ALMOST POSITIVE CURVATURE ON THE GROMOLL-MEYER SPHERE

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Abstract. Gromoll and Meyer have represented a certain exotic 7-sphere Σ^7 as a biquotient of the Lie group $G = Sp(2)$. We show for a 2-parameter family of left invariant metrics on G that the induced metric on Σ^7 has strictly positive sectional curvature at all points outside four subvarieties of codimension ≥ 1 which we describe explicitly.

1. Introduction

Let $G = Sp(2)$ be the Lie group of unitary quaternionic 2×2-matrices. Consider the subgroup $U \subset G \times G$,

$$U = \{((\begin{smallmatrix} q & \\ & 1 \end{smallmatrix}), (\begin{smallmatrix} q & \\ & q \end{smallmatrix}))\,;\ q \in Sp(1)\}, \tag{1.1}$$

which acts on G by left and right translations. D. Gromoll and W. Meyer [5] have shown that this action is free and that the orbit space $M = G/U$ is a smooth manifold which is an exotic 7-sphere (homeomorphic but not diffeomorphic to the standard 7-sphere). If G is equipped with a Riemannian metric of nonnegative sectional curvature whose isometry group contains U, then by O'Neill's formula [1] the orbit space $M = G/U$ inherits a Riemannian metric of nonnegative sectional curvature. Thus starting with the bi-invariant metric on G, Gromoll and Meyer constructed a metric of nonnegative sectional curvature on the exotic sphere M. In fact the curvature is strictly positive on some nonempty open subset of M. However, as was observed by F. Wilhelm [7], there is also an open subset with zero curvature planes in the tangent space of each of its points. But Wilhelm constructed another U-invariant metric on $Sp(2)$ (neither left nor right invariant) for which the curvature of M is strictly positive outside a subset of measure zero in M ("almost positive curvature"). In [4] the same fact was claimed for a much simpler and left invariant metric on $Sp(2)$; however, as was pointed out by the second author, the proof contains a serious mistake (see Remark 3 at the end of the present paper). The purpose of our paper is to correct this error. In fact we prove the following result, some ideas of which go back to [3] (see Theorem 4.6 for details):

Theorem 1.1. *There is a left invariant and U-invariant metric on $G = Sp(2)$ such that the induced metric on $M = G/U$ has strictly positive curvature outside a finite union of subvarieties of codimension ≥ 1. The zero curvature set $Z \subset M$ can be explicitly determined.*

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2. Cheeger metrics on Lie groups

On each Riemannian manifold, let us denote

$$\begin{aligned} \kappa(X,Y) &= \langle R(X,Y)Y,X\rangle, \\ \sec(X,Y) &= \kappa(X,Y)/|X\wedge Y|^2 \end{aligned} \tag{2.1}$$

for any two tangent vectors X,Y; the second expression is the sectional curvature of the plane σ spanned by X,Y.

Let G be a Lie group with a left invariant metric $\langle\ ,\ \rangle$ of nonnegative sectional curvature. Suppose that the metric is also right invariant with respect to a compact subgroup $K\subset G$, hence the induced metric on K is bi-invariant. The Lie algebras of G and K will be denoted $\mathfrak{g}$ and $\mathfrak{k}$. We may contract the metric on G in the direction of the K-cosets by viewing G as the homogeneous space $(G\times K)/\Delta K$ (where $\Delta K=\{(k,k);\ k\in K\}$) and choosing the metric induced from the Riemannian product metric on $G\times sK$ (*Cheeger contraction*, cf. [2], [1]) where sK is K with metric scaled by $s>0$. A vector $(X,X')\in\mathfrak{g}\times\mathfrak{k}$ is perpendicular to the ΔK-orbit ("horizontal") iff $X+sX'\perp\mathfrak{k}$, i.e. $X'=-s^{-1}X_{\mathfrak{k}}$ where $X_{\mathfrak{k}}$ is the $\mathfrak{k}$-projection of X. Using the Riemannian submersion $G\times K\to G$, $(g,k)\mapsto gk^{-1}$, a horizontal vector $(X,-s^{-1}X_{\mathfrak{k}})\in\mathfrak{g}\times\mathfrak{k}$ is mapped onto $X+s^{-1}X_{\mathfrak{k}}=X_\perp+(1+s^{-1})X_{\mathfrak{k}}\in\mathfrak{g}$ where $X_\perp=X-X_{\mathfrak{k}}\in\mathfrak{k}^\perp$. Vice versa, the horizontal lift of $X=X_\perp+X_{\mathfrak{k}}\in\mathfrak{g}$ is the horizontal vector

$$\begin{aligned} \widehat{X} &= (\tilde X,-s^{-1}\tilde X_{\mathfrak{k}}), \quad \text{where} \\ \tilde X &= X_\perp+\frac{s}{s+1}X_{\mathfrak{k}}. \end{aligned} \tag{2.2}$$

Thus the new (left invariant) metric is

$$\begin{aligned} \langle X,Y\rangle_1 &= \langle \widehat X,\widehat Y\rangle \\ &= \langle \tilde X,\tilde Y\rangle + s\,\langle s^{-1}\tilde X_{\mathfrak{k}}, s^{-1}\tilde Y_{\mathfrak{k}}\rangle \\ &= \langle \tilde X,\tilde Y\rangle + s^{-1}\langle \tilde X_{\mathfrak{k}},\tilde Y_{\mathfrak{k}}\rangle \\ &= \langle \tilde X_\perp,\tilde Y_\perp\rangle + s^{-1}(s+1)\langle \tilde X_{\mathfrak{k}},\tilde Y_{\mathfrak{k}}\rangle \\ &= \langle X_\perp,Y_\perp\rangle + s(s+1)^{-1}\langle X_{\mathfrak{k}},Y_{\mathfrak{k}}\rangle \\ &= \langle \tilde X, Y\rangle. \end{aligned} \tag{2.3}$$

For the curvature terms we have

$$\kappa(\widehat X,\widehat Y)=\kappa(\tilde X,\tilde Y)+s^{-3}\kappa(\tilde X_{\mathfrak{k}},\tilde Y_{\mathfrak{k}}). \tag{2.4}$$

Since all terms are nonnegative, the left hand side vanishes if and only if both summands on the right are zero. Thus a plane σ spanned by $X,Y\in\mathfrak{g}$ has zero curvature in the new metric, $\sec_1(\sigma)=0$, if and only if $\sec(\tilde\sigma)=0$ and $[X_{\mathfrak{k}},Y_{\mathfrak{k}}]=0$.[1]

Example 1. Suppose that the initial metric $\langle\ ,\ \rangle$ on G is bi-invariant. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the orthogonal decomposition. Consider the above metric

$$\langle X,Y\rangle_1=\langle X_{\mathfrak{p}},Y_{\mathfrak{p}}\rangle+\tilde s\langle X_{\mathfrak{k}},Y_{\mathfrak{k}}\rangle \tag{2.5}$$

with $\tilde s=\frac{s}{s+1}$. Then $\sec(\tilde\sigma)=0 \iff [\tilde X,\tilde Y]=0$, and hence $\sec_1(\sigma)=0 \iff$

$$[\tilde X,\tilde Y]=0,\quad [X_{\mathfrak{k}},Y_{\mathfrak{k}}]=0.$$

[1]The "if" statement is not obvious because of the nonnegative O'Neill term. However, in all our examples starting with a bi-invariant metric on some Lie group, the vanishing of the curvature implies that the O'Neill term also vanishes, see [3], p. 29f, Equations (1) - (4) or [8], [6]

If (G, K) is a symmetric pair, i.e. the orthogonal complement $\mathfrak{p} \subset \mathfrak{g}$ satisfies $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, then $[\tilde{X}, \tilde{Y}]_{\mathfrak{k}} = [\tilde{X}_{\mathfrak{k}}, \tilde{Y}_{\mathfrak{k}}] + [\tilde{X}_{\mathfrak{p}}, \tilde{Y}_{\mathfrak{p}}]$ and $[\tilde{X}, \tilde{Y}]_{\mathfrak{p}} = [\tilde{X}_{\mathfrak{k}}, \tilde{Y}_{\mathfrak{p}}] + [\tilde{X}_{\mathfrak{p}}, \tilde{Y}_{\mathfrak{k}}]$, hence $\sec_1(\tilde{\sigma}) = 0 \iff$

$$0 = [X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = [X_{\mathfrak{p}}, Y_{\mathfrak{p}}] = [X_{\mathfrak{k}}, Y_{\mathfrak{p}}] + [X_{\mathfrak{p}}, Y_{\mathfrak{k}}] = [X, Y]. \tag{2.6}$$

Example 2. Let $G \supset K \supset H$ a chain of subgroups and suppose that both (G, K) and (K, H) are symmetric pairs. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and $\mathfrak{k} = \mathfrak{h} + \mathfrak{q}$ be the corresponding decompositions. Start with the metric $\langle\ ,\ \rangle_1$ defined by Example 1, depending on a parameter $s > 0$, and define the metric $\langle\ ,\ \rangle_2$ by Cheeger contraction along H (depending on a new parameter $t > 0$) as in (2.3) where K is replaced by H and $\langle\ ,\ \rangle_1$ takes the role of $\langle\ ,\ \rangle$:

$$\begin{aligned} \langle X, Y\rangle_2 &= \langle X_{\mathfrak{p}}, Y_{\mathfrak{p}}\rangle_1 + \langle X_{\mathfrak{q}}, Y_{\mathfrak{q}}\rangle_1 + \tilde{t}\langle X_{\mathfrak{h}}, Y_{\mathfrak{h}}\rangle_1 \\ &= \langle X_{\mathfrak{p}}, Y_{\mathfrak{p}}\rangle + \tilde{s}\langle X_{\mathfrak{q}}, Y_{\mathfrak{q}}\rangle + \tilde{s}\tilde{t}\langle X_{\mathfrak{h}}, Y_{\mathfrak{h}}\rangle \end{aligned} \tag{2.7}$$

with $\tilde{t} = \frac{t}{t+1}$. Then $\sec_2(\sigma) = 0 \iff \sec_1(\tilde{\sigma}) = 0$ and $[\tilde{X}_{\mathfrak{h}}, \tilde{Y}_{\mathfrak{h}}] = 0 \iff$

$$0 = [\tilde{X}, \tilde{Y}] = [\tilde{X}_{\mathfrak{k}}, \tilde{Y}_{\mathfrak{k}}] = [X_{\mathfrak{p}}, Y_{\mathfrak{p}}] = [X_{\mathfrak{q}}, Y_{\mathfrak{q}}] = [X_{\mathfrak{h}}, Y_{\mathfrak{h}}], \tag{2.8}$$

where $\tilde{X} = X_{\mathfrak{p}} + X_{\mathfrak{q}} + \frac{t}{t+1}X_{\mathfrak{h}}$ and $\tilde{Y} = Y_{\mathfrak{p}} + Y_{\mathfrak{q}} + \frac{t}{t+1}Y_{\mathfrak{h}}$ like in (2.2).

3. Zero curvature planes on $Sp(2)$

Let us consider the chain $G \supset K \supset H$ for $G = Sp(2)$, $K = Sp(1) \times Sp(1)$ and $H = \Delta Sp(1) = \{\left(\begin{smallmatrix} q & \\ & q\end{smallmatrix}\right);\ q \in Sp(1)\}$. The pairs (G, K) and (K, H) are symmetric, corresponding to the rank-one symmetric spaces S^4 and S^3. We start with the bi-invariant trace metric $\langle X, Y\rangle = \mathrm{Re}\ \mathrm{trace}\ X^*Y = \mathrm{Re} \sum \overline{x_{ij}}\, y_{ij}$ on $\mathfrak{g} = \mathfrak{sp}(2)$, apply Cheeger contraction in the K-direction and Cheeger-contract again in the H-direction, defining metrics $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$ as in Example 2.

Since $G/K = S^4$ as well as $K/H = S^3$ and $H = S^3$ have positive curvature, the vanishing of the last three brackets in (2.8) means the linear dependence of the factors. In particular we may assume $Y_{\mathfrak{p}} = 0$, i.e. $\tilde{Y} = \tilde{Y}_{\mathfrak{k}} = \begin{pmatrix} y_1 & \\ & y_2 \end{pmatrix}$.

Case 1. $X_{\mathfrak{p}} = 0$, i.e. $\tilde{X} = \tilde{X}_{\mathfrak{k}} = \begin{pmatrix} x_1 & \\ & x_2 \end{pmatrix}$.

From $[\tilde{X}_{\mathfrak{k}}, \tilde{Y}_{\mathfrak{k}}] = 0$ we obtain that the imaginary quaternions x_1, y_1 as well as x_2, y_2 are linearly dependent. Moreover, from $[X_{\mathfrak{q}}, Y_{\mathfrak{q}}] = [X_{\mathfrak{h}}, Y_{\mathfrak{h}}] = 0$ we see that also $x_1 \pm x_2$ and $y_1 \pm y_2$ are linearly dependent. Putting $y = y_1$, we may assume

$$\tilde{Y} = \begin{pmatrix} y & \\ & 0 \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} 0 & \\ & y \end{pmatrix}. \tag{3.1}$$

Case 2. $X_{\mathfrak{p}} \neq 0$, i.e. $X = \begin{pmatrix} x_1 & -\bar{x} \\ x & x_2 \end{pmatrix}$ with $x \neq 0$:

Then $0 = [\tilde{X}, \tilde{Y}]_{\mathfrak{p}} = [X_{\mathfrak{p}}, \tilde{Y}] \iff y_2 = xyx^{-1}$ for $y := y_1$,
and $0 = [\tilde{X}, \tilde{Y}]_{\mathfrak{k}} = [\tilde{X}_{\mathfrak{k}}, \tilde{Y}_{\mathfrak{k}}] \iff x_1 = \alpha y_1$, $x_2 = \beta y_2$ for real numbers α, β, hence

$$\tilde{Y} = \begin{pmatrix} y & \\ & xyx^{-1} \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} \alpha y & -\bar{x} \\ x & -\alpha xyx^{-1} \end{pmatrix} \tag{3.2}$$

where $x, y \in \mathbb{H}$, y imaginary and $\alpha \in \mathbb{R}$; we have $\beta = -\alpha$ since we require $\langle \tilde{X}, \tilde{Y} \rangle = 0$.

Case 2a. $\alpha = 0$, hence

$$\tilde{Y} = \begin{pmatrix} y & \\ & xyx^{-1} \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} & -\bar{x} \\ x & \end{pmatrix}. \tag{3.3}$$

Case 2b. $\alpha \neq 0$, hence (without loss of generality) $\alpha = 1$.

Then $[X_{\mathfrak{h}}, Y_{\mathfrak{h}}] = 0$ iff $y + xyx^{-1}$ and $y - xyx^{-1}$ are proportional which means $xyx^{-1} = \beta y$. Comparing the norms on both sides we get

$$xyx^{-1} = \pm\, y, \tag{3.4}$$

and

$$\tilde{Y} = Y_{\pm} = \begin{pmatrix} y & \\ & \pm y \end{pmatrix}, \quad \tilde{X} = X_{\pm} = \begin{pmatrix} y & -\bar{x} \\ x & \mp y \end{pmatrix}. \tag{3.5}$$

Lemma 3.1. *The zero curvature planes in $\mathfrak{g} = T_eG$ for $G = Sp(2)$ and the metric $\langle\ ,\ \rangle_2$ are spanned by $X, Y \in \mathfrak{g}$ with $\tilde{X}, \tilde{Y}$ given by either (3.1) or (3.3) or (3.5).*

4. The Gromoll-Meyer sphere

The metric $\langle\ ,\ \rangle_2$ on $G = Sp(2)$ is invariant under the action of U (cf. (1.1)) and hence it induces a metric on the orbit space $M = G/U$. Consider any

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G. \tag{4.1}$$

Since g is unitary, the rows and columns are unit vectors, in particular

$$|a|^2 + |b|^2 = 1. \tag{4.2}$$

The vertical space at g of the submersion $\pi : G \to G/U$ is $T_g(U.g) = gV_g$ with $V_g = \{v_g;\ v \in \mathrm{Im}\ \mathbb{H}\}$ where

$$v_g = g^{-1} \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix} g - \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} \bar{a}va - v & \bar{a}vb \\ \bar{b}va & \bar{b}vb - v \end{pmatrix} \tag{4.3}$$

Thus according to (2.3), a vector $gX \in T_gG$ is horizontal for π iff

$$0 = \langle X, v_g \rangle_2 = \langle \tilde{X}, v_g \rangle_1 \tag{4.4}$$

for all $v \in \mathrm{Im}\ \mathbb{H}$. Note that $\langle \tilde{X}, v_g \rangle_1$ is just a multiple of $\langle \tilde{X}, v_g \rangle$ if one of the components of $\tilde{X} = \tilde{X}_{\mathfrak{p}} + \tilde{X}_{\mathfrak{k}}$ are zero. Now we discuss which of the zero curvature planes in $G = Sp(2)$ (see Lemma 3.1) can be horizontal at any $g \in G$. By a slight abuse of language, a plane $\tilde{\sigma}$ spanned by $\tilde{X}, \tilde{Y} \in \mathfrak{g}$ will be called *horizontal at g* if

$$\langle \tilde{X}, v_g \rangle_1 = \langle \tilde{Y}, v_g \rangle_1 = 0 \tag{4.5}$$

for all $v \in \mathrm{Im}\ \mathbb{H}$.

Case 1.

Lemma 4.1. *A plane of type (3.1) is nowhere horizontal.*

Proof. $\langle \tilde{Y}, v_g \rangle = \langle y, \bar{a}va - v \rangle = \langle ay\bar{a} - y, v \rangle$ vanishes for all $v \in \mathrm{Im}\ \mathbb{H}$ iff $y = ay\bar{a}$, and likewise $\langle \tilde{X}, v_g \rangle$ vanishes for all v iff $y = by\bar{b}$. But this implies $|a| = |b| = 1$ in contradiction to (4.2). $\square$

Case 2a.

Lemma 4.2. *If a plane of type (3.3) is horizontal at g then either $a = 0$ or $b = 0$ or*

$$\det(I - \mathrm{Ad}(a^{-1}) - \mathrm{Ad}(b^{-1})) = 0. \tag{4.6}$$

Proof. The matrix $\tilde{X}$ is horizontal at g if and only if

$$0 = \langle \tilde{X}, v_g \rangle = 2\langle x, \bar{b}va \rangle = 2\langle bx\bar{a}, v \rangle \tag{4.7}$$

for all $v \in \mathrm{Im}\ \mathbb{H}$. This is equivalent to $bx\bar{a} \in \mathbb{R}$. Hence, either $a = 0$ or $b = 0$ or $bx = ra$ for some non-zero $r \in \mathbb{R}$. In the latter case we have, in particular

$$\mathrm{Ad}(bx) = \mathrm{Ad}(a), \tag{4.8}$$
$$\mathrm{Ad}(x) = \mathrm{Ad}(b^{-1})\,\mathrm{Ad}(a), \tag{4.9}$$

provided that $b \neq 0$. On the other hand, the matrix $\tilde{Y}$ is horizontal at g if and only if

$$0 = \langle \tilde{Y}, v_g \rangle = \langle |a|^2\,\mathrm{Ad}(a)y - y + |b|^2\,\mathrm{Ad}(bx)y - \mathrm{Ad}(x)y, v \rangle \tag{4.10}$$

for all $v \in \mathrm{Im}\ \mathbb{H}$. Since $y \in \mathrm{Im}\ \mathbb{H}$, this means

$$\begin{aligned} 0 &= |a|^2\,\mathrm{Ad}(a)y + |b|^2\,\mathrm{Ad}(bx)y - y - \mathrm{Ad}(x)y \qquad (4.11)\\ &\overset{(4.8)}{=} \mathrm{Ad}(a)y - y - \mathrm{Ad}(x)y \\ &\overset{(4.9)}{=} \mathrm{Ad}(a)y - y - \mathrm{Ad}(b^{-1})\,\mathrm{Ad}(a)y \end{aligned}$$

where we have also used $|a|^2 + |b|^2 = 1$ (4.2). If $a \neq 0$, we obtain from the last equality

$$\mathrm{Ad}(a)y \in \ker(I - \mathrm{Ad}(a^{-1}) - \mathrm{Ad}(b^{-1}))$$

and in particular

$$\det(I - \mathrm{Ad}(a^{-1}) - \mathrm{Ad}(b^{-1}) = 0. \tag{4.6}$$

□

Lemma 4.3. *There exists a plane of type (3.3) which is horizontal at g if and only if either (4.6) holds or*

$$a = 0, \quad |\,\mathrm{Im}\ b\,| \geq \frac{1}{2} \quad \text{or} \quad b = 0, \quad |\,\mathrm{Im}\ a\,| \geq \frac{1}{2}. \tag{4.12}$$

Proof. Suppose first $a, b \neq 0$. If (4.6) is satisfied, there is a non-zero $w \in \ker(I - \mathrm{Ad}(a^{-1}) - \mathrm{Ad}(b^{-1}))$. Then defining $y = \mathrm{Ad}(a^{-1})w$ and $x = b^{-1}a$, we obtain a horizontal plane of type (3.3) at g. The converse conclusion was done before.

Now suppose $b = 0$. Then $|a| = 1$ and Equation (4.11) becomes

$$\mathrm{Ad}(a)y - y = \mathrm{Ad}(x)y. \tag{4.13}$$

Geometrically, this equality means that $\mathrm{Ad}(a)$ rotates y by the angle $\frac{\pi}{3}$ (the three vectors $\mathrm{Ad}(a)y$, y, $\mathrm{Ad}(x)y$ form the sides of an equilateral triangle). Hence (4.13) has a solution (x, y) if and only if the rotation angle of the rotation $\mathrm{Ad}(a)$ is $\geq \frac{\pi}{3}$. This in turn is equivalent to $\sphericalangle(a, 1) \geq \frac{\pi}{6}$, i.e. $|\,\mathrm{Im}\ a| \geq \frac{1}{2}$. Inserting the solution (x, y) into (3.3) defines a horizontal plane of type (3.3). The case $a = 0$ is similar. □

Case 2b.

Lemma 4.4. *If a plane of type (3.5) is horizontal at g, then*

$$|a| = |b| = 1/\sqrt{2} \tag{4.14}$$

and $w := \mathrm{Im}\; a^{-1}b$ satisfies

$$\langle w - 2a^{-1}wa, w\rangle = 0. \tag{4.15}$$

Proof.

$$\langle v_g, Y_+\rangle = \langle \bar{a}va + \bar{b}vb - 2v, y\rangle = \langle v, ay\bar{a} + by\bar{b} - 2y\rangle \tag{4.16}$$

$$\langle v_g, Y_-\rangle = \langle \bar{a}va - \bar{b}vb, y\rangle = \langle v, ay\bar{a} - by\bar{b}\rangle \tag{4.17}$$

Thus $\langle \tilde{Y}, V_g\rangle = 0$ iff one of the following equations holds:

$$\begin{aligned} ay\bar{a} + by\bar{b} &= 2y, \\ ay\bar{a} - by\bar{b} &= 0. \end{aligned}$$

The first of these equations is impossible by the triangle inequality together with (4.2):

$$|ay\bar{a} + by\bar{b}| \le |ay\bar{a}| + |by\bar{b}| \le (|a|^2 + |b|^2)|y| = |y| < |2y|.$$

Thus we are left with the second equation,

$$ay\bar{a} = by\bar{b}, \tag{4.18}$$

which implies $|a| = |b|$.

Note that we have also shown that Y_+ cannot be horizontal. Thus we need only consider $\tilde{X} = X_-$ and $\tilde{Y} = Y_-$ in (3.5), and

$$xyx^{-1} = -y \tag{4.19}$$

which means that x is imaginary and nonzero with $x \perp y$.

Now let $\tilde{X}, \tilde{Y}$ be as above spanning $\tilde{\sigma}$. By the preceding remark we have

$$\tilde{Y} = \begin{pmatrix} y & \\ & -y \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} y & x \\ x & y \end{pmatrix} \tag{4.20}$$

with $y \perp x \in \mathrm{Im}\,\mathbb{H}$. Thus according to (2.5) we get for all $v \in \mathrm{Im}\,\mathbb{H}$

$$\begin{aligned} 0 = \langle \tilde{X}, v_g\rangle_1 &= 2\langle x, \bar{b}va\rangle + \tilde{s}\langle y, \bar{a}va + \bar{b}vb - 2v\rangle \\ &= 2\langle bx\bar{a}, v\rangle + \tilde{s}\langle ay\bar{a} + by\bar{b} - 2y, v\rangle \\ &= \langle bxa^{-1} + \tilde{s}(aya^{-1} - 2y), v\rangle, \end{aligned} \tag{4.21}$$

where we have used $2\bar{a} = a^{-1}$ and $ay\bar{a} = by\bar{b} = \frac{1}{2}aya^{-1}$ from (4.14) and (4.18). Putting $p = a^{-1}b/\tilde{s}$, we obtain

$$\mathrm{Im}\; apxa^{-1} = 2y - aya^{-1}. \tag{4.22}$$

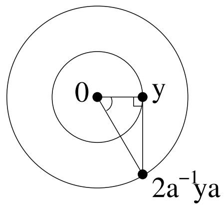
From $aya^{-1} = byb^{-1}$ we see $yp = py$, thus $p \in \mathbb{C}_y := \mathbb{R} + \mathbb{R}y$ and thus the left multiplication with p preserves $\mathbb{C}_y$ and $\mathbb{C}_y^\perp$. By (4.19) we have $x \in \mathbb{C}_y^\perp$ and therefore $px \in \mathbb{C}_y^\perp$. Conjugating (4.22) by a^{-1} we obtain

$$2a^{-1}ya - y = \mathrm{Im}\,(px) \perp y, \tag{4.23}$$

$$\langle 2a^{-1}ya - y, y\rangle = 0. \tag{4.24}$$

Since $w = \mathrm{Im}\; \tilde{s}p \in \mathbb{C}_y$ is a multiple of y, we may replace y by w in Equation (4.24) and obtain (4.15). □

Remark 1.



Geometrically, (4.24) means that the angle between y and $a^{-1}ya$ is $\pi/3 = 60^o$. Thus the rotation angle of $\mathrm{Ad}(a^{-1})$ (and of $\mathrm{Ad}(b^{-1})$, see (4.18)) must be $\geq \pi/3$, hence $\sphericalangle(1,a) \geq \pi/6$, or in other words,

$$\frac{|\,\mathrm{Im}\ a|}{|a|} \geq \frac{1}{2}. \tag{4.25}$$

Lemma 4.5. *Suppose that $a, b \in \mathbb{H}$ satisfy* (4.14), (4.15) *and* (4.25). *Then there exists a horizontal plane of type (3.5) at $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.*

Proof. First suppose that $\tilde{p} = a^{-1}b = \tilde{s}p$ is real which in view of (4.14) means $a = \pm b$. By (4.25), the rotation angle of $\mathrm{Ad}(a^{-1})$ is $\geq \pi/3$, hence there exists a nonzero $y \in \mathrm{Im}\ \mathbb{H}$ which is rotated precisely by the angle $\pi/3$ and thus satisfies (4.24). Put $x = 2a^{-1}ya - y \perp y$ and define $\tilde{X}, \tilde{Y}$ as in (4.20). This matrix pair is of type (3.5), and it is perpendicular to V_g by (4.17) and (4.21).

Now suppose that $w = \mathrm{Im}\ \tilde{p} \neq 0$; in this case (4.15) implies (4.25). Then we choose $y = w$ and $x = \mathrm{Im}\ \big(p^{-1}(2a^{-1}wa - w)\big)$, compare (4.23). Since $w - 2a^{-1}wa \in \mathbb{C}_y^\perp$ (it is imaginary and perpendicular to $w = y$), we also have $p^{-1}(w - 2a^{-1}wa) \in \mathbb{C}_y^\perp$, hence $x \perp y$ and thus $xyx^{-1} = -y$. Defining matrices $\tilde{X}, \tilde{Y}$ using (4.20), these are of type (3.5) and perpendicular to V_g by (4.17) and (4.21). $\square$

Remark 2. Clearly, the relations (4.6), (4.12), (4.14), (4.15) and (4.25) must be invariant under the action of U. In fact, if $u = (\left(\begin{smallmatrix} q & \\ & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} q & \\ & q \end{smallmatrix}\right))$, we have $u.g = \tilde{g} = \left(\begin{smallmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{smallmatrix}\right)$ with $\tilde{a} = qaq^{-1}$ and $\tilde{b} = qbq^{-1}$.

Now we have proved the following

Theorem 4.6. *Let $G = Sp(2)$ with the left invariant metric* (2.7) *and $U \subset G \times G$ defined by* (1.1). *The orbit space $M = G/U$ inherits a Riemannian metric such that the canonical projection $\pi : G \to M$ is a Riemannian submersion. Let*

$$Z = \{p \in M;\ \exists \sigma \subset T_pM :\ \sec(\sigma) = 0\}.$$

Then $Z = Z_1 \cup Z_2 \cup Z_3 \cup Z_4$ where

$$\begin{aligned}
\pi^{-1}Z_1 &= \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right);\ a, b \neq 0,\ \det(I - \mathrm{Ad}(a^{-1}) - \mathrm{Ad}(b^{-1})) = 0\},\\
\pi^{-1}Z_2 &= \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right);\ |a| = |b|,\ w := \mathrm{Im}\ a^{-1}b \perp w - 2a^{-1}wa,\ |\,\mathrm{Im}\ a| \geq |a|/2\},\\
\pi^{-1}Z_3 &= \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right);\ b = c = 0,\ |\,\mathrm{Im}\ a| \geq 1/2\},\\
\pi^{-1}Z_4 &= \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right);\ a = d = 0,\ |\,\mathrm{Im}\ b| \geq 1/2\},
\end{aligned}$$

where all matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ are supposed to belong to $Sp(2)$. $\square$

Remark 3. The mistake in [4] is in the third line of the proof of the Theorem, page 1166. The computation of $\langle v_g, X\rangle$ holds only for $X \in \mathfrak{k}$, but X may have a nonzero $\mathfrak{p}$-component as well. Thus the matrix X in (4), p. 1166, is too special and must be replaced with the more general $X = \begin{pmatrix} ry & -\bar{x} \\ x & -rxyx^{-1} \end{pmatrix}$ for arbitrary $r \in \mathbb{R}$, and instead of (5) $\mathrm{Im}\,(bx\bar{a}) = 0$ we obtain (5′) $\mathrm{Im}\,(bx\bar{a}) = r(y - ay\bar{a})$, while Equation (6) $(ay\bar{a} - y + bxyx^{-1}\bar{b} - xyx^{-1} = 0)$ remains unchanged. We have 15 variables, $(a, b) \in S^7$, $x \in \mathbb{H}$, $y \in \mathrm{Im}\,(\mathbb{H})$, $r \in \mathbb{R}$, with two arbitrary real constants (the lengths of x and y), and 6 constraint equations (5′) and (6) which reduce the number of free variables to 7. Thus the solution set is likely to project onto a subset with positive measure in the (a, b)-space S^7; this would imply that the metric considered in [4] fails to have almost positive curvature.

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