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Spectral Gap for Weil–Petersson Random Surfaces with Cusps

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We show that for any $\varepsilon > 0$, $\alpha \in [0, \frac{1}{2})$, as $g \to \infty$ a generic finite-area genus g hyperbolic surface with $n = O(g^{\alpha})$ cusps, sampled with probability arising from the Weil–Petersson metric on moduli space, has no non-zero eigenvalue of the Laplacian below $\frac{1}{4} - \left(\frac{2\alpha+1}{4}\right)^2 - \varepsilon$. For $\alpha = 0$ this gives a spectral gap of size $\frac{3}{16} - \varepsilon$ and for any $\alpha < \frac{1}{2}$ gives a uniform spectral gap of explicit size.

1 Introduction

A hyperbolic surface is a smooth, connected, orientable Riemannian surface with constant Gaussian curvature -1. Let X be a finite-area non-compact hyperbolic surface. The $L^2(X)$ spectrum of the Laplacian Δ_X consists of the following:

- A simple eigenvalue at 0 and possibly finitely many eigenvalues in $\left(0, \frac{1}{4}\right)$.
- Absolutely continuous spectrum $\left[\frac{1}{4}, \infty\right)$ with multiplicity equal to the number of cusps of X.
- Possibly infinitely many discrete eigenvalues in $\left[\frac{1}{4}, \infty\right)$, embedded in the absolutely continuous spectrum.

Spectral gap refers to the gap between the zero eigenvalue and the remaining spectrum. The spectral gap is closely related to the connectivity of a surface and the rate of mixing of the geodesic flow. We are interested in the size of the spectral gap for a random

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surface with large genus. The random model we shall consider is the Weil–Petersson model, arising from the Weil–Petersson metric on moduli space, explained in Section 3.

Motivation for this paper arises from recent results for compact surfaces. In contrast to our setting, the spectrum of a compact hyperbolic surface Y consists of eigenvalues

$$0 = \lambda_0(Y) < \lambda_1(Y) \leqslant \cdots \leqslant \lambda_k(Y) \leqslant \ldots,$$

with $\lambda_j(Y) \to \infty$ as $j \to \infty$. The first spectral gap result for Weil–Petersson random compact surfaces was due to Mirzakhani in [17], who proved the following.

Theorem 1.1 (Mirzakhani '13). The Weil–Petersson probability that a genus g compact hyperbolic surface has a non-zero Laplacian eigenvalue below $\frac{1}{4}\left(\frac{\log(2)}{2\pi+\log(2)}\right)^2 \approx 0.0024$ tends to zero as $g \to \infty$.

Recently this result was improved, independently by Wu and Xue in [31] and Lipnowski and Wright in [14] to the following.

Theorem 1.2 (Wu–Xue and Lipnowski–Wright '21). For any $\varepsilon > 0$, the Weil–Petersson probability that a genus g compact hyperbolic surface has a non-zero Laplacian eigenvalue below $\frac{3}{16} - \varepsilon$ tends to zero as $g \to \infty$.

The purpose of this paper is to extend Theorem 1.2 to non-compact finite-area surfaces. We prove the following.

Theorem 1.3. For any $0 \leqslant \alpha < \frac{1}{2}$, if $n = O(g^{\alpha})$ then for any $\varepsilon > 0$ the Weil–Petersson probability that a genus g non-compact finite-area surface with n cusps has a non-zero Laplacian eigenvalue below $\frac{1}{4} - \left(\frac{2\alpha+1}{4}\right)^2 - \varepsilon$ tends to zero as $g \to \infty$.

When $\alpha = 0$, that is the number of cusps is bounded as $g \to \infty$, Theorem 1.3 returns a spectral gap of size $\frac{3}{16} - \varepsilon$ as in Theorem 1.2. For any $\alpha < \frac{1}{2}$, Theorem 1.3 gives an explicit positive uniform spectral gap.

The hypothesis $n = O(g^{\alpha})$ for $0 \leqslant \alpha < \frac{1}{2}$ has geometric consequences in terms of Benjamini–Schramm convergence. In [19, Corollary 4.4], Monk proved that with high probability, Weil–Petersson random surfaces with genus g and $n = O(g^{\alpha})$ cusps Benjamini–Schramm converge to the hyperbolic plane. The regime $n = O(g^{\alpha})$ with $0 \leqslant \alpha < \frac{1}{2}$ is studied by Le Masson and Sahlsten in [13] where they prove a quantum ergodicity result for eigenfunctions of the Laplacian.

Remark 1.4. Due to a recent work of Shen and Wu [29], the hypothesis $n = O(g^{\alpha})$ for $0 \leqslant \alpha < \frac{1}{2}$ cannot be relaxed much further. In particular, they prove that if n satisfies $n \gg g^{\frac{1}{2}+\beta}$ for some $\beta > 0$ then for any $\varepsilon > 0$, a Weil–Petersson random surface with genus g and n cusps has a non-zero eigenvalue below ε with probability tending to 1 as $g \to \infty$. They also prove the analogous result for g fixed and $n \to \infty$.

1.1 Other related works

The first spectral gap result for random surfaces was due to Brooks and Makover [2]. They considered a random closed surface formed by gluing $2n$ copies of an ideal hyperbolic triangle with gluing determined by a random trivalent ribbon graph and then applying a compactification procedure. They proved the existence of a non-explicit constant $C > 0$ such that the first non-zero eigenvalue is greater than C with probability tending to 1 as $n \to \infty$.

1.1.1 Spectral theory in the Weil–Petersson model

The work of Monk in [18] gives estimates on the density of Laplace eigenvalues below $\frac{1}{4}$ for Weil–Petersson random compact surfaces. In [5], Gilmore, Le Masson, Sahlsten, and Thomas obtain bounds for the L^p norms of Laplace eigenfunctions for Weil–Petersson random compact surfaces.

1.1.2 Random covers

In [20], Magee and Naud introduced a model of a random surface by picking a base surface X and considering random degree n covers X_n of X, sampled uniformly. Building on work from [23], in [22], Magee, Naud, and Puder prove that for X compact, X_n has no new eigenvalues of the Laplacian below $\frac{3}{16} - \epsilon$ with probability tending towards one as $n \to \infty$. Following an intermediate result [20], Magee and Naud prove in [21] that for X conformally compact, X_n has no new resonances in any compact set $\mathcal{K} \subset \{s \mid \mathrm{Re}(s) > \frac{\delta}{2}\}$ with probability tending to 1 as $n \to \infty$, where δ is the Hausdorff dimension of the limit set of Γ_X. In contrast to our setting, a conformally compact hyperbolic surface has infinite area and no cusps.

1.1.3 Selberg's eigenvalue conjecture

Spectral theory of the Laplacian on arithmetic hyperbolic surfaces has important consequences in Number Theory; see, for example, [27]. Let $N \geqslant 1$, the principal congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of level N is

$$\Gamma(N) = \left\{T \in \mathrm{SL}_2(\mathbb{Z}) \mid T \equiv I \mod N\right\}.$$

Consider the quotient $X(N) \stackrel{\text{def}}{=} \Gamma(N)\backslash\mathbb{H}$. For $N > 2$, the quotient $X(N)$ is a finite-area non-compact hyperbolic surface with the number of cusps $n(N) > 0$ given by

$$n(N) = \frac{N^2}{2}\prod_{p|N}\left(1 - p^{-2}\right),$$

and genus

$$g(N) = 1 + \frac{(N-6)\,N^2}{24}\prod_{p|N}\left(1 - p^{-2}\right),$$

by, for example, [1, Theorem 2.12]. Letting $\lambda_1(X(N))$ denote the first non-zero eigenvalue of the Laplacian on $X(N)$, in [28] Selberg made the following conjecture.

Conjecture 1.5. For every $N \geqslant 1$,

$$\lambda_1(X(N)) \geqslant \frac{1}{4}.$$

Conjecture 1.5 remains open; however, there have been a number of results in this direction. Selberg proved in [28] that Conjecture 1.5 holds with the bound $\frac{3}{16}$. After many intermediate results [6–8, 11, 12, 26], the best known result is the following due to Kim and Sarnak [10].

Theorem 1.6 (Kim-Sarnak '03). For every $N \geqslant 1$,

$$\lambda_1(X(N)) \geqslant \frac{975}{4096}.$$

In light of this, it would be interesting to know if Theorem 1.3 can be extended to the case that the number of cusps satisfies $n \sim g^{\frac{2}{3}}$.

Question 1.7. Does a Weil–Petersson random surface with genus g and $n \sim g^{\frac{2}{3}}$ cusps have a uniform positive spectral gap as $g \to \infty$?

Remark 1.8. Since the preprint version of the current paper first appeared in July 2021, Question 1.7 has been answered in the negative by Shen and Wu [29]; c.f. Remark 1.4.

1.2 Structure of the paper

In the compact case, both proofs of Theorem 1.2, in [31] and [14], rely on Selberg's trace formula, for example, [3, 9.5.3], to relate the Laplacian eigenvalues of a surface to its length spectrum. In the non-compact finite-area setting, there is a version of Selberg's trace formula, for example, [9, Theorem 10.2], but it is more complicated with additional terms related to the absolutely continuous spectrum. It is not clear to the author how to control these additional terms. To get around this, in Section 2 we prove that if a surface $X \in \mathcal{M}_{g,n}$ has $\lambda_1(X) \leqslant \frac{3}{16}$, then $\lambda_1(X)$ satisfies an inequality (Theorem 2.1) involving the set of oriented primitive closed geodesics $\mathcal{P}(X)$, which closely resembles the form of Selberg's trace formula for compact surfaces, up to well-behaved error terms depending only the topology of the surface. Roughly, we prove that there are strictly positive functions R and f such that

$$R\left(\lambda_1(X), g, n\right) \leqslant \sum_{\gamma \in \mathcal{P}(X)} \sum_{k=1}^{\infty} \frac{l_\gamma(X)}{2\sinh\left(\frac{kl_\gamma(x)}{2}\right)} f\left(kl_\gamma(X)\right), \tag{1.1}$$

where $l_\gamma(X)$ is the length of the geodesic $\gamma \in \mathcal{P}(X)$. The proof of Theorem 2.1 relies on results from [4]. The function R is large for small $\lambda_1(X)$ and bounding the Weil–Petersson expectation of the right-hand side of (1.1) will allow us to conclude Theorem 1.3 through Markov's inequality.

After we have established Theorem 2.1, we can proceed as in the compact case, making the necessary adaptations along the way. Section 3 introduces the necessary geometric background on moduli space, the Weil–Petersson model, and Mirzakhani's integration formula. Then in Section 4 we bound the Weil–Petersson expectation of the right-hand side of (1.1), closely following the approach taken in [31]. Finally, in Section 5 we apply Markov's inequality to bound the probability that $X \in \mathcal{M}_{g,n}$ has a small eigenvalue to conclude the proof of Theorem 1.3.

In order to deduce Theorem 1.3, we need to be able to estimate expressions involving the Weil–Petersson volumes $V_{g,n}$ where n grows with g, which is the focus of the Appendix A.

1.3 Notation

For real valued functions f, h depending on a parameter g we write $f \ll h$ or $f = O(h)$ if there exists $C, G > 0$ such that $|f(g)| \leqslant Ch(g)$ for all $g > G$. We add subscripts to the $\ll$ sign if the constant C, G depend on another variable. For example, we write $f \ll_\epsilon h$ if exists $C = C(\varepsilon), G = G(\varepsilon)$ such that $|f(g)| \leqslant Ch(g)$ for all $g > G$. We write $f \sim h$ if $f \ll h$ and $h \ll f$. We write $(\underline{0}_j, a_1, \ldots, a_k)$ to denote $(0, \ldots, 0, a_1, \ldots, a_k) \in \mathbb{R}^{j+k}$ and we write $\mathbb{R}_{\geqslant 0}$ (resp. $\mathbb{Z}_{\geqslant 0}$) to denote the non-negative real numbers (resp. integers).

2 Analytic Preparations

In this section we develop the necessary analytic machinery to prove Theorem 1.3. We prove a version of Selberg's trace formula, using a pre-trace inequality in place of the usual pre-trace formula.

In Section 2.3 we exhibit a family of test functions f_T where $T = 4 \log g$, and f_T is a non-negative, even, smooth function with support exactly $(-T, T)$ whose Fourier transform $\hat{f}_T$ is non-negative on $\mathbb{R} \cup i\mathbb{R}$ with $\hat{f}_T\left(\frac{i}{2}\right) = O\left(g^2\right)$. The family of test functions f_T is defined by (2.2) with $T = 4 \log g$.

The goal of this section is to prove the following.

Theorem 2.1. For $g \geqslant 2$, let f_T be the test function defined by (2.2) with $T = 4 \log g$. For any $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that for any non-compact finite-area surface X with genus g, $n = o\left(g^{\frac{1}{2}}\right)$ cusps and $\lambda_1(X) \leqslant \frac{3}{16}$,

$$C(\varepsilon) \log(g)\, g^{4(1-\varepsilon)\sqrt{\frac{1}{4}-\lambda_1(X)}} \leqslant \sum_{\gamma \in \mathcal{P}(X)} \sum_{k=1}^{\infty} \frac{l_\gamma(X)}{2 \sinh\left(\frac{k l_\gamma(x)}{2}\right)} f_T\left(k l_\gamma(X)\right) - \hat{f}_T\left(\frac{i}{2}\right) + O(ng). \tag{2.1}$$

The left-hand side depends on $\lambda_1(X)$ and the right-hand side depends on the the length spectrum of X.

Remark 2.2. Given $\kappa > 0$, we could have stated Theorem 2.1 with the hypothesis $\lambda_1(X) \leqslant \frac{1}{4} - \kappa$, (the statement is almost the same except the constant $C(\varepsilon)$ will also depend on κ); however, our geometric estimates (Section 4) are not strong enough to prove a spectral gap larger than $\frac{3}{16}$. We therefore state Theorem 2.1 with the hypothesis $\lambda_1(X) \leqslant \frac{3}{16}$ to simplify notation.

2.1 The Laplacian on hyperbolic surfaces

Consider the upper half plane

$$\mathbb{H} = \{x + iy \mid x, y \in \mathbb{R}, y > 0\},$$

with metric given by

$$\frac{dx^2 + dy^2}{y^2}.$$

The orientation preserving isometry group of $\mathbb{H}$ is $\mathrm{PSL}_2(\mathbb{R})$, acting via Möbius transformations. The Laplacian on $\mathbb{H}$, denoted $\Delta_{\mathbb{H}}$, is given by

$$\Delta_{\mathbb{H}} = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

A non-compact finite-area hyperbolic surface can be realized as a quotient $\Gamma_X \backslash \mathbb{H}$ where Γ_X is a finitely generated discrete free subgroup of $\mathrm{PSL}_2(\mathbb{R})$, containing parabolic elements (elements with trace ± 2). $\Delta_{\mathbb{H}}$ is invariant under the action of $\mathrm{PSL}_2(\mathbb{R})$ and descends to an operator on $C_c^\infty(X)$. It extends uniquely to a non-negative unbounded self-adjoint operator on $L^2(X)$. We let Δ_X denote the Laplacian on X and write $\mathrm{spec}\left(\Delta_X\right)$ for the spectrum of Δ_X. We write $\lambda_j(X)$ to denote the jth smallest non-zero eigenvalue of Δ_X if it exists.

A parabolic cylinder is the quotient of $\mathbb{H}$ by a parabolic cyclic group. We define a cusp to be the small end of a parabolic cylinder, with boundary the unique closed horocycle of length 1. By [3, Lemma 4.4.6], in any finite-area hyperbolic surface, cusps must be pairwise disjoint. Throughout Section 2 we let $X = \Gamma_X \backslash \mathbb{H}$ be a fixed non-compact finite-area hyperbolic surface with genus g and $n = o\left(g^{\frac{1}{2}}\right)$ cusps and, for the sake of argument, $\lambda_1(X) \leqslant \frac{3}{16}$.

2.2 Fundamental domains

In this subsection we introduce a decomposition of the fundamental domain, which we will need in the proof of Theorem 2.1. We shall closely follow [9, Section 2.2], and refer the reader there for all of the notions introduced in this subsection.

We write $\mathcal{F}$ to denote a Dirichlet fundamental domain for Γ_X. Since $\mathcal{F}$ is a non-compact polygon, it has some of its vertices on $\mathbb{R} \cup \infty$ in $\mathbb{H} \cup \partial\mathbb{H}$. We call such a vertex a cuspidal vertex. By, for example, [9, Proposition 2.4], we can ensure that the cuspidal vertices are distinct modulo Γ_X. The sides of $\mathcal{F}$ can be arranged in pairs so that the side pairing motions generate Γ_X. The two sides of $\mathcal{F}$ meeting at a cuspidal vertex have to be pairs since the cuspidal vertices are distinct modulo Γ_X. The side-pairing motion has to fix the vertex and is therefore a parabolic element of Γ_X. This gives rise to a cusp in the quotient $\Gamma_X\backslash\mathbb{H}$ and each cuspidal vertex corresponds to a unique cusp in this way. We label the cuspidal vertices by $\mathfrak{a}_1, ..., \mathfrak{a}_n$. We denote the stabilizer subgroup of the vertex $\mathfrak{a}_i$ by

$$\Gamma_{\mathfrak{a}_i} \stackrel{\text{def}}{=} \{\gamma \in \Gamma_X \mid \gamma\mathfrak{a}_i = \mathfrak{a}_i\}.$$

Each $\Gamma_{\mathfrak{a}_i}$ is an infinite cyclic group generated by the parabolic element $\gamma_{\mathfrak{a}_i}$, which is the side-pairing motion at the vertex $\mathfrak{a}_i$. There exists $\sigma_{\mathfrak{a}_i} \in \mathrm{SL}_2(\mathbb{R})$ such that

$$\sigma_{\mathfrak{a}_i}^{-1}\gamma_{\mathfrak{a}_i}\sigma_{\mathfrak{a}_i} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

$\sigma_{\mathfrak{a}_i}$ is determined up to right multiplication by a translation. We choose $\sigma_{\mathfrak{a}_i}$ so that for each $l \geqslant 1$, the semi-strip

$$P(l) \stackrel{\text{def}}{=} \{z \in \mathbb{H} \mid 0 < x < 1, y \geqslant l\},$$

is mapped into $\mathcal{F}$ by $\sigma_{\mathfrak{a}_i}$.

Definition 2.3. For $i = 1, \ldots, n$ and $l \geqslant 1$, we define

$$D_{\mathfrak{a}_i}(l) \stackrel{\text{def}}{=} \sigma_{\mathfrak{a}_i} P(l),$$

and

$$D(l) \stackrel{\text{def}}{=} \mathcal{F} \backslash \bigsqcup_{i=1}^{n} D_{\mathfrak{a}_i}(l).$$

$D_{\mathfrak{a}_i}(l)$ is the part of the fundamental domain in the ith cusp bounded below by the length $\frac{1}{l}$ horocycle and $D(l)$ is a pre-compact region of $\mathcal{F}$. By, for example, [3, Lemma 4.4.6], the cusps $D_{\mathfrak{a}_i}(1)$ are pairwise disjoint and since $l \geqslant 1$, $D_{\mathfrak{a}_i}(l) \cap D_{\mathfrak{a}_j}(l) = \emptyset$ for $i \neq j$ and we can partition the fundamental domain as

$$\mathcal{F} = D(l) \sqcup \bigsqcup_{i=1}^{n} D_{\mathfrak{a}_i}(l).$$

2.3 Test functions

In this subsection we introduce the family of test functions used in Theorem 2.1.

Proposition 2.4. There exists an $f_1 \in C_c^\infty(\mathbb{R})$ with

1. $\mathrm{Supp}(f_1) = (-1, 1)$.
2. f_1 is non-negative and even.
3. The Fourier transform $\hat{f}_1$ satisfies $\hat{f}_1(\xi) \geqslant 0$ for $\xi \in \mathbb{R} \cup i\mathbb{R}$.
4. f_1 is non-increasing in $[0, 1)$.

Proposition 2.4 is based on [22, Section 2.2], with the extra condition (4) for convenience later on.

Proof of Proposition 2.4. Let ψ_0 be an even, C^∞, real valued non-negative function whose support is exactly $(-\frac{1}{2}, \frac{1}{2})$, which is non-increasing in $[0, \frac{1}{2})$. Let

$$f_1(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \psi_0(x-t)\psi_0(t)\mathrm{d}t.$$

It is proved in [22, Section 2.2] that f_1 satisfies (1) – (3). It remains to check (4). Since f_1 is even we have $f_1'(0) = 0$. If $0 < x < \frac{1}{2}$, one can calculate that

$$f_1'(x) = \int_0^{\frac{1}{2}-x} \left(\psi_0(x-z) - \psi_0(x+z)\right) \psi_0'(z)\mathrm{d}z + \int_{\frac{1}{2}-x}^{\frac{1}{2}} \psi_0(x-z)\psi_0'(z)\mathrm{d}z.$$

Since ψ_0 is positive, even and non-increasing in $[0, \frac{1}{2})$, we have $\psi_0'(z) \leqslant 0$ and $\psi_0(x-z) - \psi_0(x+z) \geqslant 0$ for all $0 \leqslant z \leqslant \frac{1}{2} - x$, so the first integrand is non-positive. The second integrand is also non-positive since ψ_0 is non-negative. Therefore, $f_1'(x) \leqslant 0$ in $[0, \frac{1}{2})$. If $\frac{1}{2} \leqslant x < 1$, then

$$f_1'(x) = \int_{x-\frac{1}{2}}^{\frac{1}{2}} \psi_0'(t)\psi_0(x-t)\mathrm{d}t \leqslant 0,$$

and f_1 is non-increasing in $[0, 1)$. ■

From here on in, we fix such a function f_1. For $T > 1$ we define

$$f_T(x) \stackrel{\text{def}}{=} f_1\left(\frac{x}{T}\right). \tag{2.2}$$

Then by Proposition 2.4, for each $T > 1$, f_T is a non-negative, even, smooth function with support exactly $(-T, T)$ whose Fourier transform $\hat{f}_T$ is non-negative on $\mathbb{R} \cup i\mathbb{R}$. We also have that f_T is non-increasing in $[0, T)$.

Let k_T denote the inverse Abel transform of f_T, that is,

$$k_T(\rho) \stackrel{\text{def}}{=} \frac{-1}{\sqrt{2}\pi} \int_\rho^\infty \frac{f_T'(u)}{\sqrt{\cosh u - \cosh \rho}} \mathrm{d}u, \tag{2.3}$$

which is well defined since f_T is compactly supported. We see that k_T is smooth, $\operatorname{Supp}(k_T) \subseteq [0, T)$ and since f_T is non-increasing in $[0, T)$, k_T is non-negative.

We now have a fixed family of test functions f_T for $T > 1$. We conclude this subsection by stating a lower bound on $\hat{f}_T$ in $i\mathbb{R}$ from [22].

Lemma 2.5 ([22, Lemma 2.4]). For any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that for all $t \in \mathbb{R}_{\geqslant 0}$ and for all $T > 1$ the Fourier transform $\hat{f}_T$ satisfies

$$\hat{f}_T(it) \geqslant TC_\varepsilon e^{T(1-\varepsilon)t}. \tag{2.4}$$

Lemma 2.4 in [22] applies for any function satisfying properties $(1) - (3)$ from Proposition 2.4 so it also applies here. Lemma 2.5 tells us that small values of λ_1 imply large values of $\hat{f}_T\left(i\sqrt{\frac{1}{4} - \lambda_1}\right)$.

2.4 Eigenfunction estimates

Now we have a family of test functions, we proceed with the proof of Theorem 2.1. For $z, w \in \mathbb{H}$, $T > 1$ we define

$$k_T(z, w) \stackrel{\text{def}}{=} k_T(d(z, w)).$$

Let $r : [0, \infty) \to \mathbb{C}$ be the function given by

$$r(x) = \begin{cases} i\sqrt{\frac{1}{4} - x} & \text{if } 0 \leqslant x \leqslant \frac{1}{4}, \\ \sqrt{x - \frac{1}{4}} & \text{if } x > \frac{1}{4}. \end{cases}$$

Let $u_j \in L^2(X)$ denote the normalized eigenfunction of the Laplacian on X corresponding to the eigenvalue λ_j. Our starting point is the following.

Lemma 2.6 (Pre-trace inequality [4, Proposition 5.2]). For all $T > 1$ and $z \in \mathbb{H}$ we have that

$$\sum_{j:\lambda_j<\frac{1}{4}} \hat{f}_T\left(r\left(\lambda_j\right)\right)|u_j(z)|^2 \leqslant \sum_{\gamma\in\Gamma_X} k_T\left(z,\gamma z\right). \tag{2.5}$$

Lemma 2.6 is immediately deduced from [4, Proposition 5.2], using the fact that $\hat{f}_T$ is non-negative on $\mathbb{R} \cup i[0, \frac{1}{2}]$ (the image of $[0,\infty)$ under r). We refer to the left-hand side of (2.5) as the spectral side and the right-hand side as the geometric side. We prove Theorem 2.1 by integrating (2.5). We cannot integrate (2.5) over the full fundamental domain as the contribution of the parabolic elements

$$\sum_{\{\gamma\in\Gamma_X\setminus\{\mathrm{Id}\}||\mathrm{tr}(\gamma)|=2\}} k_T\left(z,\gamma z\right),$$

is not absolutely integrable over the fundamental domain $\mathcal{F}$. We get around this by integrating over the region $D(l)$, as defined in Definition 2.3, with $l = 2$ (the choice $l = 2$ could be replaced by any fixed $l > 1$). This leads to another issue: we could potentially lose information on the spectral side after integrating. This could happen if an eigenfunction concentrated outside $D(2)$. The following lemma resolves this issue. From now on we write $D = D(2)$.

Lemma 2.7 ([4, Lemma 4.1]). For any $\kappa > 0$, there is a constant $c(\kappa) > 0$ such that for any u_j with $\lambda_j \leqslant \frac{1}{4} - \kappa$, we have

$$\int_D |u_j(z)|^2 \mathrm{d}\mu(z) \geqslant c(\kappa).$$

The constant c does not depend on the surface X.

The upshot is that when we integrate (2.5) over D, we obtain something bounded on the geometric side and we get a definite contribution from each eigenvalue on the spectral side.

Remark 2.8. [4, Lemma 4.1] is stated for quotients of $\mathbb{H}$ by geometrically finite subgroups of $\mathrm{SL}_2(\mathbb{Z})$. The proof extends trivially to all finite-area non-compact surfaces, as noted in [4, Footnote 10].

2.5 Proof of Theorem 2.1

We conclude this section by proving Theorem 2.1.

Proof of Theorem 2.1. Recall that X is a finite-area non-compact hyperbolic surface with genus g, $n = o\left(g^{\frac{1}{2}}\right)$ cusps. We write $\lambda_j = \lambda_j(X)$ and recall that X has first non-zero Laplacian eigenvalue $\lambda_1 \leqslant \frac{3}{16}$. Let $T = 4\log g$. By Lemma 2.6,

$$\sum_{j:\lambda_j<\frac{1}{4}} \hat{f}_T\left(r\left(\lambda_j\right)\right)|u_j(z)|^2 \leqslant \sum_{\gamma\in\Gamma_X} k_T(z,\gamma z). \tag{2.6}$$

Since $\hat{f}_T$ is non-negative on $i\mathbb{R}$, $\hat{f}_T \circ r$ is non-negative on $[0, \frac{1}{4}]$ and (2.6) still holds if we reduce the sum to just λ_0 and λ_1. Integrating (2.6) over D, we get

$$\hat{f}_T\left(r\left(\lambda_0\right)\right)\int_D |u_0(z)|^2 \mathrm{d}\mu(z) + \hat{f}_T\left(r\left(\lambda_1\right)\right)\int_D |u_1(z)|^2 \mathrm{d}\mu(z) \leqslant \int_D \sum_{\gamma\in\Gamma_X} k_T(z,\gamma z)\,\mathrm{d}\mu(z). \tag{2.7}$$

First we look at the spectral side. The eigenvalue $\lambda_0 = 0$ corresponds to the constant eigenfunction

$$u_0(z) = \frac{1}{\sqrt{\mathrm{Vol}(X)}}.$$

We have

$$\hat{f}_T\left(r\left(\lambda_0\right)\right)\int_D |u_0(z)|^2 \mathrm{d}\mu(z) = \frac{\mathrm{Vol}(D)}{\mathrm{Vol}(X)}\hat{f}_T\left(\frac{i}{2}\right).$$

Recall that

$$D = \mathcal{F}\setminus\bigsqcup_{i=1}^{n} D_{\mathfrak{a}_i}(2).$$

Since $D_{\mathfrak{a}_i}(2)$ is isometric to $\{z \in \mathbb{H} \mid 0 < x < 1, y \geqslant 2\}$, $\mathrm{Vol}\left(D_{\mathfrak{a}_i}(2)\right) = \frac{1}{2}$ for each i. By Gauss–Bonnet, $\mathrm{Vol}(X) = 2\pi(2g-2+n)$ and we see that

$$\frac{\mathrm{Vol}(D)}{\mathrm{Vol}(X)} = \frac{2\pi(2g-2+n)-\frac{n}{2}}{2\pi(2g-2+n)} = 1 + O\left(\frac{n}{g}\right).$$

For the contribution of λ_1, by Lemma 2.7 with $\kappa = \frac{1}{16}$, there is a constant $c > 0$ with

$$\hat{f}_T\left(r\left(\lambda_1\right)\right)\int_D |u_1(z)|^2 \mathrm{d}\mu(z) \geqslant c\hat{f}_T\left(r\left(\lambda_1\right)\right). \tag{2.8}$$

Let $\varepsilon > 0$ be given, then since $\lambda_1 \leqslant \frac{3}{16}$, $r(\lambda_1) = i\sqrt{\frac{1}{4}-\lambda_1}$, then by Lemma 2.5, there is a constant $C_\varepsilon > 0$ with

$$\hat{f}_T\left(r\left(\lambda_1\right)\right) \geqslant TC_\varepsilon e^{T(1-\varepsilon)\sqrt{\frac{1}{4}-\lambda_1}}. \tag{2.9}$$

Combining (2.7), (2.8), and (2.9), we see there exists a constant $C(\varepsilon) > 0$ with

$$TC(\varepsilon)e^{T(1-\varepsilon)\sqrt{\frac{1}{4}-\lambda_1}} + \left(1 + O\left(\frac{n}{g}\right)\right)\hat{f}_T\left(\frac{i}{2}\right) \leqslant \int_D \sum_{\gamma\in\Gamma_X} k_T\left(z,\gamma z\right)\mathrm{d}\mu(z). \tag{2.10}$$

We now look at the geometric side. We arrange the sum in the geometric side into the contribution from the identity, parabolic and hyperbolic elements to obtain

$$\begin{aligned}\int_D \sum_{\gamma\in\Gamma_X} k_T\left(z,\gamma z\right)\mathrm{d}\mu(z) &= \sum_{\gamma\in\Gamma_X}\int_D k_T\left(z,\gamma z\right)\mathrm{d}\mu(z)\\ &= \int_D k_T\left(z,z\right)\mathrm{d}\mu(z) + \sum_{\{\gamma\in\Gamma_X \mid |\mathrm{tr}(\gamma)|>2\}}\int_D k_T\left(z,\gamma z\right)\mathrm{d}\mu(z)\\ &\quad + \sum_{\{\gamma\in\Gamma_X\setminus\{\mathrm{Id}\} \mid |\mathrm{tr}(\gamma)|=2\}}\int_D k_T\left(z,\gamma z\right)\mathrm{d}\mu(z).\end{aligned}$$

Interchanging summation and integration is justified since D is a compact region and k_T is supported in $[0, T)$, then for each $z \in D$, $\#\{\gamma \in \Gamma_X \mid d(z,\gamma z) < T\}$ is finite and the summation is over finitely many terms.

First we treat the contribution of the identity. Since $k_T(z,w) = k_T\left(d(z,w)\right)$,

$$\int_D k_T\left(z,z\right)\mathrm{d}\mu(z) = \mathrm{Vol}(D)k_T(0).$$

A calculation involving the Abel Transform, see for example the proof of [3, Theorem 9.5.3], gives that

$$k_T(0) = \frac{1}{4\pi}\int_{-\infty}^{\infty} r\hat{f}_T(r)\tanh(\pi r)\mathrm{d}r.$$

We calculate

$$\int_{-\infty}^{\infty} r\hat{f}_T(r)\tanh(\pi r)\mathrm{d}r = T\int_{-\infty}^{\infty} r\hat{f}_1\left(Tr\right)\tanh(\pi r)\mathrm{d}r$$
$$= \frac{1}{T}\int_{-\infty}^{\infty} r'\hat{f}_1\left(r'\right)\tanh\left(\frac{\pi r'}{T}\right)\mathrm{d}r'$$
$$\leqslant \frac{2}{T}\int_{0}^{\infty} r'\hat{f}_1\left(r'\right)\mathrm{d}r' \ll \frac{1}{T},$$

where the last line follows from the fact that f_1 is compactly supported, thus $\hat{f}_1$ is a Schwartz function and decays faster that the inverse of any polynomial. Since $\mathrm{Vol}(D) = 2\pi\left(2g-2+n\right) - \frac{n}{2}$, and X has $o\left(g^{\frac{1}{2}}\right)$ cusps, this tells us that

$$\int_D k_T\left(z,z\right)\mathrm{d}\mu(z) = O\left(g\right). \tag{2.11}$$

Now we look at the hyperbolic terms. By the non-negativity of k_T,

$$\sum_{\{\gamma\in\Gamma_X \mid |\mathrm{tr}(\gamma)|>2\}} \int_D k_T\left(z,\gamma z\right)\mathrm{d}\mu(z) \leqslant \sum_{\{\gamma\in\Gamma_X \mid |\mathrm{tr}(\gamma)|>2\}} \int_{\mathcal{F}} k_T\left(z,\gamma z\right)\mathrm{d}\mu(z).$$

By arranging the sum into conjugacy classes and unfolding the integral, one can compute that

$$\sum_{\{\gamma\in\Gamma_X \mid |\mathrm{tr}(\gamma)|>2\}} \int_{\mathcal{F}} k_T\left(z,\gamma z\right)\mathrm{d}\mu(z) = \sum_{\gamma\in\mathcal{P}(X)}\sum_{k=1}^{\infty}\frac{l_\gamma\left(X\right)}{2\sinh\left(\frac{kl_\gamma(X)}{2}\right)} f\left(kl_\gamma\left(X\right)\right). \tag{2.12}$$

This computation is carried out in detail in [9, Section 10.2].

It remains to bound the contribution of the parabolic elements. Any $\gamma\in\Gamma_X\backslash\{\mathrm{Id}\}$ with $|\mathrm{tr}(\gamma)| = 2$ is conjugate to $\gamma_{\mathfrak{a}_i}^l$ for some unique pair $i\in\{1,\ldots,n\}$ and $l\in\mathbb{Z}\backslash\{0\}$. Since the centralizer of $\gamma_{\mathfrak{a}_i}^l$ in Γ_X is $\Gamma_{\mathfrak{a}_i}$, we see

$$\sum_{\{\gamma\in\Gamma_X\backslash\{\mathrm{Id}\} \mid |\mathrm{tr}(\gamma)|=2\}} \int_D k_T\left(z,\gamma z\right)\mathrm{d}\mu(z) = \sum_{i=1}^{n}\sum_{l\in\mathbb{Z}^*}\sum_{\tau\in\Gamma_{\mathfrak{a}_i}\backslash\Gamma} \int_D k_T\left(z,\tau^{-1}\gamma_{\mathfrak{a}_i}^l\tau z\right)\mathrm{d}\mu(z).$$

Since k_T and $d\mu$ are invariant under isometries, by unfolding the integral, denoting $\Gamma \cdot D \overset{\text{def}}{=} \cup_{\gamma \in \Gamma} \gamma D$, we calculate

$$\sum_{\tau \in \Gamma_{\mathfrak{a}_i} \backslash \Gamma} \int_D k_T\left(z, \tau^{-1} \gamma_{\mathfrak{a}_i}^l \tau z\right) \mathrm{d}\mu(z) = \int_{\Gamma_{\mathfrak{a}_i} \backslash \Gamma \cdot D} k_T\left(z, \gamma_{\mathfrak{a}_i}^l z\right) \mathrm{d}\mu(z).$$

We can choose a fundamental domain $\tilde{\mathcal{F}}_i$ for the action of $\Gamma_{\mathfrak{a}_i}$ on $\Gamma \cdot D$ so that

$$\tilde{\mathcal{F}}_i \subseteq \sigma_{\mathfrak{a}_i}\{z \in \mathbb{H} \mid 0 < x \leqslant 1, 0 < y \leqslant 2\},$$

and we see, recalling that $\sigma_{\mathfrak{a}_i}^{-1} \gamma_{\mathfrak{a}_i} \sigma_{\mathfrak{a}_i}(z) = z + 1$,

$$\begin{aligned}
\sum_{\tau \in \Gamma_{\mathfrak{a}_i} \backslash \Gamma} \int_D k_T(z, \tau^{-1} \gamma_{\mathfrak{a}_i}^l \tau z) \mathrm{d}\mu(z) &= \int_{\tilde{\mathcal{F}}_i} k_T(z, \gamma_{\mathfrak{a}_i}^l z) \mathrm{d}\mu(z) \\
&= \int_{\sigma_{\mathfrak{a}_i}^{-1}\left(\tilde{\mathcal{F}}_i\right)} k_T(z, z + l) \mathrm{d}\mu(z) \\
&\leqslant \int_{x=0}^{x=1} \int_{y=0}^{y=2} k_T(z, z + l) \mathrm{d}\mu(z).
\end{aligned}$$

We sum over the parabolic conjugacy classes to calculate,

$$\begin{aligned}
\sum_{\{\gamma \in \Gamma_X \backslash \{\mathrm{Id}\} | |\mathrm{tr}(\gamma)| = 2\}} \int_D k_T(z, \gamma z) \mathrm{d}\mu(z) &\leqslant n \sum_{l \in \mathbb{Z}^*} \int_0^1 \int_0^2 k_T(z, z + l) \mathrm{d}\mu(z) \\
&= n \sum_{l \in \mathbb{Z}^*} \int_0^2 k_T\left(\operatorname{arcosh}\left(1 + \frac{l^2}{2y^2}\right)\right) y^{-2} \mathrm{d}y \\
&= n \sum_{l \in \mathbb{N}} \frac{\sqrt{2}}{l} \int_{\min\left\{\operatorname{arcosh}\left(1 + \frac{l^2}{8}\right), T\right\}}^T \frac{k_T(\rho) \sinh(\rho)}{\sqrt{\cosh(\rho) - 1}} \mathrm{d}\rho. \qquad (2.13)
\end{aligned}$$

On the second line we used that $\cosh d(z, z + l) = 1 + \frac{l^2}{2y^2}$ and on the third line we used the change of variables $\rho = \operatorname{arcosh}\left(1 + \frac{l^2}{2y^2}\right)$ and that $\operatorname{Supp}\left(k_T\right) \subseteq [0, T)$. When $\operatorname{arcosh}\left(1 + \frac{l^2}{8}\right) \leqslant T$, we use that f_T is the Abel transform of k_T to see that

$$\int_{\min\left\{\operatorname{arcosh}\left(1 + \frac{l^2}{8}\right), T\right\}}^T \frac{k_T(\rho) \sinh(\rho)}{\sqrt{\cosh(\rho) - 1}} \mathrm{d}\rho \leqslant \int_0^T \frac{k_T(\rho) \sinh(\rho)}{\sqrt{\cosh(\rho) - 1}} \mathrm{d}\rho = f_T(0) = f_1(0).$$

If $\operatorname{arcosh}\left(1+\frac{l^2}{8}\right) \leqslant T$ then the contribution to the sum (2.13) is 0 and we conclude that

$$\sum_{\{\gamma\in\Gamma_X\setminus\{\mathrm{Id}\}||\mathrm{tr}(\gamma)|=2\}} \int_D k_T(z,\gamma z)\mathrm{d}\mu(z) \leqslant 2nf_1(0) \sum_{l=1}^{\lfloor\sqrt{8\cosh T}\rfloor} \frac{1}{l} \leqslant 2nf_1(0)\log\left(2\sqrt{2}e^{\frac{T}{2}}\right).$$

Thus, combining (2.10), (2.11), (2.12), and (2), we conclude that

$$TC(\varepsilon)e^{T(1-\varepsilon)\sqrt{\frac{1}{4}-\lambda_1}} + \left(1+O\left(\frac{n}{g}\right)\right)\hat{f}_T\left(\frac{i}{2}\right)$$
$$\leqslant \sum_{\gamma\in\mathcal{P}(X)}\sum_{k=1}^{\infty} \frac{l_\gamma(X)}{2\sinh\left(\frac{kl_\gamma(X)}{2}\right)} f_T\left(kl_\gamma(X)\right) + 2nf_1(0)\log\left(2\sqrt{2}e^{\frac{T}{2}}\right) + O(g).$$

Recalling that $T = 4\log g$, since f_T is even,

$$\hat{f}_T\left(\frac{i}{2}\right) = \int_0^\infty 2\cosh\left(\frac{x}{2}\right) f_T(x)\mathrm{d}x = O\left(g^2\right),$$

and we deduce that

$$C(\varepsilon)\log(g)\, g^{4(1-\varepsilon)\sqrt{\frac{1}{4}-\lambda_1}} \leqslant \sum_{\gamma\in\mathcal{P}(X)}\sum_{k=1}^{\infty} \frac{l_\gamma(X)}{2\sinh\left(\frac{kl_\gamma(x)}{2}\right)} f_T\left(kl_\gamma(X)\right) - \hat{f}_T\left(\frac{i}{2}\right) + O(ng),$$

as claimed. ■

Remark 2.9. By considering only the zero eigenvalue, the proof of Theorem 2.1 gives that there exists a constant $\nu \geqslant 0$ such that for sufficiently large g and for any $X \in \mathcal{M}_{g,n}$,

$$\sum_{\gamma\in\mathcal{P}(X)}\sum_{k=1}^{\infty} \frac{l_\gamma(X)}{2\sinh\left(\frac{kl_\gamma(x)}{2}\right)} f_T\left(kl_\gamma(X)\right) - \hat{f}_T\left(\frac{i}{2}\right) + \nu ng \geqslant 0.$$

This fact will be important in Section 5 when we want to apply Markov's inequality to the above quantity, viewed as a random variable on $\mathcal{M}_{g,n}$.

3 Geometric Background

In this section we shall introduce the necessary background on moduli space, the Weil–Petersson metric and Mirzakhani's integration formula. A detailed account of the material in this section can be found in [30].

3.1 Moduli space

Let $S_{g,n}$ denote an oriented topological surface with genus g and n labeled punctures where $2g-2+n \geqslant 1$ and $n \geqslant 0$. A marked surface of signature (g,n) is a pair (X,φ) where X is a hyperbolic surface with genus g and n cusps and $\varphi : S_{g,n} \to X$ is a homeomorphism. The Teichmüller space, denoted by $\mathcal{T}_{g,n}$, is defined by

$$\mathcal{T}_{g,n} \overset{\text{def}}{=} \{\text{Marked surfaces}(X,\varphi)\}/\sim,$$

where $(X_1,\varphi_1) \sim (X_2,\varphi_2)$ if there exists an isometry $m : X_1 \to X_2$ such that φ_2 and $m \circ \varphi_1$ are isotopic. Let $\text{Homeo}^+(S_{g,n})$ denote the group of orientation preserving homeomorphisms of $S_{g,n}$, which do not permute the punctures and let $\text{Homeo}_0^+(S_{g,n})$ denote the subgroup of homeomorphisms isotopic to the identity. The mapping class group is defined as

$$\text{MCG}_{g,n} \overset{\text{def}}{=} \text{Homeo}^+\left(S_{g,n}\right)/\text{Homeo}_0^+\left(S_{g,n}\right).$$

$\text{Homeo}^+(S_{g,n})$ acts on $\mathcal{T}_{g,n}$ by precomposition of the marking and $\text{Homeo}^+(S_{g,n})$ acts trivially hence $\text{MCG}_{g,n}$ acts on $\mathcal{T}_{g,n}$ and we define the moduli space by

$$\mathcal{M}_{g,n} \overset{\text{def}}{=} \mathcal{T}_{g,n}/\text{MCG}_{g,n}.$$

$\mathcal{M}_{g,n}$ can be thought of as the set of equivalence classes of genus g hyperbolic surfaces with n labeled cusps where two surfaces are equivalent if they are isometric by an isometry, which preserves the labeling of the cusps.

Given $\underline{l} \in \mathbb{R}^n_{\geqslant 0}$, in a similar way, we define $\mathcal{T}_{g,n}(\underline{l})$ as the Teichmüller space of genus g hyperbolic surfaces with labeled geodesic boundary components $(b_1,\dots,b_n)$ with lengths $(l_1,\dots,l_n)$. We allow $l_i=0$, then the boundary component b_i is a replaced by a cusp and we recover

$$\mathcal{M}_{g,n} = \mathcal{M}_{g,n}(0,\dots,0).$$

3.2 Weil–Petersson metric

The space $\mathcal{T}_{g,n}(\underline{l})$ carries a natural symplectic structure known as the Weil–Petersson symplectic form and is denoted by ω_{WP}. It is invariant under the action of the mapping class group and descends to a symplectic form on the quotient $\mathcal{M}_{g,n}(\underline{l})$. The form ω_{WP}

induces the volume form

$$\mathrm{dVol}_{WP} \stackrel{\mathrm{def}}{=} \frac{1}{(3g-3+n)!} \bigwedge_{i=1}^{3g-3+n} \omega_{WP},$$

which is also invariant under the action of the mapping class group and descends to a volume form on $\mathcal{M}_{g,n}(\underline{l})$. The quantity $3g-3+n$ appears as the dimension of the Teichmüller and moduli space. We write dX as shorthand for dVol_{WP}. We let $V_{g,n}(\underline{l})$ denote $\mathrm{Vol}_{WP}(\mathcal{M}_{g,n}(\underline{l}))$, the total volume of $\mathcal{M}_{g,n}(\underline{l})$, which is finite. We write $V_{g,n}$ to denote $V_{g,n}(\underline{0})$.

As in [14, 17, 31], we define a probability measure on $\mathcal{M}_{g,n}$ by normalizing dVol_{WP}. Indeed, for any Borel subset $\mathcal{B} \subseteq \mathcal{M}_{g,n}$,

$$\mathbb{P}_{WP}^{g,n}[\mathcal{B}] \stackrel{\mathrm{def}}{=} \frac{1}{V_{g,n}} \int_{\mathcal{M}_{g,n}} \mathbf{1}_{\mathcal{B}} \mathrm{d}X,$$

where

$$\mathbf{1}_{\mathcal{B}}(X) = \begin{cases} 0 & \text{if } x \notin \mathcal{B}, \\ 1 & \text{if } x \in \mathcal{B}. \end{cases}$$

is the indicator function on $\mathcal{B}$. We write $\mathbb{E}_{WP}^{g,n}$ to denote expectation with respect to $\mathbb{P}_{WP}^{g,n}$.

3.3 Mirzakhani's integration formula

We recall Mirzakhani's integration formula from [15]. We define a multi-curve to be an ordered k-tuple $(\gamma_1, \ldots, \gamma_k)$ of disjoint, simple, non-peripheral closed curves. Let $\Gamma = [\gamma_1, \ldots, \gamma_k]$ denote the homotopy class of a multi-curve. The mapping class group $\mathrm{MCG}_{g,n}$ acts naturally on homotopy classes of multi-curves and we denote the orbit containing Γ by

$$\mathcal{O}_\Gamma = \left\{ (g \cdot \gamma_1, \ldots, g \cdot \gamma_k) \mid g \in \mathrm{MCG}_{g,n} \right\}.$$

Given a function $F : \mathbb{R}_{\geqslant 0}^k \to \mathbb{R}_{\geqslant 0}$, define $F^\Gamma : \mathcal{M}_{g,n} \to \mathbb{R}$ by

$$F^\Gamma(X) = \sum_{(\alpha_1, \ldots, \alpha_k) \in \mathcal{O}_\Gamma} F\left(l_{\alpha_1}(X), \ldots, l_{\alpha_k}(X) \right),$$

where $l_{\alpha_i}(X)$ is defined for $(X,\varphi) \in \mathcal{T}_{g,n}$ as the length of the geodesic in the homotopy class of $\varphi(\alpha_i)$. Note that the function F^Γ is well defined on $\mathcal{M}_{g,n}$ since we are summing over the orbit $\mathcal{O}_\Gamma$. Let $S_{g,n}(\Gamma)$ denote the result of cutting the surface $S_{g,n}$ along $(\gamma_1,\dots,\gamma_k)$, then $S_{g,n}(\Gamma) = \sqcup_{i=1}^s S_{g_i,n_i}$ for some $\{(g_i,n_i)\}_{i=1}^s$. Each γ_i gives rise to two boundary components γ_i^1 and γ_i^2 of $S_{g,n}(\Gamma)$. Given $\underline{x} = (x_1,\dots,x_k) \in \mathbb{R}^k_{\geqslant 0}$, let $\mathcal{M}(S_{g,n}(\Gamma); l_\Gamma = \underline{x})$ be the moduli space of hyperbolic surfaces homeomorphic to $S_{g,n}(\Gamma)$ such that for $1 \leqslant i \leqslant k$, $l_{\gamma_i^1} = l_{\gamma_i^2} = x_i$. Let $\underline{x}^{(i)}$ denote the tuple of coordinates x_j of $\underline{x}$ such that γ_j is a boundary component of S_{g_i,n_i}. We have that

$$\mathcal{M}\left(S_{g,n}(\Gamma); l_\Gamma = \underline{x}\right) = \prod_{i=1}^{s} \mathcal{M}_{g_i,n_i}\left(\underline{x}^{(i)}\right),$$

and we define

$$V_{g,n}(\Gamma,\underline{x}) \stackrel{\text{def}}{=} \mathrm{Vol}_{WP}\left(\mathcal{M}\left(S_{g,n}(\Gamma); l_\Gamma = \underline{x}\right)\right) = \prod_{i=1}^{s} V_{g_i,n_i}\left(\underline{x}^{(i)}\right).$$

In terms of the above notation we have the following.

Theorem 3.1 (Mirzakhani's Integration Formula [15, Theorem 7.1]). Given $\Gamma = [\gamma_1,\dots,\gamma_k]$,

$$\int_{\mathcal{M}_{g,n}} F^\Gamma(X)\,\mathrm{d}X = C_\Gamma \int_{\mathbb{R}^k_{\geqslant 0}} F(x_1,\dots,x_k)\, V_{g,n}(\Gamma,\underline{x})\, x_1 \cdots x_k \mathrm{d}x_1 \cdots \mathrm{d}x_k,$$

where the constant $C_\Gamma \in (0,1]$ only depends on Γ. Moreover, if $g > 2$ and $\Gamma = [\gamma]$ where γ is a simple, non-separating closed curve, then $C_\Gamma = \frac{1}{2}$.

4 Geometric Estimates

Recall that the family of test functions f_T in Theorem 2.1 is defined in (2.2) with $T = 4\log g$. For $X \in \mathcal{M}_{g,n}$, $\gamma \in \mathcal{P}(X)$, $k \in \mathbb{N}$, we shall denote

$$H_{X,k}(\gamma) \stackrel{\text{def}}{=} \frac{l_\gamma(X)}{2\sinh\left(\frac{k l_\gamma(x)}{2}\right)} f_T\left(k l_\gamma(X)\right).$$

The goal of this section is to prove the following.

Theorem 4.1. For $0 \leqslant \alpha < \frac{1}{2}$, let $n = O(g^{\alpha})$. For any $\varepsilon_1 > 0$ there exists a constant $c_1(\varepsilon_1) > 0$, independent of α, with

$$\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}(X)}\sum_{k=1}^{\infty}H_{X,k}(\gamma)-\hat{f}_T\left(\frac{i}{2}\right)\right] \ll n^2 g+\log(g)^5\cdot g+c_1(\varepsilon_1)(\log g)^{\beta+1}\cdot n^2\cdot g^{1+4\epsilon_1},$$

where $\beta > 0$ is a universal constant.

Throughout Section 4, we shall always have $n = O(g^{\alpha})$ for fixed $0 \leqslant \alpha < \frac{1}{2}$.

Remark 4.2. The proof of Theorem 4.1 closely follows [31, Chapters 6 & 7], making the necessary adaptations to the case of surfaces with cusps. We therefore omit some arguments that are identical in the compact and non-compact case and instead refer the reader to the relevant place.

4.1 Method

We prove Theorem 4.1 by considering separately the contribution of different types of geodesics. As in [31], we introduce the following notation.

Definition 4.3. For $X \in \mathcal{M}_{g,n}$ we define the following:

1. $\mathcal{P}^s_{sep}(X) \overset{\text{def}}{=} \{\gamma \in \mathcal{P}(X) \mid \gamma \text{ is simple and separating}\}$.
2. $\mathcal{P}^s_{nsep}(X) \overset{\text{def}}{=} \{\gamma \in \mathcal{P}(X) \mid \gamma \text{ is simple and non-separating}\}$.
3. $\mathcal{P}^{ns}(X) \overset{\text{def}}{=} \{\gamma \in \mathcal{P}(X) \mid \gamma \text{ is non-simple}\}$.

Notice that $\mathcal{P}(X) = \mathcal{P}^s_{sep}(X) \sqcup \mathcal{P}^s_{nsep}(X) \sqcup \mathcal{P}^{ns}(X)$. We partition the sum $\sum_{\gamma\in\mathcal{P}(X)}\sum_{k=1}^{\infty}H_{X,k}(\gamma)$ as

$$\begin{aligned}\sum_{\gamma\in\mathcal{P}(X)}\sum_{k=1}^{\infty}H_{X,k}(\gamma) &= \sum_{\gamma\in\mathcal{P}(X)}H_{X,1}(\gamma)+\sum_{\gamma\in\mathcal{P}(X)}\sum_{k=2}^{\infty}H_{X,k}(\gamma)\\ &= \sum_{\gamma\in\mathcal{P}^s_{sep}(X)}H_{X,1}(\gamma)+\sum_{\gamma\in\mathcal{P}^s_{nsep}(X)}H_{X,1}(\gamma)+\sum_{\gamma\in\mathcal{P}^{ns}(X)}H_{X,1}(\gamma)\\ &\quad+\sum_{\gamma\in\mathcal{P}(X)}\sum_{k=2}^{\infty}H_{X,k}(\gamma).\end{aligned}$$

Subtracting $\hat{f}(\frac{i}{2})$ and taking Weil–Petersson expectations, we see

$$\begin{aligned}
&\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}(X)}\sum_{k=1}^{\infty}H_{X,k}(\gamma)-\hat{f}_T\left(\frac{i}{2}\right)\right]\\
&\leqslant \underbrace{\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}^s_{sep}(X)}H_{X,1}(\gamma)\right]}_{(a)}+\underbrace{\left|\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}^s_{nsep}(X)}H_{X,1}(\gamma)\right]-\hat{f}\left(\frac{i}{2}\right)\right|}_{(b)}\\
&\quad+\underbrace{\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}(X)}\sum_{k=2}^{\infty}H_{X,k}(\gamma)\right]}_{(c)}+\underbrace{\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}^{ns}(X)}H_{X,1}(\gamma)\right]}_{(d)}. \qquad (4.1)
\end{aligned}$$

The remainder of this section is dedicated to bounding terms $(a)-(d)$, from which Theorem 4.1 will follow.

- Since terms (a) and (b) depend on simple geodesics, we can bound them by applying Mirzakhani's integration formula directly.
- To bound (c) we consider geodesics with length <1 and length $\geqslant 1$ separately. The contribution of geodesics with length $\geqslant 1$ can be bounded deterministically. Any geodesic with length <1 must be simple (e.g. [3, Theorem 4.2.4]), so we can apply Mirzakhani's integration formula directly to bound their contribution.
- To bound (d), we cannot apply Mirzakhani's integration formula directly since the geodesics are not simple. Instead, we pass from non-simple geodesics to subsurfaces with simple geodesic boundary and apply Mirzakhani's integration formula to the simple boundary geodesics.

4.2 Contribution of simple separating geodesics

In this subsection we bound term (a) in (4.1), the contribution of simple separating geodesics. In particular, we prove the following.

Lemma 4.4.

$$\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}^s_{sep}(X)}H_{X,1}(\gamma)\right]\ll n^2 g.$$

Proof. We have

$$\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}_{sep}^{s}(X)} H_{X,1}(\gamma)\right] = \frac{1}{V_{g,n}}\int_{\mathcal{M}_{g,n}} \sum_{\gamma\in\mathcal{P}_{sep}^{s}(X)} H_{X,1}(\gamma)dX. \tag{4.2}$$

We shall apply Mirzakhani's integration formula, Theorem 3.1, to bound the integral in (4.2). Recall that $S_{g,n}$ is a topological surface with genus g and n labeled punctures. For $0\leqslant i\leqslant \lfloor \frac{g}{2}\rfloor$, $0\leqslant j\leqslant n$, let $\alpha_{i,j}$ be a simple closed curve in $S_{g,n}$, which separates $S_{g,n}$ into subsurfaces $S_{i,j+1}$ and $S_{g-i,n-j+1}$, each with one boundary component and j and $n-j$ punctures respectively. Then $\alpha_{i,j}$ partitions the punctures into two disjoint subsets I and J of size j and $n-j$, respectively. Let $\left[\alpha_{i,j}\right]$ denote the homotopy class of $\alpha_{i,j}$. The orbit $\mathrm{MCG}_{g,n}\cdot\left[\alpha_{i,j}\right]$ is determined by the set $\{(i,j+1,I),(g-i,n-j+1,J)\}$, since the mapping class group does not permute the punctures. Therefore, given i and j, there are $\binom{n}{j}$ $\mathrm{MCG}_{g,n}$-orbits of simple separating closed curves on $S_{g,n}$, which separate off a subsurface with genus i and with j punctures. Recalling that

$$H_{X,1}(\gamma) = \frac{l_\gamma(X)}{2\sinh\left(\frac{l_\gamma(X)}{2}\right)} f_T\left(l_\gamma(X)\right),$$

we now apply Mirzakhani's integration formula, Theorem 3.1, to see

$$\frac{1}{V_{g,n}}\int_{\mathcal{M}_{g,n}} \sum_{\gamma\in\mathcal{P}_{sep}^{s}(X)} H_{X,1}(\gamma)\mathrm{d}X$$

$$\leqslant \sum_{\substack{0\leqslant i\leqslant g,0\leqslant j\leqslant n\\ 2\leqslant 2i+j\leqslant 2g+n-2}} \int_0^\infty \binom{n}{j}\frac{x^2}{\sinh\left(\frac{x}{2}\right)} f_T(x)\frac{V_{i,j+1}\left(\underline{0}_j,x\right)V_{g-i,n-j+1}\left(\underline{0}_{n-j},x\right)}{V_{g,n}}\mathrm{d}x.$$

By Lemma A.1,

$$V_{a,b}(\underline{0}_{b-1},x)\leqslant \frac{2\sinh\left(\frac{x}{2}\right)}{x}V_{a,b},$$

giving

$$\frac{1}{V_{g,n}}\int_{\mathcal{M}_{g,n}} \sum_{\gamma\in\mathcal{P}_{sep}^{s}(X)} H_{X,1}(\gamma)\mathrm{d}X$$

$$\leqslant \frac{4}{V_{g,n}}\left(\sum_{\substack{0\leqslant i\leqslant g,0\leqslant j\leqslant n\\ 2\leqslant 2i+j\leqslant 2g+n-2}} \binom{n}{j}\cdot\frac{V_{i,j+1}V_{g-i,n-j+1}}{V_{g,n}}\right)\int_0^\infty \sinh\left(\frac{x}{2}\right)f_T(x)\mathrm{d}x.$$

Since f_T is bounded independently of T and supported in $[0, T)$, we see

$$\int_0^\infty \sinh\left(\frac{x}{2}\right) f_T(x)\mathrm{d}x \ll e^{\frac{T}{2}}.$$

By Lemma A.4,

$$\sum_{\substack{0\leqslant i\leqslant g, 0\leqslant j\leqslant n \\ 2\leqslant 2i+j\leqslant 2g+n-2}} \frac{n!}{j!\,(n-j)\,!} \cdot \frac{V_{i,j+1}V_{g-i,n-j+1}}{V_{g,n}} \ll \frac{n^2}{g},$$

giving

$$\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}_{sep}^{s}(X)} H_{X,1}(\gamma)\right] \ll \frac{n^2}{g}\cdot e^{\frac{T}{2}} \ll n^2 g,$$

as claimed. ■

4.3 Contribution of simple non-separating geodesics

In this subsection we deal with the contribution of simple non-separating geodesics (term (b) in (4.1)). We prove the following.

Lemma 4.5.

$$\left|\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}_{nsep}^{s}(X)} H_{X,1}(\gamma)\right] - \hat{f}\left(\frac{i}{2}\right)\right| \ll n^2 g + n\cdot\log(g)^2\cdot g.$$

Proof. Let α_0 be an unoriented simple non-separating closed curve in $S_{g,n}$. There is just one $\mathrm{MCG}_{g,n}$-orbit of simple non-separating closed curves on $S_{g,n}$ and we have

$$\sum_{\gamma\in\mathcal{P}_{nsep}^{s}(X)} H_{X,1}(\gamma)dX = 2\sum_{\gamma\in\mathrm{MCG}_{g,n}\cdot\alpha_0} H_{X,1}(\gamma),$$

where the factor of 2 occurs since geodesics in $\mathcal{P}(X)$ are oriented. Applying Mirzakhani's integration formula, we get

$$\int_{\mathcal{M}_{g,n}} \sum_{\gamma\in\mathcal{P}_{nsep}^{s}(X)} H_{X,1}(\gamma)\mathrm{d}X = \frac{1}{2}\int_0^\infty \frac{x^2}{\sinh(\frac{x}{2})} f_T(x) V_{g-1,n+2}\left(\underline{0}_n, x, x\right)\mathrm{d}x,$$

where the factor $\frac{1}{2}$ occurs since α_0 is simple and non-separating; c.f. Theorem 3.1. By Theorem A.3,

$$V_{g-1,n+2} = V_{g,n} \cdot \left(1 + O\left(\frac{n^2}{g}\right)\right).$$

Then we have, by applying Lemma A.1,

$$\frac{V_{g-1,n+2}\left(\underline{0}_n, x, x\right)}{V_{g,n}} = \left(\frac{2\sinh\frac{x}{2}}{x}\right)^2 \left(1 + O\left(\frac{n^2 + nx^2}{g}\right)\right).$$

This gives

$$\begin{aligned}&\frac{1}{V_{g,n}} \int_{\mathcal{M}_{g,n}} \sum_{\gamma \in \mathcal{P}^s_{nsep}(X)} \frac{l_\gamma(X)}{\sinh\left(\frac{l_\gamma(X)}{2}\right)} f_T\left(l_\gamma(X)\right) \mathrm{d}X \\ &= \int_0^T 2\sinh\left(\frac{x}{2}\right) f_T(x) \left(1 + O\left(\frac{n^2 + nx^2}{g}\right)\right) \mathrm{d}x.\end{aligned}$$

Since $\hat{f_T}\left(\frac{i}{2}\right)$ is even,

$$\hat{f_T}\left(\frac{i}{2}\right) = \int_0^T 2\cosh\left(\frac{x}{2}\right) f_T(x)\mathrm{d}x,$$

and we have

$$\begin{aligned}&\left|\mathbb{E}^{g,n}_{WP}\left[\sum_{\gamma \in \mathcal{P}^s_{nsep}(X)} H_{X,1}(\gamma)\right] - \hat{f_T}\left(\frac{i}{2}\right)\right| \\ &= \left|\int_0^T 2\sinh\left(\frac{x}{2}\right) f_T(x) \left(1 + O\left(\frac{1 + n^2 + nx^2}{g}\right)\right) \mathrm{d}x - \int_0^T 2\cosh\left(\frac{x}{2}\right) f_T(x)\mathrm{d}x\right| \\ &\ll \left|\int_0^T 2\left(\sinh\left(\frac{x}{2}\right) - \cosh\left(\frac{x}{2}\right)\right) \cdot f_T(x)\mathrm{d}x\right| + \left|\int_0^T 2\sinh\left(\frac{x}{2}\right) f_T(x) \left(\frac{n^2 + nx^2}{g}\right) \mathrm{d}x\right|.\end{aligned}$$

Using that $2\left(\cosh\left(\frac{x}{2}\right) - \sinh\left(\frac{x}{2}\right)\right) = e^{-x}$,

$$\left|\int_0^T 2\left(\sinh\left(\frac{x}{2}\right) - \cosh\left(\frac{x}{2}\right)\right) \cdot f_T(x)\mathrm{d}x\right| \ll 1.$$

Recalling $T = 4\log g$, we calculate

$$\left|\int_0^T 2\sinh\left(\frac{x}{2}\right) f_T(x)\left(\frac{1+n^2+n^2x}{g}\right)\mathrm{d}x\right| \ll \frac{e^{\frac{T}{2}}\left(n^2+nT^2\right)}{g} \ll n^2 g + n\cdot\log(g)^2\cdot g,$$

and

$$\left|\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}^s_{nsep}(X)} H_{X,1}(\gamma)\right] - \hat{f}_T\left(\frac{i}{2}\right)\right| \ll n^2 g + n\cdot\log(g)^2\cdot g,$$

as claimed. ■

4.4 Iterates of primitive geodesics

We now look at the contribution of iterates of primitive geodesics (term (c) in (4.1)). The aim of this subsection is to prove the following.

Lemma 4.6.

$$\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}(X)}\sum_{k=2}^{\infty} H_{X,k}(\gamma)\right] \ll \log(g)^2\cdot g.$$

In order to prove Lemma 4.6, we need the following soft geodesic counting bound.

Lemma 4.7. For any $X \in \mathcal{M}_{g,n}$ and any $L > 0$ we have

$$\#\{\gamma\in\mathcal{P}(X) \mid 1 \leqslant l_\gamma(X) \leqslant L\} \ll g e^L.$$

Proof. Let $\#_0(X,L)$ denote the number of closed geodesics on X with length $\leqslant L$, which are not iterates of closed geodesics of length $\leqslant 2\operatorname{arcsinh}(1)$. An immediate adaptation of the proof of [3, Lemma 6.6.4] using the non-compact version of the Collar Theorem [3, Lemma 4.4.6] tells us that

$$\#_0(X,L) \leqslant \left(g-1+\frac{n}{2}\right)e^{L+6}.$$

Lemma 4.4.6 in [3] also tells us that the number of primitive geodesics on X with length $\leqslant 4\operatorname{arcsinh}(1)$ is bounded above by $3g-3+n$. Using that $n=o\left(\sqrt{g}\right)$, we conclude that

$$\#\{\gamma \in \mathcal{P}(X) \mid 1 \leqslant l_\gamma(X) \leqslant L\} \leqslant \left(g-1+\frac{n}{2}\right) e^{L+6} + 3g-3+n \ll g e^L,$$

as claimed. ■

We now proceed with the proof of Lemma 4.6.

Proof of Lemma 4.6. Let $X \in \mathcal{M}_{g,n}$. We write

$$\sum_{\gamma\in\mathcal{P}(X)} \sum_{k=2}^{\infty} H_{X,k}(\gamma) = \sum_{\{\gamma\in\mathcal{P}(X) \mid l_\gamma(X)<1\}} \sum_{k=2}^{\infty} H_{X,k}(\gamma) + \sum_{\{\gamma\in\mathcal{P}(X) \mid l_\gamma(X)\geqslant 1\}} \sum_{k=2}^{\infty} H_{X,k}(\gamma).$$

By Lemma 4.6,

$$\#\{\gamma \in \mathcal{P}(X) \mid 1 \leqslant l_\gamma(X) \leqslant L\} \ll g e^L.$$

We then have

$$\begin{aligned}\sum_{\{\gamma\in\mathcal{P}(X) \mid l_\gamma(X)\geqslant 1\}} \sum_{k=2}^{\infty} H_{X,k}(\gamma) &\ll \sum_{\{\gamma\in\mathcal{P}(X) \mid 1\leqslant l_\gamma(X)\leqslant \frac{T}{2}\}} l_\gamma(X)\, e^{-l_\gamma(X)} \\ &\leqslant \sum_{m=1}^{\lfloor \frac{T}{2}\rfloor} m e^{-m} \cdot \#\{\gamma \in \mathcal{P}(X) \mid m \leqslant l_\gamma(X) \leqslant m+1\} \\ &\ll g \sum_{m=1}^{\lfloor \frac{T}{2}\rfloor} m \ll (\log g)^2 \cdot g.\end{aligned}$$

Taking Weil–Petersson expectations, we see

$$\begin{aligned}\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}(X)} \sum_{k=2}^{\infty} H_{X,k}(\gamma)\right] =& \mathbb{E}_{WP}^{g,n}\left[\sum_{\{\gamma\in\mathcal{P}(X) \mid l_\gamma(X)<1\}} \sum_{k=2}^{\infty} H_{X,k}(\gamma)\right] \\ &+ O\left((\log g)^2 g\right). \end{aligned} \tag{4.3}$$

For each $\gamma \in \mathcal{P}(X)$,

$$H_{X,k}(\gamma) = \frac{l_\gamma(X)}{2\sinh\left(\frac{kl_\gamma(X)}{2}\right)} f_T\left(kl_\gamma(X)\right) \leqslant f(0),$$

and if $k \geqslant \frac{T}{l_\gamma(X)}$ then $f_T\left(kl_\gamma(X)\right) = 0$. This tells us that

$$\mathbb{E}_{WP}^{g,n}\left[\sum_{\{\gamma\in\mathcal{P}(X)|l_\gamma(X)<1\}} \sum_{k=2}^{\infty} H_{X,k}(\gamma)\right] \leqslant f(0)\cdot T\cdot \mathbb{E}_{WP}^{g,n}\left[\sum_{\{\gamma\in\mathcal{P}(X)|l_\gamma(X)<1\}} \frac{1}{l_\gamma(X)}\right]. \tag{4.4}$$

It remains to bound

$$\mathbb{E}_{WP}^{g,n}\left[\sum_{\{\gamma\in\mathcal{P}(X)|l_\gamma(X)<1\}} \frac{1}{l_\gamma(X)}\right].$$

Any geodesic $\gamma \in \mathcal{P}(X)$ with length $l_\gamma(X) \leqslant 1 < 4\operatorname{arcsinh}1$ must be simple by, for example, [3, Theorem 4.2.4]. Therefore, we can apply Mirzakhani's integration formula to get

$$\begin{aligned}\mathbb{E}_{WP}^{g,n}\left[\sum_{\{\gamma\in\mathcal{P}(X)|l_\gamma(X)<1\}} \frac{1}{l_\gamma(X)}\right] &\leqslant \frac{1}{V_{g,n}}\int_0^1 V_{g-1,n+2}(\underline{0}_n,t,t)\mathrm{d}t \\ &\quad + \sum_{\substack{0\leqslant i\leqslant g,0\leqslant j\leqslant n\\ 2\leqslant 2i+j\leqslant 2g+n-2}} \frac{n!}{j!\,(n-j)\,!}\cdot\frac{V_{i,j+1}V_{g-i,n-j+1}}{V_{g,n}} \\ &\ll \frac{V_{g-1,n+2}}{V_{g,n}} + \frac{n^2}{g} \ll 1,\end{aligned} \tag{4.5}$$

where on the last line we applied Lemma A.4 and Theorem A.3. Thus, combining (4.3), (4.4) and (4.5) we see

$$\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}(X)} \sum_{k=2}^{\infty} H_{X,k}(\gamma)\right] \ll (\log g)^2\cdot g,$$

as required. ■

4.5 Non-simple geodesics

We now need to deal with the contribution of the non-simple primitive geodesics (term (d) in (4.1)). In this subsection we shall prove the following:

Lemma 4.8. There is a constant $\beta_1 > 0$ such that for any $\varepsilon_1 > 0$ there is a constant $c_1(\varepsilon_1) > 0$ such that

$$\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}^{ns}(X)} H_{X,1}(\gamma)\right] \ll (\log g)^6 \cdot g + c_1\left(\varepsilon_1\right)(\log g)^{\beta_1}\cdot n^2 \cdot g^{1+4\varepsilon_1}.$$

We prove Lemma 4.8 through a sequence of lemmas. Before we give a brief outline of the method, we need the concept of a filling closed curve.

Definition 4.9. Let X be a finite-area hyperbolic surface with possible boundary. A closed curve $\eta \subset Y$ is filling if the complement $Y\backslash\eta$ is a disjoint union of disks and cylinders such that every cylinder either deformation retracts to a boundary component of Y or is a neighbourhood of a cusp. We let $\#_{\text{fill}}(X, L)$ denote the number of oriented filling geodesics on X with lengths $\leqslant L$.

Idea of the proof of Lemma 4.8

We shall extend the method of [31, Section 7] to non-compact surfaces. The basic idea is as follows:

- Given a surface $X \in \mathcal{M}_{g,n}$ and a geodesic $\gamma \in \mathcal{P}^{ns}(X)$, we construct a subsurface $X(\gamma)$ of X with geodesic boundary (of controlled length), which is filled by γ. The multiplicity of the map $\gamma \mapsto X(\gamma)$ is bounded by the number of filling geodesics of $X(\gamma)$. This allows us to write

$$\sum_{\gamma\in\mathcal{P}^{ns}(X)} H_{X,1}(\gamma) \leqslant \sum_{\substack{Y \text{ subsurface of } X\\ Y \text{ has geodesic boundary}}} \; \sum_{\text{filling geodesics } \gamma \text{ on } Y} H_{X,1}\left(l_X(\gamma)\right).$$

- We control the length of a filling geodesic in terms of $l_X(\partial Y)$ in Lemma 4.11 and apply [31, Theorem 4] to bound the number of filling geodesics on a subsurface and show that there is an explicit function A, supported in $[0, 2T)$, with

$$\sum_{\gamma\in\mathcal{P}^{ns}(X)} H_{X,1}(\gamma) \leqslant \sum_{\substack{Y \text{ subsurface of } X\\ Y \text{ has geodesic boundary}}} A\left(l_X(\partial Y)\right).$$

- Since the boundary of each subsurface Y consists of simple closed geodesics, we can apply Mirzakhani's integration formula to bound the Weil–Petersson expectation of

$$\sum_{\{Y \text{subsurface of } X \text{ with geodesic boundary}\}} A\left(l_X(\partial Y)\right).$$

Definition 4.10. Let $X \in \mathcal{M}_{g,n}$ be a hyperbolic surface and let $\gamma \subset X$ be a non-simple closed geodesic. Let $N_\delta(\gamma)$ denote the δ-neighborhood of γ where δ is sufficiently small to ensure that $N_\delta(\gamma)$ deformation retracts to γ and that the boundary $\partial N_\delta(\gamma)$ is a disjoint union of simple closed curves. We define $X(\gamma)$ to be the connected subsurface obtained from $N_\delta(\gamma)$ as follows, for each boundary component $\xi \in N_\delta(\gamma)$:

- If ξ bounds a disc we fill the disc into $N_\delta(\gamma)$.
- If ξ is homotopically non-trivial we shrink it to the unique simple closed geodesics in its free homotopy class and deform $N_\delta(\gamma)$ accordingly.
- If two different components ξ, ξ' deform to the same geodesic then we do not glue them together, we view $X(\gamma)$ as an open subsurface of X.
- If ξ is freely homotopic to a closed horocycle bounding a cusp C_i we fill the cusp into $N_\delta(\gamma)$.

After deforming $N_\delta(\gamma)$ in this way we obtain the surface $X(\gamma)$.

The construction of $X(\gamma)$ allows us to control $\mathrm{Vol}(X(\gamma))$ and the length of $\partial X(\gamma)$ in terms of $l_\gamma(X)$, as summarized by the following lemma. Bounding $\mathrm{Vol}(X(\gamma))$ corresponds to bounding the Euler characteristic of $X(\gamma)$ by Gauss-Bonnet.

Lemma 4.11. Let $X \in \mathcal{M}_{g,n}$ and γ be a non-simple closed geodesic on X. The subsurface $X(\gamma)$ of X satisfies the following:

1. γ is a filling geodesic of $X(\gamma)$.
2. The length of the boundary satisfies

$$l(\partial X(\gamma)) \leqslant 2l_\gamma(X).$$

3. The volume satisfies

$$\mathrm{Vol}\,(X(\gamma)) \leqslant 4l_{\gamma}(X).$$

Lemma 4.11 is proved in [25, Proposition 47] for compact surfaces. The proof in our case is identical. This leads us to make the following definition.

Definition 4.12. With $T = 4\log g$, $X \in \mathcal{M}_{g,n}$, we define

$$\mathrm{Sub}(X) \stackrel{\text{def}}{=} \{Y \subset X \mid Y \text{ is a connected subsurface of } X \text{ with geodesic boundary}\},$$

and

$$\mathrm{Sub}_T(X) \stackrel{\text{def}}{=} \{Y \in \mathrm{Sub}(X) \mid l(\partial Y) \leqslant 2T, \mathrm{Vol}(Y) \leqslant 4T\},$$

where we allow two distinct simple closed geodesics on the boundary of Y to be a single simple closed geodesic in X.

Lemma 4.11 tells us that for any $X \in \mathcal{M}_{g,n}$, any non-simple geodesic γ with length $\leqslant T$ fills a subsurface $X(\gamma) \in \mathrm{Sub}_T(X)$. If any other $\gamma' \in \mathcal{P}(X)$ satisfies $X(\gamma') = X(\gamma)$ then γ' is also a filling geodesic of $X(\gamma)$ with length $\leqslant T$. We have

$$\{\gamma' \in \mathcal{P}^{ns}(X) \mid X(\gamma') = X(\gamma)\} \subseteq \{\text{oriented filling geodesics of } X(\gamma) \text{ with length } \leqslant T\}. \tag{4.6}$$

Therefore, we will need to control the number of non-simple geodesics, which fill a given subsurface. This is achieved by the following theorem.

Theorem 4.13 ([31, Theorem 4]). Let $m = 2g' - 2 + n' \geqslant 1$. For any $\varepsilon_1 > 0$ there exists a constant $c(\varepsilon_1, m)$ only depending on ε_1 and m such that for any $X \in \mathcal{M}_{g',n'}(x_1, \ldots, x_{n'})$ where $x_i \geqslant 0$, we have

$$\#_{\mathrm{fill}}(X, L) \leqslant c(\epsilon_1, m) \cdot e^{L - \frac{1-\varepsilon_1}{2}\sum_{i=1}^{n} x_i}.$$

Remark 4.14. Theorem 4 in [31] is stated in for surfaces without cusps, that is, $x_i > 0$; however, the extension to $x_i \geqslant 0$ is immediate. Indeed, [31, Theorem 4] follows from [31, Theorem 38] and [31, Lemma 10]. Theorem 38 in [31] already holds for non-compact surfaces and it is straightforward to check that the basic counting result [31, Lemma 10] generalizes to non-compact surfaces.

We can now pass from non-simple geodesics to subsurfaces with geodesic boundary. This is done in the following lemma, proved in [31, Proposition 30] for $X \in \mathcal{M}_g$. The proof is identical in our case.

Lemma 4.15. For any $\varepsilon_1 > 0$, $X \in \mathcal{M}_{g,n}$, there exists a constant $c_1(\varepsilon_1)$ only depending on ε_1 such that

$$\sum_{\gamma \in \mathcal{P}^{ns}(X)} H_{X,1}(\gamma) \ll Te^T \sum_{\substack{Y \in \mathrm{Sub}_T(X) \\ |\chi(Y)| \geqslant 34}} e^{-\frac{l(\partial Y)}{4}} + c_1(\varepsilon_1)\, T \sum_{\substack{Y \in \mathrm{Sub}_T(X) \\ 1 \leqslant |\chi(Y)| \leqslant 33}} e^{\frac{T}{2} - \frac{1-\epsilon_1}{2} l(\partial Y)}. \tag{4.7}$$

Remark 4.16. The difference between the first and second term arises because we apply Theorem 4.13 to subsurfaces with $1 \leqslant |\chi(Y)| \leqslant 34$ whereas we only apply a soft geodesic counting result, $\#_{\mathrm{fill}}(X, L) \leqslant \mathrm{Area}(X) \cdot e^{L+6}$, to subsurfaces with $|\chi(Y)| \geqslant 34$. The reason for this is that it is not clear how badly the constant $c(\epsilon_1, m)$ from Theorem 4.13 depends on the Euler characteristic m so we can only apply Theorem 4.13 to subsurfaces with uniformly bounded Euler characteristic. As a consequence of forthcoming calculations, the Weil–Petersson expectation of the number of subsurfaces $Y \in \mathrm{Sub}_T(X)$ with $|\chi(Y)| \geqslant k$ is sufficiently small for any $k \geqslant 34$ so that we can accept the loss from the soft geodesic counting.

For the remainder of the section, we assume that g is sufficiently large so that for $Y \in \mathrm{Sub}_T(X)$, the map $Y \mapsto \partial Y$ is injective. This is justified since any two distinct subsurfaces in $Y_1, Y_2 \in \mathrm{Sub}_T(X)$ with $\partial Y_1 = \partial Y_2$ must satisfy $Y_1 \cup Y_2 = X$, giving

$$\mathrm{Vol}(X) = 2\pi(2g - 2 + n) \leqslant \mathrm{Vol}(Y_1) + \mathrm{Vol}(Y_2) \leqslant 8T = 32 \log g,$$

which is not possible for sufficiently large g.

We now want to apply Mirzakhani's integration formula to bound the Weil–Petersson expectation of the right-hand side of (4.7). We introduce the following notation.

Notation 1. Let $X \in \mathcal{M}_{g,n}$. For a subsurface $Y_0 \in \mathrm{Sub}_T(X)$, we write

$$Y_0 = Y_0\left(q, (g_0, a_0, n_0), \left\{(g_1, a_1, n_1), \ldots, \left(g_q, a_q, n_q\right)\right\}\right) = Y_0\left(q, \underline{g}, \underline{a}, \underline{n}\right),$$

to indicate that Y_0 has the following properties.

- Y_0 is homeomorphic to $S_{g_0, k+a_0}$ where $k > 0$.

- Y_0 has a_0 cusps and k simple geodesic boundary components. There are $n_0 \geqslant 0$ pairs of simple geodesics in Y_0, which correspond to a single simple closed geodesic in X.
- The interior of its complement $X \backslash Y_0$ consists of $q \geqslant 1$ components $Y_1, \ldots, Y_q$ where Y_i is homeomorphic to $S_{g_i, n_i + a_i}$. We observe that $n_i \geqslant 1$ and
  - i) $\sum_{i=1}^{q} 2g_i - 2 + n_i + a_i = 2g - 2 + n - \left|\chi\left(Y_0\right)\right|$.
  - ii) $\sum_{i=1}^{q} n_i = k - 2n_0$.
  - iii) $\sum_{j=1}^{q} a_j = n - a_0$.

Given $X \in \mathcal{M}_{g,n}$ and a choice of marking, any $Y_0(q, \underline{a}, \underline{n}, \underline{g}) \in \mathrm{Sub}_T(X)$ is freely homotopic to the image under the marking of a subsurface $Y \subset S_{g,n}$ where Y is in the $\mathrm{MCG}_{g,n}$-orbit of a subsurface $\tilde{Y}_0 = \tilde{Y}_0(q, \underline{a}, \underline{n}, \underline{g}) \subset S_{g,n}$ (with $\tilde{Y}_0$ homeomorphic to $S_{g_0, k+a_0}$, where $S_{g,n} \backslash \tilde{Y}_0$ has q components $\tilde{Y}_1, \ldots, \tilde{Y}_q$ with $\tilde{Y}_i$ homeomorphic to $S_{g_i, n_i + a_i}$ with n_i boundary components and a_i punctures). We write $\left[\tilde{Y}_0\right]$ to denote the homotopy class of $\tilde{Y}_0$. Since the mapping class group does not permute the punctures of $S_{g,n}$, the number of distinct $\mathrm{MCG}_{g,n}$-orbits of subsurfaces corresponding to a given choice of $q, (g_0, n_0, g_0), \left\{(g_1, a_1, n_1), \ldots, \left(g_q, a_q, n_q\right)\right\}$ is bounded above by

$$\frac{n!}{a_0! \cdot \cdots \cdot a_q!}.$$

Lemma 4.17.

$$\mathbb{E}_{WP}^{g,n}\left[\sum_{\substack{Y \in \mathrm{Sub}_T(X) \\ |\chi(Y)| \geqslant 34}} e^{-\frac{l(\partial Y)}{4}}\right] \ll \frac{(\log g)^5}{g^3}. \tag{4.8}$$

Proof. We start by bounding the contribution of a given $\mathrm{MCG}_{g,n}$-orbit to (4.8). Let g_0, a_0, k be fixed with $m = 2g_0 - 2 + k + a_0 \geqslant 34$. By Gauss–Bonnet, we have that $m \leqslant \frac{4T}{2\pi} \leqslant \frac{5}{2} \log g$. For $n_0, n_1, \ldots, n_q, a_1, \ldots, a_q, g_1, \ldots, g_q \geqslant 0$ with $\sum_{i=1}^{q} n_i = k - 2n_0$ and $\sum_{j=1}^{q} a_j = n - a_0$, we have

$$\frac{1}{V_{g,n}} \int_{\mathcal{M}_{g,n}} \sum_{[Y] \in \mathrm{MCG}_{g,n} \cdot [\tilde{Y}_0(q, \underline{a}, \underline{n}, \underline{g})]} e^{-\frac{l(\partial Y)}{4}} \mathbf{1}_{[0,2T]}\left(l_X\left(\partial Y\right)\right) \mathrm{d}X$$

$$= \frac{1}{V_{g,n}} \int_{\mathcal{M}_{g,n}} \sum_{[\partial Y] \in \mathrm{MCG}_{g,n} \cdot [\partial \tilde{Y}_0(q, \underline{a}, \underline{n}, \underline{g})]} e^{-\frac{l(\partial Y)}{4}} \mathbf{1}_{[0,2T]}\left(l_X\left(\partial Y\right)\right) \mathrm{d}X,$$

since the map $Y \mapsto \partial Y$ is injective. By applying Mirzakhani's integration formula, one can compute that

$$\frac{1}{V_{g,n}} \int_{\mathcal{M}_{g,n}} \sum_{[Y] \in \mathrm{MCG}_{g,n} \cdot [\tilde{Y}_0(q,\underline{a},\underline{n},\underline{g})]} e^{-\frac{l(\partial Y)}{4}} \mathbf{1}_{[0,2T]} \left(l_X \left(\partial Y \right) \right) \mathrm{d}X$$
$$\ll e^{\frac{7}{2}T} \frac{V_{g_0,k+a_0} V_{g_1,n_1+a_1} \cdots V_{g_q,n_q+a_q}}{V_{g,n} \cdot n_0! \, n_1! \cdots n_q!}.$$

A near identical computation is carried out in detail in [31, Proposition 31] so we omit it here. We now sum over the $\mathrm{MCG}_{g,n}$-orbits to bound the contribution of subsurfaces in $\mathrm{Sub}_T(X)$ with a given Euler characteristic. We calculate

$$\mathbb{E}_{WP}^{g,n} \left[\sum_{\substack{Y \in \mathrm{Sub}_T(X) \\ Y \cong S_{g_0,k+a_0}}} e^{-\frac{l(\partial Y)}{4}} \right]$$
$$\leqslant \sum_{n_0=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{q=1}^{k-2a_0} \sum_{\mathcal{A}} \frac{1}{V_{g,n}} \cdot \binom{n}{a_0, \ldots, a_q} \cdot \int_{\mathcal{M}_{g,n}} \sum_{[Y] \in \mathrm{MCG}_{g,n} \cdot [\tilde{Y}_0(q,\underline{a},\underline{n},\underline{g})]} e^{-\frac{l(\partial Y)}{4}} \mathbf{1}_{[0,2T]} \left(l_X \left(\partial Y \right) \right) \mathrm{d}X$$
$$\ll e^{\frac{7}{2}T} \sum_{n_0=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{q=1}^{k-2a_0} \sum_{\{(g_j,n_j,q_j)\}_{j=1}^{q} \in \mathcal{A}} \binom{n}{a_0, \ldots, a_q} \cdot \frac{V_{g_0,k+a_0} V_{g_1,n_1+a_1} \cdots V_{g_q,n_q+a_q}}{V_{g,n} \cdot n_0! \, n_1! \cdots n_q!},$$

where for a given n_0 and q, the summation is over the set of "admissible triples" $\mathcal{A}$, whose elements we denote by $\left\{ \left(g_j, n_j, q_j \right) \right\}_{j=1}^{q}$, which we define to be the set of $\left\{ \left(g_1, a_1, n_1 \right), \ldots, \left(g_q, a_q, n_q \right) \right\}$ where $g_j, a_j \geqslant 0$, $n_j \geqslant 1$ and $2g_j + a_j + n_j \geqslant 3$ such that

i) $\sum_{i=1}^{q} \left(2g_i - 2 + n_i + a_i \right) = 2g - 2 + n - m$.
ii) $\sum_{i=1}^{q} n_i = k - 2n_0$.
iii) $\sum_{j=1}^{q} a_j = n - a_0$.

Recalling that $34 \leqslant m = 2g_0 - 2 + k + a_0 \leqslant \frac{5}{2} \log g$ is fixed, we apply lemma A.5 to see

$$\sum_{n_0=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{q=1}^{k-2n_0} \sum_{\{(g_j,n_j,a_j)\}_{j=1}^{q} \in \mathcal{A}} \frac{n!}{a_0! \cdot \cdots \cdot a_q!} \cdot \frac{V_{g_0,k+a_0} V_{g_1,n_1+a_1} \cdots V_{g_q,n_q+a_q}}{V_{g,n} \cdot n_0! \, n_1! \cdots n_q!}$$
$$\ll \sum_{n_0=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{q=1}^{k-2n_0} \left(2g_0 + k + a_0 - 3 \right)! \cdot \frac{n^{a_0}}{g^m} \ll \frac{k^2 \left(2g_0 + k + a_0 - 3 \right)!}{g^m}.$$

Summing over the possible values of g_0, a_0 and k, we calculate

$$\mathbb{E}_{WP}^{g,n}\left[\sum_{\substack{Y\in\mathrm{Sub}_T(X)\\ |\chi(Y)|\geqslant 34}} e^{-\frac{l(\partial Y)}{4}}\right]$$

$$\ll e^{\frac{7}{2}T}\sum_{0\leqslant a_0\leqslant\lceil\frac{4T}{2\pi}\rceil}\ \sum_{1\leqslant k\leqslant\lceil\frac{4T}{2\pi}\rceil+2-a_0}\ \sum_{34\leqslant 2g_0-2+k+a_0\leqslant\lceil\frac{4T}{2\pi}\rceil}\mathbb{E}_{WP}^{g,n}\left[\sum_{\substack{Y\in\mathrm{Sub}_T(X)\\ Y\cong S_{g_0,k+a_0}}} e^{-\frac{l(\partial Y)}{4}}\right]$$

$$\ll e^{\frac{7}{2}T}\sum_{0\leqslant a_0\leqslant\lceil\frac{4T}{2\pi}\rceil}\ \sum_{1\leqslant k\leqslant\lceil\frac{4T}{2\pi}\rceil+2-a_0}\ \sum_{34\leqslant 2g_0-2+k+a_0\leqslant\lceil\frac{4T}{2\pi}\rceil}\frac{k^2\left(2g_0+a_0+k-3\right)!\,n^{a_0}}{g^{2g_0+a_0+k-2}}$$

$$\ll T^5 e^{\frac{7T}{2}}\frac{1}{g^{2g_0+\frac{a_0}{2}+k-2}}\ll\frac{T^5 e^{\frac{7T}{2}}}{g^{18}},$$

since $2g_0+a_0+k\geqslant 36$ guarantees that $2g_0+\frac{a_0}{2}+k\geqslant 18$. Recalling that $T=4\log g$, we conclude that

$$\mathbb{E}_{WP}^{g,n}\left[\sum_{\substack{Y\in\mathrm{Sub}_T(X)\\ |\chi(Y)|\geqslant 34}} e^{-\frac{l(\partial Y)}{4}}\right]\ll\frac{(\log g)^5}{g^3},$$

as required. ■

Lemma 4.18. There is a constant $\beta>0$ such that for any $\varepsilon_1>0$,

$$\mathbb{E}_{WP}^{g,n}\left[\sum_{\substack{Y\in\mathrm{Sub}_T(X)\\ 1\leqslant|\chi(Y)|\leqslant 33}} e^{\frac{T}{2}-\frac{1-\epsilon_1}{2}l(\partial Y)}\right]\ll(\log g)^{\beta}\cdot n^2\cdot g^{1+4\varepsilon_1}.$$

Proof. Let $\varepsilon_1>0$, $g_0\geqslant 0$, $a_0\geqslant 0$ and $k\geqslant 1$ be fixed with $1\leqslant m=2g_0-2+k+a_0\leqslant 33$. The computation in [31, Proposition 34] gives that there exists a fixed $\beta>0$ with

$$\frac{1}{V_{g,n}}\int_{\mathcal{M}_{g,n}}\sum_{\tilde{Y}\in\mathrm{MCG}_{g,n}\cdot\tilde{Y}_0(q,\underline{a},\underline{n},\underline{g})} e^{\frac{T}{2}-\frac{1-\varepsilon_1}{2}l_X\left(\partial\tilde{Y}\right)}\mathbf{1}_{[0,2T]}\left(l_X\left(\partial\tilde{Y}\right)\right)\mathrm{d}X$$

$$\ll\frac{T^{\beta}e^{\frac{T}{2}+\varepsilon_1 T}}{V_{g,n}n_0!\cdots n_q!}V_{g_1,n_1+a_1}\cdots V_{g_q,n_q+a_q}.$$

(Note the value of β in [31, Proposition 34] is 66 and corresponds to the choice to consider $|\chi(Y)| \leqslant 16$ as opposed to our choice of 33. Here we could for example take $\beta < 135$. Fixed powers of $\log g$ will be negligible in the final calculations.) Then we see that

$$\mathbb{E}_{WP}^{g,n}\left[\sum_{\substack{Y\in \mathrm{Sub}_T(X)\\ Y\cong S_{g_0,k+a_0}}} e^{\frac{T}{2}-\frac{1-\varepsilon_1}{2}l(\partial Y)}\right] \ll T^{\beta}e^{\frac{T}{2}+\varepsilon_1 T}\sum_{n_0=0}^{\lfloor\frac{k}{2}\rfloor}\sum_{q=1}^{k-2a_0}\sum_{\mathcal{A}}\frac{n!}{a_0!\cdots a_q!}\frac{V_{g_1,n_1+a_1}\cdots V_{g_q,n_q+a_q}}{n_0!\cdots n_q!\,V_{g,n}},$$

where, as before, for given n_0 and q the summation is over the set $\mathcal{A}$ of "admissible triples" $\{(g_j,n_j,q_j)\}_{j=1}^{q}$ where $g_j,a_j\geqslant 0$, $n_j\geqslant 1$ and $2g_j+a_j+n_j\geqslant 3$ such that $\sum_{i=1}^{q}2g_i-2+n_i+a_i=2g-2+n-m$, $\sum_{i=1}^{q}n_i=k-2n_0$ and $\sum_{j=1}^{q}a_j=n-a_0$. We apply Lemma A.5 to calculate that

$$\begin{aligned}
&T^{\beta}e^{\frac{T}{2}+\varepsilon_1 T}\sum_{n_0=0}^{\lfloor\frac{k}{2}\rfloor}\sum_{q=1}^{k-2n_0}\sum_{\mathcal{A}}\frac{n!}{a_0!\cdots a_q!}\frac{V_{g_1,n_1+a_1}\cdots V_{g_q,n_q+a_q}}{n_0!\cdots n_q!\,V_{g,n}}\\
&\ll T^{\beta}e^{\frac{T}{2}+\varepsilon_1 T}\sum_{n_0=0}^{\lfloor\frac{k}{2}\rfloor}\sum_{q=1}^{k-2n_0}\frac{n^{a_0}}{g^{2g_0+a_0+k-2}}\ll T^{\beta}e^{\frac{T}{2}+\epsilon_1 T}\frac{n^{a_0}}{g^{2g_0+a_0+k-2}}.
\end{aligned}$$

We sum over possible values of g_0, a_0 and k to see that

$$\begin{aligned}
\mathbb{E}_{WP}^{g,n}\left[\sum_{\substack{Y\in \mathrm{Sub}_T(X)\\ 1\leqslant|\chi(Y)|\leqslant 33}} e^{\frac{T}{2}-\frac{1-\epsilon_1}{2}l(\partial Y)}\right] &\ll \sum_{\substack{(g_0,a_0,k)\\ 3\leqslant 2g_0+a_0+k\leqslant 35}} T^{\beta}e^{\frac{T}{2}+\varepsilon_1 T}\frac{n^{a_0}}{g^{2g_0+a_0+k-2}}\\
&\ll T^{\beta}e^{\frac{T}{2}+\varepsilon_1 T}\cdot\frac{n^2}{g}\ll(\log g)^{\beta}\,n^2 g^{1+4\varepsilon_1},
\end{aligned}$$

as claimed. ■

We can now prove Lemma 4.8.

Proof of Lemma 4.8. Combining Lemma 4.15, Lemma 4.17, and Lemma 4.18 we deduce that for any $\varepsilon_1 > 0$ there exists a constant $c_1(\varepsilon_1)$ such that

$$\begin{aligned}
&\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}^{ns}(X)} H_{X,1}(\gamma)\right] \\
&\ll e^{T} T\mathbb{E}_{WP}^{g,n}\left[\sum_{\substack{Y\in\mathrm{Sub}_T(X)\\ |\chi(Y)|\geqslant 34}} e^{-\frac{l(\partial Y)}{4}}\right] + c_1(\varepsilon_1)\, T\mathbb{E}_{WP}^{g,n}\left[\sum_{\substack{Y\in\mathrm{Sub}_T(X)\\ 1\leqslant|\chi(Y)|\leqslant 33}} e^{\frac{T}{2}-\frac{1-\varepsilon_1}{2}l(\partial Y)}\right] \\
&\ll (\log g)^6 g + c_1(\varepsilon_1)(\log g)^{\beta+1} n^2 g^{1+4\varepsilon_1},
\end{aligned}$$

concluding the proof. ■

4.6 Proof of Theorem 4.1

Finally, we conclude the section with the proof of Theorem 4.1.

Proof of Theorem 4.1. By Lemma 4.4, Lemma 4.5, Lemma 4.6, and Lemma 4.8 together with (4.1) we see that there is a constant β such that for any $\varepsilon_1 > 0$ there exists a constant $c_1(\varepsilon_1)$ with

$$\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}(X)}\sum_{k=1}^{\infty} H_{X,k}(\gamma) - \hat{f}_T\left(\frac{i}{2}\right)\right] \ll n^2 g + \log(g)^6 g + c_1(\varepsilon_1)(\log g)^{\beta+1} n^2 g^{1+4\varepsilon_1}.$$

■

5 Proof of Theorem 1.3

We now conclude with the proof of Theorem 1.3.

Proof of Theorem 1.3. Let $n = O(g^{\alpha})$ for some $0 \leqslant \alpha < \frac{1}{2}$ and let $0 < \varepsilon < \min\left\{\frac{1}{4}, \frac{1}{2} - \alpha\right\}$ be given. For $X \in \mathcal{M}_{g,n}$, we define

$$\tilde{\lambda}_1(X) \stackrel{\text{def}}{=} \begin{cases} \lambda_1(X) & \text{if it exists,} \\ \frac{1}{4} & \text{otherwise.} \end{cases}$$

Our aim is to prove that

$$\mathbb{P}_{WP}^{g,n}\left[\tilde{\lambda}_1(X) \leqslant \frac{1}{4} - \frac{(2\alpha+1)^2}{16} - \varepsilon\right] \to 0,$$

as $g \to \infty$. By Remark 2.9, there exists a constant $\nu \geqslant 0$ such that for g sufficiently large,

$$\sum_{\gamma \in \mathcal{P}(X)} \sum_{k=1}^{\infty} \frac{l_\gamma(X)}{2\sinh\left(\frac{kl_\gamma(X)}{2}\right)} f_T\left(kl_\gamma(X)\right) - \hat{f}_T\left(\frac{i}{2}\right) + \nu ng \geqslant 0,$$

for any $X \in \mathcal{M}_{g,n}$. By Theorem 4.1, for any $\varepsilon_1 > 0$ there is constant $c_1(\varepsilon_1) > 0$ with

$$\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma \in \mathcal{P}(X)} \sum_{k=1}^{\infty} H_{X,k}(\gamma) - \hat{f}_T\left(\frac{i}{2}\right) + \nu ng\right] \ll n^2 g + \log(g)^6 g + c_1(\varepsilon_1)(\log g)^{\beta+1} n^2 g^{1+4\varepsilon_1},$$

where $\beta > 0$ is a universal constant. Taking $\varepsilon_1 < \frac{\varepsilon}{8}$ and applying Markov's inequality,

$$\mathbb{P}_{WP}^{g,n}\left[\sum_{\gamma \in \mathcal{P}(X)} \sum_{k=1}^{\infty} H_{X,k}(\gamma) - \hat{f}_T\left(\frac{i}{2}\right) + \nu ng > n^2 g^{1+\varepsilon}\right] \ll_\epsilon \left(1 + \frac{\log(g)^6}{n^2} + (\log g)^{\beta+1}\right) g^{-\frac{\varepsilon}{2}}.$$

However, if $X \in \mathcal{M}_{g,n}$ has $\lambda_1(X) \leqslant \frac{1}{4} - \frac{(2\alpha+1)^2}{16} - \varepsilon$, then since $\alpha \in [0, \frac{1}{2})$ this guarantees that $\lambda_1(X) \leqslant \frac{3}{16}$ and we can apply Theorem 2.1 to see

$$C(\varepsilon)\log(g)\, g^{4(1-\varepsilon)\sqrt{\frac{1}{4}-\lambda_1(X)}} \leqslant \sum_{\gamma \in \mathcal{P}(X)} \sum_{k=1}^{\infty} H_{X,k}(\gamma) - \hat{f}_T\left(\frac{i}{2}\right) + O(ng).$$

But since $\varepsilon < \frac{1}{2} - \alpha$,

$$\sqrt{\frac{1}{4} - \lambda_1(X)} \geqslant \frac{2\alpha+1}{4} + \varepsilon,$$

and we deduce that

$$C(\varepsilon)\log(g)\, g^{4(1-\varepsilon)\sqrt{\frac{1}{4}-\lambda_1(X)}} \geqslant C(\varepsilon)\log(g)\, g^{(1-\varepsilon)((2\alpha+1)+4\varepsilon)} \gg_\epsilon g^{2\alpha+1+2\varepsilon-4\varepsilon^2} > n^2 g^{1+\varepsilon},$$

for sufficiently large g. On the last line we used that $\varepsilon < \frac{1}{4}$ and that $n = O(g^{\alpha})$. We deduce that

$$\sum_{\gamma \in \mathcal{P}(X)} \sum_{k=1}^{\infty} H_{X,k}(\gamma) - \hat{f}_T\left(\frac{i}{2}\right) > n^2 g^{1+\varepsilon},$$

for sufficiently large g. This tells us that for g sufficiently large,

$$\mathbb{P}_{WP}^{g,n}\left[\tilde{\lambda}_1(X) \leqslant \frac{1}{4} - \frac{(2\alpha+1)^2}{16} - \varepsilon\right] \leqslant \mathbb{P}_{WP}^{g,n}\left[\sum_{\gamma \in \mathcal{P}(X)} \sum_{k=1}^{\infty} H_{X,k}(\gamma) - \hat{f}_T\left(\frac{i}{2}\right) + \nu n g > n^2 g^{1+\varepsilon}\right]$$
$$\ll_{\varepsilon} \left(1 + \frac{\log(g)^6}{n} + (\log g)^{\beta+1}\right) g^{-\frac{\varepsilon}{2}} \to 0,$$

as $g \to \infty$. ■

A. Volume Estimates

The purpose of this appendix is to prove the necessary Weil–Petersson volume estimates used in the proof of Theorem 4.1. Similar estimates can be found in, for example, [5, 14, 16, 17, 25].

We need the following lemma in the proof of Lemma 4.4 and Lemma 4.5.

Lemma A.1. Let $x_1, \ldots, x_n \geqslant 0$. For $g, n \geqslant 0$, $2g - 2 + n > 0$ we have

$$\frac{V_{g,n}(x_1, \ldots, x_n)}{V_{g,n}} \leqslant \prod_{i=1}^{n} \frac{\sinh\left(\frac{x_i}{2}\right)}{\left(\frac{x_i}{2}\right)},$$

and

$$\frac{V_{g,n}(\underline{0}_{n-2}, x_1, x_2)}{V_{g,n}} = \frac{4\sinh\left(\frac{x_1}{2}\right) \cdot \sinh\left(\frac{x_2}{2}\right)}{x_1 \cdot x_2} \left(1 + O\left(\frac{n\left(x_1^2 + x_2^2\right)}{g}\right)\right),$$

as $g \to \infty$, where the implied constant is independent of n.

Remark A.2. Lemma A.1 is due to [16, Proposition 3.1] and [25, Lemma 20]. The proof of the second statement is identical to the proof of [25, Lemma 20], if one uses [14, Theorem A.1] in place of [17, p. 286].

We require estimates for $V_{g,n}$ where the number of cusps n is allowed to grow with the genus g. The starting point is the following theorem of Mirzakhani and Zograf.

Theorem A.3 ([24, Theorem 1.8]). There exists a constant $B > 0$ such that if $n = o(g^{\frac{1}{2}})$, we have

$$V_{g,n(g)} = \frac{B}{\sqrt{g}}\,(2g-3+n(g))!\left(4\pi^2\right)^{2g-3+n(g)}\left(1+O\left(\frac{1+n(g)^2}{g}\right)\right),$$

as $g \to \infty$.

In order to control the contribution of simple separating geodesics, in Lemma 4.4 we need the following lemma.

Lemma A.4. If $n = o(g^{\frac{1}{2}})$, then

$$\sum_{\substack{0\leqslant i\leqslant g,\,0\leqslant j\leqslant n\\ 2\leqslant 2i+j\leqslant 2g+n-2}} \binom{n}{j}\cdot\frac{V_{i,j+1}V_{g-i,n-j+1}}{V_{g,n}} \ll \frac{1+n^2}{g}.$$

The case that n is fixed is treated in [17, Lemma 3.3]. The fact that the number of cusps is growing with genus and the presence of the multiplicity $\binom{n}{j}$ presents the new difficulty here.

In the following, we shall frequently apply Stirling's approximation, which tells us that there exist constants $1 < c_1 < c_2 < 2$ with

$$c_1\cdot\sqrt{2\pi w}\left(\frac{w}{e}\right)^w < w! < c_2\cdot\sqrt{2\pi w}\left(\frac{w}{e}\right)^w, \tag{A.1}$$

for all $w \geqslant 1$.

Proof of Lemma A.4. By Theorem A.3, since $n = o\left(\sqrt{g}\right)$, we have

$$V_{g,n(g)} = \frac{B}{\sqrt{g}}\,(2g-3+n)!\left(4\pi^2\right)^{2g-3+n}\left(1+O\left(\frac{1+n^2}{g}\right)\right). \tag{A.2}$$

By [17, Lemma 3.2, part 3] we have that for $a, b \geqslant 0$, $2a+b \geqslant 1$,

$$V_{a,b+4} \leqslant V_{a+1,b+2}. \tag{A3}$$

Applying (A.3) iteratively, for $j \geqslant 1$,

$$V_{i,j+1} \leqslant V_{i+\lfloor\frac{j-1}{2}\rfloor,\,j+1-2\lfloor\frac{j-1}{2}\rfloor}.$$

We can then apply Theorem A.3 to see that

$$V_{i,j+1}V_{g-i,n-j+1} \ll \left(4\pi^2\right)^{2g+n}\frac{(2i+j-2)!}{\sqrt{i+\max\left\{\lfloor\frac{j-1}{2}\rfloor,0\right\}}}\cdot\frac{(2g-2i+n-j-2)!}{\sqrt{g-i+\max\left\{\lfloor\frac{n-j-1}{2}\rfloor,0\right\}}}. \tag{A.4}$$

We also observe that

$$\frac{\sqrt{g}}{\sqrt{g-i+\max\left\{\lfloor\frac{n-j-1}{2}\rfloor,0\right\}}\cdot\sqrt{i+\max\left\{\lfloor\frac{j-1}{2}\rfloor,0\right\}}}\ll 1. \tag{A.5}$$

Then applying (A.2), (A.4), and (A.5),

$$\sum_{\substack{0\leqslant i\leqslant g,0\leqslant j\leqslant n\\ 2\leqslant 2i+j\leqslant 2g+n-2}}\frac{n!}{j!\,(n-j)\,!}\cdot\frac{V_{i,j+1}V_{g-i,n-j+1}}{V_{g,n}}$$
$$\ll\sum_{\substack{0\leqslant i\leqslant g,0\leqslant j\leqslant n\\ 2\leqslant 2i+j\leqslant 2g+n-2}}\frac{n!}{j!\,(n-j)\,!}\frac{(2i+j-2)\,!\,(2g-2i+n-j-2)\,!}{(2g+n-3)\,!}.$$

If $i=0$ then $j\geqslant 2$ and we have

$$\sum_{j=2}^{n}\frac{n!}{j!\,(n-j)\,!}\cdot\frac{(j-2)\,!\,(2g+n-j-2)\,!}{(2g+n-3)\,!}=\sum_{j=2}^{n-4}\frac{n!}{j\,(j-1)\,(n-j)\,!}\cdot\frac{(2g+n-j-2)\,!}{(2g+n-3)\,!}$$
$$\ll\frac{n^2}{g}+\sum_{j=3}^{n-4}\frac{n^j}{g^{j-1}}\ll\frac{n^2}{g},$$

since $n=o\left(\sqrt{g}\right)$. By symmetry, the same calculation holds for the case that $i=g$. Similarly, if $i=1$ then $j\geqslant 0$ and we calculate

$$\sum_{j=0}^{n}\frac{n!}{j!\,(n-j)\,!}\cdot\frac{j!\,(2g+n-j-4)\,!}{(2g+n-3)\,!}\ll\sum_{j=0}^{n}\frac{n^j}{g^{j+1}}\ll\frac{1}{g}.$$

The same calculation holds in the case that $i=g-1$ by symmetry. If $2\leqslant i\leqslant g-2$, then we claim that

$$\frac{n!}{j!\,(n-j)\,!}\frac{(2i+j-2)\,!\,(2g-2i+n-j-2)\,!}{(2g+n-3)\,!}\ll g^{-3}. \tag{A.6}$$

It is a straightforward calculation to check that (A.6) holds in the case that $i=2,j=0$ and $i=2,j=1$. Now let $L=2i+j$. Then if $6\leqslant L\leq n$,

$$\frac{n!}{j!\,(n-j)\,!}\frac{(2i+j-2)\,!\,(2g-2i+n-j-2)\,!}{(2g+n-3)\,!}\ll\frac{L!\cdot n^L}{g^L}\ll\sqrt{L}\left(\frac{Ln}{ge}\right)^L,$$

by Stirling's approximation. If $L=6$, then

$$\sqrt{L}\left(\frac{Ln}{ge}\right)^L\ll\left(\frac{n}{g}\right)^6\ll g^{-3}.$$

If $6 < L \leqslant n-1$, then

$$\sqrt{L}\left(\frac{Ln}{ge}\right)^{L} = \sqrt{L}\left(\frac{Ln}{ge}\right)^{L}\left(\frac{6n}{ge}\right)^{6}\cdot\left(\frac{eg}{6n}\right)^{6} \ll \sqrt{L}g^{-3}\left(\frac{L}{6}\right)^{6}\left(\frac{Ln}{ge}\right)^{L-6}$$
$$\leqslant L^{\frac{13}{2}}e^{6-L}g^{-3}\cdot\frac{n^2}{g} \ll g^{-3}.$$

If $n \leqslant L \leqslant \frac{1}{2}(2g+n-2)$, then since

$$\binom{n}{i} \leqslant 2^n,$$

we have

$$\frac{n!}{j!\,(n-j)\,!}\frac{(2i+j-2)\,!\,(2g-2i+n-j-2)\,!}{(2g+n-3)\,!} \ll 2^n\frac{L!\,(2g+n-2-L)\,!}{(2g+n-3)\,!}$$
$$\leqslant \frac{2^n n!\,(2g-2-n)\,!}{(2g+n-3)\,!} \ll \left(\frac{2n}{g}\right)^{n} \ll g^{-3}.$$

By symmetry, the case that $2i+j \geqslant \frac{1}{2}(2g+n-2)$ is treated analogously. This establishes the claim (A.6). We can now use the rough bound

$$\#\{(i,j) \in \mathbb{Z}_{\geqslant 0} \mid 2 \leqslant i \leqslant g-2, 0 \leqslant j \leqslant n, 2 \leqslant 2i+j \leqslant 2g+n-2\} \ll ng,$$

to deduce that

$$\sum_{\substack{2\leqslant i\leqslant g-2, 0\leqslant j\leqslant n\\ 2\leqslant 2i+j\leqslant 2g+n-2}} \frac{n!}{j!\,(n-j)\,!}\frac{(2i+j-2)\,!\,(2g-2i+n-j-2)\,!}{(2g+n-3)\,!} \ll \frac{n}{g^2},$$

and the result follows. ■

In order to deal with the contribution of non-simple geodesics, we needed the following Lemma.

Lemma A.5. Let $n = o\left(\sqrt{g}\right)$ and let g_0, a_0, n_0 and k be given with $m = 2g_0 + a_0 + k - 2 \leqslant 3\log g - 2$. For $1 \leqslant q \leqslant k - 2n_0$,

$$\sum_{\{(g_j,a_j,n_j)\}_{i=1}^{q}\in\mathcal{A}} \frac{n!}{a_0!\cdots a_q!}\cdot\frac{V_{g_0,n_0+a_0}\cdots V_{g_q,n_q+a_q}}{V_{g,n}} \ll \left(2g_0+k+a_0-3\right)!\,\frac{n^{a_0}}{g^m},$$

where the summation is taken over the set $\mathcal{A}$ of all "admissible triples" $\{(g_1,a_1,n_1),\ldots,(g_q,a_q,n_q)\}$, where $g_j, a_j \geqslant 0$, $n_j \geqslant 1$ and $2g_j + a_j + n_j \geqslant 3$ such that

i) $\sum_{i=1}^{q}\left(2g_i - 2 + n_i + a_i\right) = 2g - 2 + n - m$,
ii) $\sum_{i=1}^{q} n_i = k - 2n_0$,
iii) $\sum_{j=1}^{q} a_j = n - a_0$.

This is similar to estimates proved in [31] but here we need the number of cusps to grow with genus and we have the extra multiplicity

$$\frac{n!}{a_0!\cdot\dots\cdot a_q!}.$$

We take a similar approach as in the proof of Lemma A.4. Lemma A.5 relies on a lot of computations that, for the sake of readability, are done separately in Lemma A.6.

Proof of Lemma A.5 given Lemma A.6. By [17, Lemma 3.2, part 3] we see that for each $a_i + n_i \geqslant 2$, we have

$$V_{g_i,a_i+n_i} \leqslant V_{g_i+\lfloor\frac{a_i+n_i-2}{2}\rfloor,a_i+n_i-2\lfloor\frac{a_i+n_i-2}{2}\rfloor}.$$

This allows us to apply Theorem A.3, which tells us that there exists $C_1 > 0$ with

$$V_{g_1,n_1+a_1}\cdots V_{g_q,n_q+a_q} \leqslant C_1^q \prod_{j=1}^{q} \frac{\left(4\pi^2\right)^{2g_j+a_j+n_j-3}\left(2g_j+a_j+n_j-3\right)!}{\sqrt{g_j+\max\left\{\lfloor\frac{a_j+n_j-2}{2}\rfloor,0\right\}}}, \tag{A.7}$$

where since $V_{0,3} = 1$ we interpret the product in (A.7) as only over triples with $g_j + \max\left\{\lfloor\frac{a_j+n_j-2}{2}\rfloor,0\right\} > 0$. We also see by Theorem A.3 that

$$V_{g_0,a_0+k} \leqslant C_1\left(4\pi^2\right)^{2g_0+a_0+k-3}\left(2g_0+a_0+k-3\right)!, \tag{A.8}$$

and

$$V_{g,n} = \frac{B}{\sqrt{g}}\left(2g-3+n(g)\right)!\left(4\pi^2\right)^{2g-3+n(g)}\left(1+O\left(\frac{1+n(g)^2}{g}\right)\right). \tag{A.9}$$

We introduce the notation $\overline{a_j+n_j} \overset{\text{def}}{=} \max\left\{\lfloor\frac{a_j+n_j-2}{2}\rfloor,0\right\}$. By applying (A.7), (A.8), and (A.9) and noting that $n_i! \geqslant 1$ for each i, we calculate that

$$\sum_{\mathcal{A}} \frac{n!}{a_0!\cdot\dots\cdot a_q!}\cdot\frac{V_{g_0,n_0+k}\cdot V_{g_1,n_1+a_1}\cdot\dots\cdot V_{g_q,n_q+a_q}}{V_{g,n}\cdot n_0!\,n_1!\cdots n_q!}$$

$$\ll \left(2g_0+k+a_0-3\right)!\sum_{\mathcal{A}}\frac{C_1^q\sqrt{g}}{\prod_{j=1}^q\sqrt{g_j+\overline{a_j+n_j}}}\frac{n!}{\prod_{j=0}^q a_j!}\frac{\prod_{j=1}^q\left(2g_j+a_j+n_j-3\right)!}{(2g+n-3)!}.$$

The result then follows from the fact that

$$\sum_{\mathcal{A}}\frac{C_1^q\sqrt{g}}{\prod_{j=1}^q\sqrt{g_j+\overline{a_j+n_j}}}\frac{n!}{\prod_{j=0}^q a_j!}\frac{\prod_{j=1}^q\left(2g_j+a_j+n_j-3\right)!}{(2g+n-3)!} \ll \frac{n^{a_0}}{g^m},$$

which is proved in Lemma A.6. ■

We now need to prove Lemma A.6, which is purely computational.

Lemma A.6. Let $n = o\left(\sqrt{g}\right)$, and let g_0, a_0, n_0 and k be given with $m = 2g_0 + a_0 + k - 2 \leqslant 3\log g - 2$ and $1 \leqslant q \leqslant k - 2n_0$. With $\mathcal{A}$ as in Lemma A.5, we have

$$\sum_{\mathcal{A}} \frac{C_1^q \sqrt{g}}{\prod_{j=1}^q \sqrt{g_j + a_j + n_j}} \frac{n!}{\prod_{j=0}^q a_j!} \frac{\prod_{j=1}^q \left(2g_j + a_j + n_j - 3\right)!}{(2g + n - 3)!} \ll \frac{n^{a_0}}{g^m}. \tag{A.10}$$

In the proof of Lemma A.6, we will frequently apply the following observation: if $x_i \geqslant 0$ with $\sum_{i=1}^s x_i = A$, then

$$\prod_{i=1}^s x_i! \leqslant A!\,, \tag{A.11}$$

which can be seen by the fact that the multinomial coefficient $\binom{A}{x_1,\ldots,x_s}$ is bounded below by 1.

Proof. We first note that $q \leqslant 3\log g$. For $\left\{(g_1, a_1, n_1), \ldots, \left(g_q, a_q, n_q\right)\right\} \in \mathcal{A}$, we claim that if $\max_{1\leqslant i\leqslant q}\left(2g_i + a_i + n_i - 3\right) \leqslant 2g + n - 3 - m - 8q$, then

$$\frac{C_1^q \sqrt{g}}{\prod_{j=1}^q \sqrt{g_j + a_j + n_j}} \frac{n!}{\prod_{j=0}^q a_j!} \frac{\prod_{j=1}^q \left(2g_j + a_j + n_j - 3\right)!}{(2g + n - 3)!} \ll g^{-\frac{7}{2}q}. \tag{A.12}$$

This estimate is analogous to (A.6). Once we have established (A.12) we shall apply a rough counting argument to bound the contribution of such terms to the sum (A.10).

Let $\max_{1\leqslant i\leqslant q}\left(2g_i + a_i + n_i - 3\right) = 2g + n - 3 - m - L$. First we treat the case that $L \geqslant \frac{1}{2}(2g + n - m - 3)$. We apply Stirling's approximation (A.1) to see that

$$\begin{aligned}\frac{\left(2g_i + n_i + a_i - 3\right)!}{\sqrt{g_j + a_j + n_j}} &< c_2 \frac{\sqrt{2\pi\left(2g_i + n_i + a_i - 3\right)}}{\sqrt{g_j + a_j + n_j}} \cdot \left(\frac{2g_i + a_i + n_i - 3}{e}\right)^{2g_i + a_i + n_i - 3} \\ &< 4\sqrt{\pi} \cdot \left(\frac{2g_i + a_i + n_i - 3}{e}\right)^{2g_i + a_i + n_i - 3}.\end{aligned} \tag{A.13}$$

Applying Stirling's approximation again, we see that

$$\begin{aligned}\frac{\sqrt{g}}{(2g + n - 3)!} &> \frac{1}{c_2} \frac{\sqrt{g}}{\sqrt{2\pi(2g + n - 3)}} \cdot \left(\frac{e}{2g + n - 3}\right)^{2g + n - 3} \\ &\gg \left(\frac{e}{2g + n - 3}\right)^{2g + n - 3}.\end{aligned} \tag{A.14}$$

We also note that

$$\frac{n!}{a_0!\cdots a_q!} \leqslant q^n, \tag{A.15}$$

by the multinomial theorem. By (A.13), (A.14), and (A.15),

$$\begin{aligned}
&\frac{C_1^q\sqrt{g}}{\prod_{j=1}^q \sqrt{g_j + a_j + n_j}} \frac{n!}{\prod_{j=0}^q a_j!} \frac{\prod_{j=1}^q \left(2g_j + a_j + n_j - 3\right)!}{(2g+n-3)!} \\
\ll &q^n \cdot \left(4C_1\sqrt{\pi}\right)^q \frac{\prod_{j=1}^q \left(2g_j + a_j + n_j - 3\right)^{(2g_j+a_j+n_j-3)}}{(2g+n-3)^{(2g+n-3)}}.
\end{aligned} \tag{A.16}$$

We now bound the expression in (A.16). Given s integers $x_i > 0$, Jensen's inequality for concave functions applied to the function $\log x$ tells us that

$$\frac{\sum_{i=1}^s x_i \log x_i}{\sum_{i=1}^s x_i} \leqslant \log\left(\frac{\sum_{i=1}^s x_i^2}{\sum_{i=1}^s x_i}\right).$$

If $\sum_{i=1}^s x_i = A$ and $\max_{1\leqslant i\leqslant s} x_i = B$, then

$$\begin{aligned}
\sum_{i=1}^s x_i \log x_i &\leqslant A \log\left(\frac{\sum_{i=1}^s x_i^2}{\sum_{i=1}^s x_i}\right) \\
&\leqslant A \log B,
\end{aligned}$$

and by exponentiating, we conclude that

$$\prod_{i=1}^s x_i^{x_i} \leqslant B^A. \tag{A.17}$$

Note that (A.17) also holds if instead we just require $x_i \geqslant 0$ since we can apply Jensen's inequality with only the non-zero terms. Recall that $\max_{1\leqslant i\leqslant q}\left(2g_i + a_i + n_i - 3\right) = 2g + n - 3 - m - L$ for $L \geqslant \frac{1}{2}(2g+n-m-3)$. Since $\sum_{i=1}^q \left(2g_i - 2 + n_i + a_i\right) = 2g - 2 + n - m$, then in particular, $\sum_{i=1}^q \left(2g_i - 3 + n_i + a_i\right) \leqslant 2g + n - m - 3$ and we can apply (A.17) to (A.16) to calculate that

$$\begin{aligned}
&q^n \cdot \left(4C_1\sqrt{\pi}\right)^q \frac{\prod_{j=1}^q \left(2g_j + a_j + n_j - 3\right)^{(2g_j+a_j+n_j-3)}}{(2g+n-3)^{(2g+n-3)}} \\
\ll &q^n \cdot \left(4C_1\sqrt{\pi}\right)^q \cdot \frac{(2g+n-3-m-L)^{(2g+n-3)}}{(2g+n-3)^{(2g+n-3)}} \\
\leqslant &q^n \cdot 2^{3q} C_1^q \left(\frac{1}{2}\right)^{2g+n-3} \leqslant q^n \left(\frac{1}{2}\right)^{2g+n-3-3q-q\log_2 C_1}.
\end{aligned}$$

Since $q \leqslant 3\log g$ and $n = o\left(\sqrt{g}\right)$,

$$q^n \left(\frac{1}{2}\right)^{2g+n-3-3q-q\log_2 C_1} \ll \left(\frac{1}{2}\right)^{g} = g^{-\frac{g}{\log_2 g}} \ll g^{-\frac{7}{2}q}.$$

This justifies the claim in the case that $L \geqslant \frac{1}{2}(2g+n-m-3)$.

In order to treat the remaining cases, we first make the following observation. Recalling that $\sum_{j=1}^{q}\left(2g_j + a_j + n_j - 3\right) = 2g+n-m-3-\left(q-1\right)$ and that $\overline{a_j+n_j} \overset{\text{def}}{=} \max\left\{\lfloor \frac{a_j+n_j-2}{2} \rfloor, 0\right\}$, we see that

$$\sum_{j=1}^{q}\left(g_j + \overline{a_j+n_j}\right) \geqslant \frac{1}{2}\sum_{j=1}^{q}\left(2g_j + a_j + n_j - 3\right) \geqslant \frac{2g+n-m-3-(q-1)}{2}.$$

For any q positive integers x_i, we have

$$\prod_{i=1}^{q} x_i \geqslant \sum_{i=1}^{q} x_i - (q-1).$$

Then,

$$\prod_{j=1}^{q}\left(g_j + \overline{a_j+n_j}\right) \geqslant \frac{2g+n-m-3-(q-1)}{2} - q - 1 \gg g,$$

since $n = o\left(\sqrt{g}\right)$ and $q, m = O(\log g)$. We see that

$$\frac{\sqrt{g}}{\prod_{j=1}^{q}\sqrt{g_j + \overline{a_j+n_j}}} \ll 1,$$

and therefore

$$\frac{C_1^q\sqrt{g}}{\prod_{j=1}^{q}\sqrt{g_j + \overline{a_j+n_j}}}\frac{n!}{\prod_{j=0}^{q} a_j!}\frac{\prod_{j=1}^{q}\left(2g_j + a_j + n_j - 3\right)!}{(2g+n-3)!}$$
$$\ll \frac{C_1^q n!}{\prod_{j=0}^{q} a_j!}\frac{\prod_{j=1}^{q}\left(2g_j + a_j + n_j - 3\right)!}{(2g+n-3)!}. \tag{A.18}$$

The expression in (A.18) will be easier to work with for the remaining cases. Recalling that $\max_{1\leqslant i\leqslant q}\left(2g_i + a_i + n_i - 3\right) = 2g+n-3-m-L$, we now treat the case that $8q \leqslant L \leqslant n - a_0$. Since $\max_{1\leqslant i\leqslant q}\left(2g_i + a_i + n_i - 3\right) = 2g+n-3-m-L$, this forces $\max_{1\leqslant i\leqslant q} a_i \geqslant n - a_0 - L$. Indeed if $\max_{1\leqslant i\leqslant q} a_i < n - a_0 - L$ we would have that

$$\max_{1\leqslant i\leqslant q}\left(2g_i + n_i\right) > 2g - 2g_0 - n_0,$$

which is not possible. Since there is an $1 \leqslant i \leqslant q$ such that $2g_i + a_i + n_i - 3 = 2g + n - 3 - m - L$ and we have $\sum_{j=1, j\neq q}^{q}\left(2g_i + a_i + n_i - 3\right) = L - (q-1) \leqslant L$, we apply (A.11) to see that

$$\prod_{j=1}^{q}\left(2g_j+a_j+n_j-3\right)!=(2g+n-3-m-L)!\prod_{j=1,j\neq i}^{q}\left(2g_j+a_j+n_j-3\right)!$$
$$\leqslant L!\,(2g+n-3-m-L)!\,. \tag{A.19}$$

We then use the rough bound

$$\frac{n!}{\prod_{j=0}^{q}a_j!}\leqslant\frac{n!}{\left(\max_{1\leqslant i\leqslant q}a_i\right)!}\leqslant\frac{n!}{(n-a_0-L)!}\ll n^{a_0+L}, \tag{A.20}$$

together with (A.19), to see that

$$\frac{n!}{\prod_{j=0}^{q}a_j!}\frac{\prod_{j=1}^{q}\left(2g_j+a_j+n_j-3\right)!}{(2g+n-3)!}\ll\frac{n^{a_0+L}L!\,(2g+n-3-m-L)!}{(2g+n-3)!}\ll\frac{n^{a_0+L}}{g^{m+L}}L!\,. \tag{A.21}$$

By applying Stirling's approximation (A.1),

$$\frac{n^{a_0+L}}{g^{m+L}}L!\ll\sqrt{L}\left(\frac{n\cdot L}{e\cdot g}\right)^{L}\cdot\frac{n^{a_0}}{g^{m}}.$$

If $L=8q$ then since $n=o\left(\sqrt{g}\right)$ and $q\leqslant 3\log g$,

$$C_1^q\sqrt{L}\left(\frac{n\cdot L}{e\cdot g}\right)^{L}\ll C_1^q g^{-4q}\,(8q)^{8q+\frac{1}{2}}\ll g^{\frac{-7q}{2}}. \tag{A.22}$$

Now if $8q<L\leqslant n-a_0$,

$$C_1^q\frac{n^{a_0}}{g^m}.\sqrt{L}\left(\frac{n\cdot L}{e\cdot g}\right)^{L}\ll C_1^q\frac{\sqrt{L}}{\sqrt{g}}\left(\frac{n\cdot L}{e\cdot g}\right)^{L}\cdot\left(\frac{n\cdot 8q}{e\cdot g}\right)^{8q}\cdot\left(\frac{e\cdot g}{n\cdot 8q}\right)^{8q}$$
$$\ll g^{\frac{-7q}{2}}\cdot\left(\frac{L}{8q}\right)^{8q}\cdot\left(\frac{n\cdot L}{e\cdot g}\right)^{L-8q}$$
$$\leqslant g^{\frac{-7q}{2}}\cdot e^{L-8q}\cdot\left(\frac{n\cdot L}{e\cdot g}\right)^{L-8q}\ll g^{\frac{-7q}{2}},$$

which justifies the claim (A.12) in the case that $8q\leqslant L\leqslant n-a_0$. Finally, we treat the case that $8q<n-a_0<L\leqslant\frac{2g+n-3-m}{2}$. We calculate, with (A.19) and (A.15), that

$$\frac{C_1^q n!}{\prod_{j=0}^{q}a_j!}\frac{\prod_{j=1}^{q}\left(2g_j+a_j+n_j-3\right)!}{(2g+n-3)!}\ll\frac{C_1^q\cdot q^n\cdot L!\,(2g+n-m-3-L)!}{(2g+n-3)!}$$
$$\ll\frac{C_1^q\cdot q^n\,(n-a_0)!\,(2g+a_0-m-3)!}{(2g+n-m)!}$$
$$\ll\frac{g^{3\log C_1}\,(3\log g)^{n+1}\,n^n}{(2g)^n}\ll g^{-\frac{7}{2}q}, \tag{A.23}$$

which justifies the claim (A.12) for $8q < n - a_0 < L \leqslant \frac{2g+n-3-m}{2}$. Note that in the case that $n \leqslant 8q - n_0$ we can simply apply the argument in (A.23) with $L \geqslant 8q$. The claim (A.12) is now proved.

Now we have established (A.12), we apply the very rough bound for the size of the set $\mathcal{A}$,

$$|\mathcal{A}| \ll g^{3q},$$

together with (A.12) to calculate

$$\sum_{\substack{\{(g_i,a_i,n_i)\}_{i=1}^q \in \mathcal{A} \\ \max_{1\leqslant i\leqslant q}(2g_i+a_i+n_i-3)\leqslant 2g+n-2-m-8q}} \frac{C_1^q\sqrt{g}}{\prod_{j=1}^q \sqrt{g_j+\overline{a_j+n_j}}} \frac{n!\prod_{j=1}^q \left(2g_j+a_j+n_j-3\right)!}{\prod_{j=0}^q a_j!\,(2g+n-3)!}$$

$$\ll \frac{n^{a_0}}{g^m} \sum_{\substack{\{(g_i,a_i,n_i)\}_{i=1}^q \in \mathcal{A} \\ \max_{1\leqslant i\leqslant q}(2g_i+a_i+n_i-3)\leqslant 2g+n-2-m-8q}} g^{-\frac{7}{2}q} \ll |\mathcal{A}| \cdot \frac{n^{a_0}}{g^m} \cdot g^{-\frac{7}{2}q} \ll \frac{n^{a_0}}{g^m} \cdot g^{-\frac{q}{2}}. \tag{A.24}$$

We now consider the sum

$$\sum_{\substack{\{(g_i,a_i,n_i)\}_{i=1}^q \in \mathcal{A} \\ \max_{1\leqslant i\leqslant q}(2g_i+a_i+n_i-3)> 2g+n-3-m-8q}} \frac{C_1^q\sqrt{g}}{\prod_{j=1}^q \sqrt{g_j+\overline{a_j+n_j}}} \frac{n!\prod_{j=1}^q \left(2g_j+a_j+n_j-3\right)!}{\prod_{j=0}^q a_j!\,(2g+n-3)!}. \tag{A.25}$$

Let $\max_{1\leqslant i\leqslant q}\left(2g_i+a_i+n_i-3\right) = 2g+n-3-m-L$. Since $2g_j+a_j+n_j-3 \geqslant 0$ and $\sum_{j=1}^q 2g_j+a_j+n_j-3 = 2g+n-m-3-(q-1)$, we see that $L \geqslant q-1$. By the same arguments as in (A.21) and (A.22), if $q-1 \leqslant L \leqslant 8q \leqslant 24\log g$ then

$$\frac{n!}{\prod_{j=0}^q a_j!} \frac{\prod_{j=1}^q \left(2g_j+a_j+n_j-3\right)!}{(2g+n-3)!} \ll \frac{n^{a_0+L}}{g^{m+L}} L! \ll \frac{n^{a_0}}{g^m} \cdot g^{-\frac{L}{4}}. \tag{A.26}$$

We now bound the number of $\{(g_1,a_1,n_1),\ldots,(g_q,a_q,n_q)\} \in \mathcal{A}$ with $\max_{1\leqslant i\leqslant q}(2g_i+a_i+n_i-3) = 2g+n-3-m-L$. Assume we have that $2g_1+a_1+n_1-3 = 2g+n-3-m-L$. The remaining $q-1$ triples satisfy

$$\sum_{2\leqslant i\leqslant q} \left(2g_i+a_i+n_i\right) = L+3\,(q-1)\,.$$

Since $\sum_{i=1}^q n_i = k-2n_0$ and $\sum_{j=1}^q a_j = n-a_0$, the triple $\left(g_1,a_1,n_1\right)$ is determined by the choice of $\{(g_2,a_2,n_2),\ldots,(g_q,a_q,n_q)\}$. Then the number of $\left\{\left(g_i,a_i,n_i\right)\right\}_{i=1}^q \in \mathcal{A}$ with $\max_{1\leqslant i\leqslant q}\left(2g_i+a_i+n_i-3\right) = 2g+n-3-m-L$ is therefore bounded above by

$$\binom{L+6\,(q-1)}{3\,(q-1)}. \tag{A.27}$$

Therefore, combining (A.18), (A.26), and (A.27) we see that the sum (A.25) satisfies

$$\sum_{\substack{\{(g_i,a_i,n_i)\}_{i=1}^{q}\in\mathcal{A} \\ \max_{1\leqslant i\leqslant q}(2g_i+a_i+n_i-3)>2g+n-3-m-8q}} \frac{C_1^q\sqrt{g}}{\prod_{j=1}^{q}\sqrt{g_j+a_j+n_j}} \frac{n!\prod_{j=1}^{q}\left(2g_j+a_j+n_j-3\right)!}{\prod_{j=0}^{q}a_j!\,(2g+n-3)!}$$

$$\ll \frac{n^{a_0}}{g^m}\sum_{L=q-1}^{8q}\binom{L+6(q-1)}{3(q-1)}\frac{C_1^q}{g^{\frac{L}{4}}} \ll \frac{n^{a_0}}{g^m}. \tag{A.28}$$

Combining (A.24) and (A.28), the result follows. ∎

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