THE BATCHELOR–HOWELLS–TOWNSEND SPECTRUM: THREE-DIMENSIONAL CASE

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Abstract. Given a velocity field $u(x,t)$, we consider the evolution of a passive tracer θ governed by $\partial_t\theta + u\cdot\nabla\theta = \Delta\theta + g$ with time-independent source $g(x)$. When u is small in some sense, Batchelor, Howells and Townsend (1959, J. Fluid Mech. 5:134; henceforth BHT59) predicted that the tracer spectrum scales as $|\theta_k|^2 \propto |k|^{-4}|u_k|^2$. Following our recent work for the two-dimensional case, in this paper we prove that the BHT59 scaling does hold probabilistically, asymptotically for large wavenumbers and for small enough random synthetic three-dimensional incompressible velocity fields $u(x,t)$. We also relaxed some assumptions on the velocity and tracer source, allowing finite variances for both and full power spectrum for the latter.

1. Introduction

Several theories relate the spectrum of a passive tracer to the energy spectrum of the velocity that advects it. Now the velocity (energy) spectrum is conjectured to obey the Kolmogorov–Obukhov scaling in three space dimensions or the Kraichnan–Batchelor scaling in two dimensions, in an "inertial range" between the forcing scale and some small limiting scale. Obtaining these from the Navier–Stokes equations is a major open problem, with the three-dimensional case also dependent on the resolution of the Navier–Stokes problem, so for the passive tracer problem one often assumes the existence of such an inertial range, with an energy spectrum scaling as $\mathcal{E}(k) \sim |k|^{\hat\beta}$. Then in the case of small Prandtl number, when inertial effects dominate (tracer) diffusion, which happens at larger scales for large velocity, the Obukhov–Corrsin theory [7, 22] predicts that the tracer spectrum scales as $|k|^{-(\hat\beta+5)/2}$.

On the other hand, tracer diffusion will inevitably dominate at smaller scales (but still larger than the limiting scale). In this regime, it was predicted by Batchelor, Howells and Townsend [2, henceforth BHT59] that the tracer spectrum should scale as $|k|^{-4}\mathcal{E}(k)$. An important ingredient in their argument is that, in the relevant scales, the time-dependence of the velocity can be neglected, essentially reducing the problem to a static one. Their second important assumption is that, where their scaling obtains, correlations between the tracer and the velocity can be neglected.

In [17], Kraichnan proposed using a velocity that has the conjectured spatial correlation but is white noise in time. The latter property circumvents the correlation difficulty, allowing computations of higher-order structure functions, anomalous

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scalings, etc, using tools from theoretical physics [6, 10, 16, 18]; the approach has also been applied to tracer decay, i.e. with $g = 0$ [27]. Although such white-noise velocity "is rather artificial" [17], "[i]t is believed that the results should remain valid for more general velocity fields" [18].

The model was extended to have more physical nonwhite temporal correlations in a study of the symmetry of correlation functions [25], later exploited in [23] to find anomalous scaling of the three-point correlator. Those authors also consider a frozen random velocity in finding that one-point probability distribution functions of the scalar and its gradient have exponential tails [24]. For synopses and later developments regarding the BHT spectrum, see, e.g., [11, 26, 28, 31, 32]. There is also considerable computational evidence for the BHT spectrum [4, 5, 11, 13, 33, 34].

In the complementary case of large Prandtl number, one has the viscous–advective regime, which Batchelor [1] predicted to obey the $|k|^{-1}$ spectrum. Using dynamical systems techniques they developed earlier, Bedrossian, Blumenthal and Punshon-Smith recently made this rigorous in an important work [3], with velocity fields arising from actual Navier–Stokes equations with stochastic forcing, plus hyper-diffusion in the three-dimensional case.

Following our earlier work on the two-dimensional case [14], in this paper we make rigorous the intuitive arguments of BHT59 in three dimensions. Our synthetic velocity is slowly-varying, like BHT and unlike Kraichnan's and subsequent works. At leading order, we recover the BHT spectrum in a probabilistic sense, as expected and in agreement with earlier works using both BHT and Kraichnan velocity. Unable to obtain a probabilistic bound on the remainder (that part of the tracer beyond leading order, asserted to be small in BHT59) due to correlations, we bound it analytically. Unfortunately our bound of the remainder requires an energy spectrum no shallower than $|k|^{-2}$, which rules out an application to the conjectured Kolmogorov $|k|^{-5/3}$ energy spectrum. We expect that a probabilistic bound, if possible, would admit shallower energy spectra.

As in [14], we also confirm the intuition of [2] that this holds for non-constant velocity, as long as it does not vary too rapidly (in a precise scale-dependent manner), and we give higher-order corrections to account for this. Relaxing some assumptions in [14], here we allow a more general modal random variable, only requiring it to be circular, and a more general tracer (variance) source, only requiring sufficient Sobolev regularity.

Our general approach is similar to [14], with some differences: As in the two-dimensional case, our approach is to compute exactly the expectation of the spectrum for the first iterate of a fixed point iteration of the tracer, and show that the error from the actual tracer can be made small by taking the velocity small. This, coupled with a certain bound on the variance, establish the BHT scaling. Unlike the 2d case, however, here we consider a random tracer source at all scales, except for the bound on the variance. Also different is that rather than randomizing the phases of the streamfunction in Fourier space, here we randomize components of the coefficients in the Craya–Herring 3d basis [8, 12, 15].

Through its obvious relationship to various Sobolev norms, the tracer spectrum is related to the degree of tracer mixing and of the efficiency of mixing by the advecting velocity [9, 19–21, 29, 30]. A steep tracer spectrum such as the BHT suggests poor mixing, either throughout the entire range for small velocity (treated here), or beyond the so-called diffusive wavenumber [31, (8.108)] for larger velocity (with the Obukhov–Corrsin tracer spectrum at larger scales, for which no rigorous results are currently available).

Charlie Doering's influence goes beyond mixing, turbulence, and science in general. We remember him fondly in dedicating this paper to him.

2. Preliminaries

We consider the evolution of a passive scalar $\theta(x,t)$ under a prescribed velocity field $u(x,t)$ and source $g(x)$,

$$\partial_t\theta + u\cdot\nabla\theta = \Delta\theta + g. \tag{2.1}$$

For simplicity, we take $x \in D := [0,2\pi]^3$ and assume periodic boundary conditions in all directions. With no loss of generality, we assume that, for all t

$$\int_D u(x,t)\,\mathrm{d}x = 0 \quad \text{and} \quad \int_D \theta(x,t)\,\mathrm{d}x = 0. \tag{2.2}$$

We note that for the latter to hold for all $t > 0$, we must impose the same condition on g and $\theta(\cdot,0)$.

We expand $\theta(x,t)$ in Fourier series as

$$\theta(x,t) = \textstyle\sum_k' \theta_k(t)\mathrm{e}^{\mathrm{i}k\cdot x}, \tag{2.3}$$

where the prime indicates that the sum is taken over $k \in \mathbb{Z}^3\backslash\{0\}$ to satisfy (2.2), and denote spectral projection by

$$(\mathsf{P}_{\kappa,\kappa'}\theta)(x,t) := \textstyle\sum_{\kappa\le|k|<\kappa'} \theta_k(t)\mathrm{e}^{\mathrm{i}k\cdot x}. \tag{2.4}$$

For the tracer (variance) source, we take the deterministic

$$(\Delta^{-1}g)(x) = \textstyle\sum_k' \gamma_k \mathrm{e}^{\mathrm{i}k\cdot x} \tag{2.5}$$

where $\gamma_k \in \mathbb{C}$ with $\gamma_{-k} = \overline{\gamma_k}$, $\gamma_0 = 0$ and

$$|\gamma_k| \le c_g|k|^\alpha \qquad \text{when } |k| \ge \kappa_g \tag{2.6}$$

for some constants $c_g \ge 0$, $\kappa_g > 1$ and $\alpha < 0$. The case $c_g = 0$ gives the bandwidth-limited source considered in [14], for which somewhat tighter estimates can be obtained below. Alternately, and without altering the conclusion (see below), one may also consider the random

$$(\Delta^{-1}g_r)(x) = \textstyle\sum_k' \gamma_k Z_k \mathrm{e}^{\mathrm{i}k\cdot x}. \tag{2.7}$$

One could generalise further, making g to depend on t as well as x, but this would introduce another timescale that would enter into the computations, so we forego this in the interest of clarity. In any case, we expect our results to carry over *mutatis mutandis* with $g(x,t)$, with the obvious stipulation that it be independent of u.

The complex random variable Z_k is constructed as follows. For a fixed $k \in \mathbb{Z}^3$, we write $Z_k = R_k\mathrm{e}^{\mathrm{i}\zeta_k}$ where the random phase $\zeta_k \sim \mathcal{U}(0,2\pi)$, implying that Z_k is *circular*, i.e. $\mathsf{E}(\mathrm{e}^{\mathrm{i}\phi}Z_k) = \mathsf{E}Z_k$ for any deterministic real ϕ. This in turn implies that $\mathsf{E}Z_k^n = 0$ for any integer $n \neq 0$. We constrain the random modulus R_k so that

$$\Xi := \sup\{s : P(R_k > s) > 0\} < \infty, \tag{2.8}$$

implying that Z_k is bounded, $|Z_k| \le \Xi$. (With circularity, this means that Z_k is a *proper random variable*.) With no loss of generality, we put as its variance and fourth moment

$$\mathsf{E}|Z_k|^2 = 1 \quad \text{and} \quad \mathsf{E}|Z_k|^4 = \varrho. \tag{2.9}$$

We note that by the Cauchy–Schwarz inequality, $1 = (\mathsf{E}|Z_k|^2)^2 \le \mathsf{E}|Z_k|^4 = \varrho$, with equality (i.e. $\varrho = 1$) attained iff $|Z_k| = 1$ a.s. We denote $Z_k \sim \mathcal{R}_\varrho$.

Now for $g(x,t)$ to be real-valued, we must require that $Z_{-k} = \overline{Z_k}$, but otherwise Z_k are assumed to be uncorrelated, so $\mathsf{E}Z_j\overline{Z_k} = \delta_{jk}$ and $\mathsf{E}Z_jZ_k = \delta_{j,-k}$. A convenient tool to handle this reality constraint is the wavenumber half-space

$$\mathbb{Z}^3_+ := \{(l,m,n) : n > 0\} \cup \{(l,m,0) : m > 0\} \cup \{(l,0,0) : l > 0\} \tag{2.10}$$

with l, m, $n \in \mathbb{Z}$; we thus have $\mathbb{Z}^3_+ \cup (-\mathbb{Z}^3_+) = \mathbb{Z}^3\backslash\{0\}$. With this, we can write $\mathsf{E}Z_jZ_k = 0$ and $\mathsf{E}Z_j\overline{Z_k} = \delta_{jk}$ for all j, $k \in \mathbb{Z}^3_+$.

To set up our velocity, we recall the Craya–Herring basis [8, 12, 15]: Writing a wavevector $k = (k_x, k_y, k_z)$, defining $k_h := (k_x, k_y, 0)$ and using spherical coordinates $(1, \varphi, \phi)$, we define the (k-dependent) orthonormal vectors

$$\begin{aligned}
d_k &= \frac{k}{|k|} &&= \frac{(k_x,k_y,k_z)}{|k|} &&= (\sin\varphi\cos\phi, \sin\varphi\sin\phi, \cos\varphi),\\
e_k &= \frac{k\times\hat z}{|k\times\hat z|} &&= \frac{(k_y,-k_x,0)}{|k_h|} &&= (\sin\phi, -\cos\phi, 0),\\
f_k &= \frac{k\times k\times\hat z}{|k\times k\times\hat z|} &&= \frac{(k_xk_z, k_yk_z, -|k_h|^2)}{|k|\,|k_h|} &&= (\cos\varphi\cos\phi, \cos\varphi\sin\phi, -\sin\varphi)\;,
\end{aligned}$$

where $\hat z = (0,0,1)$. With these basis vectors, any velocity field $v(x,t)$ can be written as

$$v(x,t) = \textstyle\sum_k' [\tilde U_d(k,t)d_k + \tilde U_e(k,t)e_k + \tilde U_f(k,t)f_k]\mathrm{e}^{\mathrm{i}k\cdot x} \tag{2.11}$$

for some $(\tilde U_d, \tilde U_e, \tilde U_f) \in \mathbb{C}^3$. Now since $\operatorname{div} v(x,t) = \mathrm{i}\sum_k \tilde U_d(k,t)|k|\mathrm{e}^{\mathrm{i}k\cdot x}$, for $v(\cdot,t)$ to be incompressible we must have $\tilde U_d(k,t) \equiv 0$. We thus write our incompressible velocity field as

$$u(x,t) = \textstyle\sum_k' |k|^\beta[U_e e_k V_k(t) + U_f f_k W_k(t)]\,\mathrm{e}^{\mathrm{i}k\cdot x} \tag{2.12}$$

where $\beta < 0$, U_e and U_f are real constants, and $V_k(t)$ and $W_k(t)$ complex random processes whose time behaviour will be precised below.[1] For now, we require that, for each fixed t, $V_k(t)$ and $W_k(t) \sim \mathcal{R}_\varsigma$, proper random variables with unit variance and $\mathsf{E}|V_k(t)|^4 = \mathsf{E}|W_k(t)|^4 = \varsigma$, and bounded as $|V_k(t)|$, $|W_k(t)| \le \Xi$. As with $g(x)$, for $u(x,t)$ to be real-valued, we must require that $V_{-k}(t) = \overline{V_k(t)}$ and $W_{-k}(t) = \overline{W_k(t)}$. Aside from this constraint, we assume that $V_k(t)$ and $W_k(t)$ are uncorrelated, so $\mathsf{E}V_j(t)\overline{V_k(t)} = \mathsf{E}W_j(t)\overline{W_k(t)} = \delta_{jk}$ and $\mathsf{E}V_j(t)\overline{W_k(t)} = 0$ for all j, $k \in \mathbb{Z}^3_+$. Unlike d_k, which gives the divergent component of u, the e_k and f_k components have no special meaning when u is isotropic, although they do carry physical significance in, e.g., stratified flows (as the "vortex" and "wave" components, respectively).

We turn to the energy spectrum. First, we compute [suppressing dependence on t where no confusion may arise]

$$\begin{aligned}
&\|\mathsf{P}_{\kappa,2\kappa}u(\cdot,t)\|^2_{L^2}\\
&\quad= \big(\textstyle\sum_j |j|^\beta(U_e e_j V_j + U_f f_j W_j)\mathrm{e}^{\mathrm{i}j\cdot x}, \sum_k |k|^\beta(U_e e_k V_k + U_f f_k W_k)\mathrm{e}^{\mathrm{i}k\cdot x}\big)_{L^2}\\
&\quad= 8\pi^3 \textstyle\sum_k |k|^{2\beta}(U_e^2|V_k|^2 + U_f^2|W_k|^2),
\end{aligned} \tag{2.13}$$

[1] We note that in this paper $|u_k| \sim |k|^\beta$ whereas in [14] $|u_k| \sim |k|^{\beta+1}$, so the βs are different; in hindsight, we feel the present notation more natural.

where for the second equality we have used the facts that $(\mathrm{e}^{\mathrm{i}j\cdot x}, \mathrm{e}^{\mathrm{i}k\cdot x})_{L^2} = (2\pi)^3\delta_{jk}$, $e_k \cdot e_k = f_k \cdot f_k = 1$ and $e_k \cdot f_k = 0$. Unlike in our previous work on the 2d case [14], here $\|\mathsf{P}_{\kappa,2\kappa}u\|^2$ contains the random variables $|V_k|^2$ and $|W_k|^2$, so we compute

$$\begin{aligned} \mathsf{E}\|\mathsf{P}_{\kappa,2\kappa}u(\cdot,t)\|_{L^2}^2 &= 8\pi^3 \textstyle\sum_k |k|^{2\beta}(U_e^2\mathsf{E}|V_k|^2 + U_f^2\mathsf{E}|W_k|^2) \\ &= 8\pi^3(U_e^2+U_f^2)\textstyle\sum_{\kappa\le|k|<2\kappa}|k|^{2\beta}. \end{aligned} \tag{2.14}$$

Approximating the sum as an integral over the corresponding region in $\mathbb{R}^d$, we find this scales as $\kappa^{2\beta+d}$ for sufficiently large κ, so for the classical Kolmogorov $-\frac{5}{3}$ spectrum in $d=3$ (i.e. $2\beta+d-1=-\frac{5}{3}$), we must take $\beta=-\frac{11}{6}$. Next, we compute the variance $\mathsf{var}\|\mathsf{P}_{\kappa,2\kappa}u\|_{L^2}^2$, by first using (2.9) to obtain

$$\begin{aligned} (2\pi)^{-6}\mathsf{E}\|\mathsf{P}_{\kappa,2\kappa}u\|_{L^2}^4 &= \mathsf{E}\big(\textstyle\sum_k |k|^{2\beta}(U_e^2|V_k|^2 + U_f^2|W_k|^2)\big)^2 \\ &= \textstyle\sum_{jk}|j|^{2\beta}|k|^{2\beta}(U_e^4\mathsf{E}|V_j|^2|V_k|^2 + 2U_e^2U_f^2\mathsf{E}|V_j|^2|W_k|^2 + U_f^4\mathsf{E}|W_j|^2|W_k|^2) \\ &= \textstyle\sum_{j\ne k}|j|^{2\beta}|k|^{2\beta}(U_e^4\mathsf{E}|V_j|^2|V_k|^2 + U_f^4\mathsf{E}|W_j|^2|W_k|^2) \\ &\quad + \textstyle\sum_k |k|^{4\beta}(U_e^4\mathsf{E}|V_k|^4 + U_f^4\mathsf{E}|W_k|^4) + 2\sum_{jk}|j|^{2\beta}|k|^{2\beta}U_e^2U_f^2\mathsf{E}|V_j|^2|W_k|^2 \\ &= \textstyle\sum_{j\ne k}|j|^{2\beta}|k|^{2\beta}(U_e^4+U_f^4) + \varsigma\sum_{j=k}|j|^{2\beta}|k|^{2\beta}(U_e^4+U_f^4) \\ &\quad + 2\textstyle\sum_{jk}|j|^{2\beta}|k|^{2\beta}U_e^2U_f^2 \\ &= \big((U_e^2+U_f^2)\textstyle\sum_k|k|^{2\beta}\big)^2 + (\varsigma-1)(U_e^4+U_f^4)\sum_k|k|^{4\beta}, \end{aligned}$$

whence

$$\begin{aligned} \mathsf{var}\|\mathsf{P}_{\kappa,2\kappa}u\|_{L^2}^2 &= \mathsf{E}\|\mathsf{P}_{\kappa,2\kappa}u\|_{L^2}^4 - \big(\mathsf{E}\|\mathsf{P}_{\kappa,2\kappa}u\|_{L^2}^2\big)^2 \\ &= (2\pi)^6(\varsigma-1)(U_e^4+U_f^4)\textstyle\sum_{\kappa\le|k|<2\kappa}|k|^{4\beta}. \end{aligned} \tag{2.15}$$

For large κ, this scales as $\kappa^{4\beta+d}$, so $(\mathsf{var}\|\mathsf{P}_{\kappa,2\kappa}u\|^2)^{1/2}/\mathsf{E}\|\mathsf{P}_{\kappa,2\kappa}u\|^2 \propto \kappa^{-d/2}$, giving asymptotic convergence (over dyads) to an energy spectrum that is $\kappa^{2\beta+d-1}$.

For the time dependence, we assume that, for all j, $k\in\mathbb{Z}_+^3$,

$$\mathsf{E}V_j(s)\overline{V_k(t)} = \mathsf{E}W_j(s)\overline{W_k(t)} = \delta_{jk}\Phi_k(s-t). \tag{2.16}$$

We take a time correlation function of the form

$$\Phi_k(t) = \Phi(\chi_k|t|) \tag{2.17}$$

with $\Phi\in C^n(\mathbb{R}_+)$ for some $n\ge 2$ and $\Phi(0)=1$, where the correlation timescale χ_k^{-1} is assumed not to grow too rapidly with $|k|$,

$$\lim_{|k|\to\infty}\chi_k|k|^{-2} = 0. \tag{2.18}$$

Using the Cauchy–Schwarz inequality, we have

$$\begin{aligned} |\Phi(h)| &= |\mathsf{E}\,V_k(s)\overline{V_k(s+h)}| \\ &\le (\mathsf{E}\,|V_k(s)|^2)^{1/2}(\mathsf{E}\,|V_k(s+h)|^2)^{1/2} = \Phi(0) = 1. \end{aligned} \tag{2.19}$$

We also assume that $V_k(t)$ has sufficient smoothness in t for the usual Riemann integral to be defined. As before, $V_j(s)$ and $W_k(t)$ are uncorrelated proper random variables for any j, $k\in\mathbb{Z}_+^3$, s and t.

3. Main Result and Discussion

As in the 2d case, it is both convenient and instructive to first consider the static case

$$u \cdot \nabla\theta = \Delta\theta + g. \tag{3.1}$$

Here the time-independent random velocity is

$$u(x) = \textstyle\sum_k' |k|^\beta [U_e e_k V_k + U_f f_k W_k]\, \mathrm{e}^{\mathrm{i}k\cdot x} \tag{3.2}$$

with V_k and $W_k \sim \mathcal{R}_\varsigma$ i.i.d. satisfying the usual reality constraints.

The strategy, as in [14], is to solve (3.1) by a fixed-point iteration, showing that the error from the first iterate ϑ to the solution is, at most, of the same order, as $\kappa \to \infty$ as $\mathsf{E}\|\mathsf{P}_{\kappa,2\kappa}\vartheta\|_{L^2}^2$. The latter will satisfy the BHT spectrum and the *relative* error can be made arbitrarily small by taking $U/U_{\max}$ small enough. We put $\theta^{(0)} = -\Delta^{-1}g$ and

$$\theta^{(n+1)} = \Delta^{-1}(u \cdot \nabla\theta^{(n)} - g). \tag{3.3}$$

We seek to prove that this iteration converges under some assumptions, and that the limit $\theta^{(\infty)}$ asymptotes, dyad-wise as $\kappa \to \infty$, probabilistically to the BHT spectrum. Unlike in [14], however, here our source g may have a full spectrum, so $\vartheta := \theta^{(1)} - \theta^{(0)} = -\Delta^{-1}(u \cdot \nabla\Delta^{-1}g)$ has a remainder arising from high-frequency parts of g.

Denoting $\|f\|_{l_1} := \sum_k |f_k|$ and putting $U_e = U_f = U$, we have the following:

Theorem 1. *Let g be given by (2.5)–(2.6), g_r by (2.7)–(2.6) with $\alpha < 2\min\{\beta, -d\} - 1$ and $\kappa_g \ge 16$; and with $\beta < -2$, let u be given by (3.2) satisfying $\varepsilon := U/U_{max} < 1$ for some $U_{max}(\beta, \kappa_g, \Xi)$. Then for $\kappa \ge 4\kappa_g^2$ the static problem (3.1) has a unique solution $\theta = -\Delta^{-1}g + \vartheta + \delta\theta$ where*

$$\mathsf{E}\|\mathsf{P}_{\kappa,2\kappa}\vartheta\|_{L^2}^2 = \kappa^{2\beta-1}\frac{8\pi U^2}{3}\,\frac{2^{2\beta-1}-1}{2\beta-1}\,\|\mathsf{P}_{1,\kappa^{1/2}}\nabla^{-1}g\|_{L^2}^2 + \mathcal{E}(\kappa) \tag{3.4}$$

and the remainder terms are bounded as,

$$|\mathcal{E}(\kappa)| \le c_g^2 U^2 c(\alpha,\beta)\,\kappa^\alpha + c(\beta)U^2\|\nabla^{-1}g\|_{L^2}^2\kappa^{2\beta-3/2}, \tag{3.5}$$

$$\|\mathsf{P}_{\kappa,2\kappa}\delta\theta\|_{L^2}^2 \le \varepsilon^2 c(g,\alpha,\beta)U^2\Xi^2\,\kappa^{2\beta-1}. \tag{3.6}$$

With finite-mode source, $c_g = 0$ in (2.6), the variance is bounded from above as

$$\mathsf{var}\|\mathsf{P}_{\kappa,2\kappa}\vartheta\|_{L^2}^2 \lesssim \kappa^{4\beta-5}\,16\pi U^4\,\frac{2^{4\beta-5}-1}{4\beta-5}\|\nabla^{-1}g\|_{L^2}^2\big\{\|\nabla^{-1}g\|_{l_1}^2 + (\varsigma-1)\|\nabla^{-1}g\|_{L^2}^2\big\}. \tag{3.7}$$

As noted after (2.14), $\mathsf{E}\|P_{\kappa,2\kappa}u\|_{L^2}^2$ scales as $\kappa^{2\beta+3}$ so that

$$\mathsf{E}\|\mathsf{P}_{\kappa,2\kappa}\vartheta\|_{L^2}^2/\mathsf{E}\|\mathsf{P}_{\kappa,2\kappa}u\|_{L^2}^2 \propto |k|^{-4}\ .$$

As in [14], by $f_1(\kappa) \simeq f_2(\kappa)$ we mean that $\lim_{\kappa\to\infty} f_1(\kappa)/f_2(\kappa) = 1$. Thus, "$\simeq$" arises either from lattice effect, when we approximate sums over subsets of $\mathbb{Z}^d$ by the corresponding integrals over subsets of $\mathbb{R}^d$, or from dropping terms of (relative) order κ_g/κ. The same convention will be used for "$\lesssim$". As a consequence, absolute constants are included in such relations. We note that with finite-mode sources, $\mathsf{P}_{1,\kappa^{1/2}}\nabla^{-1}g = \nabla^{-1}g$ in (3.4), while in (3.5) the first term vanishes and the second term can be improved to $\mathsf{O}(\kappa^{2\beta-2})$. These results are stated for the isotropic case, $U_e = U_f$, but we have kept U_e and U_f when computing (the main part of) $\mathsf{E}\|\mathsf{P}_{\kappa,2\kappa}\vartheta\|^2$ for readers interested in the effect of non-isotropic velocity. We see no conceptual difficulty to extend (3.7) to sources with full spectra following the approach for $\mathsf{E}\|\mathsf{P}_{\kappa,2\kappa}\vartheta\|^2$, but did not attempt this in order to keep the proof readable. The explicit expression for U_{max} is rather messy, given in (4.67).

For the time-dependent case, we write the solution $\theta(x,t)$ of (2.1) as the limit of iterates $\theta^{(n)}(x,t)$ defined by

$$\theta^{(0)} = -\Delta^{-1}g, \tag{3.8}$$

$$\theta^{(n+1)}(\cdot,t) = -\Delta^{-1}g - \int_0^t \mathrm{e}^{(t-s)\Delta}[u(\cdot,s)\cdot\nabla\theta^{(n)}(\cdot,s)]\,\mathrm{d}s. \tag{3.9}$$

Here $\mathrm{e}^{-t\Delta}$ is the heat kernel, i.e. $\theta^{(n+1)}$ is the solution of

$$(\partial_t - \Delta)\theta^{(n+1)} = g - u\cdot\nabla\theta^{(n)} \quad \text{with} \quad \theta^{(n+1)}(\cdot,0) = -\Delta^{-1}g. \tag{3.10}$$

Our main result is that this iteration converges, and that the limit obeys the BHT scaling in the following sense:

Theorem 2. *Let the source $g(x)$ be given by* (2.5)–(2.6) *or* (2.7)–(2.6) *with $\alpha < 2\min\{\beta,-d\}-1$. Let the incompressible velocity $u(x,t)$ be given by* (2.12) *and* (2.16) *with $\beta < -2$ and U satisfying the hypotheses of Theorem 1 and, in addition,*

$$8\pi^3 c_3(d)U^3\Xi^2 < |2\beta+d+1| \tag{3.11}$$

for an absolute constant $c_3(d)$. Then the solution of (2.1) *can be written as $\theta(x,t) = -\Delta^{-1}g + \vartheta + \delta\theta$ where $\vartheta(x,t)$ satisfies*

$$\begin{aligned}\lim_{t\to\infty}\mathsf{E}|\vartheta_k(t)|^2 &= |k|^{-4}{\textstyle\sum_j'}\big[U_e^2(e_{k-j}\cdot j)^2 + U_f^2(f_{k-j}\cdot j)^2\big]\,|k-j|^{2\beta}|\gamma_j|^2\times\\ &\Big[1+\frac{\chi_{k-j}}{|k|^2}\Phi'(0)+\cdots+\frac{\chi_{k-j}^{n-1}}{|k|^{2n}}\int_0^\infty \mathrm{e}^{-s|k|^2/\chi_{k-j}}\Phi^{(n)}(s)\,\mathrm{d}s\Big].\end{aligned} \tag{3.12}$$

When $\sup_k\{\chi_k\}/\kappa^2 \ll 1$, this reduces to the static case in Theorem 1, up to further lower-order remainders.

4. Proofs

Proof of Theorem 1. This consists of three main parts. In the first part, we compute ϑ and show that it satisfies (3.4) and (3.5). We then bound $\mathsf{var}\|\mathsf{P}_{\kappa,2\kappa}\vartheta\|^2$. In the final part, we estimate $\theta^{(\infty)} - \theta^{(1)}$ to obtain (3.6).

4.1. Computing ϑ.

We start with the computation of $\vartheta = \theta^{(1)} - \theta^{(0)} = -\Delta^{-1}(u\cdot\nabla\Delta^{-1}g)$, which, when combined with the scaling in (2.14), shows that it satisfies the BHT scaling up to small remainders. We use g_r in (2.7), as will be apparent shortly, with no loss of generality. In some expressions (notably as exponents), we write $d=3$ and $\omega_3 = 4\pi$, to give a hint of how the analogues would appear in two dimensions. From (2.7) and (2.12), we have

$$\vartheta_k = \mathrm{i}\,|k|^{-2}\sum_j{}' \,|k-j|^\beta\gamma_j[U_e(e_{k-j}\cdot j)V_{k-j} + U_f(f_{k-j}\cdot j)W_{k-j}]Z_j\,. \tag{4.1}$$

In computing $\mathsf{E}\vartheta_k\overline{\vartheta_k}$, we find factors of $\mathsf{E}V_{k-j}\overline{V_{k-i}}$, which is nonzero if and only if $k-j = k-i$, i.e. $j=i$. An analogous reasoning applies to $\mathsf{E}W_{k-j}\overline{W_{k\pm i}}$, so we have $\mathsf{E}V_{k-i}\overline{V_{k-j}} = \mathsf{E}W_{k-i}\overline{W_{k-j}} = \delta_{ij}$. Recalling that $\mathsf{E}V_j\overline{W_k} = 0\ \forall j,k$ and, by independence of V_j and Z_k, $\mathsf{E}|V_j|^2|Z_k|^2 = \mathsf{E}|V_j|^2\mathsf{E}|Z_k|^2$, we arrive at

$$\mathsf{E}\vartheta_k\overline{\vartheta_k} = |k|^{-4}{\textstyle\sum_j'}|k-j|^{2\beta}|\gamma_j|^2(U_e^2\xi_{kj}^2 + U_f^2 \upsilon_{kj}^2)\mathsf{E}|Z_j|^2 =: |k|^{-4}S_k \tag{4.2}$$

where $\xi_{kj} := e_{k-j}\cdot j$ and $\upsilon_{ki} := f_{k-i}\cdot i$. Since $\mathsf{E}|Z_j|^2 = 1$, it is clear that this expression applies to both deterministic g and random g_r.

We fix some $r\in(0,1)$; for concreteness, we put $r=\frac12$ in Theorem 1, but write r in this proof to indicate possible optimisation. Consider any wavenumber dyad $[\kappa,2\kappa)$ with $\kappa>(2\kappa_g)^{1/r}$. For any k within this dyad, $\kappa\le|k|<2\kappa$, we split the sum in (4.2) into (here and below S_k denotes a "temporary variable" with no global significance),

$$S_k = \textstyle\sum_{1\le|j|<\kappa^r} + \sum_{\kappa^r\le|j|} =: S_k^{\ll} + S_k^{\gtrsim}. \tag{4.3}$$

We start with the last sum $S_k^{\gtrsim}$, where by (2.6) and e_{k-j} and f_{k-j} being unit vectors,

$$|\gamma_j|^2 \le c_g^2|j|^{2\alpha}, \qquad (e_{k-j}\cdot j)^2\le|j|^2 \quad\text{and}\quad (f_{k-j}\cdot j)^2\le|j|^2. \tag{4.4}$$

From $|j|\ge\kappa^r$, we have $|j|^{2\alpha+2}\le\kappa^{(2\alpha+2)r}$, so writing $m:=k-j$, we then replace the sum over $|j|\ge\kappa^r$ with one over $m\in\mathbb{Z}^3\backslash\{0\}$, giving

$$\sum_{\kappa^r\le|j|} c_g^2U^2|j|^{2\alpha+2}|k-j|^{2\beta} \le c_g^2U^2\kappa^{(2\alpha+2)r}\sum_m{}' \,|m|^{2\beta} \le c_g^2U^2\kappa^{(2\alpha+2)r}\frac{\omega_d}{|2\beta+d|} \tag{4.5}$$

since $2\beta+d<0$. In the case of bandwidth-limited source, $c_g=0$ in (2.6), so this remainder term is zero. For g with full spectrum, we sum over our dyad to obtain

$$\textstyle\sum_{\kappa\le|k|<2\kappa}|k|^{-4}S_k^{\gtrsim} \le c_g^2U^2c(\alpha,\beta,d)\kappa^{(2\alpha+2)r-4+d}. \tag{4.6}$$

For this to be dominated by $\kappa^{2\beta-1}$, we need $(2\alpha+2)r-4+d<2\beta-1$. Putting $d=3$ and $r=\frac12$ gives the first term in (3.5).

The first sum in (4.3) is more delicate, requiring tight upper and lower bounds. We start with a couple of preliminary estimates. Writing $m:=k-j$ again, we bound

$$\begin{aligned} &|j|<\kappa^r\le|k|^r\le\tfrac12|k|\\ \Rightarrow\;&|k-j|\ge|k|-|j|\ge\tfrac12|k|\\ \Rightarrow\;&|j|\le|k|^r\le(2|k-j|)^r=2^r|m|^r. \end{aligned} \tag{4.7}$$

We then bound $|k|^{-4}=|m+j|^{-4}$ from above and below subject to the constraints on $|j|$. Noting that for $x\in(0,1)$, by convexity we can estimate

$$(1+x)^{-4}\ge1-4x\;. \tag{4.8}$$

This and (4.7) give us

$$\begin{aligned} |m+j|^{-4} &\ge (|m|+|j|)^{-4} &&= |m|^{-4}(1+|j|/|m|)^{-4}\\ &\ge |m|^{-4}(1-4|j|/|m|) &&\ge |m|^{-4}(1-2^{2+r}|m|^{r-1}). \end{aligned}$$

For the upper bound, we use the fact (readily seen by convexity), that for $x\in(0,\frac12]$

$$(1-x)^{-4}\le1+30x. \tag{4.9}$$

Analogous reasoning then gives us

$$\begin{aligned} |m+j|^{-4} &\le (|m|-|j|)^{-4} &&= |m|^{-4}(1-|j|/|m|)^{-4}\\ &\le |m|^{-4}(1+30|j|/|m|) &&\le |m|^{-4}(1+30\cdot2^r|m|^{r-1}). \end{aligned} \tag{4.10}$$

From $\kappa \ge 4\kappa_g^2$, we have $|j| < \kappa^r \le \kappa/4 \le \frac{1}{4}|k| < \frac{1}{3}|k|$, so $\frac{3}{2}|j| \le \frac{1}{2}|k|$ and $|j| \le \frac{1}{2}(|k|-|j|) \le \frac{1}{2}|k-j| = \frac{1}{2}|m|$, so we can use (4.9) in (4.10). We have thus shown that

$$|m|^{-4} - 8\,|m|^{r-5} \le |k|^{-4} \le |m|^{-4} + 60\,|m|^{r-5}. \tag{4.11}$$

This can be improved slightly by taking $r = (\beta-1)/(\alpha+1)$ instead of $\frac{1}{2}$ and adjusting the constants. We note that with finite-mode sources, there is no need to split S_k and $|m+j|^{-4}$ is bounded by $|m|^{-4} \pm 4\kappa_g|m|^{-5}$.

Instead of computing individual $S_k^{\ll}$, we proceed directly to the dyadic sum

$$\sum_{\kappa\le|k|<2\kappa} |k|^{-4} S_k^{\ll} = \sum_{1\le|j|<\kappa^r} |\gamma_j|^2 \sum_{\kappa\le|k|<2\kappa} |k|^{-4}|k-j|^{2\beta}(U_e^2\xi_{kj}^2 + U_f^2\upsilon_{kj}^2). \tag{4.12}$$

Defining spherical coordinates (ρ,φ,ϕ) w.r.t. m, i.e.

$$m = \rho(\sin\varphi\cos\phi, \sin\varphi\sin\phi, \cos\varphi) =: \rho\hat{m},$$

we compute

$$e_m\cdot j = j_x\sin\phi - j_y\cos\phi \text{ and } f_m\cdot j = j_x\cos\varphi\cos\phi + j_y\cos\varphi\sin\phi - j_z\sin\varphi. \tag{4.13}$$

We approximate the k-sum by an integral over m, and in view of (4.10), replace $|k|^{-4}$ by $|m|^{-4}$, so that

$$\begin{aligned} &\textstyle\sum_{\kappa\le|k|<2\kappa} |k|^{-4}|k-j|^{2\beta}[U_e^2\xi_{kj}^2 + U_f^2\upsilon_{kj}^2] = \mathcal{E}_1(\kappa) + \\ &\int_0^{2\pi}\int_0^{\pi}\int_{r_j(\kappa,\varphi,\phi)}^{r_j(2\kappa,\varphi,\phi)} [U_f^2(j_x\cos\varphi\cos\phi + j_y\cos\varphi\sin\phi - j_z\sin\varphi)^2 \\ &\qquad\qquad + U_e^2(j_x\sin\phi - j_y\cos\phi)^2]\;\rho^{2\beta-2}\,\mathrm{d}\rho\sin\varphi\,\mathrm{d}\varphi\,\mathrm{d}\phi \\ &=: \mathcal{E}_1(\kappa) + I_1(\kappa) \end{aligned} \tag{4.14}$$

where $\mathcal{E}_1$ is a remainder to be bounded below, and where the radial limit $r_j(\lambda,\varphi,\phi)$, with $\lambda\in\{\kappa,2\kappa\}$, is determined by solving $|m+j|^2 = \lambda^2$ for $\rho = |m|$,

$$\begin{aligned} r_j(\lambda,\varphi,\phi) &= -j\cdot\hat{m} + \sqrt{\lambda^2 - |j_\perp|^2} &&\text{with } |j_\perp|^2 = |j|^2 - (j\cdot\hat{m})^2, \\ &= \lambda - j\cdot\hat{m} - |j_\perp|^2/(2\lambda) + \cdots &&\text{for } \lambda \gg |j|. \end{aligned} \tag{4.15}$$

Finally, we modify the region of integration, replacing the ρ-limit $r_j(\lambda,\varphi,\phi)$ by λ,

$$\begin{aligned} &\int_0^{2\pi}\int_0^{\pi}\int_{\kappa}^{2\kappa} [U_f^2(j_x\cos\varphi\cos\phi + j_y\cos\varphi\sin\phi - j_z\sin\varphi)^2 \\ &\qquad\qquad + U_e^2(j_x\sin\phi - j_y\cos\phi)^2]\;\rho^{2\beta-2}\,\mathrm{d}\rho\sin\varphi\,\mathrm{d}\varphi\,\mathrm{d}\phi \\ &\quad = \kappa^{2\beta-1} i_2(2\beta-1)\Big[2\pi|j_h|^2U_e^2 + \Big(\frac{2\pi}{3}|j|^2 + 2\pi j_z^2\Big)U_f^2\Big] =: I_2(\kappa) \end{aligned} \tag{4.16}$$

where $i_2(s) := (2^s-1)/s$. We write $\mathcal{E}_2(\kappa) := I_1(\kappa) - I_2(\kappa)$.

Our approximation for $\mathsf{E}\|\mathsf{P}_{\kappa,2\kappa}\vartheta\|_{L^2}^2$ is obtained by using $I_2(\kappa)$ in (4.12),

$$\sum_{|j|<\kappa^r} |\gamma_j|^2 I_2 = \kappa^{2\beta-1} i_2(2\beta-1) \sum_{|j|<\kappa^r} |\gamma_j|^2 \Big[2\pi|j_h|^2U_e^2 + \Big(\frac{2\pi}{3}|j|^2 + 2\pi j_z^2\Big)U_f^2\Big]. \tag{4.17}$$

In the isotropic case, $U_e = U_f \equiv U$, this reduces to

$$\sum_{\kappa\le|k|<2\kappa} |k|^{-4}S_k^{\ll} = \kappa^{2\beta-1}\frac{U^2}{3\pi^2} i_2(2\beta-1)\|\mathsf{P}_{1,\kappa^r}\nabla^{-1}g\|_{L^2}^2 + \sum_{1\le|j|<\kappa^r} |\gamma_j|^2(\mathcal{E}_1+\mathcal{E}_2) \tag{4.18}$$

We now bound the remainders $\mathcal{E}_1$ and $\mathcal{E}_2$. The remainder $\mathcal{E}_2$ was incurred by replacing $r_j(\lambda,\varphi,\phi)$ in (4.14) by λ in (4.16). Now from (4.15) since $\lambda \gg |j|$, we can bound

$$|\lambda - r_j| \le \lambda + j\cdot\hat{m} - \sqrt{\lambda^2 + (j\cdot\hat{m})^2 - |j|^2} \le \lambda + |j| - \sqrt{\lambda^2 - |j|^2} \le 2|j|, \tag{4.19}$$

so we can bound $\mathcal{E}_2$ by integrating (a bound on the integrand) over two spherical shells of thickness $4|j| \le 4\kappa^r$ at $\lambda = \kappa$ and 2κ. Bounding the integrand by $U^2|j|^2\rho^{2\beta-2}$, which is largest (since $\beta - 1 < 0$) for smallest ρ, we have

$$\begin{aligned}
|\mathcal{E}_2(\kappa)| &\le \omega_d \sum\nolimits_{\lambda\in\{\kappa,2\kappa\}} \int_{\lambda-2\kappa^r}^{\lambda+2\kappa^r} U^2|j|^2\rho^{2\beta-2}\,\mathrm{d}\rho \\
&\le \omega_d U^2|j|^2 4\kappa^r \left[(\kappa-2\kappa^r)^{2\beta-2} + (2\kappa-2\kappa^r)^{2\beta-2}\right] \\
&\le \omega_d U^2|j|^2 8\kappa^r(\kappa-2\kappa^r)^{2\beta-2} \\
&\le 2^{5-2\beta}\omega_d U^2|j|^2\kappa^{2\beta-2+r}.
\end{aligned}$$

Therefore, bounding $\|\mathsf{P}_{1,\kappa^r}\nabla^{-1}g\|_{L^2}^2 \le \|\nabla^{-1}g\|_{L^2}^2$,

$$\sum\nolimits_{1\le|j|<\kappa^r} |\gamma_j|^2|\mathcal{E}_2(\kappa)| \le \frac{2^{2-2\beta}}{\pi^3}\omega_d U^2\kappa^{2\beta-2+r}\|\nabla^{-1}g\|_{L^2}^2. \tag{4.20}$$

Next, the remainder $\mathcal{E}_1$ incurred in (4.14) is bounded by replacing $|m|^{-4}$ there by $60|m|^{r-5}$, giving [cf. the first term in (4.18)]

$$\sum\nolimits'_{|j|<\kappa^r} |\gamma_j|^2\mathcal{E}_1(\kappa) \le c(\beta,d,r)\kappa^{2\beta+r-2}U^2\|\nabla^{-1}g\|_{L^2}^2. \tag{4.21}$$

Together (4.20)–(4.21) give the second term in (3.5). We note that this took more work than in two dimensions, where the simpler "geometric term" $k\wedge j = k_xj_y - k_yj_x$ in [14] allowed direct integration in k rather than having to shift to $m = k - j$.

4.2. **Upper Bound for $\mathsf{var}\|\mathsf{P}_{\kappa,2\kappa}\vartheta\|^2$.** For this, we take $U_e = U_f = U$. To bound the variance, we first compute

$$\mathsf{E}\,\|\mathsf{P}_{\kappa,2\kappa}\vartheta\|_{L^2}^4 = \mathsf{E}\sum\nolimits_{kl}|\vartheta_k|^2|\vartheta_l|^2 =: U^4\sum\nolimits_{kl}|k|^{-4}|l|^{-4}\mathsf{E}\,|\varphi_k|^2|\varphi_l|^2 \tag{4.22}$$

where, here and in the rest of this subsection, $\sum_{kl}$ is taken over $|k|$, $|l| \in [\kappa, 2\kappa)$. Assuming g is deterministic, we have

$$\begin{aligned}
\mathsf{E}\,|\varphi_k|^2|\varphi_l|^2 = &\sum\nolimits'_{ijmn}|k-i|^\beta|l-j|^\beta|l-m|^\beta|k-n|^\beta \\
&\big\{\,\xi_{ki}\xi_{lj}\xi_{lm}\xi_{kn}\,\mathsf{E}\,V_{k-i}V_{l-j}\overline{V_{l-m}V_{k-n}}\,\gamma_i\gamma_j\overline{\gamma_m\gamma_n} + (*)' \\
&+ \xi_{ki}\xi_{lj}\upsilon_{lm}\upsilon_{kn}\,\mathsf{E}\,V_{k-i}V_{l-j}\overline{W_{l-m}W_{k-n}}\,\gamma_i\gamma_j\overline{\gamma_m\gamma_n} + (*)' \\
&+ \xi_{ki}\upsilon_{lj}\xi_{lm}\upsilon_{kn}\,\mathsf{E}\,V_{k-i}W_{l-j}\overline{V_{l-m}W_{k-n}}\,\gamma_i\gamma_j\overline{\gamma_m\gamma_n} + (*)' \\
&+ \xi_{ki}\upsilon_{lj}\upsilon_{lm}\xi_{kn}\,\mathsf{E}\,V_{k-i}W_{l-j}\overline{W_{l-m}V_{k-n}}\,\gamma_i\gamma_j\overline{\gamma_m\gamma_n} + (*)'\big\}
\end{aligned} \tag{4.23}$$

where $(*)'$ denotes the preceeding term with $\xi \leftrightarrow \upsilon$ and $V \leftrightarrow W$ swapped (but not their indices).

We start with the last term: here $\mathsf{E}\,V_{k-i}W_{l-j}\overline{W_{l-m}V_{k-n}} = \mathsf{E}\,V_{k-i}\overline{V_{k-n}}\,\mathsf{E}\,W_{l-j}\overline{W_{k-n}} \neq 0$ only when $k-i = k-n$ and $l-j = l-m \Leftrightarrow n = i$ and $j = m$. This last term then contributes

$$S_{kl}^{(p)} = \sum\nolimits'_{ij}|k-i|^{2\beta}|l-j|^{2\beta}\xi_{ki}^2\upsilon_{lj}^2|\gamma_i|^2|\gamma_j|^2. \tag{4.24}$$

To reduce clutter, we now write $q := k - l$ and $r := k + l$. In the penultimate term, $\mathsf{E}\cdots = \mathsf{E}V_{k-i}\overline{V_{l-m}}\,\mathsf{E}W_{l-j}\overline{W_{k-n}} \neq 0$ only if $k - i = l - m \Leftrightarrow m = i - q$ and $l - j = k - n \Leftrightarrow n = j + q$, thus contributing

$$S^{(o)}_{kl} = {\textstyle\sum}'_{ij}\,|k-i|^{2\beta}|l-j|^{2\beta}\xi_{ki}\xi_{l,i-q}\upsilon_{lj}\upsilon_{k,j+q}\,\gamma_i\gamma_j\overline{\gamma_{i-q}\gamma_{j+q}}. \tag{4.25}$$

Similarly, in the second term $\mathsf{E}\cdots = \mathsf{E}V_{k-i}V_{l-j}\,\mathsf{E}\overline{W_{l-m}W_{k-n}} \neq 0$ only if $j - l = k - i \Leftrightarrow j = -i + r$ and $n - k = l - m \Leftrightarrow n = -m + r$, contributing (upon relabelling $m \mapsto j$)

$$S^{(h)}_{kl} = {\textstyle\sum}'_{ij}\,|k-i|^{2\beta}|l-j|^{2\beta}\xi_{ki}\xi_{l,-i+r}\upsilon_{lj}\upsilon_{k,-j+r}\,\gamma_i\overline{\gamma_{i-r}\gamma_j}\gamma_{j-r}. \tag{4.26}$$

The first term is the most involved. Denoting $-y \neq x \neq y$ by $x \neq_\pm y$, the factor $\mathsf{E}V_{k-i}V_{l-j}\overline{V_{l-m}V_{k-n}} \neq 0$ only in the following cases:

(a) $k-i = k-n \neq_\pm l-j = l-m \quad\Leftrightarrow\quad i = n \neq j+q = m+q,$

(b) $k-i = l-m \neq_\pm l-j = k-n \quad\Leftrightarrow\quad i = m+q \neq n = j+q,$

(c) $k-i = j-l \neq_\pm k-n = m-l \quad\Leftrightarrow\quad j = -i+r \neq m = -n+r,$

(d) $k-i = k-n = l-j = l-m \quad\Leftrightarrow\quad i = n = j+q = m+q,$

(e) $k-i = j-l = k-n = m-l \quad\Leftrightarrow\quad j = m = -i+r = -n+r,$

(f) $k-i = l-m = j-l = n-k \quad\Rightarrow\quad m = i-q,\ j = -i+r,\ n = -i+2k.$

In cases (a)–(c), the $\mathsf{E}\cdots = 1$, while in cases (d)–(f), the $\mathsf{E}\cdots = \varsigma$. Imposing these conditions in (4.23), the first term is $S^{(a)}_{kl} + S^{(b)}_{kl} + S^{(c)}_{kl} + \varsigma S^{(d)}_{kl} + \varsigma S^{(e)}_{kl} + \varsigma S^{(f)}_{kl}$, where (in all these sums, k, l, $q = k - l$ and $r = k + l$ are fixed)

$$S^{(a)}_{kl} = {\textstyle\sum}'_{|k-i|\neq|l-j|}\,|k-i|^{2\beta}|l-j|^{2\beta}\xi^2_{ki}\xi^2_{lj}|\gamma_i|^2|\gamma_j|^2 \tag{4.27}$$

$$S^{(b)}_{kl} = {\textstyle\sum}'_{|k-i|\neq|l-j|}\,|k-i|^{2\beta}|l-j|^{2\beta}\xi_{ki}\xi_{k,j+q}\xi_{lj}\xi_{l,i-q}\,\gamma_i\gamma_j\overline{\gamma_{i-q}\gamma_{j+q}} \tag{4.28}$$

$$\begin{aligned} S^{(c)}_{kl} &= {\textstyle\sum}'_{|k-i|\neq|l-m|}\,|k-i|^{2\beta}|l-m|^{2\beta}\xi_{ki}\xi_{k,r-m}\xi_{lm}\xi_{l,r-i}\,\gamma_i\gamma_{r-i}\overline{\gamma_m\gamma_{r-m}} \\ &= {\textstyle\sum}'_{|k-i|\neq|l-j|}\,|k-i|^{2\beta}|l-j|^{2\beta}\xi_{ki}\xi_{l,r-i}\xi_{k,r-j}\xi_{lj}\,\gamma_i\overline{\gamma_{i-r}\gamma_j}\gamma_{j-r}, \end{aligned} \tag{4.29}$$

$$S^{(d)}_{kl} = {\textstyle\sum}'_{i}\,|k-i|^{4\beta}\xi^2_{ki}\xi^2_{l,i-q}\,|\gamma_i|^2|\gamma_{i-q}|^2 \tag{4.30}$$

$$S^{(e)}_{kl} = {\textstyle\sum}'_{i}\,|k-i|^{4\beta}\xi^2_{ki}\xi^2_{l,r-i}\,|\gamma_i|^2|\gamma_{r-i}|^2 \tag{4.31}$$

$$S^{(f)}_{kl} = {\textstyle\sum}'_{i}\,|k-i|^{4\beta}\xi_{ki}\xi_{l,r-i}\xi_{l,i-q}\xi_{k,2k-i}\,\gamma_i\gamma_{i-2k}\overline{\gamma_{i-q}\gamma_{i-r}}. \tag{4.32}$$

Analogously, the first $(*)'$ in (4.23) is $S^{(a')}_{kl} + \cdots + \varsigma S^{(f')}_{kl}$ with υ replacing ξ.

Returning to the variance, we have

$$\begin{aligned} \mathsf{var}\|\mathsf{P}_{\kappa,2\kappa}\vartheta\|^2_{L^2} &= \mathsf{E}\,\|\mathsf{P}_{\kappa,2\kappa}\vartheta\|^4_{L^2} - \big(\mathsf{E}\,\|\mathsf{P}_{\kappa,2\kappa}\vartheta\|^2_{L^2}\big)^2 \\ &= U^4{\textstyle\sum}_{kl}\,|k|^{-4}|l|^{-4}\big(\mathsf{E}\,|\varphi_k|^2|\varphi_l|^2 - \mathsf{E}|\varphi_k|^2\mathsf{E}|\varphi_l|^2\big). \end{aligned} \tag{4.33}$$

Now

$$\mathsf{E}|\varphi_k|^2\,\mathsf{E}|\varphi_l|^2 = S^{(a)}_{kl} + S^{(d)}_{kl} + S^{(e)}_{kl} + S^{(a')}_{kl} + S^{(d')}_{kl} + S^{(e')}_{kl} + 2S^{(p)}_{kl} \tag{4.34}$$

where the factor of 2 on $S^{(p)}_{kl}$ came from its $(*)'$. This gives us

$$\begin{aligned} \mathsf{var}\|\mathsf{P}_{\kappa,2\kappa}\vartheta\|^2_{L^2} = U^4{\textstyle\sum}_{kl}\,|k|^{-4}|l|^{-4}\big(&S^{(b)}_{kl} + S^{(c)}_{kl} + S^{(b')}_{kl} + S^{(c')}_{kl} \\ &+ (\varsigma-1)\big[S^{(d)}_{kl} + S^{(e)}_{kl} + S^{(d')}_{kl} + S^{(e')}_{kl}\big] + \varsigma\big[S^{(f)}_{kl} + S^{(f')}_{kl}\big] \\ &+ S^{(h)}_{kl} + S^{(o)}_{kl} + S^{(h')}_{kl} + S^{(o')}_{kl}\big) \end{aligned} \tag{4.35}$$

where $S^{(h')}_{kl}$ is $S^{(h)}_{kl}$ with ξ and υ (but not their indices) swapped, arising from the $(*)'$ of the (h) term in (4.23), and analogously for $S^{(o')}_{kl}$. So far, no approximation has been made, nor has the finite-mode source assumption been used.

We now invoke the assumption that $\gamma_j = 0$ whenever $|j| \ge \kappa_g$. Since $|k| \gg \kappa_g$, only one of γ_i and γ_{i-2k} can be non-zero, so the factor $\gamma_i\gamma_{i-2k}$ in $S^{(f)}_{kl}$ vanishes, killing the term; obviously $S^{(f')}_{kl} = 0$ as well.

Next, we treat the contribution of $S^{(d)}_{kl}$: due to the terms $|\gamma_i|^2|\gamma_{i-q}|^2$, we must have $|q| < 2\kappa_g$ for the terms containing it to be non-zero. Rewriting the l-sum over $q = k - l$ and using the fact that $|k| \gg \kappa_g$, we approximate $|k-q| \simeq |k| \simeq |k-i|$ and bound $|\xi_{ki}| = |e_{k-i}\cdot i| \le |i|$ to get

$$\begin{aligned}
\textstyle\sum_{kl} |k|^{-4}|l|^{-4}\, S^{(d)}_{kl} &= \textstyle\sum_{kq} |k|^{-4}|k-q|^{-4}\sum'_i |k-i|^{4\beta}\xi_{ki}^2\xi_{k-q,i-q}^2\, |\gamma_i|^2|\gamma_{i-q}|^2 \\
&\lesssim \textstyle\sum_k |k|^{4\beta-8}\sum'_{iq} |i|^2|i-q|^2|\gamma_i|^2|\gamma_{i-q}|^2 \\
&= \textstyle\sum_k |k|^{4\beta-8}\sum'_i |i|^2|\gamma_i|^2\sum'_q|i-q|^2|\gamma_{i-q}|^2 \\
&= \Bigl(\textstyle\sum_k |k|^{4\beta-8}\Bigr)\Bigl(\sum'_i |i|^2|\gamma_i|^2\Bigr)^2 \\
&= \frac{\omega_3}{(2\pi)^6} i_2(4\beta+d-8)\kappa^{4\beta+d-8}\|\nabla^{-1}g\|^4_{L^2}.
\end{aligned}\tag{4.36}$$

where for the penultimate equality we have changed the last $\sum_q$ to go over $n = i - q$ and re-labelled. Since we can bound $|\upsilon_{ki}| = |f_{k-i}\cdot i| \le |i|$ as with $|\xi_{ki}|$, this bound also holds for the contribution of $S^{(d')}_{kl}$. An analogous argument gives us the bound

$$\textstyle\sum_{kl} |k|^{-4}|l|^{-4}\bigl(S^{(e)}_{kl} + S^{(e')}_{kl}\bigr) \lesssim \dfrac{2\omega_3}{(2\pi)^6} i_2(4\beta+d-8)\kappa^{4\beta+d-8}\|\nabla^{-1}g\|^4_{L^2}.\tag{4.37}$$

We bound the contribution of $S^{(b)}_{kl}$ as follows:

$$\begin{aligned}
&\textstyle\sum_{kl} |k|^{-4}|l|^{-4}\sum'_{ij} |k-i|^{2\beta}|l-j|^{2\beta}\xi_{ki}\xi_{k,j+q}\xi_{lj}\xi_{l,i-q}\,\gamma_i\gamma_j\overline{\gamma_{i-q}\gamma_{j+q}} \\
&\quad= \textstyle\sum_{kq} |k|^{-4}|k-q|^{-4}\sum'_{ij} |k-i|^{2\beta}|k-q-j|^{2\beta}\cdots \\
&\quad\lesssim \textstyle\sum_k |k|^{4\beta-8}\sum'_{ij} |i|\,|\gamma_i|\,|j|\,|\gamma_j|\sum'_q|i-q|\,|\gamma_{i-q}|\,|j+q|\,|\gamma_{j+q}| \\
&\quad\le \tfrac12 \textstyle\sum_k |k|^{4\beta-8}\sum'_{ij} |i|\,|\gamma_i|\,|j|\,|\gamma_j|\sum'_q\bigl(|i-q|^2|\gamma_{i-q}|^2 + |j+q|^2|\gamma_{j+q}|^2\bigr) \\
&\quad= \textstyle\sum_k |k|^{4\beta-8}\sum'_i |i|\,|\gamma_i|\sum'_j |j|\,|\gamma_j|\sum'_n |n|^2|\gamma_n|^2 \\
&\quad= \frac{\omega_3}{(2\pi)^3} i_2(4\beta+d-8)\kappa^{4\beta+d-8}\,\|\nabla^{-1}g\|^2_{l_1}\|\nabla^{-1}g\|^2_{L^2}\,.
\end{aligned}\tag{4.38}$$

Obviously this bound also applied to the contribution of $S^{(b')}_{kl}$, and by inspection, also to those of $S^{(o)}_{kl}$ and $S^{(o')}_{kl}$. Similarly, we bound

$$\begin{aligned}
&\textstyle\sum_{kl} |k|^{-4}|l|^{-4}\sum'_{ij} |k-i|^{2\beta}|l-j|^{2\beta}\xi_{ki}\xi_{l,r-i}\xi_{lj}\xi_{k,r-j}\,\gamma_i\overline{\gamma_{i-r}\gamma_j}\gamma_{j-r} \\
&\quad= \textstyle\sum_{kr} |k|^{-4}|r-k|^{-4}\sum'_{ij} |k-i|^{2\beta}|r-k-j|^{2\beta}\cdots \\
&\quad\lesssim \textstyle\sum_k |k|^{4\beta-8}\sum'_{ij} |i|\,|\gamma_i|\,|j|\,|\gamma_j|\sum'_r|i-r|\,|\gamma_{i-r}|\,|j-r|\,|\gamma_{j-r}| \\
&\quad\le \tfrac12 \textstyle\sum_k |k|^{4\beta-8}\sum'_{ij} |i|\,|\gamma_i|\,|j|\,|\gamma_j|\sum'_r\bigl(|i-r|^2|\gamma_{i-r}|^2 + |j-r|^2|\gamma_{j-r}|^2\bigr) \\
&\quad= \textstyle\sum_k |k|^{4\beta-8}\sum'_i |i|\,|\gamma_i|\sum'_j |j|\,|\gamma_j|\sum'_n |n|^2|\gamma_n|^2 \\
&\quad= \frac{\omega_3}{(2\pi)^3} i_2(4\beta+d-8)\kappa^{4\beta+d-8}\,\|\nabla^{-1}g\|^2_{l_1}\|\nabla^{-1}g\|^2_{L^2}\,,
\end{aligned}\tag{4.39}$$

with the same bound applying for $S^{(c')}_{kl}$, $S^{(h)}_{kl}$ and $S^{(h')}_{kl}$. Putting everything together gives

$$\mathsf{var}\|\mathsf{P}_{\kappa,2\kappa}\vartheta\|^2_{L^2} \lesssim U^4\frac{\omega_3}{(2\pi)^3}i_2(4\beta{-}5)\kappa^{4\beta-5}\|\nabla^{-1}g\|^2_{L^2}\{8\|\nabla^{-1}g\|^2_{l_1}+\frac{4}{(2\pi)^3}(\varsigma{-}1)\|\nabla^{-} \tag{4.40}$$

whence follows (3.7).

4.3. **Bounding $\theta^{(\infty)}-\theta^{(1)}$.** Finally, we bound the remainder $\theta^{(\infty)}-\theta^{(1)}$ in each dyad. As before, we write $d=3$ and $\omega_d=4\pi$ to make it easier to adapt the proof to two dimensions. We start by obtaining a bound for $|\vartheta_k|$. From (4.1), we have

$$|\vartheta_k| \le 2U\Xi|k|^{-2}{\textstyle\sum'_j}\,|k-j|^\beta|j|\,|\gamma_j| =: 2U\Xi|k|^{-2}S_k. \tag{4.41}$$

When $|k|<2\kappa_g$, we have using (2.5) and bounding $|k-j|^\beta\le 1$,

$$S_k \le {\textstyle\sum'_j}|j|\,|\gamma_j| = \|\nabla^{-1}g\|_{l_1}. \tag{4.42}$$

For the case $|k|\ge 2\kappa_g$, we split the sum into four parts [cf. (4.3)]

$$S_k = {\textstyle\sum'_{|j|<\kappa_g}}+{\textstyle\sum_{\kappa_g\le|j|<|k|^r}}+{\textstyle\sum_{|k|^r\le|j|<2|k|}}+{\textstyle\sum_{2|k|\le|j|}} =: S^g_k+S^{\ll}_k+S^{\simeq}_k+S^{>}_k. \tag{4.43}$$

For S^g_k, since $|j|<\kappa_g$ and $|k|\ge 2\kappa_g$ we have $|k-j|\ge|k|-|j|\ge\frac12|k|$ and thus $|k-j|^\beta\le 2^{-\beta}|k|^\beta$; this gives

$$S^g_k \le 2^{-\beta}|k|^\beta{\textstyle\sum'_{|j|<\kappa_g}}\,|j|\,|\gamma_j| \le 2^{-\beta}|k|^\beta\|\nabla^{-1}g\|_{l_1}. \tag{4.44}$$

Similarly for $S^{\ll}_k$, since $|j|<|k|^r\le\frac12|k|$ (the latter holds since $2\le\kappa_g^{1-r}$), we again have $|k-j|\ge|k|-|j|\ge\frac12|k|$ and, since $\alpha+1+d<0$,

$$S^{\ll}_k \le 2^{-\beta}|k|^\beta c_g{\textstyle\sum_{\kappa_g\le|j|<|k|^r}}\,|j|^{\alpha+1} \le 2^{-\beta}|k|^\beta c_g\omega_d\kappa_g^{\alpha+1+d}/|\alpha+1+d|. \tag{4.45}$$

For $S^{\simeq}_k$, we use $|j|<2|k|$ to bound $|k-j|\le|k|+|j|<3|k|$ and change the sum over j to (a larger) one over $m=k-j$; using $|j|\ge|k|^r$ to bound $|j|^{\alpha+1}\le|k|^{(\alpha+1)r}$, we then get (assuming $\beta+d\ne0$, a harmless special case)

$$S^{\simeq}_k \le c_g|k|^{(\alpha+1)r}\sum_{1\le|m|<3|k|}|m|^\beta \le c_g\omega_d|k|^{(\alpha+1)r}\frac{(3|k|)^{\beta+d}-1}{\beta+d}. \tag{4.46}$$

If $\beta+d>0$, the fraction is bounded by $(3|k|)^{\beta+d}/(\beta+d)$, and for the rhs to be $\mathsf{O}(|k|^\beta)$, we need $(\alpha+1)r+d\le0$. If $\beta+d<0$, the fraction is bounded by $1/|\beta+d|$, and for the rhs to be $\mathsf{O}(|k|^\beta)$, we need $(\alpha+1)r\le\beta$. Either way, we need $(\alpha+1)r\le\min\{\beta,-d\}$. For $S^{>}_k$, we use $|j|\ge2|k|$ to bound $|j-k|\ge|j|-|k|\ge\frac12|j|$ and $|k-j|^\beta\le2^{-\beta}|j|^\beta$, to get

$$S^{>}_k \le 2^{-\beta}c_g{\textstyle\sum_{2|k|\le|j|}}\,|j|^{\alpha+\beta+1} \le 2^{-\beta}c_g\omega_d\,(2|k|)^{\alpha+\beta+d+1}/|\alpha+\beta+d+1| \tag{4.47}$$

since $\alpha+\beta+d+1<0$; for the rhs to be $\mathsf{O}(|k|^\beta)$, we need $\alpha+d+1\le0$. Putting these together, we can write

$$|\vartheta_k| \le U\Xi c_1(g,\beta,d)|k|^{-2}K_\beta(|k|) \tag{4.48}$$

where

$$K_\beta(|k|) := \min\{1,(2\kappa_g)^{-\beta}|k|^\beta\} \tag{4.49}$$

is continuous non-increasing for all $|k|\ge0$ and monotone decreasing for $|k|\ge2\kappa_g$. Summing over k, this gives $\|\mathsf{P}_{\kappa,2\kappa}\vartheta\|^2_{L^2}\le c\,\kappa^{2\beta-4+d}U^2\Xi^4\|\nabla^{-1}g\|^2_{l_1}$ for large κ, i.e. the same $\kappa^{2\beta-1}$ dependence as $\mathsf{E}\|\mathsf{P}_{\kappa,2\kappa}\vartheta\|_{L^2}^2$ albeit with a worse constant and dependence on $\|\nabla^{-1}g\|_{l_1}^2$ instead of $\|\nabla^{-1}g\|_{L^2}^2$.

Writing $\delta\theta^{(n)} := \theta^{(n)} - \theta^{(n-1)}$ with $\delta\theta^{(1)} = \vartheta$, we have from (3.3)

$$\begin{aligned} &\delta\theta^{(n+1)} = \Delta^{-1}(u\cdot\nabla\delta\theta^{(n)}) \\ \Rightarrow\quad &|\delta\theta_k^{(n+1)}| \le 2U\Xi\,|k|^{-2}{\textstyle\sum_j'}\,|k-j|^\beta|j|\,|\delta\theta_j^{(n)}|. \end{aligned}\tag{4.50}$$

Therefore, since $\delta\theta^{(1)} = \vartheta$ is already bounded in (4.48), if we can show that

$$S_k := {\textstyle\sum_j'}\,|k-j|^\beta|j|^{-1}K_\beta(|j|) \le M_1(\beta,\kappa_g)K_\beta(|k|),\tag{4.51}$$

we can, by choosing

$$4U\Xi M_1(\beta,\kappa_g) =: U/U_{max}(\beta,\kappa_g,\Xi) =: \varepsilon < 1,\tag{4.52}$$

ensure the mode-wise convergence

$$\begin{aligned} |\theta_k^{(1)} - \theta_k^{(\infty)}| &= |\delta\theta_k^{(2)} + \delta\theta_k^{(3)} + \cdots| \le |\delta\theta_k^{(2)}| + |\delta\theta_k^{(3)}| + \cdots \\ &\le \frac{c_1\varepsilon}{2}|k|^{-2}K_\beta(|k|) + \frac{c_1\varepsilon^2}{4}|k|^{-2}K_\beta(|k|) + \cdots \le c_1\varepsilon\,|k|^{-2}K_\beta(|k|). \end{aligned}\tag{4.53}$$

To show (4.51), we first consider $|k| < 4\kappa_g$. We split the sum as

$${\textstyle\sum_j'}\,|k-j|^\beta|j|^{-1}K_\beta(|j|) = {\textstyle\sum_{|j|<8\kappa_g}'} + {\textstyle\sum_{8\kappa_g\le|j|}} =: S_k^g + S_k^>.\tag{4.54}$$

For S_k^g, we first use $|k-j|^\beta K_\beta(|j|) \le 1$ to get

$$S_k^g \le {\textstyle\sum_{|j|<8\kappa_g}'}\,|j|^{-1} \le \omega_d(8\kappa_g)^{d-1}/(d-1) = 32\,\omega_d\kappa_g^2.\tag{4.55}$$

As for $S_k^>$, $|k| < 4\kappa_g$ and $|j| \ge 8\kappa_g$ implies that $|j-k| \ge |j| - |k| \ge \frac12|j|$ and thus $|k-j|^\beta \le 2^{-\beta}|j|^\beta$, leading to

$$\begin{aligned} S_k^> &\le 2^{-\beta}{\textstyle\sum_{8\kappa_g\le|j|}}\,|j|^{\beta-1} \\ &\le 2^{-\beta}\omega_d(8\kappa_g)^{2\beta+d-1}/|2\beta+d-1| = 2^{5\beta+5}\kappa_g^{2\beta+2}\omega_d/|\beta+1| \end{aligned}\tag{4.56}$$

since $2\beta + d - 1 < 0$. Since $K_\beta(\cdot)$ is non-increasing, $K_\beta(s) \ge K_\beta(4\kappa_g) = 2^\beta$ for $s \le 4\kappa_g$, giving

$$\begin{aligned} S_k = S_k^g + S_k^> &\le 2\omega_d\max\bigl\{32\kappa_g^2, 2^{5\beta+5}\kappa_g^{2\beta+2}/|\beta+1|\bigr\} \\ &\le 2^{1-\beta}\omega_d\max\bigl\{32\kappa_g^2, 2^{5\beta+5}\kappa_g^{2\beta+2}/|\beta+1|\bigr\}\,K_\beta(|k|)\quad\text{for all } |k| < 4\kappa_g. \end{aligned}\tag{4.57}$$

For the case $|k| \ge 4\kappa_g$, we split the sum four ways

$$\begin{aligned} S_k &= {\textstyle\sum_{|j|<2\kappa_g}'} + {\textstyle\sum_{2\kappa_g\le|j|<\frac12|k|}} + {\textstyle\sum_{\frac12|k|\le|j|<2|k|}} + {\textstyle\sum_{2|k|\le|j|}} \\ &=: S_k^1 + S_k^< + S_k^\simeq + S_k^>. \end{aligned}\tag{4.58}$$

For S_k^1, since $|j| < 2\kappa_g$ and $|k| \ge 4\kappa_g$, we have $|k-j| \ge |k| - |j| \ge \frac12|k|$ and $|k-j|^\beta \le 2^{-\beta}|k|^\beta$, giving

$$S_k^1 \le 2^{-\beta}|k|^\beta{\textstyle\sum_{|j|<2\kappa_g}'}|j|^{-1} \le 2^{-\beta}|k|^\beta\omega_d(2\kappa_g)^{d-1}/|d-1|.\tag{4.59}$$

Next, for $S_k^<$, since $|j| < \frac12|k|$ and thus $|k-j| \ge |k| - |j| \ge \frac12|k|$, we have

$$S_k^< \le (4\kappa_g)^{-\beta}|k|^\beta{\textstyle\sum_{2\kappa_g\le|j|}}\,|j|^{\beta-1} \le (4\kappa_g)^{-\beta}|k|^\beta\omega_d(2\kappa_g)^{\beta+d-1}/|\beta+d-1|\tag{4.60}$$

assuming that $\beta+d-1<0$. For $S_k^{\simeq}$, we change the summation variable from j to $m=k-j$ as in (4.46) and use $|j|\ge\frac12|k|$ and thus $|j|^{\beta-1}\le 2^{1-\beta}|k|^{\beta-1}$, to obtain (again assuming $\beta+d\neq 0$)

$$S_k^{\simeq}\le 2\,(4\kappa_g)^{-\beta}|k|^{\beta-1}{\sum}'_{|m|<3|k|}|m|^\beta\le 2\,(4\kappa_g)^{-\beta}|k|^{\beta-1}\omega_d\frac{(3|k|)^{\beta+d}-1}{\beta+d}. \tag{4.61}$$

If $\beta+d>0$, the rhs is $\mathsf{O}(|k|^\beta)$ if $\beta+d-1\le 0$. Finally, for $S_k^{>}$, we use $|j|\ge 2|k|$ to bound $|j-k|\ge|j|-|k|\ge\frac12|j|$ and $|k-j|^\beta\le 2^{-\beta}|j|^\beta$; this gives us

$$S_k^{>}\le(4\kappa_g)^{-\beta}{\sum}_{2|k|\le|j|}|j|^{2\beta-1}\le(4\kappa_g)^{-\beta}\omega_d(2|k|)^{2\beta+d-1}/|2\beta+d-1|. \tag{4.62}$$

For the rhs to be $\mathsf{O}(|k|^\beta)$, we need $\beta+d-1\le 0$. We have thus established (4.51) subject to the following assumptions:

$$4\le\kappa_g^{1-r}, \tag{4.63}$$

$$(\alpha+1)r\le\min\{\beta,-d\}\qquad\qquad\text{(obviating }\alpha+d+1<0), \tag{4.64}$$

$$\beta+d-1<0. \tag{4.65}$$

After extracting $K_\beta(|k|)$, and dropping terms <1. we have

$$\begin{aligned}M_1(\beta,\kappa_g):=\omega_d\,\max\bigl\{&2^{1-\beta}32\kappa_g^2+1/|\beta+1|\;,\;2^{d-1}/|d-1|\\&+2^{-\beta}/|\beta+d-1|+2^{1-\beta}3^{\beta+d}/|\beta+d|+1/|2\beta+d-1|\bigr\}\end{aligned} \tag{4.66}$$

$$\Rightarrow\quad U_{max}(\beta,\kappa_g,\Xi):=1/(4\Xi M_1). \tag{4.67}$$

Summing (4.53), we have

$$\|\mathsf{P}_{\kappa,2\kappa}\delta\theta\|_{L^2}^2\le 8\pi^3\varepsilon^2c_1^2\sum_{\kappa\le|k|<2\kappa}|k|^{-4}K_\beta^2(|k|)^2\le c_2(\cdots)|k|^{2\beta+d-4} \tag{4.68}$$

with the second inequality valid since $|k|\ge 2\kappa_g$. □

Proof of Theorem 2. The proof of the theorem is similar to that of the 2d case [14], with some improvements (e.g., requiring less regularity on u). As in the proof of Theorem 1, we shall often write $d=3$ to help possible adaptation to $d=2$.

Thanks to the boundedness of V_k and W_k, we have from (2.12)

$$\begin{aligned}\|u(\cdot,t)\|_{H^{1/2}}^2&\le 8\pi^3{\sum}'_k|k|^{2\beta+1}(U_e^2|V_k(t)|^2+U_f^2|W_k(t)|^2)\\&\le 8\pi^3U^2\Xi^2{\sum}'_k|k|^{2\beta+1}\le 8\pi^3U^2\Xi^2/|2\beta+d+1|\end{aligned} \tag{4.69}$$

for all t, assuming that $2\beta+d+1<0$.

Considering the iteration (3.9) as a mapping $T:\theta^{(n)}\mapsto\theta^{(n+1)}$, convergence of the iterations (3.8)–(3.9) would follow from the contractivity of T. To prove the latter, we write $\delta\theta^{(n)}:=\theta^{(n)}-\theta^{(n-1)}$ and observe that it satisfies

$$(\partial_t-\Delta)\delta\theta^{(n)}=-(u\cdot\nabla)\delta\theta^{(n-1)}\quad\text{with}\quad\delta\theta^{(n)}(\cdot,0)=0. \tag{4.70}$$

Multiplying this by $\delta\theta^{(n)}$ in $L^2(D)$, we find

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\delta\theta^{(n)}\|_{L^2}^2+\|\nabla\delta\theta^{(n)}\|_{L^2}^2=-((u\cdot\nabla)\delta\theta^{(n-1)}),\delta\theta^{(n)})_{L^2}. \tag{4.71}$$

We next bound the contribution from the advected term as

$$\begin{aligned}\bigl|(u\cdot\nabla\delta\theta^{(n-1)},\delta\theta^{(n)})_{L^2}\bigr|&\le\|u\|_{L^3}\|\nabla\delta\theta^{(n-1)}\|_{L^2}\|\delta\theta^{(n)}\|_{L^6}\\&\le c\,\|u\|_{H^{1/2}}\|\nabla\delta\theta^{(n-1)}\|_{L^2}\|\nabla\delta\theta^{(n)}\|_{L^2}\\&\le\tfrac12\|\nabla\delta\theta^{(n)}\|_{L^2}^2+c\,\|u\|_{H^{1/2}}^2\|\nabla\delta\theta^{(n-1)}\|_{L^2}^2\,.\end{aligned} \tag{4.72}$$

Integrating (4.71) in time, we find

$$\begin{aligned}
&\|\delta\theta^{(n)}(t)\|_{L^2}^2 + \int_0^t \|\nabla\delta\theta^{(n)}(s)\|_{L^2}^2 \,\mathrm{d}s \\
&\qquad\le c_3\,\|u\|^2_{L^\infty([0,t],H^{1/2}(D))} \int_0^t \|\delta\theta^{(n-1)}(s)\|_{L^2}^2 \,\mathrm{d}s \\
&\qquad\le c_3\,\|u\|^2_{L^\infty([0,t],H^{1/2}(D))} \int_0^t \|\nabla\delta\theta^{(n-1)}(s)\|_{L^2}^2 \,\mathrm{d}s,
\end{aligned}\tag{4.73}$$

so (pathwise) convergence of $\theta^{(n)}$ in $L^2([0,t],H^1(D))$ would follow from

$$c_3\,\|u\|^2_{L^\infty(0,\infty;H^{1/2}(D))} = 8\pi^3 c_3 U^2\Xi^2/|2\beta+d+1| < 1.\tag{4.74}$$

We now turn our attention to ϑ, given by

$$\vartheta(t) = \theta^{(1)}(t) + \Delta^{-1}g = \int_0^t \mathrm{e}^{(t-s)\Delta}u(s)\cdot\nabla\Delta^{-1}g\,\mathrm{d}s,\tag{4.75}$$

and whose Fourier coefficients satisfy (taking the general *deterministic* g)

$$\begin{aligned}
\vartheta_k(t) &= \int_0^t \mathrm{e}^{(s-t)|k|^2} {\textstyle\sum_j'}\, |k-j|^\beta [\xi_{kj}V_{k-j}(s) + \upsilon_{kj}W_{k-j}(s)]\gamma_j\,\mathrm{d}s \\
&= {\textstyle\sum_j'}\, |k-j|^\beta\gamma_j \int_0^t \mathrm{e}^{(s-t)|k|^2}[U_e\xi_{kj}V_{k-j}(s) + U_f\upsilon_{kj}W_{k-j}(s)]\,\mathrm{d}s
\end{aligned}\tag{4.76}$$

where, as in the proof of Theorem 1, $\xi_{kj} := e_{k-j}\cdot j$ and $\upsilon_{kj} := f_{k-j}\cdot j$. Since $\mathsf{E}V_j(s)\overline{W_k(r)} = 0$, for clarity we put $U_f = 0$ temporarily, restoring it in (4.80). We compute

$$\mathsf{E}\vartheta_k(t)\overline{\vartheta_k(t)} = U_e^2 {\textstyle\sum_{ij}'}\, |k-i|^\beta|k-j|^\beta\gamma_j\overline{\gamma_i}\,\mathsf{E}\int_0^t \{\cdots\}_j\,\mathrm{d}s\int_0^t \overline{\{\cdots\}_i}\,\mathrm{d}r.\tag{4.77}$$

Now

$$\begin{aligned}
\mathsf{E}\int_0^t \{\cdots\}_j\,\mathrm{d}s\int_0^t \overline{\{\cdots\}_i}\,\mathrm{d}r &= \int_0^t\!\!\int_0^t \mathrm{e}^{(s+r-2t)|k|^2}\,\mathsf{E}V_{k-j}(s)\overline{V_{k-i}(r)}\,\mathrm{d}r\,\mathrm{d}s \\
&= \int_0^t\!\!\int_0^t \mathrm{e}^{(s+r-2t)|k|^2}\,\Phi_{k-j}(s-r)\delta_{ij}\,\mathrm{d}r\,\mathrm{d}s.
\end{aligned}\tag{4.78}$$

As in the static case, the sum over i,j then collapses to one over j:

$$\mathsf{E}|\vartheta_k(t)|^2 = U_e^2 {\textstyle\sum_j'} |k-j|^{2\beta}\xi_{kj}^2|\gamma_j|^2 \int_0^t\!\!\int_0^t \mathrm{e}^{(s+r-2t)|k|^2}\,\Phi_{k-j}(s-r)\,\mathrm{d}r\,\mathrm{d}s.\tag{4.79}$$

We note that, except for the time integrals, the sum is exactly that in (4.2), having *not* assumed stochastic g. Restoring U_f, we have

$$\mathsf{E}|\vartheta_k(t)|^2 = {\textstyle\sum_j'}|k-j|^{2\beta}(U_e^2\xi_{kj}^2 + U_f^2\upsilon_{kj}^2)|\gamma_j|^2 \int_0^t\!\!\int_0^t \mathrm{e}^{(s+r-2t)|k|^2}\,\Phi_{k-j}(s-r)\,\mathrm{d}r\,\mathrm{d}s.\tag{4.80}$$

To handle the integrals, we recall the following result proved in [14, (3.20)–(3.26)]: with Φ as in (2.17),

$$\begin{aligned}
&|k|^4 \lim_{t\to\infty}\int_0^t\int_0^t \mathrm{e}^{(s+r-2t)|k|^2}\,\Phi(\chi|s-r|)\,\mathrm{d}s\,\mathrm{d}r \\
&\qquad= 1 + \frac{\chi}{|k|^2}\Phi'(0) + \cdots + \frac{\chi^{n-1}}{|k|^{2n}}\int_0^\infty \mathrm{e}^{-s|k|^2/\chi}\Phi^{(n)}(s)\,\mathrm{d}s.
\end{aligned}\tag{4.81}$$

We note that there is a spurious factor of $\frac{1}{2}$ in the last line of (3.24) in [14], but (3.25) which we used here is correct.

Using this in (4.80), we thus have

$$\lim_{t\to\infty} \mathsf{E}|\vartheta_k(t)|^2 = |k|^{-4}\textstyle\sum_j' |k-j|^{2\beta}|\gamma_j|^2(U_e^2\xi_{kj}^2+U_f^2\upsilon_{kj}^2)\{1+\chi_{k-j}\Phi'(0)/|k|^2+\cdots\}. \tag{4.82}$$

The first term (the 1) of the bracket, being exactly (4.2), recovers the static case. The higher-order terms, as in (4.81), give smaller remainders. □

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