, 68] or outdated lattice results [90, 91].

- A first non-perturbative determination of the matrix elements of the dimension-seven four-quark operators in order to test the validity of VIA. A similar endeavour has already been performed for B_s mixing [92].

Overall, we find that the HQE can describe inclusive charm observables, in which no pronounced GIM cancellation arises,[19] albeit with very large uncertainties. We therefore do not observe a clear signal for a breakdown of the HQE in the charm sector or of violations of quark hadron duality, see e.g. ref. [94] and we presented a long list of potential theoretical improvements, which might shed further light into the convergence properties of the HQE in the charm sector.

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[19]See e.g. ref. [93] for a recent discussion of the extreme GIM cancellations in mixing of neutral D mesons.

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A Numerical input

We use five-loop running for $\alpha_s(\mu)$ [7] with four active flavours at the scale $\mu \sim m_c$, and the most recent value [1]

$$\alpha_s(M_Z) = 0.1179 \pm 0.0010. \tag{A.1}$$

For the CKM matrix elements we apply the standard parametrisation in terms of $\theta_{12}, \theta_{13}, \theta_{22}, \delta$ and use as an input [95] (online update)

$$|V_{us}| = 0.224834^{+0.000252}_{-0.000059}\,, \tag{A.2}$$

$$\frac{|V_{ub}|}{|V_{cb}|} = 0.088496^{+0.001885}_{-0.002244}\,, \tag{A.3}$$

$$|V_{cb}| = 0.04162^{+0.00026}_{-0.00080}\,, \tag{A.4}$$

$$\delta = \left(65.80^{+0.94}_{-1.29}\right)^{\circ}. \tag{A.5}$$

For the c-quark mass, we use different values depending of the scheme. In the $\overline{\text{MS}}$-scheme we take [1]:

$$\overline{m}_c(\overline{m}_c) = (1.27 \pm 0.02)\,\text{GeV}, \tag{A.6}$$

in the kinetic scheme we employ (at NLO) [14]:

$$m_c^{\text{kin}}(0.5\,\text{GeV}) = (1.363 \pm 0.02)\,\text{GeV}, \tag{A.7}$$

and in the $1S$-scheme (see eq. (1.13)) we use $m_c^{1S} = 1.44\,\text{GeV}$ [7].

For the s-quark mass we take the value [1]

$$m_s = \left(93^{+11}_{-5}\right)\,\text{MeV}. \tag{A.8}$$

The masses of D-mesons are known very precisely [1]:

$$M_{D^0} = 1.86493\,\text{GeV}, \qquad M_{D^+} = 1.86965\,\text{GeV}, \qquad M_{D_s^+} = 1.96834\,\text{GeV}.$$

The values of the non-perturbative parameters used in the analysis are shown in tables 16 and 17.

B Expressions for the coefficients of the Darwin operator

The coefficients $C^{(q_1 q_2)}_{\rho_D, mn}(\rho, \mu_0)$ including full $\rho = m_s^2/m_c^2$ dependence are given by the expressions:

$$\mathcal{C}^{(d\bar{d})}_{\rho_D,11} = 6 + 8\,\log\left(\frac{\mu_0^2}{m_c^2}\right), \tag{B.1}$$

Parameter	$D^{+,0}$	Source	D_s^+	Source
f_D [GeV]	0.2120 ± 0.0007	Lattice QCD [78]	0.2499 ± 0.0005	Lattice QCD [78]
$\mu_\pi^2(D)$ [GeV^2]	0.48 ± 0.20	Exp. fit [82] and HQ symmetry	0.57 ± 0.23	Exp. fit [44], SU(3)$_f$-breaking [86] and HQ symmetry
$\mu_G^2(D)$ [GeV^2]	0.34 ± 0.10	Spectroscopy relations [83]	0.36 ± 0.10	Spectroscopy relations [83]
$\rho_D^3(D)$ [GeV^3]	0.082 ± 0.035	Exp. fit [82] and E.O.M relation	0.119 ± 0.052	Exp. fit [82] and E.O.M relation

Table 16. Numerical values of the non-perturbative parameters used in our analysis.

HQET, $\mu_0 = 1.5\,\text{GeV}$	$\tilde{B}_1$	$\tilde{B}_2$	$\tilde{\epsilon}_1$	$\tilde{\epsilon}_2$
$D^{+,0}$	$1.0026^{+0.0198}_{-0.0106}$	$0.9982^{+0.0052}_{-0.0066}$	$-0.0165^{+0.0209}_{-0.0346}$	$-0.0004^{+0.0200}_{-0.0326}$
D_s^+	$1.0022^{+0.0185}_{-0.0099}$	$0.9983^{+0.0052}_{-0.0067}$	$-0.0104^{+0.0202}_{-0.0330}$	$0.0001^{+0.0199}_{-0.0324}$
HQET, $\mu_0 = 1.5\,\text{GeV}$	$\tilde{\delta}_1$	$\tilde{\delta}_2$	$\tilde{\delta}_3$	$\tilde{\delta}_4$
$\langle D_q \vert \tilde{O}^q \vert D_q \rangle$	$0.0026^{+0.0142}_{-0.0092}$	$-0.0018^{+0.0047}_{-0.0072}$	$-0.0004^{+0.0015}_{-0.0024}$	$0.0003^{+0.0012}_{-0.0008}$
$\langle D_s \vert \tilde{O}^q \vert D_s \rangle$	$0.0025^{+0.0144}_{-0.0093}$	$-0.0018^{+0.0047}_{-0.0072}$	$-0.0004^{+0.0015}_{-0.0024}$	$0.0003^{+0.0012}_{-0.0008}$
$\langle D_q \vert \tilde{O}^s \vert D_q \rangle$	$0.0023^{+0.0140}_{-0.0091}$	$-0.0017^{+0.0046}_{-0.0070}$	$-0.0004^{+0.0015}_{-0.0023}$	$0.0003^{+0.0012}_{-0.0008}$

Table 17. Numerical values of the HQET Bag parameters [66, 68] evaluated through a traditional HQET sum rule.

$$\mathcal{C}^{(d\bar{d})}_{\rho_D,12} = -\frac{34}{3}, \tag{B.2}$$

$$\mathcal{C}^{(d\bar{d})}_{\rho_D,22} = 6 + 8\,\log\left(\frac{\mu_0^2}{m_c^2}\right), \tag{B.3}$$

$$\begin{aligned}\mathcal{C}^{(d\bar{s})}_{\rho_D,11} = {} & \frac{2}{3}(1-\rho)\Big[9 + 11\rho - 12\rho^2\log(\rho) - 24\left(1-\rho^2\right)\log(1-\rho) - 25\rho^2 + 5\rho^3\Big] \\ & + 8\,(1-\rho)(1-\rho^2)\log\left(\frac{\mu_0^2}{m_c^2}\right),\end{aligned} \tag{B.4}$$

$$\begin{aligned}\mathcal{C}^{(d\bar{s})}_{\rho_D,12} = {} & -\frac{2}{3}\Big[17 + 12\rho\left(5 + 2\rho - 2\rho^2\right)\log(\rho) + 48(1-\rho)(1-\rho^2)\log(1-\rho) \\ & - 26\rho + 18\rho^2 - 38\rho^3 + 5\rho^4 + 24\,\rho\,(1+\rho-\rho^2)\log\left(\frac{\mu_0^2}{m_c^2}\right)\Big],\end{aligned} \tag{B.5}$$

$$\begin{aligned}\mathcal{C}^{(d\bar{s})}_{\rho_D,22} = {} & \frac{2}{3}(1-\rho)\Big[9 + 11\rho - 12\rho^2\log(\rho) - 24\left(1-\rho^2\right)\log(1-\rho) - 25\rho^2 + 5\rho^3\Big], \\ & + 8\,(1-\rho)(1-\rho^2)\log\left(\frac{\mu_0^2}{m_c^2}\right),\end{aligned} \tag{B.6}$$

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$$\mathcal{C}^{(s\bar{d})}_{\rho_D,11} = \frac{2}{3}\Bigg[9 - 16\rho - 12\rho^2 + 16\rho^3 - 5\rho^4 + 12\log\left(\frac{\mu_0^2}{m_c^2}\right)\Bigg], \tag{B.7}$$

$$\begin{aligned}\mathcal{C}^{(s\bar{d})}_{\rho_D,12} &= -\frac{2}{3}\Bigg[17 + 12\,\rho^2\,(3-\rho)\log(\rho) - 24(1-\rho)^3\log(1-\rho)\\ &\quad -50\rho + 90\rho^2 - 54\rho^3 + 5\rho^4 - 12\rho\,(3 - 3\rho + \rho^2)\log\left(\frac{\mu_0^2}{m_c^2}\right)\Bigg],\end{aligned} \tag{B.8}$$

$$\begin{aligned}\mathcal{C}^{(s\bar{d})}_{\rho_D,22} &= \frac{2}{3}(1-\rho)\Bigg[9 + 11\rho - 12\rho^2\log(\rho) - 24\left(1-\rho^2\right)\log(1-\rho)\\ &\quad -25\rho^2 + 5\rho^3 + 12\,(1-\rho^2)\log\left(\frac{\mu_0^2}{m_c^2}\right)\Bigg],\end{aligned} \tag{B.9}$$

$$\begin{aligned}\mathcal{C}^{(s\bar{s})}_{\rho_D,11} &= \frac{2}{3}\Bigg[\sqrt{1-4\rho}\left(17 + 8\rho - 22\rho^2 - 60\rho^3\right) - 4\left(2 - 3\rho + \rho^3\right) +\\ &\quad -12\left(1 - \rho - 2\rho^2 + 2\rho^3 + 10\rho^4\right)\log\left(\frac{1+\sqrt{1-4\rho}}{1-\sqrt{1-4\rho}}\right)\\ &\quad -12\,(1-\rho)(1-\rho^2)\left(\log(\rho) - \log\left(\frac{\mu_0^2}{m_c^2}\right)\right)\Bigg],\end{aligned} \tag{B.10}$$

$$\begin{aligned}\mathcal{C}^{(s\bar{s})}_{\rho_D,12} &= \frac{2}{3}\Bigg[\sqrt{1-4\rho}\left(-33 + 24\,\log(\rho) - 24\,\log(1-4\rho) + 46\rho - 106\rho^2 - 60\rho^3\right)\\ &\quad +12\left(3 - 2\rho + 4\rho^2 - 16\rho^3 - 10\rho^4\right)\log\left(\frac{1+\sqrt{1-4\rho}}{1-\sqrt{1-4\rho}}\right)\\ &\quad +4\,(1-\rho)^2\,(4 + 3(1-\rho)\log(\rho) - \rho)\\ &\quad -12\,\left(1 - \sqrt{1-4\rho} - 3\rho + 3\rho^2 - \rho^3\right)\,\log\left(\frac{\mu_0^2}{m_c^2}\right)\Bigg],\end{aligned} \tag{B.11}$$

$$\begin{aligned}\mathcal{C}^{(s\bar{s})}_{\rho_D,22} &= \frac{2}{3}\Bigg[\sqrt{1-4\rho}\left(9 + 24\,\log(\rho) - 24\,\log(1-4\rho) + 22\rho - 34\rho^2 - 60\rho^3\right)\\ &\quad +24\left(1 - 2\rho - \rho^2 - 2\rho^3 - 5\rho^4\right)\log\left(\frac{1+\sqrt{1-4\rho}}{1-\sqrt{1-4\rho}}\right)\\ &\quad +12\,\sqrt{1-4\rho}\,\log\left(\frac{\mu_0^2}{m_c^2}\right)\Bigg].\end{aligned} \tag{B.12}$$

The numerical values of the above coefficients for $\rho = 0.006$ are shown in table 18.

C Parametrisation of the matrix element of four-quark operators

The matrix elements of the dimension-six operators in QCD are parametrised in the following way

$$\langle D_q|O_i^q\,|D_q\rangle = A_i\,f_{D_q}^2 m_{D_q}^2\,B_i^q, \tag{C.1}$$

$$\langle D_q|O_i^{q'}\,|D_q\rangle = A_i\,f_{D_q}^2 m_{D_q}^2\,\delta_i^{qq'}, \qquad q \neq q'\,, \tag{C.2}$$

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	$3\,C_1^2$	$2\,C_1C_2$	$3\,C_2^2$
$c \to d\bar{d}u$	6	-11.33	6
$c \to d\bar{s}u$	6.10	-9.81	6.10
$c \to s\bar{d}u$	5.94	-11.23	6.10
$c \to s\bar{s}u$	6.04	-9.70	6.21

Table 18. Numerical values of $C^{(q_1q_2)}_{\rho_D,nm}$ for $\rho = 0.006$ and $\mu_0 = m_c$.

where

$$A_1^q = A_3^q = 1, \qquad A_2^q = A_4^q = \frac{m_D^2}{(m_c + m_q)^2}.$$

In VIA the Bag parameters reduce to $B_1^q = B_2^q = 1$ and $B_3^q = \epsilon_1^q = 0$, $B_4^q = \epsilon_2^q = 0$ and all $\delta_i^{qq'} = 0$.

The matrix elements of the dimension-seven four-quark operators in eqs. (2.61)–(2.69) in HQET are parametrised in the following way:

$$\langle D_q|\tilde{P}_1^q|D_q\rangle = -m_q F^2(\mu_0)\, m_D\, \tilde{B}^q_{P,1}\,, \tag{C.3}$$
$$\langle D_q|\tilde{P}_2^q|D_q\rangle = -F^2(\mu_0)\, m_D\, \bar{\Lambda}\, \tilde{B}^q_{P,2}\,, \tag{C.4}$$
$$\langle D_q|\tilde{P}_3^q|D_q\rangle = -F^2(\mu_0)\, m_D\, \bar{\Lambda}\, \tilde{B}^q_{P,3}\,, \tag{C.5}$$
$$\langle D_q|\tilde{R}_1^q|D_q\rangle = -F^2(\mu_0)\, m_D\, (\bar{\Lambda} - m_q)\, \tilde{B}^q_{R,1}\,, \tag{C.6}$$
$$\langle D_q|\tilde{R}_2^q|D_q\rangle = F^2(\mu_0)\, m_D\, (\bar{\Lambda} - m_q)\, \tilde{B}^q_{R,1}\,, \tag{C.7}$$

with $\bar{\Lambda} = m_D - m_c$, and

$$\langle D_q|\tilde{M}^q_{1,\pi}|D_q\rangle = 2\,F^2(\mu_0)\, m_D\, G_1(\mu_0)\, \tilde{L}^q_{1,\pi}\,, \tag{C.8}$$
$$\langle D_q|\tilde{M}^q_{2,\pi}|D_q\rangle = 2\,F^2(\mu_0)\, m_D\, G_1(\mu_0)\, \tilde{L}^q_{2,\pi}\,, \tag{C.9}$$
$$\langle D_q|\tilde{M}^q_{1,G}|D_q\rangle = 12\,F^2(\mu_0)\, m_D\, G_2(\mu_0)\, \tilde{L}^q_{1,G}\,, \tag{C.10}$$
$$\langle D_q|\tilde{M}^q_{2,G}|D_q\rangle = 12\,F^2(\mu_0)\, m_D\, G_2(\mu_0)\, \tilde{L}^q_{2,G}\,, \tag{C.11}$$

and similar expressions for the colour-octet operators. Again, in VIA, the dimension-seven Bag parameters are $\tilde{B}^q_{P,i} = 1$, $\tilde{B}^q_{R,i} = 1$, and $\tilde{L}^q_{1,\pi} = 1$, $\tilde{L}^q_{1,G} = 1$ and the corresponding colour-octet Bag parameters vanish.

The expressions in eqs. (C.3)–(C.7) can be obtained using a general parametrisation of matrix elements of the HQET quark currents with a heavy pseudo-scalar meson $\mathcal{M}$ (see e.g. ref. [71]):

$$\langle 0|\bar{q}\,\Gamma\, h_v|M(v)\rangle = \frac{i}{2} F(\mu)\,\mathrm{Tr}[\Gamma \mathcal{M}(v)]\,, \tag{C.12}$$
$$\langle 0|\bar{q}\,\Gamma\, iD_\alpha h_v|M(v)\rangle = -\frac{i}{6}(\bar{\Lambda} - m_q) F(\mu)\,\mathrm{Tr}[(v_\alpha + \gamma_\alpha)\Gamma \mathcal{M}(v)]\,, \tag{C.13}$$
$$\langle 0|\bar{q}(-i\overleftarrow{D}_\alpha)\,\Gamma\, h_v|M(v)\rangle = -\frac{i}{6} F(\mu)\,\mathrm{Tr}[((4\bar{\Lambda} - m_q)v_\alpha + (\bar{\Lambda} - m_q)\gamma_\alpha)\Gamma \mathcal{M}(v)], \tag{C.14}$$

and for the non-local operators:

$$\langle 0|i\int d^4y\, T\left[(\bar{q}\,\Gamma\, h_v)(0), (\bar{h}_v(iD)^2 h_v)(y)\right]|\mathcal{M}(v)\rangle = F(\mu)\, G_1(\mu)\, \mathrm{Tr}[\Gamma\mathcal{M}(v)], \tag{C.15}$$

$$\langle 0|i\int d^4y\, T\left[(\bar{q}\,\Gamma\, h_v)(0), \frac{1}{2}g_s\left(\bar{h}_v\sigma_{\alpha\beta}G^{\alpha\beta}h_v\right)(y)\right]|\mathcal{M}(v)\rangle = 6\, F(\mu)\, G_2(\mu)\, \mathrm{Tr}[\Gamma\mathcal{M}(v)], \tag{C.16}$$

where Γ is a generic Dirac structure, and

$$\mathcal{M}(v) = -\sqrt{m_D}\,\frac{(1+\not{v})}{2}\gamma_5\,. \tag{C.17}$$

Since we are limited to LO-QCD for the dimension-seven contribution, we can just replace the HQET decay constant $F(\mu)$ by the full QCD one f_D, using $F(\mu) = f_D\sqrt{m_D}$.

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