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Parameterised temporal exploration problems ☆

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ABSTRACT

We study the fixed-parameter tractability of the problem of deciding whether a given temporal graph admits a temporal walk that visits all vertices (temporal exploration) or, in some variants, a certain subset of the vertices. In the strict variant, edges must be traversed in strictly increasing timesteps; in the non-strict variant, any number of edges can be traversed in each timestep. For both variants, we give FPT algorithms for finding a temporal walk that visits a given set X of vertices, parameterised by $|X|$, and for finding a temporal walk that visits at least k distinct vertices, parameterised by k. We also show W[2]-hardness for a set version of temporal exploration. For the non-strict variant, we give an FPT algorithm for temporal exploration parameterised by the lifetime, and show that temporal exploration can be solved in polynomial time if the graph in each timestep has at most two connected components.

1. Introduction

The problem of computing a series of consecutive edge-traversals in a static (i.e., classical discrete) graph G, such that each vertex of G is an endpoint of at least one traversed edge, is a fundamental problem in algorithmic graph theory, and an early formulation was provided by Shannon [1]. Such a sequence of edge-traversals might be referred to as an *exploration* or *search* of G and, from a computational standpoint, it is easy to check whether a given graph G admits such an exploration and easy to compute one if the answer is yes – we simply carry out a depth-first search starting at an arbitrary start vertex in $V(G)$ and check whether every vertex of G is reached. We consider in this paper a decidedly more complex variant of the problem, in which we try to find an exploration of a *temporal graph*. A temporal graph $\mathcal{G} = \langle G_1, \ldots, G_L \rangle$ is a sequence of static graphs G_t such that $V(G_t) = V(G)$ and $E(G_t) \subseteq E(G)$ for any *timestep* $t \in [L]$ and some fixed *underlying graph* G.

A concerted effort to tackle algorithmic problems defined for temporal graphs has been made in recent years. With the addition of time to a graph's structure comes more freedom when defining a problem. Hence, many studies have focused on temporal variants of classical graph problems: for example, the travelling salesperson problem [2]; shortest paths [3]; vertex cover [4]; maximum matching [5]; network flow problems [6]; and a number of others. For more examples, we point the reader to the works of Molter [7] or Michail [2]. One seemingly common trait of the problems that many of these studies consider is the following: *Problems that are easy for static graphs often become hard on temporal graphs, and hard problems for static*

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Table 1
Overview of results. The parameters are: L = lifetime, γ = maximum number of connected components per step, k = number of vertices to be visited.

Problem	Parameter	strict	non-strict
TEXP	L	FPT Corollary 14	FPT Theorem 34
TEXP	γ	NPC for $\gamma = 1$ Observation 12	poly for $\gamma = 1, 2$ Theorem 28
k-FIXED TEXP	k	FPT Theorem 13	FPT Corollary 21
k-ARBITRARY TEXP	k	FPT Theorems 15, 17	FPT Corollary 22
SET-TEXP	L	W[2]-hard Theorem 19	W[2]-hard Theorem 37

graphs remain hard on temporal graphs. This certainly holds true for the problem of deciding whether a given temporal graph $\mathcal{G}$ admits a *temporal walk* W – roughly speaking, a sequence of edges traversed consecutively and during strictly increasing timesteps – such that every vertex of $\mathcal{G}$ is an endpoint of at least one edge of W (any temporal walk with this property is known as an *exploration schedule*). Indeed, Michail and Spirakis [8] showed that this problem, TEMPORAL EXPLORATION or TEXP for short, is NP-complete. In this paper, we consider variants of the TEXP problem from a fixed-parameter perspective and under both *strict* and *non-strict* settings. More specifically, we consider problem variants in which we look for *strict* temporal walks, which traverse each consecutive edge at a timestep strictly larger than the previous, as well as variants that ask for *non-strict* temporal walks, which allow an unlimited but finite number of edges to be traversed in each timestep.

1.1. Contribution

An overview of our results is shown in Table 1. After presenting preliminaries and problem definitions in Section 2, we show in Section 3 for the strict setting that two natural parameterised variants of TEXP are in FPT. Firstly, we parameterise by the size k of a fixed subset of the vertex set and ask for an exploration schedule that visits at least these vertices, providing an $O(2^k kLn^2)$-time algorithm. Secondly, we parameterise by only an integer k and ask that a computed solution visits at least k arbitrary vertices – in this case we specify, for any $\varepsilon > 0$, a randomised algorithm (based on the colour-coding technique first introduced by Alon et al. [9]) with running time $O((2e)^k Ln^3 \log \frac{1}{\varepsilon})$. A now-standard derandomisation technique [9,10] is then utilised in order to obtain a deterministic $(2e)^k k^{O(\log k)} Ln^3 \log n$-time algorithm. Furthermore, we show that a generalised variant, SET TEXP, in which we are supplied with m subsets of the input temporal graph's vertex set and are asked to decide whether there exists a strict temporal walk that visits at least one vertex belonging to each set, is W[2]-hard.

In Section 4, we consider the non-strict variant known as NON-STRICT TEMPORAL EXPLORATION, or NS-TEXP, which was introduced in [11]. Here, a candidate exploration schedule is permitted to traverse an unlimited but finite number of edges during each timestep, and it is not too hard to see that this change alters the problem's structure quite drastically (more details in Sections 2.2 and 4). We therefore use a different model of temporal graphs to the one considered in Section 3, which we properly define later. In this model, an exploration schedule may exist even if the lifetime L is much smaller than the number n of vertices. Nevertheless, we show that NS-TEXP parameterised by L is FPT by giving an $O(L(L!)^2 n)$-time recursive search-tree algorithm. Furthermore, we show that the FPT algorithms for visiting k fixed vertices or k arbitrary vertices, where k is taken as the parameter, can be adapted from the strict to the non-strict case, while saving a factor of n in the running-time. For the case that the maximum number of components in each step is bounded by 2, we show that all four non-strict problem variants can be solved in polynomial time. For the non-strict variant of SET TEXP, we show W[2]-hardness.

1.2. Related work

We refer the interested reader to Casteigts et al. [12] for a study of various models of dynamic graphs, and to Michail [2] for an introduction to temporal graphs and some of their associated combinatorial problems. Brodén et al. [13] considered the TEMPORAL TRAVELLING SALESPERSON PROBLEM for complete temporal graphs with n vertices. The costs of edges are allowed to differ between 1 and 2 in each timestep. They showed that when an edge's cost changes at most k times during the input graph's lifetime, the problem is NP-complete, but provided a $(2 - \frac{2}{3k})$-approximation. For the same problem, Michail and Spirakis [8] proved APX-hardness and provided a $(1.7 + \epsilon)$-approximation. Bui-Xuan et al. [14] proposed multiple objectives for optimisation when computing temporal walks/paths: e.g., *fastest* (fewest number of timesteps used) and *foremost* (arriving at the destination at the earliest time possible).

Michail and Spirakis [8] introduced the TEXP problem, which asks whether or not a given temporal graph admits a temporal walk that visits all vertices at least once. The problem was shown to be NP-complete when no restrictions are placed on the input, and they proposed considering the problem under the *always-connected* assumption as a means of ensuring that exploration is possible (provided the lifetime of the input graph is sufficiently long). Erlebach et al. [15] considered the problem of computing foremost exploration schedules under the always-connected assumption, proving $O(n^{1-\varepsilon})$-inapproximability (for any $\varepsilon > 0$). They also showed that subquadratic exploration schedules exist for temporal graphs whose underlying graph is planar, has bounded treewidth, or is a $2 \times n$ grid. Furthermore, they proved that cycles with at most one chord can be explored in $O(n)$ steps. For always-connected cycles, it had already been shown earlier by Ilcinkas and Wade [16] that $O(n)$ steps always suffice. Bodlaender and van der Zanden [17] examined the TEXP problem when restricted to always-connected temporal graphs whose underlying graph has pathwidth at most 2, showing the problem to be NP-complete in this case.

Later, Erlebach et al. [18] showed that temporal graphs can be explored in $O(n^{1.75})$ steps if the graph in each step admits a spanning-tree of bounded degree or if one is allowed to traverse two edges per step. Taghian Alamouti [19] showed that a cycle with k chords can be explored in $O(k^2 \cdot k! \cdot (2e)^k \cdot n)$ timesteps. Adamson et al. [20] improved this bound for cycles with k chords to $O(kn)$ timesteps. They also improved the bounds on the worst-case exploration time for temporal graphs whose underlying graph is planar or has bounded treewidth.

Akrida et al. [21] considered a TEXP variant called Return-To-Base TEXP, in which the underlying graph is a star and a candidate solution must return to the vertex from which it initially departed (the star's centre). They proved various hardness results and provided polynomial-time algorithms for some special cases. Casteigts et al. [22] studied the fixed-parameter tractability of the problem of finding temporal paths between a source and destination that wait no longer than Δ consecutive timesteps at any intermediate vertex. Bumpus and Meeks [23] considered, again from a fixed-parameter perspective, a temporal graph exploration variant in which the goal is no longer to visit all of the input graph's vertices at least once, but to traverse all edges of its underlying graph exactly once (i.e., computing a temporal Eulerian circuit). They also resolved the complexity of the two cases of the Return-To-Base TEXP problem that had been left open by [21].

The problem of Non-Strict Temporal Exploration was introduced and studied in [11]. Here, a computed walk may make an unlimited number of edge-traversals in each given timestep. Amongst other things, NP-completeness of the general problem was shown, as well as $O(n^{1/2-\varepsilon})$ and $O(n^{1-\varepsilon})$-inapproximability for the problem of minimising the arrival time of a temporal exploration in the cases where the number of timesteps required to reach any vertex v from any vertex u is bounded by $c = 2$ and $c = 3$, respectively. Notions of strict/non-strict paths which respectively allow for a single edge/unlimited number of edge(s) to be crossed in any timestep have been considered before, notably by Kempe et al. [24] and Zschoche et al. [25].

2. Preliminaries

For a pair of integers x, y with $x \leq y$ we denote by $[x, y]$ the set $\{z : x \leq z \leq y\}$; if $x = 1$ we write $[y]$ instead. We use standard terminology from graph theory [26], and we assume any static graph $G = (V, E)$ to be simple and undirected. A parameterised problem is a language $L \subseteq \Sigma^* \times \mathbb{N}$, where Σ is a finite alphabet. For an instance $(I, k) \in \Sigma^* \times \mathbb{N}$, k is called the parameter. The problem is in FPT (fixed-parameter tractable) if there is an algorithm that solves every instance in time $f(k) \times |I|^{O(1)}$ for some computable function f. A proof that a problem is hard for complexity class W[r] for some integer $r \geq 1$ is seen as evidence that the problem is unlikely to be contained in FPT. For more on parameterised complexity, including definitions of the complexity classes W[r], we refer to [27,28].

2.1. Temporal exploration with strict temporal walks

The relevant concepts and problem definitions for strict temporal walks are as follows. We begin with the definition of a temporal graph:

Definition 1 *(Temporal graph).* A temporal graph $\mathcal{G}$ with *underlying graph* $G = (V, E)$, lifetime L and order n is a sequence of simple undirected graphs $\mathcal{G} = \langle G_1, G_2, \ldots, G_L \rangle$ such that $|V| = n$ and $G_t = (V, E_t)$ (where $E_t \subseteq E$) for all $t \in [L]$.

For a temporal graph $\mathcal{G} = \langle G_1, \ldots, G_L \rangle$, the subscripts $t \in [L]$ indexing the graphs in the sequence are referred to as *timesteps* (or *steps*) and we call G_t the t-th *layer*. A tuple (e, t) with $e \in E(G)$ is an *edge-time pair* (or *time edge*) of $\mathcal{G}$ if $e \in E_t$. Note that the size of any temporal graph (i.e., the maximum number of time edges) is bounded by $O(Ln^2)$.

Definition 2 *(Strict temporal walk).* A strict temporal walk W in $\mathcal{G}$ is a tuple $W = (t_0, S)$, consisting of a start time t_0 and an alternating sequence of vertices and edge-time pairs $S = \langle v_1, (e_1, t_1), v_2, (e_2, t_2), \ldots, v_{l-1}, (e_{l-1}, t_{l-1}), v_l \rangle$ such that $e_i = \{v_i, v_{i+1}\}$, $e_i \in G_{t_i}$ for $i \in [l-1]$ and $1 \leq t_0 \leq t_1 < t_2 < \cdots < t_{l-1} \leq L$.

We say that a strict temporal walk $W = (t_0, S)$ *visits* any vertex that is included in S. Further, W *traverses* edge e_i at time t_i for all $i \in [l-1]$ and is said to *depart from* (or start at) $v_1 \in V(\mathcal{G})$ at timestep t_0 and *arrive at* (or finish at) $v_l \in V(\mathcal{G})$ at the end of timestep t_{l-1} (or, equivalently, at the beginning of timestep $t_{l-1} + 1$). Its *arrival time* is defined to be $t_{l-1} + 1$. It is assumed that W is positioned at v_1 at the start of timestep $t_0 \in [t_1]$ and waits at v_1 until edge e_1 is traversed during timestep t_1. The quantity $|W| = t_{l-1} - t_0 + 1$ is called the *duration* of W. Observe that the arrival time of a strict temporal walk equals its start time plus its duration. We remark that a walk with arrival time t that finishes at a vertex v and a walk with start time t (or later) that departs from v can be combined into a single walk in the obvious way.

We denote by $sp(u, v, t)$ the duration of a shortest (i.e., having minimum arrival time) temporal walk in $\mathcal{G}$ that starts at $u \in V(\mathcal{G})$ in timestep t and ends at $v \in V(\mathcal{G})$. If $u = v$, $sp(u, v, t) = 0$. We note that there is no guarantee that a walk between a pair of vertices u, v exists; in such cases we let $sp(u, v, t) = \infty$. The algorithms that we present in Section 3 will repeatedly require us to compute such shortest walks for specific pairs of vertices $u, v \in V(\mathcal{G})$ and a timestep $t \in [L]$ – the following theorem allows us to do this:

Theorem 3 *(Wu et al. [3]). Let $\mathcal{G} = \langle G_1, \ldots, G_L \rangle$ be an arbitrary temporal graph. Then, for any $u \in V(\mathcal{G})$ and $t \in [L]$, one can compute in $O(Ln^2)$ time for all $v \in V(\mathcal{G})$ the value $sp(u, v, t)$. For any $v \in V(\mathcal{G})$ for which $sp(u, v, t)$ is finite, a temporal walk that starts at u at time t, ends at v, and has duration $sp(u, v, t)$ can then be determined in time proportional to the number of time-edges of that walk.*

The following two definitions will be used to describe the sets of candidate solutions for several of the problems that we consider in this paper.

Definition 4 *((v, t, X)-tour).* A (v, t, X)-tour W in a given temporal graph $\mathcal{G}$ is a strict temporal walk that starts at some vertex $v \in V(\mathcal{G})$ in timestep t and visits (at least) all vertices in $X \subseteq V(\mathcal{G})$. We can assume that the walk ends as soon as all vertices in X have been visited, so we take the arrival time $\alpha(W)$ of a (v, t, X)-tour W to be the timestep after the timestep at the end of which W has for the first time visited all vertices in X.

Definition 5 *((v, t, k)-tour).* A (v, t, k)-tour W in a given temporal graph $\mathcal{G}$ is a (v, t, X)-tour for some subset $X \subseteq V(\mathcal{G})$ that satisfies $|X| = k$. The arrival time $\alpha(W)$ of a (v, t, k)-tour W is the timestep after the timestep at the end of which W has for the first time visited all vertices in X.

A (v, t, X)-tour W ((v, t, k)-tour W^*) in a temporal graph $\mathcal{G}$ is said to be *foremost* if $\alpha(W) \le \alpha(W')$ ($\alpha(W^*) \le \alpha(W^{*\prime})$) for any other (v, t, X)-tour W' (any other (v, t, k)-tour $W^{*\prime}$). We now formally define the main problems of interest: For a given temporal graph $\mathcal{G}$ with start vertex $s \in V(\mathcal{G})$, an $(s, 1, V)$-tour is also called an *exploration schedule*. The standard temporal exploration problem is defined as follows:

Definition 6 *(TEXP).* An instance of TEXP is given as a tuple $(\mathcal{G}, s)$, where $\mathcal{G}$ is an arbitrary temporal graph with underlying graph $G = (V, E)$ and lifetime L; and s is a start vertex in $V(\mathcal{G})$. The problem then asks that we decide if there exists an exploration schedule in $\mathcal{G}$.

Instead of visiting all vertices, we may be interested in visiting all vertices in a given set of k vertices, or even an arbitrary set of k vertices. These problems are captured by the following two definitions.

Definition 7 *(k-fixed TEXP).* An instance of the k-FIXED TEXP problem is given as a tuple $(\mathcal{G}, s, X, k)$ where $\mathcal{G} = \langle G_1, \ldots, G_L \rangle$ is an arbitrary temporal graph with underlying graph G and lifetime L; s is a start vertex in $V(\mathcal{G})$; and $X \subseteq V(\mathcal{G})$ is a set of target vertices such that $|X| = k$. The problem then asks that we decide if there exists an $(s, 1, X)$-tour W in $\mathcal{G}$.

Definition 8 *(k-arbitrary TEXP).* An instance of the k-ARBITRARY TEXP problem is given as a tuple $(\mathcal{G}, s, k)$ where $\mathcal{G} = \langle G_1, \ldots, G_L \rangle$ is an arbitrary temporal graph with underlying graph G and lifetime L; s is a start vertex in $V(\mathcal{G})$; and $k \in \mathbb{N}$. The problem then asks that we decide whether there exists an $(s, 1, k)$-tour W in $\mathcal{G}$.

Finally, we may be given a family of subsets of the vertex set, and our goal may be to visit at least one vertex in each subset. This leads to the following problem, whose definition is analogous to the GENERALIZED TSP problem [29] (also known by various other names including SET TSP, GROUP TSP, and MULTIPLE-CHOICE TSP).

Definition 9 *(Set TEXP).* An instance of SET TEXP is given as a tuple $(\mathcal{G}, s, \mathcal{X})$, where $\mathcal{G}$ is an arbitrary temporal graph with lifetime L, $s \in V(\mathcal{G})$ is a start vertex, and $\mathcal{X} = \{X_1, \ldots, X_m\}$ is a set of subsets $X_i \subseteq V(\mathcal{G})$. The problem then asks whether or not there exists a set $X \subseteq V(\mathcal{G})$ and an $(s, 1, X)$-tour in $\mathcal{G}$ with $X \cap X_i \neq \emptyset$ for all $i \in [m]$.

For yes-instances of all the problems defined above, a tour with minimum arrival time (among all tours of the type sought) is called an *optimal solution*.

2.2. Temporal exploration with non-strict temporal walks

When we consider the non-strict version of TEXP, a walk is allowed to traverse an unlimited number of edges in every timestep. As mentioned in the introduction, this changes the nature of the problem significantly. In particular, it means that a temporal walk positioned at a vertex v in timestep t is able to visit, during timestep t, any other vertex contained in the same connected component C as v and move to an arbitrary vertex $u \in C$, beginning timestep $t+1$ positioned at vertex u. As such, it is no longer necessary to know the edge structure of the input temporal graph during each timestep, and we can focus only on the connected components of each layer. This leads to the following definition:

Definition 10 *(Non-strict temporal graph, $\mathcal{G}$).* A non-strict temporal graph $\mathcal{G} = \langle G_1, \dots, G_L \rangle$ with vertex set $V := V(\mathcal{G})$ and lifetime L is an indexed sequence of partitions (layers) $G_t = \{C_{t,1}, \dots, C_{t,\gamma_t}\}$ of V for $t \in [L]$. For all $t \in [L]$, each $v \in V$ satisfies $v \in C_{t,j}$ for a unique $j \in [\gamma_t]$. The integer γ_t denotes the number of components in layer G_t; clearly we have $\gamma_t \in [n]$.

For a given non-strict temporal graph with lifetime L and γ_t components per step for $t \in [L]$, we define $\gamma = \max_{t \in [L]} \gamma_t$ to be the *maximum number of components per step*. A non-strict temporal walk is defined as follows:

Definition 11 *(Non-strict temporal walk, W).* A non-strict temporal walk W starting at vertex v at time t_1 in a non-strict temporal graph $\mathcal{G} = \langle G_1, \dots, G_L \rangle$ is a sequence $W = C_{t_1,j_1}, C_{t_2,j_2}, \dots, C_{t_l,j_l}$ of components C_{t_i,j_i} $(i \in [l])$ with $1 \le t_1 \le t_l \le L$ such that: $t_i + 1 = t_{i+1}$ for all $i \in [1, l-1]$; $C_{t_i,j_i} \in G_{t_i}$ and $j_i \in [\gamma_{t_i}]$ for all $i \in [l]$; $C_{t_i,j_i} \cap C_{t_{i+1},j_{i+1}} \neq \emptyset$ for all $i \in [l-1]$; and $v \in C_{t_1,j_1}$. Its arrival time is defined to be t_l.

Let $W = C_{t_1,j_1}, C_{t_2,j_2}, \dots, C_{t_l,j_l}$ be a non-strict temporal walk in some non-strict temporal graph $\mathcal{G}$ starting at some vertex $s \in C_{t_1,j_1}$. We refer to $l-1$ as the *duration* of W. The walk W is said to start at vertex $s \in C_{t_1,j_1}$ in timestep t_1 and finish at component C_{t_l,j_l} (or sometimes at some $v \in C_{t_l,j_l}$) in timestep t_l. Furthermore, W *visits* the set of vertices $\bigcup_{i \in [l]} C_{t_i,j_i}$. Note that W visits exactly one component in each of the l timesteps from t_1 to t_l. We call W *non-strict exploration schedule starting at s* with *arrival time l* if $t_1 = 1$ and $\bigcup_{i \in [l]} C_{t_i,j_i} = V(\mathcal{G})$. A non-strict temporal walk W_1 that finishes in component $C_{t,j}$ and a non-strict temporal walk W_2 that starts at a vertex v in $C_{t,j}$ at time t can be combined into a single non-strict temporal walk in the obvious way. This is why the arrival time of W_1 is defined to be t rather than $t+1$, as one might have expected in analogy with the case of strict temporal walks. Furthermore, note that the arrival time of a non-strict temporal walk equals its start time plus its duration.

A *non-strict (v,t,X)-tour* is a non-strict temporal walk that starts at v at time t and visits at least all vertices in X. A *non-strict (v,t,k)-tour* is a non-strict (v,t,X)-tour for some $X \subseteq V$ with $|X| = k$.

The problems TEXP, k-fixed TEXP, k-arbitrary TEXP, and Set TEXP that have been defined for strict temporal walks then translate into the corresponding problems for non-strict temporal walks, which we call NS-TEXP, k-fixed NS-TEXP, k-arbitrary NS-TEXP, and Set NS-TEXP, respectively.

3. Strict TEXP parameterisations

In this section, we consider temporal exploration problems in the strict setting. First, we observe that we cannot hope for an FPT algorithm for TEXP for parameter γ, the maximum number of connected components per step, unless P = NP: It was shown in [15, Theorem 3.5] that TEXP is NP-hard even if the graph in each timestep is the same connected planar graph of maximum degree 3, which implies the following:

Observation 12. TEXP is NP-hard even if $\gamma = 1$.

In the remainder of this section, we first give an FPT algorithm for k-fixed TEXP in Section 3.1. In Section 3.2, we first give a randomised FPT algorithm for k-arbitrary TEXP and then show how to derandomise it. In Section 3.3, we show that Set TEXP is W[2]-hard for parameter L.

3.1. An FPT algorithm for k-fixed TEXP

In this section we provide a deterministic FPT algorithm for k-fixed TEXP. Let $(\mathcal{G}, s, X, k)$ be an instance of k-fixed TEXP. For a given order $(v_1, v_2, \dots, v_k)$ of k vertices, one can use Theorem 3 to check in polynomial time whether it is possible to visit the vertices in that order: We find the earliest arrival time for reaching v_1 from s, then the earliest arrival time for reaching v_2 from v_1 if we start at v_1 at the arrival time of the first walk, and so on. In this way we obtain a walk that visits the vertices in the given order, if one exists, and that walk has earliest arrival time among all such walks. Therefore, one approach to obtaining an FPT algorithm for k-fixed TEXP would be to enumerate all $k!$ possible orders in which to visit the k vertices, and to determine for each order using Theorem 3 whether it is possible to visit the vertices in that order. In the following, we design an FPT algorithm for k-FIXED TEXP whose running-time has a better dependency on k, namely, $2^k k$ instead of $k!$.

Our algorithm looks for an earliest arrival time $(s, 1, X)$-tour of $\mathcal{G}$ via a dynamic programming (DP) approach. We note that the approach is essentially an adaptation of an algorithm proposed (independently by Bellman [30] and Held & Karp [31]) for the classic Travelling Salesperson Problem to the parameterised problem for temporal graphs.

Theorem 13. *It is possible to decide any instance $I = (\mathcal{G}, s, X, k)$ of k-FIXED TEXP, and return an optimal solution if I is a yes-instance, in time $O(2^k kLn^2)$, where $n = |V(\mathcal{G})|$ and L is $\mathcal{G}$'s lifetime.*

Proof. First we describe our algorithm before proving its correctness and analysing its running time. We begin by specifying a dynamic programming formula for $F(S, v)$, by which we denote the minimum arrival time of any temporal walk in $\mathcal{G}$ that starts at vertex $s \in V(\mathcal{G})$ in timestep 1, visits all vertices in $S \subseteq X$, and finishes at vertex $v \in S$. One can compute $F(S, v)$ via the following formula:

$$F(S, v) = \begin{cases} 1 + sp(s, v, 1) & (|S| = 1) \\ \min\limits_{u \in S - \{v\}} [F(S - \{v\}, u) + sp(u, v, F(S - \{v\}, u))] & (|S| > 1) \end{cases} \tag{1}$$

Note that to compute $F(S, v)$ when $|S| > 1$, Equation (1) states that we need only consider values $F(S', u)$ with $u \in S'$ and $|S'| = |S| - 1$, and so we begin by computing all values $F(S', u)$ such that $S' \subseteq X$ satisfies $|S'| = 1$ and $u \in S'$, before computing all values such that $|S'| = 2$ and $u \in S'$ and so on, until we have computed all values $F(X, u)$ where $u \in X$ (i.e., values $F(S', u)$ with $|S'| = k = |X|$). Once all necessary values have been obtained, computing the following value gives the arrival time of an optimal $(s, 1, X)$-tour:

$$F^* = \min_{v \in X} F(X, v). \tag{2}$$

If, whenever we compute a value $F(S, v)$ with $|S| > 1$, we also store alongside $F(S, v)$ a single pointer

$$p(S, v) = \arg \min_{u \in S - \{v\}} [F(S - \{v\}, u) + sp(u, v, F(S - \{v\}, u))],$$

then once we have computed F^* we can use a traceback procedure to reconstruct the walk with arrival time F^*. More specifically, let $u_1 = \arg\min_{u \in X} F(X, u)$ and $u_i = p(X - \{u_1, \ldots, u_{i-2}\}, u_{i-1})$ for all $i \in [2, k]$. To complete the algorithm, we then check if F^* is finite: If so, then there must be a $(s, 1, X)$-tour W in $\mathcal{G}$ with $\alpha(W) = F^*$ that visits the vertices $u_k, \ldots, u_1$ in that order. We can reconstruct W by concatenating the k shortest walks obtained by starting at s in timestep 1 and computing a shortest walk from s to u_k, then computing a shortest walk from u_k to u_{k-1} starting at the timestep at which u_k was reached, and so on, until u_1 is reached; once constructed, return W. If, on the other hand, $F^* = \infty$ (which is possible by the definition of $sp(u, v, t)$) then return no.

Correctness The correctness of Equation (1) can be shown via induction on $|S|$: The base case (i.e., when $|S| = 1$) is correct since the arrival time of the foremost temporal walk that starts at s in timestep 1 and ends at a specific vertex $v \in X$ is clearly equal to one plus the duration of the foremost temporal walk between s and v starting at timestep 1.

For the general case (when $|S| > 1$), assume first that the formula holds for any set S' such that $|S'| = l$ and any vertex $u \in S'$. To see that the formula holds for all sets S with $|S| = l + 1$ and vertices $v \in S$, consider any walk W that starts in timestep 1, visits all vertices in some set S with $|S| = l + 1$ and ends at v. Let $x_1, \ldots, x_{l+1}$ be the order in which the vertices $x_i \in S$ are reached by W for the first time; let $x = x_{l+1} = v$ and $x' = x_l$. Note that the subwalk W' of W that begins in timestep 1 and finishes at the end of the timestep in which W arrives at x' for the first time is surely an $(s, 1, S - \{v\})$-tour, since W' visits every vertex in $S - \{x\} = S - \{v\}$. Then, by the induction hypothesis we have $\alpha(W') \geq F(S - \{v\}, x')$ because $|S - \{v\}| = l$, and since W ends at v we have

$$\begin{aligned} \alpha(W) &\geq \alpha(W') + sp(x', v, \alpha(W')) \\ &\geq F(S - \{v\}, x') + sp(x', v, F(S - \{v\}, x')). \end{aligned}$$

More generally, we can say that any $(s, 1, S)$-tour W that starts at s in timestep 1, visits all vertices in S (where $|S| = l + 1$), and finishes at $v \in S$ satisfies the above inequality for some $x' \in S - \{v\}$. Note that for any $u \in S - \{v\}$, $F(S - \{v\}, u) + sp(u, v, F(S - \{v\}, u))$ corresponds to the arrival time of a valid $(s, 1, S)$-tour, obtained by concatenating an earliest arrival time $(s, 1, S - \{v\})$-tour that ends at u and a shortest walk between u and v starting at time $F(S - \{v\}, u)$. Therefore, to compute $F(S, v)$ it suffices to compute the minimum value of $F(S - \{v\}, u) + sp(u, v, F(S - \{v\}, u))$ over all $u \in S - \{v\}$; note that this is exactly Equation (1) in the case that $|S| > 1$.

To establish the correctness of Equation (2) recall that, by Definition 4, the arrival time of any $(s, 1, X)$-tour in $\mathcal{G}$ is equal to the timestep after the timestep in which it traverses a time edge to reach the final unvisited vertex of X for the first time. Assume that I is a yes-instance and let $x^* \in X$ be the k-th unique vertex in X that is visited by some foremost

$(s, 1, X)$-tour W; then, by the analysis in the previous paragraph, we must have $\alpha(W) = F(X, x^*)$ since W is foremost, so $x^* = \arg\min_{v \in X} F(X, v)$ and thus $\alpha(W) = F(X, x^*) = \min_{v \in X} F(X, v) = F^*$, as required.

The fact that the answer returned by the algorithm is correct follows from the correctness of Equations (1) and (2) and the traceback procedure, together with the fact that I is a no-instance if and only if $F^* = \infty$. The details of this second claim are not difficult to see and are omitted, but we note that it is indeed possible that $F^* = \infty$ since F^* is the summation of a number of values $sp(u, v, t)$, some of which may satisfy $sp(u, v, t) = \infty$ by definition.

Runtime analysis Since we only compute values of $F(S, v)$ such that $v \in S$ and $1 \le |S| \le k$, in total we compute $O(\sum_{i=1}^{k} \binom{k}{i} i) = O(2^k k)$ values. Note that, to compute any value $F(S, v)$ with $|S| = i > 1$, Equation (1) requires that we consider the values $F(S - \{v\}, u) + sp(u, v, F(S - \{v\}, u))$ with $u \in S - \{v\}$, of which there are exactly $i - 1$. We therefore use Theorem 3 to compute (and store temporarily), for each S' with $|S'| = i - 1$ and $x \in S'$, in $O(Ln^2)$ time the value of $sp(x, y, F(S', x))$ for all $y \in V(\mathcal{G})$ immediately after computing all $F(S', x)$, and use these precomputed shortest walk durations to compute $F(S, v)$ for any S with $|S| = i$ and $v \in S$ in time $O(i) = O(k)$. Thus, we spend $O(k) + O(Ln^2) = O(Ln^2)$ (since $k \le n$) time for each of $O(2^k k)$ values $F(S, v)$. This yields an overall time of $O(2^k kLn^2)$. Note that F^* can be computed using Equation (2) in $O(k)$ time since we take the minimum of $O(k)$ values; also note that a $(v, 1, X)$-tour with arrival time F^* can be reconstructed in time $O(kLn^2)$ using the aforedescribed traceback procedure, since we need to recompute $O(k)$ shortest walks, spending $O(Ln^2)$ time on each walk. Hence the overall running time of the algorithm is bounded by $O(2^k kLn^2)$, as claimed. □

We remark that k-FIXED TEXP is also in FPT when parameterised by the lifetime L: If $L < k - 1$, the instance is clearly a no-instance, and if $L \ge k - 1$, the FPT algorithm for k-FIXED TEXP with parameter k is also an FPT algorithm for parameter L.

As k-FIXED TEXP becomes TEXP when $X = V(\mathcal{G})$, we get the following corollary.

Corollary 14. TEXP *is in FPT when parameterised by the number of vertices n or by the lifetime L.*

3.2. FPT *algorithms for k-*ARBITRARY TEXP

The main result of this section is a randomised FPT algorithm for k-ARBITRARY TEXP that utilises the *colour-coding* technique originally presented by Alon et al. [9]. There, they employed the technique primarily to detect the existence of a k-vertex simple path in a given undirected graph G. More generally, it has proven useful as a technique for finding fixed motifs (i.e., prespecified subgraphs) in static graphs/networks. We provide a high-level description of the technique and the way that we apply it at the beginning of Section 3.2.1. A standard derandomisation technique (originating from [9,10]) is then utilised in Section 3.2.2 to obtain a deterministic algorithm for k-ARBITRARY TEXP with a worse, but still FPT, running time.

3.2.1. A randomised algorithm

The algorithm of this section employs the colour-coding technique of Alon et al. [9]. First, we informally sketch the structure of the algorithm behind Theorem 15: We colour the vertices of an input temporal graph uniformly at random, then by means of a DP subroutine we look for a temporal walk that begins at some start vertex s in timestep 1 and visits k vertices with distinct colours by the earliest time possible. Notice that if such a walk is found then it must be a (v, t, k)-tour, since the k vertices are distinctly coloured and therefore must be distinct. Then, the idea is to repeatedly (1) randomly colour the input graph $\mathcal{G}$'s vertices; then (2) run the DP subroutine on each coloured version of $\mathcal{G}$. We repeat these steps enough times to ensure that, with high probability, the vertices of an optimal $(s, 1, k)$-tour are coloured with distinct colours at least once over all colourings – if this happens then the DP subroutine will surely return an optimal $(s, 1, k)$-tour. With this high-level description in mind, we now present/analyse the algorithm:

Theorem 15. *For every $\varepsilon > 0$, there exists a Monte Carlo algorithm that, with probability $1 - \varepsilon$, decides a given instance $I = (\mathcal{G}, s, k)$ of k-*ARBITRARY TEXP*, and returns an optimal solution if I is a yes-instance, in time $O((2e)^k Ln^3 \log \frac{1}{\varepsilon})$, where $n = |V(\mathcal{G})|$ and L is $\mathcal{G}$'s lifetime.*

Proof. Let $V := V(\mathcal{G})$. We now describe our algorithm before proving it correct and analysing its running time. Let $c : V \to [k]$ be a colouring of the vertices $v \in V$. Let a walk W in $\mathcal{G}$ that starts at s and visits a vertex coloured with each colour in $D \subseteq [k]$ be known as a *D-colourful walk*; let the timestep after the timestep at the end of which W has for the first time visited vertices with k distinct colours be known as the *arrival time* of W, denoted by $\alpha(W)$. The algorithm employs a subroutine that computes, should one exist, a $[k]$-colourful walk W in $\mathcal{G}$ with earliest arrival time. Note that a D-colourful walk ($D \subseteq [k]$) in $\mathcal{G}$ is by definition an $(s, 1, |D|)$-tour in $\mathcal{G}$.

Define $H(D, v)$ to be the earliest arrival time of any D-colourful walk (where $D \subseteq [k]$) in $\mathcal{G}$ that ends at a vertex v with $c(v) \in D$. The value of $H(D, v)$ for any $D \subseteq [k]$ and v with $c(v) \in D$ can be computed via the following dynamic programming formula (within the formula we denote by $D^-_{c(v)}$ the set $D - \{c(v)\}$):

$$H(D,v)=\begin{cases}1+sp(s,v,1) & (|D|=1)\\ \min\limits_{u\in V:c(u)\in D^-_{c(v)}}[H(D^-_{c(v)},u)+sp(u,v,H(D^-_{c(v)},u))] & (|D|>1)\end{cases} \tag{3}$$

In order to compute $H(D,v)$ for any $D\subseteq[k]$ and vertex v with $c(v)\in D$, Equation (3) requires that we consider values $H(D-\{c(v)\},u)$ such that $c(u)\in D-\{c(v)\}$, and so we begin by computing $H(D',v)$ for all D' with $|D'|=1$ and v with $c(v)\in D'$, then for all D' with $|D'|=2$ and v with $c(v)\in D'$, and so on, until all values $H([k],v)$ have been obtained. The earliest arrival time of any $[k]$-colourful walk in $\mathcal{G}$ is then given by

$$H^*=\min_{u\in V(\mathcal{G})} H([k],u). \tag{4}$$

Once H^* has been computed, we check whether its value is finite or equal to ∞. If H^* is finite then we can use a pointer system and traceback procedure (almost identical to those used in the proof of Theorem 13) to reconstruct an $(s,1,k)$-tour with arrival time H^* if one exists; otherwise we return no. This concludes the description of the dynamic programming subroutine.

Let $r=\lceil\frac{1}{\varepsilon}\rceil$ and let W^* initially be the trivial walk that starts and finishes at vertex s in timestep 1. Perform the following two steps for $e^k\ln r$ iterations:

1. Assign colours in $[k]$ to the vertices of V uniformly at random.
2. Run the DP subroutine in order to find an optimal $[k]$-colourful walk W in $\mathcal{G}$ if one exists. If such a W is found then check if $\alpha(W)<\alpha(W^*)$ or W^* starts and ends at s in timestep 1 (i.e., still has its initial value), and in either case set $W^*=W$; otherwise the DP subroutine returned no and we make no change to W^*.

Once all iterations of the above steps are over, check if W^* is still equal to the walk that starts and finishes at s in timestep 1; if not then return W^*, otherwise return no. This concludes the algorithm's description.

Correctness We focus on proving the randomised aspect of the algorithm correct and omit correctness proofs for Equations (3) and (4) since the arguments are similar to those provided in Theorem 13's proof.

If I is a no-instance then in no iteration will the DP subroutine find an $(s,1,k)$-tour in $\mathcal{G}$. Hence in the final step the algorithm will find that W^* is equal to the walk that starts and ends at s in timestep 1 (by the correctness of Equations (3) and (4)) and return no, which is clearly correct. Assume then that I is yes-instance. Let W be an $(s,1,k)$-tour in $\mathcal{G}$ with earliest arrival time, and let $X\subseteq V$ be the set of k vertices visited by W. Then, if during one of the $e^k\ln r$ iterations of steps 1 and 2 we colour the vertices of V in such a way that X is well-coloured (we say that a set of vertices $U\subseteq V$ is *well-coloured* by colouring c if $c(u)\neq c(v)$ for every pair of vertices $u,v\in U$), W will induce an optimal $[k]$-colourful walk in $\mathcal{G}$. The DP subroutine will then return W or some other optimal $[k]$-colourful walk W' with $\alpha(W)=\alpha(W')$ that visits a well-coloured subset of vertices X'; note that the arrival time of the best tour found in any iteration so far will then surely be $\alpha(W)$, since W has earliest arrival time.

Observe that if we colour the vertices of V with k colours uniformly at random, then, since $|X|=k$, there are k^k ways to colour the vertices in $X\subseteq V$, of which $k!$ constitute well-colourings of X. Hence after a single colouring of V we have

$$\Pr[X \text{ is well-coloured}]=\frac{k!}{k^k}>\frac{1}{e^k},$$

where the inequality follows from the fact that $k!/k^k>\sqrt{2\pi}k^{\frac{1}{2}}e^{\frac{1}{12k+1}}/e^k$ (this inequality is due to Robbins [32] and is related to Stirling's formula). Hence, after $e^k\ln r$ colourings, we have (using the standard inequality $(1-\frac{1}{x})^x\le\frac{1}{e}$ for all $x\ge1$):

$$\Pr[X \text{ is not well-coloured in any colouring}]\le\left(1-\frac{1}{e^k}\right)^{e^k\ln r}\le 1/r\le\varepsilon.$$

Thus, the probability that X is well-coloured at least once after $e^k\ln r$ colourings is at least $1-\varepsilon$. It follows that, with probability $\ge 1-\varepsilon$, the earliest arrival $[k]$-colourful walk returned by the algorithm after all iterations is in fact an optimal $(s,1,k)$-tour in $\mathcal{G}$, since either W or some other $(s,1,k)$-tour with equal arrival time will eventually be returned.

Runtime analysis Note that the DP subroutine computes exactly the values $H(D,v)$ such that $D\subseteq[k]$ and v satisfies $c(v)\in D$. Hence there are at most $\binom{k}{i}n$ values $H(D,v)$ such that $|D|=i$, for all $i\in[k]$; this gives a total of $\sum_{i\in[k]}\binom{k}{i}n=O(2^kn)$ values. In order to compute $H(D,v)$ for any D with $|D|=i>1$, Equation (3) requires us to consider the value of $H(D-\{c(v)\},u)+sp(u,v,H(D-\{c(v)\},u))$ for all u such that $c(u)\in D-\{c(v)\}$. Therefore, similar to the algorithm in the proof of Theorem 13, we compute and store, immediately after computing each value $H(D',x)$ with $|D'|=i-1$ and $c(x)\in D'$, the value of $sp(x,y,H(D',x))$ for all $y\in V(\mathcal{G})$ in $O(Ln^2)$ time (Theorem 3). Note that there can be at most n vertices u such that $c(u)\in D-\{c(v)\}$, and so in total we spend $O(n)+O(Ln^2)=O(Ln^2)$ time on each of $O(2^kn)$ values of $H(D,v)$, giving an overall time of $O(2^kLn^3)$. We can compute H^* in $O(n)$ time since we take the minimum of $O(n)$ values, and the traceback procedure can be performed in $O(kLn^2) = O(Ln^3)$ time since we concatenate k walks obtained using Theorem 3. Thus the overall time spent carrying out one execution of the DP subroutine is $O(2^k Ln^3)$.

Since the running time of each iteration of the main algorithm is dominated by the running time of the DP subroutine and there are $e^k \ln r = O(e^k \log \frac{1}{\varepsilon})$ iterations in total, we conclude that the overall running time of the algorithm is $O((2e)^k Ln^3 \log \frac{1}{\varepsilon})$, as claimed. This completes the proof. □

3.2.2. Derandomising the algorithm of Theorem 15

The randomised colour-coding algorithm of Theorem 15 can be derandomised at the expense of incurring a $k^{O(\log k)} \log n$ factor in the running time. We employ a standard derandomisation technique, presented initially in [9], which involves the enumeration of a *k-perfect family of hash functions* from $[n]$ to $[k]$. The functions in such a family will be viewed as colourings of the vertex set of the temporal graph given as input to the k-ARBITRARY TEXP problem.

Formally, a family $\mathcal{H}$ of hash functions from $[n]$ to $[k]$ is *k-perfect* if, for every subset $S \subseteq [n]$ with $|S| = k$, there exists a function $f \in \mathcal{H}$ such that f restricted to S is bijective (i.e., one-to-one). The following theorem of Naor et al. [10] enables one to construct such a family $\mathcal{H}$ in time linear in the size of $\mathcal{H}$:

Theorem 16 *(Naor, Schulman and Srinivasan [10]). A k-perfect family $\mathcal{H}$ of hash functions f_i from $[n]$ to $[k]$, with size $e^k k^{O(\log k)} \log n$, can be computed in $e^k k^{O(\log k)} \log n$ time.*

We note that the value of $f_i(x)$ for any $f_i \in \mathcal{H}$ and $x \in [n]$ can be evaluated in $O(1)$ time.

To solve an instance of k-ARBITRARY TEXP, we can now use the algorithm from the proof of Theorem 15, but instead of iterating over $e^k \ln r$ random colourings, we iterate over the $e^k k^{O(\log k)} \log n$ hash functions in the k-perfect family of hash functions constructed using Theorem 16. This ensures that the set X of k vertices visited by an optimal $(s, 1, k)$-tour is well-coloured in at least one iteration, and we obtain the following theorem.

Theorem 17. *There is a deterministic algorithm that can solve a given instance $(\mathcal{G}, s, k)$ of k-ARBITRARY TEXP in $(2e)^k k^{O(\log k)} Ln^3 \log n$ time, where $n = |V(\mathcal{G})|$. If the instance is a yes-instance, the algorithm also returns an optimal solution.*

Similar to the case of k-FIXED TEXP, we can remark that k-ARBITRARY TEXP is also in FPT when parameterised by the lifetime L: If $L < k - 1$, the instance is clearly a no-instance, and if $L \geq k - 1$, the FPT algorithm for k-ARBITRARY TEXP with parameter k from Theorem 17 is also an FPT algorithm for parameter L.

3.3. W[2]-hardness of SET TEXP for parameter L

The NP-complete HITTING SET problem is defined as follows [33].

Definition 18 *(Hitting Set).* An instance of HITTING SET is given as a tuple $(U, \mathcal{S}, k)$, where $U = \{a_1, \ldots, a_n\}$ is the ground set and $\mathcal{S} = \{S_1, \ldots, S_m\}$ is a set of subsets $S_i \subseteq U$. The problem then asks whether or not there exists a subset $U' \subseteq U$ of size at most k such that, for all $i \in [m]$, there exists an $u \in U'$ such that $u \in S_i$.

It is known that HITTING SET is W[2]-hard when parameterised by k [27].

Theorem 19. SET TEXP *parameterised by L (the lifetime of the input temporal graph) is* W[2]*-hard.*

Proof. We give a parameterised reduction from the HITTING SET problem with parameter k to the SET TEXP problem with parameter L. Given an instance $I = (U, \mathcal{S}, k)$ of HITTING SET, we construct an instance $I' = (\mathcal{G}, s, \mathcal{X})$ of SET TEXP as follows: The lifetime of $\mathcal{G}$ is set to $L = k$. In each of the L steps, the graph is a complete graph with vertex set $U \cup \{s\}$, where s is a start vertex that is assumed not to be in U. Finally, we set $\mathcal{X} = \mathcal{S}$. We proceed to show that I is yes-instance if and only if I' is a yes-instance.

If I is a yes-instance, let $U' = \{u_1, u_2, \ldots, u_k\}$ be a hitting set of size k. Then the walk that moves from s to u_1 in step 1 and then from u_{i-1} to u_i in step i for $2 \leq i \leq k$ is an $(s, 1, U')$-tour that visits at least one vertex from each set in $\mathcal{X}$. Therefore, I' is a yes-instance.

If I' is a yes-instance, let W be a strict temporal walk that visits at least one vertex from each set in $\mathcal{X}$. Let U' be the set of at most $L = k$ vertices that this walk visits in addition to the start vertex s. Then U' is a hitting set for I. Hence, I is a yes-instance. □

4. Non-strict TEXP parameterisations

In this section, we study temporal exploration problems in the non-strict setting. Let $\mathcal{G} = \langle G_1, \ldots, G_L \rangle$ be the given non-strict temporal graph, and let $s \in V(\mathcal{G})$ be the given start vertex. When analysing running-times in this section, we assume that the non-strict temporal graph is given by providing, for each timestep t, a list of the vertex sets (with each of these sets given as a list of vertices) of the components in that timestep. This representation has size $\Theta(Ln)$. If the graph was given in the same form as a strict temporal graph, this representation could be computed by a pre-processing step that runs in time $O(Ln^2)$.

First, we show in Section 4.1 that FPT algorithms for k-FIXED NS-TEXP and k-ARBITRARY NS-TEXP can be derived using similar techniques as in Section 3. After that, we show that NS-TEXP and its variants can all be solved in polynomial time if γ (the maximum number of connected components in any layer of $\mathcal{G}$) is bounded by 2 (Section 4.2) and that NS-TEXP is in FPT when parameterised by the lifetime L (Section 4.3). Finally, we prove W[2]-hardness for the SET NS-TEXP problem when the same parameter is considered (Section 4.4).

4.1. k-FIXED NS-TEXP and k-ARBITRARY NS-TEXP

We now define $sp(u, v, t)$ as the duration of a shortest (i.e., having minimum arrival time) *non-strict* temporal walk in $\mathcal{G}$ that starts at $u \in V(\mathcal{G})$ in timestep t and ends at $v \in V(\mathcal{G})$. If $u = v$ or if u and v are in the same component in step t, then $sp(u, v, t) = 0$. If there is no such non-strict temporal walk, we let $sp(u, v, t) = \infty$.

Lemma 20. *For given u and t, one can compute the values $sp(u, v, t)$ for all $v \in V(\mathcal{G})$ in $O(Ln)$ time. Once this computation has been completed and the relevant data kept in memory, one can then, for each $v \in V(\mathcal{G})$, determine a shortest walk starting at u at time t and reaching v in time proportional to $1 + sp(u, v, t)$.*

Proof. Let $V = V(\mathcal{G})$. For each $w \in V$, maintain a label $r(w)$ to represent whether w is reachable by the time step under consideration, and a label $a(w)$ to represent the earliest arrival time at w if w is reachable. In addition, we will remember a predecessor $p(w)$ for every reachable vertex. Initialise the current time to $t_c = t$; set $r(w) = \mathsf{true}$, $a(w) = t_c$ and $p(w) = u$ for all w in the component of u at time t_c; set $r(w) = \mathsf{false}$ and $a(w) = \infty$ for all other vertices. This takes $O(n)$ time.

Then repeat the following step until either all vertices are reachable or t_c equals the lifetime of the graph: Increase t_c by one. For each component B of step t_c, check whether B contains a vertex w with $r(w) = \mathsf{true}$ and, if so, mark B and remember w as p_B. For each vertex w with $r(w) = \mathsf{false}$ in any marked component B of step t_c, we then set $r(w) = \mathsf{true}$, $a(w) = t_c$ and $p(w) = p_B$. Each execution of this step takes $O(n)$ time.

Finally, for each vertex $v \in V$, we set $sp(u, v, t) = a(v) - t$.

To construct the shortest temporal walk corresponding to a value $sp(u, v, t)$, we trace back the vertices (and their components) starting with v (visited at time $t' = t + sp(u, v, t)$), $p(v)$ (visited at time $a(p(v)) \le t' - 1$), $p(p(v))$, and so on.

It is clear that the running-time is $O(Ln)$. Correctness can be shown by induction: When the step for value t_c has been completed, a vertex w satisfies $r(w) = \mathsf{true}$ if and only if w is reachable from u with arrival time at most t_c, and in that case $a(w) = t'$ is the earliest arrival time at w and, if $t' > t$, $p(w)$ is a vertex that is reachable with arrival time at most $t' - 1$ and from which w can be reached in step t'. □

Next, we observe that it is easy to see that Equations (1) and (2) from the proof of Theorem 13 remain valid in the non-strict case, as the arguments for correctness remain the same. The factor Ln^2 in the running-time of Theorem 13 improves to Ln in the non-strict case as, by Lemma 20, it takes only $O(Ln)$ time to compute $sp(u, v, t)$ for all $v \in V$ right after $F(S', u) = t$ has been computed for some set S' and $u \in S'$. Thus, we obtain:

Corollary 21. *It is possible to decide any instance $I = (\mathcal{G}, s, X, k)$ of k-FIXED NS-TEXP, and return an optimal solution if I is a yes-instance, in time $O(2^k kLn)$, where $n = |V(\mathcal{G})|$ and L is $\mathcal{G}$'s lifetime.*

Similarly, Equations (3) and (4) from the proof of Theorem 15 remain valid, and the derandomisation used in the proof of Theorem 17 works for the non-strict case without any alterations. Thus, we obtain the following corollary of Theorems 15 and 17, where again we save a factor of n in the running-time because we can use Lemma 20 instead of Theorem 3.

Corollary 22. *For every $\varepsilon > 0$, there exists a Monte Carlo algorithm that, with probability $1 - \varepsilon$, decides a given instance $I = (\mathcal{G}, s, k)$ of k-ARBITRARY NS-TEXP, and returns an optimal solution if I is a yes-instance, in time $O((2e)^k Ln^2 \log \frac{1}{\varepsilon})$, where $n = |V(\mathcal{G})|$ and L is $\mathcal{G}$'s lifetime. Furthermore, there is a deterministic algorithm that can solve a given instance $(\mathcal{G}, s, k)$ of k-ARBITRARY NS-TEXP in $(2e)^k k^{O(\log k)} Ln^2 \log n$ time. If the instance is a yes-instance, the algorithm also returns an optimal solution.*

4.2. Non-strict exploration with at most two components per step

Let $\mathcal{G} = \langle G_1, \ldots, G_L \rangle$ be the given non-strict temporal graph. If there is a step t in which the partition G_t consists of a single component $C_{t,1}$, then it is trivially possible to visit all vertices: We simply wait at the start vertex until step t, and then visit all vertices in step t. Therefore, for all four problem variants (NS-TEXP, k-FIXED NS-TEXP, k-ARBITRARY NS-TEXP, and SET NS-TEXP), instances where the maximum number of components per step is $\gamma = 1$ are trivially yes-instances, and instances with $\gamma = 2$ are also yes-instances if at least one step has a single component. In the remainder of this section, we

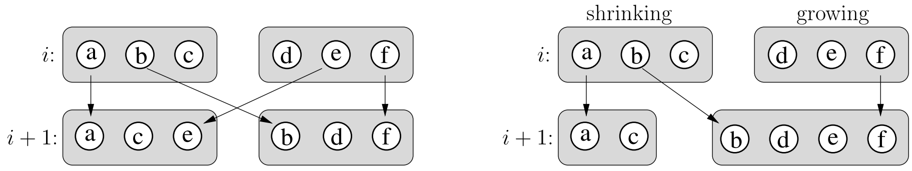

Fig. 1. Free transition (left) and restricted transition (right).

therefore consider the case $\gamma = 2$ under the assumption that the partition in every step consists of exactly two components. Furthermore, we can assume without loss of generality that no two consecutive steps have the same two components: Any number of consecutive steps that all have the same two components could be replaced by a single step without changing the answer to any of the four variants of the NS-TEXP problem.

First, we are interested in the movements that the partitions in two consecutive steps allow. We refer to two consecutive steps i and $i+1$ as a *transition*.

Definition 23. A transition between step i with partition $G_i = (A_i, B_i)$ and step $i+1$ with partition $G_{i+1} = (A_{i+1}, B_{i+1})$ is called *free* if the four sets $A_i \cap A_{i+1}$, $A_i \cap B_{i+1}$, $B_i \cap A_{i+1}$, $B_i \cap B_{i+1}$ are all non-empty. If exactly one of these sets is empty, the transition is called *restricted*.

See Fig. 1 for an illustration. In a free transition, a walk can reach any of the two components in step $i+1$ no matter which component the walk visits in step i. In a restricted transition, there is one component in step i such that one component in step $i+1$ cannot be reached from it. We show next that these are the only possible types of transitions.

Lemma 24. *Every transition is either free or restricted.*

Proof. Assume that the transition from G_i to G_{i+1} is not free. Assume without loss of generality that $A_i \cap B_{i+1}$ is empty. This means that every vertex of A_i must be contained in A_{i+1}. As we assume that the partitions of consecutive steps are different, we get that $A_i \subset A_{i+1}$ and, hence, $B_i \supset B_{i+1}$. This implies that at least one vertex from B_i is in A_{i+1}. Furthermore, neither A_{i+1} nor B_{i+1} can be empty, so there must also be a vertex in $B_i \cap B_{i+1}$. Hence, the transition is restricted. □

The proof of Lemma 24 shows that in a restricted transition there is one component that shrinks (gets replaced by a strict subset) and one that grows (gets replaced by a strict superset). We call the former the *shrinking component* and the latter the *growing component* (as indicated in Fig. 1).

Lemma 25. *If there is a restricted transition from step i to $i+1$, a walk that visits the shrinking component in step i can visit all vertices of the graph in steps i and $i+1$.*

Proof. The walk can visit all vertices of the shrinking component in step i and then end step i at a vertex that leaves the shrinking component. In step $i+1$, the walk then visits all vertices in the component that has grown. It is easy to see that every vertex is contained in the two components visited by the walk. □

Lemma 26. *If a restricted transition follows a free transition, the whole graph can be explored.*

Proof. Assume that there is a free transition from step $i-1$ to step i and a restricted transition from step i to step $i+1$. Let B_i be the shrinking component in the restricted transition. Then a walk can visit B_i in step i (because the free transition allows it to reach B_i) and then, by Lemma 25, visit all remaining unvisited vertices in step $i+1$. □

Lemma 27. *In $1 + \log_2 n$ consecutive free transitions, the whole graph can be explored.*

Proof. Let A be the component that the walk visits in the first step of the first free transition. In each of the $1 + \log_2 n$ free transitions, we can choose as component to visit in the next step the one that contains more of the previously unvisited vertices. In this way, we are guaranteed to visit at least half of all the remaining unvisited vertices in each of these $1+\log_2 n$ steps. The number of unvisited vertices remaining at the end of these $1 + \log_2 n$ steps is hence at most $n/2^{1+\log_2 n} < 1$. □

Theorem 28. *There is an algorithm that solves instances of* NS-TEXP *with $\gamma = 2$ in $O(Ln + n^2 \log n)$ time.*

Proof. In $O(Ln)$ time, we can check whether there is a step in which there is a single component (in that case, we output "yes" and terminate). In the same time bound, we also preprocess the graph to ensure that no two consecutive steps have the same partition and determine for each transition whether it is free or restricted.

If a restricted transition follows a free transition, we can output "yes" by Lemma 26. Otherwise, there must be an initial (possibly empty) sequence $\mathcal{R}$ of restricted transitions, followed by a (possibly empty) sequence $\mathcal{F}$ of free transitions.

If the start vertex s is in the shrinking component in one of the restricted transitions $\mathcal{R}$, then we can visit all vertices of the graph by Lemma 25, so we output "yes". Otherwise, the start vertex s must be in the growing component in all the restricted transitions $\mathcal{R}$. In this case, it is impossible to leave that component. No decision needs to be made during $\mathcal{R}$, and the walk must visit the component containing s in the first time step of the first free transition.

If the number of free transitions in $\mathcal{S}$ is greater than $1 + \log_2 n$, the answer is "yes" by Lemma 27. Otherwise, there are at most $1 + \log_2 n$ free transitions. Then, all possible choices for the next component to visit during each of the at most $1 + \log_2 n$ free transitions can be enumerated in $O(2^{1+\log_2 n}) = O(n)$ time. Furthermore, for each of these possibilities, one can check in $O(n \log n)$ time whether the corresponding walk visits all vertices of the graph. □

Corollary 29. *For each of the problems k-FIXED NS-TEXP, k-ARBITRARY NS-TEXP, and SET NS-TEXP, there is an algorithm that solves instances with $\gamma = 2$ in $O(Ln + n^2 \log n)$ time.*

Proof. First, assume that there is a step with a single component, or that a restricted transition follows a free transition, or that the vertex s is ever contained in the shrinking component of a restricted transition, or that the number of free transitions is greater than $1 + \log_2 n$. In all these cases, as argued in the proof of Theorem 28, all vertices of the input graph can be visited, and hence the given instance is a yes-instance also of the three problem variants under consideration here.

Now, assume that the temporal graph consists of an initial (possibly empty) sequence $\mathcal{R}$ of restricted transitions such that s is always contained in the growing component, followed by a sequence $\mathcal{F}$ of at most $1 + \log_2 n$ free transitions. Then there are at most $2^{1+\log_2 n} = O(n)$ possible non-strict temporal walks in the graph, and we can simply enumerate them all and check for each of them in $O(n \log n)$ time whether it is a solution to the given variant of NS-TEXP. □

We leave open the complexity of NS-TEXP and its variants in the case where γ is a fixed constant greater than 2.

4.3. *An* FPT *algorithm for* NS-TEXP *with parameter L*

We now consider NS-TEXP parameterised by the lifetime L of the input temporal graph $\mathcal{G}$. Let an instance of NS-TEXP be given as a tuple $(\mathcal{G}, s, L)$. We prove that NS-TEXP is in FPT for parameter L by specifying a bounded search tree-based FPT algorithm.

Let $\mathcal{G} = \langle G_1, \dots, G_L \rangle$ be some non-strict temporal graph. Throughout this section we let $\mathcal{C}(\mathcal{G}) := \bigcup_{t \in [L]} G_t$, i.e., $\mathcal{C}(\mathcal{G})$ is the set of all components belonging to some layer of $\mathcal{G}$. We implicitly assume that each component $C \in \mathcal{C}(\mathcal{G})$ is *associated* with a unique layer G_t of $\mathcal{G}$ in which it is contained. If a component (seen as just a set of vertices) occurs in several layers, we thus treat these occurrences as different elements of $\mathcal{C}(\mathcal{G})$ (or of any subset thereof) because they are associated with different layers. If Q is a set of components in $\mathcal{C}(\mathcal{G})$ that are associated with distinct layers (i.e., no two components in Q are associated with the same layer G_t of $\mathcal{G}$), then we say that the components in Q *originate from unique layers of* $\mathcal{G}$. For a set Q of components that originate from unique layers of $\mathcal{G}$, we let $D(Q) := \bigcup_{C \in Q} C$ be the union of the vertex sets of the components in Q. For any such set Q, we also let $T(Q) = \{t \in [L] : \text{there is a } C \in Q \text{ associated with layer } G_t\}$.

Within the following, we assume that $\mathcal{G}$ admits a non-strict exploration schedule W.

Observation 30. Let Q ($|Q| \in [0, L-1]$) be a subset of the components visited by the exploration schedule W. Then there exists $C \in \mathcal{C}(\mathcal{G}) - Q$ with $C \in G_t$ ($t \in [L] - T(Q)$) such that $|C - D(Q)| \geq (n - |D(Q)|)/(L - |T(Q)|)$.

Observation 30 follows since, otherwise, W visits at most $L - |T(Q)|$ components $C \in \mathcal{C}(\mathcal{G}) - Q$ that each contain $|C - D(Q)| < (n - |D(Q)|)/(L - |T(Q)|)$ of the vertices $v \notin D(Q)$, and so the total number of vertices visited by W is strictly less than $|D(Q)| + (L - |T(Q)|) \cdot (n - |D(Q)|)/(L - |T(Q)|) = n$, a contradiction.

We briefly outline the main idea of our FPT result: We use a search tree algorithm that maintains a set Q of components that a potential exploration schedule could visit, starting with the empty set. Then the algorithm repeatedly tries all possibilities for adding a component (from some so far untouched layer) that contains at least $(n - |D(Q)|)/(L - |T(Q)|)$ unvisited vertices (whose existence is guaranteed by Observation 30 if there exists an exploration schedule). It is clear that the search tree has depth L, and the main further ingredient is an argument showing that the number of candidates for the component to be added is bounded by a function of L, namely, by $(L - |T(Q)|)^2$: This is because each of the $L - |T(Q)|$ untouched layers can contain at most $L - |T(Q)|$ components that each contain at least $(n - |D(Q)|)/(L - |T(Q)|)$ unvisited vertices. We now proceed to describe the details of the algorithm and its analysis. First, we state the following corollary of Lemma 20.

Corollary 31. *Let $\mathcal{G} = \langle G_1, \dots, G_L \rangle$ be an arbitrary order-n non-strict temporal graph. Then, for components $C_{t_1, j_1} \in G_{t_1}$ and $C_{t_2, j_2} \in G_{t_2}$ (with $1 \leq t_1 \leq t_2 \leq L$) one can decide, in $O((t_2 - t_1 + 1)n)$ time, whether there exists a non-strict temporal walk beginning at any vertex contained in C_{t_1, j_1} in timestep t_1 and finishing at C_{t_2, j_2} in timestep t_2.*

Algorithm 1: Recursive function $g(\mathcal{G}, s, Q)$.

```
1  if |Q| = L or |D(Q)| = n then
2      if |D(Q)| = n then return W^?_G(s, Q)
3      else return no
4  else
5      C' ← {C ∈ C(G) − Q : |C − D(Q)| ≥ (n − |D(Q)|)/(L − |T(Q)|)}
6      C* ← C' − {C ∈ C' : C ∈ G_t, t ∈ T(Q)}.
7      if |C*| = 0 then  return no
8      for C ∈ C* do
9          if g(G, s, Q ∪ {C}) = yes then return yes
10     end
11     return no
12 end
```

Proof. We construct the non-strict temporal graph $\mathcal{G}'$ that consists of the layers $\langle G_{t_1}, G_{t_1+1}, \dots, G_{t_2}\rangle$ of $\mathcal{G}$ and has lifetime $L' = t_2 - t_1 + 1$. Then, we pick arbitrary vertices $u \in C_{t_1,j_1}$ and $v \in C_{t_2,j_2}$ and apply the algorithm from Lemma 20 to determine whether $\mathcal{G}'$ contains a non-strict temporal walk from u to v. Both steps take $O(L'n)$ time. □

Let Q be a set of components originating from unique layers of $\mathcal{G}$, and let $W^?_{\mathcal{G}}(s, Q) = \text{yes}$ if and only if there exists a non-strict temporal walk in $\mathcal{G}$ that starts at $s \in V(\mathcal{G})$ in timestep 1 and visits at least the components contained in Q, and no otherwise.

Lemma 32. *For any order-n non-strict temporal graph $\mathcal{G} = \langle G_1, \dots, G_L\rangle$, any $s \in V(\mathcal{G})$, and any set Q of components originating from unique layers of $\mathcal{G}$, $W^?_{\mathcal{G}}(s, Q)$ can be computed in $O(Ln)$ time.*

Proof. Let $C_{s_1}, C_{s_2}, \dots, C_{s_{|Q|}}$ be an index-ordered sequence of the components in Q, with the indices $s_i \in [L]$ satisfying $C_{s_i} \in G_{s_i}$ (for all $i \in [|Q|]$) and $s_i < s_{i+1}$ (for all $i \in [|Q| - 1]$). Let $C_s \in G_1$ be the unique component in layer 1 such that $s \in C_s$ (note that we may have $C_{s_1} = C_s$). Now, apply the algorithm of Corollary 31 with $C_{t_1,j_1} = C_s$ and $C_{t_2,j_2} = C_{s_1}$, and then with $C_{t_1,j_1} = C_{s_i}$ and $C_{t_2,j_2} = C_{s_{i+1}}$ for all $i \in [|Q| - 1]$. If the return value of any application of the algorithm of Corollary 31 is no, then we return $W^?_{\mathcal{G}}(s, Q) = \text{no}$; otherwise we return $W^?_{\mathcal{G}}(s, Q) = \text{yes}$. This concludes the algorithm's description.

Since each component C_{s_i} can only be visited in timestep s_i it is clear that any walk that visits all components of Q must visit them in the specified order. The algorithm sets $W^?_{\mathcal{G}}(s, Q) = \text{yes}$ if the components of Q can be visited in the specified order. On the other hand, if the algorithm of Corollary 31 returns no for at least one pair of input components $C_{s_i}, C_{s_{i+1}}$ (or C_s, C_{s_1}), then it must be that the components cannot be visited in this order, and thus the algorithm sets $W^?_{\mathcal{G}}(s, Q) = \text{no}$. Thus, the algorithm's correctness follows from the correctness of Corollary 31's algorithm. To see that the running-time of the algorithm is bounded by $O(Ln)$, recall that each application of Corollary 31's algorithm to start/finish components C_{s_i} and $C_{s_{i+1}}$ takes $c(s_{i+1} - s_i + 1)n$ time (for a constant c hidden in the bound of Corollary 31). Thus the total amount of time spent over all applications is $c(s_1 - 1 + 1)n + \sum_{i \in [|Q|-1]} c(s_{i+1} - s_i + 1)n = cn(s_{|Q|} + |Q| - 1) \le cn(2L - 1) = O(Ln)$, where the last inequality holds since $|Q|, s_{|Q|} \le L$. □

Now, let $\mathcal{G}$ be some input graph, and let Q be some set of components originating from unique layers of $\mathcal{G}$. For any $s \in V(\mathcal{G})$, the recursive function $g(\mathcal{G}, s, Q)$ (Algorithm 1) returns yes if and only if there exists a non-strict exploration schedule of $\mathcal{G}$ that starts at s and visits (at least) the components contained in Q, and returns no otherwise. We prove the correctness of Algorithm 1 in Lemma 33.

Lemma 33. *For any non-strict temporal graph $\mathcal{G}$, any $s \in V(\mathcal{G})$, and any set Q (with $|Q| \in [0, L]$) containing components originating from unique layers of $\mathcal{G}$, Algorithm 1 correctly computes $g(\mathcal{G}, s, Q)$.*

Proof. We first show that $g(\mathcal{G}, s, Q)$ is correct in the base case, i.e., when $|Q| = L$ or $|D(Q)| = n$. If we have $|D(Q)| = n$, then any non-strict temporal walk that starts at s in timestep 1 and visits all components in Q is an exploration schedule. Thus, the correctness of line 2 follows from the definition of the return value $W^?_{\mathcal{G}}(s, Q)$ (which can be computed using Lemma 32). If $|Q| = L$ and $|D(Q)| < n$, i.e., we have reached line 3, then there must exist no exploration schedule that visits each of the components in Q, since any non-strict temporal walk in a temporal graph with lifetime L can visit at most L components, but at least one additional component $C \notin Q$ needs to be visited to cover at least one vertex $v \notin D(Q)$ – thus it is correct to return no in this case.

Otherwise, we have $|Q| < L$ and $|D(Q)| < n$, and are in the recursive case. Then, by Observation 30, any non-strict exploration schedule that visits all components in Q must visit at least one other component $C \in \mathcal{C}(\mathcal{G}) - Q$ such that $|C - D(Q)| \ge (n - |D(Q)|)/(L - |T(Q)|)$. Line 5 computes the set C' consisting of all such components, line 6 forms from C' the set C^* by removing from C' any components that originate from layers G_t such that $C \in G_t$ for some $C \in Q$ (since only one component can be visited in each timestep, and thus we want Q to be a set of components originating from unique layers of $\mathcal{G}$). We remark that a more efficient implementation could skip layers G_t with $t \in T(Q)$ already when constructing C' in line 5, but the asymptotic running-time of the overall algorithm would not be affected by this change. The correctness of line 7 follows from Observation 30. To complete the proof, we claim that the value yes is returned by line 9 if and only if there exists a non-strict temporal exploration schedule starting at s that visits all the components contained in Q; we proceed by reverse induction on $|Q|$. Assume first that the return value of $g(\mathcal{G}, s, Q')$ is correct for any Q' with $|Q'| = k$ ($k \in [L]$) and let $|Q| = k - 1$. Now assume that, during the execution of $g(\mathcal{G}, s, Q)$, line 9 returns yes; it follows that $g(\mathcal{G}, s, Q') =$ yes for some $Q' = Q \cup C$ with $C \in C^*$ and thus it follows from the induction hypothesis that there exists a non-strict temporal exploration schedule that starts at s and visits all the components contained in Q, as required. In the other direction, assume that there exists some non-strict exploration schedule W that starts at s in timestep 1 and visits all the components in Q. Note that, since the execution has reached line 9, we surely have $|C^*| > 0$; since we also have $|Q| < L$ and $|D(Q)| < n$ it follows from Observation 30 that W visits at least one additional component $C \in C^*$. Then, by the induction hypothesis, we must have $g(\mathcal{G}, s, Q \cup \{C\}) =$ yes; thus when the loop of lines 8–10 processes $C \in C^*$ the algorithm will return yes as required. □

Theorem 34. *There is an algorithm that decides any instance $I = (\mathcal{G}, s, L)$ of* NS-TEXP *in $O(L(L!)^2 n)$ time.*

Proof. The algorithm simply returns the value of function call $g(\mathcal{G}, s, \emptyset)$ (Algorithm 1).

By Lemma 33, $g(\mathcal{G}, v, Q)$ returns yes if and only if $\mathcal{G}$ admits a non-strict exploration schedule that starts at v and visits at least the components contained in the set Q (which contains $|Q| \in [0, L]$ components originating from unique layers of $\mathcal{G}$), and returns no otherwise. Thus the correctness of the above follows immediately.

In order to bound the running time of the above algorithm, it suffices to bound the running time of Algorithm 1, i.e., the recursive function g. The initial call is $g(\mathcal{G}, s, \emptyset)$, and each recursive call is of the form $g(\mathcal{G}, s, Q)$ where Q is a set of components with size one more than the input set of the parent call. Hence, line 1 ensures that there are at most L levels of recursion in total (not including the level containing the initial call). For a call at level $i \geq 0$, the set C^* constructed in line 5 has size at most $(L - i)^2$, since at most $L - i$ components can cover at least $(n - |D(Q)|)/(L - i)$ of the vertices in $V(\mathcal{G}) - D(Q)$ during each of the $L - i$ steps $t \in [L] - T(Q)$. Thus each call at level $i \geq 0$ makes at most $(L - i)^2$ recursive calls. The tree of recursive calls thus has at most $(L!)^2$ nodes at depth L, and hence $O((L!)^2)$ nodes in total. It follows that the overall number of calls is bounded by $O((L!)^2)$.

Next, note that if some level-i call $g(\mathcal{G}, s, Q)$ is such that $|Q| < L$ and $|D(Q)| < n$, then line 5 computes the set C', which can be achieved in $O(Ln)$ time by, for each $t \in [L]$, scanning over the components $C \in G_t$ (which collectively contain n vertices) and adding a component $C \in G_t$ to C' if and only if $|C - D(Q)| \geq (n - |D(Q)|)/(L - i)$. (Note that we can maintain a map from V to $\{0, 1\}$ that records for each vertex v whether $v \in D(Q)$, and hence the value $|C - D(Q)|$ can be computed in $O(|C|)$ time.) To compute the set C^* in line 6 we can follow a similar approach: for each $t \in [L] - T(Q)$ ($|[L] - T(Q)| = L - i$), add a component $C \in G_t$ to C^* if and only if it satisfies $C \in C'$. This requires $O((L - i)n) = O(Ln)$ time, and thus lines 5–6 take $O(Ln)$ time in total. Additionally, the return value of each recursive call is checked by the for-loop (line 9) of its parent call in $O(1)$ time – this contributes an extra $O((L!)^2)$ time over all recursive calls. On the other hand, if a call $g(\mathcal{G}, s, Q)$ is such that $|Q| = L$ or $|D(Q)| = n$, then line 2 computes $W^?_{\mathcal{G}}(s, Q)$ in $O(Ln)$ time using Lemma 32. Thus in all cases the overall work per recursive call is $O(Ln)$, and the total amount of time spent before $g(\mathcal{G}, s, \emptyset)$ is returned is $O((L!)^2) \cdot O(Ln) = O(L(L!)^2 n)$, as claimed. □

We remark that the algorithm of Theorem 34 can be adapted to k-fixed NS-TEXP in a straightforward way: If we are only interested in visiting the vertices in a given set X with $|X| = k$, an observation analogous to Observation 30 shows the existence of a component C that contains at least a $1/(L - |T(Q)|$ fraction of the unvisited vertices in X, i.e., $|(C - D(Q)) \cap X| \geq (k - |D(Q) \cap X|)/(L - |T(Q)|)$. In Algorithm 1, we only need to replace the condition $|D(Q)| = n$ in lines 1 and 2 by $|D(Q) \cap X| = k$, and the selection criterion for components in line 5 by $|(C - D(Q)) \cap X| \geq (k - |D(Q) \cap X|)/(L - |T(Q)|)$.

Corollary 35. *k-fixed* NS-TEXP *with parameter L is in* FPT.

4.4. W[2]-*hardness of* Set NS-TEXP *for parameter L*

Our aim in this section is to show that the Set NS-TEXP problem is W[2]-hard when parameterised by the lifetime L of the input graph. The reduction is from the well-known Set Cover problem with parameter k – the maximum number of sets allowed in a candidate solution. Set Cover is known to be W[2]-hard for this parameterisation [34].

Definition 36 *(Set Cover).* An instance of Set Cover is given as a tuple $(U, \mathcal{S}, k)$, where $U = \{a_1, \ldots, a_n\}$ is the ground set and $\mathcal{S} = \{S_1, \ldots, S_m\}$ is a set of subsets $S_i \subseteq U$. The problem then asks whether or not there exists a subset $\mathcal{S}' \subseteq \mathcal{S}$ of size at most k such that, for all $i \in [n]$, there exists an $S \in \mathcal{S}'$ such that $a_i \in S$.

For any instance I of Set Cover that we consider, we will w.l.o.g. assume that for each $i \in [n]$ we have $a_i \in S_j$ for some $j \in [m]$.

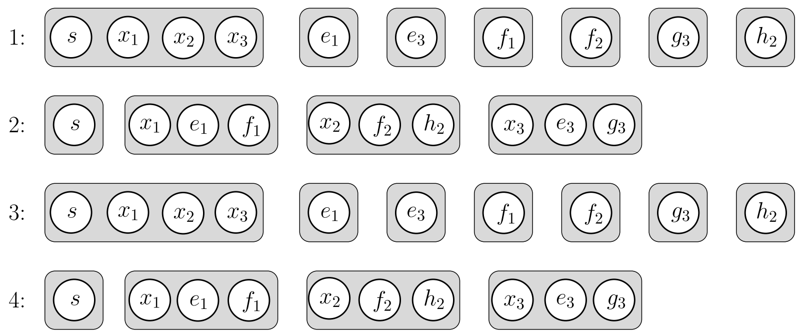

Fig. 2. Instance of Set NS-TEXP constructed from the instance of Set Cover with $k = 2$ given by $U = \{e, f, g, h\}$ and $\mathcal{S} = \{S_1, S_2, S_3\}$ with $S_1 = \{e, f\}$, $S_2 = \{f, h\}$, $S_3 = \{e, g\}$. The set $\mathcal{X}$ of vertex subsets that must be visited is $\{\{e_1, e_3\}, \{f_1, f_2\}, \{g_3\}, \{h_2\}\}$.

Theorem 37. Set NS-TEXP *parameterised by L (the lifetime of the input non-strict temporal graph) is* W[2]*-hard.*

Proof. Let $I = (U = \{a_1, \ldots, a_n\}, \mathcal{S} = \{S_1, \ldots, S_m\}, k)$ be an arbitrary instance of Set Cover parameterised by k. We construct a corresponding instance $I' = (\mathcal{G}, s, \mathcal{X})$ of Set NS-TEXP as follows: Let $V(\mathcal{G}) = \{s\} \cup \{x_j : j \in [m]\} \cup \{y_{i,j} : j \in [m], a_i \in S_j\}$, and define $X_i = \{y_{i,j} \in V(\mathcal{G}) : j \in [m]\}$ $(i \in [n])$ and $\mathcal{X} = \bigcup_{i \in [n]} \{X_i\}$. We set the lifetime L of $\mathcal{G}$ to $L = 2k$ and specify the components for each timestep $t \in [2k]$ as follows: In all odd steps let one component be $\{s\} \cup \{x_j : j \in [m]\}$ and let all other vertices belong to components of size 1. In even steps, for each $j \in [m]$ let there be a component $\{y_{i,j} \in V(\mathcal{G}) : i \in [n]\} \cup \{x_j\}$ and let s form a component of size 1. An example of the construction is shown in Fig. 2. (In the figure, for the sake of readability, the elements of U are denoted by e, f, g, h instead of a_1, a_2, a_3, a_4 and the elements of X_2 are denoted by f_2, h_2 instead of $y_{2,2}, y_{4,2}$, and similarly for X_1 and X_3.) Since $|V(\mathcal{G})| \leq 1 + m + mn = O(mn)$, $|\bigcup_{i \in [n]} X_i| = O(mn)$ and $L = 2k$ we have that the size of instance I' is $|I'| = O(kmn)$ and the parameter L is bounded solely by a function of instance I's parameter k, as required. To complete the proof, we argue that I is a yes-instance if and only if I' is a yes-instance:

($\Longrightarrow$) Assume that I is a yes-instance; then there exists a collection of sets $\mathcal{S}' \subseteq \mathcal{S}$ of size $|\mathcal{S}'| = k' \leq k$ and, for all $i \in [n]$, there exists $S \in \mathcal{S}'$ with $a_i \in S$. Let $S_{j_1}, S_{j_2}, \ldots, S_{j_{k'}}$ be an arbitrary ordering of the sets in $\mathcal{S}'$; note that $j_i \leq m$ for all $i \in [k']$. We construct a non-strict temporal walk W in $\mathcal{G}$ as follows: Starting at vertex s, for every $l \in [1, k']$, during timestep $t = 2l - 1$ visit all vertices in the current component then finish timestep $2l - 1$ positioned at x_{j_l}. The component occupied during step $2l$ will be the one containing x_{j_l} – explore all vertices contained in that component and finish step $2l$ positioned at x_{j_l}. If $k' < k$, then spend the steps of the interval $[2k' + 1, 2k]$ positioned in an arbitrary component. We claim that W visits at least one vertex in X_i for all $i \in [n]$. To see this, first note that for every $i \in [n]$ there exists an $S_j \in \mathcal{S}'$ such that $a_i \in S_j$. Hence, by our reduction, it follows that a vertex $y_{i,j}$ is contained in the component containing x_j during timestep $2l$ for every $l \in [k]$ and, by its construction, W visits the component containing x_j (and thus visits $y_{i,j} \in X_i$) during timestep $2l^*$ for some l^* such that $j_{l^*} = j$. Since this holds for all $i \in [n]$ it follows that W is a feasible solution and I' is a yes-instance.

($\Longleftarrow$) Assume that I' is a yes-instance and that we have some non-strict temporal walk W that visits at least one vertex in X_i for all $i \in [n]$. We first claim that W visits any vertex of the form $y_{i,j}$ for the first time during an even step. To see this, observe that every $y_{i,j}$ lies disconnected in its own component in every odd step t, and so to visit any $y_{i,j}$ in an odd step W would need to occupy the component containing $y_{i,j}$ during step $t - 1$ and finish step $t - 1$ positioned at $y_{i,j}$; hence $y_{i,j}$ was already visited in step $t - 1$, which is even. Therefore, in order for W to visit any $y_{i,j}$ it must be positioned, during at least one even step, at the component containing x_j. Now, to construct a collection of subsets $\mathcal{S}' \subseteq \mathcal{S}$ with size $x \leq k$, let $\mathcal{S}' = \{S_j : W$ visits the component containing x_j during some even timestep$\}$. To see that $\mathcal{S}'$ is a cover of U with size $x \leq k$, observe that W visits at least one vertex $y_{i,j}$ for every $i \in [n]$; thus, by the reduction, for every $i \in [n]$ the element a_i is contained in set S_j for some $S_j \in \mathcal{S}'$. It follows that the union of $\mathcal{S}'$'s elements covers U, and so I is a yes-instance. □

5. Conclusion

In this paper we have initiated the study of temporal exploration problems from the viewpoint of parameterised complexity. For both strict and non-strict temporal walks, we have shown several variants of the exploration problem to be in FPT. For the variant where we are given a family of vertex subsets and need to visit only one vertex from each subset, we have shown W[2]-hardness for both the strict and the non-strict model for parameter L. For non-strict temporal exploration, we have shown that the problem can be solved in polynomial time if γ, the maximum number of connected components per step, is bounded by 2. An interesting question for future work is to determine whether NS-TEXP with parameter γ is in FPT or at least in XP (i.e., admits a polynomial-time algorithm for each fixed value of γ). Another interesting question is whether k-arbitrary NS-TEXP is in FPT for parameter L.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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