A NOTE ON TOTALLY GEODESIC EMBEDDINGS OF ESCHENBURG SPACES INTO BAZAIKIN SPACES

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ABSTRACT. In this note it is shown that every 7-dimensional Eschenburg space can be totally geodesically embedded into infinitely many topologically distinct 13-dimensional Bazaikin spaces. Furthermore, examples are given which show that, under the known construction, it is not always possible to totally geodesically embed a positively curved Eschenburg space into a Bazaikin space with positive curvature.

Dedicated to Wolfgang Ziller on the occasion of his sixtieth birthday

In dimensions 7 and 13 there are two very special families of closed Riemannian manifolds, namely the Eschenburg and Bazaikin spaces. These are defined as quotients of $\mathrm{SU}(3)$ and $\mathrm{SU}(5)/\mathrm{Sp}(2)$ by free, isometric circle actions (see Section 2 and [AW], [Baz], [DE], [Es1], [Ke], [Zi]), where $\mathrm{SU}(k)$ has been equipped with a left-invariant, right $\mathrm{U}(k-1)$-invariant metric. In each case there are infinitely many (distinct homotopy types of) family members admitting positive sectional curvature. Given that there are so few known examples of closed manifolds with positive sectional curvature, these families have been studied extensively (see, for example, [Baz], [CEZ], [DE], [Es1], [Es2], [FZ], [GSZ], [Ke], [Kr], [Zi]).

One particularly intriguing observation, made in [Ta], is that to each Bazaikin space there can be associated a totally geodesic, embedded Eschenburg space. In fact, as demonstrated in [DE], a Bazaikin space generically contains ten mutually distinct, totally geodesic, embedded Eschenburg spaces. This led to the question: given an Eschenburg space, does there exist a (non-singular) Bazaikin space containing it as a totally geodesic, embedded submanifold?

Theorem A. *For any given Eschenburg space $E^7_{a,b}$, there exist infinitely many mutually non-homotopy equivalent Bazaikin spaces into which $E^7_{a,b}$ can be embedded as a totally geodesic submanifold.*

In [DE] it has also been proven that a Bazaikin space is positively curved if and only if each of the ten embedded, totally geodesic Eschenburg spaces it contains are also positively curved. On the other hand, it is well-known (see [DE], [Zi]) that all positively curved Aloff-Wallach spaces (see [AW]), that is, the subfamily of homogeneous Eschenburg spaces, and all positively curved cohomogeneity-one Eschenburg spaces (see [GSZ]) can be embedded as totally geodesic submanifolds of positively curved Bazaikin spaces. This raises the question of whether this is true in general. However, at least for the known construction of a totally geodesic embedding, there exist counter-examples already in cohomogeneity-two, see Table 1 in Section 5.

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The article is organised as follows. In Section 1 the basic notation and definitions for biquotients are reviewed. Section 2 provides a brief summary of Eschenburg and Bazaikin spaces, while Section 3 recalls how to each Bazaikin space there correspond ten totally geodesic, embedded Eschenburg spaces. In Section 4 the proof of Theorem A is given and, finally, in Section 5 the ability to embed a positively curved Eschenburg space as a totally geodesic submanifold of a positively curved Bazaikin space is discussed.

1. Biquotients and induced metrics

The following section provides a review of some material from [Es1] which establishes the basic ideas and notation which will be used throughout the remainder of the article.

Let G be a compact Lie group, $U \subset G \times G$ a closed subgroup, and let U act (effectively) on G via

$$(u_1, u_2) \star g = u_1 g u_2^{-1}, \;\; g \in G, (u_1, u_2) \in U. \tag{1.1}$$

The resulting quotient $G/\!\!/U$ is called a *biquotient*. The action (1.1) is free (hence $G/\!\!/U$ is a manifold) if and only if, for all non-trivial $(u_1, u_2) \in U$, u_1 is never conjugate to u_2 in G. In the event that the action of U is ineffective, the quotient $G/\!\!/U$ will be a manifold so long as any element $(u_1, u_2) \in U$ which fixes some $g \in G$ fixes all of G, that is, (u_1, u_2) lies in the ineffective kernel. A biquotient is *non-singular* if the action (1.1) is free (modulo any ineffective kernel), and *singular* otherwise.

Let $K \subset G$ be a closed subgroup. Consider a biquotient $G/\!\!/U$, where $U \subset G \times K \subset G \times G$ and G is equipped with a left-invariant, right K-invariant metric $\langle\, , \rangle$. U acts by isometries on G and therefore the submersion $G \to G/\!\!/U$ induces a metric on $G/\!\!/U$ from the metric on G. This is encoded in the notation $(G, \langle\, , \rangle)/\!\!/U$.

Now, for $g \in G$ define

$$\begin{aligned} U_L^g &:= \{(g u_1 g^{-1}, u_2) \mid (u_1, u_2) \in U\}, \\ U_R^g &:= \{(u_1, g u_2 g^{-1}) \mid (u_1, u_2) \in U\}, \;\; \text{and} \\ \widehat{U} &:= \{(u_2, u_1) \mid (u_1, u_2) \in U\}. \end{aligned}$$

Then U_L^g, U_R^g and $\widehat{U}$ act freely on G, and $G/\!\!/U$ is isometric to $G/\!\!/U_L^g$, diffeomorphic to $G/\!\!/U_R^g$ (isometric if $g \in K$), and diffeomorphic to $G/\!\!/\widehat{U}$ (isometric if $U \subset K \times K$).

In the case of U_L^g this follows from the fact that left-translation $L_g : G \to G$ is an isometry which satisfies $g u_1 g^{-1}(L_g g') u_2^{-1} = L_g(u_1 g' u_2^{-1})$. Therefore L_g induces an isometry of the orbit spaces $G/\!\!/U$ and $G/\!\!/U_L^g$. Similarly, $R_{g^{-1}}$ induces a diffeomorphism between $G/\!\!/U$ and $G/\!\!/U_R^g$, which is an isometry if $g \in K$.

Consider now $\widehat{U}$. The actions of U and $\widehat{U}$ are equivariant under the diffeomorphism $\tau : G \to G$, $\tau(g) := g^{-1}$. That is, $u_1 \tau(g) u_2^{-1} = \tau(u_2 g u_1^{-1})$. Notice that this is an isometry only if $U \subset K \times K$. In general $G/\!\!/U$ and $G/\!\!/\widehat{U}$ are therefore diffeomorphic but not isometric.

For completeness one should note that it is, in fact, simple to equip G with a left-invariant, right K-invariant metric. Given $K \subset G$, let $\mathfrak{k} \subset \mathfrak{g}$ be the corresponding Lie algebras and let $\langle\, , \rangle_0$ be a bi-invariant metric on G. One can write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with respect to $\langle\, , \rangle_0$. Recall that $G \cong (G \times K)/\Delta K$ via $(g, k) \mapsto g k^{-1}$, where $\Delta K = \{(k, k) \mid k \in K\}$ acts diagonally on the right of $G \times K$. Thus a left-invariant, right K-invariant metric $\langle\, , \rangle$

on G can be defined via the Riemannian submersion

$$(G \times K, \langle\,,\,\rangle_0 \oplus t\langle\,,\,\rangle_0|_{\mathfrak{k}}) \to (G, \langle\,,\,\rangle)$$
$$(g,k) \mapsto gk^{-1},$$

where $t > 0$ and

$$\langle\,,\,\rangle = \langle\,,\,\rangle_0|_{\mathfrak{p}} + \lambda\langle\,,\,\rangle_0|_{\mathfrak{k}}, \;\; \lambda = \frac{t}{t+1} \in (0,1). \tag{1.2}$$

2. Eschenburg and Bazaikin spaces

Given $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3) \in \mathbb{Z}^3$, with $\sum a_i = \sum b_i$, recall that the Eschenburg biquotients (see [AW], [Es1]) are defined as $E^7_{a,b} := (\mathrm{SU}(3), \langle\,,\,\rangle)/\!\!/S^1_{a,b}$, where $S^1_{a,b}$ acts isometrically on $(\mathrm{SU}(3), \langle\,,\,\rangle)$ via

$$z \star A = \mathrm{diag}(z^{a_1}, z^{a_2}, z^{a_3}) \cdot A \cdot \mathrm{diag}(\bar{z}^{b_1}, \bar{z}^{b_2}, \bar{z}^{b_3}), \;\; A \in \mathrm{SU}(3), z \in S^1,$$

and the left-invariant, right $\mathrm{U}(2)$-invariant metric $\langle\,,\,\rangle$ (with $\sec \geqslant 0$) on $\mathrm{SU}(3)$ is defined as in (1.2), where $\mathrm{U}(2) \hookrightarrow \mathrm{SU}(3)$ via

$$A \in \mathrm{U}(2) \mapsto \mathrm{diag}(\overline{\det(A)}, A) \in \mathrm{SU}(3).$$

The action is free if and only if

$$\gcd(a_1 - b_{\sigma(1)}, a_2 - b_{\sigma(2)}) = 1 \;\text{ for all permutations }\; \sigma \in S_3, \tag{2.1}$$

in which case $E^7_{a,b}$ is called an *Eschenburg space*.

It is important to remark that the above defined circle subgroup $S^1_{a,b}$ is not, in general, a subgroup of $\mathrm{SU}(3) \times \mathrm{SU}(3)$. Indeed, $S^1_{a,b} \subset \mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(3)) := \{(A, B) \in \mathrm{U}(3) \times \mathrm{U}(3) \mid \det A = \det B\}$. This is not a problem, however, since the bi-invariant metric on $\mathrm{SU}(3)$ can be thought of as the restriction of the analogously defined bi-invariant metric on $\mathrm{U}(3)$. Hence, an element of $(A, B) \in \mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(3))$ maps $(\mathrm{SU}(3), \langle\,,\,\rangle_0)$ isometrically to itself via $X \in \mathrm{SU}(3) \mapsto AXB^{-1}$. In particular, conjugation by an element of the centre of $\mathrm{U}(3)$ is an isometry (namely, the identity map) of $(\mathrm{SU}(3), \langle\,,\,\rangle_0)$, and remains an isometry with respect to the new metric $\langle\,,\,\rangle$. Therefore the Eschenburg biquotient $E^7_{a',b'}$ defined by the action of the circle $S^1_{a',b'}$, where $a' = (a_1+c, a_2+c, a_3+c)$ and $b' = (b_1+c, b_2+c, b_3+c)$, with $c \in \mathbb{Z}$, is isometric to $E^7_{a,b}$. Furthermore, introducing an ineffective kernel to the circle action will not alter the isometry class of the biquotient. Thus $E^7_{\tilde{a},\tilde{b}}$ defined by $\tilde{a} = (ka_1, ka_2, ka_3)$ and $\tilde{b} = (kb_1, kb_2, kb_3)$ is isometric to $E^7_{a,b}$. In particular, it follows that a circle action by $S^1_{a,b} \subset \mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(3))$ can then be rewritten as the action of a circle subgroup of $\mathrm{SU}(3) \times \mathrm{SU}(3)$ via the change of parameters $(a_1, a_2, a_3, b_1, b_2, b_3) \mapsto (3a_1 - \kappa, 3a_2 - \kappa, 3a_3 - \kappa, 3b_1 - \kappa, 3b_2 - \kappa, 3b_3 - \kappa)$, where $\kappa := \sum a_i = \sum b_i$, without changing the isometry class.

From Section 1 it is clear that, for the $S^1_{a,b}$-action, permuting the a_i (via the action of the Weyl group of $\mathrm{SU}(3)$) and permuting b_2, b_3 are isometries, while permuting all of the b_i and swapping a, b are diffeomorphisms. Indeed, given the fixed choice of embedding $\mathrm{U}(2) \hookrightarrow \mathrm{SU}(3)$, cyclic permutations of the b_i (and, similarly, swapping a and b and considering cyclic permutations of the a_i) induce, in general, non-isometric metrics on the quotient $E^7_{a,b}$.

It was shown in [Es1] that an Eschenburg biquotient $E^7_{a,b} = (\mathrm{SU}(3), \langle\,,\,\rangle)/\!\!/S^1_{a,b}$ has positive curvature if and only if $b_i \notin [\underline{a}, \overline{a}]$ for all $i = 1, 2, 3$, where $\underline{a} := \min\{a_1, a_2, a_3\}$, $\overline{a} := \max\{a_1, a_2, a_3\}$.

In order to define the Bazaikin spaces (see [Baz], [Zi], [DE]), first let

$$\mathrm{Sp}(2)\cdot S^1_{q_1,\dots,q_5} = (\mathrm{Sp}(2)\times S^1_{q_1,\dots,q_5})/\mathbb{Z}_2, \quad \mathbb{Z}_2 = \{\pm(1,I)\},$$

where $q_1,\dots,q_5 \in \mathbb{Z}$ and $\mathrm{Sp}(2)$ is considered as a subgroup of $\mathrm{SU}(4)$ via the inclusion $\mathrm{Sp}(2)\hookrightarrow \mathrm{SU}(4)$ given by

$$A = S + Tj \in \mathrm{Sp}(2) \mapsto \hat{A} = \begin{pmatrix} S & T \\ -\bar{T} & \bar{S}\end{pmatrix} \in \mathrm{SU}(4), \quad S,T\in M_2(\mathbb{C}). \tag{2.2}$$

Equip $\mathrm{SU}(5)$ with a left-invariant and right $\mathrm{U}(4)$-invariant metric $\langle\,,\,\rangle$ as defined in (1.2), where $\mathrm{U}(4)\hookrightarrow \mathrm{SU}(5)$ via $A\in \mathrm{U}(4)\mapsto \mathrm{diag}(\overline{\det A}, A)\in \mathrm{SU}(5)$.

Then $\mathrm{Sp}(2)\cdot S^1_{q_1,\dots,q_5}$ acts effectively and isometrically on $(\mathrm{SU}(5),\langle\,,\,\rangle)$ via

$$[A,z]\star B = \mathrm{diag}(z^{q_1},\dots,z^{q_5})\cdot B\cdot \mathrm{diag}(\bar{z}^q, \hat{A}^{-1}),$$

with $q := \sum q_i$, $z\in S^1$, $B\in\mathrm{SU}(5)$, and $A\in \mathrm{Sp}(2)\subset \mathrm{SU}(4)$. The quotient $B^{13}_{q_1,\dots,q_5} := (\mathrm{SU}(5),\langle\,,\,\rangle)/\!\!/\,\mathrm{Sp}(2)\cdot S^1_{q_1,\dots,q_5}$ is called a Bazaikin biquotient.

It is not difficult to show that the action of $\mathrm{Sp}(2)\cdot S^1_{q_1,\dots,q_5}$ is free (hence $B^{13}_{q_1,\dots,q_5}$ is a *Bazaikin space*) if and only all $q_1,\dots,q_5$ are odd and

$$\gcd(q_{\sigma(1)}+q_{\sigma(2)}, q_{\sigma(3)}+q_{\sigma(4)}) = 2 \ \text{ for all permutations } \ \sigma\in S_5. \tag{2.3}$$

From the discussion in Section 1 it follows that permuting the q_i is an isometry of $B^{13}_{q_1,\dots,q_5}$. Furthermore, it is well-known (see [Zi], [DE]) that a general Bazaikin biquotient $B^{13}_{q_1,\dots,q_5}$ admits positive curvature if and only if $q_i+q_j>0$ (or <0) for all $1\leqslant i<j\leqslant 5$.

3. Totally geodesic submanifolds of Bazaikin spaces

Consider the Lie group $\mathrm{SU}(5)$ equipped with the left-invariant, right $\mathrm{U}(4)$-invariant metric $\langle\,,\,\rangle$ as described in Section 2. Let $\sigma : (\mathrm{SU}(5),\langle\,,\,\rangle)\to(\mathrm{SU}(5),\langle\,,\,\rangle)$ be the isometric involution defined by

$$\sigma(A) = \mathrm{diag}(1,1,1,-1,-1)\cdot A\cdot \mathrm{diag}(1,1,1,-1,-1), \quad A\in \mathrm{SU}(5). \tag{3.1}$$

The fixed point set of this involution is totally geodesic in $(\mathrm{SU}(5),\langle\,,\,\rangle)$ and given by

$$\mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(2)) = \{\mathrm{diag}(B,C)\mid B\in\mathrm{U}(3), C\in\mathrm{U}(2), \det B = \overline{\det C}\}.$$

Given the embedding (2.2) of $\mathrm{Sp}(2)$ into $\mathrm{SU}(5)$, it follows that the intersection $\mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(2))\cap\mathrm{Sp}(2)$ is $\Delta\,\mathrm{U}(2) := \{(\mathrm{diag}(1,C,\bar{C})\mid C\in\mathrm{U}(2)\}$.

Hence the image of $\mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(2))$ in the quotient $\mathrm{SU}(5)/\mathrm{Sp}(2)$ is the totally geodesic submanifold $\mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(2))/\Delta\,\mathrm{U}(2)$. Every element of $\mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(2))/\Delta\,\mathrm{U}(2)$ has a unique representative of the form $\mathrm{diag}(B,I)$, where

$$B = X\left(\begin{smallmatrix}1 & \\ & Y^t\end{smallmatrix}\right)\in\mathrm{SU}(3), \quad \text{with } X\in\mathrm{U}(3), Y\in\mathrm{U}(2) \text{ and } \det Y = \overline{\det X}. \tag{3.2}$$

Therefore $\mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(2))/\Delta\,\mathrm{U}(2)$ is a totally geodesic copy of $\mathrm{SU}(3)$ inside $\mathrm{SU}(5)/\mathrm{Sp}(2)$.

The induced metric on $\mathrm{SU}(3)\cong \mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(2))/\Delta\,\mathrm{U}(2)\subset\mathrm{SU}(5)/\mathrm{Sp}(2)$ is invariant under $\mathrm{SU}(3)\times\mathrm{U}(2)$ and, after the identification (3.2), one may consider this metric as a left-invariant, right $\mathrm{U}(2)$-invariant metric on $\mathrm{SU}(3)$ of the form described in Section

2. Indeed, the identity component of the total isometry group of $\mathrm{SU}(3) \cong \mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(2))/\Delta\,\mathrm{U}(2)$ is given by

$$\begin{aligned}\mathrm{Isom}(\mathrm{SU}(3)) &= \left\{ \left(\begin{pmatrix} Z & \\ & W \end{pmatrix}, \begin{pmatrix} w & \\ & I \end{pmatrix} \right) \mid (Z,W) \in U(3) \times U(2), w = (\det Z)(\det W) \right\} \\ &\cong \mathrm{U}(3) \times \mathrm{U}(2)\end{aligned}$$

and acts on $\mathrm{SU}(3) \cong \mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(2))/\Delta\,\mathrm{U}(2)$ via

$$\begin{aligned}\left(\begin{pmatrix} Z & \\ & W \end{pmatrix}, \begin{pmatrix} w & \\ & I \end{pmatrix} \right) \star \left[\begin{pmatrix} B & \\ & I \end{pmatrix} \right] &= \left[\begin{pmatrix} ZB \left(\begin{smallmatrix} \bar{w} & \\ & I \end{smallmatrix}\right) & \\ & W \end{pmatrix} \right] \\ &= \left[\begin{pmatrix} ZB \left(\begin{smallmatrix} \bar{w} & \\ & W^t \end{smallmatrix}\right) & \\ & I \end{pmatrix} \right] \end{aligned} \tag{3.3}$$

where $[\mathrm{diag}(B, I)] \in \mathrm{SU}(3)$, $(Z, W) \in \mathrm{U}(3) \times \mathrm{U}(2)$ and $w = (\det Z)(\det W)$.

Now, if the isometric left-action of $S^1_{q_1,\dots,q_5}$ on $\mathrm{SU}(5)/\mathrm{Sp}(2)$ is free with ineffective kernel $\{\pm 1\}$, the same must be true of the induced action on the totally geodesic submanifold $\mathrm{SU}(3) \cong \mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(2))/\Delta\,\mathrm{U}(2)$. From (3.3) it is clear that the action on $\mathrm{SU}(3)$ is given by

$$z \star B = \mathrm{diag}(z^{q_1}, z^{q_2}, z^{q_3}) B \,\mathrm{diag}(\bar{z}^q, z^{q_4}, z^{q_5}), \quad B \in \mathrm{SU}(3), z \in S^1,$$

where $q = \sum q_i$. The quotient is an Eschenburg space $\mathrm{SU}(3)/\!\!/S^1_{q_1,\dots,q_5}$ totally geodesically embedded in the Bazaikin space $B^{13}_{q_1,\dots,q_5}$. The circle action defining the Eschenburg space can be made effective. Indeed, since the q_i are odd and from the discussion of reparameterisations in Section 2, it follows that the Eschenburg space $\mathrm{SU}(3)/\!\!/S^1_{q_1,\dots,q_5}$ is isometric to the Eschenburg space $E^7_{a,b}$, where $a = (a_1, a_2, a_3) = (\frac{1}{2}(q_1 - 1), \frac{1}{2}(q_2 - 1), \frac{1}{2}(q_3 - 1))$ and $b = (b_1, b_2, b_3) = (\frac{1}{2}(q - 1), -\frac{1}{2}(q_4 + 1), -\frac{1}{2}(q_5 + 1))$.

It is now simple to recover the remaining nine totally geodesic, embedded Eschenburg spaces in $B^{13}_{q_1,\dots,q_5}$. As every permutation $\sigma \in S_5$ of the q_i is an isometry, it follows that $B^{13}_{q_1,\dots,q_5} = B^{13}_{q_{\sigma(1)},\dots,q_{\sigma(5)}}$ and hence, by the same reasoning as before, the Eschenburg space $E^7_{a_\sigma,b_\sigma}$, where $a_\sigma = (\frac{1}{2}(q_{\sigma(1)} - 1), \frac{1}{2}(q_{\sigma(2)} - 1), \frac{1}{2}(q_{\sigma(3)} - 1))$ and $b_\sigma = (\frac{1}{2}(q - 1), -\frac{1}{2}(q_{\sigma(4)} + 1), -\frac{1}{2}(q_{\sigma(5)} + 1))$, is a totally geodesic, embedded submanifold. That generically there are ten such submanifolds follows since permutations of the entries of a_σ, and of the last two entries of b_σ, are isometries. This, of course, is equivalent to fixing the order of the q_i and permuting the signs of the entries along the diagonal in the involution (3.1) acting on $\mathrm{SU}(5)$, thus achieving ten (isometric) copies of $\mathrm{SU}(3)$ which quotient to the desired Eschenburg spaces (see [DE]).

4. TOTALLY GEODESIC EMBEDDINGS OF ESCHENBURG SPACES

In Section 3 it was shown that every Bazaikin space contains a totally geodesically embedded Eschenburg space. The converse statement, namely that every Eschenburg space can be totally geodesically embedded into a Bazaikin space, is also true. Indeed, it will now be shown that every Eschenburg space can be totally geodesically embedded into infinitely many Bazaikin spaces.

Let $E^7_{a,b}$ be the Eschenburg space given by $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3) \in \mathbb{Z}^3$ with $\sum a_i = \sum b_i$ and satisfying the freeness condition (2.1):

$$\gcd(a_1 - b_{\sigma(1)}, a_2 - b_{\sigma(2)}) = 1 \text{ for all permutations } \sigma \in S_3.$$

By the discussion in Section 3, a candidate for a Bazaikin space into which to embed is given by the 5-tuple $(q_1, \dots, q_5) := (2a_1+1, 2a_2+1, 2a_3+1, -(2b_2+1), -(2b_3+1))$, with $q = \sum q_i = 2b_1 + 1$. As each of the q_i is odd, one need only check the condition (2.3):

$$\gcd(q_{\sigma(1)} + q_{\sigma(2)}, q_{\sigma(3)} + q_{\sigma(4)}) = 2 \text{ for all permutations } \sigma \in S_5.$$

It follows from the Eschenburg freeness condition (2.1) (and from $\sum a_i = \sum b_i$) that $\gcd(q_4 + q_i, q_5 + q_j) = 2$ for all $i \neq j \in \{1,2,3\}$. Furthermore, if $\{i,j,k\} = \{1,2,3\}$ and $\{\ell, m\} = \{4,5\}$, then since $q = \sum q_i$ it follows that

$$\begin{aligned} \gcd(q_i + q_j, q_k + q_\ell) &= \gcd(q_i + q_j, q - q_m) \\ &= \gcd(q - q_m, q_k + q_\ell) \\ \text{and that} \quad \gcd(q_i + q_j, q_4 + q_5) &= \gcd(q_i + q_j, q - q_k). \end{aligned}$$

Hence the 5-tuple $(q_1, \dots, q_5) = (2a_1+1, 2a_2+1, 2a_3+1, -(2b_2+1), -(2b_3+1))$ defines a Bazaikin space if and only if

$$d_{k\ell} := \gcd(a_i + a_j + 1, a_k - b_\ell) = 1 \text{ for all } \ell \in \{i,j,k\} = \{1,2,3\}. \tag{4.1}$$

Lemma 4.1. *Let $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3) \in \mathbb{Z}^3$ satisfy (2.1) and $\sum a_i = \sum b_i$. Suppose that p is a prime divisor of $\gcd(a_i + a_j + 1, a_k - b_\ell)$, where $\ell \in \{i,j,k\} = \{1,2,3\}$. Then p is odd.*

Proof. By (2.1) together with $\sum a_i = \sum b_i$ it follows that either all a_i or all b_i have the same parity. Hence either all $a_i + a_j + 1$ or all $b_m + b_n + 1$ are odd, where $1 \leqslant i < j \leqslant 3$, $1 \leqslant m < n \leqslant 3$. Since $\gcd(a_i + a_j + 1, a_k - b_\ell) = \gcd(b_m + b_n + 1, a_k - b_\ell)$, where $\{i,j,k\} = \{\ell, m, n\} = \{1,2,3\}$, the conclusion follows. □

Recall now that one can (isometrically) rewrite the Eschenburg space $E^7_{a,b}$ as $E^7_{a_c,b_c}$, where $a_c = (a_1 + c, a_2 + c, a_3 + c)$ and $b_c = (b_1 + c, b_2 + c, b_3 + c)$, for any $c \in \mathbb{Z}$. In this case the candidate Bazaikin biquotient is given by the 5-tuple $(q_1^c, \dots, q_5^c) := (2(a_1+c)+1, 2(a_2+c)+1, 2(a_3+c)+1, -(2(b_2+c)+1), -(2(b_3+c)+1))$, with $q^c = \sum q_i^c = 2(b_1+c)+1$. Analogously to above, this Bazaikin biquotient is non-singular if and only if

$$d^c_{k\ell} := \gcd(a_i + a_j + 1 + 2c, a_k - b_\ell) = 1 \text{ for all } \ell \in \{i,j,k\} = \{1,2,3\}. \tag{4.2}$$

Lemma 4.2. *Let $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3) \in \mathbb{Z}^3$ satisfy (2.1) and $\sum a_i = \sum b_i$. For all $\ell \in \{i,j,k\} = \{1,2,3\}$, let $p_{k\ell 1}, \dots, p_{k\ell r_{k\ell}}$ be the prime divisors, if any, of $a_k - b_\ell$ which satisfy $\gcd(p_{k\ell\tau}, a_i + a_j + 1) = 1$, for all $1 \leqslant \tau \leqslant r_{k\ell}$. Set*

$$c = c_\mu := \pm 2^{\mu-1} \left(\prod_{k,\ell=1}^{3} \prod_{\tau=1}^{r_{k\ell}} p_{k\ell\tau} \right)^{\mu}, \tag{4.3}$$

where μ is an arbitrary positive integer. Then $d^c_{k\ell} = \gcd(a_i + a_j + 1 + 2c, a_k - b_\ell) = 1$.

Proof. Suppose that p is a prime divisor of $d^c_{k\ell}$. As p divides $a_k - b_\ell$, then p divides either $a_i + a_j + 1$ or c. But p divides $a_i + a_j + 1 + 2c$, hence must divide both $a_i + a_j + 1$ and c, and furthermore must be odd by Lemma 4.1. Now, by definition of c, p must divide $a_t - b_u$, for some $(t,u) \neq (k,\ell)$, and must satisfy $\gcd(p, a_r + a_s + 1) = 1$, where $\{r,s,t\} = \{1,2,3\}$. However, because of the freeness condition (2.1) for Eschenburg spaces, either $t = k$ or $u = \ell$. If $t = k$, then $1 = \gcd(p, a_r + a_s + 1) = \gcd(p, a_i + a_j + 1) = p$. On the other hand, if $u = \ell$ then $k \neq t \in \{1,2,3\}$, that is, it may be assumed without loss of generality that $(t,u) = (i,\ell)$. As p divides $a_i + a_j + 1$, $a_k - b_\ell$ and $a_i - b_\ell$, it follows that

$1 = \gcd(p, a_r + a_s + 1) = \gcd(p, a_j + a_k + 1) = \gcd(p, (a_i + a_j + 1) + (a_k - b_\ell) - (a_i - b_\ell)) = p$. □

Corollary 4.3. *With notation as above, for all $c = c_\mu$, $\mu \in \mathbb{Z}$, $\mu > 0$, the Bazaikin biquotient $B^{13}_{q_1^c,\ldots,q_5^c}$ is non-singular and contains $E^7_{a,b} = E^7_{a_c,b_c}$ as a totally geodesic submanifold.*

Remark 4.4. The conditions (4.1) and (2.1) ensuring that the Bazaikin biquotient is non-singular are precisely the conditions which ensure that all ten of the totally geodesic, embedded Eschenburg biquotients are themselves non-singular. The equivalence of the non-singularity of the Bazaikin biquotient and these ten submanifolds was already observed in [DE].

Remark 4.5. Given a Bazaikin space $B^{13}_{q_1,\ldots,q_5}$, with $(q_1, \ldots, q_5) = (2a_1 + 1, 2a_2 + 1, 2a_3 + 1, -(2b_2 + 1), -(2b_3 + 1))$ and $q = \sum q_i = 2b_1 + 1$, into which the Eschenburg space $E^7_{a,b}$ has a totally geodesic embedding, that is, condition (4.1) holds, then the diffeomorphic, but in general non-isometric, Eschenburg space $E^7_{b,a}$ totally geodesically embeds into the Bazaikin space $B^{13}_{q,-q_4,-q_5,-q_2,-q_3}$. This follows easily from condition (4.1) together with $\sum a_i = \sum b_i$. The Bazaikin spaces $B^{13}_{q_1,\ldots,q_5}$ and $B^{13}_{q,-q_4,-q_5,-q_2,-q_3}$ are diffeomorphic but not isometric in general (see [EKS], [FZ]).

The integral cohomology rings of Eschenburg spaces and Bazaikin spaces are well-known (see [Es2], [CEZ], [Baz], [FZ]). In particular, if $\sigma_i(x) := \sigma_i(x_1, \ldots, x_m)$ denotes the i^{th} symmetric polynomial in $x = (x_1, \ldots, x_m)$, then $H^4(E^7_{a,b}) = \mathbb{Z}_r$, where $r = |\sigma_2(a) - \sigma_2(b)|$ is always odd [Kr, Remark 1.3], and $H^6(B^{13}_{q_1,\ldots,q_5}) = \mathbb{Z}_s$, where $s = \frac{1}{8}|\sigma_3(q_1, q_2, q_3, q_4, q_5, -\sum q_i)|$.

Lemma 4.6. *Let $a := (a_1, a_2, a_3), b := (b_1, b_2, b_3) \in \mathbb{Z}^3$, with $\sum a_i = \sum b_i$, and for $c \in \mathbb{Z}$ define the 6-tuple*

$$q_c := (2(a_1+c)+1, 2(a_2+c)+1, 2(a_3+c)+1, -2(b_1+c)-1, -2(b_2+c)-1, -2(b_3+c)-1).$$

If $c \neq d \in \mathbb{Z}$, then $|\sigma_3(q_c)| = |\sigma_3(q_d)|$ if and only if either $\sigma_2(a) = \sigma_2(b)$ or

$$c + d = \frac{\sigma_3(a) - \sigma_3(b)}{\sigma_2(a) - \sigma_2(b)} - \sigma_1(a) - 1.$$

Proof. Recall that $\sigma_i(x)$, the i^{th} symmetric polynomial in $x = (x_1, \ldots, x_m)$, is defined as the coefficient of y^{m-i} in the product $\prod_{j=1}^m (y + x_j)$. Then, with $y_+ = y + 1 + 2c$ and $y_- = y - 1 - 2c$, $\sigma_3(q_c)$ is given by the coefficient of y^3 in the product

$$\prod_{j=1}^3 (y + 2a_j + 1 + 2c) \prod_{j=1}^3 (y - 2b_j - 1 - 2c) = \prod_{j=1}^3 (y_+ + 2a_j) \prod_{j=1}^3 (y_- - 2b_j) = \left(\sum_{j=0}^3 \sigma_j(a) y_+^{3-j}\right) \left(\sum_{j=0}^3 \sigma_j(b) y_-^{3-j}\right).$$

Hence, since $\sigma_1(a) = \sum a_j = \sum b_j = \sigma_1(b)$,

$$\sigma_3(q_c) = 8(\sigma_3(a) - \sigma_3(b)) - 8(\sigma_1(a) + 2c + 1)(\sigma_2(a) - \sigma_2(b)).$$

Now $|\sigma_3(q_c)| = |\sigma_3(q_d)|$ if and only if $\sigma_3(q_c) = \pm\sigma_3(q_d)$, that is, if and only if either

$$(c - d)(\sigma_2(a) - \sigma_2(b)) = 0 \quad \text{or}$$

$$(c + d)(\sigma_2(a) - \sigma_2(b)) = (\sigma_3(a) - \sigma_3(b)) - (\sigma_1(a) + 1)(\sigma_2(a) - \sigma_2(b)),$$

from which the claim follows. □

Corollary 4.7. *The collection of Bazaikin spaces into which a particular Eschenburg space can be totally geodesically embedded consists of infinitely many distinct homotopy types.*

Proof. Given an Eschenburg space $E^7_{a,b}$ and c of the form (4.3), it has been shown that there is a totally geodesic embedding of $E^7_{a,b}$ into each Bazaikin space $B^{13}_{q_1^c,\dots,q_5^c}$, where $(q_1^c,\dots,q_5^c) = (2a_1+2c+1, 2a_2+2c+1, 2a_3+2c+1, -2b_2-2c-1, -2b_3-2c-1)$. Fix one such value of c, that is, fix some $\mu > 0$. Since $2^{\mu-1} > 1$ for $\mu > 1$, we may assume without loss of generality that $|c| > 1$. Let $c_1 = c^{\alpha_1}$ and $c_2 = c^{\alpha_2}$ for some $0 < \alpha_1 < \alpha_2$. With the same notation as in Lemma 4.6, the order of H^6 for the Bazaikin space corresponding to c_i is given by $s_i = \frac{1}{8}|\sigma_3(q_{c_i})|$, for $i = 1, 2$ respectively. By Lemma 4.6, $s_1 = s_2$ if and only if either $\sigma_2(a) - \sigma_2(b) = 0$ or

$$c_1 + c_2 = \frac{\sigma_3(a) - \sigma_3(b)}{\sigma_2(a) - \sigma_2(b)} - \sigma_1(a) - 1. \tag{4.4}$$

But, by [Kr, Remark 1.3], $\sigma_2(a) - \sigma_2(b)$ is odd, hence non-zero. Therefore (4.4) must hold. However, as the right-hand side is constant, it is clear that (4.4) can hold for at most one pair $0 < \alpha_1 < \alpha_2$. □

The following example shows that the c values given in Lemma 4.2 do not achieve all Bazaikin spaces into which an Eschenburg space can be totally geodesically embedded.

Example 4.8. Consider the positively curved Eschenburg space $E^7_{a,b}$ given by $a = (2, 0, 0)$ and $b = (15, -2, -11)$. The corresponding candidate for a Bazaikin space is given by $q = (5, 1, 1, 3, 21)$, which is singular because $\gcd(q_1 + q_2, q_4 + q_5) = 6$.

By Corollary 4.3, $E^7_{a,b}$ can be totally geodesically embedded into each of the Bazaikin spaces given by $(5 + 2c, 1 + 2c, 1 + 2c, 3 - 2c, 21 - 2c)$, with $c = \pm 2^{\mu-1}(2^3 \cdot 5^2 \cdot 11^2 \cdot 13^2)^\mu$, for $\mu > 0$. None of these spaces is positively curved, since it is clear that any value of $\mu > 0$ leads to $q_i + q_j$ of mixed signs.

On the other hand, note that when $c = -1$ the Eschenburg space $E^7_{a,b}$ can be rewritten as $E^7_{\tilde a,\tilde b}$, with $\tilde a = (1, -1, -1)$ and $\tilde b = (14, -3, -12)$, and the corresponding Bazaikin biquotient $B^{13}_{\tilde q}$ with $\tilde q = (3, -1, -1, 5, 23)$ is non-singular. Note that $B^{13}_{\tilde q}$ does not have positive curvature.

However, when $c = 2$ and when $c = 5$, the resulting biquotients are both non-singular and positively curved, namely the Bazaikin spaces $B^{13}_{q'}$, with $q' = (9, 5, 5, -1, 17)$ (corresponding to $a' = (4, 2, 2)$, $b' = (17, 0, -9)$), and $B^{13}_{q''}$, with $q'' = (15, 11, 11, -7, 11)$ (corresponding to $a' = (7, 5, 5)$, $b' = (20, 3, -6)$).

Finally, notice that the order of H^6 is 503, 1541 and 2579 for $B^{13}_{\tilde q}$, $B^{13}_{q'}$ and $B^{13}_{q''}$ respectively, hence these spaces are not even homotopy equivalent.

5. Embeddings inducing positive sectional curvature

Given that every Eschenburg space can be totally geodesically embedded into infinitely many Bazaikin spaces, it is natural to ask whether a positively curved Eschenburg space admits a totally geodesic embedding into a positively curved Bazaikin space.

Recall that an Eschenburg space $E^7_{a,b}$ has positive curvature if and only if $b_i \notin [\underline{a}, \overline{a}]$ for all $i = 1, 2, 3$, where $\underline{a} := \min\{a_1, a_2, a_3\}$, $\overline{a} := \max\{a_1, a_2, a_3\}$. Because $\sum a_j = \sum b_j$, this is equivalent to the requirement that two of the b_i lie on one side of $[\underline{a}, \overline{a}]$, and one on the other. Indeed, given the metric on $E^7_{a,b}$ defined in Section 2, it turns out that b_2 and b_3 must lie on the same side of $[\underline{a}, \overline{a}]$, see [Es1], [Ke]. Since permuting the a_i

and permuting b_2 and b_3 are isometries, the condition for positive curvature is, after relabelling if necessary, equivalent to

$$b_3 \leqslant b_2 < a_3 \leqslant a_2 \leqslant a_1 < b_1 \quad \text{or} \quad b_1 < a_1 \leqslant a_2 \leqslant a_3 < b_2 \leqslant b_3. \tag{5.1}$$

In fact, the second chain of inqualities is equivalent to the first via the reparametrization of $S^1_{a,b}$ via $z \mapsto \bar{z}$, and therefore one may restrict attention to the first chain.

Now $E^7_{a,b}$ can be totally geodesically embedded into the (possibly singular) Bazaikin biquotient $B^{13}_{q_1,\dots,q_5}$, with $(q_1,\dots,q_5) = (2a_1+1, 2a_2+1, 2a_3+1, -(2b_2+1), -(2b_3+1))$. As discussed in Section 2, $B^{13}_{q_1,\dots,q_5}$ has positive curvature if and only if $q_i + q_j > 0$ (or < 0) for all $1 \leqslant i < j \leqslant 5$. Since the first chain of inequalities in (5.1) implies that $q_i + q_j = 2(a_i - b_{j-2}) > 0$, for all $i = 1, 2, 3$, $j = 4, 5$, it follows that $B^{13}_{q_1,\dots,q_5}$ has positive curvature if $a_i + a_j + 1 > 0$, for all $1 \leqslant i < j \leqslant 3$, and $b_2 + b_3 + 1 < 0$. But $a_3 \leqslant a_2 \leqslant a_1$, hence $B^{13}_{q_1,\dots,q_5}$ has positive curvature if and only if

$$b_2 + b_3 + 1 < 0 < a_2 + a_3 + 1. \tag{5.2}$$

In particular, it is necessary that $0 \leqslant a_2 \leqslant a_1 < b_1$ and $b_3 \leqslant -1$. Clearly the special case $b_2 < 0 \leqslant a_3$ ensures that the inequalities in (5.2) are satisfied.

If $B^{13}_{q_1,\dots,q_5}$ does not have positive curvature, then one can find a $c \in \mathbb{Z}$ such that the Bazaikin biquotient $B^{13}_{q_1^c,\dots,q_5^c}$, with

$$(q_1^c,\dots,q_5^c) := (2(a_1+c)+1, 2(a_2+c)+1, 2(a_3+c)+1, -(2(b_2+c)+1), -(2(b_3+c)+1)),$$

is positively curved. Indeed, $B^{13}_{q_1^c,\dots,q_5^c}$ has positive curvature if and only if $c \in Z$ satisfies

$$-\frac{1}{2}(a_2 + a_3 + 1) < c < -\frac{1}{2}(b_2 + b_3 + 1). \tag{5.3}$$

It then remains only to examine the values of c given by (5.3) to determine which, if any, of the Bazaikin biquotients $B^{13}_{q_1^c,\dots,q_5^c}$ are non-singular.

One particular example, where an embedding into a positively curved Bazaikin space exists, is that of a positively curved Eschenburg space of cohomogeneity-one, given by $a = (p, 1, 1)$, $b = (p+2, 0, 0)$, with $p \geqslant 1$. Here (5.3) yields $-1 \leqslant c \leqslant 0$. Indeed, $c = -1$ ensures that $B^{13}_{q_1^c,\dots,q_5^c}$ is non-singular and positively curved. This was first observed by W. Ziller [Zi] and appeared in [DE].

Note that in this example $c = -a_3 = -1$. Indeed, for an arbitrary Eschenburg space $E^7_{a,b}$, the choice $c = -a_3$ ensures $B^{13}_{q_1^c,\dots,q_5^c}$ has positive curvature, although in general one cannot hope that this space will be non-singular. Example 4.8 illustrates this phenomenon and, furthermore, that the values of c suggested by the expression (4.3) are not of much use when it comes to finding a Bazaikin space of positive curvature.

It turns out, in fact, that there are examples of positively curved Eschenburg spaces for which none of the values of c coming from (5.3) yield a non-singular Bazaikin biquotient, that is, a totally geodesic embedding constructed as in Section 4 cannot be into a positively curved Bazaikin space. A list of such examples is given in Table 1. Notice, in particular, that the first and last examples in Table 1 are Eschenburg spaces of cohomogeneity-two (see [GSZ]). In fact, there are infinitely many such cohomogeneity-two examples, for example, the infinite families given by $a = (15015k+39, 0, 0)$, $b = (15015k+55, -3, -13)$, $k \geqslant 0$, and $a = (15015k+12909, 0, 0)$, $b = (15015k+12925, -3, -13)$, $k \geqslant 0$, respectively.

Remark 5.1. One might hope that introducing an ineffective kernel into the $S^1_{a,b}$ action defining a positively curved Eschenburg space $E^7_{a,b}$ would yield a Bazaikin space with positive curvature into which it can be totally geodesically embedded. However, such a modification reduces to the case discussed above. Indeed, if $\tilde{a} = (\lambda a_1 + d, \lambda a_2 + d, \lambda a_3 + d)$ and $\tilde{b} = (\lambda b_1 + d, \lambda b_2 + d, \lambda b_3 + d)$, where without loss of generality $\lambda > 0$, then, by (2.1), $\gcd(\tilde{a}_i - \tilde{b}_\ell, \tilde{a}_j - \tilde{b}_m) = \lambda$. Hence the candidate for a positively curved Bazaikin space $B^{13}_{\tilde{q}}$ is given by $\tilde{q} = (2\tilde{a}_1 + 1, 2\tilde{a}_2 + 1, 2\tilde{a}_3 + 1, -(2\tilde{b}_2 + 1), -(2\tilde{b}_3 + 1))$, where $\gcd(\tilde{a}_i + \tilde{a}_j + 1, \tilde{a}_k - \tilde{b}_\ell) = \lambda$, for all $\ell \in \{i, j, k\} = \{1, 2, 3\}$, and $\tilde{a}_2 + \tilde{a}_3 + 1 > 0 > \tilde{b}_2 + \tilde{b}_3 + 1$. Consequently λ must be odd and there is $c \in \mathbb{Z}$ such that $2d + 1 = \lambda(2c + 1)$, in which case $B^{13}_{\tilde{q}}$ is non-singular if and only if (2.1) and (4.2) hold, and has positive curvature if and only if (5.3) is satisfied.

$\mathbf{E^7_{a,b}}$		$\mathbf{B^{13}_{q^c_1,\ldots,q^c_5}}$	
$\mathbf{a}$	$\mathbf{b}$	$(\mathbf{q^c_1}, \ldots, \mathbf{q^c_5})$	$\sec > \mathbf{0}$
$(39, 0, 0)$	$(55, -3, -13)$	$(79 + 2c, 1 + 2c, 1 + 2c, 5 - 2c, 25 - 2c)$	$0 \leqslant c \leqslant 7$
$(77, 2, 0)$	$(93, -3, -11)$	$(155 + 2c, 5 + 2c, 1 + 2c, 5 - 2c, 21 - 2c)$	$-1 \leqslant c \leqslant 6$
$(171, 2, 0)$	$(187, -3, -11)$	$(343 + 2c, 5 + 2c, 1 + 2c, 5 - 2c, 21 - 2c)$	$-1 \leqslant c \leqslant 6$
$(225, 4, 0)$	$(247, -5, -13)$	$(451 + 2c, 9 + 2c, 1 + 2c, 9 - 2c, 25 - 2c)$	$-2 \leqslant c \leqslant 8$
$(281, 3, 0)$	$(294, -2, -8)$	$(563 + 2c, 7 + 2c, 1 + 2c, 3 - 2c, 15 - 2c)$	$-1 \leqslant c \leqslant 4$
$(309, 6, 0)$	$(323, -3, -5)$	$(619 + 2c, 13 + 2c, 1 + 2c, 5 - 2c, 9 - 2c)$	$-3 \leqslant c \leqslant 3$
$(664, 2, 0)$	$(678, -3, -9)$	$(1329 + 2c, 5 + 2c, 1 + 2c, 5 - 2c, 17 - 2c)$	$-1 \leqslant c \leqslant 5$
$(827, 4, 0)$	$(843, -3, -9)$	$(1655 + 2c, 9 + 2c, 1 + 2c, 5 - 2c, 17 - 2c)$	$-2 \leqslant c \leqslant 5$
$(12909, 0, 0)$	$(12925, -3, -13)$	$(25819 + 2c, 1 + 2c, 1 + 2c, 5 - 2c, 25 - 2c)$	$0 \leqslant c \leqslant 7$

TABLE 1. Positively curved Eschenburg spaces which do not totally geodesically embed into a non-singular Bazaikin biquotient with positively curvature ($\sec > 0$) via the construction of Section 4.

REFERENCES

[AW] S. Aloff and N. Wallach, *An infinite family of distinct 7-manifolds admitting positively curved Riemannian structures*, Bull. Amer. Math. Soc. **81** (1975), 93–97

[Baz] Y. V. Bazaĭkin, *On a family of 13-dimensional closed Riemannian manifolds of positive curvature*, Sibirsk. Mat. Zh. **37** (1996), 1219–1237; translation in Siberian Math. J. **37** (1996)

[CEZ] T. Chinburg, C. Escher and W. Ziller, *Topological properties of Eschenburg spaces and 3-Sasakian manifolds*, Math. Ann. **339** (2007), 3–20

[DE] O. Dearricott and J.-H. Eschenburg, *Totally geodesic embeddings of 7-manifolds in positively curved 13-manifolds*, Manuscripta Math. **114** (2004), 447–456

[Es1] J.-H. Eschenburg, *Freie isometrische Aktionen auf kompakten Liegruppen mit positiv gekrümmten Orbiträumen*, Schriftenreihe Math. Inst. Univ. Münster (2) 32 (1984)

[Es2] J.-H. Eschenburg, *Cohomology of biquotients*, Manuscripta Math. **75** (1992), 151–166

[EKS] J.-H. Eschenburg, A. Kollross and K. Shankar, *Free isometric circle actions on compact symmetric spaces*, Geom. Dedicata **103** (2003), 35–44

[FZ] L. Florit and W. Ziller, *On the topology of positively curved Bazaikin spaces*, J. Europ. Math. Soc. **11** (2009), 189–205

[GSZ] K. Grove, K. Shankar and W. Ziller, *Symmetries of Eschenburg spaces and the Chern problem*, Asian J. Math. **10** (2006), 647–662

[Ke] M. Kerin, *On the curvature of biquotients*, Math. Ann. **352** (2012), 155–178

[Kr] B. Kruggel, *Kreck-Stolz invariants, normal invariants and the homotopy classification of generalized Wallach spaces*, Quart. J. Math. Oxford Ser. (2) **49** (1998), 469–485

[Ta] I. A. Taĭmanov, *On totally geodesic embeddings of* 7*-dimensional manifolds in* 13*-dimensional manifolds of positive sectional curvature*, Mat. Sb. **187** (1996),no. 12, 121–136; translation in Sb. Math. **187** (1996), no. 12, 1853–1867

[Zi] W. Ziller, *Homogeneous spaces, biquotients and manifolds with positive curvature*, Lecture Notes (1998), U. Penn., unpublished

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