Partitioning H-Free Graphs of Bounded Diameter*

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────── **Abstract** ──────

A natural way of increasing our understanding of NP-complete graph problems is to restrict the input to a special graph class. Classes of H-free graphs, that is, graphs that do not contain some graph H as an induced subgraph, have proven to be an ideal testbed for such a complexity study. However, if the forbidden graph H contains a cycle or claw, then these problems often stay NP-complete. A recent complexity study (MFCS 2019) on the k-COLOURING problem shows that we may still obtain tractable results if we also bound the diameter of the H-free input graph. We continue this line of research by initiating a complexity study on the impact of bounding the diameter for a variety of classical vertex partitioning problems restricted to H-free graphs. We prove that bounding the diameter does not help for INDEPENDENT SET, but leads to new tractable cases for problems closely related to 3-COLOURING. That is, we show that NEAR-BIPARTITENESS, INDEPENDENT FEEDBACK VERTEX SET, INDEPENDENT ODD CYCLE TRANSVERSAL, ACYCLIC 3-COLOURING and STAR 3-COLOURING are all polynomial-time solvable for chair-free graphs of bounded diameter. To obtain these results we exploit a new structural property of 3-colourable chair-free graphs.

1 Introduction

Many well-known graph problems are NP-complete in general but become polynomial-time solvable under input restrictions. We focus on problems that partition the vertex set V of a graph G into sets $V_1, \ldots, V_k$ such that each V_i satisfies some property π_i and where V_1 might have the extra condition of being large or being small. For instance, the k-COLOURING problem is to decide if V can be partitioned into sets $V_1, \ldots, V_k$, called *colour classes*, such that each V_i is an independent set. To give another example, the INDEPENDENT SET problem is to decide if V can be partitioned into sets V_1 and V_2 where V_1 is independent and $|V_1| \geq p$

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for some given integer p. Our underlying goal is to understand which graph properties ensure tractability of these problems and which properties cause the computational hardness. In the literature, input is restricted in various ways. In particular, *hereditary* graph classes have been considered.

Hereditary graph classes are the classes of graphs closed under vertex deletion. They form a natural and rich framework that cover many well-known graph classes (see, for example, [7]). Moreover, they enable a systematic study on the computational complexity of graph problems under input restrictions. The reason is that a graph class $\mathcal{G}$ is hereditary if and only if $\mathcal{G}$ can be characterized by a set $\mathcal{F}_\mathcal{G}$ of forbidden induced subgraphs; we also say that $\mathcal{G}$ is $\mathcal{F}_\mathcal{G}$-*free*. A natural starting point for a systematic study is the case where $\mathcal{F}_\mathcal{G}$ has size 1, say $\mathcal{F}_\mathcal{G} = \{H\}$ for some graph H. In this case the graphs in $\mathcal{G}$ are said to be H-*free*. In other words, no graph $G \in \mathcal{G}$ can be modified into H by a sequence of vertex deletions.

In the literature, there are extensive studies on H-free graphs; for example, on bull-free graphs [11] and claw-free graphs [12, 18]. There also exist several surveys on graph problems or graph parameters for hereditary graph classes that are characterized by a small set of forbidden induced subgraphs, for example, on COLOURING [17, 28] and clique-width [13].

A well-known dichotomy on COLOURING restricted to H-free graphs is due to Král', Kratochvíl, Tuza, and Woeginger [20]. Namely, COLOURING on H-free graphs is polynomial-time solvable if H is an induced subgraph of P_4 (the 4-vertex path) or of $P_1 + P_3$ (the disjoint union of P_1 and P_3) and it is NP-complete otherwise. Recently, similar but almost-complete dichotomies (up to one missing case each) were established for ACYCLIC COLOURING, STAR COLOURING and INJECTIVE COLOURING [4]. In particular, all these problems stay NP-complete if the forbidden induced subgraph H has a cycle or claw (the 4-vertex star $K_{1,3}$). Moreover, the latter holds even if the number of colours k is fixed, i.e., not part of the input.

Several other vertex partitioning problems on H-free graphs stay NP-complete as well if H has a cycle or claw. Examples of such problems include (INDEPENDENT) FEEDBACK VERTEX SET [6, 24, 27], (INDEPENDENT) ODD CYCLE TRANSVERSAL [6, 10] and EVEN CYCLE TRANSVERSAL [25]. Hence, for all these problems, if H is a cycle or claw, then we need to add more structure to the class of input graphs in order to find tractable results for H-free graphs. One way of doing this is to bound the *diameter* of the input graph G for some problem. Our research question then becomes:

Does bounding the diameter of an H-free graph lead to new tractability results?

We note that graph classes of diameter at most d are hereditary if and only if $d \leq 1$. Many graph problems, such as COLOURING, ACYCLIC COLOURING, STAR COLOURING, CLIQUE and INDEPENDENT SET stay NP-complete even for graphs of diameter 2. The reason is that we can take an arbitrary instance (G, k) from such a problem and add a dominating vertex to obtain a new graph G': (G, k) is a yes-instance if and only if $(G', k+1)$ is a yes-instance.

This approach of adding a dominating vertex does not work if we consider 3-COLOURING. Mertzios and Spirakis [23] proved in a highly nontrivial way that 3-COLOURING is NP-complete even for triangle-free graphs of diameter at most 3. However, determining the complexity of 3-COLOURING for graphs of diameter 2 is a notoriously open problem (see [3, 9, 21, 23, 26]); we refer to [23] and [15] for subexponential-time algorithms for LIST 3-COLOURING on graphs of diameter at most 2. It is also known that ACYCLIC 3-COLOURING and STAR COLOURING, restricted to graphs of diameter at most d, are polynomial-time solvable if $d = 2$ or $d = 3$, respectively, but NP-complete if $d = 5$ or $d = 8$, respectively [8]. Moreover, the related problems NEAR BIPARTITENESS and INDEPENDENT FEEDBACK VERTEX SET, which we define below, are polynomial-time solvable for $d = 2$ but NP-complete for $d = 3$ [5]. These
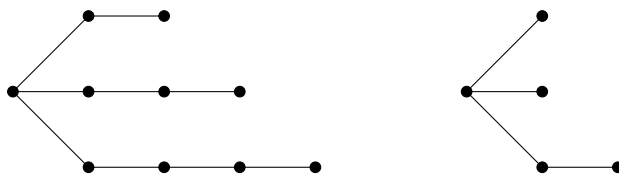
■ **Figure 1** Left: the graph $S_{2,3,4}$. Right: the graph $S_{1,1,2}$ also known as the chair or fork.

results also show that bounding the diameter on its own (without forbidding any induced graph H) does not suffice.

We refer to [21, 22] for a number of results on 3-COLOURING and LIST 3-COLOURING for H-free graphs of bounded diameter, where H is a cycle or a *polyad*, which is a tree where exactly one vertex has degree at least 3 (polyads are also known as *subdivided stars*). One crucial observation in [21], based on an application of Ramsey's Theorem, was the starting point of this investigation: for all integers d, k, the k-COLOURING problem is constant-time solvable on claw-free graphs of diameter at most d. In the same paper [21], this result was generalized to the case where H is the *chair*, which is the graph obtained from the claw after subdividing exactly one of its edges (see also Figure 1). The chair is also known as the *fork*.

Our Results

We first consider the INDEPENDENT SET problem. For this problem we will prove new NP-hardness results that show that bounding the diameter does not help. To explain this, let the graph $S_{h,i,j}$, for $1 \leq h \leq i \leq j$, be the *subdivided claw*, which is the tree with one vertex x of degree 3 and exactly three leaves, which are at distance h, i and j from x, respectively; see Figure 1 for an example. Note that $S_{1,1,1}$ is the claw $K_{1,3}$, the graph $S_{1,1,2}$ is the chair and that every subdivided claw is a polyad. Let $\mathcal{S}$ be the set of graphs, each connected component of which is a subdivided claw or a path. Alekseev [1] proved that for every finite set of graphs $\mathcal{F}$, if no graph from $\mathcal{F}$ belongs to $\mathcal{S}$, then INDEPENDENT SET is NP-complete for the class of $\mathcal{F}$-free graphs. In Section 2, we show that exactly the same NP-completeness result holds for the class of $\mathcal{F}$-free graphs *of diameter* 2 if and only if $|\mathcal{F}| = 1$.

We then turn to the class of H-free graphs where H is a polyad. First, we focus on the case where H is the chair. In Section 3, we prove that for every integer $d \geq 1$, a number of vertex partitioning problems that require yes-instances to be 3-colourable become polynomial-time solvable on chair-free graphs of diameter at most d. The problems are ACYCLIC 3-COLOURING, STAR 3-COLOURING, NEAR-BIPARTITENESS, INDEPENDENT FEEDBACK VERTEX SET and INDEPENDENT ODD CYCLE TRANSVERSAL. We define these problems below.

Our proof is based on a common strategy. Namely, we determine the following for every chair-free 3-colourable non-bipartite input graph G of bounded diameter: either G has a constant number of 3-colourings or there exists a set S such that $G - S$ has this property. We prove that we can let S be the set of private neighbours of some vertex u of a triangle on vertices u, v, w, that is, the vertices of S are adjacent to neither v nor w. We then consider each constructed 3-colouring c and determine in polynomial time if we can extend c to a solution for the vertex partitioning problem under consideration.

In Section 4 we prove that there is little hope of a full extension from the chair to arbitrary polyads H. To be more precise, we prove that for ACYCLIC 3-COLOURING, STAR 3-COLOURING and INDEPENDENT ODD CYCLE TRANSVERSAL, there exists a polyad H and a constant d such that each of these problems is NP-complete for the class of H-free graphs of diameter at most d. We finish our paper with some relevant open problems in Section 5.

Additional Terminology

Let $G = (V, E)$ be a graph. A *colouring* of G is a mapping $c : V \to \{1, 2, \ldots\}$ with $c(u) \neq c(v)$ for every pair of adjacent vertices u and v in G. If $c(u) \in \{1, \ldots, k\}$ for every $u \in V$, then c is also called a *k-colouring*. Note that the sets $\{u \in V(G) \mid c(u) = i\}$ for $i \in \{1, \ldots, k\}$ are the corresponding colour classes. If a k-colouring exists, then G is said to be *k-colourable*.

A graph is *acyclic 3-colourable* or *star 3-colourable* if it is 3-colourable and every two colour classes induce a forest or a star forest, respectively (in this context, the P_1 and P_2 are seen as stars). The corresponding decision problems are ACYCLIC 3-COLOURING and STAR 3-COLOURING. A graph G is *near-bipartite* if its vertex set can be partitioned into an independent set I and a forest F; we also say that I is an *independent feedback vertex set* of G. The problems NEAR-BIPARTITENESS and INDEPENDENT FEEDBACK VERTEX SET are to decide if a graph is near-bipartite or has an independent feedback vertex set of size at most k for some given integer k. A subset $S \subseteq V$ of a graph $G = (V, E)$ is an *independent odd cycle transversal* if S is independent and $G - S$ is bipartite. Note that a graph is 3-colourable if and only if it has an independent odd cycle transversal. The INDEPENDENT ODD CYCLE TRANSVERSAL problem is to decide if a given graph has an independent odd cycle transversal of size at most k for some given integer k.

Let C_r, P_r and K_r be the cycle, path and complete graph on r vertices. The graph $G + H = (V(G) \cup V(H), E(G) \cup E(H))$ is the disjoint union of graphs G and H, and sG is the disjoint union of s copies of G. A graph G is *$\mathcal{H}$-free* if G is H-free for every $H \in \mathcal{H}$.

Let G be a graph. For a vertex $u \in V(G)$, we let $N(u) = \{v \mid uv \in E(G)\}$ denote the *neighbourhood* of u in G. For a subset $U \subseteq V(G)$, we write $N(U) = \bigcup_{u \in U} N(u) \setminus U$.

2 Independent Set

We let $\mathcal{S}$ denote the set of graphs, each connected component of which is either a subdivided claw or a path. The following well-known result is due to Alekseev.

▶ **Theorem 1** ([1])**.** *Let $\mathcal{F}$ be a finite set of graphs. If no graph from $\mathcal{F}$ belongs to $\mathcal{S}$, then* INDEPENDENT SET *is* NP-*complete for $\mathcal{F}$-free graphs.*

We strengthen Theorem 1 to $\mathcal{F}$-free graphs of diameter 2 if $|\mathcal{F}| = 1$. We need two lemmas.
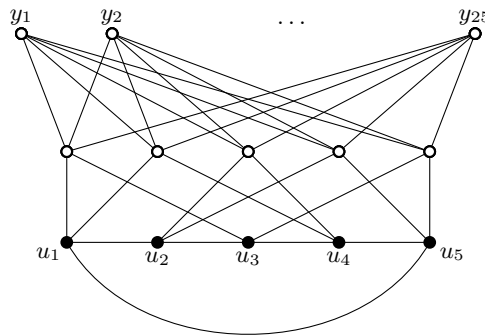
Figure 2 The graph G' in the proof of Lemma 2 if $G = C_5$. The black vertices are those of G and the white vertices in the middle are $x_{u_1 u_3}, x_{u_1 u_4}, x_{u_2 u_4}, x_{u_2 u_5}$, and $x_{u_3 u_5}$ from left to right.

▶ **Lemma 2.** *For every $r \geq 3$,* INDEPENDENT SET *is* NP-*complete for C_r-free graphs of diameter 2.*

Proof. First suppose that $r = 3$. By Theorem 1, Independent Set is NP-complete for C_3-free graphs. Let (G, k) be an instance of Independent Set, where G is an n-vertex C_3-free graph. We may assume without loss of generality that $n \geq 2$ and $k \geq 2$; otherwise, the problem is trivial. We also assume that G has no dominating vertex; else, if u is a dominating vertex of G, then G is a star with centre u and the leaves of G form a maximum independent set of size $n - 1$. From G we now construct a graph G' as follows; see also Figure 2.

- Construct a copy of G.
- For every pair $\{u, v\}$ of nonadjacent vertices of G,
  - construct an arbitrary inclusion maximal independent set I_{uv} of G containing u and v,
  - construct a vertex x_{uv} and make x_{uv} adjacent to every vertex of I_{uv}.
  Denote by X the set of all vertices x_{uv} constructed for the pairs of nonadjacent u and v.
- Construct an independent set of n^2 vertices Y and make every vertex of Y adjacent to every vertex of X.

Observe that for every pair $\{u, v\}$ of nonadjacent vertices of G, I_{uv} can be constructed by the straightforward greedy procedure starting from the set $\{u, v\}$. This implies that G can be constructed in polynomial time. Our construction immediately implies that G' is triangle free, because the initial graph G is triangle-free and the neighbourhood of every vertex from $X \cup Y$ is an independent set. We claim that the diameter of G' is at most 2.

Consider two distinct vertices $u, v \in V(G')$. We show that either u and v are adjacent or they have a common neighbour z. This is easy to see if $u, v \in X \cup Y$. Suppose that $u \in V(G)$ and v is not adjacent to u. If $v \in V(G)$, then G' contains x_{uv} that is adjacent to both u and v as required. Let $v \in X$. Then $v = x_{u'v'}$ is constructed for some nonadjacent $u', v' \in V(G)$. We have that $u \notin I_{u'v'}$, because u and v are not adjacent. We also have that $I_{u'v'} \cup \{u\}$ is not an independent set, because $I_{u'v'}$ is inclusion maximal. Hence, u is adjacent to some $z \in I_{u'v'}$ which is a common neighbour of u and v. Assume that $v \in Y$. Since G' has no universal vertex, u is not adjacent to some $v' \in V(G)$. Therefore, there is $x_{uv'} \in X$. Clearly, this vertex is adjacent to u and v. This completes the case analysis. Since u and v are arbitrary, we conclude that the diameter of G' is at most two.

Let $k' = n^2 + k$. We show that G has an independent set of size at least k if and only if G' has an independent set of size at least k'.

Assume that G has an independent set U of size at least k. Consider this set in the copy of G in G' and set $W = U \cup Y$. Clearly, W is an independent set of G' and $|W| = |U| + |Y| \geq n^2 + k = k'$. For the opposite direction, let W be an independent set of G' of size at least k'. Observe that $W \cap Y \neq \emptyset$. Otherwise, $W \subseteq V(G) \cup X$ and $|W| \leq |V(G)| + |X| \leq n + \binom{n}{2} < n^2$, because $n \geq 2$. Let $y \in W \cap Y$. Because $N_{G'}(y) = X$, $W \cap X = \emptyset$. Let $U = W \cap V(G)$. We obtain that $|U| = |W| - |W \cap Y| \geq |W| - n^2 \geq k' - n^2 = k$. Hence, U is an independent set of G of size at least k.

Now suppose that $r \geq 4$. By Theorem 1, Independent Set is NP-complete for C_r-free graphs. Let (G, k) be an instance of Independent Set, where G is a C_r-free graph. We add a dominating vertex u to G. This yields a graph G', which is also C_r-free but which has diameter 2. Moreover, G has an independent set of size at least k if and only if G' has an independent set of size at least k. This completes the proof. ◄

▶ **Lemma 3.** Independent Set *is* NP*-complete for $K_{1,4}$-free graphs of diameter* 2.

Proof. By Theorem 1, Independent Set is NP-hard for $(C_3, K_{1,4})$-free graphs. Let (G, k) be an instance of Independent Set, where G is a $(C_3, K_{1,4})$-free graph of order n and size

m. We may assume without loss of generality that G is connected. Note that G is subcubic, that is, has maximum degree at most 3. We may assume without loss of generality that G has no vertices of degree 1 (as we can pick such vertices to be in the independent set and remove their neighbours from G until this operation can no longer be applied). To G, we add for every pair of edges e_1 and e_2 of G that do not share an end-vertex, a new vertex x_{e_1,e_2} and an edge between x_{e_1,e_2} and the two end-vertices of both e_1 and e_2. We also add a new vertex y and all edges such that $\{x_{e_1,e_2} : e_1, e_2\} \cup \{y\}$ induces a complete subgraph. Let G' be the resulting graph and let $X := \{x_{e_1,e_2} : e_1, e_2\} \cup \{y\}$. See also Figure 3.
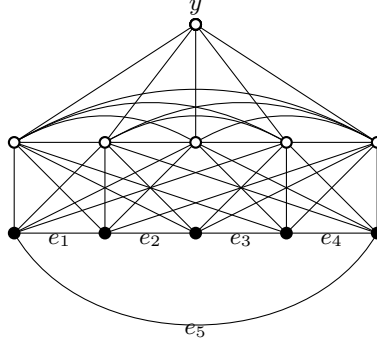
Figure 3 The graph G' in the proof of Lemma 3 if $G = C_5$. The black vertices are those of G and the white vertices in the middle are $x_{e_1 e_3}, x_{e_2 e_5}, x_{e_1 e_4}, x_{e_3 e_5}$, and $x_{e_2 e_4}$ from left to right.

We claim that G' is $K_{1,4}$-free. For a contradiction, assume that there is some induced $K_{1,4}$ in G', say z is its centre vertex and z_1, z_2, z_3, z_4 are its leafs. Since $N_{G'}(x)$ can be partitioned into at most three cliques for each $x \in X$, we have $z \in V(G)$. Furthermore, since G is subcubic and X is a clique, X contains at least one and at most one vertex of $\{z_1, z_2, z_3, z_4\}$. As we see now, $N_G(z) \subseteq \{z_1, z_2, z_3, z_4\}$. However, each vertex of X that is adjacent to z has a neighbour in $N_G(z)$, which contradicts our assumption that z is the centre vertex of an induced $K_{1,4}$ in G'.

To show that G' has diameter 2, first consider an arbitrary vertex u of G. As G is connected, there is a vertex $v \in N_G(u)$. As G has minimum degree at least 2 and G is triangle-free, there are vertices $u' \in N_G(u) \setminus \{v\}$ and $v' \in N_G(v) \setminus \{u, u'\}$. For $e_1 = uu'$ and $e_2 = vv'$, e_1 and e_2 do not have a common end-vertex, $u, x \in N_{G'}[x_{e_1,e_2}]$, and $dist_{G'}(u, x) \leq 2$ for each $x \in X$. Additionally, if $w \in V(G) \setminus \{u\}$ is a vertex with $dist_G(u, w) \geq 3$, then, since G is connected, there are edges $e_3, e_4 \in E(G)$ such that u is incident to e_3, w is incident to e_4, and the two edges e_3, e_4 have no common incident vertex. Thus, $u, w \in N_G(x_{e_3,e_4})$ and so $dist_{G'}(u, w) \leq 2$. As X is a clique, we conclude that G' has diameter 2.

We observe that $I \cup \{y\}$ is an independent set of size $k + 1$ in G' if I is an independent set of size k in G. Vice versa, given an independent set I' of size $k + 1$ in G', at most one of its vertices belongs to X (as X is a clique) and thus $I' \setminus X$ is an independent set of size at least k in G.

Finally, we observe that the graph G' can be constructed in time $\mathcal{O}(m^2)$ and has at most $n + m^2$ vertices. ◀

We now strengthen Theorem 1, but can only do this for the case where $|\mathcal{F}| = 1$. For example, if $\mathcal{F} = \{C_3, K_{1,4}\}$, every $\mathcal{F}$-free graph has maximum degree 3. Then every $\mathcal{F}$-free graph with bounded diameter has constant size. Hence, INDEPENDENT SET is NP-complete for $\mathcal{F}$-free graphs by Theorem 1 but constant-time solvable for $\mathcal{F}$-free graphs of bounded diameter.

▶ **Theorem 4.** *Let H be a graph. If $H \notin \mathcal{S}$, then* INDEPENDENT SET *is* NP-*complete for H-free graphs of diameter at most* 2. *If $H \in \mathcal{S}$, then* INDEPENDENT SET *for H-free graphs is polynomially equivalent to* INDEPENDENT SET *for H-free graphs of diameter at most* 2.

Proof. Let H be a graph. First assume that $H \notin \mathcal{S}$. If H contains an induced cycle C_r for some $r \geq 3$, then the class of H-free graphs contains the class of C_r-free graphs, and we use Lemma 2. Otherwise H is a forest with either an induced $K_{1,4}$ or a connected component that has at least two vertices of degree 3. In the first case, we use Lemma 3. In the second case, we reduce from INDEPENDENT SET for H-free graphs, which is NP-complete by Theorem 1. Let (G,k) be an instance of INDEPENDENT SET, where G is an H-free graph. We add a dominating vertex to G to obtain a graph G'. As H has no dominating vertex, G' is H-free. We also note that (G,k) and (G',k) are equivalent instances.

Now assume that $H \in \mathcal{S}$. Any polynomial-time algorithm for INDEPENDENT SET on H-free graphs can be used on H-free graphs of diameter 2. As INDEPENDENT SET is polynomial-time solvable for $K_{1,3}$-free graphs [29], we may assume that $H \notin \{K_{1,3}, P_1, P_2, P_3\}$. Any polynomial-time algorithm for INDEPENDENT SET on H-free graphs of diameter 2 can be used on H-free graphs of arbitrary diameter as follows. To the H-free input graph G we add a dominating vertex. As $H \in \mathcal{S} \setminus \{K_{1,3}, P_1, P_2, P_3\}$ and $K_{1,3}$, P_1, P_2, P_3 are the only graphs in $\mathcal{S}$ with a dominating vertex, this yields an H-free graph G' of diameter 2. We then observe that G has an independent set of size at least k if and only if G' has an independent set of size at least k. ◀

3   Chair-Free Graphs of Bounded Diameter

It follows from Ramsey's Theorem that every k-colourable $K_{1,r}$-free graph of diameter at most d has order bounded by a function in d, k, r; see [21] for a proof of this observation. As a consequence, every problem that has the property that all its yes-instances are k-colourable for some constant k is constant-time solvable on $K_{1,r}$-free graphs of diameter at most d. We aim to extend the above observation to H-free graphs of bounded diameter when H is obtained from a star after at least one edge subdivision. In [21], a number of results are given for 3-COLOURING for such graph classes. We consider a variety of problems that require all the yes-instances to be 3-colourable. We focus on the first interesting case which is where H is the chair $S_{1,1,2}$ (recall that the chair is obtained from the claw after subdividing one edge).

We need the following characterization of bipartite chair-free graphs, due to Alekseev. A *complex* is a bipartite graph that can be obtained by removing the edges of a possibly empty matching from a complete bipartite graph.

▶ **Theorem 5** ([2]). *If G is a connected bipartite chair-free graph, then G is a cycle or a path or a complex.*

It is well-known (cf. [19]) that finding the components of a graph by breadth-first search takes $O(n+m)$ time. Let p be the number of components of a graph G, n be its order, and m be its size. Then G is a forest if and only if $p = n - m$, which implies the following result.

▶ **Observation 6.** *If G is a graph, then we can decide if G is a forest in $O(n+m)$ time.*

If T is a tree of order n, then its diameter is at most 2 if and only if its maximum degree equals $n - 1$. Therefore, we can decide whether a given graph is a forest each component of which is of diameter at most 2 in $O(n+m)$ time. When working with vertex labellings, our findings imply the following observation.

▶ **Observation 7.** *If G is a graph and ℓ is a vertex labelling of G with labels $1, 2,$ and 3, then we can decide whether ℓ is a 3-colouring, star 3-colouring, or acyclic 3-colouring of G in $O(n + m)$ time.*

It is also well-known that we can use breadth-first search for deciding whether a given graph G is bipartite and, if so, we are in a position to determine its parts in the same time. By this fact, we obtain the following result.

▶ **Observation 8.** *If G is a graph, k is an integer, and ℓ is a vertex labelling of G with labels $1, 2,$ and 3, then we can decide whether one colour class of ℓ is an independent feedback vertex set or an independent odd cycle transversal (of size at most k) in $O(n + m)$ time.*

To prove our results we need some more terminology. A *list assignment* of a graph G is a function L that gives each vertex $u \in V(G)$ a (finite) *list of admissible colours* $L(u) \subseteq \{1, 2, \ldots\}$. A colouring c *respects* L if $c(u) \in L(u)$ for every $u \in V(G)$. If $|L(u)| \leq 2$ for each $u \in V(G)$, then L is a 2-*list assignment*. The 2-LIST COLOURING problem is to decide if a graph G with a 2-list assignment L has a colouring that respects L.

We use the following well-known result, which is due to Edwards (see the proof of Lemma 2.4 in [16]).

▶ **Lemma 9** ([16])**.** *The 2-LIST COLOURING problem is solvable in $O(n + m)$ time on graphs with n vertices and m edges.*

We now introduce some additional terminology. Let G be a graph of diameter d for some $d \geq 1$ and S be some set of vertices of G. A vertex $u \notin S$ is a *private* neighbour of a vertex $v \in S$ *with respect to S* if u is adjacent to v but non-adjacent to any other vertex of S. We let $P(v)$ be the set of private neighbours of v with respect to S. Let $N_0 = S$ and, for $i \in \{1, \ldots d\}$, let N_i be the set of vertices that do not belong to $N_0 \cup \ldots \cup N_{i-1}$ but that do have a neighbour in N_{i-1}; in particular $N_1 = N(S)$. As G has diameter at most d, we partition $V(G)$ into the sets $N_0, N_1, \ldots, N_d$ (where some sets might be empty). We say that we *partition $V(G)$ from S*. Note that this partitioning takes $O(n + m)$ time by breadth-first search on the graph G' which is obtained from G by adding a new vertex u and all edges from u to every vertex of S.

In the next lemma we show that a sufficiently large connected chair-free graph contains a triangle T, which we can find in linear time. In later proofs we will partition from $V(T)$.

▶ **Lemma 10.** *Let $d \geq 1$ and G be a chair-free non-bipartite graph of diameter d with n vertices and m edges. If G has at least $2d + 2$ vertices, then G contains at least one triangle, which is computable in $O(n + m)$ time.*

Proof. For a contradiction, assume that G is triangle-free. As G is not bipartite, there is an odd cycle in G. Let $x_1 x_2 \ldots x_p x_1$ be a shortest one. As G is triangle-free and of diameter d, we find that $5 \leq p \leq 2d + 1$. Moreover, as G is of order at least $2d + 2$, there is some vertex outside this cycle that has a neighbour on this cycle. Without loss of generality let us assume y with $y \notin \{x_1, x_2, \ldots, x_p\}$ is adjacent to x_1. As G is triangle-free, y does not have two consecutive neighbours on $x_1 x_2 \ldots x_p x_1$. As G is chair-free and y is neither adjacent to x_2 nor to x_p, we find that y must be adjacent to x_3. We repeat this argument and obtain that y is adjacent to x_{2q+1} for every $0 \leq q \leq \lfloor \frac{p}{2} \rfloor$. In particular y is adjacent to the two consecutive vertices x_1 and x_p, a contradiction. We conclude that our assumption is false and that G contains a triangle.

We continue and show that we can compute a triangle of G in $O(n + m)$ time. Let u be a vertex of G. We partition $V(G)$ from $\{u\}$ and note that breadth-first search computes

a breadth-first tree F, that is, F is a spanning tree of G such that each vertex of N_i has distance i to u in F for any i. As G is not bipartite, there has to be an edge e and an integer i such that e is incident to two vertices of N_i. We can compute such an edge that additionally minimizes i in $O(n+m)$ time. By adding this edge to F, we find an odd cycle C in G. As F is of diameter at most $2d$, we find that C has at most $2d+1$ vertices. Hence, we can determine in constant time an induced odd cycle, say C', in $G[V(C)]$. We check in constant time whether C' is a triangle. If not, then C' is of order at least 5. As G is of order at least $2d+2$, there is a vertex outside C' that has a neighbour on C'. We compute such a vertex, say y, in $O(n+m)$ time. As shown above, y has two consecutive neighbours on C'. As C' has at most $2d+1$ vertices, we can find such two vertices, and thus a triangle in G, in constant time. ◀

In the next two lemmas we show two necessary conditions for a chair-free graph G with a triangle T to be 3-colourable. To prove the second lemma, we partition $V(G)$ from $V(T)$.

▶ **Lemma 11.** *Let G be a 3-colourable chair-free graph, and let T be a triangle in G with vertex set $\{x, y, z\}$. If T contains at least two vertices that have a private neighbour, then $|P(x) \cup P(y) \cup P(z)| \le 6$.*

Proof. For a contradiction, assume that $|P(x) \cup P(y) \cup P(z)| > 6$. By the pigeonhole principle, one vertex of T, say x, has at least three private neighbours. As G is 3-colourable, and thus K_4-free, there are two non-adjacent vertices in $P(x)$, say u and v. As at least two vertices of T have a private neighbour, we may assume without loss of generality that y has a private neighbour w. Then $\{u, v, w, x, y\}$ induces a chair unless w is adjacent to at least one of u and v. If w is adjacent to exactly one of u, v, then $\{u, v, w, x, z\}$ induces a chair. If w is adjacent to both u and v, then $\{u, v, w, y, z\}$ induces a chair. In all three cases, we obtained a contradiction. ◀

▶ **Lemma 12.** *Let G be a 3-colourable chair-free graph of diameter d for some $d \ge 1$, and let T be a triangle in G with vertex set $\{x, y, z\}$. Then $G - N(T)$ has at most $9 \cdot 2^d + 2$ vertices.*

Proof. For a contradiction, assume that G has at least $9 \cdot 2^d + 3$ vertices. We partition $V(G)$ from $V(T)$. As G is 3-colourable, G is K_4-free. Hence, every vertex of N_1 has at least one non-neighbour on T. Let $i \ge 1$ and let u be a vertex of N_i. As G is chair-free, the neighbours of u in N_{i+1} form a clique. Note that for $i = 1$ this holds since u must have at least one neighbour and at least one non-neighbour on T. As G is K_4-free, this means that the neighbourhood of u in N_{i+1} is a clique of size at most 2. It follows that

$$3 + 9 \cdot 2^d \le |N_0| + |N_2| + |N_3| + \ldots + |N_d| \le 3 + |N_2| \cdot \sum_{i=2}^{d} 2^{i-2} < 3 + |N_2| \cdot 2^{d-1}.$$

Hence, $|N_2| > 18$. We let N_1^* be the set of all vertices of N_1 that have two neighbours on T, and we let N_2^* be the set of neighbours of N_1^* in N_2. Consider the set N_{xy} of common neighbours of x and y in N_1^*. Then N_{xy} is an independent set, as otherwise two adjacent vertices in N_{xy} form, together with x and y, a K_4, contradicting the K_4-freeness of G.

Every vertex $u \in N_2^*$ with a neighbour in N_{xy} must be adjacent to every vertex in N_{xy}, as G is chair-free and N_{xy} is an independent set. Recall that every vertex of N_1, and thus every vertex of N_{xy}, has at most two neighbours in N_2, which belong to N_2^* by definition. Hence, there are at most two vertices in N_2^* that are adjacent to the vertices of N_{xy}. By applying the same reasoning for every other pair of vertices of T, we find that N_2^* has size at most 6. Thus, $|N_2 \setminus N_2^\star| > 12$. As every vertex of N_1 has at most two neighbours in N_2, it

follows that $|P(x) \cup P(y) \cup P(z)| > 6$. By Lemma 11, we obtain that there is exactly one vertex, say x, of T which has private neighbours. As G is 3-colourable, $G[P(x)]$ is bipartite.

As $G[P(x)]$ is bipartite, we can partition $P(x)$ into two independent sets A and B (one of these two sets might be empty). As G is chair-free and as A is independent, the vertices of A share the same set of neighbours in N_2. Similarly, the vertices of B share the same set of neighbours in N_2. Recall that the neighbourhood of every vertex of $A \cup B$ in N_2 is a clique of size at most 2. We conclude that the total number of vertices in N_2 with a neighbour in $P(x)$ is at most 4, a contradiction as $|N_2 \setminus N_2^*| > 12$. We conclude that $G - N_1$ has at most $9 \cdot 2^d + 2$ vertices.   ◀

In [21], it was shown that 3-COLOURING is polynomial-time solvable for chair-free graphs of bounded diameter. By giving more precise arguments, we show in the first statement of Theorem 13 that we can even obtain a linear-time algorithm for 3-COLOURING on chair-free graphs of bounded diameter. Recall that we wish to prove a similar result for a number of vertex partitioning problems that require yes-instances to be 3-colourable. In order to do this (in Theorem 15), we first need to prove a new result on the structure of 3-colourable chair-free (non-bipartite) graphs of bounded diameter and on how to find this structure. These graphs might have an exponential number of 3-colourings (for example, take a triangle T and add a number of pendant vertices to one of the vertices of T). However, if there are too many, then we find a certain induced subgraph that has a bounded number of 3-colourings. This is shown in the second statement of Theorem 13. The proof of Theorem 13 uses Lemmas 10–12.

▶ **Theorem 13.** *For an integer $d \geq 1$ and a chair-free non-bipartite graph G of diameter d with n vertices and m edges, it is possible to find the following in $O(n + m)$ time:*
1. *whether or not G is 3-colourable, and*
2. *should G be 3-colourable, either all 3-colourings of G, or a triangle T with exactly one vertex x that has private neighbours and all 3-colourings of $G - P(x)$ that can be extended to 3-colourings of G; in both cases, the number of 3-colourings found is at most $3^{9 \cdot 2^d + 8}$.*

Proof. We first check in constant time whether G is of order at most $2d + 1$. If so, then we can determine in constant time all 3-colourings of G and these are at most 3^{2d+1}. Note that $3^{2d+1} < 3^{9 \cdot 2^d + 8}$. In what follows, we assume that G is of order at least $2d + 2$.

By Lemma 10, G contains a triangle and we can compute such a triangle, say T, in $O(n+m)$ time. Let x, y, z be the vertices of T. We partition $V(G)$ from $\{x, y, z\}$ in $O(n+m)$ time. We additionally determine all private neighbours of the vertices of T and all vertices of N_1 that are adjacent to all vertices of T in linear time. If there is a vertex of the latter type, then G is not 3-colourable. Thus, we focus on the case where each vertex of N_1 is adjacent to at most two vertices of T. For simplicity, let $S = P(x) \cup P(y) \cup P(z)$.

We compute the set N_1^* of all vertices of N_1 that have two neighbours on T. Clearly, $S = N_1 \setminus N_1^*$ and the computation of N_1^* and S takes $O(n+m)$ time. We proceed by considering $G - N_1$. We determine $|V(G - N_1)|$ in $O(n + m)$ time and check if $|V(G - N_1)| \geq 9 \cdot 2^d + 3$. If so, then by Lemma 12 we find that G is not 3-colourable. Assume that $G - N_1$ has at most $9 \cdot 2^d + 2$ vertices.

We consider every vertex labelling of $G - N_1$ with labels $1, 2, 3$. In what follows we determine in $O(n + m)$ time which ones lead to a 3-colouring of G. First of all, we discard those labellings which are not a 3-colouring of $G - N_1$. Given a 3-colouring of $G - N_1$, each vertex of N_1^* receives the remaining available label that is not used for its neighbours on T. Note that this assignment takes linear time. We discard such a labelling if it does not lead to a 3-colouring of $G - S$. Next we assign lists to the vertices of G as follows: we set $L(u) = \{i\}$, where i is the label of u, if $u \notin S$ and we set $L(u) = \{1, 2, 3\} \setminus \{i\}$, where i is the label of

the unique neighbour of u on T, if $u \in S$. Thus, checking whether a given 3-colouring of $G - N_1$ leads to a 3-colouring of G takes $O(n + m)$ time as (G, L) is an instance of 2-LIST COLOURING (cf. Lemma 9). We discard those 3-colourings of $G - N_1$ which do not lead to a 3-colouring of G. As there are at most $3^{9 \cdot 2^d + 2}$ vertex labellings of $G - N_1$, we repeat this procedure constantly many times. Note that we get all 3-colourings of $G - S$ as a by-product.

If all 3-colourings of $G - N_1$ do not lead to a 3-colouring of G, then G is not 3-colourable. Consequently, we can decide whether G is 3-colourable in $O(n + m)$ time. As Theorem 13.1 is now proven, we proceed with our proof for Theorem 13.2 by assuming that G is 3-colourable.

As there are at most $3^{9 \cdot 2^d + 2}$ vertex labellings of $G - N_1$, there are at most $3^{9 \cdot 2^d + 2}$ 3-colourings of $G - S$. If $S = \emptyset$, then $G - S$ equals G and Theorem 13.2 follows. If all vertices of S are adjacent to a single vertex, say x, of T, we conclude that $S = P(x)$ and Theorem 13.2 follows. Hence, we may assume that there are at least two vertices of T which have a private neighbour. As G is 3-colourable, we find that $|S| \leq 6$ by Lemma 11. Thus, as we have at most $3^{9 \cdot 2^d + 2}$ 3-colourings of $G - S$, we find at most $3^{9 \cdot 2^d + 8}$ 3-colourings of G and their computation takes $O(n + m)$ time, which completes our proof. ◀

The double-exponential bound in Theorem 13 cannot be improved to single-exponential. In order to see this we define a series $G_1, G_2, \ldots$ of connected chair-free graphs as follows (see Figure 4 for a drawing of the graph G_4):

- G_1 is a triangle with vertex set $\{v_1^1, v_2^1, v_3^1\}$.
- G_{d+1} is obtained from G_d by adding $3 \cdot 2^d$ new vertices $\{v_1^{d+1}, v_2^{d+1}, \ldots, v_{3 \cdot 2^d}^{d+1}\}$ and adding the edges $v_i^d v_{2i-1}^{d+1}, v_i^d v_{2i}^{d+1}, v_{2i-1}^{d+1} v_{2i}^{d+1}$ for each $i = 1, 2, \ldots, 3 \cdot 2^{d-1}$.

We note that G_d is of diameter $2d - 1$ for each d. Let $a(d)$ be the number of 3-colourings of G_d. We claim $a(d) = 6 \cdot 2^{3 \cdot (2^{d-1} - 1)}$ and prove this by induction. Obviously, $a(1) = 6$. For each 3-colouring of G_d and each $i = 1, 2, \ldots, 3 \cdot 2^{d-1}$, we have two possibilities to colour the vertices v_{2i-1}^{d+1} and v_{2i}^{d+1}. Thus, using induction hypothesis, we find that $a(d + 1) = a(d) \cdot 2^{3 \cdot 2^{d-1}} = \left(6 \cdot 2^{3 \cdot (2^{d-1} - 1)} \right) \cdot 2^{3 \cdot 2^{d-1}} = 6 \cdot 2^{3 \cdot (2^d - 1)}$.

Furthermore, the only triangles with exactly one vertex that has private neighbours are those on vertex set $\{v_i^{d-1}, v_{2i-1}^d, v_{2i}^d\}$ for $i = 1, 2, \ldots, 3 \cdot 2^{d-1}$. For such a triangle, the vertex with private neighbours is v_i^{d-1}. As every two distinct 3-colourings of G_d differ on at least one vertex of $V(G_d - P(v_i^{d-1}))$, there are $a(d)$ 3-colourings of $G_d - P(v_i^{d-1})$ that can be extended to 3-colourings of G_d.

We now use Theorem 13 to prove that apart from 3-COLOURING (Theorem 13.1), the problems ACYCLIC 3-COLOURING, STAR 3-COLOURING, INDEPENDENT ODD CYCLE TRANSVERSAL, INDEPENDENT FEEDBACK VERTEX SET, and NEAR-BIPARTITENESS can be solved in linear time for chair-free graphs of bounded diameter.

We first deal, in the next lemma, with the case that the chair-free input graph of bounded diameter is bipartite.

▶ **Lemma 14.** *For every integer $d \geq 1$, ACYCLIC 3-COLOURING, STAR 3-COLOURING, INDEPENDENT ODD CYCLE TRANSVERSAL, INDEPENDENT FEEDBACK VERTEX SET, and NEAR-BIPARTITENESS can be solved in $O(n + m)$ time for chair-free bipartite graphs of diameter at most d with n vertices and m edges.*

Proof. Note that a bipartite graph is near-bipartite and has an independent odd cycle transversal of size at most k for every integer k. Hence, it remains to consider the problems ACYCLIC 3-COLOURING, STAR 3-COLOURING and INDEPENDENT FEEDBACK VERTEX SET.
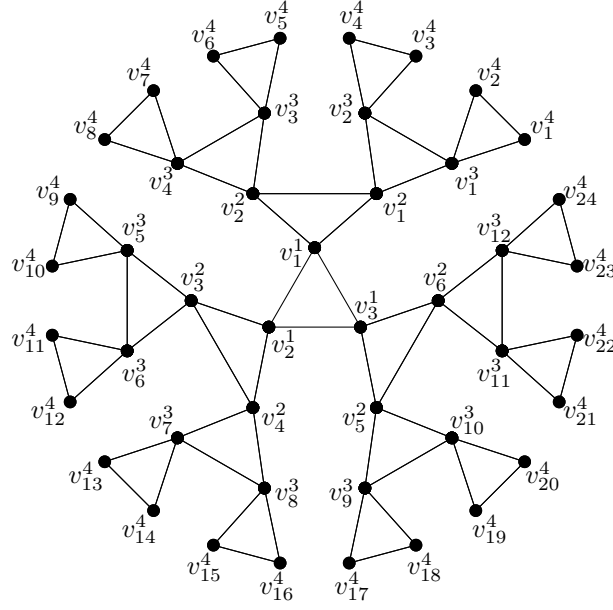
Figure 4 The graph G^4.

Let G be a chair-free bipartite graph of diameter at most d with n vertices and m edges. We may assume without loss of generality that G is connected. We first determine the two partition classes S_1 and S_2 of G in $O(n+m)$ time. We may assume without loss of generality that $|S_1| \geq |S_2|$. We check in constant time whether $|S_1| + |S_2| \leq \max\{8, 2d\}$ and if so, then we can solve each of our problems in constant time. Otherwise, we find that $|S_1| \geq 5$. As bipartite graphs of maximum degree at most 2 and diameter at most d are paths or cycles of at most $2d$ vertices, we find that G has a vertex of degree at least 3, and so G is a complex by Theorem 5.

We first claim that in the case where G is a complex with $|S_1| \geq 5$, G is star 3-colourable if $|S_2| \leq 2$ and acyclic 3-colourable only if $|S_2| \leq 2$. This claim suffices to prove the lemma for ACYCLIC 3-COLOURING and STAR 3-COLOURING as we can decide whether $|S_2| \leq 2$ or not in constant time and moreover, every star 3-colouring of a graph is acyclic.

We prove the above claim as follows: If $|S_2| \leq 2$, then, for any $s \in S_2$, $G - s$ is a forest each component of which is of diameter at most 2, and thus G is star 3-colourable with colour classes $S_1, S_2 \setminus \{s\}$, and $\{s\}$. If $|S_2| \geq 3$, then let c be an arbitrary 3-colouring of G. By the pigeonhole principle there exists a colour class X of c that contains at least two vertices of S_1, and so $X \cap S_2 = \emptyset$. As $|S_2| \geq 3$, there are two vertices $s_2, s_2' \in S_2$ that are coloured alike. As $|S_1| \geq 5$, and as s_2 and s_2' are of degree at least $|S_1| - 1$, we find that s_2 and s_2' have at least three common neighbours in S_1 two of which, say s_1 and s_1', are coloured alike. Hence, $s_1 s_2 s_1' s_2' s_1$ is a bichromatic 4-cycle. We conclude that every 3-colouring of G is not acyclic, which completes the proof of our claim.

It remains to consider INDEPENDENT FEEDBACK VERTEX SET. Let k be an arbitrary integer. We claim that if G is a complex with $|S_1| \geq 5$, then G has an independent feedback vertex set of size at most k if and only if $k \geq |S_2| - 1$. Note that the latter can be decided in linear time.

We prove the above claim as follows: If $|S_2| \leq 2$, then $G - s$ is a forest for any $s \in S_2$ and G has an independent feedback vertex set of size at most k. Hence, we may assume

$|S_2| \geq 3$. Let I be a minimum independent feedback vertex set in G. Such a set exists as G is bipartite. As $S_2 \setminus \{s\}$ is independent and as $G[S_1 \cup \{s\}]$ is a forest for each vertex $s \in S_2$, we find $|I| \leq |S_2| - 1$. For the sake of a contradiction, let us assume $|I| \leq |S_2| - 2$. Hence, any two vertices of $S_2 \setminus I$ have at least $|S_1| - 2$ common neighbours in S_1, and so $|I \cap S_1| \geq |S_1| - 3 \geq 2$. Moreover, $I = I \cap S_1$ as every vertex of S_2 has a neighbour in $I \cap S_1$ and as I is independent. As I is an independent feedback vertex set with $|I| \leq |S_1| - 2$, any two vertices of $S_1 \setminus I$ do not have two common neighbours in S_2 and so $|S_2| \leq 3$. Hence, $5 \leq |S_1| \leq |I| + 3 \leq |S_2| + 1 \leq 4$, a contradiction. As $|I| = |S_2| - 1$, the proof of the lemma is complete. ◀

We are now ready to prove the main result of this section, which is the same statement as Lemma 14 except that we now also include the non-bipartite graphs.

▶ **Theorem 15.** *For every integer $d \geq 1$,* Acyclic 3-Colouring, Star 3-Colouring, Independent Odd Cycle Transversal, Independent Feedback Vertex Set, *and* Near-Bipartiteness *can be solved in $O(n + m)$ time for chair-free graphs of diameter at most d with n vertices and m edges.*

Proof. Let G be a chair-free graph of diameter at most d with n vertices and m edges. Recall that G is acyclic 3-colourable or star 3-colourable only if G is 3-colourable. Moreover, if I is an independent set of G for which $G - I$ is a bipartite graph, then G is 3-colourable. Hence, our problems require all the yes-instances to be 3-colourable. If $d = 1$, then G is 3-colourable if and only if G has at most 3 vertices, and so each of our problems can be solved in constant time. We proceed by assuming $d \geq 2$. We first check in $O(n + m)$ time whether G is bipartite. If so, then we apply Lemma 14. In the remainder of our proof we assume that G is not bipartite.

Outline. As our problems require all the yes-instances to be 3-colourable, we check first whether G is 3-colourable. If so, then we compute an induced subgraph H of G and determine the set $\mathcal{C}$ of all its 3-colourings that can be extended to 3-colourings of G. As we compute H by applying Theorem 13, we find that $|\mathcal{C}| \leq 3^{9 \cdot 2^d + 8}$. We then distinguish some subcases. In some of them we further branch by extending our 3-colourings. However, in some of them we find that H equals G, and so Observations 7 and 8 imply that our six problems are solvable in $O(n + m)$ time as $\mathcal{C}$ is of constant size. As an implicit step, we apply this finding whenever H is the whole graph G.

Full Proof. We first apply Theorem 13. We continue by assuming that G is 3-colourable. In fact, the only remaining case is that where the lemma provides a triangle T on vertex set $\{x, y, z\}$, a vertex x of T that has private neighbours, and the set of all 3-colourings of $G - P(x)$ that can be extended to 3-colourings of G. Note that we have at most $3^{9 \cdot 2^d + 8}$ such 3-colourings. We partition $V(G)$ from $V(T)$.

We find that $G[P(x)]$ is bipartite, as G is 3-colourable, but not necessarily connected. We extend each 3-colouring of $G - P(x)$ that can be extended to a 3-colouring of G to some vertices of $P(x)$. Let c be an arbitrary 3-colouring of $G - P(x)$ that can be extended to a 3-colouring of G. For $i \in \{0, 1, 2\}$, we compute in $O(n + m)$ time the set S_i of all vertices of $P(x)$ which have i available colours with respect to c, that is, S_i is the set of all vertices of $P(x)$ which have neighbours in $3 - i$ colours. As c can be extended to a 3-colouring of G, we find that S_0 is empty. It takes $O(n + m)$ time to determine the available colour of each vertex in S_1. Furthermore, we can extend c by breadth-first search in the same time to the vertices of those components of $G[P(x)]$ that contain at least one vertex of S_1.

Let S_c be the set of vertices that induce those components of $G[P(x)]$ that do not contain a vertex of S_1. Note that S_c can be computed in $O(n + m)$ time and that all neighbours of all vertices of S_c in $V(G) \setminus S_c$ are coloured alike. Moreover, every vertex of S_c has its neighbours in $[N(y) \cap N(z)] \cup N_2 \cup S_c \cup \{x\}$ by definition. As c can be extended to a 3-colouring of G, we find that our approach leads to a 3-colouring, say c', of $G - S_c$. As there are at most $3^{9 \cdot 2^6 + 2}$ 3-colourings of $G - P(x)$, we find at most $3^{9 \cdot 2^6 + 2}$ such triples (c, c', S_c). Furthermore, for each 3-colouring c_s of $G - P(x)$, there exists a triple (c_s, c_s', S_{c_s}) if c_s can be extended to a 3-colouring of G. We proceed by considering the case where $S_c \neq \emptyset$ as otherwise $G = G - S_c$. We continue by distinguishing on the problems we are considering. Recall that G is 3-colourable.

Case 1. Acyclic 3-Colouring and Star 3-Colouring.
We check whether for some triple (c, c', S_c), the 3-colouring c' of $G - S_c$ that can be extended to an acyclic 3-colouring or star 3-colouring of G. By this approach, we clearly solve Acyclic 3-Colouring and Star 3-Colouring.

Let (c, c', S_c) be an arbitrary triple as defined above. Recall that a star 3-colouring of a graph is acyclic. In time $O(n + m)$, we can determine the components of $G[S_c]$ and check whether $G[S_c]$ is a forest. If not, then $G[S_c \cup \{x\}]$, and thus G, is not acyclic 3-colourable. We continue and assume that $G[S_c]$ is a forest. We check in $O(n + m)$ time if a vertex of S_c has a neighbour in $N(y) \cap N(z)$. If so, say $s \in S_c$ is adjacent to $v \in N(y) \cap N(z)$, then c' cannot be extended to an acyclic 3-colouring of c as either s and x are coloured alike or one of $\{svyxs, svzxs\}$ is a bichromatic 4-cycle.

We proceed by assuming that S_c has its neighbours in $N_2 \cup S_c \cup \{x\}$. As G is chair-free, every two non-adjacent vertices of S_c share the same neighbours in N_2 and, if there exists such a neighbour, then these two vertices have to be coloured differently to avoid a bichromatic 4-cycle. Therefore, in any acyclic extension of c' to G, each of the two colour classes in S_c either has size at most 1 or has no neighbour in N_2. We check in constant time if S_c is of size at most 2. If so, then there are at most 4 possibilities to extend c' to a 3-colouring of G and for each we apply Observation 7. Hence, we may assume $|S_c| \geq 3$. We check in $O(n + m)$ time if a vertex of S_c has a neighbour in N_2.

Let us consider the subcase where $s \in S_c$ has a neighbour, say v, in N_2. Let G_s be the component of $G[P(x)]$ that contains s. Note that there are at most two possibilities to extend c' to the vertices of G_s. We check in linear time if $S_c \setminus V(G_s)$ is of size at least 2. If so, say $s_1, s_2 \in S_c \setminus V(G_s)$, then v is a neighbour of s, s_1, and s_2. Thus, $xs_1'vs_2'x$ is a bichromatic 4-cycle for two vertices s_1' and s_2' of $\{s, s_1, s_2\}$. We conclude that c' cannot be extended to an acyclic 3-colouring of G. Hence, we may assume $|S_c \setminus V(G_s)| \leq 1$, and so there are at most four possibilities to extend c' to a 3-colouring of G each of which can be obtained in $O(n + m)$ time. We apply Observation 7 for each.

We proceed by assuming that no vertex of S_c has a neighbour in N_2. In other words, each vertex of S_c has its neighbours in $S_c \cup \{x\}$. As x is a cut-vertex of G, any extension of c' to a 3-colouring of G is acyclic if and only if c' is acyclic. Hence, we apply Observation 7 on $G - S_c$ and c' in order to solve Acyclic 3-Colouring.

We now check in $O(n + m)$ time if each component of $G[S_c]$ is of diameter at most 2. If not, then $G[S_c \cup \{x\}]$, and thus G is not star 3-colourable. Let us proceed by assuming that each component of $G[S_c]$ is of diameter at most 2. We find that every 3-colouring of $G[S_c \cup \{x\}]$ is a star 3-colouring. In other words, we can restrict ourselves to those 3-colouring extensions of c' to G that assign one colour to all vertices of S_c if S_c is independent, and an arbitrary 3-colouring extensions of c' to G if S_c is not independent. Note that we can check in $O(n + m)$ time whether S_c is independent. We find in both subcases at most two

extensions of c' to G and apply Observation 7 for each in order to solve Star 3-Colouring.

Case 2. Independent Odd Cycle Transversal.
Let k be an arbitrary integer. We check whether some triple (c, c', S_c) consists of a 3-colouring c' of $G - S_c$ that can be extended to a 3-colouring of G whose one colour class is an independent odd cycle transversal of size at most k. As all the yes-instances require G to be 3-colourable, this approach clearly solves Independent Odd Cycle Transversal.

Let (c, c', S_c) be an arbitrary triple as defined above. Moreover, let X, Y, Z be the colour classes of c' with $x \in X$, $y \in Y$, and $z \in Z$. Clearly, X, Y, and Z can be computed in linear time. We decide in linear time which of $\{Y, Z\}$ is of smaller size, say $|Y| \leq |Z|$.

Recall that all vertices of S_c have their neighbours in $S_c \cup X$. Note that c' can be extended to a 3-colouring of G by 2-colourings of $G[S_c]$ on the colours that c' assigns to y and z, and these are the only possibilities. We find that the smallest possible colour class of a 3-colouring of G that extends c' consists of the vertices either in X or in $Y \cup W$, where W is the smallest possible colour class of a 2-colouring of $G[S_c]$. As we can compute the components of $G[S_c]$ and its parts in $O(n + m)$ time, we find W in the same time. Hence, the smallest possible independent odd cycle transversal of G that is a colour class of an extension of c' to a 3-colouring of G is of size $\min\{|X|, |Y \cup W|\}$. We can compare the sizes of X and $Y \cup W$ with k in linear time.

Case 3. Independent Feedback Vertex Set and Near-Bipartiteness.
Let k be an arbitrary integer. We check whether some triple (c, c', S_c) consists of a 3-colouring c' of $G - S_c$ that can be extended to a 3-colouring of G whose one colour class is an independent feedback vertex set (of size at most k). As all the yes-instances require G to be 3-colourable, this approach clearly solves Independent Feedback Vertex Set and Near-Bipartiteness.

Let (c, c', S_c) be an arbitrary triple as defined above. Moreover, let X, Y, Z be the colour classes of c' with $x \in X$, $y \in Y$, and $z \in Z$. Clearly, X, Y, and Z can be computed in linear time. We check first whether $G - X$ is a forest in $O(n + m)$ time. If so, then we find that X is an independent feedback vertex set of G and we can determine its size in linear time. Hence, we proceed by assuming that $G - X$ contains a cycle or $|X| > k$. As we aim to find an extension of c' to a 3-colouring of G whose one colour class is an independent feedback vertex set (of size at most k), we find that such a set consists of the vertices of Y or of Z, and the vertices of some set $A \subseteq S_c$.

Recall that all vertices of S_c have their neighbours in $[N(y) \cap N(z)] \cup N_2 \cup S_c \cup \{x\}$ and their neighbours in $[N(y) \cap N(z)] \cup N_2 \cup \{x\}$ form an independent set. Note that c' can be extended to a 3-colouring of G by 2-colourings of $G[S_c]$ on the colours that c' assigns to y and z, and these are the only possibilities. If $G[S_c]$ is connected, which can be tested in $O(n + m)$ time, then there are at most two such possibilities, and so we apply Observation 8 for each. We proceed by assuming that $G[S_c]$ is disconnected, and so $|S_c| \geq 2$.

We claim that all vertices of S_c have the same neighbours in N_2. Let us assume that v is an arbitrary vertex of N_2 that is adjacent to some vertex of S_c. Let S_v be the set of neighbours of v in S_c. By definition, we find that S_v is non-empty. As G is chair-free, we obtain that every vertex of S_v is adjacent to every vertex of $S_c \setminus S_v$ as otherwise $\{s_1, s_2, v, x, y\}$ would induce a chair for some possible vertices $s_1 \in S_v$ and $s_2 \in S_c \setminus S_v$. As $G[S_c]$ is disconnected, we find that $S_c \setminus S_v = \emptyset$, which completes the proof of our claim as v is arbitrarily chosen.

We can check if there is a vertex in $N(y) \cap N(z)$ in $O(n + m)$ time. First assume there is such a vertex, say w. As $\{s_1, s_2, w, x, y\}$ does not induce a chair for each two vertices s_1, s_2 of an independent set I of $G[S_c]$, we find that w is adjacent to all but at most one vertex of I. As $G[S_c]$ is bipartite, it follows that w has at least $|S_c| - 2$ neighbours in S_c. For each

$s \in N(w) \cap S_c$, we find $s \in A$ as $sxyws$ and $sxzws$ are 4-cycles. Note that $N(w) \cap S_c$ can be computed in $O(n + m)$ time. As $|N(w) \cap S_c| \geq |S_c| - 2$, we find at most eight possibilities to extend c' to a 3-colouring of G by a 2-colouring of $G[S_c]$ in which one colour class contains all the vertices of $N(w) \cap S_c$. We apply Observation 8 for each. Hence, we may assume that $N(y) \cap N(z) = \emptyset$, and so every two vertices of S_c share the same neighbours in $V(G) \setminus S_c$.

If no vertex of N_2 has a neighbour in S_c, then x is a cut-vertex. In this case we find that G has an independent feedback vertex set of size at most k if and only if $G - S_c$ has an independent feedback vertex set (of size at most $k - |W|$, where W is the smallest possible colour class of a 2-colouring of $G[S_c]$. As W can be computed in $O(n + m)$ time, we apply Observation 8 for $G - S_c$ and c'.

We proceed by considering the situation where $v \in N_2$ has a neighbour in S_c. Recall that all vertices of S_c are adjacent to v. As xs_1vs_2x is a 4-cycle for any two vertices $s_1, s_2 \in S_c$, we find that A has size at least $|S_c| - 1$. In other words, we aim for such a 2-colouring of $G[S_c]$ whose one colour class is of size at most 1. If S_c is not independent, we have at most two such possibilities, and each leads to a 3-colouring of G. We apply Observation 8 for each. Now suppose that S_c is independent. We find that any two vertices of S_c have the same neighbours in G. Let us fix one vertex, say, s of S_c. As there is at most one vertex of S_c that is not in the independent feedback vertex set, we may assume that s is that vertex. We have four ways of colouring the vertices of S_c such that all vertices of $S_c \setminus \{s\}$ receive the same colour. It remains to apply Observation 8 for each case.   ◄

4   Polyad-Free Graphs of Bounded Diameter

In this section we show that for bounded diameter tractability, we should not seek to extend some of the results from the previous section to omission of arbitrary polyads (subdivided stars). We let $K_{1,r}^{\ell}$ denote the *ℓ-subdivided r-star*, which is the graph obtained from a star $K_{1,r}$ by subdividing *one* edge of $K_{1,r}$ exactly ℓ times.

The problems for which we show NP-hardness when restricted to H-free graphs for some polyad H are INDEPENDENT ODD CYCLE TRANSVERSAL, ACYCLIC 3-COLOURING and STAR 3-COLOURING. We show this in Theorems 17–19, respectively.

In our proofs we reduce from two special variants of the NOT-ALL-EQUAL 3-SATISFIABILITY problem, which is well-known to be NP-complete [30]. The problem is defined as follows. Given a CNF formula ϕ that consists of a set $X = \{x_1, x_2, ..., x_n\}$ of logical variables, and a set $C = \{C_1, C_2, ..., C_m\}$ of three-literal clauses over X, does there exist a truth assignment for X such that each C_j contains at least one true literal and at least one false literal? If such a truth assignment exists, then ϕ is *not-all-equal satisfiable*.

Before explaining the two variants, we need to introduce some additional terminology. Let ϕ be an instance of NOT-ALL-EQUAL 3-SATISFIABILITY in which every clause still has three literals, but each literal appears in at most two clauses. A *collection* $\mathcal{C}$ of ϕ is a set of pairs (z, C) such that

- z is a literal of the clause C, and
- for every z that appears in at least one clause of ϕ, there exists exactly one pair $(z, C) \in \mathcal{C}$.

A clause C is *uncovered* by $\mathcal{C}$ if $(z, C) \notin \mathcal{C}$ for each literal z that appears in C. The collection $\mathcal{C}$ is *covering* if no clause of ϕ is uncovered.

We can now discuss the aforementioned two variants of NOT-ALL-EQUAL 3-SATISFIABILITY, which we call variant A and B, respectively. In variant A, every clauses still has three literals, but each literal appears in at most two clauses. Furthermore, variant A requires as additional

input a covering collection. In variant B, every clause also still has three literals, but each clause has only positive literals and each literal occurs in at most four different clauses. For variant B we do not require the input to include a covering collection. Variants A and B are both NP-complete. For variant A, we prove this in our next lemma, whereas the NP-completeness of variant B was shown by Darmann and Döcke [14].

▶ **Lemma 16.** *Variant A of* NOT-ALL-EQUAL 3-SATISFIABILITY *is* NP-*complete.*

Proof. We reduce from NOT-ALL-EQUAL 3-SATISFIABILITY in two steps.

Step 1. Let ϕ be a CNF formula with m clauses and n variables. Suppose that literal z appears in $k \geq 3$ clauses, say z appears in clauses $C_1, C_2, \ldots, C_k$. From ϕ we first construct a CNF formula ϕ' as follows:

- Add three new variables y_z, z_1, and z_2.
- Add three new clauses $(y_z, z_1, \bar{z}_2)$, $(y_z, \bar{z}_1, z_2)$ and $(z, \bar{z}_1, \bar{z}_2)$.
- For $i = 1, 2$, replace z by z_i in C_i.

We note that ϕ is not-all-equal satisfiable if and only if ϕ' is not-all-equal-satisfiable. The reason is that z_1, z_2 and z have the same value in any truth assignment of ϕ' for which each clause contains at least one true literal and at least one false literal. We also note that $y_z, z_1, \bar{z}_1, z_2, \bar{z}_2$ each appear in at most two clauses of ϕ'. Furthermore, z appears in at most $k - 1$ clauses of ϕ', namely, in $C_3, \ldots, C_k$ and $(z, \bar{z}_1, \bar{z}_2)$. Hence, applying this reduction recursively (at most $O(n \cdot m)$ times) leads to an equivalent CNF formula of NOT-ALL-EQUAL 3-SATISFIABILITY of polynomial length, in which each literal appears in at most two clauses.

Step 2. Let ϕ be an instance of NOT-ALL-EQUAL 3-SATISFIABILITY with m clauses and n variables such that each literal appears in at most two clauses.

For every literal z that appears in at least one clause, we fix one clause, say C_z. Let $\mathcal{C}$ be the set of all pairs (z, C_z). Note that $\mathcal{C}$ is a collection of ϕ and that its computation takes $O(nm)$ time. Let $\gamma_\phi(\mathcal{C})$ be the number of clauses in ϕ which are uncovered.

If $\gamma_\phi(\mathcal{C}) > 0$, then let C_1 be an uncovered clause and z be a literal that appears in C_1. By construction, there is a second clause C_2 in which z appears as literal and $(z, C_2) \in \mathcal{C}$. From $(\phi, \mathcal{C})$ we first construct a CNF formula ϕ' and a collection $\mathcal{C}'$ as follows:

- Add three new variables y_z, z_1, and z_2.
- Add three new clauses $(y_z, z_1, \bar{z}_2)$, $(y_z, \bar{z}_1, z_2)$, $(z, \bar{z}_1, \bar{z}_2)$.
- For $i = 1, 2$, replace z by z_i in C_i.
- Remove the pair (z, C_2).
- Add six new pairs $(y_z, (y_z, \bar{z}_1, z_2))$, $(z, (z, \bar{z}_1, \bar{z}_2))$, (z_1, C_1), (z_2, C_2), $(\bar{z}_1, (z, \bar{z}_1, \bar{z}_2))$, and $(\bar{z}_2, (y_z, z_1, \bar{z}_2))$.

By repeating the arguments from the proof of Step 1, we find that ϕ is not-all-equal satisfiable if and only if ϕ' is not-all-equal-satisfiable. As before, it is readily seen that ϕ is an instance of NOT-ALL-EQUAL 3-SATISFIABILITY in which each literal appears in at most two clauses. Moreover, $\mathcal{C}'$ is a collection of ϕ'. This follows from the fact that $\mathcal{C}$ is a collection of ϕ and our construction: we replaced (z, C_2) by a new pair that contains z and added exactly one pair for the literals $y_z, z_1, z_2, \bar{z}_1, \bar{z}_2$. We did this in such a way that $\mathcal{C}'$ covers all the new clauses as well as C_1 and C_2. Thus, $\gamma_{\phi'}(\mathcal{C}') < \gamma_\phi(\mathcal{C})$. Hence, applying this reduction recursively (at most $O(m)$ times) leads to an equivalent CNF formula of NOT-ALL-EQUAL 3-SATISFIABILITY of polynomial length, in which each literal appears in at most two clauses and for which a covering collection is fixed.   ◀

▶ **Theorem 17.** INDEPENDENT ODD CYCLE TRANSVERSAL *is* NP-*complete for $K_{1,4}^3$-free graphs of diameter* 4.

Proof. We deploy an argument previously used in [21]. Let us recall first the standard reduction to 3-COLOURING from NOT-ALL-EQUAL 3-SAT. We use variant A. Hence, we are given a CNF formula ϕ with clauses $C_1, \dots, C_m$ and variables $x_1, \dots, x_n$ such that each literal appears in at most two clauses. From ϕ, we construct a graph G as follows (see also Figure 5):

- Add two vertices v_{x_i} and $v_{\bar{x}_i}$ for each variable x_i.
- Add an edge between v_{x_i} and $v_{\bar{x}_i}$ for $i = 1, \dots, n$.
- Add a new vertex z and make z adjacent to every v_{x_i} and every $v_{\bar{x}_i}$
- For each clause C_i, add a triangle T_i with three new vertices $c_{i_1}, c_{i_2}, c_{i_3}$. Fix an arbitrary order of the literals of C_i, say $y_{i_1}, y_{i_2}, y_{i_3}$ where $y_{i_j} \in \{x_{i_j}, \bar{x}_{i_j}\}$, and add the edge $v_{y_{i_j}} c_{i_j}$ for $j = 1, 2, 3$.
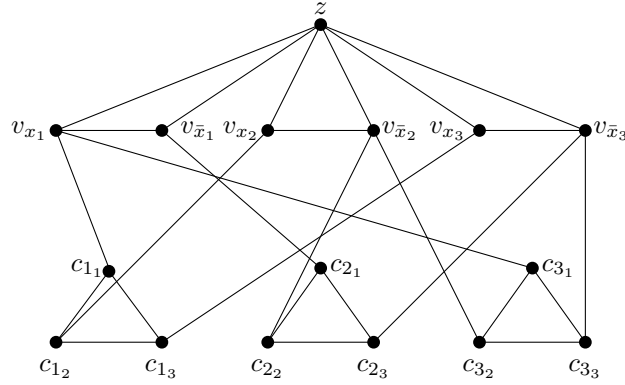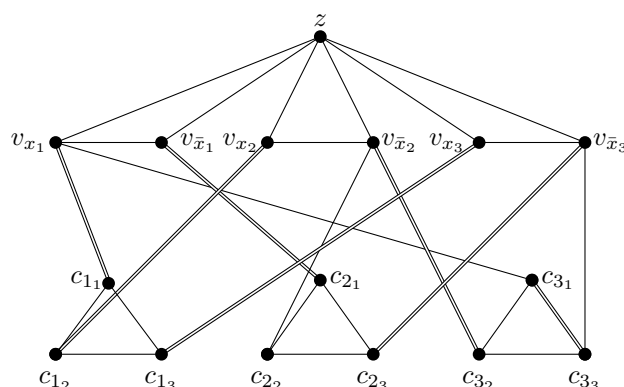

■ **Figure 5** The reduction from Variant A of NOT-ALL-EQUAL 3-SATISFIABILITY to INDEPENDENT ODD CYCLE TRANSVERSAL on the instance $\phi = (x_1, x_2, x_3), (\bar{x}_1, \bar{x}_2, \bar{x}_3), (x_1, \bar{x}_2, \bar{x}_3)$.

We now show that G has diameter at most 4. First note that any literal vertex is adjacent to z and any clause vertex is adjacent to some literal vertex, so any vertex is at distance at most 2 from z. Therefore any two vertices are at distance at most 4.

Next, we show that G is $K_{1,4}^3$-free. Any literal vertex has degree at most 4 since it appears in at most two clauses. However it has at most three independent neighbours since two of its neighbours, its negation and z, are adjacent. Each clause vertex has at most three neighbours. So the only vertex with four independent neighbours is z. The longest induced path with end-vertex z has order at most 4 since any such path contains at most one literal and at most two vertices of any T_i. Therefore G is $K_{1,4}^3$-free.

Finally, we prove that G has an independent odd cycle transversal of size $m + 1$ if and only if ϕ is not-all-equal satisfiable.

First suppose that G has an independent odd cycle transversal S of size $m + 1$. Then $G - S$ is bipartite, say with partition classes A and B, whereas S is an independent set. Hence G has a 3-colouring with colour classes A, B and S. Assume that z is assigned colour 1. Then each literal vertex is assigned either colour 2 or colour 3. If, for some clause C_i, the vertices corresponding to the literals of C_i are all assigned the same colour, then T_i cannot be coloured. Therefore, if we set literals whose vertices are coloured with colour 2 to be true

Figure 6 The reduction from Variant A of Not-all-Equal 3-Satisfiability to Acyclic 3-Colouring on the instance $\phi = (x_1, x_2, x_3), (\bar{x}_1, \bar{x}_2, \bar{x}_3), (x_1, \bar{x}_2, \bar{x}_3)$ given a covering collection $\mathcal{C} = \{(x_1, (x_1, x_2, x_3)), (\bar{x}_1, (\bar{x}_1, \bar{x}_2, \bar{x}_3)), (x_2, (x_1, x_2, x_3)), (\bar{x}_2, (x_1, \bar{x}_2, \bar{x}_3)), (x_3, (x_1, x_2, x_3)), (\bar{x}_3, (\bar{x}_1, \bar{x}_2, \bar{x}_3))\}$. Double lines are edges that are substitution instances of $K_{2,3}$ where the endpoints come from the partition of size 2.

and those coloured with colour 3 to be false, each clause must contain at least one true literal and at least one false literal.

Now suppose that ϕ is not-all-equal satisfiable. Then we can colour vertex z with colour 1, each true literal with colour 2 and each false literal with colour 3. Since each clause has at least one true literal and at least one false literal, each triangle has neighbours in two different colours. This implies that each triangle is 3-colourable. Hence, G is 3-colourable, and in particular we coloured a set S of $m+1$ vertices of G with colour 1 (namely, one vertex of each of the m triangles T_i and the vertex z). Consequently, S is an independent odd cycle transversal of size $m + 1$. ◀

▶ **Theorem 18.** Acyclic 3-Colouring *is* NP-*complete for* $K_{1,6}^5$-*free graphs of diameter* 6.

Proof. Suppose now we attempt to frame a similar argument for Acyclic 3-Colouring by putting the construction used in the proof of Theorem 17 through a process by which we map edges to bipartite graphs $K_{2,3}$ (which is the standard reduction from 3-Colouring to Acyclic 3-Colouring). That is, for Acyclic 3-Colouring, we replace each edge $u_1 u_2$ by three new vertices w_1, w_2, w_3 and edges $u_i w_j$ for $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$. Alas, using vertex z, which is the only vertex of unbounded degree, we can find arbitrarily large polyads. However, we reduce from variant A and only substitute the edges of a certain matching.

Let ϕ be an instance of variant A and $\mathcal{C}$ be a given covering collection of ϕ. We only substitute edges $w_1 w_2$ in the construction used in the proof of Theorem 17 for which

- $w_1 w_2 = v_{z_1} c_{i_j}$ such that $(z_1, C_i) \in \mathcal{C}$; or
- $w_1 w_2 = c_{i_h} c_{i_j}$ such that there are two literals z_1 and z_2 with $(z_1, C_i), (z_2, C_i) \notin \mathcal{C}$, and c_{i_h} and c_{i_j} are the two vertices of T_i adjacent to v_{z_1} and v_{z_2}, respectively.

This is drawn in Figure 6. We note that the set of edges we substitute is indeed a matching. We claim that the resulting graph G has diameter at most 6. Indeed, there is path of length at most 3 from any vertex to z. We further claim that G is $K_{1,6}^5$-free, which follows from the fact that any induced $K_{1,6}^5$ in the graph would have to involve the vertex z as its centre (as the other vertices have at most 5 independent neighbours) and all paths of order 7 starting from z must involve two vertices adjacent to z.

Finally, we claim that G is acyclically 3-colourable if and only if ϕ is not-all-equal satisfiable.

First suppose that G is acyclically 3-colourable. Given an acyclic 3-colouring of G, assume z is assigned colour 1. Then each literal vertex is assigned either colour 2 or colour 3. The argument here concludes as it does in the reduction to 3-Colouring, since in any acyclic 3-colouring of the gadget $K_{2,3}$ the vertices in the partition of size 2 must receive distinct colours.

Now suppose that ϕ is not-all-equal satisfiable. Then, we can colour vertex z with colour 1, each true literal v_{x_i} with colour 2 and each false literal v_{x_i} with colour 3. The proof concludes as in the reduction to 3-Colouring except that we must argue that there are no bichromatic cycles. For a contradiction, assume that there is a bichromatic cycle, say C. Bearing in mind that in a substitution instance of $K_{2,3}$, the vertices of the partition of size 2 are coloured distinct, we note that C cannot contain any edge that belongs to such a gadget. As C is a bichromatic cycle, we find that it contains at least two adjacent vertices v_{z_1} and $c_{i_{j_1}}$. Moreover, C contains exactly one other vertex, say $c_{i_{j_2}}$, from $\{c_{i_1}, c_{i_2}, c_{i_3}\}$, and this vertex is adjacent to $c_{i_{j_1}}$. It follows that some v_{z_2} for $z_2 \neq z_1$ is the second neighbour of $c_{i_{j_2}}$ in C. But now $(z_1, C_i), (z_2, C_i) \notin \mathcal{C}$ and $c_{i_{j_1}} c_{i_{j_2}}$ is an edge of G, which contradicts our substitution rule. ◀

▶ **Theorem 19.** Star 3-Colouring *is* NP-*complete for $K_{1,6}^{14}$-free graphs of diameter* 14.

Proof. Our construction for Acyclic 3-Colouring in the proof of Theorem 19 cannot be directly adapted for Star 3-Colouring. Instead of this, we reduce from Variant B of Not-All-Equal 3-Satisfiability. Hence, we are given a CNF formula ϕ with clauses $C_1, \ldots, C_m$ and variables $x_1, \ldots, x_n$ such that each C_i consists of three positive literals and each literal occurs in at most four different clauses. From ϕ, we construct a graph G as follows (see also Figure 7):

- Add a vertex v_{x_i} for each variable x_i.
- Add a vertex z adjacent to each vertex v_{x_i}.
- Add two new vertices z', z'' in a triangle with z.
- Add vertices $p_{x_i}^1, p_{x_i}^2, p_{x_i}^3, p_{x_i}^4$ for each instance of a variable x_i with edges from each of these to v_{x_i}.
- Add vertices $q_{x_i}^1, q_{x_i}^2, q_{x_i}^3, q_{x_i}^4$ for each instance of a variable x_i with edges from each $q_{x_i}^j$ to $p_{x_i}^j$ which are substitution instances of $K_{2,2}$ and the endpoints come from the same part.
- For each clause C_i add a triangle T_i with vertices $c_{i_1}, c_{i_2}, c_{i_3}$ where edges are substitution instance of $K_{2,2}$ and the endpoints come from the same part.
- Fix an arbitrary order of the literals of every C_i, $x_{i_1}, x_{i_2}, x_{i_3}$. Assign every pair (i, j) a vertex of $q_{x_{i_j}}^1, q_{x_{i_j}}^2, q_{x_{i_j}}^3$ and make this vertex adjacent to c_{i_j}, such that this assignment is injective. Let each of the new edges be a substitution instance of $K_{2,2}$, where the endpoints come from the same part.

We draw the special edges that are in fact built from instances $K_{2,2}$ with double lines in Figure 7. We claim that G has diameter at most 14. Indeed, every vertex has a path of length at most 7 from it to z. We also claim that G is $K_{1,6}^{14}$-free. Vertices that are not equal to some c_{i_j} or z have degree at most 5 so cannot be the centre of an induced $K_{1,6}^{14}$. Moreover, every c_{i_j} has degree 6, but for every c_{i_j} there is no vertex that is adjacent to exactly one of the neighbours of c_{i_j}. Hence, no c_{i_j} can be the centre of an induced $K_{1,6}^{14}$ either. Finally, z cannot be the centre of an induced $K_{1,6}^{14}$ either, as every induced path that starts from z has length at most 14. Hence, G is indeed $K_{1,6}^{14}$-free.
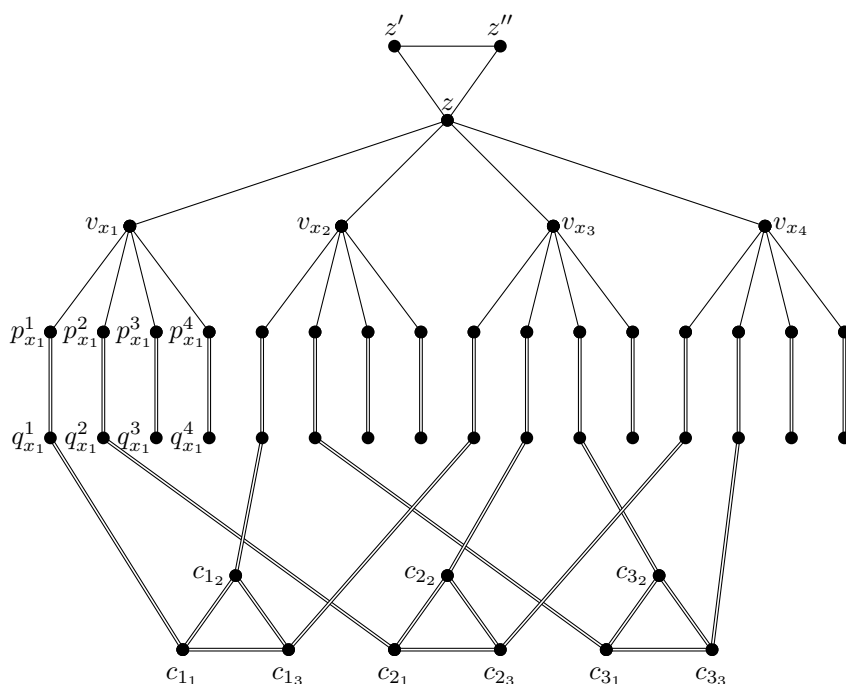
Figure 7 The reduction from variant B of Not-All-Equal 3-Satisfiability to Star 3-Colouring on the instance $\phi = (x_1, x_2, x_3), (x_1, x_3, x_4), (x_2, x_3, x_4)$. Double lines are edges that are substitution instances of $K_{2,2}$ where the endpoints come from same partition.

Finally, we claim that G is star 3-colourable if and only if ϕ is not-all-equal satisfiable. First suppose that G is star 3-colourable. Given a star 3-colouring of G, assume z is assigned colour 1. Then each vertex v_{x_i} is assigned either colour 2 or colour 3. Now each of the vertices $p_{x_i}^1, p_{x_i}^2, p_{x_i}^3$ is assigned the same colour from $\{2,3\}$ (this is enforced by the fact that $\{z', z''\}$ must be coloured $\{2,3\}$ which forbids any possibility that colour 1 is used). Furthermore each of the vertices $q_{x_i}^1, q_{x_i}^2, q_{x_i}^3$ is assigned precisely the colour from $\{2,3\}$ that $p_{x_i}^1, p_{x_i}^2, p_{x_i}^3$ was not assigned. The argument here concludes as it does in the reduction to 3-Colouring.

Now suppose that ϕ is not-all-equal satisfiable, then we can colour vertex z with colour 1, the vertices $q_{x_i}^1, q_{x_i}^2, q_{x_i}^3$ of each true literal with colour 2 and the vertices $q_{x_i}^1, q_{x_i}^2, q_{x_i}^3$ of each false literal with colour 3. Then, since each clause has at least one true literal and at least one false literal, each triangle has neighbours in two different colours. This implies that each triangle is 3-colourable. The argument here concludes as it does in the reduction to 3-Colouring. ◄

5   Conclusions

We showed that bounding the diameter does not help for Independent Set for H-free graphs. We proved that this does help for some problems related to 3-Colouring if H is the chair. Whether these results can be extended to larger polyads H is an interesting but challenging task. For three of these problems we gave a polyad H such that they are NP-complete for H-free graphs of diameter d for some constant d. Such a polyad H was already known to exist for 3-Colouring of graphs of diameter at most 4 [21]. We ask if there exists a polyad H and an integer d such that Near-Bipartiteness and Independent Feedback Vertex Set are NP-complete for H-free graphs of diameter at most d.

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