

Interacting conformal scalar in a wedge

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ABSTRACT: We study a class of two-point functions in a conformal field theory near a wedge. This is a set-up with two boundaries intersecting at an angle θ . We compute it as a solution to the Dyson-Schwinger equation of motion for a quartic interaction in the $d = 4 - \epsilon$ bulk and on one of the boundaries in $d = 3 - \epsilon$, up to order $\mathcal{O}(\epsilon)$. We have extracted the anomalous dimensions from such correlators, which we complemented with Feynman diagrams computations.

KEYWORDS: Boundary Quantum Field Theory, Scale and Conformal Symmetries

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1 Introduction and outlook

Conformal field theories (CFT's) play an important role in understanding the physics of critical phenomena. In particular, critical exponents are related to the quantum numbers of local operators in a CFT. However, if we would like to make connection with real world systems, we cannot avoid discussing the presence of boundaries or defects. Another reason to study these instances come from string theory, where the appearance of D-branes as boundaries is central in this context. The presence of boundaries and/or defects reduces the amount of symmetry preserved, but the residual symmetry is still powerful enough to constrain the structure of correlators, making them more intricate but richer. In particular it is possible to define an additional operator product expansion (OPE), the so called bulk-to-boundary or bulk-to-defect OPE, which allow us to expand an operator in the presence of a boundary/defect as an infinite sum of boundary/defect operators.

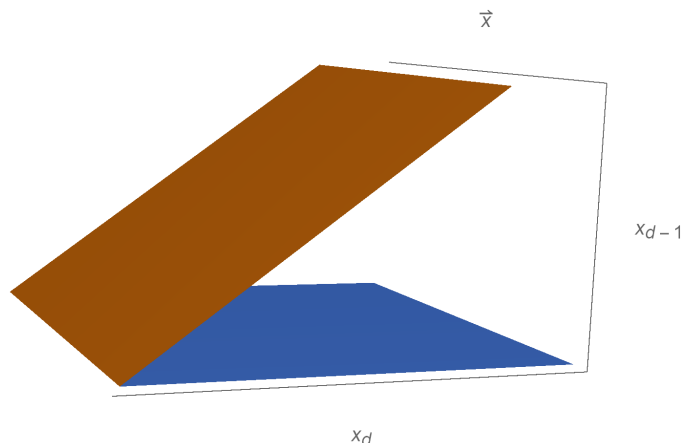


Figure 1. An illustration of the setup we consider, where the blue plane is the wall, and the orange is the ramp. The wedge is the intersection of the two boundaries.

In this paper we consider a d -dimensional CFT living in a wedge-shaped region bounded by two $(d-1)$ -dimensional half-planes intersecting at an angle $\theta \in (0, 2\pi)$, as in figure 1. To be concrete, we take one half-plane, which we call *wall*, located at $x_{d-1} = 0$, and the other one, the *ramp*, defined by $x_{d-1} = \tan \theta x_d$. With a little abuse of terminology, we will call *wedge* the intersection of the two boundaries while we will refer to this entire system as a *wedge conformal field theory* (WCFT).

The WCFT setup naturally breaks the bulk full conformal symmetry $SO(d+1, 1)$, in particular translations in the normal directions. The preserved amount of symmetry is then the conformal group in $(d-2)$ dimensions, namely $SO(d-1, 1)$. Despite the conformal symmetry is significantly reduced from a global point of view, from an OPE perspective we can consider the system as containing four CFT's of different dimensionalities: one in the bulk with $SO(d+1, 1)$ symmetry, one on each codimension-one boundary preserving $SO(d, 1)$, and one on the codimension-two wedge with $SO(d-1, 1)$ symmetry.

Quantum numbers of local operators in a WCFT are related to the surface and wedge critical exponents in addition to the bulk ones. Probing such CFT's not only constrains the surface and wedge observables, but also gives access to a part of the bulk observables in a homogeneous CFT (without boundaries). In particular, while the quantum numbers associated to the wall and the ramp are universal, the ones associated with the wedge depend on the angle θ . Systems with wedges have been studied in the older literature [1–7] and they are of interest to condensed matter physics. In addition, we also notice that the identification of the two boundaries, $\theta \rightarrow 2\pi$, leads to a theory in the presence of a conical singularity located on the wedge. Quantum field theories in the presence of such conical defects have been widely discussed in the literature, see e.g. [8–11]. They play an important role in cosmology and string theory (cosmic strings) [12, 13], and in the entanglement literature where they are used as a computational tool for studying Rényi and entanglement entropies [14, 15].

One powerful method to study and to characterise CFT's is the conformal bootstrap. This approach is used to constrain the space of possible conformal dimensions and OPE

coefficients, by enforcing a few mild assumptions based mostly on symmetries. Since its revival in [16], there has been huge progress in using the conformal bootstrap in its numerical and analytic incarnation applied in several contexts, see [17, 18] for recent reviews on the topics. Recently, there has been an interest in studying WCFT's using the methods of conformal bootstrap [19]. This is a non-perturbative approach and refers to no Lagrangian description for the underlying microscopic theory. The idea is to focus on the residual symmetries preserved by the wedge and the boundaries, and impose consistency conditions on the correlation functions. Although the symmetry is reduced from the full conformal group, it is still powerful enough to constrain the observables. This is a nice playground to study the bulk properties of the CFT in addition to the boundary and wedge ones. Bulk one-point functions are non-trivial observables in this case and admit an expansion into wall- and ramp-channel conformal blocks [19]. Moreover, two-point correlation functions of a bulk and an edge scalar operator carry information about the dynamical content of the CFT living on the wedge. Those correlators can also be decomposed into a wall- and a ramp-channel [19], and the dynamical content of the WCFT data entering in these bootstrap equations is encoded in the bulk, boundary and wedge scaling dimensions in addition to the corresponding boundary operator product expansion (BOE) coefficients.

Motivated by these developments, in this paper we focus on the study of a scalar field theory with global $O(N)$ symmetry in the free case first, and successively in the presence of either bulk or boundary interactions. Since we are interested in the situation with the maximal amount of symmetry, we will only impose conformal boundary conditions (b.c.'s) both on the wall and ramp. In particular, there are three distinct possible choices of b.c.'s that we can consider:

- NN: both boundaries are equipped with Neumann b.c.'s.
- DD: both boundaries are equipped with Dirichlet b.c.'s.
- DN: one boundary (for example the wall) is equipped with Dirichlet b.c., and the other (the ramp) with Neumann b.c.

In this setup, we will study, among other things, the two-point functions involving one bulk and one wedge operator. More specifically, we compute the solution to the Dyson-Schwinger (DS) equation for a quartic interaction in the $d = 4 - \epsilon$ bulk and on one of the boundaries in $d = 3 - \epsilon$, up to order $\mathcal{O}(\epsilon)$, along the lines of [20]. We have extracted the anomalous dimensions from such correlators, which we have complemented with Feynman diagrams computations, finding perfect agreement. We have also used the OPE to extract BOE coefficients. At order $\mathcal{O}(\epsilon)$ the structure of the result is quite neat, and it contains a logarithmic term as expected at this order in perturbation theory.

There are several directions that would be interesting to pursue.

- *Discontinuity as building block.* In homogeneous CFT's there has been progress in understanding four-point correlators using a single variable dispersion relation by exploiting the analytic structure of the bootstrap equation [21, 22]. The idea is to use the analytic structure together with the crossing symmetry of the correlator to

constrain the CFT data. The correlators in a boundary conformal field theory (BCFT) can also be studied using bootstrap methods. The bootstrap approach for BCFT's was initiated in [20]. In [23], the analytic structure of the conformal blocks was exploited to constrain the BCFT data. This method was further modified in [24] for CFT's with interfaces (where there is a bulk theory on each side of the codimension one defect). It would be interesting to generalise the approach presented recently in [25, 26] to the wedge configuration to be able to understand how to systematise the study of one- and two-point functions. As a consequence, it would be interesting to further explore the connections found in [27] to more general cases, such as the one presented in this paper.

- *Bootstrap equations.* Another interesting direction to pursue is to exploit the power of conformal symmetry and use the conformal bootstrap approach, for instance adapting [19] to higher order in perturbation theory. This approach would allow us to study more general interactions and to constrain different OPE data, using the different OPEs involving operators belonging to the wall/ramp/bulk. In addition, it would be interesting to complement the WCFT bootstrap equations with the ones coming from the BCFT and the homogeneous case. The study of the compatibility of the mixed system could potentially constrain even more the CFT data.
- *Holographic description.* Holography in the form of the AdS/CFT correspondence [28–30] is one of the main tools to approach quantum field theories at strong coupling. The AdS space-time can be endowed with suitable branes which can be used to add boundaries and/or defects to the CFT living on the asymptotic boundary of the holographic space-time. In particular, a bottom-up construction has been proposed in [31, 32] where an end-of-the-world brane introduces a boundary in the CFT side of the correspondence. In this framework, it would be interesting to first find solutions to the Einstein equations describing CFT's confined in wedge-shaped regions,¹ and successively study their correlation functions and other relevant quantities related to quantum information. The holographic approach is particularly convenient to obtain analytic and numerical results for the entanglement entropy by using the Ryu-Takayanagi formula [34, 35], which can also be applied to the AdS/BCFT case [36–39]. The entanglement entropy has been computed in a free-scalar WCFT in [40] (see also [41] for a numerical approach).
- *Conformal anomalies and RG flows.* Conformal anomalies and their associated coefficients, sometimes called central charges, play a major role in the classification of CFT's and in defining monotonic quantities along the RG flow, providing strong constraints on the possible outcomes of the latter. While in a CFT conformal anomalies are present only in even space dimensions, the situation is richer when boundaries and/or defects occur, and the possibility of having anomalies localised on them considerably extends the possible anomaly contributions [42–46]. Conformal anomalies of boundaries and defects have been widely studied in the literature, where monotonicity

¹In the context of the AdS₃/CFT₂ duality, the AdS-bulk dual to a wedge in $d = 2$ has been found in [33].

theorems have been proven [47–51] or conjectured [52], and various relations to boundary and bulk correlation functions have been found [53–58]. We find it interesting to investigate them in the context of the WCFT setup, which is a natural playground where anomalies localised on manifolds of different dimensions may relate to each other through either bulk, boundary or wedge RG flows.

The structure of the paper is as follows. In section 2 we study a free scalar field theory confined in the wedge-shaped region described above in generic bulk dimension d . We first obtain the mode expansion of the field (eq. (2.10)) in the canonical formulation for all the possible combinations of b.c.’s, and by taking a suitable wedge limit we define some of the wedge primaries of the theory. After that, we find the propagator reported in eq.’s (2.16), (2.19), from which we obtain the relevant correlators we will use in the rest of the paper, namely the bulk one-point function of the fundamental field (2.26), (2.27), (2.29) and of the stress tensor (2.35), (2.36), (2.37). As regarding the latter, in eq. (2.34) we also report its general form consistent with conservation and traceness. We then compute the bulk-wedge and wedge-wedge two-point functions that can be found in eq. (2.38) and (2.39) respectively. We finally discuss a wedge-RG flow triggered by wedge primaries corresponding to a mode singular in the wedge limit which can be present in the expansion of the bulk fundamental field.

In section 3 we study the bulk-wedge two-point correlation function of scalar operators using the equation of motion (e.o.m.) for different b.c.’s. In this way, we are able to obtain the first-order correction of the bulk-wedge correlators (eq.’s (3.22)) when the ϕ^4 bulk-interaction is taken into account, and the anomalous dimension of the wedge primaries corresponding to either bulk- or boundary-interactions (see eq. (3.21) and (3.38) respectively).

In section 4 we extract anomalous dimensions of wall and ramp fields by studying the respective boundary limits of the bulk-wedge correlator, and comparing it with the expected result found from the BOE. Moreover, as was showed in [19], bulk-wedge correlators can be decomposed in wall- and ramp-conformal blocks. We find that these blocks satisfy the orthogonality relation (4.14), which we use to read off the BOE coefficients entering in the bootstrap eq. (up to $\mathcal{O}(\epsilon)$).

Part of the computations has been relegated to the appendices. In appendix A we report the details of the derivation of the propagator in the free theory case. In appendix B we provide an alternative derivation of the wedge anomalous dimensions, and finally in appendix C we give some detail on the expansion of conformal blocks at first order in ϵ .

2 Free scalar in a wedge

In this section we consider the theory of a single scalar confined in a wedge. We take the classical Euclidean action of the free scalar in d dimension to be

$$S = \frac{1}{2} \int_{\mathcal{M}} d^d x \sqrt{g} \left(g_{\mu\nu} \nabla^\mu \phi \nabla^\nu \phi + \frac{d-2}{4(d-1)} \mathcal{R} \phi^2 \right), \quad (2.1)$$

where the non-minimal coupling to the metric is required by Weyl invariance. From this action we obtain the e.o.m.

$$\left(-\nabla^2 + \frac{d-2}{4(d-1)}\mathcal{R}\right)\phi = 0, \quad (2.2)$$

and the stress tensor

$$T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{d-2}{4(d-1)} \left[\nabla_\mu \nabla_\nu + \frac{g_{\mu\nu}}{d-2} \nabla^2 - R_{\mu\nu} \right] \phi^2. \quad (2.3)$$

The goal of this section is to solve the theory living in a wedge for generic d . We will use canonical quantisation to obtain the mode expansion of the field, thus we will switch to Lorentzian signature. However, in the rest of the paper, we will use interchangeably the Lorentzian and Euclidean formulations, which are simply related by the replacement $t \rightarrow -i\tau$. In order to take advantage of the symmetry of the problem, we find it convenient to adopt polar coordinates in the (x_{d-1}, x_d) plane, namely

$$ds^2 = -dt^2 + d\mathbf{x}^2 + d\rho^2 + \rho^2 d\varphi^2, \quad x_d^2 + x_{d-1}^2 = \rho^2, \quad \frac{x_{d-1}}{x_d} = \tan \varphi, \quad (2.4)$$

with $\rho > 0$ and $\varphi \in (0, \theta)$. Furthermore we will denote the $d-2$ coordinates parallel to the boundaries as $x_\parallel = (t, \mathbf{x})$.

In this coordinate system the wedge is defined by $\mathcal{M} = \{x_\parallel \in \mathbb{R}^{d-2}, \rho > 0, \varphi \in (0, \theta)\}$ where θ is the opening angle. We are interested in breaking the minimum amount of symmetries. In particular, this means we will impose only conformal b.c.'s. In the case of a free scalar, we can impose either Dirichlet ($\phi = 0$ at the boundaries) or Neumann ($\partial_\perp \phi = 0$ at the boundary) b.c.'s.

The e.o.m. in flat space-time reads

$$\frac{1}{\rho} \partial_\rho (\rho \partial_\rho \phi) + \left[-\partial_t^2 + \frac{\partial_\varphi^2}{\rho^2} + \partial_\mathbf{x}^2 \right] \phi = 0, \quad (2.5)$$

which can be solved by the following ansatz (see for example [1])

$$\phi \sim e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} h(\rho) g(\varphi), \quad g_\alpha(\varphi) \equiv A \sin(\alpha \varphi) + B \cos(\alpha \varphi), \quad (2.6)$$

where the coefficients A and B as well as the parameter α need to be fixed by imposing the b.c.'s (both on the wall and the ramp) and from the normalisation of the angular part $A^2 + B^2 = 1$.

As discussed in section 1, since we are free to impose different b.c.'s on the two boundaries we have 3 distinct possibilities: Dirichlet-Dirichlet (DD), Neumann-Neumann (NN), and Dirichlet-Neumann (DN). Imposing such conditions gives

$$\begin{cases} A = 1, & B = 0, & \alpha = \frac{m\pi}{\theta}, & \text{DD,} \\ A = 0, & B = 1, & \alpha = \frac{m\pi}{\theta}, & \text{NN,} \\ A = 1, & B = 0, & \alpha = \frac{\left(m + \frac{1}{2}\right)\pi}{\theta}, & \text{DN.} \end{cases} \quad (2.7)$$

where m is an integer (non-zero for DD).

This ansatz leads to an ordinary differential equation for $h(\rho)$, namely

$$\rho^2 h'' + \rho h' + [(\omega^2 - \mathbf{k}^2)\rho^2 - \alpha^2] h = 0, \quad (2.8)$$

whose solutions are the Bessel functions

$$h = J_{\pm\alpha} \left(\sqrt{\omega^2 - \mathbf{k}^2} \rho \right), \quad h = Y_{\pm\alpha} \left(\sqrt{\omega^2 - \mathbf{k}^2} \rho \right). \quad (2.9)$$

Regularity of the solution at $\rho \rightarrow 0$ for any value of $\theta \in (0, 2\pi)$ excludes $Y_{\pm\alpha}$, and imposes $\pm\alpha \geq 0$ for the Bessel $J_{\pm\alpha}$.² This requirement leads to

$$\begin{aligned} \phi(x) = \sum_{m \geq 0}' \int_{\mathbb{R}^{d-2}} d^{d-2} \mathbf{k} \int_0^\infty dk_\rho \frac{\sqrt{k_\rho}}{(\sqrt{2}\pi)^{d-3} \sqrt{2}\omega} \sqrt{\frac{2}{\theta}} \\ \times \left[a_{\{k\}} g_{\alpha(m)}(\varphi) e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} + h.c. \right] J_{\alpha(m)}(k_\rho \rho), \end{aligned} \quad (2.10)$$

where we defined $\{k\} \equiv (\mathbf{k}, k_\rho, m)$ and the primed sum means that the zero mode needs to be halved, i.e. its normalisation has an additional $1/\sqrt{2}$. Note that in order to quantise the field, we found it convenient to change integration variable setting $k_\rho = \sqrt{\omega^2 - \mathbf{k}^2}$. The usual canonical commutation relation $[\phi, \Pi] = [\phi, \dot{\phi}] = i\delta$ implies the following commutation relations for the operators a and a^\dagger ,³

$$[a_{\{k\}}, a_{\{k'\}}^\dagger] = \delta(\mathbf{k} - \mathbf{k}') \delta(k_\rho - k'_\rho) \delta_{m,m'}. \quad (2.11)$$

By applying the bulk-boundary BOE, followed by the boundary-wedge BOE, we can write the bulk field in terms of wedge modes [19]. For Dirichlet or Neumann b.c.'s there is only one boundary primary in the bulk-boundary BOE [20]. This means that we can write the fundamental field (2.10) in terms of wedge primaries, which we denote \hat{O}_α , as⁴

$$\phi(x) = \sum_{m \geq 0}' \rho^\alpha B_{\hat{\Delta}_\alpha}^{d-1} \left(\rho^2 \partial_\parallel^2 \right) \hat{O}_\alpha(x_\parallel) g_\alpha(\varphi), \quad (2.12)$$

where we defined

$$\hat{O}_\alpha \equiv \frac{1}{2^\alpha \Gamma(\alpha + 1)} \sqrt{\frac{2}{\theta}} \int \frac{d^{d-1} k}{(\sqrt{2}\pi)^{d-3}} \frac{\sqrt{k_\rho}}{\sqrt{2}\omega} k_\rho^\alpha \left(a_{\{k\}} e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} + h.c. \right). \quad (2.13)$$

²In section 2.5, we discuss about the possibility of having mildly divergent modes in the wedge limit.

³In the computation we use the following orthogonality property of the Bessel functions

$$\int_0^\infty d\rho \rho J_\beta(\rho v) J_\beta(\rho u) = \frac{\delta(u - v)}{u}.$$

⁴The differential operator generating the boundary descendants gives us $g_\alpha(\varphi)$.

The differential operator is defined as⁵

$$B_{\hat{\Delta}_\alpha}^{d-1}(\rho^2 \partial_\parallel^2) \equiv \sum_{j=0}^{+\infty} \frac{(-4)^{-j} (\rho^2 \partial_\parallel^2)^j}{j! (1 + \hat{\Delta}_\alpha - \Delta_\phi)_j}, \quad (2.14)$$

where $(a)_j \equiv a(a+1) \dots (a+j-1)$ and $(a)_0 \equiv 1$ is the Pochhammer symbol. The quantity $\hat{\Delta}_\alpha$ is

$$\hat{\Delta}_\alpha = \Delta_\phi + \alpha, \quad \Delta_\phi = \frac{d-2}{2}, \quad (2.15)$$

and it corresponds to the engineering conformal dimension of the operator \hat{O}_α .

Thus, we see that at least in the specific case of the free theory we can reconstruct the bulk field ϕ in terms of wedge primaries. This is in agreement with the discussion in [59] for a codimension-2 defect. However, we find it important to highlight that the simple form of eq. (2.12) in the present discussion comes from the fact that the linearity of the e.o.m. allows us to factorise the angular dependence even though the wedge breaks the rotational symmetry around the θ -direction. A more general approach, which is discussed in [19], is expanding the bulk operator in terms of boundary operators, and successively expand the latter ones in terms of operators living on the intersection between the boundaries.

It is straightforward to generalise the theory of a single real scalar to the one of N scalars with $O(N)$ symmetry. This can be achieved by introducing an internal index $i \in \{1, \dots, N\}$ and requiring that the commutation relations eq. (2.11) are diagonal. From now on, we will report the result corresponding to the more general $O(N)$ model.

2.1 The bulk-bulk propagator

The bulk-bulk propagator can be computed directly from the mode expansion (2.10). The computation is analysed in detail in appendix A. Here we just report the results. For DD and NN b.c.'s we find

$$\langle \phi^i(x_\parallel, \varphi, \rho) \phi^j(0, \varphi', \rho') \rangle = \delta^{ij} \frac{\pi}{\theta} \sum_{m \in \mathbb{Z}} G_S^{(|\frac{\pi m}{\theta}|)}(x_\parallel, \rho; x'_\parallel, \rho') \left(e^{i \frac{\pi}{\theta} m(\varphi - \varphi')} \pm e^{i \frac{\pi}{\theta} m(\varphi + \varphi')} \right), \quad (2.16)$$

⁵That this differential operator reconstructs the full bulk field $\phi(x)$ can be seen by observing that the Bessel functions can be expanded as

$$J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{j \geq 0} \frac{(-1)^j}{j! \Gamma(j+1+\alpha)} \left(\frac{x}{2}\right)^{2j},$$

which implies

$$\frac{J_\alpha(x)}{x^\alpha} = \frac{1}{2^\alpha \Gamma(1+\alpha)} B_{\hat{\Delta}_\alpha}^{d-1}(\partial_x^2) e^{\pm i x}.$$

where $+$ corresponds to NN while $-$ to DD, and we remind the reader we have promoted the fields to be invariant under $O(N)$, with indices $i, j \in \{1, \dots, N\}$. The function $G_s^{(\nu)}$ is⁶

$$G_S^{(\nu)}(x, x') = \frac{\Gamma(\Delta_\phi + \nu)}{4\pi^{\frac{d}{2}}\Gamma(\nu+1)} \left(\frac{1}{\rho\rho'}\right)^{\Delta_\phi} \left(\frac{\xi}{2}\right)^{\Delta_\phi + \nu} {}_2F_1\left(\frac{\Delta_\phi + \nu}{2}, \frac{\Delta_\phi + \nu + 1}{2}; \nu + 1; \xi^2\right), \quad (2.17)$$

where Δ_ϕ is the scaling dimension of the bulk scalar given by (2.15), and ξ is the cross-ratio

$$\xi \equiv \frac{2\rho\rho'}{\rho^2 + \rho'^2 + (x_\parallel - x'_\parallel)^2}. \quad (2.18)$$

In the DN case we find

$$\begin{aligned} \langle \phi^i(x_\parallel, \varphi, \rho) \phi^j(0, \varphi', \rho') \rangle &= \frac{4\pi\delta^{ij}}{\theta} \sum_{m \in \mathbb{Z}} G_S^{(\frac{\pi}{\theta}(m+\frac{1}{2}))}(x_\parallel, \rho; x'_\parallel, \rho') \times \\ &\times \sin\left[\left(m + \frac{1}{2}\right) \frac{\pi}{\theta} \varphi\right] \sin\left[\left(m + \frac{1}{2}\right) \frac{\pi}{\theta} \varphi'\right]. \end{aligned} \quad (2.19)$$

$d = 4$ case. Interestingly, for $d = 4$ the hypergeometric function appearing in (2.17) simplifies to

$${}_2F_1\left(\frac{\nu+1}{2}, \frac{\nu+2}{2}; \nu+1; \xi^2\right) = 2^\nu \frac{(\sqrt{1-\xi^2} + 1)^{-\nu}}{\sqrt{1-\xi^2}}. \quad (2.20)$$

This makes it possible to perform the sum over m finding a result in closed form. For DD and NN we obtain

$$\begin{aligned} \langle \phi^i(x) \phi^j(x') \rangle &= \frac{\delta^{ij}}{4\pi^2} \frac{\pi}{\theta} \left(\frac{\xi}{2\rho\rho'}\right) \sum_{m=0}^{+\infty} \frac{\chi^{-\frac{\pi}{\theta}|m|}}{\sqrt{1-\xi^2}} \left(e^{i\frac{\pi}{\theta}m(\varphi-\varphi')} \pm e^{i\frac{\pi}{\theta}m(\varphi+\varphi')}\right) \\ &= \frac{\delta^{ij}}{8\pi\theta} \frac{1}{\rho\rho'} \frac{\xi}{\sqrt{1-\xi^2}} \frac{\chi^{\frac{2\pi}{\theta}} - 1}{1 + \chi^{\frac{2\pi}{\theta}} - 2\chi^{\frac{\pi}{\theta}} \cos\left(\frac{\pi(\varphi-\varphi')}{\theta}\right)} \pm \left[(\varphi-\varphi') \rightarrow (\varphi+\varphi')\right], \end{aligned} \quad (2.21)$$

where we defined

$$\chi \equiv \frac{\sqrt{1-\xi^2} + 1}{\xi}, \quad (2.22)$$

while for DN b.c.'s we get

$$\langle \phi^i(x) \phi^j(x') \rangle = \frac{\delta^{ij}}{4\pi\theta} \frac{1}{\rho\rho'} \frac{\xi}{\sqrt{1-\xi^2}} \frac{\chi^{\frac{\pi}{2\theta}} \left(\chi^{\frac{\pi}{\theta}} - 1\right) \cos\left(\frac{\pi(\varphi-\varphi')}{2\theta}\right)}{1 + \chi^{\frac{2\pi}{\theta}} - 2\chi^{\frac{\pi}{\theta}} \cos\left(\frac{\pi(\varphi-\varphi')}{\theta}\right)} - \left[(\varphi-\varphi') \rightarrow (\varphi+\varphi')\right]. \quad (2.23)$$

⁶We note that the functions in eq. (2.17) correspond to the defect conformal blocks found in [60] (see also [59, 61]) in the case of monodromy defects. In particular, eq. (2.16) can be seen as an expansion in defect (or, more precisely, wedge) conformal blocks. However, we want to stress that in our case this precise form might be a consequence of the choice of quadratic boundary conditions, and we do not know whether this structure would persist in the presence of boundary interactions.

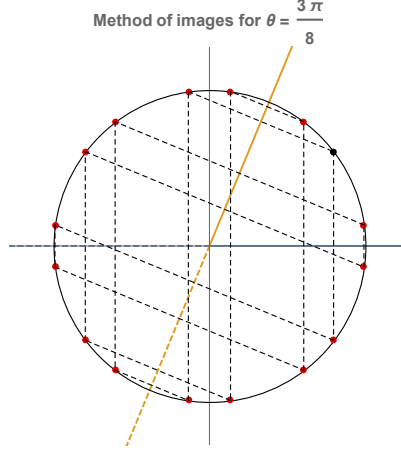


Figure 2. The 15 image points (in red) required to solve the free theory DS equations for $\theta = \frac{3\pi}{8}$. The blue line is the wall, and the orange is the ramp. The dashed lines clarify how the bulk point (in black) is reflected through the two boundaries. All of the image points lie on a circle.

$\theta = \frac{\pi}{n}$ case in generic d . Another case in which the sum over m simplifies is when $\theta = \frac{\pi}{n}$, with $n \in \mathbb{Z}_{\geq 0}$ for DD and NN b.c.'s. Also in this case the details of the derivation are in appendix A. The result is

$$\begin{aligned} \langle \phi^i(x_{\parallel}, \varphi, \rho) \phi^j(0, \varphi', \rho') \rangle &= \delta^{ij} \frac{\Gamma(\Delta_{\phi})}{4\pi^{\frac{d}{2}}} \sum_{k=0}^{n-1} \left(\frac{1}{\left[\rho^2 + \rho'^2 - 2\rho\rho' \cos\left(\frac{2k\pi}{n} + \varphi - \varphi'\right) + x_{\parallel}^2 \right]^{\Delta_{\phi}}} \right. \\ &\quad \left. \pm \frac{1}{\left[\rho^2 + \rho'^2 - 2\rho\rho' \cos\left(\frac{2k\pi}{n} + \varphi + \varphi'\right) + x_{\parallel}^2 \right]^{\Delta_{\phi}}} \right). \end{aligned} \quad (2.24)$$

Notice that this expression can also be derived directly by using the method of images [62–64]. More generally, we can use method of images for angles $\theta = \frac{a\pi}{b}$ with $a, b \in \mathbb{Z}_{\geq 0}$. Then if $\theta \in (0, \pi]$ the number of image point required is $2 \text{numerator}\left(\frac{b}{a}\right) - 1$.⁷ In figure 2 we have an illustrative example.

2.2 The one-point function of ϕ^2

From the bulk-bulk propagator we can compute the vacuum expectation value (v.e.v.) of ϕ^2 in the case of $d = 4$ for generic θ , and in the case of $\theta = \frac{\pi}{n}$ for generic d .

$d = 4$ case. One way to proceed is by taking the coincident limit of the propagators (2.21), (2.23). In the case of DD and NN b.c.'s we find

$$\lim_{y \rightarrow x} \langle \phi^i(x) \phi^j(y) \rangle = \delta^{ij} \left(\frac{1}{4\pi^2 \varepsilon^2} + \frac{1}{48\pi^2} \frac{\pi^2 - \theta^2}{\theta^2} \frac{1}{\rho^2} \pm \frac{\csc^2\left(\frac{\pi\varphi}{\theta}\right)}{16\theta^2} \frac{1}{\rho^2} + \mathcal{O}(\varepsilon^2) \right), \quad (2.25)$$

where ε is the regulator in the point-splitting procedure. The divergent piece proportional to ε^{-2} represents the identity contribution in the ϕ - ϕ bulk OPE that can be safely removed

⁷In the case of $\pi < \theta < 2\pi$, we need a finite $2 \text{numerator}\left(\frac{b}{a-b}\right) - 1$ number of image points.

by subtracting the propagator in absence of boundaries. Thus, we obtain

$$\langle \phi^2(x) \rangle = \frac{1}{48\pi^2} \frac{\pi^2 - \theta^2}{\theta^2} \frac{1}{\rho^2} \pm \frac{\csc^2\left(\frac{\pi\varphi}{\theta}\right)}{16\theta^2} \frac{1}{\rho^2}. \quad (2.26)$$

It is interesting to observe that the first term corresponds to the v.e.v. of the field in the presence of a conical singularity (see for example [61, 65]), while the second contribution is a modification due to the presence of the two boundaries.

The same procedure in the case of DN b.c.'s yields to

$$\langle \phi^2(x) \rangle = -\frac{2\theta^2 + \pi^2 [1 + 6 \cot\left(\frac{\pi\varphi}{\theta}\right) \csc\left(\frac{\pi\varphi}{\theta}\right)]}{96\pi^2\theta^2\rho^2}. \quad (2.27)$$

The results for DD and NN have been found for the first time in [1], while the result for DN is to the best of our knowledge novel.

Finally, as a limiting case we consider $\theta = \pi$, which gives

$$\langle \phi^2(x) \rangle = -\frac{2 \cot \varphi \csc \varphi + 1}{32\pi^2\rho^2}, \quad \text{if } \theta = \pi. \quad (2.28)$$

This correlator shows a non-trivial dependence on the coordinate x_d , which is parallel to the boundary. In addition, as a sanity check, we can see that in the near-boundary limit this one-point function behaves as the one with purely Neumann boundary condition if $x_d < 0$ (or $\varphi > \frac{\pi}{2}$), and as the one with purely Dirichlet b.c.'s if $x_d > 0$ (or $\varphi < \frac{\pi}{2}$) [66]. This suggests the presence of an interface on top of the boundary at $x_d = 0$ (or $\varphi = \frac{\pi}{2}$) separating the two phases.

$\theta = \frac{\pi}{n}$ case in generic d . In this case we obtain

$$\langle \phi^2(x) \rangle = \frac{\Gamma(\Delta_\phi)}{2^{\frac{d}{2}+1}\pi^{\frac{d}{2}}} \left\{ \sum_{k=1}^{n-1} \frac{1}{[1 - \cos\left(\frac{2k\pi}{n}\right)]^{\Delta_\phi}} \pm \sum_{k=0}^{n-1} \frac{1}{[1 - \cos\left(\frac{2k\pi}{n} + 2\varphi\right)]^{\Delta_\phi}} \right\} \frac{1}{\rho^{2\Delta_\phi}}, \quad (2.29)$$

where Δ_ϕ is given by (2.15). Unfortunately, we are not able to sum over k in generic d . On the other hand, we can check the result (2.26) valid for $d = 4$. The sums can then be analytically performed and we get

$$\langle \phi^2 \rangle = \frac{n^2 - 1 \pm 3n^2 \csc^2(n\varphi)}{48\pi^2} \frac{1}{\rho^2}, \quad d = 4, \quad (2.30)$$

which is exactly (2.26) for $\theta = \frac{\pi}{n}$.

2.3 The one-point function of $T_{\mu\nu}$

Another interesting correlator we can compute is the one-point function of the stress tensor. Conformal symmetry and conservation impose strict constraints to the possible form of this correlator. In particular, as we show below, we only need to find one particular component. In fact, by imposing scale invariance, rotations in the parallel directions of the wedge, and employing the fact that the stress tensor is symmetric we can write the one-point function as

$$\langle T^{ij} \rangle = \frac{f(\varphi)}{\rho^d} \delta^{ij}, \quad \langle T^{\rho\rho} \rangle = \frac{g_{\rho\rho}(\varphi)}{\rho^d}, \quad \langle T^{\varphi\varphi} \rangle = \frac{g_{\varphi\varphi}(\varphi)}{\rho^{d+2}}, \quad \langle T^{\varphi\rho} \rangle = \frac{g_{\varphi\rho}(\varphi)}{\rho^{d+1}}. \quad (2.31)$$

In addition, conservation leads to the following differential equations

$$\begin{aligned} g'_{\varphi\varphi}(\varphi) - (d-2)g_{\rho\varphi}(\varphi) &= 0, \\ (1-d)g_{\rho\rho}(\varphi) - g_{\varphi\varphi}(\varphi) + g'_{\rho\varphi}(\varphi) &= 0, \end{aligned} \quad (2.32)$$

while the condition of being traceless gives

$$(d-2)f(\varphi) + g_{\rho\rho}(\varphi) + g_{\varphi\varphi}(\varphi) = 0. \quad (2.33)$$

Thus, we find that everything is fixed up to a single function $g(\varphi)$ as⁸

$$\begin{aligned} \langle T^{ij} \rangle &= -\frac{1}{d-1} \left[g(\varphi) + \frac{g''(\varphi)}{(d-2)^2} \right] \frac{1}{\rho^d} \delta^{ij}, & \langle T^{\rho\rho} \rangle &= -\frac{1}{d-1} \left[g(\varphi) - \frac{g''(\varphi)}{d-2} \right] \frac{1}{\rho^d}, \\ \langle T^{\varphi\varphi} \rangle &= \frac{g(\varphi)}{\rho^{d+2}}, & \langle T^{\varphi\rho} \rangle &= \frac{1}{d-2} \frac{g'(\varphi)}{\rho^{d+1}}. \end{aligned} \quad (2.34)$$

For this reason we find it convenient to compute the component $T_{\varphi\varphi}$. Its v.e.v. can be computed by using eq. (2.3) and taking the coincident limit of $\langle \phi\phi \rangle$. As we did for ϕ^2 , we first compute the correlator when $d = 4$, and then we consider the case in generic dimension but with $\theta = \frac{\pi}{n}$.

$d = 4$ case. For DD and NN the one-point function reads

$$\langle T_{\varphi\varphi} \rangle = -\frac{1}{480\pi^2} \frac{\pi^4 - \theta^4}{\theta^4} \frac{1}{\rho^2}, \quad (2.35)$$

while in the DN case we obtain

$$\langle T_{\varphi\varphi} \rangle = \frac{1}{3840\pi^2} \frac{8\theta^4 + 7\pi^4}{\theta^4} \frac{1}{\rho^2}. \quad (2.36)$$

Note that in both the cases this correlator is independent of the angle φ . This means that also all the others components will be a function of ρ only. Interestingly, we note that the sign of the coefficient of the stress tensor one-point function changes sign at $\theta = \pi$ only in the DD and NN cases, while in the DN case it remains negative for all the possible values of θ . Also in this case, the results for DD and NN have been found for the first time in [1, 67], while the result for DN is to the best of our knowledge novel.⁹

$\theta = \frac{\pi}{n}$ case in generic d . In this case, for the DD and NN case we find

$$\langle T_{\varphi\varphi} \rangle = -\frac{\Gamma\left(\frac{d}{2}\right)}{2^{d+2}\pi^{\frac{d}{2}}} \frac{1}{\rho^{d-2}} \sum_{k=1}^{n-1} \frac{(d-2)\cos\left(\frac{2\pi k}{n}\right) + d}{\sin^d\left(\frac{\pi k}{n}\right)}. \quad (2.37)$$

⁸We notice that the general form we obtained in eq. (2.34) is in conflict with the one reported in [1], which does not seem to be conserved to us. Regarding the specific case of the free scalar we find perfect matching.

⁹J.S. Dowker pointed out to us that, for free scalars, integrated quantities with DN b.c.'s can be obtained from the ones with DD and NN b.c.'s by applying simple algebraic identities [68]. By applying that method to the correlator $\langle T_{\varphi\varphi} \rangle \theta$ we find exactly eq. (2.36).

2.4 Correlators of wedge operators

From the general form of the propagators, we can compute the bulk-wedge correlators by employing the explicit form of the operators \hat{O}_α reported in eq. (2.13) (now promoted to $O(N)$). The computation is analogous to the one done for the full bulk-bulk propagator, thus we only give the result. We find

$$\langle \phi^i(x) \hat{O}_\alpha^j(x'_\parallel) \rangle = \frac{\Gamma(\hat{\Delta}_\alpha)}{\pi^{\Delta_\phi} \theta \Gamma(\hat{\Delta}_\alpha - \Delta_\phi + 1)} \frac{\delta^{ij}}{\rho^{\Delta_\phi - \hat{\Delta}_\alpha} (s_\parallel^2 + \rho^2)^{\hat{\Delta}_\alpha}} \begin{cases} \sin(\alpha \varphi), & \text{DD and DN,} \\ \cos(\alpha \varphi), & \text{NN,} \end{cases} \quad (2.38)$$

where we remind the reader that $\Delta_\phi = (d-2)/2$. Note also that even though the DD and DN cases seem identical they differ because of the possible values of α , which are given in eq. (2.7).

Now we consider the wedge-wedge correlator

$$\langle \hat{O}_\alpha^i(x_\parallel) \hat{O}_{\alpha'}^j(x'_\parallel) \rangle = \frac{\Gamma(\hat{\Delta}_\alpha)}{\pi^{\Delta_\phi} \theta \Gamma(\hat{\Delta}_\alpha - \Delta_\phi + 1)} \frac{\delta^{ij} \delta_{\alpha\alpha'}}{|x_\parallel - x'_\parallel|^{2\hat{\Delta}_\alpha}}, \quad (2.39)$$

where $\hat{\Delta}_\alpha$ is exactly the one introduced in eq. (2.15).

2.5 Singular modes and RG flows

As anticipated it is possible to relax the requirement that ϕ is regular in the wedge limit $\rho \rightarrow 0$. In fact, this is allowed by unitarity in the following way: considering the wedge as a $(d-2)$ -dimensional defect we can write the unitarity bound as [59]

$$\begin{aligned} \hat{\Delta} &\geq \frac{d}{2} - 2, & \text{if } d > 4, \\ \hat{\Delta} &\geq 0, & \text{if } d \leq 4. \end{aligned} \quad (2.40)$$

Note that the bound above is meant for operators different from the identity ($\hat{\Delta} = 0$). By employing the conformal dimensions (2.15) we get

$$\begin{aligned} \alpha &\geq -1, & \text{if } d > 4, \\ \alpha &\geq 1 - \frac{d}{2}, & \text{if } d \leq 4. \end{aligned} \quad (2.41)$$

Let us now focus for simplicity on the case $d = 4$ with either DD or NN b.c.'s. In this case we see that there is a possible singular mode in the range $\theta \in (\pi, 2\pi)$ given by $\alpha^{(-)} = -\frac{\pi}{\theta}$. Now we can trade the regular mode $\alpha^{(+)} = +\frac{\pi}{\theta}$ for $\alpha^{(-)}$. If we consider this new choice for the mode $|m| = 1$, the spectrum of wedge operators will get modified in that we will have the operator $\hat{O}_{-\frac{\pi}{\theta}}$ instead of $\hat{O}_{+\frac{\pi}{\theta}}$. This, in turn, will modify the bulk-bulk propagator in the following way

$$G_S^{(+\frac{\pi}{\theta})} \Rightarrow G_S^{(-\frac{\pi}{\theta})}. \quad (2.42)$$

This modification also affects the one-point function of ϕ^2

$$\langle \phi^2 \rangle^{(-)} - \langle \phi^2 \rangle^{(+)} = \frac{\Gamma(\Delta_\phi + \frac{\pi}{\theta}) \Gamma(\Delta_\phi - \frac{\pi}{\theta}) \sin\left(\frac{\pi^2}{\theta}\right)}{2^{d-3} \pi^{\frac{d+1}{2}} \Gamma\left(\frac{d-1}{2}\right)} \frac{1}{\rho^{d-2}} \begin{cases} \sin^2\left(\frac{\pi}{\theta}\varphi\right), & \text{DD}, \\ \cos^2\left(\frac{\pi}{\theta}\varphi\right), & \text{NN}, \end{cases} \quad (2.43)$$

while for the stress tensor we find

$$\langle T_{\varphi\varphi} \rangle^{(-)} - \langle T_{\varphi\varphi} \rangle^{(+)} = \frac{(d-1) \sin\left(\frac{\pi^2}{\theta}\right) \Gamma(\Delta_\phi + \frac{\pi}{\theta}) \Gamma(\Delta_\phi - \frac{\pi}{\theta})}{2^{d-1} \pi^{\frac{d-5}{2}} \theta^3 \Gamma\left(\frac{d+1}{2}\right)} \frac{1}{\rho^{d-2}}. \quad (2.44)$$

Another interesting feature of this singular mode is that if we consider the quadratic relevant wedge perturbation

$$S \ni h \int d^{d-2} x_{\parallel} \hat{O}_{-\frac{\pi}{\theta}} \hat{O}_{-\frac{\pi}{\theta}}, \quad (2.45)$$

then we trigger an RG flow from the theory with the mode $\alpha^{(-)}$ to the one with the regular one $\alpha^{(+)}$. An analogous situation appears for a scalar in the presence of a monodromy defect as studied in [59, 61, 69, 70]. In that case, unitarity allows for some modes which are mild divergent in the defect limit and provide different b.c.'s to the solution of the e.o.m for the field ϕ . Different b.c.'s are shown to be related by defect RG flows, and in particular in [61] the authors computed explicitly the bulk-bulk propagator along the defect RG flow. For the case of our interest, a very analogous computation can be performed, and we obtain for $d = 4$,¹⁰

$$\begin{aligned} \Delta \langle \phi^i(x_1) \phi^j(x_2) \rangle &\equiv \langle \phi^i(x_1) \phi^j(x_2) \rangle^{(+)} - \langle \phi^i(x_1) \phi^j(x_2) \rangle^{(-)} \\ &= -4 \delta^{ij} \frac{\sin\left(\frac{\pi^2}{\theta}\right)}{\pi \theta} \int \frac{d^2 k_{\parallel}}{(2\pi)^2} \frac{c h}{c h + k_{\parallel}^{\frac{2\pi}{\theta}}} K_{\frac{\pi}{\theta}}(k_{\parallel} \rho_1) K_{\frac{\pi}{\theta}}(k_{\parallel} \rho_2) \times \\ &\quad \times e^{i k_{\parallel} (x_{\parallel}^1 - x_{\parallel}^2)} \begin{cases} \sin\left(\frac{\pi}{\theta}\varphi_1\right) \sin\left(\frac{\pi}{\theta}\varphi_2\right), & \text{DD}, \\ \cos\left(\frac{\pi}{\theta}\varphi_1\right) \cos\left(\frac{\pi}{\theta}\varphi_2\right), & \text{NN}, \end{cases} \end{aligned} \quad (2.46)$$

where

$$c \equiv \frac{4^{\frac{\pi}{\theta}} \Gamma\left(\frac{\pi}{\theta}\right)}{\theta \Gamma\left(1 - \frac{\pi}{\theta}\right)}. \quad (2.47)$$

We see that in the $h \rightarrow +\infty$ limit we obtain

$$\begin{aligned} \Delta \langle \phi^i(x_1) \phi^j(x_2) \rangle &= -4 \delta^{ij} \frac{\sin\left(\frac{\pi^2}{\theta}\right)}{\pi \theta} \int \frac{d^2 k_{\parallel}}{(2\pi)^2} K_{\frac{\pi}{\theta}}(k_{\parallel} \rho_1) K_{\frac{\pi}{\theta}}(k_{\parallel} \rho_2) e^{i k_{\parallel} (x_{\parallel}^1 - x_{\parallel}^2)} \times \\ &\quad \times \begin{cases} \sin\left(\frac{\pi}{\theta}\varphi_1\right) \sin\left(\frac{\pi}{\theta}\varphi_2\right), & \text{DD}, \\ \cos\left(\frac{\pi}{\theta}\varphi_1\right) \cos\left(\frac{\pi}{\theta}\varphi_2\right), & \text{NN}. \end{cases} \end{aligned} \quad (2.48)$$

¹⁰We refer the reader to [61] for details on the computation.

In this limit the integral can be performed analytically

$$\Delta \langle \phi^i(x_1) \phi^j(x_2) \rangle = -4 \delta^{ij} \frac{\pi}{\theta} \left(\frac{1}{\rho_1 \rho_2} \right) \left[\mathcal{F} \left(-\frac{\pi}{\theta} \right) - \mathcal{F} \left(\frac{\pi}{\theta} \right) \right] \begin{cases} \sin \left(\frac{\pi}{\theta} \varphi_1 \right) \sin \left(\frac{\pi}{\theta} \varphi_2 \right), & \text{DD,} \\ \cos \left(\frac{\pi}{\theta} \varphi_1 \right) \cos \left(\frac{\pi}{\theta} \varphi_2 \right), & \text{NN,} \end{cases} \quad (2.49)$$

where we defined

$$\mathcal{F}(\alpha) \equiv \frac{1}{4\pi^2} \left(\frac{\xi}{2} \right)^{1+\alpha} {}_2F_1 \left(\frac{1+\alpha}{2}, 1 + \frac{\alpha}{2}, \alpha; \xi^2 \right). \quad (2.50)$$

We notice that eq. (2.49) produces exactly the substitution of eq. (2.42), proving that the two different b.c.'s are related by a wedge RG flow.

3 The bulk-wedge correlator from the equation of motion

In this section, we study the bulk-wedge two-point functions of scalar operators, and how they can be found from the DS equations up to $\mathcal{O}(\epsilon)$ with a quartic interaction in the bulk in $d = 4 - \epsilon$, or a quartic boundary interaction in $d = 3 - \epsilon$. This approach has been used in [71, 72] in the context of a single boundary with a bulk interaction in $d = 4 - \epsilon$.

In a WCFT there are four OPE's in play: two for each boundary, which allow us to express bulk operators in terms of the boundary operators, and two more which express the boundary operators on the wall or the ramp in terms of wedge operators. We call the former two the bulk-boundary BOE's, and the latter two boundary-wedge BOE's. Using these OPE's we find that the bulk-wedge correlator for scalars is determined up to a function

$$\langle \phi^i(x_{\parallel}, x_{d-1}, x_d) \hat{O}^j(y_{\parallel}) \rangle = \frac{\delta^{ij} f(\varphi)}{\rho^{\Delta_{\phi} - \hat{\Delta}} (s_{\parallel}^2 + \rho^2)^{\hat{\Delta}}}, \quad s_{\parallel} \equiv x_{\parallel} - y_{\parallel}. \quad (3.1)$$

Here φ is the single cross-ratio

$$\varphi = \tan^{-1} \frac{x_{d-1}}{x_d}. \quad (3.2)$$

We will find $f(\varphi)$ by solving perturbatively the associated DS equations together with the different b.c.'s, which read¹¹

$$\begin{aligned} \text{DD:} \quad & f^{(0)}(0) = f^{(0)}(\theta) = 0, \\ \text{NN:} \quad & \partial_{\varphi} f^{(0)}(0) = \partial_{\varphi} f^{(0)}(\theta) = 0, \\ \text{DN:} \quad & f^{(0)}(0) = \partial_{\varphi} f^{(0)}(\theta) = 0. \end{aligned} \quad (3.3)$$

The strategy we will follow is to expand all the relevant quantities such as the function f and the scaling dimensions in terms of the dimensionless parameter ϵ . Up to first order we write

$$\begin{aligned} f(\varphi) &= f^{(0)}(\varphi) + \epsilon f^{(1)}(\varphi) + \mathcal{O}(\epsilon^2), \\ \Delta_{\phi} &= \Delta_{\phi}^{(0)} + \epsilon \gamma_{\phi} + \mathcal{O}(\epsilon^2), \\ \hat{\Delta} &= \hat{\Delta}^{(0)} + \epsilon \hat{\gamma} + \mathcal{O}(\epsilon^2). \end{aligned} \quad (3.4)$$

¹¹Note that the derivatives in r in the directional derivatives w.r.t. the boundary normals vanish in the respective boundary limits.

Then, we need to plug the expansion (3.4) into the DS equation for the field bulk field $\phi(x)$ and solve the differential equation order by order in ϵ . Finally, the last step is to suitably impose the b.c.'s (3.3), as explained in detail later.

3.1 Free theory

To start we analyse the free theory case. A massless scalar satisfies the Klein-Gordon equation

$$\square_x \phi^i = 0, \quad (3.5)$$

which gives us the DS equation for the bulk-wedge correlator

$$\square_x \langle \phi^i(x_{\parallel}, x_{d-1}, x_d) \hat{O}^j(y_{\parallel}) \rangle = 0. \quad (3.6)$$

When applied to (3.1) we find the following differential equation

$$\delta^{ij} \frac{\rho^{-\Delta_{\phi}^{(0)} + \hat{\Delta}^{(0)} - 2}}{(s_{\parallel}^2 + \rho^2)^{\hat{\Delta}^{(0)}}} \left[\partial_{\varphi}^2 f^{(0)}(\varphi) + f^{(0)}(\varphi) \left(\left(\hat{\Delta}^{(0)} - \Delta_{\phi}^{(0)} \right)^2 - \frac{2\rho^2 \hat{\Delta}^{(0)} (d-2-2\Delta_{\phi}^{(0)})}{\rho^2 + s_{\parallel}^2} \right) \right] = 0. \quad (3.7)$$

Since, as discussed above, conformal symmetry implies that f is a function of φ only, the second term inside the bracket of (3.7) needs to vanish, which in turn implies

$$\Delta_{\phi}^{(0)} = \frac{d-2}{2}. \quad (3.8)$$

We notice that this is indeed the scaling dimension of the fundamental free bulk scalar field in (2.15). At this point we are left with the following differential equation for $f^{(0)}(\varphi)$

$$\partial_{\varphi}^2 f^{(0)}(\varphi) + \left(\hat{\Delta}^{(0)} - \Delta_{\phi}^{(0)} \right)^2 f^{(0)}(\varphi) = 0, \quad (3.9)$$

with the general solution

$$f^{(0)}(\varphi) = A \cos \left[\left(\hat{\Delta}^{(0)} - \Delta_{\phi} \right) \varphi \right] + B \sin \left[\left(\hat{\Delta}^{(0)} - \Delta_{\phi} \right) \varphi \right]. \quad (3.10)$$

Imposing the wall b.c. fixes one of the constants A or B to zero. The ramp b.c. will in turn give us the wedge scaling dimension

$$\hat{\Delta}_{\alpha}^{(0)} = \frac{d-2}{2} + \alpha, \quad \alpha = \frac{\pi}{\theta} \begin{cases} m, & \text{DD, NN,} \\ m + \frac{1}{2}, & \text{ND,} \end{cases} \quad (3.11)$$

with $m \in \mathbb{Z}$ (non-zero for DD). This agrees with (2.15). The remaining constant can be fixed by normalization, which we choose in such a way that we find agreement with (2.38)

$$f^{(0)}(\varphi) = \frac{\Gamma(\hat{\Delta}_{\alpha})}{\pi^{\Delta_{\phi}} \theta \Gamma(\alpha + 1)} \begin{cases} \sin(\alpha \varphi), & \text{DD, DN,} \\ \cos(\alpha \varphi), & \text{NN.} \end{cases} \quad (3.12)$$

So far we have discussed the free scalar field in generic dimensions. Now we move on to discuss interacting Wilson-Fisher theories.

3.2 Interacting theory: ϕ^4 -deformation in the bulk in $d = 4 - \epsilon$

In $d = 4 - \epsilon$ we consider a quartic interaction in the bulk

$$S \ni \int_{\mathbb{R}^d} d^d x \frac{\lambda}{8} (\phi^2)^2, \quad (3.13)$$

from which we have the e.o.m.

$$\square_x \phi^i = \frac{\lambda}{2} \phi^2 \phi^i, \quad (3.14)$$

where we denoted $\phi^2 \equiv (\phi^i)^2$. In the conformal case we have to specify the coupling at the RG fixed point

$$\lambda_* = \frac{(4\pi)^2}{N+8} \epsilon + \mathcal{O}(\epsilon^2). \quad (3.15)$$

The e.o.m. applied to the bulk-edge correlator (2.38) is now given by

$$\square_x \langle \phi^i(x_{\parallel}, x_{d-1}, x_d) \hat{\mathcal{O}}_{\alpha}^j(y_{\parallel}) \rangle = \frac{\lambda_*}{2} \langle \phi^2 \phi^i(x_{\parallel}, x_{d-1}, x_d) \hat{\mathcal{O}}_{\alpha}^j(y_{\parallel}) \rangle. \quad (3.16)$$

Since the coupling starts at $\mathcal{O}(\epsilon)$, it is sufficient to consider the free theory correlators on the r.h.s. Applying Wick's theorem

$$\square_x \langle \phi^i(x_{\parallel}, x_{d-1}, x_d) \hat{\mathcal{O}}_{\alpha}^j(y_{\parallel}) \rangle_1 = \frac{(N+2)\lambda_*}{2} \langle \phi^i(x_{\parallel}, x_{d-1}, x_d) \hat{\mathcal{O}}_{\alpha}^j(y_{\parallel}) \rangle_0 \langle \phi^2(x_{\parallel}, x_{d-1}, x_d) \rangle_0, \quad (3.17)$$

where we use the subscript 0 to denote the free theory correlator, and 1 is its first-order correction in the interacting theory. E.g. the differential equation at $\mathcal{O}(\epsilon)$ for DD is given by

$$\begin{aligned} \partial_{\varphi}^2 f^{(1)} + \alpha^2 f^{(1)}(\varphi) + \gamma_{\phi} \frac{4x_d^2(\alpha+1)\sin(\alpha\varphi)}{\pi\theta(x_d^2 + s_{\parallel}^2 \cos^2 \varphi)} + \left(\theta^2 + 3\pi^2 \csc^2 \frac{\pi\varphi}{\theta} - \pi^2 \right) \frac{\sin(\alpha\varphi)}{3\pi\theta^3} \\ + \frac{2\alpha(\gamma_{\phi} - \hat{\gamma})}{\pi\theta} \frac{\csc \varphi}{\tan \varphi} \sin((\alpha-1)\varphi) + \frac{(2\alpha-1)(\gamma_{\phi} - \hat{\gamma})}{\pi\theta} \frac{\sin(\alpha\varphi)}{\sin^2 \varphi} = 0 \end{aligned} \quad (3.18)$$

For it to only depend on φ we have

$$\gamma_{\phi} = 0. \quad (3.19)$$

Similarly, from the differential equation at $\mathcal{O}(\epsilon)$ for NN and DN we find that this holds. Using this condition, the differential equation at $\mathcal{O}(\epsilon)$ reads

$$\partial_{\varphi}^2 f^{(1)}(\varphi) + \alpha^2 f^{(1)}(\varphi) + h(\varphi) = 0, \quad (3.20)$$

where

$$h(\varphi) = \begin{cases} \sin(\alpha\varphi) \left(\frac{2\pi\alpha}{\theta} \hat{\gamma} + \left(\theta^2 + 3\pi^2 \csc^2 \frac{\pi\varphi}{\theta} - \pi^2 \right) \frac{N+2}{6(N+8)\pi\theta^3} \right), & \text{DD,} \\ \cos(\alpha\varphi) \left(\frac{2\pi\alpha}{\theta} \hat{\gamma} + \left(\theta^2 - 3\pi^2 \csc^2 \frac{\pi\varphi}{\theta} - \pi^2 \right) \frac{N+2}{6(N+8)\pi\theta^3} \right), & \text{NN,} \\ \sin(\alpha\varphi) \left(\frac{2\pi\alpha}{\theta} \hat{\gamma} + \left(\theta^2 + 3\pi^2 \csc \frac{\pi\varphi}{\theta} \cot \frac{\pi\varphi}{\theta} + \frac{\pi^2}{2} \right) \frac{N+2}{6(N+8)\pi\theta^3} \right), & \text{DN.} \end{cases}$$

Note in particular that the $f^{(1)}(\varphi)$ -term vanishes in eq. (3.20) for NN when $\alpha = 0$. This means that this solution has to be studied on its own, which we will do in the next subsection.

For $\alpha \neq 0$, these equations can be solved for different values of α as given in (3.11). Similar to the free theory, the solution is fixed up to two constants. One of these constants is zero which is seen by implementing the wall b.c.¹² By implementing the ramp b.c. we fix the edge anomalous dimension to be¹³

$$\hat{\gamma} = \begin{cases} \frac{(N+2)(\pi^2 - 6\pi\alpha\theta - \theta^2)}{12(N+8)\alpha\theta^2}, & \text{DD, NN,} \\ -\frac{(N+2)(\pi^2 + 12\pi\alpha\theta + 2\theta^2)}{24(N+8)\alpha\theta^2}, & \text{DN.} \end{cases} \quad (3.21)$$

The anomalous dimension for DD with $\alpha = \frac{\pi}{\theta}$ agrees with the older literature [2]. We double check all of these anomalous dimensions using Feynman diagrams in appendix B.1. The final results are

$$\begin{aligned} f_{m \geq 1}^{(1)DD}(\varphi) = & -\pi^2 \left(\frac{N+2}{2(N+8)\pi\theta} \log \left[\sin \left(\frac{\pi\varphi}{\theta} \right) \right] - \mathcal{A} \right) \sin(\alpha\varphi) + \\ & + \pi \frac{N+2}{2(N+8)m\theta} \sum_{i=1}^m \left(\frac{m}{i} - 1 \right) \sin \left(\frac{\pi(2i-m)\varphi}{\theta} \right), \end{aligned} \quad (3.22)$$

$$\begin{aligned} f_{m \geq 1}^{(1)NN}(\varphi) = & -\pi^2 \left(\frac{N+2}{2(N+8)\pi\theta} \log \left[\sin \left(\frac{\pi\varphi}{\theta} \right) \right] - \mathcal{A} \right) \cos(\alpha\varphi) + \\ & - \pi^2 \frac{N+2}{2(N+8)\alpha\theta^2} \sum_{i=1}^m \left(\frac{m}{i} - 1 \right) \cos \left(\frac{\pi(2i-m)\varphi}{\theta} \right) \\ & - \pi^2 \frac{N+2}{2(N+8)\alpha\theta^2} \csc \left(\frac{\pi\varphi}{\theta} \right) \sin \left(\frac{\pi(m-1)\varphi}{\theta} \right), \end{aligned} \quad (3.23)$$

$$f_{m=0}^{(1)ND}(\varphi) = -\pi^2 \left(\frac{N+2}{2(N+8)\pi\theta} \log \left[\sin \left(\frac{\pi\varphi}{\theta} \right) \right] - \mathcal{A} \right) \sin(\alpha\varphi), \quad (3.24)$$

$$\begin{aligned} f_{m \geq 1}^{(1)ND}(\varphi) = & -\pi^2 \left(\frac{N+2}{2(N+8)\pi\theta} \log \left[\sin \left(\frac{\pi\varphi}{\theta} \right) \right] - \mathcal{A} \right) \sin(\alpha\varphi) + \\ & + \pi^2 \frac{N+2}{4(N+8)\alpha\theta^2} \sum_{i=1}^m \left(\frac{2i-2m-1}{i} \right) \sin \left[\left(m - 2i + \frac{1}{2} \right) \frac{\pi\varphi}{\theta} \right] \\ & - \pi^2 \frac{N+2}{4(N+8)\alpha\theta^2} \tan \left(\frac{\pi\varphi}{2\theta} \right) \cos \left[\left(m + \frac{1}{2} \right) \frac{\pi\varphi}{\theta} \right]. \end{aligned} \quad (3.25)$$

\mathcal{A} is an undetermined constant which we will relate to a BOE coefficient in section 4.

NN $\alpha = 0$. Let us now solve the differential equation (3.20) for the special case of NN with $\alpha = 0$. It has the solution

$$f^{(1)}(\varphi) = \mathcal{A} + \mathcal{B}\varphi + \frac{(N+2)(\pi^2 - \theta^2)\lambda_*}{192\pi^3\theta^3\epsilon} \varphi^2 - \frac{(N+2)\lambda_*}{32\pi^3\theta\epsilon} \log \sin \left(\frac{\pi\varphi}{\theta} \right). \quad (3.26)$$

¹²For Neumann b.c.'s there is a single pole in φ , which corresponds to the BOE exchange of $\hat{\phi}$. This term does not need to be zero. The b.c. is implemented by setting the φ^0 -term to zero. Similarly for NN, we encounter a single pole in $\theta - \varphi$ in the ramp limit of $\partial_\varphi \phi$ which should not be tuned to zero.

¹³We had to solve DN for $m = \frac{1}{2}$ and $m \geq \frac{3}{2}$ (in α of (3.11)) as two separate cases.

In the respective boundary limits of its normal derivative we find

$$\begin{aligned}\partial_\varphi f^{(1)}(\varphi) &= -\frac{(N+2)\lambda_*}{32\pi^3\theta\epsilon\varphi} + \mathcal{B} + \mathcal{O}(\varphi), \\ \partial_\varphi f^{(1)}(\varphi) &= -\frac{(N+2)\lambda_*}{32\pi^3\theta\epsilon(\varphi-\theta)} + \mathcal{B} + \frac{(N+2)(\pi^2-\theta^2)\lambda_*}{96\pi^3\theta^2\epsilon} + \mathcal{O}(\varphi-\theta).\end{aligned}\tag{3.27}$$

As mentioned previously, the poles in φ and $\varphi - \theta$ correspond to the BOE exchange of $\hat{\phi}$ and should not be set to zero. Tuning the $\mathcal{O}(\varphi^0)$ and $\mathcal{O}((\varphi - \theta)^0)$ terms to zero yields

$$\mathcal{B} = 0, \quad \theta = \pi.\tag{3.28}$$

This means that the wedge operator $\hat{\phi}$ (ϕ confined to the edge), is only conformal in the unfolding limit $\theta \rightarrow \pi$. Adding (3.27) to the free theory result (3.12) gives us

$$f(\varphi) = \frac{1}{\pi^{\frac{d}{2}}\Gamma(\Delta_\phi^{(0)})} - \epsilon \left(\frac{(N+2)\lambda_*}{32\pi^4\epsilon} \log \sin \varphi - \mathcal{A} \right) + \mathcal{O}(\epsilon^2).\tag{3.29}$$

This corresponds to the expansion of a bulk-boundary correlator in a BCFT [73]

$$\langle \phi^i(x) \hat{\phi}^j(0) \rangle = \delta^{ij} \frac{\mu_{\hat{\phi}}^\phi}{x_{d-1}^{\Delta_\phi - \hat{\Delta}_0} (s_{||}^2 + x_{d-1}^2)^{\hat{\Delta}_0}},\tag{3.30}$$

where the factor in front of the logarithm of (3.29) is the boundary anomalous dimension of $\hat{\phi}$ and originates from the expansion of $x_{d-1}^{-\Delta_\phi + \hat{\Delta}_0}$. I.e. the wedge operator $\hat{\phi}$ is nothing else than $\hat{\phi}$ at the origin

$$\hat{\gamma} = \gamma = -\frac{(N+2)\lambda_*}{32\pi^2\epsilon} = -\frac{N+2}{2(N+8)}.\tag{3.31}$$

This anomalous dimension agrees with the result in older literature [66]. Note also that the wedge anomalous dimensions (3.21) equal the boundary anomalous dimension above in the unfolding limit $\theta \rightarrow \pi$.

3.3 Interacting theory: $\hat{\phi}^4$ -deformation on the boundary in $d = 3 - \epsilon$

In $d = 3 - \epsilon$ we can consider a quartic interaction on the boundary [74], see also [75]. In our case we will consider it on the ramp

$$S \ni \int_{\mathbb{R}^{d-1}} d^{d-1}x_{||} \frac{g}{8} (\hat{\phi}^2)^2,\tag{3.32}$$

which gives us a modified Neumann b.c.'s on the ramp¹⁴

$$\frac{1}{\rho} \partial_\varphi \phi^i \Big|_{\varphi=\theta} = -\frac{g}{2} \hat{\phi}^2 \hat{\phi}^i.\tag{3.33}$$

¹⁴As seen by varying the fields.

Here we denoted $\hat{\phi}^2 \equiv (\hat{\phi}^i)^2$. In the conformal case we have to specify the coupling to the RG fixed point

$$g_* = \frac{4\pi}{N+8}\epsilon + \mathcal{O}(\epsilon^2). \quad (3.34)$$

Also in this case we will focus on the correlator (3.1), where we are interested in the first order in the ϵ -expansion. Since the bulk DS equation is not affected by the boundary perturbation, the solution to the function $f(\varphi)$ is the same as we found in eq. (3.10). The only difference is in the b.c. on the ramp, which now reads

$$\frac{1}{\rho} \partial_\varphi \langle \phi^i(x_\parallel, \rho, \varphi) \hat{O}_\alpha^j(0) \rangle \Big|_{\varphi=\theta} = -\delta^{ij} \frac{(N+2)g_*}{2} \langle \hat{\phi}^i(x_\parallel, \rho) \hat{O}_\alpha^j(0) \rangle_0 \langle \hat{\phi}^{i2}(x_\parallel, \rho) \rangle_0. \quad (3.35)$$

As clear from the equation above, we need the boundary-wedge correlator and the one-point function of $\hat{\phi}^2$ in the free limit. While the former can be found in any dimension from the bulk-wedge correlator (2.38), we are able to report the form of the coefficient of the one-point function for $d=3$ only in the special case $\theta = \frac{\pi}{n}$. In particular, we have

$$\langle \hat{\phi}^i(x) \hat{O}_\alpha^j(y_\parallel) \rangle = \delta^{ij} \frac{\Gamma(\alpha + \frac{1}{2})}{\pi^{\frac{3}{2}} \theta \Gamma(\alpha + 1)} \frac{\rho^\alpha}{(\rho^2 + (s_\parallel)^2)^{\alpha + \frac{1}{2}}}, \quad (3.36)$$

and

$$\langle \hat{\phi}^2 \rangle = \frac{1}{2\sqrt{2}\pi} \frac{\kappa_n}{\rho}, \quad \kappa_n \equiv \sum_{k=1}^{n-1} \frac{1}{\sqrt{1 - \cos\left(\frac{2k\pi}{n}\right)}}. \quad (3.37)$$

By imposing the purely Neumann b.c. on the wall boundary fixes the coefficient $B=0$. As in the free case, the other coefficient (A) is related to the normalisation of the wedge operators, and the b.c. on the ramp in eq. (3.35) fixes the conformal dimension of the wedge operator. By plugging the correlators in (3.36) and (3.37) back to eq. (3.35), and setting $d=3-\epsilon$, we obtain the anomalous dimension

$$\hat{\gamma} = \frac{N+2}{N+8} \frac{1}{\sqrt{2}\alpha\theta} \kappa_n. \quad (3.38)$$

To check this result, we found it worth deriving it also from the standard diagrammatic approach in appendix B.2.

4 CFT data at order ϵ

In this section, we will explain how the BOE coefficients at $\mathcal{O}(\epsilon)$ can be found from the bulk-wedge correlators found in section 3.2. This method relies on the conformal block decomposition from [19], and in the process we will also find the anomalous dimension of the single boundary operator from the free theory ($\hat{\phi}$ or $\partial_\perp \hat{\phi}$).

4.1 Boundary anomalous dimensions

Using the bulk-boundary and boundary-wedge BOE's (for the wall) we can write the bulk-wedge correlator as

$$\langle \phi^i(x) \hat{\mathcal{O}}^j(y_{\parallel}) \rangle = \sum_{\hat{\mathcal{O}}} \frac{\mu^{\phi_{\hat{\mathcal{O}}}}}{|x_{d-1}|^{\Delta_{\phi}-\hat{\Delta}}} B_{\hat{\Delta}}^d(x_{d-1}^2) \langle \hat{\mathcal{O}}^i(x_{\parallel}, r) \hat{\mathcal{O}}^j(y_{\parallel}) \rangle. \quad (4.1)$$

The sum runs over all boundary primaries. The boundary-wedge correlator is fixed by conformal symmetry (in the same way as a bulk-boundary correlator in a BCFT)

$$\langle \hat{\mathcal{O}}^i(x_{\parallel}, r) \hat{\mathcal{O}}^j(y_{\parallel}) \rangle = \delta^{ij} \frac{\hat{\mu}^{\hat{\mathcal{O}}}}{\rho^{\hat{\Delta}-\hat{\Delta}}(s_{\parallel}^2 + \rho^2)^{\hat{\Delta}}}. \quad (4.2)$$

$B_{\hat{\Delta}}^d$ is a differential operator generating towers of descendants for each boundary primary

$$B_{\hat{\Delta}}^d(x_{d-1}^2) = \sum_{m \geq 0} \frac{(-1)^m x_{d-1}^{2m}}{m! \left(\hat{\Delta} - \frac{d-3}{2} \right)_m} (\partial_{x_{\parallel}}^2 + \partial_{x_d}^2)^m. \quad (4.3)$$

If we perform the summation over m we find the conformal blocks [19]. However, for the purpose of finding the boundary anomalous dimensions from (3.22) it is enough to study eq. (4.1) at the lowest order in ρ

$$\langle \phi^i(x) \hat{\mathcal{O}}^j(y_{\parallel}) \rangle = \delta^{ij} \frac{c_{\hat{\Delta}} \sin^{\hat{\Delta}-\Delta_{\phi}} \varphi}{\rho^{\Delta_{\phi}-\hat{\Delta}}(s_{\parallel}^2 + r^2)^{\hat{\Delta}}} + \dots, \quad c_{\hat{\Delta}} = \mu^{\phi_{\hat{\mathcal{O}}_f}} \hat{\mu}^{\hat{\mathcal{O}}_f}_{\hat{\mathcal{O}}}, \quad (4.4)$$

where $\hat{\mathcal{O}}_f \in \{\hat{\phi}, \partial_{\perp} \hat{\phi}\}$ is the boundary operator from the free theory. By comparing this with (3.1) we find

$$f(\varphi) = c_{\hat{\Delta}} \sin^{\hat{\Delta}-\Delta_{\phi}} \varphi + \dots. \quad (4.5)$$

In the ϵ -expansion (3.4) with¹⁵

$$\begin{aligned} \hat{\Delta}_n &= \Delta_{\phi}^{(0)} + n + \epsilon \hat{\gamma}_n + \mathcal{O}(\epsilon^2), \\ c_{\hat{\Delta}_n} &= c_{\hat{\Delta}_n}^{(0)} + \epsilon c_{\hat{\Delta}_n}^{(1)} + \mathcal{O}(\epsilon^2), \end{aligned} \quad (4.6)$$

we have to the lowest order in φ

$$\begin{aligned} f^{(1)}(\varphi) &= c_{\hat{\Delta}_n}^{(0)} \varphi^{\hat{\Delta}_n^{(0)} - \Delta_{\phi}^{(0)}} + \dots, \\ f^{(1)}(\varphi) &= \varphi^{\hat{\Delta}_n^{(0)} - \Delta_{\phi}^{(0)}} (c_{\hat{\Delta}_n}^{(1)} + c_{\hat{\Delta}_n}^{(0)} (\hat{\gamma}_n - \gamma_{\phi}) \log \varphi) + \dots. \end{aligned} \quad (4.7)$$

If we were to use the BOE's for the ramp in (4.1) we simply exchange $\varphi \rightarrow \theta - \varphi$.

By expanding our results of $f(\varphi)$ from section 3 in φ (either around zero or θ) we can extract the CFT data of the boundary operator from the free theory. For the three different

¹⁵The boundary anomalous dimensions correspond to normal derivatives. This choice of scaling dimensions is motivated by the generalized free theory solutions [20].

combinations of b.c.'s we find the boundary anomalous dimensions (3.31) (it is the same on the wall and the ramp) consistent with the BCFT literature. Remarkably, we have now found all of the anomalous dimensions appearing at $\mathcal{O}(\epsilon)$ (bulk, boundary and wedge) only assuming that the bulk field satisfies the e.o.m. (3.14).

The correction to the first BOE coefficient, $c_{\hat{\Delta}}^{(1)}$, differs for different b.c.'s, and e.g. for DD it is given by

$$c_{\hat{\Delta}}^{(1)} = \alpha \pi^2 \mathcal{A} + \frac{(N+2)\lambda_*}{32 \pi^2 \theta^2 \epsilon} \left(m \left(2 - \log \left(\frac{\pi}{\theta} \right) - \sum_{k=1}^{m-1} \frac{1}{k} \right) - \frac{3}{2} \right), \quad (4.8)$$

where we also see how this is related to the undetermined coefficient \mathcal{A} from (3.22). It is related in a similar way for the other b.c.'s.

4.2 BOE coefficients

Let us now study how we can find the other BOE coefficients that appear at $\mathcal{O}(\epsilon)$ from (3.22). As previously mentioned, by performing the summation in the differential operator in (4.1) we find the conformal blocks, and in turn also the bootstrap equation [19]

$$\begin{aligned} f(\varphi) &= \sin^{\hat{\Delta}_\alpha - \Delta_\phi}(\varphi) \sum_n c_n \mathcal{G}_n(\hat{\Delta}_n, \hat{\Delta}_\alpha, \tan \varphi) \\ &= \sin^{\hat{\Delta}_\alpha - \Delta_\phi}(\theta - \varphi) \sum_m c'_m \mathcal{G}_m(\hat{\Delta}'_m, \hat{\Delta}_\alpha, \tan(\theta - \varphi)), \end{aligned} \quad (4.9)$$

with the conformal blocks

$$\mathcal{G}_n(\hat{\Delta}_n, \hat{\Delta}_\alpha, \eta) = \eta^{\hat{\Delta}_n - \hat{\Delta}_\alpha} {}_2F_1 \left(\frac{\hat{\Delta}_n - \hat{\Delta}_\alpha}{2}, \frac{\hat{\Delta}_n - \hat{\Delta}_\alpha + 1}{2}, \hat{\Delta}_n - \frac{d-3}{2}, -\eta^2 \right). \quad (4.10)$$

Note that the bootstrap equation (4.9) is symmetric under $\varphi \rightarrow \theta - \varphi$, which is the same as switching the two boundaries with each other: wall \longleftrightarrow ramp. This explains why DN b.c.'s is the same as ND (Neumann on the wall and Dirichlet on the ramp).

In the ϵ -expansion (3.4), (4.6) we write

$$\begin{aligned} \mathcal{G}_n(\hat{\Delta}_n, \hat{\Delta}_\alpha, \eta) &= \mathcal{G}_n^{(0)}(\hat{\Delta}_n^{(0)}, \hat{\Delta}_\alpha^{(0)}, \eta) + \epsilon \mathcal{G}_n^{(1)}(\hat{\Delta}_n^{(0)}, \hat{\Delta}_\alpha^{(0)}, \hat{\gamma}_n, \hat{\gamma}_\alpha, \eta), \\ \mathcal{G}_n^{(0)}(\hat{\Delta}_n^{(0)}, \hat{\Delta}_\alpha^{(0)}, \eta) &= \mathcal{G}_n(\hat{\Delta}_n^{(0)}, \hat{\Delta}_\alpha^{(0)}, \eta), \end{aligned} \quad (4.11)$$

which gives us the wall block decomposition

$$f^{(0)}(\varphi) = \sin^\alpha \varphi \sum_n c_n^{(0)} \mathcal{G}_n^{(0)}(\hat{\Delta}_n^{(0)}, \hat{\Delta}_\alpha^{(0)}, \tan \varphi), \quad (4.12)$$

$$\begin{aligned} f^{(1)}(\varphi) &= \sin^\alpha \varphi \sum_n \left[c_n^{(1)} \mathcal{G}_n^{(0)}(\hat{\Delta}_n^{(0)}, \hat{\Delta}_\alpha^{(0)}, \tan \varphi) + \right. \\ &\quad + c_n^{(0)} \mathcal{G}_n^{(1)}(\hat{\Delta}_n^{(0)}, \hat{\Delta}_\alpha^{(0)}, \hat{\gamma}_{0/1}, \hat{\gamma}_\alpha, \tan \varphi) + \\ &\quad \left. + (\hat{\gamma} - \gamma_\phi) \log(\sin \varphi) c_n^{(0)} \mathcal{G}_n^{(0)}(\hat{\Delta}_n^{(0)}, \hat{\Delta}_\alpha^{(0)}, \tan \varphi) \right]. \end{aligned} \quad (4.13)$$

The conformal block in the free theory is orthogonal¹⁶

$$\oint_{|\eta|=\tilde{\epsilon}} \frac{d\eta}{2\pi i} \Xi(\hat{\Delta}_m^{(0)}, \hat{\Delta}_\alpha^{(0)}, \eta) \mathcal{G}_n^{(0)}(\hat{\Delta}_n^{(0)}, \hat{\Delta}_\alpha^{(0)}, \eta) = \delta_{mn}. \quad (4.14)$$

The integration is over a small circle with radius $0 < \tilde{\epsilon} \ll 1$, i.e. we consider the residue at $\eta = 0$, and the orthogonality weight function is given by

$$\Xi(\hat{\Delta}_m^{(0)}, \hat{\Delta}_\alpha^{(0)}, \eta) = \eta^{\hat{\Delta}_\alpha^{(0)} - \hat{\Delta}_m^{(0)} - 1} {}_2F_1\left(\frac{\hat{\Delta}_\alpha^{(0)} - \hat{\Delta}_m^{(0)} - 1}{2}, \frac{\hat{\Delta}_\alpha^{(0)} - \hat{\Delta}_m^{(0)}}{2} - 1, \frac{d+1}{2} - \hat{\Delta}_m^{(0)}, -\eta^2\right).$$

If we apply this orthogonality relation to $f^{(0)}(\varphi)$ in (4.12) we find

$$c_n^{(0)} = \oint_{|\varphi|=\tilde{\epsilon}} \frac{d\varphi}{2\pi i} \frac{\Xi(\hat{\Delta}_n^{(0)}, \hat{\Delta}_\alpha^{(0)}, \tan \varphi)}{\cos^2 \varphi \sin^\alpha \varphi} f^{(0)}(\varphi). \quad (4.15)$$

For the ramp-channel BOE coefficients we exchange $\cos^{-2} \varphi \rightarrow -\cos^{-2}(\theta - \varphi)$ and take the residue at $\varphi = \theta$. Using the result of $f^{(0)}$ from section 3.1 we find for the different b.c.'s

$$\begin{aligned} \text{DD:} \quad c_n^{(0)} &= \frac{\Gamma(\hat{\Delta}_\alpha^{(0)})}{\pi^{\Delta_\phi^{(0)}} \theta \Gamma(\alpha)} \delta_{n1}, & c_n^{\prime(0)} &= \frac{(-1)^{\alpha+1} \Gamma(\hat{\Delta}_\alpha^{(0)})}{\pi^{\Delta_\phi^{(0)}} \theta \Gamma(\alpha)} \delta_{n1}, \\ \text{NN:} \quad c_n^{(0)} &= \frac{\Gamma(\hat{\Delta}_\alpha^{(0)})}{\pi^{\Delta_\phi^{(0)}} \theta \alpha \Gamma(\alpha)} \delta_{n0}, & c_n^{\prime(0)} &= \frac{(-1)^\alpha \Gamma(\hat{\Delta}_\alpha^{(0)})}{\pi^{\Delta_\phi^{(0)}} \theta \alpha \Gamma(\alpha)} \delta_{n0}, \\ \text{DN:} \quad c_n^{(0)} &= \frac{\Gamma(\hat{\Delta}_\alpha^{(0)})}{\pi^{\Delta_\phi^{(0)}} \theta \Gamma(\alpha)} \delta_{n1}, & c_n^{\prime(0)} &= -\frac{i^{2\alpha+1} \Gamma(\hat{\Delta}_\alpha^{(0)})}{\pi^{\Delta_\phi^{(0)}} \theta \alpha \Gamma(\alpha)} \delta_{n0}, \end{aligned} \quad (4.16)$$

where m is the integer in α . This means that at $\mathcal{O}(\epsilon)$ we only need the ϵ -expansion of the two conformal blocks with $n \in \{0, 1\}$, i.e. $\mathcal{G}_{0/1}^{(1)}$. In appendix C we outline how this is done. Applying the orthogonality relation (4.14) to (4.13) gives us

$$\begin{aligned} c_n^{(1)} = \oint_{|\varphi|=\tilde{\epsilon}} \frac{d\varphi}{2\pi i} \frac{\Xi(\hat{\Delta}_n^{(0)}, \hat{\Delta}_\alpha^{(0)}, \tan \varphi)}{\cos^2 \varphi \sin^\alpha \varphi} & \left[f^{(1)}(\varphi) - c_{0/1}^{(0)} \mathcal{G}^{(1)}(\hat{\Delta}_{0/1}^{(0)}, \hat{\Delta}_\alpha^{(0)}, \hat{\gamma}_{0/1}, \hat{\gamma}_\alpha, \tan \varphi) + \right. \\ & \left. - (\hat{\gamma} - \gamma_\phi) \log(\sin \varphi) c_{0/1}^{(0)} \mathcal{G}^{(0)}(\hat{\Delta}_{0/1}^{(0)}, \hat{\Delta}_\alpha^{(0)}, \tan \varphi) \right]. \end{aligned} \quad (4.17)$$

This integral is difficult to do in general, but it can be done for specific values of n . Only odd (Dirichlet) or even (Neumann) BOE coefficients (w.r.t. n) are non-zero, which agrees with the results in a BCFT [23]. Unfortunately we were not able to find the general form for the BOE coefficients, and we list the first few BOE coefficients for DD. The first one, $n = 1$, agrees with (4.8), and the next non-trivial one is given by

$$c_5^{(1)} = -\frac{(N+2)m(\pi^4 - \theta^4)\lambda_*}{9600\pi^2\theta^6\epsilon}. \quad (4.18)$$

Here m is the same as in (3.11).

¹⁶This orthogonality is found in the same way as in [23, 24].

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A Computation of the propagator

In this appendix we give the details regarding the computation of the free bulk-bulk propagator reported in the main text (see also [1, 76] for previous discussions). We need to compute

$$\begin{aligned} \langle \phi^i(\mathbf{x}, \varphi, \rho) \phi^j(0, \varphi', \rho') \rangle &= \frac{\delta^{ij}}{2\theta} \sum_{m \in \mathbb{Z}} \int d^{d-1}k \frac{k_\rho}{2\omega(2\pi)^{d-3}} e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} J_{|\frac{\pi m}{\theta}|}(k_\rho \rho) J_{|\frac{\pi m}{\theta}|}(k_\rho \rho') \times \\ &\times \left(e^{i\frac{\pi}{\theta} m(\varphi - \varphi')} \pm e^{i\frac{\pi}{\theta} m(\varphi + \varphi')} \right), \end{aligned}$$

where we promoted the fields to be invariant under $O(N)$. In the Euclidean formulation this may be rewritten as

$$\langle \phi(\mathbf{x}, \varphi, \rho) \phi(0, \varphi', \rho') \rangle = \frac{\pi \delta^{ij}}{\theta} \sum_{m \in \mathbb{Z}} G_S^{(\frac{\pi m}{\theta})}(x_{||}, \rho; x'_{||}, \rho') \left(e^{i\frac{\pi}{\theta} m(\varphi - \varphi')} \pm e^{i\frac{\pi}{\theta} m(\varphi + \varphi')} \right), \quad (\text{A.1})$$

where we defined

$$G_S^{(\nu)}(x_{||}, \rho; x'_{||}, \rho') \equiv \int d^{d-3}\mathbf{k} dk_\rho dk_\tau \frac{k_\rho}{(2\pi)^{d-1}} \frac{e^{-ik_\tau(\tau - \tau') + i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')}}{k_\rho^2 + \mathbf{k}^2 + k_\tau^2} J_\nu(k_\rho \rho) J_\nu(k_\rho \rho'). \quad (\text{A.2})$$

By employing a Schwinger parametrisation

$$\frac{1}{A^n} = \int_0^\infty ds \frac{s^{n-1}}{\Gamma(n)} e^{-sA}, \quad A > 0, \quad (\text{A.3})$$

and performing the Gaussian integration over \mathbf{k} and k_τ , we obtain

$$\begin{aligned} G_S^{(\nu)}(x, x') &= \int_0^{+\infty} ds \int_0^{+\infty} dk_\rho \frac{k_\rho}{2^{d-1} \pi^{d/2}} \frac{1}{s^{d/2-1}} e^{-\frac{(\mathbf{x} - \mathbf{x}')^2 + (\tau - \tau')^2}{4s}} e^{-k_\rho^2 s} \\ &\times J_\nu(k_\rho \rho) J_\nu(k_\rho \rho'). \end{aligned} \quad (\text{A.4})$$

The integral over k_ρ in eq. (A.4) can be performed analytically (see for example [77])

$$G_S^{(\nu)}(x, x') = \frac{1}{2^d \pi^{d/2}} \int_0^\infty ds s^{\frac{d}{2}-2} e^{-s \frac{\rho^2 + \rho'^2 + (\mathbf{x} - \mathbf{x}')^2}{4}} I_\nu \left(\frac{s \rho \rho'}{2} \right). \quad (\text{A.5})$$

At this point also the integral over s in eq. (A.5) can be done, and the result is expressed in equation (2.17).¹⁷

¹⁷We found it convenient to use the relation

$$I_\beta(z) = e^{\mp i\beta\pi/2} J_\beta \left(ze^{\pm i\pi/2} \right),$$

before computing the integral.

A case of interest is when $\theta = \frac{\pi}{n}$. Indeed with this assumption, the sum over m simplifies and can be done. In this case the propagator reads

$$\begin{aligned} \langle \phi(\mathbf{x}, \varphi, \rho) \phi(0, \varphi', \rho') \rangle &= \frac{n \delta^{ij}}{2^{d-2} \pi^{\frac{d}{2}}} \int_0^\infty ds s^{\frac{d}{2}-2} e^{-s \frac{\rho^2 + \rho'^2 + (\mathbf{x} - \mathbf{x}')^2}{4}} \times \\ &\times \sum_{m \geq 0} 'I_{m,n} \left(\frac{s \rho \rho'}{2} \right) \begin{cases} \sin(m n \varphi) \sin(m n \varphi') , & \text{DD,} \\ \cos(m n \varphi) \cos(m n \varphi') , & \text{NN,} \end{cases} \end{aligned} \quad (\text{A.6})$$

where the sums can be expressed in terms of elementary functions as

$$\begin{aligned} \sum_{m \geq 0} \sin(n m \varphi) \sin(n m \varphi') I_{m,n}(z) &= \\ \frac{1}{4n} \sum_{k=0}^{n-1} \left\{ \exp \left[z \cos \left(\frac{2k\pi + n(\varphi - \varphi')}{n} \right) \right] - \exp \left[z \cos \left(\frac{2k\pi + n(\varphi + \varphi')}{n} \right) \right] \right\}, \end{aligned} \quad (\text{A.7})$$

and

$$\begin{aligned} \sum_{m \geq 0} ' \cos(n m \varphi) \cos(n m \varphi') I_{m,n}(z) &= \\ \frac{1}{4n} \sum_{k=0}^{n-1} \left\{ \exp \left[z \cos \left(\frac{2k\pi + n(\varphi - \varphi')}{n} \right) \right] + \exp \left[z \cos \left(\frac{2k\pi + n(\varphi + \varphi')}{n} \right) \right] \right\}. \end{aligned} \quad (\text{A.8})$$

By plugging those identities back to eq. (A.6) and performing the integral over s we obtain eq. (2.24).

B Feynman diagrams

In this appendix we double check the results on the wedge anomalous dimensions by calculating Feynman diagrams. We do not find the full correlator in this approach, making the method in section 3 stronger. At the first order in perturbation theory, they can be extracted from the logarithmic contribution to the relevant Feynman diagram. In addition, since we are only interested in the leading order in ϵ , we will just evaluate the relevant integrals at $d = 4$ in section B.1 and at $d = 3$ in section B.2. By conformal symmetry along the edge, the two-point function of defect primaries has the form

$$\langle \hat{O}_\alpha^i(x_{||}) \hat{O}_\beta^j(y_{||}) \rangle = \delta^{ij} \delta_{\alpha\beta} \frac{C_{\hat{\Delta}}}{|x_{||}|^{2\hat{\Delta}}}, \quad (\text{B.1})$$

where $\hat{\Delta}$ is the conformal dimension of the defect primary \hat{O}_α . In a perturbative framework we may write

$$\hat{\Delta} = \hat{\Delta}^{(0)} + \epsilon \hat{\gamma} + \mathcal{O}(\epsilon^2), \quad C_{\hat{\Delta}} = C_{\hat{\Delta}}^{(0)} + \epsilon C_{\hat{\Delta}}^{(1)} + \mathcal{O}(\epsilon^2). \quad (\text{B.2})$$

By plugging this into eq. (B.1) and expanding in λ we obtain

$$\langle \hat{O}_\alpha^i(x_{||}) \hat{O}_\beta^j(y_{||}) \rangle = \frac{\delta^{ij} \delta_{\alpha\beta}}{|s_{||}|^{2\hat{\Delta}^{(0)}}} \left[C_{\hat{\Delta}}^{(0)} + \left(C_{\hat{\Delta}}^{(1)} - 2C_{\hat{\Delta}}^{(0)} \hat{\gamma}^{(1)} \log |s_{||}| \right) \epsilon \right] + \mathcal{O}(\epsilon^2). \quad (\text{B.3})$$

This means that, at $\mathcal{O}(\lambda)$, we can extract the anomalous dimension, $\hat{\gamma}$, from the coefficient of the logarithmic divergence, A_{\log} , as

$$\hat{\gamma} = -\frac{A_{\log}}{2 C_{\hat{\Delta}}^{(0)} \epsilon}. \quad (\text{B.4})$$

B.1 Wedge anomalous dimension in $d = 4 - \epsilon$

The leading contribution to the anomalous dimension comes from the following integral

$$\begin{aligned} \langle \hat{O}_\alpha^i(x_{||}) \hat{O}_\alpha^j(y_{||}) \rangle_1 &= -\delta^{ij} \frac{(N+2)\lambda}{2} \int_0^{+\infty} d\rho \rho \int_0^\theta d\varphi \int_{\mathbb{R}^{d-2}} d^{d-2}z_{||} \langle \hat{O}_\alpha^k(x_{||}) \phi^k(z) \rangle_0 \times \\ &\times \langle \phi^{k^2}(z) \rangle_0 \langle \phi^k(z) \hat{O}_\alpha^k(y_{||}) \rangle_0. \end{aligned}$$

Here k represents only one of the possible N fields, and in the integrand we have the correlators from the free theory reported in section 2. We are interested in the case when $\alpha > 0$. We find it convenient to rewrite the integral as

$$\langle \hat{O}_\alpha^i(x_{||}) \hat{O}_\alpha^j(y_{||}) \rangle_1 = -\delta^{ij} \frac{(N+2)\lambda}{64 \pi^4 \theta^4} \mathcal{I}(\theta) \mathcal{J}(s_{||}), \quad (\text{B.5})$$

where we defined

$$\begin{aligned} \mathcal{I}(\theta) &\equiv \int_0^\theta d\varphi g_\alpha(\varphi) h(\varphi), \\ \mathcal{J}(s_{||}) &\equiv \int_0^{+\infty} d\rho \int d^2z_{||} \frac{1}{\rho^3} \left(\frac{\rho}{\rho^2 + (s_{||} + z_{||})^2} \right)^{\alpha+1} \left(\frac{\rho}{\rho^2 + z_{||}^2} \right)^{\alpha+1}. \end{aligned} \quad (\text{B.6})$$

The angular integral can be done directly by introducing an angular cutoff ϑ s.t. $\varphi \in (\vartheta, \theta - \vartheta)$

$$\mathcal{I}(\theta) = \begin{cases} \frac{(\pi^2 - \theta^2)\theta}{2\theta^3} - 3\pi^2 m \theta, & DD, \\ \frac{3\pi\vartheta(\theta - \vartheta)}{3\theta^3} + \frac{(\pi^2 - \theta^2)\theta}{2} - 3\pi^2 m \theta, & NN, \\ \frac{\theta - \vartheta}{\theta - \vartheta} - \frac{\theta^2}{2} - \frac{\pi^2\theta(12m + 7)}{4}, & DN. \end{cases} \quad (\text{B.7})$$

For the purpose of finding the wedge anomalous dimension we only need to care about the finite part of $\mathcal{I}(\theta)$, and discard the non-universal power-law divergence. Note that the result corresponding to DD and NN are the same, while the DN case differs. The poles in ϑ or $\theta - \vartheta$ corresponds to the $\hat{\phi}$ exchange on its respective boundary for Neumann b.c.

We are left with analysing the function $\mathcal{J}(s_{||})$. After a Feynman parametrisation, and a shift $z_{||} \rightarrow z_{||} + u s_{||}$

$$\mathcal{J}(s_{||}) = \frac{\Gamma(2(\alpha+1))}{\Gamma(\alpha+1)^2} \int_0^{+\infty} d\rho \int_0^1 du \int_{\mathbb{R}^{d-2}} d^{d-2}z_{||} \rho^{2\alpha-1} \frac{[u(1-u)]^\alpha}{[\rho^2 + z_{||}^2 + s_{||}^2 u(1-u)]^{2(\alpha+1)}}. \quad (\text{B.8})$$

The integral over $z_{||}$ can now be done using a Schwinger parametrization (A.3)

$$\mathcal{J}(s_{||}) = \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2} \frac{\pi}{|s_{||}|^{2(2\alpha+1)}} \int_0^{+\infty} d\rho \int_0^1 du \rho^{2\alpha-1} \frac{[u(1-u)]^\alpha}{\left(\frac{\rho^2}{s_{||}^2} + u(1-u)\right)^{2\alpha+1}}. \quad (\text{B.9})$$

At this point, if we perform the integral over ρ first, we see that the result is finite without the introduction of any cutoff. However, the integral over u turns out to be logarithmic divergent. To better expose this divergence we first rewrite the ρ -integral in terms of ρ^2 and then we introduce a cutoff ϱ s.t. $\rho^2 \geq \varrho^2$. We get

$$\begin{aligned} \mathcal{J}(s_{||}) &= \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2} \frac{\pi}{2|s_{||}|^{2(2\alpha+1)}} \int_{\varrho^2}^{+\infty} d\rho^2 \int_0^1 du (\rho^2)^{\alpha-1} \frac{[u(1-u)]^\alpha}{\left(\frac{\rho^2}{s_{||}^2} + u(1-u)\right)^{2\alpha+1}} \\ &= \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2} \frac{\pi}{2|s_{||}|^{2(2\alpha+1)}} \int_0^{+\infty} d\rho^2 \int_0^1 du (\rho^2)^{\alpha-1} \frac{[u(1-u)]^\alpha}{\left(\frac{\rho^2}{s_{||}^2} + \frac{\varrho^2}{s_{||}^2} + u(1-u)\right)^{2\alpha+1}} + \dots \end{aligned}$$

Note that in the second step we did the shift $\rho^2 \rightarrow \rho^2 + \varrho^2$. In an expansion in the cutoff we can neglect the ϱ -contribution from the ρ^2 -term of the numerator. Performing the integral over ρ^2 gives

$$\mathcal{J}(s_{||}) = \frac{\theta}{2m|s_{||}|^{2(\alpha+1)}} \int_0^1 du \frac{[u(1-u)]^\alpha}{\left(\frac{\varrho^2}{s_{||}^2} + u(1-u)\right)^{\alpha+1}}. \quad (\text{B.10})$$

Thanks to the cutoff ϱ , now also the integral over u is convergent, and it can be performed analytically. The result is a hypergeometric function which we do not report here. By expanding the result in the cutoff we find

$$\mathcal{J}(s_{||}) = \frac{2\pi}{\alpha|x_{||}|^{2(\alpha+1)}} \left[\log\left(\frac{|s_{||}|}{\varrho}\right) - \frac{1}{2\alpha} - \frac{\psi(\alpha)}{2} - \frac{\gamma_E}{2} \right] + \mathcal{O}(\varrho), \quad (\text{B.11})$$

where $\psi(\alpha)$ is the digamma function, and γ_E is the Euler-Mascheroni constant. As discussed above, the correction to the conformal dimension can be found from the coefficient of the logarithmic term in the correlator (B.5).

All and all, this gives

$$C_{\hat{\Delta}}^{(0)} = \frac{1}{\pi\theta}, \quad A_{\log} = \frac{(N+2)\lambda}{96\pi^4\theta^2} \begin{cases} \frac{\pi^2(6m-1) + \theta^2}{m}, & \text{DD, NN,} \\ \frac{\pi^2(12m+7) + 2\theta^2}{2m+1}, & \text{DN.} \end{cases} \quad (\text{B.12})$$

The formula (B.4) then reproduces the wedge anomalous dimensions in (3.21).

B.2 Wedge anomalous dimension in $d = 3 - \epsilon$

In this case we need to compute

$$\begin{aligned} \langle \hat{O}_\alpha^i(x_{||}) \hat{O}_\alpha^j(y_{||}) \rangle_1 = \\ - \delta^{ij} \frac{(N+2)g_*}{2} \int_0^{+\infty} d\rho \int_{\mathbb{R}^{d-2}} dz_{||} \langle \hat{O}_\alpha^k(x_{||}) \hat{\phi}^k(z) \rangle_0 \langle \hat{\phi}^2(z) \rangle_0 \langle \hat{\phi}^k(z) \hat{O}_\alpha^k(y_{||}) \rangle_0, \end{aligned}$$

where again k represents only one of the N possible fields. Analogously to the computation done above, we need the boundary-wedge propagator and the one-point function of ϕ^2 when $d = 3$. Those correlators have been reported in (3.36), (3.37).

Thus, the integral to compute is

$$\begin{aligned} \langle \hat{O}_\alpha^i(x_{||}) \hat{O}_\alpha^j(y_{||}) \rangle_1 = - \delta^{ij} \frac{(N+2)g_*\kappa_n\Gamma\left(\alpha + \frac{1}{2}\right)^2}{4\sqrt{2}\pi^4\theta^2\Gamma(\alpha+1)^2} \times \\ \times \int_0^{+\infty} d\rho \int dz_{||} \rho^{2\alpha-1} \left(\frac{1}{\rho^2 + (s_{||} + z_{||})^2} \right)^{\alpha+\frac{1}{2}} \left(\frac{1}{\rho^2 + z_{||}^2} \right)^{\alpha+\frac{1}{2}}. \end{aligned}$$

By doing the same steps as in the case of section B.1 we obtain

$$\langle \hat{O}_\alpha^i(x_{||}) \hat{O}_\alpha^j(y_{||}) \rangle_1 = \delta^{ij} \frac{(N+2)g_*\kappa_n\Gamma(2\alpha)}{2^{2\alpha+\frac{3}{2}}\pi\theta^2\Gamma(\alpha+1)^2s_{||}^{2\alpha+1}} \left[H_{\alpha-\frac{1}{2}} - 2\log\left(\frac{|s_{||}|}{\varrho}\right) \right] + \mathcal{O}(\varrho), \quad (\text{B.13})$$

where H_x is the analytic continuation of the harmonic number to real values. From the above equation we can extract the anomalous dimension, which gives us (3.38).¹⁸

C Expansion of the conformal blocks

In this appendix we explain how the conformal blocks from the free theory can be expanded in ϵ . For Dirichlet b.c. we wish to expand

$$\mathcal{G}_0(\hat{\Delta}_0, \hat{\Delta}_\alpha, z) \propto {}_2F_1\left(-\frac{\alpha}{2} + a\epsilon, \frac{1-\alpha}{2} + a\epsilon, \frac{1}{2} + b\epsilon, z\right), \quad (\text{C.1})$$

and for Neumann b.c.

$$\mathcal{G}_1(\hat{\Delta}_0, \hat{\Delta}_\alpha, z) \propto {}_2F_1\left(\frac{1-\alpha}{2} + a\epsilon, 1 - \frac{\alpha}{2} + a\epsilon, \frac{3}{2} + b\epsilon, z\right). \quad (\text{C.2})$$

The coefficients a and b are given by

$$a = \frac{\hat{\gamma} - \hat{\gamma}}{2}, \quad b = \hat{\gamma}. \quad (\text{C.3})$$

These two blocks satisfy the relation

$${}_2F_1(a, b, c, z) = (\sqrt{z} + 1)^{-2a} {}_2F_1\left(2a, c - \frac{1}{2}, 2c - 1, \frac{2\sqrt{z}}{\sqrt{z} + 1}\right), \quad (\text{C.4})$$

¹⁸Had we added the same interaction on both of the boundaries, there is an extra factor of 2 in the wedge anomalous dimension.

which holds if $b = a + \frac{1}{2}$. Applying this relation, and for the Dirichlet block also the following one of Kummer's relations

$${}_2F_1(a, b, c, z) = \frac{\Gamma(1-a)\Gamma(1-b)\Gamma(c)}{\Gamma(2-c)\Gamma(c-a)\Gamma(c-b)} \frac{{}_2F_1(a-c+1, b-c+1, 2-c, z)}{z^{c-1}} + \frac{\Gamma(1-a)\Gamma(1-b)}{\Gamma(1-c)\Gamma(c-a-b+1)} \frac{{}_2F_1(c-a, c-b, c-a-b+1, z)}{(1-z)^{a+b-c}}, \quad (\text{C.5})$$

we find ${}_2F_1(a, b, c, z)$'s in the conformal blocks that all satisfy $c > b > \mathcal{O}(\epsilon)$. This means that we can use the following real-line integral representation (on all of the ${}_2F_1$'s)¹⁹

$${}_2F_1(a, b, c, z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a}. \quad (\text{C.6})$$

The integrand can then be expanded in ϵ . This gives us integrals over different kinds of logarithms at $\mathcal{O}(\epsilon)$, which we performed using the 'Rubi' package for Mathematica [78]. We have then found the ϵ -expansions of the ${}_2F_1$ in the blocks²⁰

$$\begin{aligned} \mathcal{G}_0 \propto & \frac{(1-\sqrt{z})^k + (1+\sqrt{z})^k}{2} + \epsilon \left\{ (a-c) \frac{(1+\sqrt{z})^k - (1-\sqrt{z})^k}{k} + \right. \\ & + (c-a) (1+\sqrt{z})^k \log(1+\sqrt{z}) - a (1-\sqrt{z})^k \log(1-\sqrt{z}) + \\ & - c (\sqrt{z}+1)^k \log(2\sqrt{z}) + (c-a) H_{k-1} \left[(1-\sqrt{z})^k - (1+\sqrt{z})^k \right] + \\ & + H_k \left[(a-c) (1-\sqrt{z})^k - a (\sqrt{z}+1)^k \right] + \\ & - \frac{c(1-\sqrt{z})^{k+1}}{(k+1)(1+\sqrt{z})} {}_2F_1\left(1, k+1; k+2; \frac{1-\sqrt{z}}{1+\sqrt{z}}\right) + \\ & + \frac{ck\sqrt{z}}{1+\sqrt{z}} \left[(1+\sqrt{z})^k {}_3F_2\left(1, 1, 1-k; 2, 2; \frac{2\sqrt{z}}{\sqrt{z}+1}\right) + \right. \\ & \left. \left. - (1-\sqrt{z})^k {}_3F_2\left(1, 1, 1-k; 2, 2; \frac{2\sqrt{z}}{\sqrt{z}-1}\right) \right] \right\}, \quad (\text{C.7}) \end{aligned}$$

$$\begin{aligned} \mathcal{G}_1 \propto & \frac{(1+\sqrt{z})^k - (1-\sqrt{z})^k}{2k\sqrt{z}} + \epsilon \left\{ (a+ck) \frac{(1+\sqrt{z})^k - (1-\sqrt{z})^k}{k^2\sqrt{z}} + \right. \\ & + \frac{a(1-\sqrt{z})^k \log(1-\sqrt{z})}{k\sqrt{z}} - \frac{a(1+\sqrt{z})^k \log(1+\sqrt{z})}{k\sqrt{z}} + \\ & - c \left[(1-\sqrt{z})^{k-1} {}_3F_2\left(1, 1, 1-k; 2, 2; \frac{2\sqrt{z}}{\sqrt{z}-1}\right) + \right. \\ & \left. \left. + (1+\sqrt{z})^{k-1} {}_3F_2\left(1, 1, 1-k; 2, 2; \frac{2\sqrt{z}}{1+\sqrt{z}}\right) \right] \right\}. \quad (\text{C.8}) \end{aligned}$$

¹⁹This integral is convergent for $b > a > 0$.

²⁰These can also be found in the supplementary Mathematica file attached to this paper.

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